

Part A(Short Answer Type Questions)

Answer *all* the questions

(Each Questions has weightage *one*)

Time: 3Hours

Max. Weightage: 36

1. Define Hamming weight $wt(u)$ of a word u and distance $d(u, v)$ between two binary words u and v in B^n . Let $u = 1101010111$ and $v = 0111001110$. Find (i) $wt(u)$, (ii) $wt(v)$, (iii) $d(u, v)$.
2. Find the order of the element $(3, 6, 12, 16)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20} \times \mathbb{Z}_{24}$.
3. Find all abelian groups, upto isomorphism of order 1089.
4. Let $\phi : \mathbb{Z}_{18} \rightarrow \mathbb{Z}_{12}$ be the homomorphism such that $\phi(1) = 10$. Find the kernel K of ϕ . Write the cosets of \mathbb{Z}_{18}/K and list the elements in each coset.
5. Show that if a finite group G contains a proper subgroup of index 2 in G , then G is not simple.
6. Find an isomorphic refinement of the series

$$\{0\} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z} \quad \text{and}$$

$$\{0\} < 9\mathbb{Z} < \mathbb{Z}.$$
7. Show that \mathbb{Z} has no composition series.
8. Show that S_3 is solvable.
9. Show that if H and K are normal subgroups of a group G such that $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H$ and $k \in K$.
10. Show that no group of order 20 is simple.
11. Compute $(i + 3j)(4 + 2j - k)$ in the Quaternions \mathbb{Q} .
12. If F is a field and $a \neq 0$ is a zero of $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ in $F[x]$, show that $\frac{1}{a}$ is a zero of $a_n + a_{n-1}x + a_{n-2}x^2 + \dots + a_0x^n$.
13. Factorize $x^4 + 4$ in $\mathbb{Z}_5[x]$.
14. Show that the ring $M_2(\mathbb{R})$ of all 2×2 matrices with entries in \mathbb{R} , the field of real numbers, is not isomorphic to the field \mathbb{C} of complex numbers.

Part B (Paragraph Type Questions)

Answer any Seven questions

(Each question has weightage *two*)

15. Show that B^n with the operation of word addition is a group.
16. Show that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to \mathbb{Z}_{mn} if and only if m and n are relatively prime.

17. Prove that M is a maximal normal subgroup of a group G if and only if G/M is simple.
18. Compute the factor group $\mathbb{Z}_4 \times \mathbb{Z}_6 / \langle (2, 3) \rangle$.
19. Find the number of distinguishable ways the edges of an equilateral triangle can be painted if four different colours of paint are available, assuming only one colour is used on each edge and the same colour may be used on different edges.
20. Obtain the class equation of D_4 .
21. Show that for a prime number p , every group G of order p^2 is abelian.
22. State and prove second isomorphism theorem.
23. Define group presentation. Write a presentation of \mathbb{Z}_6 .
24. Show that the multiplicative group of all non-zero elements of a finite field is cyclic.

Part C (Essay Type Questions)

Answer Any Two Questions

(Each Question has weightage Four)

25. (a) Let X be a G -set and let $x \in X$. Show that $|Gx| = (G : G_x)$.
- (b) Find the number of orbits in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ under the subgroup of S_8 generated by $\langle (1, 3, 5, 6) \rangle$.
26. (a) State and prove the third Sylow theorem.
- (b) Show that a group G of order 30 is not simple.
27. (a) State and prove Eisenstein condition for irreducibility.
- (b) Show that $x^4 - 2x^2 + 8x + 1$ is irreducible over \mathbb{Q} .
28. (a) Give the addition and multiplication tables for the group algebra $\mathbb{Z}_2(G)$, where $G = \{e, a\}$ is a cyclic group of order 2.
- (b) Let H be a subring of the ring R . Prove that multiplication of additive cosets of H is well defined by the equation $(a + H)(b + H) = ab + H$ if and only if $ah \in H$ and $hb \in H$ for all $a, b \in R$ and $h \in H$.