D 52400 **(Pages : 3)** **Name...................................**

Reg. No................................

THIRD SEMESTER P.G. (CCSS) DEGREE EXAMINATION, NOVEMBER 2023

Mathematics

MAT 3 14—COMPLEX ANALYSIS

(2022 Admission onwards)

Time : Three Hours Maximum Marks : 50

Part A

Answer **all** *questions.*
Each question carries 1 mark.

1. Prove that an absolutely convergent series is convergent.
2. Evaluate the cross ratio $(7 + i, 1, 0, \infty)$.
3. Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation. If P and Q are partitions of $[a, b]$ and if $P \subset Q$, then prove that $v(\gamma;P) \leq v(\gamma;Q)$.
4. Prove that every bounded entire function is constant.
5. Prove that every convex set is star shaped.
6. What is meant by a simply connected set ?
7. Give the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ in ann(0; 0, 1).
8. If $|a| < 1$, then prove that Möbius transformation ϕ_a defined by :

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}$$

is a one-one map of $D = \{z : |z| < 1\}$ onto itself.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 marks.

9. For the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, let the number R, $0 \leq R \leq \infty$ be defined by $\dfrac{1}{R} = \limsup |a_n|^{1/n}$. Prove that if $|z-a| < R$, then the series converges absolutely.

Turn over

10. Let $f(z) = \sum a_n (z-a)^n$ have radius of convergence $R > 0$. Prove that the function f is infinitely differentiable on $B(a; R)$.

11. Prove that the cross ratio is invariant under any Möbius transformation.

12. If γ is piecewise smooth and $f : [a,b] \to \mathbb{C}$ is continuous then prove that

$$\int_a^b f d\gamma = \int_a^b f(t)\gamma'(t)(dt).$$

13. Prove that $\int_0^{2\pi} \frac{e^{is}}{e^{is} - z} ds = 2\pi$ for $|z| < 1$.

14. If γ_0 and γ_1 are two rectifiable curves in G from a to b and γ_0, γ_1 are FEP homotopic, then prove that $\int_{\gamma_0} f = \int_{\gamma_1} f$ for any function f analytic in G.

15. Let G be a region and suppose that f is a nonconstant analytic function on G. For every open set U in G, prove that f(U) is open.

16. Let $z = a$ be an isolated singularity of f, which is analytic in the annulus ann $(a; 0, R)$ and let $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ be its Laurent expansion in the punctured disk. Prove that $z = a$ is a removable singularity if and only if $a_n = 0$ for $n \leq -1$.

17. Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose there is a constant M such that

$$\limsup_{z \to a} |f(z)| \leq M$$

for all a in $\partial_\infty G$. Prove that $|f(z)| \leq M$ for all z in G.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 marks.

18. (a) Briefly describe the extended plane and its spherical representation.

(b) Let u and v be real-valued functions defined on a region G and suppose that u and v have continuous partial derivatives such that u and v satisfy the Cauchy-Riemann equations. Prove that $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic.

19. (a) Define a conformal map. If f is analytic and $f'(z) \neq 0$ for any z, then prove that f is conformal.

(b) If $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth, then prove that γ is of bounded variation and $V(\gamma) = \int_a^b |\gamma'(t)|\, dt.$

20. (a) If f and g are analytic on a region G, then prove that $f \equiv g$ if and only if $\{z \in G : f(z) = g(z)\}$ has a limit point in G.

(b) Let γ be a closed rectifiable curve in $\mathbb{C}$. Prove that $n(\gamma : a)$ is a constant for a belonging to a component of $\mathbb{C} - \{\gamma\}$.

21. (a) Let f be analytic on B(0; 1) and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show that $|f'(0)| \leq 1$.

(b) If G is simple connected and $f : G \to \mathbb{C}$ is analytic in G, then prove that f has a primitive in G.

22. (a) Suppose f and g are meromorphic in a neighbourhood of $\overline{B}(a; R)$ with no zeroes or poles on the circle $\gamma = \{z : |z - a| = R\}$. If Z_f, Z_g are the number of zeroes of P_f, P_g are the number of poles of f and g respectively inside γ counted according to their multiplicities and if $|f(z) + g(z)| < |f(z)| + |g(z)|$ on γ, then prove that $Z_f - P_f = Z_g - P_g$.

(b) State and prove Schwarz's Lemma.

(3 × 8 = 24 marks)

D 52397 **(Pages : 3)** **Name..................................**

Reg. No...............................

THIRD SEMESTER P.G. (CCSS) DEGREE EXAMINATION, NOVEMBER 2023

Mathematics

MAT 3C 11—COMPLEX ANALYSIS

(2019—2021 Admissions)

Time : Three Hours Maximum Marks : 80

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Which subsets of S correspond to the real and imaginary axes in $\mathbb{C}$?

2. Show that the real part of the function $f(z) = z^{\frac{1}{2}}$ is always positive.

3. Show that if $\gamma : [a, b] \to \mathbb{R}$ is a Lipchitz function then γ is of bounded variation.

4. Define winding number.

5. Define fixed end point homotopic.

6. Evaluate $\int_\gamma \frac{dz}{z}$ where $\gamma(t) = 1 + 2t + it^2, t \in [0, 1]$.

7. Evaluate the residues of $\frac{\sin(3z)}{z}$ at singularity zero.

8. Determine the nature of singularity of $f(z) = \frac{z^2 + 1}{z(z-1)}$ at $z = 0$. If it is a removable singularity define $f(0)$ so that f is analytic at $z = 0$; if it is a pole find the singular part ; if it is an essential singularity determine $f(\{z : 0 < |z| < \delta\})$ for arbitrarily small values of δ.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence R > 0. Show that for each $k \geq 1$ the series $\sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)a_n(z-a)^{n-k}$ has radius of convergence R.

Turn over

10. Define Möbius map. Show that if S is a Möbius transformation then S is the composition of translations, dilations and the inversion.

11. Let γ and σ be the two polygons $[1, i]$ and $[1, 1 + i, i]$. Express γ and σ as paths and calculate $\int_\gamma f$ and $\int_\sigma$ where $f(z) = |z|^2$.

12. State and prove independence of path theorem.

13. State and prove Casorati-Weierstrass theorem.

14. Give the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ in the annuli : $ann(0; 2, \infty)$.

(4 × 4 = 16 marks)

Part C

Answer **all** *questions.*
Each question carries 12 marks.

15. (a) (i) Show that if G is open and connected and $f : G \mapsto \mathbb{C}$ is differentiable with $f'(z) = 0$ for all z in G, then f is constant.

(6 marks)

(ii) If $Tz = \dfrac{az+b}{cz+d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$ where Γ is the unit circle $\{z : |z| = 1\}$.

(6 marks)

Or

(b) (i) Discuss the Stereographic projection. (5 marks)

(ii) State and prove the angle preserving property of an analytic function. (7 marks)

16. (a) (i) Show that if γ is piecewise continuous and $f : [a, b] \mapsto \mathbb{C}$ is continuous then $\int_\gamma f\, d\gamma = \int_a^b f(t)\gamma'(t)\, dt$.

(4 marks)

(ii) Let G be open in $\mathbb{C}$ and let γ be a rectifiable path in G with initial and end points α and β respectively. Show that if $f : G \mapsto \mathbb{C}$ is a continuous function with a primitive $F : G \mapsto \mathbb{C}$, then $\int_\gamma = F(\beta) - F(\alpha)$.

(8 marks)

Or

(b) (i) Show that if γ be a closed rectifiable curve in $\mathbb{C}$ then $n(\gamma : a)$ is constant for a belonging to a component of $G = \mathbb{C} - \{\gamma\}$.

(4 marks)

(ii) Let γ_0 and γ_1 be two closed rectifiable curves in G suth that γ_0 is homotopic to γ_1. Show that $\int_{\gamma_0} f = \int_{\gamma_1} f$ for every function f analytic in G.

(8 marks)

17. (a) (i) Show that if G is simply connected and $f : G \mapsto \mathbb{C}$ is analytic in G then f has a primitive in G.

(6 marks)

(ii) State and prove Open mapping theorem. (6 marks)

Or

(b) (i) Show that G be an open set and $f : G \mapsto \mathbb{C}$ be a differentiable function then f is analytic on G.

(6 marks)

(ii) Let G be a region and let $f : G \mapsto \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path in G, prove that f is analytic in G.

(6 marks)

18. (a) (i) State and prove Rouche's theorem. (6 marks)

(ii) Show that $\int_0^\infty \frac{\log x}{1+x^2}\,dx = 0.$ (6 marks)

Or

(b) (i) Prove that one zero of $z^5 + 15z + 1$ belongs to $B\left(0, \frac{3}{2}\right)$ and four zeroes of $z^5 + 15z + 1$ belong to $ann\left(0; \frac{3}{2}, 2\right)$.

(6 marks)

(ii) State and prove Schwarz's lemma. (6 marks)

[4 × 12 = 48 marks]

D 52398 **(Pages : 3)** **Name..................................**

Reg. No...............................

THIRD SEMESTER P.G. (CCSS) DEGREE EXAMINATION, NOVEMBER 2023

Mathematics

MAT 3C 12—FUNCTIONAL ANALYSIS

(2019—2021 Admissions)

Time : Three Hours Maximum Marks : 80

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Give an example to show that in a noncomplete metric space, the intersection of a denumerable number of dense open subsets need not be dense.
2. Let m denote the Lebesgue measure on $\mathbb{R}$ and E be a measurable of $\mathbb{R}$. Give an example to show that if $m(\mathrm{E}) = \infty$, there may be no inclusion relation among the spaces $\mathrm{L}^p(\mathrm{E})$.
3. Let X be a linear space over $\mathbb{C}$. Regarding X as a linear space over $\mathbb{R}$, consider a real linear functional $u : \mathrm{X} \to \mathbb{R}$. Define $f(x) = u(x) - iu(ix), x \in \mathrm{X}$. Show that f is a complex linear functional on X.
4. Let X and Y be normed spaces. If X is infinite dimensional and $\mathrm{Y} \neq \{0\}$. Then show that there is a discontinuous linear map from X to Y.
5. Let X be a normed space and Y be a Banach space. Let X_0 be a dense subspace of X and $\mathrm{F}_0 \in \mathrm{BL}(\mathrm{X}_0, \mathrm{Y})$. Prove that there is a unique $\mathrm{F} \in \mathrm{BL}(\mathrm{X},\mathrm{Y})$ such that $\mathrm{F}_{|\mathrm{X}_0} = \mathrm{F}_0$.
6. State the uniform boundedness principle.
7. Give an example to show that the completeness condition in the domain of the map under consideration cannot be omitted in the open mapping theorem.
8. Let X and Y be Banach space. Let $\mathrm{F} \in \mathrm{BL}(\mathrm{X},\mathrm{Y})$ be bijective, then show that $\mathrm{F}^{-1} \in \mathrm{BL}(\mathrm{Y}, \mathrm{X})$.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Let X be a normed spaces, Y be a closed subspace of X and $\mathrm{Y} \neq \mathrm{X}$. Let r be a real number such that 0 < r < 1. Show that there exists some, $x_r \in \mathrm{X}$ such that $\|x_r\| = 1$ and $r \leq dist(x_r, \mathrm{Y}) \leq 1$.

Turn over

10. Consider a measurable subset E of $\mathbb{R}$. If E = $[a, b]$ and $1 \leq p < \infty$, then prove that the set of all step functions on E is dense $L^p([a,b])$.

11. Give an example to show that a linear map on a linear space X may be continuous with respect to some norm on X, but discontinuous with respect to another norm on X.

12. Let E be a nonempty convex subset of a normed space X over $\mathbb{K}$. Prove that if $a \in X$ but $a \notin \overline{E}$, then there are $f \in X'$ and $t \in \mathbb{R}$ such that $Re f(x) \leq t < Re f(a)$ for all $x \in \overline{E}$.

13. Prove that a Banach space cannot have a denumerable Hamel basis.

14. Let X be a Banach space in the norm $\| \ \|$. Then prove that a norm $\| \ \|'$ on the linear space X is equivalent to the norm $\| \ \|$ if and only if X is also a Banach space in the norm $\| \ \|'$ and the norm $\| \ \|'$ is comparable to the norm $\| \ \|$.

(4 × 4 = 16 marks)

Part C

Answer either A **or** B *of the following questions.*
Each question carries 12 *marks.*

Unit 1

15. A Prove that for $1 \leq p \leq \infty$, the metric space l^p complete.

B Let $\| \ \|$ and $\| \ \|'$ be norms on a linear space X. Then prove that the norm $\| \ \|$ is stronger than $\| \ \|'$ if and only if there is some $\alpha > 0$ such that $\|x\|' \leq \alpha \|x\|$ for all $x \in X$. Also, prove that $\| \ \|$ is equivalent to the norm $\| \ \|'$ if and only if there are $\alpha > 0$ and $\beta > 0$ such that $\beta \|x\| \leq \|x\|' \leq \alpha \|x\|$ for all $x \in X$. Give an example for equivalent norms.

Unit 2

16. A Let B(T) denote the set of scalar valued bounded functions on a set T and let X be a subspace of B(T) with the sup norm, $1 \in X$ and f be a linear functional on X. If f is continuous and $\|f\| = f(1)$, then prove that f is positive. Conversely, if $\mathrm{Re}\, x \in X$ whenever $x \in X$ and if f positive, then prove that f is continuous and $\|f\| = f(1)$.

B State and prove that Hahn Banach separation theorem.

Unit 3

17. A Prove that the dual X′ of every normed space X is a Banach space.

 B Prove that a normed space X is a Banach space if and only if every absolutely summable series of elements in X is summable in X.

Unit 4

18 A State and prove the Open mapping theorem.

 B Prove that there are scalars $k_n \in \mathbb{K}, n = 0, \pm 1, \pm 2, \ldots,$ such that $k_n \to 0$ as $n \to \pm\infty$, but there is no $x \in L^1([-\pi, \pi])$ such that $\tilde{x}(n) = k_n$ for all $n = 0, \pm 1, \pm 2, \ldots.$

(4 × 12 = 48 marks)

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D 52401 (Pages : 3) Name.......................................

Reg. No..................................

THIRD SEMESTER P.G. DEGREE EXAMINATION
NOVEMBER 2023

(CCSS)

Mathematics

MAT 315—FUNCTIONAL ANALYSIS

(2022 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question is of 1 *mark.*

1. Show that a bounded sequence in a metric space need not be a Cauchy sequence.
2. Define norm on a linear space.
3. Show that the operation of differentiation from $C^1([0,1])$ to $C([0,1])$ both with *sup* norm, is discontinuous.
4. Let X be a normed space and let $0 \neq a \in X$. Show that there is some $f \in X'$. such that $\| f(a) \| = \| a \|$ and $\| f \| = 1$.
5. What is the geometrical significance of the uniform boundedness principle ?
6. Show that the linear space c_{00} is not a Banach space in any norm.
7. Show that if a closed map is bijective then its inverse is closed.
8. State the two norm theorem.

(8 × 1 = 8 marks)

Turn over

Part B

Answer any **six** *questions.*
Each question is of 3 marks.

9. Let E be a measurable subset of $\mathbb{R}$. Show that the set of all simple measurable functions on E is dense in $L^{\infty}(E)$.

10. Show that the closed unit ball in l^2 is convex and bounded, but not compact.

11. Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that the range of F is finite dimensional. Show that F is continuous if and only if the zero space $Z(F)$ of F is closed in X.

12. Let $X = K^2$ with norm $\| \|_{\infty}$ and let $Y = \{(x_1, x_2) \in X : x_2 = 0\}$. Define $g \in Y'$ by $g(x_1, x_2) = x_1$. Determine the Halm-Banach extension of g to X.

13. Show that a normed space can be embedded as a dense subspace of Banach space.

14. Let X be a normed space and E be a subset of X. Show that E is bounded in X if and only if $f(E)$ is bounded for every $f \in X'$.

15. Let X and Y be normed spaces and $F : X \to Y$ be linear. Let $\tilde{F} : X/Z(F) \to Y$ be defined by $\tilde{F}(x + Z(F)) = F(x), x \in X$. Show that F is a closed map if and only if $Z(F)$ is closed in X and $\tilde{F}$ is a closed map.

16. Let X and Y be Banach spaces and $F : X \to Y$ be a linear map which is closed and surjective. Show that F is continuous and open.

17. Let $\| \|'$ be a complete norm on $C([a, b])$ such that if $\| x_n - x \|' \to 0$ then $x(t) \to x(t)$ for every $t \in [a, b]$. Show that $\| \|'$ is equivalent to the *sup* norm on $C([a, b])$.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question is of 8 *marks.*

18. (a) Show that for $1 \le p \le \infty$, the metric space l^p is complete.

    (b) Let Y be a subspace of a normed space X. Show that Y and its closure $\bar{Y}$ are normed spaces with the induced norm.

19. (a) Show that every linear map from a finite dimensional normed space X to a normed space Y is continuous.

    (b) Let X and Y be normed spaces and $X \ne \{0\}$. Show that $BL(X, Y)$ is a Banach space in the operator norm if and only if Y is a Banach space.

20. (a) Let X be a normed space. Y be a subspace of X and $g \in Y'$. Show that there is some $f \in X'$ such that $f|Y = g$ and $\| f \| = \| g \|$.

    (b) Let X, Y be normed spaces and let $F : X \to Y$ be linear. Show that F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$ there is some $x \in X$ with $F(x) = y$ and $\| x \| \le \gamma \| y \|$.

21. (a) Let $\{a_1, a_2, ..., a_n\}$ be a linearly independent set in a normed space X. Show that there are $f_1, f_2, ..., f_n$ in X' such that $f_j(a_i) = \delta_{ij}$ for $1 \le i, j \le n$.

    (b) State and prove uniform boundedness principle.

22. (a) Show that a projection P on a normed space X is a closed map if and only if the subspaces $R(P)$ and $Z(P)$ are closed in X.

    (b) Let Y be a finite dimensional subspace of a normed space X. Show that there is a continuous projection P defined on X such that $R(P) = Y$.

(3 × 8 = 24 marks)

D 52404 **(Pages : 3)** **Name..................................**

Reg. No...............................

THIRD SEMESTER P.G. (CCSS) DEGREE EXAMINATION, NOVEMBER 2023

Mathematics

MAT 3 18—LINEAR PROGRAMMING AND ITS APPLICATIONS

(2022 Admission onwards)

Time : Three Hours Maximum Marks : 50

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Show that the sum of two convex functions is a convex function.
2. What is meant by feasible solution of a linear programming problem ?
3. Prove that the dual of the dual is the primal.
4. What is meant by degeneracy in a transportation problem.
5. Define the centre of a graph.
6. What is meant by a mixed integer programming problem ?
7. Describe the effect of introducing new constraint on the optimal solutin of an LP problem.
8. A factory can manufacture two products A and B. The profit of a unit of A is Rs. 80 and of B is Rs. 40. The maximum demand of A is 6 units per week and of B it is 8 units. The manufacturer has set up a goal of achieving a profit of Rs. 640 per week. Formulate the problem as a goal programming problem.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Let $X \in E_n$ and let $f(X) = X'AX$ be a quadratic form. If $f(X)$ is positive semidefinite, then prove that $f(X)$ is a convex function.
10. Solve graphically :

$$\begin{aligned} \text{Maximize} \quad & 5x_1 + 3x_2 \\ \text{subject to} \quad & 4x_1 + 5x_2 \leq 10 \\ & 5x_1 + 2x_2 \leq 10 \\ & 3x_1 + 8x_2 \leq 12 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Turn over

11. Briefly describe the simplex method to solve a linear programming problem.
12. Describe briefly the applications of using duality theory in solving a linear programming problem.
13. Solve the following transportation problem for minimum cost starting with the degenerate solution $x_{12} = 30, x_{21} = 40, x_{32} = 20, x_{43} = 60$:

	D_1	D_2	D_3	
O_1	4	5	2	30
O_2	4	1	3	40
O_3	3	6	2	20
O_4	2	3	7	60
	40	50	60	

14. State and prove max-flow min-cut theorem.
15. Briefly describe the cutting plane method to solve an integer programming problem.
16. For the problem :

Maximize $f = x_1 - x_2 + 2x_3$

subject to
$$x_1 - x_2 + x_3 \le 4$$
$$x_1 + x_2 - x_3 \le 3$$
$$2x_1 - 2x_2 + 3x_3 \le 15$$
$$x_1, x_2, x_3 \ge 0.$$

Assuming x_4, x_5, x_6 respectively as the slack variables for the three constraints, the optimal table is the following :—

Basis	Values	x_1	x_2	x_3	x_4	x_5	x_6
x_3	21	4		1		2	1
x_4	7	2			1	1	0
x_2	24	5	1			3	1
$-f$	18	2				1	1

Carry out sensitivity analysis when the objective function changes to $3x_1 + x_2 + 5x_3$.

17. Show that the optimal solution of the following problem for $\lambda = 0$ remains optimal for, $0 \le \lambda \le 2/3$, and find the solution :

Maximize $3x_1 + 6x_2$

subject to
$$(1 + 2\lambda)\, x_1 \le 4$$
$$3(1 - \lambda)\, x_1 + 2x_2 \le 18$$
$$x_1 \ge 0, x_2 \ge 0.$$

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. (a) Let $f(X)$ be defined in a convex domain $K \subseteq E_n$ and be differentiable. Prove that $f(X)$ is a convex function if and only if

$$f(X_2) - f(X_1) \geq (X_2 - X_1)' \nabla f(X_1)$$

for all X_1, X_2 in K.

(b) Prove that a vertex of the set S_F of feasible solutions is a basic feasible solution.

19. Solve using revised simplex method the problem :

$$\begin{aligned} \text{Maximize} \quad & 4x_1 + 5x_2 \\ \text{subject to} \quad & x_1 - 2x_2 \leq 2 \\ & 2x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 5 \\ & -x_1 + x_2 \leq 2 \\ & x_1 + x_2 \geq 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

20. (a) Provet that the optimum value of $f(X)$ of the primal, if it exists, is equal to the optimum value of $\phi(Y)$ of the dual.

(b) Proe that the transportation problem has a triangular basis.

21. (a) Briefly describe the Caterer problem.

(b) Solve the following problem by branch and bound method :—

$$\begin{aligned} \text{Maximize} \quad & 11x_1 + 21x_2 \\ \text{subject to} \quad & 4x_1 + 7x_2 + x_3 = 13 \end{aligned}$$

x_1, x_2, x_3 non-negative integers.

22. Solve the problem :

$$\begin{aligned} \text{Maximize} \quad & (2 + \lambda)x_1 + (1 + 4\lambda)x_2 \\ \text{subject to} \quad & 4x_1 + 3x_2 \geq 6 \\ & 3x_1 + x_2 \geq 3 \\ & x_1 + 2x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

for $\lambda = 0$ by the dual simplex method and then do its parametric analysis for $\lambda \geq 0$.

(3 × 8 = 24 marks)

D 52399 **(Pages : 4)** **Name..................................**

Reg. No...............................

THIRD SEMESTER P.G. (CCSS) DEGREE EXAMINATION, NOVEMBER 2023

Mathematics

MAT 3C 13—PDE AND INTEGRAL EQUATIONS

(2019—2021 Admissions)

Time : Three Hours Maximum Marks : 80

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. What is the general form of a first order quasilinear equation in two independent variable ? Given example of a quasilinear equation.
2. Define shock and entropy condition.
3. Classify the differential equation $u_{xx} - 2\sin(x)u_{xy} - \cos^2(x)u_{yy} - \cos(x)u_y = 0$.
4. Describe Domain of dependence and region of influence.
5. Show that if u be a function of $C^2(D)$ satisfying the mean value property at every point in D then u is harmonic in D.
6. Prove that every harmonic function in D are infinitely differentiable on D.
7. State four properties of Green's function.
8. Convert the following differential equation into an integral equation :

$$y'' + \lambda xy = f(x);\ y(0) = 1\ ;\ y'(0) = 0.$$

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Solve the equation $u_x + 3y^{\frac{2}{3}}u_y = 2$ subject to the initial condition $u(x, 1) = 1 + x$.
10. Use the Lagrange method to find a function $u\ (x\ ,y)$ that solves

$$uu_x + u_y = 1;\ u(3x, 2) = 4 - 3x,\ -\infty < x < \infty.$$

11. Using Green's theorem derive the solution of non-homogeneous infinite string.

Turn over

12. Prove the uniqueness of the solution of Dirichlet problem for the vibration string of finite length L.

13. Derive the Poisson formula for the Neumann problem in a disk.

14. Find the iterated kernels of the kernel $K(x, \xi) = e^{x} \cos(\xi)$, on $[0, \pi]$.

(4 × 4 = 16 marks)

Part C

Answer any **all** *questions.*
Each question carries 12 *marks.*

15. (A) (i) Solve the equation $u_x + u_y + u = 1$, subject to the initial condition $u = \sin(x)$, on $y = x + x^2, x > 0$.

(7 marks)

(ii) Solve the eikonal equation $u_x^2 + u_y^2 = n^2$ for a medium with a constant refraction index $n = n_0$, and initial condition $u(x, 2x) = 1$.

(5 marks)

Or

(B) (i) Consider the Cauchy problem $u_x + u_y = 1, u(x, x) = x$. Show that it has infinitely many solutions.

(5 marks)

(ii) Find a function $u(x, y)$ that solves the Cauchy problem :

$$x^2 u_x + y^2 u_y = u^2, \; u(x, 2x) = x^2 \;\; x \in \mathbb{R}.$$

(a) Check whether the transversality condition holds.

(b) Draw the projections on the (x, y) plane of the initial curve and the characteristic curves that start at the points (1, 2, 1) and (0, 0, 0).

(c) Is the solution you found in part defined for all x and y ?

(7 marks)

16. (A) Consider the equation $u_{xx} - 6u_{xy} + 9u_{yy} = xy^2$:

(a) Find the co-ordinates system (s, t) in which the equation has the form :

$$9v_{tt} = \frac{1}{3}(s - t)t^2.$$

(b) Find the general solution $u(x, y)$.

(c) Find a solution of the equation which satisfies the initial conditions $u(x,0) = \sin(x), u_y(x,0) = \cos(x)$ for all $x \in \mathbb{R}$.

Or

(B) Consider the Cauchy problem

$$\begin{aligned} &u_{tt} - 4u_{xx} = F_{(x,t)} && -\infty < x < \infty, t > 0 \\ &u(x,0) = f(x), u_t(x,0) = g(x) && -\infty < x < \infty, \end{aligned}$$

where

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 1 & 1 < x < 2 \\ 3 - x & 2 < x < 3 \\ 0 & x > 3, x < 0 \end{cases}$$

$$g(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

and $F(x,t) = -4e^x$ on $t > 0, -\infty < x < \infty$.

(a) Is the d'Alembert solution of the problem a classical solution ? If your answer is negative, find all the points where the solution is singular.

(b) Evaluate the solution at (1, 1).

17. (A) Solve the equation by separation of variable :

$$\begin{aligned} &u_t - 4u_{xx} = 0 && 0 < x < \pi, t > 0 \\ &u(0,t) = u(\pi,t) = 0 && t \geq 0 \\ &u(x,0) = f(x) && f(x) = \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \pi \end{cases} \end{aligned}$$

(a) Discuss about the converges of solution and verify that the solution satisfies the equation.

(b) Is that the solution smooth for the given $f(x)$? Justify your answer.

Or

Turn over

(B) (i) State and prove the mean value principle. (4 marks)

(ii) (a) Compute the Laplace equation in a polar co-ordinate system.

(b) Find a function u, harmonic in the disk $x^2 + y^2 < 6$, and satisfying $u(x, y) = y + y^2$ on the disk's boundary. Write your answer in a Cartesian co-ordinate system.

(8 marks)

18 (A) Solve the following integral equation and discuss all its possible cases :

$$y(x) = f(x) + \lambda \int_0^1 (1 - 3x\xi)\, y(\xi)\, d\xi.$$

Or

(B) Let the Green's function for $\mathcal{L}_y(x) :\equiv \dfrac{d}{dx}\left(p(x) \dfrac{dy}{dx} \right) + q(x)y = 0$ be given by :

$$G(x, \xi) = \begin{cases} -\dfrac{1}{A} u(x) v(\xi), & a \le x \le \xi \\ -\dfrac{1}{A} u(\xi) v(\xi), & \xi \le x \le b \end{cases}$$

where A is constant, independent of x and ξ. Show that $y(x)$ is a solution of the boundary value problem :

$$\mathcal{L}_y(x) + \phi(x) = 0,\ \ \alpha_1 y(a) + \beta_1 y'(a) = 0,\ \alpha_2 y(b) + \beta_2 y'(b) = 0$$

if and only if $y(x) = \int_a^b G(x, \xi)\, \phi(\xi)\, d\xi$. Assume that at least one of α_1, β_1 and one of α_2, β_2 aer non-zero in the boundary condition.

(4 × 12 = 48 marks)

D 52402 **(Pages : 4)** **Name.......................................**

Reg. No..................................

THIRD SEMESTER P.G. DEGREE EXAMINATION
NOVEMBER 2023

(CCSS)

Mathematics

MAT 316—PDE AND INTEGRAL EQUATIONS

(2022 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question is of 1 *mark.*

1. What is transversality condition ?

2. Consider the equation $xu_x + u_y = 1$. Find a characteristic curve passing through the point $(1, 1, 1)$.

3. Find a mapping $q = q(x, y), r = r(x, y)$ that transforms the Tricumi equation $u_{xx} + xu_{yy} = 0, x < 0$ into its canonical form.

4. Consider the Cauchy problem : $u_{tt} = u_{xx}, -\infty < x < \infty, t > 0,$

$$u(x,0) = \begin{cases} 0, -\infty < x < 1 \\ x+1, -1 \le x \le 0 \\ 1-x, 0 \le x \le 1 \\ 0, 1 < x < \infty \end{cases}, \quad u_t(x,0) = \begin{cases} 0, -\infty < x < -1 \\ 1, -1 \le x \le 1 \\ 0, 1 < x < \infty \end{cases}.$$

Evaluate u at the point $(1, 1/2)$.

5. What is Neumann problem ?

Turn over

6. Show that the Dirichlet problem in a bounded domain D :

$$\Delta u = f(x, y), (x, y) \in D : u(x, y) = g(x, y), (x, y) \in \partial D$$

has at most one solution.

7. Define separable kernel and give an example of it.

8. Show that the kernel $K(x, \xi) = 1 + 3x\xi$ has a double characteristic number associated with $(-1, 1)$ with two independent characteristic functions.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question is of 3 *marks.*

9. Solve the equation $u_x + u_y = 2$ subject to the initial condition $u(x, 0) = x^2$.

10. Solve the eikonal equation $u_x^2 + u_y^2 = n_0^2$ with initial condition $u(x, 2x) = 1$.

11. Find the canonical form of the equation $u_{xx} - 6u_{xy} + 9u_{yy} = xy^2$.

12. Prove the stability of the Cauchy problem

$$u_{tt} - c^2 u_{xx} = 0, -\infty < x < \infty, t > 0 ; u(x, 0) = f(x), u_t(x, 0) = g(x).$$

13. Obtain the Cable equation.

14. Find a harmonic function $u(x, y)$ in the square $0 < x, y < \pi$ satisfying the Neumann boundary conditions :

$$u_y(x, \pi) = x - \frac{\pi}{2}, u_x(0, y) = u_x(\pi, y) = u_y(x, 0) = 0.$$

15. Transform the problem

$$\frac{d^2 y}{dx^2} + y = x, y(0) = 1, y'(1) = 0.$$

to a Fredholm integral equation.

16. Show that if $y_m(x)$ and $y_n(x)$ are characteristic functions corresponding to distinct characteristic numbers of the Fredholm integral equation $y(x) = \lambda \int_a^b K(x, \xi)\, y(\xi)\, d\xi$ where the kernel $K(x, \xi)$ is symmetric, then $y_m(x)$ and $y_n(x)$ are orthogonal over the interval (a, b).

17. Determine the resolvent kernel associated with $K(x, \xi) = x + \xi$ in the interval $(0, 1)$ in the from of a power series in λ obtaining the first three terms.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question is of 8 *marks.*

18. (a) Solve the Cauchy problem

$$u_x^2 + u_y^2 = 1 \,;\, u(\cos x, \sin x) = 0,\, 0 \le x \le 2\pi.$$

(b) Find the canonical form and the general solution $u(x, y)$ of the equation $uxx + 6u_{xy} - 16u_{yy} = 0$.

19. (a) Find the general solution of the equation $yu_x = xu_y$ using Lagrange's method.

(b) Explain the iterative method for solving the Fredholm equation of the second kind

$$y(x) = F(x) + \lambda \int_a^b K(x, \xi)\, y(\xi)\, d\xi$$

where F and K are continuous.

20. (a) Obtain the d'Alembert's solution for the one dimensional homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = 0,\ -\infty < x < \infty, t > 0 \,;\, u(x, 0) = f(x), u_t(x, 0) = g(x).$$

(b) Solve the Laplace equation in the unit disk subject to the boundary conditions $w(r, \theta) = y^2$ on $r = 1$.

Turn over

21. (a) Using the method of separation of variables find a solution of the problem

$$u_t - ku_{xx} = 0, 0 < x < L, t > 0$$
$$u(0, t) = u(L, t) = 0, t \geq 0$$
$$u(x, 0) = f(x), 0 \leq x \leq L$$

Where f is a given initial condition and k is a positive constant.

(b) Show that if $y(x)$ satisfies the differential equation $\frac{d^2y}{dx^2} + xy = 1$ and the conditions $y(0) = y'(0) = 0$ then y satisfies the Volterra equation

$$y(x) = \int_0^x \xi(x - \xi)\, y(\xi)\, d\xi + \frac{1}{2}x^2.$$

22. (a) Use the method of eigen function expansion to solve the problem :

$$u_{tt} - u_{xx} = \cos 2\pi x \cos 2\pi t, - 0 < x < 1, t > 0$$
$$u_x(0, t) = u_x(1, t) = 0, t \geq 0$$
$$u(x, 0) = \cos^2 \pi x\, f(x), 0 \leq x \leq 1$$
$$u_t(x, 0) = 2 \cos 2\pi x, 0 \leq x \leq 1.$$

(b) Determine the characteristic values and the corresponding characteristic functions of the equation

$$y(x) = \lambda \int_0^{2\pi} \sin(x + \xi)\, y(\xi)\, d\xi.$$

(3 × 8 = 24 marks)