LAPLACE AUTOREGRESSIVE TIME SERIES MODELS

Thesis Submitted to the University of Calicut for the degree of

DOCTOR OF PHILOSOPHY

in Statistics

By

KUTTYKRISHNAN.A.P.

Under the supervision of

Dr.K. JAYAKUMAR

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALICUT
KERALA, INDIA

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CERTIFICATE

This is to certify that the work reported in this thesis entitled "Laplace Autoregressive Time Series Models" submitted to the University of Calicut for the award of degree of Doctor of Philosophy in Statistics is a bonafide research work carried out by Mr. Kuttykrishnan.A.P. under my supervision and guidance in the Department of statistics, University of Calicut. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

Dr. K. Jayakumar

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DECLARATION

I hereby declare that the matter embodied in this thesis is the result of investigations carried out by me in the Department of Statistics, University of Calicut, under the supervision and guidance of Dr.K.Jayakumar, Lecturer Senior Scale, Department of Statistics, University of Calicut. This thesis contains no material, which has been accepted for the award of any degree or diploma in any university or institute and to the best of my knowledge and belief, it contains no material previously published by any other person, except where the due references are made in the text of the thesis.

Calicut University,

October 19, 2006.

Kuttykrishnan.A.P.

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Chapter-I

Introduction

1.1. Introduction

A time series is a series of observations made sequentially in time. The primary objective of time series analysis is to reveal the probability law that governs the observed time series. Thus, if we wish to explain a particular pattern of fluctuations in a series, we need to construct a mathematical model that explains the random features of the series of observations. The construction of such mathematical model is the main objective in time series analysis. If we obtain a satisfactory model for our time series, it may provide knowledge of the physical mechanism generating the data, and it can be used for forecast and thereby control future values of the series. A model is selected in such a way that the selected model is a simple one and reasonably reflects the physical law that governs the data. In selecting a model, one first identifies the salient features from the observed series and then chooses an appropriate model that possesses such features. After estimating the parameters of the model, one verifies whether the model fits the data reasonably well and looks for further improvement whenever possible.

When we analyze a time series using formal statistical methods, we view the collection of observations $\{x_n, n = 1,2,...\}$ as a particular realization of the stochastic process $\{X_t\}$. Hence a complete description of a time series, observed as a collection of n random variables at arbitrary time points $t_1, t_2, \ldots, t_n$, is provided by

the joint distribution function $F(x_1, x_2, ..., x_n) = P(X_{t_1} \le x_1, X_{t_2} \le x_2, ..., X_{t_n} \le x_n)$. This multidimensional distribution function can be written easily provided the random variables are jointly normal. Generally such an assumption is unlikely to be appropriate for many time series, and so the multidimensional distribution cannot be written in a simple form. A simple and more useful way of describing a time series is to give one dimensional distribution function $F(x_1) = P(X_{t_1} \le x_1)$ and mean value function $\mu_t = E(X_t)$, if it exists. The dependence between two adjacent values X_{t+k} and X_t can be assessed numerically by the autocovariance function, which is defined as $\gamma(k) = E((X_t - \mu_t)(X_{t+k} - \mu_{t+k}))$. It may be noted that $\gamma(k)$ of a time series measures the linear dependence between two points on the same series at different times. As in the classical set up, it is more convenient to define autocorrelation function $\rho(k) = \frac{\gamma(k)}{\sqrt{Var(X_t)\ Var(X_{t+k})}}$ to describe the association between two values on the same series at different times.

A special class of time series, which encountered in most of the practical case, is stationary time series. By stationarity, we mean that the series looks much the same over all time periods and so the statistical properties of $\{X_t\}$ are the same if we change the time origin. If the joint probability distribution of X_t at any set of times $t_1, t_2, ..., t_n$ is the same as the joint probability distribution at times $t_1 + k, t_2 + k, ..., t_n + k$, where k is any arbitrary shift in time , then $\{X_t\}$ is called a strictly stationary time series. These conditions are too strong, hence a milder

version of stationarity, namely weak stationarity, is introduced to apply in practical problems. A time series $\{X_t\}$ is said to be weakly stationary if (i) $E(X_t^2) < \infty$, (ii) $E(X_t) = \mu$ for all t and (iii) $\gamma(k) = \gamma(k+h)$ for all k and h. It may be noted that $\{X_t\}$ is weakly stationary if it is strictly stationary with finite second moments.

In 1927, Yule developed linear time series models, defined in terms of linear difference equation, with the intention of explaining the dynamic relationship of the sunspot series (see Chatfield (1989)). These linear time series models have now become an integral part of time series literature. The most popular class of linear time series models consists of autoregressive moving average (ARMA) models, including purely autoregressive (AR) and purely moving average (MA) models as special cases. Adding non-stationary models to the mixed ARMA models leads to the autoregressive integrated moving average (ARIMA) model that are popularized and studied by Box and Jenkins (1970).

An autoregressive model is a time series model that can be extremely useful in the representation of certain practically occurring series. An Autoregressive model of order $p \geq 1$, abbreviated as AR (p), is defined as

$$X_n = \rho_1 X_{n-1} + \rho_2 X_{n-2} + \ldots + \rho_p X_{n-p} + \varepsilon_n \qquad (1.1.1)$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables, and $\rho_1, \rho_2, \ldots, \rho_p$ are constants. The autoregressive model (1.1.1)

represents the current value X_n of the process through its immediate p past values $X_{n-1}, X_{n-2}, \ldots, X_{n-p}$ and a random shock ε_n in a linear regression form.

If we define an autoregressive operator of order p by

$$\rho(B) = 1 - \rho_1 B - \rho_2 B^2 - \ldots - \rho_p B^p$$

where B is the back shift operator defined as $B^k X_n = X_{n-k}$, $k = 0,1,\ldots$, then the AR(p) process (1.1.1) may be written as $\rho(B) X_n = \varepsilon_n$. If we define $\psi(B) = \rho^{-1}(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots$ it is equivalent to $X_n = \psi(B)\varepsilon_n$.

An autoregressive process may be stationary or non-stationary. For the process to be stationary, the 'ρ's must be chosen so that the $\psi(B)$ form a convergent series.

The most simple form of an autoregressive model is an AR (1) and given by the linear difference equation

$$X_n = \rho X_{n-1} + \varepsilon_n \qquad (1.1.2)$$

and it may be verified that the parameter ρ must satisfy the condition $|\rho| < 1$ to ensure stationarity of the process . The autocovariance function of the process (1.1.2) satisfied the equation $\gamma(k) = \rho\,\gamma(k-1)$, for all $k > 0$ and consequently $\rho(k) = \rho^k$.

Another kind of model that has very great practical importance in the representation of the series is the moving average model. The standard form of a moving average model of order $q \geq 1$, denoted by MA (q) is given by

$$X_n = \theta_1 \varepsilon_{n-1} + \theta_2 \varepsilon_{n-2} + \ldots + \theta_q \varepsilon_{n-q} + \varepsilon_n \qquad (1.1.3)$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables, and $\theta_1, \theta_2, \ldots, \theta_q$ are constants. Here we make the current value X_n linearly dependent on previous q values of ε_n's.

If we define a moving average operator of order q by

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q$$

then the MA(q) process (1.1.3) may be written as $X_n = \theta(B)\, \varepsilon_n$. The simplest form of moving average process, first order moving average process, is given by the difference equation

$$X_n = \theta\, \varepsilon_{n-1} + \varepsilon_n . \qquad (1.1.4)$$

The general linear time series model is obtained by including both autoregressive and moving average terms in the model. Such models are known as autoregressive moving average models, denoted by ARMA (p, q) and has the form

$$X_n - \rho_1 X_{n-1} - \ldots - \rho_p X_{n-p} = \varepsilon_n + \theta_1 \varepsilon_{n-1} + \ldots + \theta_q \varepsilon_{n-q} \qquad (1.1.5)$$

where $\{\rho_i\}$ and $\{\theta_j\}, i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q$ are sequences of constants and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables. One can write ARMA (p, q) process (1.1.5) as $\rho(B) X_n = \theta(B) \varepsilon_n$ using autoregressive

($\rho(B)$) and moving average ($\theta(B)$) operators. It can be seen that AR (p) model is same as ARMA (p, 0) model and a MA (q) model is same as an ARMA (0,q) model. The mathematical details of these models are available in Box and Jenkins (1970), Box *et al.* (1994), Chatfield (1975) and Brockwell and Davis (1991).

Until quite recently much of time series modeling has been limited to ARMA models with the assumption that $\{\varepsilon_n\}$ is a sequence of independent and identically distributed Gaussian random variables. Then the process $\{X_n\}$ is known as Gaussian process. If $\{X_n\}$ is Gaussian then the conditional mean of X_n given the past values will be linear, the conditional variance of X_n given the past values will be a constant and the process will be time reversible. But, there are many naturally occurring time series data sets that cannot be modeled using the assumption that the variables are distributed as Gaussian.

1.2. Non –Gaussian autoregressive time series models

It is well known that the modeling and interpretation of time series data plays a significant role in almost every field of modern research. Many time series models are developed and studied by several researchers using the assumption that the sequence of observations follows Gaussian distribution. However, time series in which observations are of non-Gaussian nature are very common in many areas. In modeling such non-Gaussian time series, the usual practice is to make suitable transformation to remove skewness of the data and fit a Gaussian model. Such technique for a class of non-Gaussian time series has been discussed in Granger and

Newbold (1976) and Janacek and Swift (1990). But in most of the cases, the assumption that transformed data follows Gaussian is unlikely to be true in practice (see Sim (1994)).

In the last two decades, several time series models with non-Gaussian marginal distribution have been introduced and studied by various authors. The need for such models arises from the fact that many naturally occurring time series are clearly non-Gaussian. Some researchers have been trying to find model such that the variables have a given marginal distribution and a given correlation structure corresponding to an ARMA process. One class of non-Gaussian linear time series models that appears to be particularly useful is the first order autoregressive process. Cox (1981) and Bondesson (1981), discussed the problem of existence of solution of an autoregressive model (1.1.2) and established that the linear autoregressive model is properly defined if and only if the marginal distribution belongs to class L. Similar problems were also discussed in Hart (1984), Jayakumar and Pillai (1992) and Pillai and Jayakumar (1994). Pillai and Jose (1994) considered an autoregressive process with structure

$$X_n = I_n \; X_{n-1} + \varepsilon_n,$$

where $P(I_n = 0) = p = 1 - P(I_n = 1), 0 < p < 1$ and obtained a necessary and sufficient condition on the marginal distribution for the process to be properly defined. They showed that the process is properly defined if and only if the marginal distribution is geometrically infinitely divisible.

The first order autoregressive process (1.1.2) for positive variables is properly defined only if $\phi_\varepsilon(t)=\frac{\phi_X(t)}{\phi_X(\rho t)}$ is a proper characteristic function for $0 \le \rho < 1$, where $\phi_X(t)$ is the characteristic function of stationary process $\{X_n\}$. Gaver and Lewis (1980) introduced and studied first order autoregressive process of the form (1.1.2) using exponential marginal distributions and obtained

$$X_n = \begin{cases} \rho X_{n-1} & \text{w.p.} \quad \rho \\ \rho X_{n-1} + E_n & \text{w.p.} \quad 1-\rho \end{cases} \tag{1.2.1}$$

where $0 \le \rho < 1$ and $\{E_n\}$ is a sequence of independent and identically distributed exponential random variables and $X_0 \stackrel{d}{=} E_1$. This model, known as EAR (1), has autocorrelation $\rho(h) = \rho^h$ and generates sample paths in which large values are followed by runs of falling values. However the degeneracy at ρ, which is known as "zero defect", limit the broad applicability of the model. Corresponding to EAR (1) model Lawrance and Lewis (1977) introduced a first order moving average model EMA (1). Jacobs and Lewis (1977) linked the two models into a first order autoregressive-moving average model EARMA (1,1). Lawrance and Lewis (1980) generalized this model and developed an exponential autoregressive-moving average model of order (p,q).

Lawrance and Lewis (1981) proposed a new autoregressive process with structure

$$X_n = \begin{cases} \rho X_{n-1} + \varepsilon_n & \text{w.p.} \quad \gamma \\ \varepsilon_n & \text{w.p.} \quad 1-\gamma \end{cases} . \qquad (1.2.2)$$

If $\{X_n\}$ is stationary with exponential distribution, they identified that the innovation sequence $\{\varepsilon_n\}$ as a sequence of independent and identically distributed random variables given by

$$\varepsilon_n = \begin{cases} E_n & \text{w.p.} \quad \dfrac{1-\rho}{1-(1-\gamma)\rho} \\ (1-\gamma)\rho E_n & \text{w.p.} \quad \dfrac{\gamma\rho}{1-(1-\gamma)\rho}, \end{cases} \qquad (1.2.3)$$

where $\{E_n\}$ is a sequence of independent and identically distributed exponential random variables. The stationary autoregressive process $\{X_n\}$ with structure (1.2.2) and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables given by (1.2.3) with $X_0 \stackrel{d}{=} E_1$ is known as NEAR (1) model. If $\gamma = 0$ or $\rho = 0$, $\{X_n\}$ is a sequence of independent and identically distributed exponential random variables, whereas $\gamma = 1$, the model reduced to structure given by (1.2.1) where ρ is replaced by $1-\rho$. The autocorrelation function of the NEAR (1) process is obtained as $\rho(h) = (\rho(1-\gamma))^h$.

Jayakumar and Pillai (1993) developed and studied a general form of process discussed in Gaver and Lewis (1980) using semi-Mittag-Leffler distribution as marginal distribution. Subsequently number of time series models for positive variables have been studied and constructed using different marginal distributions

such as inverse Gaussian, beta, Weibull, gamma, uniform etc. in Abraham and Balakrishna (1999), Mc Kenzie (1982,1985), Sim (1986,1990), Lawrance (1982, 1992), Lewis *et al.* (1989), Walker (2000) and Ristić and Popović (2000). Some of the applications of such autoregressive process are discussed in Lawrance and Kottegoda (1977), Lawrance and Lewis (1985), Bell and Smith (1986), Sim (1987) and Hutton (1990).

Recently number of time series models have been studied and constructed for real valued observations using different marginal distributions by several researchers. Lawrance (1978), Andel (1983) and Dewald and Lewis (1985) developed and studied time series models for real valued variables using Laplace marginal distribution. The applications of such models in the fields of environmental studies, communication theory etc. is given in Gibson (1986) and Damsleth and El-Shaarawi (1989). Anderson and Arnold (1993) and Jayakumar *et al.* (1995) studied a generalization of the Laplace process using Linnik / α-Laplace marginal distribution and discussed the applications in the fields of financial mathematics. Autoregressive models using marginal distributions such as logistic, semi α-Laplace, Cauchy and Pakes are studied in Sim (1993), Jayakumar (1997), Balakrishna and Nampoothiri (2003) and Seetha Lekshmi and Jose (2006) respectively.

Tavares (1980) introduced and studied an autoregressive process with structure $X_n = k \min(X_{n-1}, \varepsilon_n)$ where $n \geq 1$ and $k > 1$ is a constant. The innovation sequence $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random

variables chosen to ensure that $\{X_n\}$ is a stationary process with a given marginal distribution. The process of this type is known as minification process. Tavares (1980) discussed the minification process using exponential marginal distribution. Yeh *et al.* (1988), Arnold and Robertson (1989), Pillai (1991), Arnold (1993), Pillai *et al.* (1995), Kuttykrishnan and Jayakumar (2001) and Ristić (2005) studied minification processes using different marginal distributions and different structures.

1.3. Autoregressive process using Laplace variables

One of the most well-known and widely used symmetric distributions for modeling data with heavier tails than normal is the classical Laplace distribution. The probability density function of Laplace random variable with mean zero and variance $2\sigma^2$ is given by

$$f(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}, -\infty < x < \infty, \sigma > 0 \qquad (1.3.1)$$

and the characteristic function is

$$\phi_X(t) = \frac{1}{1+\sigma^2 t^2}. \qquad (1.3.2)$$

In this case we write $X \underline{d} L(\sigma)$.

It is only in recent years that Laplace distribution, together with its various generalizations, has been revived and is now being used in a variety of fields, including biology, economics, environmental science and financial mathematics. Several properties and applications of the Laplace distribution have been reported

demonstrating that it is a natural and sometimes superior alternative to the Gaussian distribution (for more details see Kotz *et al.* (2001)).

Although the theory and applications of Laplace distribution are well developed and have appeared in literature in recent years, their applications in time series modeling have not developed that much. Lawrance (1978) discussed autoregressive process using Laplace variables. Andel (1983) studied a first order autoregressive time series model of the form (1.1.2) where $|\rho|<1$ and with Laplace distribution as marginal. The stationary process using Laplace random variables has the structure

$$X_n = \begin{cases} \rho X_{n-1} & \text{w.p.} \quad \rho^2 \\ \rho X_{n-1} + L_n & \text{w.p.} \quad 1-\rho^2 \end{cases} \tag{1.3.3}$$

where $\{L_n\}$ is a sequence of independent and identically distributed Laplace random variables with characteristic function (1.3.2). This model, known as LAR (1), has autocorrelation $\rho(h) = \rho^h$ and hence exhibits positive and negative autocorrelation. However, as in the case of EAR (1) process LAR (1) process also has the problem of "zero defect" that limit the broad applicability of the model. Dewald and Lewis (1985) discussed the process $\{X_n\}$ with structure (1.2.2) and Laplace distribution as the marginal distribution. The resulting Laplace model is free from the drawback "zero defect" of the Laplace model (1.3.3). They have shown that the innovation sequence $\{\varepsilon_n\}$ is a convex mixture of Laplace random variables.

The main goal of the research presented here is to develop and study different types of autoregressive time series models for the set of real valued observations using Laplace random variables.

1.4. Basic concepts

In this Section, we give some preliminary concepts needed for the discussion in the subsequent Chapters.

1.4.1. Self – decomposability

A random variable X is said to be self-decomposable if for every $\alpha \in (0,1)$, it can be written (in distribution) as

$$X \stackrel{d}{=} \alpha X + X_\alpha, \tag{1.4.1}$$

where X and X_α are independent random variables.

It may be noted that a distribution is self-decomposable if and only if for every $\alpha \in (0,1)$, there exists a characteristic function $\phi_{X_\alpha}(t)$ such that $\phi_{X_\alpha}(t) = \frac{\phi_X(t)}{\phi_X(\alpha t)}$ where $\phi_X(t)$ is the characteristic function of the random variable X.

It will be very simple to show that any non-degenerate self-decomposable distribution is absolutely continuous. Some important self-decomposable distributions are normal, exponential, gamma, Cauchy, Laplace etc. The class of self-decomposable distributions is closed under scale transformation, convolution

and weak convergence. Also it can be proved that a self-decomposable distribution on the real line is infinitely divisible. For more details and discussions on self-decomposability, see Steutel and van Harn (2004). Application of self-decomposability of distributions in time series modeling is discussed in Gaver and Lewis (1980) and Bondesson (1981).

By replacing the ordinary product αX in (1.4.1) by the product $\alpha \oplus X$, Steutel and van Harn (1979) defined a meaningful concept of self-decomposability for discrete random variables. The product $\alpha \oplus X$ is defined as $\alpha \oplus X = \sum_{i=1}^{X} Y_i$ where $P(Y_i = 1) = \alpha = 1 - P(Y_i = 0)$ and $\{Y_i\}$ is a sequence of independent and identically distributed random variables independent of X. The self-decomposability property of a discrete random variable can be explained using the probability generating function $P_X(z)$ of X. A discrete random variable X is self-decomposable if and only if for every $\alpha \in (0,1)$, there exists a probability generating function $P_{X_\alpha}(z)$ such that $P_{X_\alpha}(z) = \frac{P_X(z)}{P_X(1-\alpha+\alpha z)}, |z| \le 1$.

1.4.2. Stable and semi-stable distributions

Stable distributions are very important class of distributions that allow skewness and heavy tails and have many interesting mathematical properties. The lack of closed formulas for probability density and distribution functions for all but a few stable distributions (normal, Cauchy and Levy distributions) has been a major drawback to the use of stable distributions by practitioners.

A random variable X with distribution function F(x) is said to be stable if for each $n \in \{1,2,...\}$ there exists $d_n \in (-\infty,\infty)$ and $c_n > 0$ such that

$$X \overset{d}{=} c_n (X_1 + X_2 + ... + X_n) + d_n \tag{1.4.2}$$

where $X_1, X_2, ..., X_n$ are independently and identically distributed random variables with distribution function F(x). The random variable X is called strictly stable if (1.4.2) holds with $d_n = 0$. Using characteristic function $\phi_X(t)$ of a strictly stable random variable we can write (1.4.2) as

$$\phi_X(t) = (\phi_X(c_n t))^n .$$

Now it is simple to obtain following results related to stable distribution.

(1) A distribution with characteristic function $\phi_X(t)$ is stable if and only if

$\phi_X(t) = (\phi_X(n^{-1/\alpha}t))^n, n > 0$.

The positive constant α is called the exponent of stability of the random variable X and it can be verified that $0 < \alpha \leq 2$.

It will be clear that for $\alpha > 0$, the characteristic function $\phi_X(t)$ is stable with exponent α if and only if the constants $c_n = n^{-1/\alpha}$.

(2) A random variable X is stable with exponent α if and only if for every $n \in \{1,2,...\}$ it can be written as $X \overset{d}{=} n^{-1/\alpha}(X_1 + X_2 + \cdots + X_n)$ where $X_1, X_2, ..., X_n$ are independent with $X_i \overset{d}{=} X$ for all i.

(3) A stable random variable is infinitely divisible and self-decomposable.

The most concrete way to describe all possible stable distribution is through the following representation of characteristic function.

A random variable X is said to have stable distribution if there are parameters $0<\alpha\le 2, \sigma\ge 0, -1\le\beta\le 1$ and $\mu\in(-\infty,\infty)$ such that its characteristic function has the following form

$$\phi_X(t)=\begin{cases}\exp\left\{-\sigma^\alpha|t|^\alpha\left(1-i\beta\tan\frac{\pi\alpha}{2}(\text{sign } t)\right)+i\mu t\right\} & \text{if } \alpha\neq 1\\ \exp\left\{-\sigma|t|\left(1+i\beta\frac{2}{\pi}\ln|t|(\text{sign } t)\right)+i\mu t\right\} & \text{if } \alpha=1\end{cases} \qquad (1.4.3)$$

where $\text{sign}(t)=\begin{cases}1 & \text{if } t>0\\ 0 & \text{if } t=0\\ -1 & \text{if } t<0\end{cases}$.

The representation (1.4.3) shows that a general stable distribution requires four parameters $\alpha\in(0,2]$, known as index of stability or characteristic exponent, skewness parameter $\beta\in[-1,1]$, scale parameter $\sigma\ge 0$ and location parameter $\mu\in(-\infty,\infty)$. If X is a random variable with characteristic function (1.4.3), then we denote it by $X\sim S_\alpha(\sigma,\beta,\mu)$. These distributions are symmetric around zero if $\beta=\mu=0$ and so the characteristic function of symmetric stable is $\phi_X(t)=\exp(-\sigma^\alpha|t|^\alpha)$. The various properties and applications of stable distributions are discussed in Samorodnitsky and Taqqu (1994).

Note that stable distribution becomes

(i) normal distribution if $\alpha=2,\beta=0$

(ii) Cauchy distribution when $\alpha = 1, \beta = 0$

(iii) Levy distribution when $\alpha = \frac{1}{2}, \beta = 1$.

A class of distributions containing stable distributions, namely semi-stable distributions is studied in Kagan *et al.* (1973).

A class of distributions is called semi-stable if

$$\phi(t) = (\phi(bt))^a, a > 1, 0 < |b| < 1, \tag{1.4.4}$$

where $\phi(t)$ is the characteristic function of the distribution.

Pillai (1968) proved that a semi-stable distribution is infinitely divisible. For more properties and discussions, see Pillai (1971).

1.4.3. Geometric infinite divisibility

A random variable X is infinitely divisible if for each $n \geq 1$, there exist independent and identically distributed random variables $X_1, X_2, ..., X_n$ such that $X \stackrel{d}{=} X_1 + X_2 + ... + X_n$. Klebanov *et al.* (1984) introduced the concept of geometric infinite divisibility.

Let N_p be a geometric random variable independent of X_i's and with probability mass function

$$P(N_p = k) = p(1-p)^{k-1}; k = 1, 2, ... \tag{1.4.5}$$

Then a real valued random variable X is said to have a geometrically infinitely divisible distribution if for any $p \in (0,1)$, there exists a sequence of independent and identically distributed real valued random variables $\left\{X_i^{(p)}\right\}$ such that

$$X \underline{\underline{d}} \sum_{i=1}^{N_p} X_i^{(p)}, \tag{1.4.6}$$

where N_p and $\left\{X_i^{(p)}\right\}$ are independent.

In terms of characteristic function, the relation (1.4.6) can be written as

$$\phi_X(t) = \frac{p\phi_{X_i^{(p)}}(t)}{1-(1-p)\phi_{X_i^{(p)}}(t)}, \tag{1.4.7}$$

where $\phi_X(t)$ and $\phi_{X_i^{(p)}}(t)$ are the characteristic functions of X and $X_i^{(p)}$ respectively.

Klebanov *et al.* (1984), Pillai (1990a), Mohan *et al.* (1993) and Fujita (1993) obtained several characterizations and properties of geometrically infinitely divisible distributions.

Some important results related to geometrically infinitely divisible distributions are

(1) A random variable X with characteristic function $\phi(t)$ is geometrically infinitely divisible if and only if $\phi(t) = \lim_{n\to\infty} \frac{1}{1+\theta_n(1-\phi_n(t))}$, where $\{\theta_n\}$ is a sequence of positive numbers and $\{\phi_n(t)\}$ is a sequence of characteristic functions.

(2) A random variable X with characteristic function $\phi(t)$ is geometrically infinitely divisible if and only if $\psi(t)=\exp\left\{1-\frac{1}{\phi(t)}\right\}$ represents characteristic function of an infinitely divisible distribution.

(3) The class of geometrically infinitely divisible distributions forms a subclass of infinitely divisible distributions.

The applications of geometrically infinitely divisible distributions in the field of time series modeling are discussed in Pillai and Jose (1994).

1.4.4. Geometric stable distribution

Geometric stable distributions are the weak limits of appropriately normalized geometric random sums of independent and identically distributed random variables $X_1, X_2, \ldots$. Let N_p be a geometric random variable independent of X_i's and with probability mass function (1.4.5). If there exists a weak limit of

$$S_p = a(p)\sum_{i=1}^{N_p}(X_i + b(p)), \qquad (1.4.8)$$

when $p \to 0$, where $a(p) > 0$ and $b(p) \in (-\infty, \infty)$, then the class of limiting distributions is called geometric stable distributions. If $b(p) = 0$, then the limiting distribution is called strictly geometric stable.

Since the sums such as (1.4.8) frequently appear in many applied problems in various areas (see Gnedenko and Korolev (1996)), these classes of distributions

have a wide variety of applications especially in the field of reliability, biology, economics, financial mathematics etc.

The class of geometric stable distributions is a four-parameter family denoted by $GS_{\alpha}(\sigma,\beta,\mu)$ and conveniently described in terms of characteristic function

$$\phi_X(t)=\frac{1}{1+\sigma^{\alpha}|t|^{\alpha}\varpi_{\alpha,\beta}(t)-i\mu t}, \tag{1.4.9}$$

where

$$\varpi_{\alpha,\beta}(t)=\begin{cases}1-i\beta\,\mathrm{sign}(t)\tan(\pi\alpha/2) & \text{if } \alpha\neq 1\\ 1+i\beta\dfrac{2}{\pi}\,\mathrm{sign}(t)\log|t| & \text{if } \alpha=1.\end{cases} \tag{1.4.10}$$

The parameter $\alpha\in(0,2]$ is the index of stability and determines the tail of the distribution, $\beta\in[-1,1]$ determines the skewness of the distribution. The parameters $\mu\in(-\infty,\infty)$ and $\sigma\geq 0$ correspond to location and scale. As the tail $P(X>x)$ of a geometric stable random variable X is regularly varying at infinity, this class of distributions seems to be appropriate for modeling heavy tailed asymmetric data. One such area of application is mathematical finance, where price change over a certain period of time can be regarded as the sum of changes over shorter periods of time.

Special cases of geometric stable distributions include exponential, Mittag-Leffler, Laplace, asymmetric Laplace and Linnik distributions.

Note that geometric stable distribution becomes

(i) exponential distribution if $\sigma = 0$

(ii) symmetric Laplace distribution if $\alpha = 2, \mu = 0$

(iii) asymmetric Laplace distribution if $\alpha = 2, \mu \neq 0$

(iv) symmetric Linnik distribution if $\alpha \in (0,2), \beta = \mu = 0$

(v) Mittag-Leffler distribution with Laplace transform $\frac{1}{1+\sigma^{\alpha}s^{\alpha}}$ if $0 < \alpha \leq 1$, $\beta = \mu = 0$ and X is defined on $(0,\infty)$.

Various properties and representations of geometric stable distribution are discussed in Mittinik and Rachev (1991), Ramachandran (1997), and Kozubowski and Rachev (1999). Aly and Bouzar (2000) studied geometric infinity divisibility and geometric stability properties of distributions and developed first order stationary autoregressive process with geometric stable marginal distributions. Some important properties of geometric stable distribution are as follows:

(1) A random variable X is geometric stable if and only if its characteristic function $\phi(t)$ has the form

$$\phi(t) = \frac{1}{1 - \ln \psi(t)}, \tag{1.4.11}$$

where $\psi(t)$ is the characteristic function of a stable distribution.

(2) If X is a $GS_{\alpha}(\sigma, \beta, \mu)$ random variable then

$$X \underline{d} \begin{cases} \mu Z + Z^{1/\alpha} \sigma Y & \text{if } \alpha \neq 1 \\ \mu Z + Z\sigma Y + \sigma Z \beta (2/\pi) \ln(\sigma Z) & \text{if } \alpha = 1, \end{cases} \tag{1.4.12}$$

where $Y \sim S_{\alpha}(1,\beta,0)$, Z has standard exponential distribution, and Y and Z are independent.

(3) If X is a $GS_{\alpha}(\sigma,\beta,\mu)$ random variable, then for any $\tau > 0$, absolute moments $E|X|^{\tau}$ exists if and only if $\tau < \alpha$,.where $0 < \alpha < 2$.

This result does not apply to the exponential ($\sigma = 0$) and to the Laplace ($\alpha = 2$) distributions, which have moments of all orders.

(4) The geometric stable distribution is self-decomposable

(i) if $\alpha = 2$ without restriction on μ and β

(ii) if $0 < \alpha < 1$ or $1 < \alpha < 2$ for $\mu = 0$ and

(iii) if $\alpha = 1$ for $\mu = \beta = 0$.

Kozubowski (2000) discussed method of simulation of geometric stable random variable using the representation (1.4.12). The applications of the geometric stable distribution in the field of financial mathematics are discussed in Kozubowski and Rachev (1994).

1.4.5. Asymmetric Laplace distribution

The Laplace distribution is symmetric, and there were several asymmetric extensions in generalizing the Laplace distribution. In the last several decades, different forms of skewed Laplace distributions have been introduced and studied by various authors (see Holla and Bhattacharya (1968) and Yu and Zhang (2004)). Kozubowski and Podgórski (2000) studied asymmetric Laplace distribution having probability density function

$$f(x;\mu,\sigma)=\frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}\begin{cases}\exp(-\frac{\kappa}{\sigma}x) & \text{if } x\geq 0\\ \exp(\frac{1}{\kappa\sigma}x) & \text{if } x<0,\end{cases} \tag{1.4.13}$$

where $\sigma>0, -\infty<\mu<\infty$.

The additional parameter $\kappa>0$ is related to μ and σ as $\frac{1}{\kappa}-\kappa=\frac{\mu}{\sigma}$ and hence

$$\kappa=\frac{2}{\frac{\mu}{\sigma}+\sqrt{4+(\frac{\mu}{\sigma})^2}}.$$

If X is a random variable with probability density function (1.4.13), then we represent it as $X \underline{d} AL(\mu,\sigma)$.

The n^{th} arbitrary moment of $X \underline{d} AL(\mu,\sigma)$ is given by

$$E(X^n)=n!\,(\frac{\sigma}{k})^n\ \frac{1+(-1)^n k^{2(n+1)}}{1+k^2}.$$

The characteristic function of $AL(\mu,\sigma)$ random variable is given by

$$\phi_X(t)=\frac{1}{1+\sigma^2 t^2-i\mu t},\ \sigma>0, -\infty<\mu<\infty. \tag{1.4.14}$$

The class of asymmetric Laplace distributions with characteristic function (1.4.14) arises as limiting distribution of a random (geometric) sum of independent and identically distributed random variables with finite second moments. Hence the class of asymmetric Laplace distributions forms a subclass of geometric stable

distributions where the geometric stable distributions, similar to stable laws, have tail behavior governed by the index of stability $\alpha \in (0,2]$. The class of distributions with characteristic function (1.4.14) corresponds to the geometric stable subclass with $\alpha = 2$ (see Kozubowski and Rachev (1999)). By specifying $\sigma = 0$ and $\mu > 0$, we have an exponential distribution with mean μ and obtain symmetric Laplace distribution if $\mu = 0$ and $\sigma \neq 0$. The asymmetric Laplace distribution plays an analogues role among geometric stable laws as Gaussian distributions do among the stable laws. The applications of asymmetric Laplace distribution in the fields of biology, financial mathematics, environmental science etc are well established by different authors (see Kotz *et al.* (2001) and Julia and Vives-Rego (2005))

1.4.6. Semi α -Laplace distribution

Linnik (1963) considered a family of symmetric distributions defined on $(-\infty, \infty)$ with the characteristic function

$$\phi(t) = \frac{1}{1+|t|^{\alpha}}, 0 < \alpha \leq 2, \tag{1.4.15}$$

and several researchers discussed the properties and method of generation of random variables belong to this family of distributions (see Devroye (1990), Lin (1994, 1998) and Kotz *et al* .(2001).

When $\alpha = 2$, the function (1.4.15) corresponds to the characteristic function of symmetric Laplace distribution and hence Pillai (1985) termed the distribution with characteristic function (1.4.15) as α-Laplace distribution. Note that this

distribution is also known in the literature as Linnik distribution. Lin (1998) proved that Linnik distribution is geometric infinitely divisible and discussed characterization of Linnik distribution using geometric summation. There are no closed- form expressions for distribution and density functions for Linnik random variable except for $\alpha = 2$, which correspond to the Laplace distribution. The probability density function of the Linnik random variable with characteristic function (1.4.15) has the following representation

$$f_\alpha(x) = \frac{\sin\frac{\pi\alpha}{2}}{\pi} \int_0^\infty \frac{\upsilon^\alpha \exp(-\upsilon|x|)}{1+\upsilon^{2\alpha}+2\upsilon^\alpha \cos\frac{\pi\alpha}{2}} d\upsilon \ , \text{if } x > 0$$

and

$$f_\alpha(x) = f_\alpha(-x) \ , \text{if } x < 0 .$$

Sabu George and Pillai (1987) derived the probability density function of α-Laplace distribution in terms of the Meijer's G function and studied multivariate extension of α-Laplace distributions.

Unlike Laplace distribution, the Linnik distribution has infinite variance and the mean is finite only for $1 < \alpha < 2$. The absolute moments of the distribution are given by

$$E|X|^\delta = \frac{2^\delta \Gamma(1+\frac{\delta}{\alpha})\Gamma(1-\frac{\delta}{\alpha})\Gamma((1+\delta)/2)}{\sqrt{\pi}\,\Gamma(1-\frac{\delta}{2})}, \ 0 < \delta < \alpha, 0 < \alpha \le 2 .$$

Various authors discussed the applications of the Linnik distribution in different fields that ranges from engineering to finance (see Devroye (1986) and Anderson and Arnold (1993)). The Linnik distribution is considered as the best choice for modeling whenever data exhibit both high kurtosis and heavy tails than Gaussian tails. Christoph and Schreiber (1998) studied the discrete version of Linnik distribution, namely discrete Linnik distribution.

Pillai (1985) has introduced a general class of distributions, termed semi α-Laplace distributions, of which α-Laplace distribution is a special case.

Let $\phi(t)$ be the characteristic function, which is never zero, defined by

$$\phi(t) = \frac{1}{1+\psi(t)}. \tag{1.4.16}$$

Then the distribution with characteristic function (1.4.16) is called semi α-Laplace distribution if $\psi(t)$ has the property

$$\psi(t) = a\ \psi(bt), 0 < b < 1, \tag{1.4.17}$$

where a is the unique solution of the equation

$$a\, b^{\alpha} = 1, 0 < \alpha \leq 2, \tag{1.4.18}$$

where b and α are known as order and exponent of the semi α-Laplace distribution, respectively. It can be noted that the solution of the functional equation $\psi(t) = a\ \psi(bt), 0 < b < 1, a > 0$ is

$$\psi(t) = |t|^{\alpha} h(t) \tag{1.4.19}$$

where h(t) is periodic in $\ln|t|$ (see Pillai (1985)).

It may be noted that when h(t) is a constant the semi α-Laplace distribution reduces to α-Laplace distribution.

Pillai (1985) and Divanji (1988) discussed properties of the semi α-Laplace distribution and some of the properties are

(1) A random variable X is a semi α-Laplace random variable with order b if and only if the distribution function F(x) satisfies the equation

$$F(x) = pF_1(x) + (1-p)F_2(x) \qquad (1.4.20)$$

for some $p \in (0,1)$, where $F_1(x)$ is the distribution of bX and $F_2 = F * F_1$.

(2) The semi α-Laplace distribution is geometrically infinitely divisible.

(3) For suitable choice of A and $\alpha < 1$, the characteristic function of a semi α-Laplace can be written as $\phi(t) = \dfrac{1}{1+|t|^{\alpha}(1-A\cos(k\ln|t|))}$ where

$k = \dfrac{2\pi}{\ln b}, 0 < b < 1$.

(4) The absolute moments $E|X|^{\delta}$ for the semi α-Laplace distribution with exponent α exist if and only if $0 \le \delta < \alpha$.

1.5. Outline of the thesis

The thesis is concerned with autoregressive time series models using Laplace random variables. In our study we focus on first order autoregressive process and we develop such models using the self- decomposability and geometric infinite divisibility properties of the variables. In the present work, we introduce and study the properties of first order autoregressive process using Laplace, asymmetric Laplace, geometric stable and discrete Laplace variables. Bivariate time series models with first order autoregressive structure are developed corresponding to the bivariate Laplace, marginal Laplace and Linnik distributions and bivariate semi α-Laplace distributions. The properties of such models are discussed and obtained autocorrelation structure of the process. The technique of estimating model and distribution parameters of the autoregressive process is given. An application of the process is discussed.

The thesis is structured in seven chapters. In the introductory chapter, Chapter-I, we discussed the general aspect and development of non-Gaussian autoregressive time series models. The applications and relevance of the study of such models are pointed out there. Some basic concepts with regard to distribution theory that is needed for constructing time series models are also given in this Chapter.

In the second Chapter, we consider first order autoregressive model using Laplace, asymmetric Laplace and generalized Laplace variables and the existence of such processes are established. The models are developed using the self-

decomposability property of these distributions. Empirical analysis of some important time series data shows that asymmetric and heavy tailed distributions are more suitable for modeling the data. Hence we have given emphasis to first order asymmetric Laplace autoregressive process. It is identified that the distribution of the innovation sequence of the process is a convex mixture of exponential random variables with different parameters. The random coefficient representation of the process is given. . Properties of the model such as autocorrelation, joint distribution and asymptotic behavior are studied. Also the problem of estimation of parameters of the model and distribution is addressed. The sample path behavior of the process for various parameters is given here. But the processes discussed here have the problem of "zero defect" that causes successive values of the process to be fixed multiples of the previous values.

In Chapter-III, we introduce a two-parameter time series model that is free from the drawback "zero defect" using asymmetric Laplace marginal distribution. The innovation sequence of the process is obtained as a sum of two independent random variables U_n and V_n where U_n is a convex mixture of exponential random variables with different parameters and V_n is an asymmetric Laplace random variable. The properties of the process are studied. Also we develop a tractable one-parameter model using geometric infinitely divisible property of the asymmetric Laplace distribution and discuss the properties of the model. An autoregressive process with geometric stable distribution as marginal distribution is developed. Also we considered some important cases by which the geometric stable

autoregressive process is reduced to some well known autoregressive processes. We introduce the concept of tailed asymmetric Laplace distribution and an autoregressive process using tailed asymmetric Laplace marginal distribution is developed. Also we developed a three-parameter autoregressive model using Laplace, asymmetric Laplace and semi α-Laplace variables. If the marginal distribution of the stationary three- parameter autoregressive process is Laplace then we obtain the solution of innovation sequence as a convex mixture of Laplace random variables. Further we establish the condition for existence of stationary solution of the three-parameter autoregressive process when the marginal distribution is semi α-Laplace distribution. The condition for existence of solution as a mixture of semi α-Laplace with negative weights is given. The autocorrelation function, joint characteristic function and sample path behavior of the process are obtained. A second order autoregressive process with Laplace marginal distribution is introduced and the distribution of the innovation sequence in this case is obtained. The first order moving average process with Laplace marginal distribution is developed in this Chapter.

In Chapter-IV, we define a bivariate process with Laplace marginal distribution and establish the existence of the process. We discuss autocorrelation structure of the process and consider the question of obtaining bivariate Laplace autoregressive process with negative correlation. The autocorrelation structures of the marginal processes of the bivariate Laplace process are discussed and the conclusion is that the marginal process has the autocorrelation of ARMA(p,q) process with $p \leq 2$ and $q \leq 1$ and some particular cases are examined.

The Chapter-V is about a class of bivariate distribution, namely marginal Laplace and Linnik distribution, which is a particular case of operator geometric stable distributions. Here we study some properties and characterizations of marginal Laplace and Linnik distributions and present a first order autoregressive marginal Laplace and Linnik process. The autocorrelation and asymptotic behavior of the process are derived in this Chapter. We consider a generalized class of distributions, namely marginal asymmetric Laplace and asymmetric Linnik distribution by introducing asymmetric component in the marginal distributions and time series model corresponding to this distribution is also developed.

Also we have introduced a new class of bivariate distributions called bivariate semi α-Laplace distribution, containing bivariate Laplace distributions. Some characterizations of bivariate semi α-Laplace distribution are obtained. Relation of bivariate semi α-Laplace distribution with bivariate semi stable distribution is established. An autoregressive model with bivariate semi α-Laplace distribution as marginal distributions is developed and its properties are studied.

The Chapter-VI is devoted to discrete Laplace distribution and time series model using this distribution. Since the discrete Laplace distribution is geometrically infinitely divisible it is possible to construct an autoregressive model equivalent to one-parameter model developed in Chapter-III. The distribution of innovation sequence of the stationary process when variables follow discrete Laplace distribution is identified in this chapter and hence we develop a time series model using discrete Laplace distribution. We extended this result to the skewed discrete

Laplace distribution and constructed a time series model corresponding to this skewed variables. The properties of the models are also studied.

In Chapter-VII we discuss an application of the asymmetric Laplace autoregressive process.

LAPLACE AUTOREGRESSIVE TIME SERIES MODELS

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Chapter-II

Laplace Autoregressive Models

2.1. Introduction

In the last two decades, there has been increasing interest in developing time series models for real valued observations using non-Gaussian distributions. The studies of such models are necessitated by the fact that many naturally occurring time series observations show a tendency to follow asymmetric and heavy tailed distributions. Various authors discussed the applications of the Laplace distribution in different fields that ranges from engineering to finance and Laplace distribution is considered as the best choice for modeling whenever data exhibit heavier tails than Gaussian tails. Lawrance (1978), Andel (1983), Dewald and Lewis (1985) and Damsleth and El-Shaarawi (1989) developed and studied autoregressive process using Laplace distribution and discussed the applications of the process in different fields such as communication theory, environmental science etc.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined by the autoregressive equation (1.1.2) with $|\rho| < 1$ and $\{\varepsilon_n\}$ be a sequence of independent and identically distributed random variables, such that $\{X_n\}$ and $\{\varepsilon_n\}$ are semi-independent, that is, X_m and ε_n are independent if $m < n$.

Let the characteristic functions of $\{X_n\}$ and $\{\varepsilon_n\}$ are denoted by $\phi_{X_n}(t)$ and $\phi_{\varepsilon_n}(t)$ respectively. Then from (1.1.2) we get

$$\phi_{X_n}(t) = \phi_{X_{n-1}}(\rho t)\phi_{\varepsilon_n}(t). \tag{2.1.1}$$

Assume that $\{X_n\}$ is stationary with Laplace marginal distribution having characteristic function (1.3.2). From (2.1.1), we have

$$\phi_{\varepsilon_n}(t) = \frac{\phi_X(t)}{\phi_X(\rho t)}$$

Since Laplace distribution is self-decomposable, $\phi_{\varepsilon_n}(t)$ is a properly defined characteristic function and is obtained as

$$\phi_{\varepsilon_n}(t) = \frac{1+\sigma^2\rho^2 t^2}{1+\sigma^2 t^2}$$

$$= \left[\rho^2 + \frac{1}{2}(1-\rho^2)\frac{1}{1-i\sigma t} + \frac{1}{2}(1-\rho^2)\frac{1}{1+i\sigma t}\right]. \tag{2.1.2}$$

Hence the innovation sequence $\{\varepsilon_n\}$ of the first order autoregressive process (1.1.2) is the convex mixture of random variables given by

$$\varepsilon_n = \begin{cases} 0 & \text{w.p.} \quad \rho^2 \\ E_n & \text{w.p.} \quad \frac{1}{2}(1-\rho^2) \\ -E_n & \text{w.p.} \quad \frac{1}{2}(1-\rho^2), \end{cases} \tag{2.1.3}$$

where $\{E_n\}$ is a sequence of independent exponential random variables with mean σ.

Also it can be verified that, if $X_0 \overset{d}{=} L(\sigma)$ and $\{\varepsilon_n\}$ is an independent and identically distributed sequence of convex mixture of exponential random variables given by (2.1.3), the first order autoregressive process (1.1.2) is stationary with Laplace marginal distribution.

Thus we have the following theorem.

Theorem 2.1.1.

Let $\{\varepsilon_n\}$ be a sequence of independent and identically distributed random variables defined as in (2.1.3). Then relation $X_n = \rho X_{n-1} + \varepsilon_n$, $|\rho| < 1$ with $X_0 \overset{d}{=} L(\sigma)$ defines a stationary autoregressive process with Laplace marginal distribution. □

We call the process defined in (1.1.2) with $X_0 \overset{d}{=} L(\sigma)$ and ε_n as in (2.1.3) as the first order Laplace autoregressive (LAR (1)) process.

The structure of the first order Laplace autoregressive process is as follows:

Let $X_0 \stackrel{d}{=} L(\sigma)$ and for $n = 1,2,...$

$$X_n = \begin{cases} \rho X_{n-1} & \text{w.p.} \quad \rho^2 \\ \rho X_{n-1} + E_n & \text{w.p.} \quad \frac{1}{2}(1-\rho^2) \\ \rho X_{n-1} - E_n & \text{w.p.} \quad \frac{1}{2}(1-\rho^2). \end{cases} \tag{2.1.4}$$

On simplification of equation (2.1.2), we get

$$\phi_{\varepsilon_n}(t) = \left[\rho^2 + (1-\rho^2)\frac{1}{1+\sigma^2 t^2}\right].$$

Hence

$$X_n = \begin{cases} \rho X_{n-1} & \text{w.p.} \quad \rho^2 \\ \rho X_{n-1} + L_n & \text{w.p.} \quad 1-\rho^2, \end{cases} \tag{2.1.5}$$

where $\{L_n\}$ is a sequence of independent and identically distributed Laplace random variables with characteristic function $\frac{1}{1+\sigma^2 t^2}$. This model is equivalent to the first order Laplace autoregressive process defined in Lawrance (1978) and Andel (1983).

The average length of up-run sequences of the process is related to the probability of X_n greater than X_{n-1}. Hence a simple quantification of sample path behavior is obtained by finding $P(X_n > X_{n-1})$.

By using the definition of X_n given in (2.1.5),

$$P(X_n > X_{n-1}) = \rho^2 P(\rho X_{n-1} > X_{n-1}) + (1-\rho^2) P(\rho X_{n-1} + L_n > X_{n-1})$$

$$= (1-\rho^2) \int_x P(L_n > x(1-\rho)) f(x) dx$$

$$= (1-\rho^2) \int_{-\infty}^{0} (1 - \frac{1}{2} e^{x(1-\rho)/\sigma}) \frac{1}{2\sigma} e^{x/\sigma} dx$$

$$+ (1-\rho^2) \int_0^{\infty} \frac{1}{2} e^{-x(1-\rho)/\sigma} \frac{1}{2\sigma} e^{-x/\sigma} dx$$

On simplification, we get

$$P(X_n > X_{n-1}) = \frac{1-\rho^2}{2}. \qquad (2.1.6)$$

Dewald and Lewis (1985) studied the LAR (1) process and established that it has similar properties as the EAR (1) model discussed in Gaver and Lewis (1980). The autocorrelation function of the process is $\rho(h) = \rho^h$. Hence the process is negatively correlated if $-1 < \rho < 0$ for odd lags and positively correlated for all other cases. This is the basic difference of LAR (1) model with EAR (1) model discussed in Gaver and Lewis (1980). Also note that the LAR (1) process has "zero defect" property as in the case of EAR (1) model. The "zero defect" property is an undesirable property and we have to modify the process in such a way that the "zero defect" is eliminated.

2.2. Asymmetric Laplace autoregressive process

Since the Laplace distribution is symmetric, it is not appropriate for modeling data with asymmetric empirical distribution. So, in this case the most appropriate skewed generalization of the Laplace distribution is considered. Kozubowski and Podgórski (1999, 2000) studied the class of asymmetric Laplace distributions with characteristic function (1.4.14). The class of asymmetric Laplace distributions is well suited for modeling phenomena where the variable of interest results from a large random number of independent observations and the empirical distribution appears to be asymmetric, with steep peak and tails heavier than those allowed by normal distribution. It can be used for modeling observations in various fields such as mathematical finance, communication theory, environmental science, biology etc.

Now we develop a first order autoregressive process using asymmetric Laplace distribution.

Theorem 2.2.1.

The first order autoregressive process

$$X_n = \rho X_{n-1} + \varepsilon_n,\ 0 \le \rho < 1,\ n = 1,2,... \qquad (2.2.1)$$

is stationary with asymmetric Laplace marginal distribution if and only if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables such that

$$\varepsilon_n = \begin{cases} 0 & \text{w.p.} \quad \rho^2 \\ E_{n1} & \text{w.p.} \quad (1-\rho)\left(\rho + \dfrac{1-\rho}{1+\kappa^2}\right) \\ -E_{n2} & \text{w.p.} \quad (1-\rho)\left(\rho + \dfrac{(1-\rho)\kappa^2}{1+\kappa^2}\right), \end{cases} \tag{2.2.2}$$

where E_{n1} and E_{n2} are two independent exponential random variables with means σ/κ and $\kappa\,\sigma$ respectively, provided $X_0 \overset{d}{=} AL(\mu,\sigma)$.

Proof:

Assume that $\{X_n\}$ is stationary with asymmetric Laplace marginal distribution having characteristic function (1.4.14). Then, using equation (2.1.1), we obtain

$$\phi_{\varepsilon_n}(t) = \frac{\phi_X(t)}{\phi_X(\rho t)}$$

$$= \frac{1+\sigma^2\rho^2 t^2 - i\mu\rho t}{1+\sigma^2 t^2 - i\mu t}$$

$$= \rho^2 + (1-\rho)\left(\rho + \frac{1-\rho}{1+\kappa^2}\right)\frac{1}{1-i\dfrac{\sigma}{\kappa}t} + (1-\rho)\left(\rho + \frac{(1-\rho)\kappa^2}{1+\kappa^2}\right)\frac{1}{1+i\sigma\kappa t},$$

where $\dfrac{1}{\kappa} - \kappa = \dfrac{\mu}{\sigma}$.

Hence $\{\varepsilon_n\}$ is a sequence of convex mixture of exponential random variables given by

$$\varepsilon_n = \begin{cases} 0 & \text{w.p.} \quad \rho^2 \\ E_{n1} & \text{w.p.} \quad (1-\rho)\left(\rho + \dfrac{1-\rho}{1+\kappa^2}\right) \\ -E_{n2} & \text{w.p.} \quad (1-\rho)\left(\rho + \dfrac{(1-\rho)\kappa^2}{1+\kappa^2}\right), \end{cases}$$

where E_{n1} and E_{n2} are two independent exponential random variables with means σ/κ and $\kappa\,\sigma$ respectively.

Conversely, if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variable given by (2.2.2) and $X_0 \stackrel{d}{=} AL(\mu,\sigma)$, then from (2.2.1) when $n=1$, we have

$$\phi_{X_1}(t) = \phi_{X_0}(\rho t)\phi_{\varepsilon_1}(t)$$

$$= \frac{1}{1+\sigma^2\rho^2 t^2 - i\mu\rho t}\,\frac{1+\sigma^2\rho^2 t^2 - i\mu\rho t}{1+\sigma^2 t^2 - i\mu t}$$

$$= \frac{1}{1+\sigma^2 t^2 - i\mu t}.$$

Similarly, if we assume $X_{n-1} \stackrel{d}{=} AL(\mu,\sigma)$ then we get $X_n \stackrel{d}{=} AL(\mu,\sigma)$.

Thus $\{X_n\}$ is a stationary process with asymmetric Laplace marginal distribution.

Hence the theorem.

Thus the first order autoregressive asymmetric Laplace (ALAR (1)) process defined as $X_0 \overset{d}{=} AL(\mu,\sigma)$ and for $n = 1,2,...$

$$X_n = \begin{cases} \rho X_{n-1} & \text{w.p.} \quad \rho^2 \\ \rho X_{n-1} + E_{n1} & \text{w.p.} \quad (1-\rho)\left(\rho + \dfrac{1-\rho}{1+\kappa^2}\right) \\ \rho X_{n-1} - E_{n2} & \text{w.p.} \quad (1-\rho)\left(\rho + \dfrac{(1-\rho)\kappa^2}{1+\kappa^2}\right), \end{cases} \tag{2.2.3}$$

where $\{E_{n1}\}$ and $\{E_{n2}\}$ are two independent sequence of exponential random variables with means σ/κ and $\kappa\,\sigma$ respectively.

Remark 2.2.1.

The asymmetric Laplace distribution is not self-decomposable if $-1 < \rho < 0$. Hence the characteristic function of innovation sequence $\{\varepsilon_n\}$ of the autoregressive equation (1.1.2) is not properly defined. Therefore the first order autoregressive process (1.1.2) using asymmetric Laplace marginal distribution can be defined only when the condition $0 \le \rho < 1$ is satisfied.

Remark 2.2.2.

When $\kappa = 1$, the asymmetric Laplace distribution is reduced to symmetric Laplace distribution with characteristic function (1.3.2). Hence in this case, the model (2.2.3) is reduced to LAR (1) model (2.1.5).

2.2.1. Properties of ALAR (1) process

From the autoregressive equation (2.2.1), we can write

$$X_n = \rho^n X_0 + \sum_{k=0}^{n-1} \rho^k \varepsilon_{n-k} . \tag{2.2.4}$$

So the characteristic function $\phi_{X_n}(t)$ can be expressed as

$$\phi_{X_n}(t) = \phi_{X_0}(\rho^n t) \prod_{k=0}^{n-1} \phi_{\varepsilon_{n-k}}(\rho^k t) . \tag{2.2.5}$$

If X_0 is distributed arbitrary and $\{\varepsilon_n\}$ is an independent and identically distributed sequence of convex mixture of exponential random variables given by (2.2.2), then by (2.2.5) we have

$$\phi_{X_n}(t) = \phi_{X_0}(\rho^n t) \prod_{k=0}^{n-1} \frac{1+\sigma^2 \rho^{2k+2} t^2 - i\mu\rho^{k+1} t}{1+\sigma^2 \rho^{2k} t^2 - i\mu\rho^k t} .$$

When $n \to \infty$, $\phi_{X_n}(t) = \dfrac{1}{1+\sigma^2 t^2 - i\mu t}$.

Hence, if X_0 has an arbitrary distribution, then the autoregressive process is asymptotically stationary with asymmetric Laplace marginal distribution.

For the ALAR (1) process, we have $E(X_n) = \mu$ and from equation (2.2.1) it is easily verified that $E(\varepsilon_n) = (1-\rho)\mu$.

Now $E(X_n / X_{n-1} = x) = E(\rho X_{n-1} + \varepsilon_n / X_{n-1} = x)$

$$= \rho x + (1-\rho)\mu. \tag{2.2.6}$$

Hence the conditional expectation of the process is linear in x.

The autocovariance function $\gamma(h)$ of the process

$$\gamma(h) = E(X_n X_{n-h}) - E(X_n)E(X_{n-h})$$

$$= E\left((\rho X_{n-1} + \varepsilon_n) X_{n-h}\right) - E(X_n)\, E(X_{n-h})$$

$$= \rho^h \, Cov(X_{n-h}, X_{n-h}).$$

Hence, the autocorrelation function is given by

$$\rho(h) = \rho^h. \tag{2.2.7}$$

Since $0 \le \rho < 1$, the autocorrelation function of the ALAR (1) process is always positive. Hence the ALAR (1) process is positively correlated.

The joint distribution function of (X_{n-1}, X_n) is obtained by finding the joint characteristic function of (X_{n-1}, X_n) and it is of the form

$$\phi_{X_{n-1},X_n}(t,s) = E\left(\exp(i\,t\,X_{n-1} + i\,s\,X_n)\right)$$

$$= E\left(\exp(i\,(t+\rho s)X_{n-1} + is\varepsilon_n)\right)$$

$$= \phi_{X_{n-1}}(t+\rho s)\;\phi_{\varepsilon_n}(s)$$

$$= \frac{1}{1+\sigma^2\,(t+\rho\,s)^2 - i\,\mu\,(t+\rho s)} \frac{1+\sigma^2\,\rho^2 s^2 - i\,\mu\,\rho s}{1+\sigma^2 s^2 - i\,\mu\,s}.$$

The joint characteristic function is not symmetric in t and s. Hence the ALAR (1) process is not time reversible.

A simple quantification of sample path behavior is given by $P(X_n > X_{n-1})$, which is related to the average length of up-run sequences. By using the definition of X_n given in (2.2.3), the probability $P(X_n > X_{n-1})$ is given by

$$P(X_n > X_{n-1}) = \rho^2\,P(\rho X_{n-1} > X_{n-1}) + (1-\rho)\left(\rho + \frac{1-\rho}{1+\kappa^2}\right)P(\rho X_{n-1} + E_{n1} > X_{n-1})$$

$$+ (1-\rho)\left(\rho + \frac{(1-\rho)\kappa^2}{1+\kappa^2}\right)P(\rho X_{n-1} - E_{n2} > X_{n-1})$$

$$= (1-\rho)\left(\rho + \frac{1-\rho}{1+\kappa^2}\right)\int_x P\left(E_{n1} > (1-\rho)x\right) f(x)\,dx$$

$$+ (1-\rho)\left(\rho + \frac{(1-\rho)\kappa^2}{1+\kappa^2}\right)\int_x P\left(E_{n2} < -(1-\rho)x\right) f(x)\,dx$$

$$= (1-\rho)\left(\rho + \frac{1-\rho}{1+\kappa^2}\right)\int_0^\infty P\left(E_{n1} > (1-\rho)x\right)\frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}e^{-(\kappa/\sigma)x}\,dx$$

$$+ (1-\rho)\left(\rho + \frac{(1-\rho)\kappa^2}{1+\kappa^2}\right)\int_{-\infty}^0 P\left(E_{n2} < -(1-\rho)x\right)\frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}e^{(1/\kappa\sigma)x}\,dx$$

$$= (1-\rho)\left(\rho + \frac{1-\rho}{1+\kappa^2}\right)\int_0^\infty e^{-(\kappa/\sigma)(1-\rho)x}\frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}e^{-(\kappa/\sigma)x}dx$$

$$+ (1-\rho)\left(\rho + \frac{(1-\rho)\kappa^2}{1+\kappa^2}\right)\int_{-\infty}^0 (1-e^{(1/\kappa\sigma)(1-\rho)x})\frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}e^{(1/\kappa\sigma)x}dx$$

$$= \frac{1}{(1+\kappa^2)}\frac{1}{2-\rho}\left((1-\rho)\left(\rho + \frac{1-\rho}{1+\kappa^2}\right) + (1-\rho)\left(\rho + \frac{(1-\rho)\kappa^2}{1+\kappa^2}\right)\kappa^2(1-\rho)\right).$$

Simplifying, we have

$$P(X_n > X_{n-1}) = \frac{(1-\rho)\left[1+\rho\kappa^2(2-\rho)+\kappa^4(1-\rho)\right]}{(1+\kappa^2)^2(2-\rho)}. \qquad (2.2.8)$$

In Figure 2.2.1 we present the sample path behavior of the ALAR (1) process for different values of parameters.

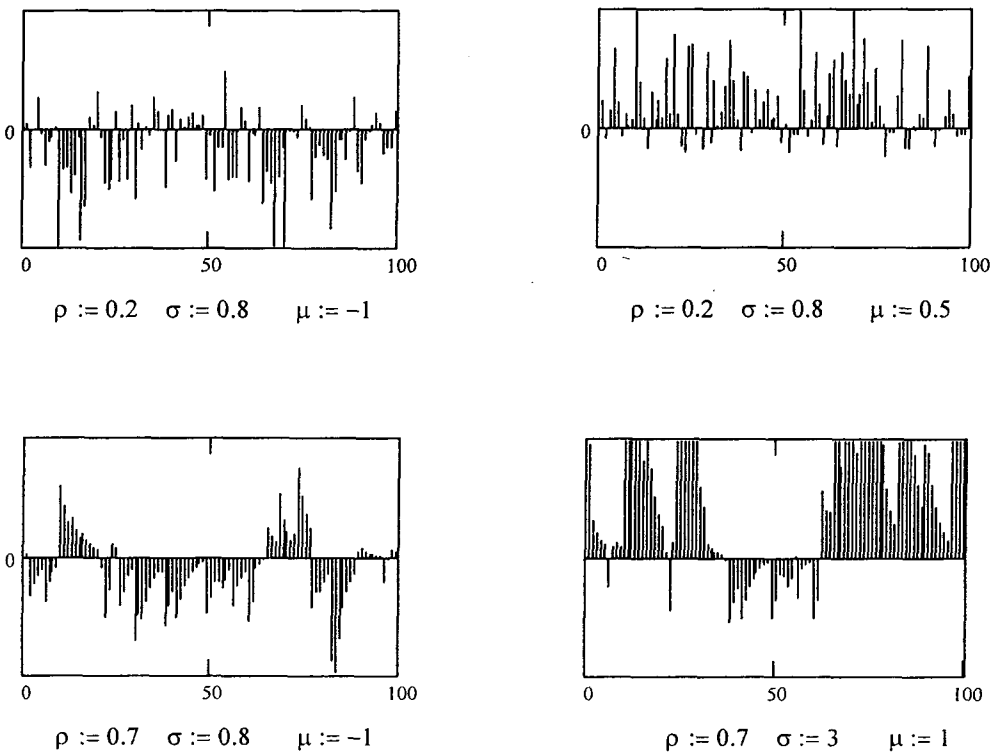

Figure 2.2.1

Sample path of ALAR (1) process

The sample path behavior of the process seems to be distinctive and is adjustable through the parameters ρ, μ and σ. Runs of increasing values or runs of decreasing values or both are noticed and this makes the model very rich.

2.2.2. Estimation of parameters of ALAR (1) process

Let $\{x_0, x_1, ..., x_n\}$ be a given set of observations of an ALAR (1) process with unknown parameters $\theta = (\rho, \mu, \sigma)$. Then the conditional least square estimate (CLS) of θ is obtained by minimizing the sum of squares

$$S_n(\theta) = \sum_{i=1}^{n} (x_i - E(X_i / X_{i-1} = x_{i-1}))^2$$

$$= \sum_{i=1}^{n} (x_i - (\rho x_{i-1} + (1-\rho)\mu))^2.$$

Hence, CLS estimates of ρ and μ are given by solving $\frac{\partial S_n(\theta)}{\partial \rho} = 0$ and $\frac{\partial S_n(\theta)}{\partial \mu} = 0$.

Now

$$\frac{\partial S_n(\theta)}{\partial \rho} = 0 \Rightarrow \sum_{i=1}^{N} x_i(-x_{i-1} + \mu) = \rho \sum_{i=1}^{N} x_{i-1}(-x_{i-1} + \mu) + (1-\rho)\mu \sum_{i=1}^{N} (-x_{i-1} + \mu)$$

Similarly

$$\frac{\partial S_n(\theta)}{\partial \mu} = 0 \Rightarrow \sum_{i=1}^{n} x_i = \rho \sum_{i=1}^{n} x_{i-1} + (1-\rho)\mu n.$$

Solving the equations we obtain the CLS estimates of ρ and μ as

$$\hat{\rho} = \frac{n\sum_{i=1}^{n} x_i\, x_{i-1} - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_{i-1}\right)}{n\sum_{i=1}^{N} x_{i-1}^2 - \left(\sum_{i=1}^{n} x_{i-1}\right)^2} \tag{2.2.10}$$

and

$$\hat{\mu} = \frac{\sum_{i=1}^{n} x_i - \hat{\rho}\sum_{i=1}^{n} x_{i-1}}{(1-\hat{\rho})\, n} \tag{2.2.11}$$

respectively.

The maximum likelihood estimate of the parameter σ is obtained by maximizing the likelihood function. The likelihood equation $L(x_0, x_1, ..., x_n; \rho, \mu, \sigma)$ is given by

$$L(x_0, x_1, ..., x_n; \rho, \mu, \sigma) = f_{X_0}(x_0) \prod_{j=1}^{n} f_{X_j / X_{j-1}}(x_j / x_{j-1})$$

where the conditional probability density function is

$$\begin{aligned} f_{X_j / X_{j-1}}(x_j / x_{j-1}) &= \pi_1 \delta(x_j - \rho x_{j-1}) \\ &+ \pi_2 \frac{\kappa}{\sigma} \exp\left(-\frac{\kappa}{\sigma}(x_j - \rho x_{j-1})\right) I_{(0,\infty)}(x_j - \rho x_{j-1}) \\ &+ \pi_3 \frac{1}{\kappa\sigma} \exp\left(\frac{x_j - \rho\, x_{j-1}}{\kappa\sigma}\right) I_{(-\infty,0)}(x_j - \rho\, x_{j-1}). \end{aligned} \tag{2.2.12}$$

Note that

$$\delta(x_j - \rho x_{j-1}) = \begin{cases} 1 & \text{if } x_j - \rho x_{j-1} = 0 \\ 0 & \text{if } x_j - \rho x_{j-1} \neq 0 \end{cases},$$

$$I_{(-\infty,0)}(x_j - \rho x_{j-1}) = \begin{cases} 1 & \text{if } x_j - \rho x_{j-1} < 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$I_{(0,\infty)}(x_j - \rho x_{j-1}) = \begin{cases} 1 & \text{if } x_j - \rho x_{j-1} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

For a given set of observations $\{x_0, x_1, ..., x_n\}$ from ALAR (1) process, the estimates of ρ and μ are obtained by solving equations (2.2.10) and (2.2.11). The estimate of σ is obtained by the method of maximum likelihood estimation.

The likelihood function of the process is

$$L(x_0, x_1, ..., x_n; \sigma) = \begin{cases} \frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}\exp(-\frac{\kappa}{\sigma}x_0)\prod_{j=}^{n} f_{X_j/X_{j-1}}(x_j/x_{j-1}) & \text{if } x_0 > 0 \\ \frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}\exp(\frac{1}{\kappa\sigma}x_0)\prod_{j=}^{n} f_{X_j/X_{j-1}}(x_j/x_{j-1}) & \text{if } x_0 < 0, \end{cases} \quad (2.2.13)$$

where $f_{X_j/X_{j-1}}(x_j/x_{j-1})$ is given by (2.2.12).

The maximum likelihood estimate of the unknown parameter σ is obtained from (2.2.13) by numerical method.

The procedure for finding estimates ρ, μ and σ of ALAR (1) process is given below.

(1) Assume initial values for ρ, μ and σ.

(2) Using these initial values, generate an independent and identically distributed sequence $\{\varepsilon_n\}$ given by (2.2.2) and an asymmetric Laplace random variable X_0 such that $X_0 \stackrel{d}{=} AL(\mu, \sigma)$. Hence we obtain a sequence $\{X_n\}$ using the relation $X_n = \rho X_{n-1} + \varepsilon_n$, $n = 1, 2, ...$

To avoid initial anomalies, discard first fifty observations of the sequence generated and thus we obtain a pivotal sequence $X_0, X_1, ..., X_n$.

(3) Using CLS estimation technique obtain the estimates $\hat{\rho}$ and $\hat{\mu}$ of ρ and μ respectively, corresponding to the generated sequence $X_0, X_1, ..., X_n$.

(4) Obtain an initial value σ_0 of σ.

To get the initial value σ_0, we consider the second moment of the asymmetric Laplace distribution $E(X^2) = 2\mu^2 + 2\sigma^2$ and equate this to the sample moment $\frac{1}{n+1}\sum_{i=0}^{n} x_i^2$. Hence we choose σ_0 as the initial value from the

equation $\sigma_0 = \dfrac{\frac{1}{n+1}\sum_{i=0}^{n} x_i^2 - 2\hat{\mu}^2}{2}$ where $\hat{\mu}$ is the CLS estimate of μ obtained from the simulated sample.

(5) Obtain the MLE of σ, by numerically maximizing the likelihood function (2.2.13) and choose the initial value of σ as σ_0. Let $\hat{\sigma}_1$ be the estimate.

(6) Repeat this procedure 100 times by changing the random samples and take the average of $\hat{\sigma}_i$ and let $\hat{\sigma}$ be the average. Then choose $\hat{\sigma}$ as the MLE of σ.

A simulation study is performed to provide a better understanding of the numerical method to find the estimates of ρ, μ and σ. The simulation is done with a set of initial values $(\rho, \mu, \sigma) = (0.2, 0.2, 0.2)$.

Simulate a sequence of 200 observations of ALAR (1) process $\{X_n\}$ given by (2.2.1) and (2.2.2) using the set of initial values and an arbitrary asymmetric Laplace random variable X_0. Discard first fifty observations to avoid initial anomalies and use the remaining 150 observations to obtain CLS estimates of ρ and μ using the equations (2.2.10) and (2.2.11). To get the maximum likelihood estimate of σ choose an initial value σ_0. We choose the value σ_0 based on the second moment of the asymmetric Laplace distribution as described above.

Corresponding to the set of initial values $(\rho, \mu, \sigma) = (0.2, 0.2, 0.2)$ we have $\sigma_0 = 0.31$.

Now, the MLE of σ is obtained by numerically maximizing the likelihood function

(2.2.13) and let $\hat{\sigma}_i$ be the estimate. Repeat this procedure 100 times by changing the random samples and take the average of $\hat{\sigma}_i$ and let $\hat{\sigma}$ be the average. Treat this $\hat{\sigma}$ as the maximum likelihood estimate of σ. Hence we obtain the estimates of (ρ,μ,σ) using the simulated sequence $\{X_n\}$.

Repeat the same procedure with different sets of values (ρ,μ,σ). We obtain the estimated values and corresponding standard error of the estimates. The estimate value and corresponding standard error in respect of different sets of values are listed in the following tables.

Table 2.1

Simulation results when $n = 200$ and $\sigma = 0.2$

Initial Value of (ρ,μ,σ)	Estimate of ρ	Std.error of ρ	Estimate of μ	Std.error of μ	Estimate of σ	Std.error of σ
(0.2,0.2,0.2)	0.2013	0.0402	0.1998	0.0189	0.1867	0.0166
(0.2,0.4,0.2)	0.1994	0.0427	0.397	0.0259	0.1996	0.0419
(0.2,0.6,0.2)	0.2017	0.0443	0.5979	0.0364	0.1898	0.0401
(0.2,0.8,0.2)	0.2028	0.0431	0.797	0.0484	0.1989	0.0213
(0.5,0.2,0.2)	0.4946	0.0374	0.1974	0.025	0.2062	0.0613
(0.5,0.4,0.2)	0.4962	0.0383	0.393	0.0338	0.1891	0.0415
(0.5,0.6,0.2)	0.4961	0.0392	0.5878	0.0451	0.2101	0.0654
(0.5,0.8,0.2)	0.4957	0.0403	0.7841	0.0556	0.1859	0.0756

Table 2.2

Simulation results when $n = 200$ and $\sigma = 0.6$

Initial Value of (ρ,μ,σ)	Estimate of ρ	Std.error of ρ	Estimate of μ	Std.error of μ	Estimate of σ	Std.error of σ
(0.2,0.2,0.6)	0.1978	0.0422	0.2018	0.0529	0.6256	0.0442
(0.2,0.4,0.6)	0.2012	0.0419	0.4013	0.0537	0.5914	0.1523
(0.2,0.6,0.6)	0.2013	0.0402	0.5994	0.0566	0.6256	0.1653
(0.2,0.8,0.6)	0.2002	0.0421	0.7986	0.0641	0.5763	0.1582
(0.5,0.2,0.6)	0.4937	0.037	0.195	0.0666	0.5623	0.1489
(0.5,0.4,0.6)	0.4931	0.0371	0.3923	0.0689	0.6245	0.2013
(0.5,0.6,0.6)	0.4946	0.0374	0.5922	0.0751	0.5896	0.1856
(0.5,0.8,0.6)	0.4951	0.0375	0.7889	0.0834	0.6235	0.2136

When $\kappa = 1$, the ALAR (1) process is reduced to LAR (1) process. In such a situation we can estimate the parameters as follows:

Let $\{x_0, x_1, ..., x_n\}$ be a given set of observations of a LAR (1) process with unknown parameters $\theta = (\rho, \sigma)$. Then the conditional least square estimate (CLS) of θ is obtained by minimizing the sum of squares

$$S_n(\theta) = \sum_{i=1}^{n} (x_i - E(X_i / X_{i-1} = x_{i-1}))^2$$

$$= \sum_{i=1}^{n} (x_i - \rho x_{i-1})^2 .$$

Hence

$$\frac{\partial S_n(\theta)}{\partial \rho} = 0 \Rightarrow \sum_{i=1}^{n} x_i x_{i-1} - \rho \sum_{i=1}^{n} x_{i-1}^2 = 0 .$$

Therefore, the CLS estimate of ρ is given by

$$\hat{\rho} = \frac{\sum_{i=1}^{n} x_i x_{i-1}}{\sum_{i=1}^{n} x_{i-1}^2} .$$

The likelihood function is given by

$$L(x_0, x_1, ..., x_n; \sigma) = f_{X_0}(x_0) \prod_{j=1}^{n} f_{X_j / X_{j-1}}(x_j / x_{j-1})$$

where the conditional probability density function of X_j given $X_{j-1} = x_{j-1}$ is given by

$$f_{X_j / X_{j-1}}(x_j / x_{j-1}) = \rho^2 \delta(x_j - \rho x_{j-1}) + (1 - \rho^2) \frac{1}{2\sigma} \exp(-|x_j - \rho x_{j-1}| / \sigma) .$$

The maximum likelihood estimate of the parameter σ is obtained by maximizing the log-likelihood function

$$\log L(x_0, x_1, ..., x_n; \sigma) = \log f_X(x_0) + \sum_{j=1}^{n} \log f_{X_j / X_{j-1}}(x_j / x_{j-1}) .$$

From the above log-likelihood function equation, Sim (1994) obtained the maximum likelihood estimate of σ as $\hat{\sigma} = \frac{1}{n'}\sum_{i=1}^{n'} \left| y_i \right|$ where $y_i = x_i - \rho x_{i-1}$ and n' is the number of pairs of (x_{i-1}, x_i) such that $x_i - \rho x_{i-1} \neq 0$. Hence for the LAR (1) process the model parameter ρ is estimated by CLS method and the estimate σ is given by the method of maximum likelihood.

Remark 2.2.3.

An alternative way of estimating the model parameter ρ of ALAR (1) process is possible, using the estimates of distribution parameters $\hat{\mu}$, $\hat{\sigma}$ and estimate of probability of up or down runs of the observations.

Let $\hat{P}$ be the estimate of $P(X_n > X_{n-1})$, probability of up-runs, then the estimate is obtained from the set observations and is given by

$$\hat{P} = \frac{1}{N}\sum_{i=1}^{N} I(X_i > X_{i-1})$$

where $I(X_i > X_{i-1}) = \begin{cases} 1 & \text{if } X_i > X_{i-1} \\ 0 & \text{otherwise.} \end{cases}$

Since the estimates of distribution parameters $\hat{\mu}$, $\hat{\sigma}$ are available, the estimate $\hat{\rho}$ of ρ is obtained by substituting estimates $\hat{P}$, $\hat{\mu}$ and $\hat{\sigma}$ for $P(X_n > X_{n-1})$, μ and σ respectively in the expression (2.2.8).

2.3. Generalized asymmetric Laplace process

Mathai (1993a) introduced and studied a class of distributions, namely generalized Laplace distribution, with characteristic function

$$\phi(t)=\left(\frac{1}{1+\sigma^2 t^2}\right)^{\tau}, \sigma \geq 0, \tau \geq 0. \qquad (2.3.1)$$

The applications of generalized Laplace distributions in different contexts such as the production of a chemical called meltatonin in human body, solar neutrino fluxes in cosmos, growth decay mechanism like formation of sand dunes in nature etc. were discussed in Mathai (1993 a, 1993 b, 1994, 2000). The applications of generalized Laplace distribution in the field of time series modeling is discussed in Seetha Lekshmi *et al.* (2003) and they developed first order autoregressive process with generalized Laplace distribution as the marginal distribution.

Kotz *et al.* (2001) developed a general class of asymmetric distributions by introducing asymmetric component in the generalized Laplace distributions. The resulting class of asymmetric distributions is called generalized asymmetric Laplace (GAL) distributions.

Definition 2.3.1.

A random variable X is said to have a generalized asymmetric Laplace distribution if its characteristic function is given by

$$\psi(t)=\left(\frac{1}{1+\sigma^2 t^2-i\mu t}\right)^{\tau},\ -\infty<\mu<\infty,\sigma,\tau\geq 0 \qquad (2.3.2)$$

and we write it as $X \sim GAL(\mu,\sigma,\tau)$.

Note that the characteristic function (2.3.2) becomes characteristic function of

(i) a symmetric Laplace distribution for $\tau=1,\mu=0$

(ii) an asymmetric Laplace random variable when $\tau=1$ and

(iii) a gamma distribution for $\sigma=0$.

The probability density function corresponding to (2.3.2) can be represented in terms of Bessel function of third kind and hence this class of distributions is referred to as Bessel function distribution. Again since this class of distributions arises as a mixture of normal distribution with stochastic variance having gamma distribution, it is called variance gamma model in literature.

If $X \sim GAL(\mu,\sigma,\tau)$ then we obtain the representation of the random variable as

$$X \underline{d} \mu W+\sigma\sqrt{2W}\,Z$$

where Z and W are independent random variables such that Z is a standard normal random variable and W is gamma random variable with probability density function

$$f(w)=\frac{1}{\Gamma(\tau)}w^{\tau-1}e^{-w};\ w>0,\tau>0\ .$$

From this representation it is simple to find the mean and variance of the random variable and they are given by $E(X) = \tau\mu$ and $V(X) = \tau(\mu^2 + 2\sigma^2)$

Let X be a $GAL(\mu, \sigma, \tau)$ random variable, where $\tau = n$, a positive integer then we can represent

$$X \overset{d}{=} \sum_{i=1}^{n} X_i,$$

where X_i's are independent and identically distributed $AL(\mu, \sigma)$ random variables. Since asymmetric Laplace distribution is self-decomposable, from this representation we can conclude that $GAL(\mu, \sigma, \tau = n)$ is self-decomposable as well. Thus we can define a first order stationary autoregressive process with $GAL(\mu, \sigma, \tau = n)$ marginal distribution.

For the construction of the process, let us define a bivariate random variable (δ_1, δ_2) having values $(0,0)$, $(1,0)$ and $(0,1)$ with respective probabilities

$$P(\delta_1 = 0, \delta_2 = 0) = \pi_1,\ P(\delta_1 = 1, \delta_2 = 0) = \pi_2 \text{ and } P(\delta_1 = 0, \delta_2 = 1) = \pi_3 \qquad (2.3.3)$$

where $\pi_1 = \rho^2, \pi_2 = (1-\rho)\left(\rho + \frac{1-\rho}{1+\kappa^2}\right),\ \pi_3 = (1-\rho)\left(\rho + \frac{(1-\rho)\kappa^2}{1+\kappa^2}\right)$.

Theorem 2.3.1.

The first order autoregressive process $\{X_n\}$ given by (2.2.1) is a stationary time series with $GAL(\mu, \sigma, \tau = n)$ marginal distribution if and only if the innovation

sequence $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables given by

$$\varepsilon_n = \sum_{i=1}^{n} \varepsilon_{ni} \tag{2.3.4}$$

where

$$\varepsilon_{ni} \stackrel{d}{=} \sigma\left(\frac{1}{\kappa}\delta_1 Z_{1i} - \kappa\delta_2 Z_{2i}\right) \tag{2.3.5}$$

and Z_{1i}, Z_{2i} are two independent standard exponential random variables such that Z_{1i}, Z_{2i} and (δ_1, δ_2) are mutually independent, provided $X_0 \stackrel{d}{=} GAL(\mu, \sigma, n)$.

Proof:

From equation (2.2.1), the characteristic function of $\{\varepsilon_n\}$ is given by

$$\phi_{\varepsilon_n}(t) = \frac{\phi_{X_n}(t)}{\phi_{X_{n-1}}(\rho t)}.$$

Suppose $\{X_n\}$ is stationary with $GAL(\mu, \sigma, \tau = n)$ marginal distribution, then

$$\phi_{\varepsilon_n}(t) = \left[\frac{1+\sigma^2\rho^2 t^2 - i\mu\rho t}{1+\sigma^2 t^2 - i\mu t}\right]^n.$$

It can be expressed as

$$\phi_{\varepsilon_n}(t)=\left[\rho^2+(1-\rho)\left(\rho+\frac{1-\rho}{1+\kappa^2}\right)\frac{1}{1-i\frac{\sigma}{\kappa}t}+(1-\rho)\left(\rho+\frac{(1-\rho)\kappa^2}{1+\kappa^2}\right)\frac{1}{1+i\sigma\kappa t}\right]^n \quad .(2.3.6)$$

Suppose Z_{1i}, Z_{2i} are independent standard exponential variables with characteristic function $\frac{1}{1-it}$ and the bivariate probability of (δ_1,δ_2) is given by (2.3.3) and let Z_{1i}, Z_{2i} and (δ_1,δ_2) are mutually independent.

Then from the representation (2.3.6), we can write

$$\varepsilon_n=\sum_{i=1}^{n}\varepsilon_{ni},$$

where random variable ε_{ni} is distributed as

$$\varepsilon_{ni}\stackrel{d}{=}\sigma\left(\frac{1}{\kappa}\delta_1 Z_{1i}-\kappa\delta_2 Z_{2i}\right),$$

Conversely, we can prove that the process is stationary with $GAL(\mu,\sigma,\tau=n)$ marginal distribution if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables given by (2.3.4) and (2.3.5) and $X_0\stackrel{d}{=}GAL(\mu,\sigma,n)$ using an inductive argument very easily.

Hence the theorem.

Using the above theorem we can generate an autoregressive process with $GAL(\mu,\sigma,\tau=n)$ marginal distribution by using an independent and identically distributed sequence $\{\varepsilon_n\}$ as given by (2.3.4) and (2.3.5). Such an autoregressive process is called a first order generalized asymmetric Laplace (GALAR (1)) process.

Consider a simple case, $\tau=2$.

Then the marginal distribution of GALAR (1) process is $GAL(\mu,\sigma,\tau=2)$. The characteristic function of the innovation sequence $\{\varepsilon_n\}$ is given by

$$\phi_{\varepsilon_n}(t)=\left[\frac{1+\sigma^2\rho^2t^2-i\mu\rho t}{1+\sigma^2t^2-i\mu t}\right]^2$$

$$=\left[\pi_1+\pi_2\frac{1}{1-i\frac{\sigma}{\kappa}t}+\pi_3\frac{1}{1+i\sigma\kappa t}\right]^2$$

$$\begin{aligned}\phi_{\varepsilon_n}(t)=\pi_1^2+\pi_2^2\left(\frac{1}{1-i\frac{\sigma}{\kappa}t}\right)^2+\pi_3^2\left(\frac{1}{1+i\sigma\kappa t}\right)^2\\ +2\pi_1\pi_2\frac{1}{1-i\frac{\sigma}{\kappa}t}+2\pi_1\pi_3\frac{1}{1+i\sigma\kappa t}+2\pi_2\pi_3\frac{1}{1-i\frac{\sigma}{\kappa}t}\frac{1}{1+i\sigma\kappa t}.\end{aligned} \tag{2.3.7}$$

Let E_{n1}, E_{n2} be two exponential random variables; G_{n1}, G_{n2} be two gamma random variables and AL_n be an asymmetric Laplace random variable and assume they are mutually independent.

From the representation (2.3.7), we get

$$\phi_{\varepsilon_n}(t) = \pi_1^2 + \pi_2^2 \phi_{G_{n1}}(t) + \pi_3^2 \phi_{-G_{n2}}(t)$$

$$+ 2\pi_1\pi_2 \phi_{E_{n1}}(t) + 2\pi_1\pi_3 \phi_{-E_{n2}}(t) + 2\pi_2\pi_3 \phi_{AL_n}(t)$$

where

$$E_{n1} \sim \exp(\frac{\kappa}{\sigma}), E_{n2} \sim \exp(\frac{1}{\kappa\sigma})$$

$$G_{n1} \sim G(\frac{\kappa}{\sigma}, 2),\ G_{n2} \sim G(\frac{1}{\kappa\sigma}, 2) \text{ and}$$

$$AL_n \sim AL(\mu, \sigma).$$

Hence the innovation sequence $\{\varepsilon_n\}$ can be considered as a convex mixture of exponential, gamma and asymmetric Laplace random variables.

LAPLACE AUTOREGRESSIVE TIME SERIES MODELS

Thesis Submitted to the University of Calicut for the degree of

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in Statistics

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Chapter-III

A New Autoregressive Time Series Model Using Laplace Variables

3.1. Introduction

The autoregressive models discussed in the previous Chapter have the problem of "zero defect", caused successive values of the process to be fixed multiples of the previous values. Lawrance and Lewis (1981) removed this type of drawback from the exponential autoregressive process of Gaver and Lewis (1980) by defining two-parameter first order autoregressive model. Such model, which is free from "zero defect", is named as first order new exponential autoregressive (NEAR (1)) model. The structure of such two-parameter autoregressive model is given by

$$X_n = \begin{cases} \varepsilon_n & \text{w.p.} \quad p \\ \rho X_{n-1} + \varepsilon_n & \text{w.p.} \quad 1-p, \end{cases} \tag{3.1.1}$$

where $0 < p < 1$.

Dewald and Lewis (1985) discussed and studied the autoregressive process given by the equation (3.1.1) with $|\rho| < 1$, using Laplace random variables. Such autoregressive model is termed as new Laplace autoregressive process of order one

(NLAR (1)) and we can easily notice that the model is free from the "zero defect" with the help of the following theorem.

Theorem 3.1.1.

Let $\{\varepsilon_n\}$ be a sequence of independent and identically distributed random variables given by

$$\varepsilon_n = \begin{cases} L_n & \text{w.p.} \quad p_1 \\ \sqrt{p}\,|\rho| L_n & \text{w.p.} \quad 1-p_1, \end{cases} \tag{3.1.2}$$

where $p_1 = \dfrac{1-\rho^2}{1-\rho^2 p}$ and $\{L_n\}$ is a sequence of independently and identically distributed Laplace random variables. Then the relation (3.1.1) with $X_0 \overset{d}{=} L(\sigma)$ defines a stationary autoregressive process with Laplace marginal distribution.

In the following Section we develop a new two-parameter autoregressive process having structure (3.1.1) using asymmetric Laplace distribution.

3.2. A new autoregressive process using asymmetric Laplace variables

Since the asymmetric Laplace distribution is self-decomposable only if $0 \le \rho < 1$, a two-parameter first order autoregressive process with the structure (3.1.1) using asymmetric Laplace variables can be constructed by assuming the value of the

parameter $\rho \in [0,1)$. Hence we define a first order autoregressive process $\{X_n\}$ given by (3.1.1) with $0 < p < 1, 0 \le \rho < 1$ where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables selected in such a way that $\{X_n\}$ is a stationary process with asymmetric Laplace marginal distribution.

Theorem 3.2.1.

The first order autoregressive process (3.1.1) with $X_0 \overset{d}{=} AL(\mu, \sigma)$ is stationary with asymmetric Laplace marginal distribution if and only if $\varepsilon_n = U_n + V_n$ where U_n and V_n are two independent random variables with

$$U_n = \begin{cases} 0 & \text{w.p.} \quad \rho^2 \\ E_{n1} & \text{w.p.} \quad (1-\rho)\left(\rho + \dfrac{1-\rho}{1+\kappa^2}\right) \\ -E_{n2} & \text{w.p.} \quad (1-\rho)\left(\rho + \dfrac{(1-\rho)\kappa^2}{1+\kappa^2}\right), \end{cases} \tag{3.2.1}$$

and

$$V_n = AL_n \,, \; AL_n \overset{d}{=} AL(\mu\rho p, \sigma\rho\sqrt{p}) \tag{3.2.2}$$

where $\{E_{n1}\}$ and $\{E_{n2}\}$ are sequences of independent and identically distributed exponential random variables with means σ/κ and $\kappa\sigma$ respectively.

Proof:

In terms of characteristic function, equation (3.1.1) becomes

$$\phi_{\varepsilon_n}(t) = \frac{\phi_{X_n}(t)}{p + (1-p)\phi_{X_n}(\rho t)} \tag{3.2.3}$$

If we assume $\{X_n\}$ is stationary with asymmetric Laplace marginal distribution, then (3.2.3) implies

$$\phi_{\varepsilon_n}(t) = \frac{1+\sigma^2\rho^2t^2 - i\mu\rho t}{1+\sigma^2t^2 - i\mu t} \frac{1}{1+\sigma^2\rho^2pt^2 - i\mu\rho pt}$$

$$= \left(\pi_1 + \pi_2 \frac{1}{1 - i\frac{\sigma}{\kappa}t} + \pi_3 \frac{1}{1+i\sigma\kappa t}\right) \frac{1}{1+\sigma^2\rho^2pt - i\mu\rho pt},$$

where $\pi_1 = \rho^2$, $\pi_2 = (1-\rho)\left(\rho + \frac{1-\rho}{1+\kappa^2}\right)$, $\pi_3 = (1-\rho)\left(\rho + \frac{(1-\rho)\kappa^2}{1+\kappa^2}\right)$ and note that $\frac{1}{\kappa} - \kappa = \frac{\mu}{\sigma}$.

If U_n and V_n are two independent random variables, then we can write

$$\phi_{\varepsilon_n}(t) = \phi_{U_n}(t)\,\phi_{V_n}(t),$$

where

$$U_n = \begin{cases} 0 & \text{w.p. } \pi_1 \\ E_{n1} & \text{w.p. } \pi_2 \\ -E_{n2} & \text{w.p. } \pi_3, \end{cases}$$

and

$$V_n = AL_n \ , \quad AL_n \stackrel{d}{=} AL(\mu\rho p, \sigma\rho\sqrt{p}) \ .$$

Conversely, if $\varepsilon_n = U_n + V_n$, where U_n and V_n given by (3.2.1) and (3.2.2) and $X_0 \stackrel{d}{=} AL(\mu,\sigma)$, then from (3.1.1), when $n=1$, we have

$$\phi_{X_1}(t) = \phi_{\varepsilon_1}(t)\left[p + (1-p)\phi_{X_0}(t)\right]$$

$$= \frac{1}{1+\sigma^2\rho^2 t^2 - i\mu\rho t} \ \frac{1+\sigma^2\rho^2 t^2 - i\mu\rho t}{1+\sigma^2 t^2 - i\mu t}\left[p + (1-p)\frac{1}{1+\sigma^2\rho^2 t^2 - i\mu\rho t}\right]$$

$$= \frac{1}{1+\sigma^2 t^2 - i\mu t} \ .$$

If $X_{n-1} \stackrel{d}{=} AL(\mu,\sigma)$ then we get $X_n \stackrel{d}{=} AL(\mu,\sigma)$.

Thus $\{X_n\}$ is a stationary process with asymmetric Laplace marginal distribution.

Hence the theorem.

We call the first order autoregressive process $\{X_n\}$ given by

$$X_n = \begin{cases} \varepsilon_n & \text{w.p.} \quad p \\ \rho X_{n-1} + \varepsilon_n & \text{w.p.} \quad 1-p, \end{cases}$$

where $0 \le \rho < 1$ and $0 < p < 1$ with $X_0 \stackrel{d}{=} AL(\mu, \sigma)$ and $\varepsilon_n = U_n + V_n$, U_n and V_n are two independent random variables defined in (3.2.1) and (3.2.2) as the new asymmetric Laplace autoregressive process of first order ((NALAR (1)).

It may be noted that when $p = 0$, the NALAR (1) model is equivalent to the ALAR (1) model.

Remark 3.2.1.

When $\kappa = 1$ (symmetric case), representation (3.2.1) and (3.2.2) reduces to the form

$$U_n = \begin{cases} 0 & \text{w.p.} \quad \rho^2 \\ L_n(\sigma) & \text{w.p.} \quad 1-\rho^2 \end{cases}$$

and

$$V_n = L_n(\sigma \rho \sqrt{p}) ,$$

where $\{L_n(\sigma)\}$ is a sequence of symmetric Laplace random variables with characteristic function (1.3.2). Hence, if $\{X_n\}$ is stationary with symmetric Laplace marginal distribution then the innovation sequence $\{\varepsilon_n\}$ of the model (3.1.1) can be regarded as the sum of two independent random variables U_n and V_n.

3.2.1. Properties of NALAR (1) process

From the representation (3.2.1) and (3.2.2), it is verified that

$$E(U_n) = \mu(1-\rho)$$

and

$$E(V_n) = p\rho\mu.$$

Hence

$$E(\varepsilon_n) = (1-\rho)\mu + p\rho\mu, \tag{3.2.4}$$

because $\varepsilon_n = U_n + V_n$.

The conditional expectation

$$E(X_n / X_{n-1} = x) = (1-p)\rho x + E(\varepsilon_n)$$

$$= (1-p)\rho x + \mu - \rho\mu(1-p)$$

$$= (1-p)\ \rho\,(x-\mu) + \mu.$$

That is, the conditional expectation $E(X_n / X_{n-1} = x)$ of the NALAR (1) process is linear in x.

The autocorrelation function of the NALAR (1) process is obtained as follows:

Consider $E(X_n X_{n-1}) = E\left(E(X_n X_{n-1} / X_{n-1} = x)\right)$,

where $E(X_n X_{n-1} / X_{n-1} = x) = (1-p)\rho x^2 + x\,E(\varepsilon_n)$

$= (1-p)\rho x^2 + x\,(1-\rho)\,\mu + x\,p\,\rho\,\mu$, using (3.2.4).

Hence

$$E(X_n X_{n-1}) = E\left((1-p)\,\rho\,X^2 + (1-\rho)\mu X + p\,\rho\,\mu X\right)$$

$$= (1-p)\rho\left(2\mu^2 + 2\sigma^2\right) + (1-\rho)\,\mu^2 + p\rho\mu^2,$$

because $X_n \underline{d} AL(\mu,\sigma)$ implies $E(X_n) = \mu$, $E(X_n^2) = 2\mu^2 + 2\sigma^2$.

Therefore

$$E(X_n X_{n-1}) = \mu^2\left(1 + (1-p)\,\rho\right) + (1-p)\,\rho\,2\sigma^2.$$

This implies

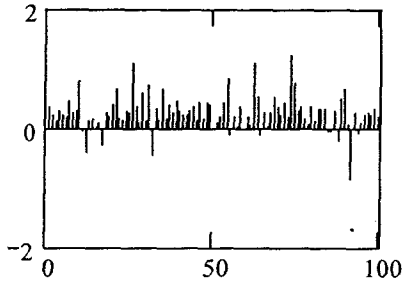
$$Cov(X_n X_{n-1}) = (1-p)\,\rho\left(\mu^2 + 2\sigma^2\right)$$

Hence, the first order autocorrelation function of the NALAR (1) process is given by

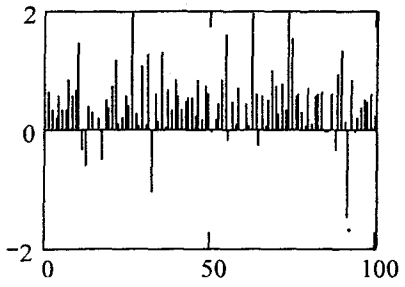
$$\rho(1) = \frac{(1-p)\,\rho\left(\mu^2 + 2\sigma^2\right)}{(\mu^2 + 2\sigma^2)} = \rho(1-p).$$

Using similar arguments we can prove that the h^{th} order autocorrelation function of the process is given by

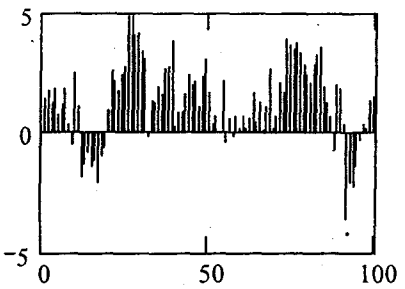
$$\rho(h) = \left(\rho(1-p)\right)^h. \tag{3.2.5}$$



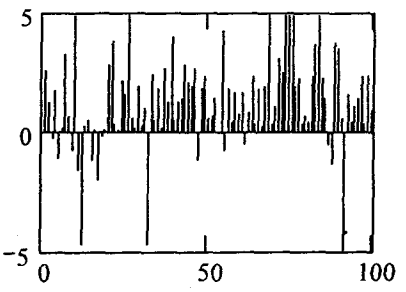
$\rho = 0.2, p = 0.2$



$\rho = 0.2, p = 0.7$



$\rho = 0.8, p = 0.2$



$\rho = 0.8, p = 0.7$

Fig.3.2.1

Sample path of NALAR (1) process

The simulated sample paths of NALAR (1) process for various model parameters ρ and p when $\mu = 0.5$ and $\sigma = 2$ are given in the above figures.

3.2.2. One parameter NALAR (1) model

In this Section we consider a tractable form of autoregressive model that is free from "zero defect" using asymmetric Laplace distribution with structure similar to the one-parameter TEAR (1) model discussed in Lawrance and Lewis (1981).

Let $\{X_n\}$ be an autoregressive process defined by

$$X_n = \begin{cases} \varepsilon_n & \text{w.p.} \quad p \\ X_{n-1} + \varepsilon_n & \text{w.p.} \quad 1-p \end{cases} \tag{3.2.6}$$

where $0 < p < 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables. It may be noted that the model (3.2.6) has only one parameter p and it is a particular case of (3.1.1) with $\rho = 1$.

Pillai and Jose (1994) established that the model given by the equation (3.2.6) is defined if and only if stationary distribution of the process is geometrically infinitely divisible. Since asymmetric Laplace distribution belongs to the class of geometrically infinitely divisible distributions, it is possible to construct a stationary process $\{X_n\}$ with structure given by (3.2.6) for the asymmetric Laplace distribution.

Theorem 3.2.2.

The first order autoregressive process defined by (3.2.6) is stationary with $AL(\mu,\sigma)$ marginal if and only if $\{\varepsilon_n\}$ is distributed as $AL(\mu p,\sigma\sqrt{p})$ and $X_0 \stackrel{d}{=} AL(\mu,\sigma)$.

Proof:

Using characteristic functions, equation (3.2.6) gives

$$\phi_{X_n}(t) = p\phi_{\varepsilon_n}(t) + (1-p)\ \phi_{X_{n-1}}(t)\phi_{\varepsilon_n}(t). \tag{3.2.7}$$

Assuming stationarity, from equation (3.2.7) we obtain

$$\phi_{\varepsilon_n}(t) = \frac{\phi_X(t)}{p + (1-p)\phi_X(t)}. \tag{3.2.8}$$

If $X_n \stackrel{d}{=} AL(\mu,\sigma)$ then $\phi_X(t) = \dfrac{1}{1+\sigma^2t^2 - i\mu t}$.

Substituting this in (3.2.8), we get

$$\phi_{\varepsilon_n}(t) = \frac{1}{1+\sigma^2 pt^2 - i\mu pt}.$$

Hence

$$\varepsilon_n \stackrel{d}{=} AL(\mu p,\sigma\sqrt{p}).$$

Conversely, if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $AL(\mu p, \sigma\sqrt{p})$ random variables and $X_0 \stackrel{d}{=} AL(\mu,\sigma)$, then from (3.2.7), when $n=1$, we have

$$\phi_{X_1}(t) = p\,\frac{1}{1+\sigma^2 pt^2 - i\mu pt} + (1-p)\,\frac{1}{1+\sigma^2 t^2 - i\mu t}\,\frac{1}{1+\sigma^2 pt^2 - i\mu pt}$$

$$= \frac{1}{1+\sigma^2 t^2 - i\mu t}.$$

If $X_{n-1} \stackrel{d}{=} AL(\mu,\sigma)$ then we get $X_n \stackrel{d}{=} AL(\mu,\sigma)$.

Thus $\{X_n\}$ is a stationary process with asymmetric Laplace marginal distribution.

Hence the theorem. □

We call the autoregressive process defined in (3.2.6) with $X_0 \stackrel{d}{=} AL(\mu,\sigma)$ as the TALAR (1) process.

From the definition (3.2.6) of the model it is easily verified that

$$\phi_{X_n}(t) = p\,\phi_{\varepsilon_n}(t)\,\frac{1-(1-p)^n \phi^n_{\varepsilon_n}(t)}{1-(1-p)\phi_{\varepsilon_n}(t)} + (1-p)^n\,\phi_{X_0}(t)\,\phi^n_{\varepsilon_n}(t). \qquad (3.2.9)$$

Hence

$$\phi_{X_n}(t) = p\,\phi_{\varepsilon_n}(t)\frac{1}{1-(1-p)\phi_{\varepsilon_n}(t)}, \text{ when } n \to \infty. \tag{3.2.10}$$

If X_0 has an arbitrary distribution and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables such that $\varepsilon_n \stackrel{d}{=} AL(\mu p, \sigma\sqrt{p})$ then from (3.2.10) we get

$$\phi_{X_n}(t) = \frac{1}{1+\sigma^2 t^2 - i\mu t}, \text{ when } n \to \infty.$$

Hence, if X_0 has an arbitrary distribution then the autoregressive process is asymptotically stationary with $AL(\mu, \sigma)$ marginal distribution.

When $\rho = 1$, the NALAR (1) model reduced to the TALAR (1) model. Hence, for the TALAR (1) model, the regression of X_n on X_{n-1} is

$$E(X_n / X_{n-1} = x) = (1-p)\,x + p\mu .$$

and the h^{th} order autocorrelation function $\rho(h)$ is

$$\rho(h) = (1-p)^h. \tag{3.2.11}$$

The joint characteristic function of (X_{n-1}, X_n) of the TALAR (1) process is given by

$$\phi_{X_{n-1},X_n}(t,s) = \frac{1}{1+\sigma^2 s^2 - i\mu s}\left[\frac{p}{1+\sigma^2 t^2 - i\mu t} + \frac{1-p}{1+\sigma^2 (t+s)^2 - i\mu (t+s)}\right]$$

From the expression it is clear that $\phi_{X_{n-1},X_n}(t,s) \neq \phi_{X_{n-1},X_n}(s,t)$ and hence the process is not time reversible.

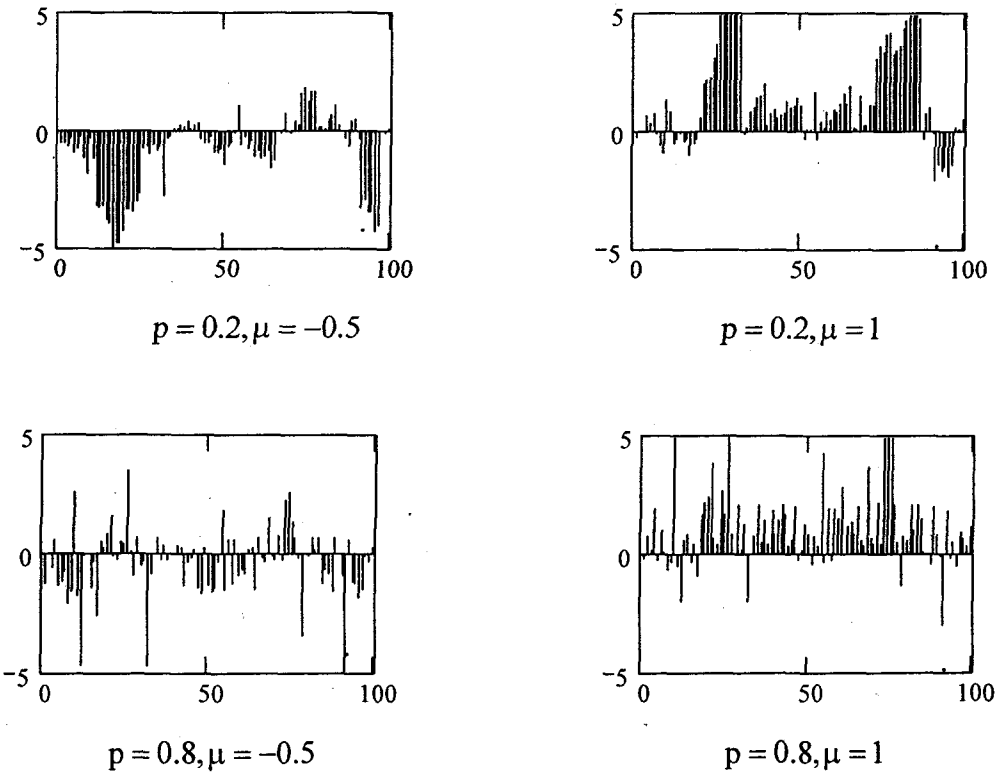
Construction of usual higher order autoregressive models of the form (1.1.1) using asymmetric Laplace distribution is a complicated problem. This is because it is a difficult task to establish the distribution of innovation sequence $\{\varepsilon_n\}$ so as to ensure that $\{X_n\}$ is stationary with asymmetric marginal distribution if $\{X_n\}$ is given by (1.1.1). Here we present a tractable form of higher order autoregressive model using asymmetric Laplace distribution.

Let $\{X_n\}$ be a an autoregressive process defined by

$$X_n = \begin{cases} \varepsilon_n & \text{w.p.} \quad p_0 \\ X_{n-1} + \varepsilon_n & \text{w.p.} \quad p_1 \\ \vdots & \vdots \quad \vdots \\ X_{n-k} + \varepsilon_n & \text{w.p.} \quad p_k, \end{cases} \tag{3.2.12}$$

where $\sum_{i=0}^{k} p_i = 1$, $0 < p_i < 1$, $i = 0,1,...,k$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables. Then $\{X_n\}$ given by (3.2.12) defines a stationary autoregressive process of order k with $X_0 \stackrel{d}{=} AL(\mu,\sigma)$ if and only if $\{\varepsilon_n\}$ is distributed as $AL\left(\mu p_0, \sigma\sqrt{p_0}\right)$.

The sample path of TALAR (1) process for various values of μ and p when $\sigma = 1$ is given in the following figures.



$p = 0.2, \mu = -0.5$ $\quad$ $p = 0.2, \mu = 1$

$p = 0.8, \mu = -0.5$ $\quad$ $p = 0.8, \mu = 1$

Fig.3.2.2

Sample path of TALAR (1) process.

3.3. First order autoregressive geometric stable processes

Geometric stable distributions received much attention in recent years and their applications in different fields such as reliability, biology, economics, insurance mathematics etc. are well established by several researchers. It can be used in modeling peaked and heavy tailed observations that may be a result of a random number of independent innovations. The theory and applications of geometric stable distributions have been studied by several researchers but their applications in time series modeling are not developed. In this Section, we introduce a time series model with geometric stable distribution as marginal distribution, which can be used for modeling heavy tailed asymmetric time series observations.

Theorem 3.3.1.

Let $\{X_n\}$ be an autoregressive process given by (3.2.6) with $X_0 \stackrel{d}{=} GS_\alpha(\sigma,\beta,\mu)$. Then $\{X_n\}$ is a stationary process if and only if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables with $GS_\alpha\left(\sigma p^{1/\alpha},\beta,\mu p\right)$ distribution.

Proof:

Let $\{X_n\}$ be a stationary process with $GS_\alpha(\sigma,\beta,\mu)$ marginal distribution.

Substituting the characteristic function (1.4.9) in (3.2.8), we have

$$\phi_{\varepsilon_n}(t) = \frac{\dfrac{1}{1+\sigma^{\alpha}|t|^{\alpha}\varpi_{\alpha,\beta}(t)-i\mu t}}{p+(1-p)\dfrac{1}{1+\sigma^{\alpha}|t|^{\alpha}\varpi_{\alpha,\beta}(t)-i\mu t}}$$

$$= \frac{1}{1+\sigma^{\alpha}p|t|^{\alpha}\varpi_{\alpha,\beta}(t)-i\mu pt}$$

Hence

$$\varepsilon_n \stackrel{d}{=} GS_{\alpha}\left(\sigma p^{1/\alpha}, \beta, \mu p\right). \tag{3.3.1}$$

Conversely, if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $GS_{\alpha}\left(\sigma p^{1/\alpha}, \beta, \mu p\right)$ random variables and let $X_0 \stackrel{d}{=} GS_{\alpha}(\sigma, \beta, \mu)$, then from (3.2.7) when $n = 1$, we have

$$\phi_{X_1}(t) = p\,\frac{1}{1+\sigma^{\alpha}p|t|^{\alpha}\varpi_{\alpha,\beta}(t)-i\mu pt} + (1-p)\,\frac{1}{1+\sigma^{\alpha}p|t|^{\alpha}\varpi_{\alpha,\beta}(t)-i\mu pt}\,\frac{1}{1+\sigma^{\alpha}|t|^{\alpha}\varpi_{\alpha,\beta}(t)-i\mu t}$$

$$= \frac{1}{1+\sigma^{\alpha}|t|^{\alpha}\varpi_{\alpha,\beta}(t)-i\mu t}.$$

If $X_{n-1} \stackrel{d}{=} GS_{\alpha}(\sigma,\beta,\mu)$ then we get $X_n \stackrel{d}{=} GS_{\alpha}(\sigma,\beta,\mu)$.

Thus, $\{X_n\}$ is a stationary process with geometric stable distribution as marginal distribution.

Hence the theorem. □

We call the process defined by (3.2.6) with $X_0 \stackrel{d}{=} GS_{\alpha}(\sigma,\beta,\mu)$ and $\varepsilon_n \stackrel{d}{=} GS_{\alpha}\left(\sigma p^{1/\alpha},\beta,\mu p\right)$ as first order geometric stable autoregressive (GSAR (1)) process.

If X_0 is distributed arbitrary and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables given by (3.3.1), then from equation (3.2.10) we get

$$\phi_{X_n}(t) = \frac{1}{1+\sigma^{\alpha}|t|^{\alpha}\varpi_{\alpha,\beta}(t)-i\mu t}, \text{ when } n \to \infty.$$

Hence, if X_0 has an arbitrary distribution then the autoregressive process is asymptotically stationary with geometric stable marginal distribution.

When the parameters σ, β and μ assume different values, we get the following autoregressive process as particular cases of GSAR (1) process.

Case (i): Let $\sigma = 0$.

In this case the geometric stable distribution is reduced to exponential distribution with characteristic function $\phi(t) = \frac{1}{1 - i\mu t}$. Then the GSAR (1) process is stationary with exponential marginal distribution. This model, known as TEAR (1) model, is discussed in Lawrance and Lewis (1981).

Case (ii): Let $\alpha = 2, \mu = 0$.

In this case the geometric stable distribution is reduced to symmetric Laplace distribution with characteristic function $\phi(t) = \frac{1}{1 + \sigma^2 t^2}$. Now the GSAR (1) process is stationary with symmetric Laplace distribution as marginal distribution if and only if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed Laplace variables such that $\varepsilon_n \stackrel{d}{=} p^{1/2} L_n$, where $\{L_n\}$ is a sequence of symmetric Laplace variables.

Case (iii): Let $\alpha = 2, \mu \neq 0$.

When $\alpha = 2$ and μ is any non-zero real number then the geometric stable distribution is reduced to asymmetric Laplace distribution with characteristic function $\phi(t) = \frac{1}{1 + \sigma^2 t^2 - i\mu t}$. Then the GSAR (1) process is stationary with asymmetric Laplace distribution ($AL(\mu, \sigma)$) if and only if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed asymmetric Laplace random variables $AL(\mu p, \sigma p^{1/2})$.

Case (iv): Let $\alpha \in (0,2), \mu = 0$.

When $\alpha \in (0,2)$ and $\mu = 0$, then the geometric stable distribution is reduced to the Linnik distribution with characteristic function $\phi(t) = \frac{1}{1+\sigma^{\alpha}|t|^{\alpha}}$ (see Kozubowski and Rachev (1999)). Then the GSAR (1) process defines a first order stationary autoregressive process with Linnik distribution as marginal distribution if and only if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed Linnik variables such that $\varepsilon_n \underline{d} = p^{1/\alpha} L_{\alpha,n}$, where $\{L_{\alpha,n}\}$ is a sequence of Linnik random variables with characteristic function $\phi(t) = \frac{1}{1+\sigma^{\alpha}|t|^{\alpha}}$.

The model discussed in this case is similar to the past innovations process of Anderson and Arnold (1993). They studied different forms of autoregressive process with Linnik distribution as marginal distribution and discussed their applications in modeling stock price changes.

Case (v): Let $\mu = 0,\ 0 < \alpha \leq 1$.

For $\mu = 0$ and $0 < \alpha \leq 1$, the geometric stable distribution on $(0,\infty)$ has the Laplace transform $f(s) = \frac{1}{1+\sigma^{\alpha}s^{\alpha}}$, $s \geq 0$ (see Kozubowski and Rachev (1999)). In this case, the geometric stable distribution reduces to the Mittag-Leffler distribution discussed in Pillai (1990 (b)). It is a generalization of exponential distribution, to which

it reduces for $\alpha = 1$ (for more properties and applications of Mittag-Leffler distribution see Pillai and Sabu George (1984), Jayakumar and Pillai (1996), Weron and Kotulski (1996) and Lin (1998, 2001)). Now the GSAR (1) process defines a stationary process with Mittag-Leffler distribution as marginal if and only if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed Mittag-Leffler variables, such that $\varepsilon_n \stackrel{d}{=} p^{1/\alpha} M_n$, where $\{M_n\}$ is a sequence of Mittag-Leffler variables with Laplace transform $\frac{1}{1+\sigma^{\alpha} s^{\alpha}}$.

Jayakumar and Pillai (1993) introduced and studied first order autoregressive process $X_n = \rho X_{n-1} + \varepsilon_n, 0 \le \rho < 1$, using semi Mittag-Leffler marginal distributions. The semi-Mittag-Leffler distribution is a class of distribution defined on positive real line with Laplace transform $\frac{1}{1+\eta(s)}$ where $\eta(s)$ satisfies the functional equation $\eta(s) = a\,\eta(bs), 0 < b < 1$ and a is the unique solution of $a\,b^{\alpha} = 1, 0 \le \alpha < 1$. The Mittag-Leffler distribution is a special case of semi Mittag-Leffler distribution. They established that the first order autoregressive equation

$$X_n = \begin{cases} \rho X_{n-1} & \text{w.p. } \rho^{\alpha} \\ \rho X_{n-1} + SM_n & \text{w.p. } 1-\rho^{\alpha} \end{cases}$$

defines a stationary process if and only if SM_n's are semi Mittag-Leffler and with $X_0 \stackrel{d}{=} SM_n$.

3.4. Tailed Laplace distribution and process

The concept of tailed distributions was introduced in Littlejohn (1994) while discussing non-Gaussian time series models. Muralidharan (1999) considered tailed distribution in the context of mixing a degenerate distribution (degenerate at zero) and a continuous distribution such as exponential, Weibull, gamma etc. and discussed inference problems related to such models. The tailed distributions are used as a model in several situations where the variable can take either zero value with certain probability or a random variable with remaining probability. Hence the tailed distribution can be used as a model in different areas like dose response in the medical field, flow of water in a river which is dry for a part of the year, economics that show dull behavior in money circulation etc.

Definition 3.4.1.

Let X be a random variable with characteristic function $\phi_X(t)$. Then a random variable Y with characteristic function

$$\phi_Y(t) = \theta + (1-\theta)\phi_X(t) \qquad (3.4.1)$$

is called a tail random variable associated with X, where $\theta \in (0,1)$ is the tail probability.

Example 3.4.1.

Suppose X be a symmetric Laplace random variable with characteristic function $\phi_X(t) = \frac{1}{1+\sigma^2 t^2}$. Then a tailed random variable Y associated with X and with tail probability θ has the characteristic function

$$\phi_Y(t) = \frac{1+\theta\sigma^2 t^2}{1+\sigma^2 t^2}. \tag{3.4.2}$$

The mean and variance of the tailed Laplace distribution are 0 and $2(1-\theta)\sigma^2$ respectively.

Example 3.4.2.

Suppose X be an asymmetric Laplace random variable with characteristic function $\phi_X(t) = \frac{1}{1+\sigma^2 t^2 - i\mu t}$. Then a tailed random variable Y associated with X and with tail probability θ has the characteristic function

$$\phi_Y(t) = \frac{1+\theta\sigma^2 t^2 - i\theta\mu t}{1+\sigma^2 t^2 - i\mu t}. \tag{3.4.3}$$

If Y is a random variable with characteristic function (3.4.3) then we represent it as $Y \underline{d} ALT(\mu, \sigma, \theta)$.

The mean and variance of the tailed asymmetric Laplace distribution are $\mu(1-\theta)$ and $\mu^2(1-\theta^2)+2(1-\theta)\sigma^2$ respectively.

Now we develop an autoregressive model with tailed marginal distribution using the model discussed in Lawrance and Lewis (1981).

Theorem 3.4.1.

The first order autoregressive equation (3.2.6) defines a stationary first order autoregressive process with tailed asymmetric Laplace marginal distribution if and only if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $ALT\left(\mu\gamma, \sigma\sqrt{\gamma}, \frac{\theta}{\gamma}\right)$ random variables with $X_0 \underline{d} ALT(\mu, \sigma, \theta)$, where $\gamma = p+(1-p)\theta$.

Proof:

Assume $\{X_n\}$ is stationary with tailed asymmetric Laplace distribution having characteristic function (3.4.3). Then, from (3.2.8) using characteristic function we get

$$\phi_{\varepsilon_n}(t) = \frac{\frac{1+\sigma^2\theta t^2 - i\mu\theta t}{1+\sigma^2 t^2 - i\mu t}}{p + (1-p)\frac{1+\sigma^2\theta t^2 - i\mu\theta t}{1+\sigma^2 t^2 - i\mu t}}$$

Simplifying, we get

$$\phi_{\varepsilon_n}(t) = \frac{1+\sigma^2\theta\, t^2 - i\mu\theta t}{1+\sigma^2\gamma\, t^2 - i\mu\gamma\, t}, \quad \text{where } \gamma = p+(1-p)\theta.$$

$$= \frac{\theta}{\gamma} + (1-\frac{\theta}{\gamma})\frac{1}{1+\sigma^2\gamma t^2 - i\mu\gamma t}.$$

Hence

$$\varepsilon_n \overset{d}{\equiv} \text{ALT}\left(\mu\gamma, \sigma\sqrt{\gamma}, \frac{\theta}{\gamma}\right). \tag{3.4.4}$$

Conversely, if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $\text{ALT}\left(\mu\gamma, \sigma\sqrt{\gamma}, \frac{\theta}{\gamma}\right)$ random variables and $X_0 \overset{d}{\equiv} \text{ALT}(\mu,\sigma,\theta)$, then from (3.2.7) when $n=1$, we have

$$\phi_{X_1}(t) = p\,\frac{1+\sigma^2\theta\, t^2 - i\mu\theta t}{1+\sigma^2\gamma\, t^2 - i\mu\gamma\, t} + (1-p)\,\frac{1+\sigma^2\theta\, t^2 - i\mu\theta t}{1+\sigma^2 t^2 - i\mu t}\,\frac{1+\sigma^2\theta\, t^2 - i\mu\theta t}{1+\sigma^2\gamma\, t^2 - i\mu\gamma\, t}$$

$$= \frac{1+\sigma^2\theta\, t^2 - i\mu\theta\, t}{1+\sigma^2 t^2 - i\mu t}.$$

If $X_{n-1} \overset{d}{\equiv} \text{ALT}(\mu,\sigma,\theta)$ then we get $X_n \overset{d}{\equiv} \text{ALT}\,(\mu,\sigma,\theta)$.

Thus $\{X_n\}$ is stationary with tailed asymmetric Laplace marginal distribution.

Hence the theorem. □

We call the first order autoregressive process given by the structure (3.2.6) with $X_0 \overset{d}{=} ALT(\mu,\sigma,\theta)$ and $\varepsilon_n \overset{d}{=} ALT\left(\mu\gamma, \sigma\sqrt{\gamma}, \frac{\theta}{\gamma}\right)$ as the fist order tailed asymmetric Laplace autoregressive (ALTAR (1)) process.

If X_0 is distributed arbitrary and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $ALT\left(\mu\gamma, \sigma\sqrt{\gamma}, \frac{\theta}{\gamma}\right)$ random variables then from (3.2.10) we get

$$\phi_{X_n}(t) = \frac{1+\sigma^2\theta t^2 - i\mu\theta t}{1+\sigma^2 t^2 - i\mu t}, \text{ when } n \to \infty.$$

Hence if X_0 has an arbitrary distribution then the autoregressive process is asymptotically stationary with $ALT(\mu,\sigma,\theta)$ marginal distribution.

For the ALTAR (1) process $\{X_n\}$,

$$E(X_n) = \mu(1-\theta),$$

$$Var(X_n) = \mu^2(1-\theta^2) + 2(1-\theta)\sigma^2$$

and

$$E(\varepsilon_n) = \mu(\gamma-\theta).$$

Therefore, regression of X_n on X_{n-1} is linear (in x) and given by

$$E(X_n / X_{n-1} = x) = (1-p)\, x + \mu(\gamma - \theta).$$

The covariance between X_n and X_{n-h} is obtained on considering the representation (3.2.6) and simple computations show that

$$Cov(X_n, X_{n-h}) = (1-p)Cov(X_{n-1}, X_{n-h}) = (1-p)^h Cov(X_{n-h}, X_{n-h}).$$

Hence, the autocorrelation function of the ALTAR (1) process is given by

$$\rho(h) = (1-p)^h.$$

Note that $\rho(h)$ is always positive and so the variables are positively correlated.

It is easily verified that the joint characteristic function of (X_{n-1}, X_n) is not symmetric in t and s. Hence the ALTAR (1) process is not time reversible.

3.5. A general first order autoregressive process

In this Section, we consider a general first order autoregressive model with three parameters given by the structure

$$X_n = \begin{cases} \varepsilon_n & \text{w.p} \quad p_0 \\ \rho X_{n-1} + \varepsilon_n & \text{w.p} \quad p_1 \\ \rho X_{n-1} & \text{w.p} \quad p_2, \end{cases} \tag{3.5.1}$$

where p_0, p_1 and p_2 are probabilities with $p_0 + p_1 + p_2 = 1$, $0 \le \rho < 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables.

The equation (3.5.1) have a direct physical interpretation as the observed process X_n at time n, is one of three possibilities:

(i) with probability p_0, there is a scatter of values of $X_n = \varepsilon_n$, independent of X_{n-1} .

(ii) with probability p_1, $X_n = \rho X_{n-1} + \varepsilon_n$, and is always above or below the line $X_n = \rho X_{n-1}$.

(iii) with probability p_2, the values of X_n are in the line $X_n = \rho X_{n-1}$.

When $p_2 = 0$, we obtain the autoregressive model with structure (3.1.1) and when $p_0 = p_2 = 0$, the model reduced to (2.2.1). Hence, this model is a general case of (2.2.1) and (3.1.1).

The first order autoregressive model (3.5.1) can be expressed in the form of an additive, linear, random coefficient autoregressive process as

$$X_n = A_n X_{n-1} + I_n \varepsilon_n$$

where $\{(A_n, I_n), n \geq 1\}$, is a sequence of independent and identically distributed discrete random variables with distribution given by

$$(A_n, I_n) = \begin{cases} (0,1) & \text{w.p. } p_0 \\ (\rho,1) & \text{w.p. } p_1 \\ (\rho,0) & \text{w.p. } p_2 \end{cases}.$$

Under stationary assumption and $X_0 \stackrel{d}{=} \varepsilon_1$, the characteristic function $\phi_X(t)$ of the process $\{X_n\}$ is

$$\phi_X(t) = \frac{\theta \phi_X(\rho t)}{1-(1-\theta)\phi_X(\rho t)}, \text{ where } \theta = \frac{p_2}{1-p_0}.$$

Hence, the stationary process $\{X_n\}$ given by the structure (3.5.1) is geometrically infinitely divisible.

Jevremović (1990) studied a first order autoregressive model with structure (3.5.1) using mixed exponential marginal distributions and Jevremović and Mališić (1993) developed a moving average model similar to this structure using exponential marginal distribution.

3.5.1. First order Laplace generalized autoregressive process

Here we develop a stationary process $\{X_n\}$ given by (3.5.1) using Laplace marginal distribution. In the following Theorem, we show that that the innovation sequence $\{\varepsilon_n\}$ is a sequence of convex mixture of Laplace distributed random variables so as to ensure $\{X_n\}$ is stationary with Laplace distribution as marginal distribution.

Theorem 3.5.1.

Let $X_0 \stackrel{d}{=} L(\sigma)$. Then the first order autoregressive equation given by (3.5.1) defines a stationary autoregressive process with Laplace marginal distribution if and only if innovations are of the form

$$\varepsilon_n = \begin{cases} L_n & \text{w.p. } \theta \\ \rho\left(\frac{p_0}{1-p_2}\right)^{1/2} L_n & \text{w.p. } 1-\theta. \end{cases} \tag{3.5.2}$$

where $\{L_n\}$ is a sequence of independent and identically distributed Laplace random variables with characteristic function (1.3.2) and $\theta = \frac{1-\rho^2}{1-p_2-\rho^2 p_0}$.

Proof:

Using characteristic functions, equation (3.5.1) gives

$$\phi_{X_n}(t) = p_0\,\phi_{\varepsilon_n}(t) + p_1\,\phi_{X_{n-1}}(\rho t)\,\phi_{\varepsilon_n}(t) + p_2\,\phi_{X_{n-1}}(\rho t). \qquad (3.5.3)$$

If we assume $\{X_n\}$ is stationary with Laplace marginal distribution, then (3.5.3) implies

$$\phi_{\varepsilon_n}(t) = \frac{\dfrac{1}{1+\sigma^2 t^2} - p_2 \dfrac{1}{1+\sigma^2\rho^2 t^2}}{p_0 + p_1 \dfrac{1}{1+\sigma^2\rho^2 t^2}}$$

$$= \frac{1+\sigma^2 t^2 \dfrac{\rho^2 - p_2}{1-p_2}}{(1+\sigma^2 t^2)\left(1+\sigma^2 t^2 \dfrac{\rho^2 p_0}{1-p_2}\right)}$$

$$= \frac{\theta}{1+\sigma^2 t^2} + \frac{1-\theta}{1+\sigma^2 t^2\left(\dfrac{\rho^2 p_0}{1-p_2}\right)}, \text{ where } \theta = \frac{1-\rho^2}{1-p_2-\rho^2 p_0}.$$

Hence, ε_n is a convex mixture of Laplace distributed random variables given by

$$\varepsilon_n = \begin{cases} L_n & \text{w.p. } \theta \\ \rho\left(\dfrac{p_0}{1-p_2}\right)^{1/2} L_n & \text{w.p. } 1-\theta. \end{cases}$$

Thus $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables that has probability distribution, which can be generated as the $(\theta, 1-\theta)$ mixture of Laplace random variables.

Conversely, if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables given by (3.5.2) and $X_0 \stackrel{d}{=} L(\sigma)$ then from (3.5.3), when $n=1$, we have

$$\phi_{X_1}(t) = p_0\, \phi_{\varepsilon_1}(t) + p_1\, \phi_{X_0}(\rho t)\, \phi_{\varepsilon_1}(t) + p_2\, \phi_{X_0}(\rho t)$$

$$= \left(p_0 + p_1 \frac{1}{1+\sigma^2\rho^2 t^2}\right)\left(\frac{\theta}{1+\sigma^2 t^2} + \frac{1-\theta}{1+\sigma^2 t^2\left(\frac{\rho^2 p_0}{1-p_2}\right)}\right) + p_2 \frac{1}{1+\sigma^2\rho^2 t^2},$$

where $\theta = \dfrac{1-\rho^2}{1-p_2-\rho^2 p_0}$.

Hence

$$\phi_{X_1}(t) = \frac{1}{1+\sigma^2 t^2}.$$

Suppose $X_{n-1} \stackrel{d}{=} L(\sigma)$, then it can be shown that $X_n \stackrel{d}{=} L(\sigma)$.

Therefore, the process (3.5.1) is stationary with Laplace marginal distribution.

Hence the theorem. □

Based on the above theorem, we can define first order Laplace general autoregressive process (LGAR (1)) as follows:

Let $X_0 \overset{d}{=} L(\sigma)$ and for $n = 1, 2, ...$

$$X_n = \begin{cases} \varepsilon_n & \text{w.p } p_0 \\ \rho X_{n-1} + \varepsilon_n & \text{w.p } p_1 \\ \rho X_{n-1} & \text{w.p } p_2 \end{cases}$$

where $0 < p_0, p_1, p_2 < 1$ with $p_0 + p_1 + p_2 = 1$, $0 \le \rho < 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed convex mixture of Laplace random variables defined in (3.5.2).

Remark 3.5.1.

It is easy to verify that

(i) if $\rho^2 > \frac{1-p_2}{p_0}$ then $\theta < 0$ and

(ii) if $\rho^2 < \frac{p_2}{1-p_0}$ then $\theta > 1$.

In a situation like this the distribution of the innovation sequence is a mixture of Laplace distribution with negative weights. Jevremović (1991) demonstrated the use of autoregressive process to simulate a sequence of values of the random variable whose

probability density function is of the form $\sum_{j=1}^{n} a_j f_j(x)$ with negative weights a_j. Hence, using the LGAR (1) process one can obtain the mixture of Laplace random variables with negative weights by selecting the parameters in such a way that $\frac{p_2}{1-p_0} \leq \rho^2 \leq \frac{1-p_2}{p_0}$ does not hold.

Now consider some particular case of the model (3.5.1).

Case (i): Let $p_0 = p_2 = 0$

When $p_0 = p_2 = 0$, the model is $X_n = \rho X_{n-1} + \varepsilon_n$.

Then the innovation sequence $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables such that $\varepsilon_n = \begin{cases} 0 & \text{w.p.} \quad \rho^2 \\ L_n & \text{w.p.} \quad 1-\rho^2 \end{cases}$.

Case (ii): Let $p_2 = 0$

When $p_2 = 0$ the model is equivalent to the NLAR (1) model discussed in Dewald and Lewis (1985) and in such a situation $\{\varepsilon_n\}$ is a convex mixture of two Laplace random variables such that

$$\varepsilon_n = \begin{cases} L_n & \text{w.p.} \quad \theta \\ \rho p_0^{1/2} L_n & \text{w.p.} \quad 1-\theta, \end{cases}$$

where $\theta = \dfrac{1-\rho^2}{1-\rho^2 p_0}$.

Case (iii): Let $\rho = 1$

When $\rho = 1$, the innovation sequence $\{\varepsilon_n\}$ is an independent and identically distributed sequence of scaled Laplace random variables $\left(\dfrac{p_0}{1-p_2}\right)^{1/2} L_n$ provided $p_2 \neq 1$.

The autocorrelation function $\rho(h)$ of the LGAR (1) process is obtained as follows:

Consider

$$E(X_n X_{n-h}) = E\big(E(X_n X_{n-h} / X_{n-h} = x)\big),$$

where

$$E(X_n X_{n-h} / X_{n-h} = x) = p_0 \, x \, E(\varepsilon_n) + p_1 (\rho E(X_{n-1} X_{n-h} / X_{n-h} = x) + x E(\varepsilon_n))$$

$$+ p_2 \rho E(X_{n-1} X_{n-h} / X_{n-h} = x)$$

$$= (p_1 + p_2) \rho \, x \, E(X_{n-1}), \text{ because } E(\varepsilon_n) = 0.$$

$$= ((p_1 + p_2) \rho)^h x^2, \text{ by repeatedly using the procedure.}$$

Hence

$$E(X_n X_{n-h}) = \big(\rho (p_1 + p_2)\big)^h E(X^2).$$

Therefore

$$\rho(h) = \left(\rho\,(p_1 + p_2)\right)^h. \tag{3.5.4}$$

Hence, LGAR (1) process is positively correlated and autocorrelation function is geometrically decreasing. Varying the parameters of the process, one can obtain different values for the autocorrelation function.

The joint characteristic function of the LGAR (1) process is given by

$$\phi_{X_{n-1},X_n}(t,s) = p_0\,\phi_{X_{n-1}}(t)\phi_{\varepsilon_n}(s) + p_1\,\phi_{X_{n-1}}(t+\rho s)\,\phi_{\varepsilon_{n-1}}(s) + p_2\,\phi_{X_{n-1}}(t+\rho s)$$

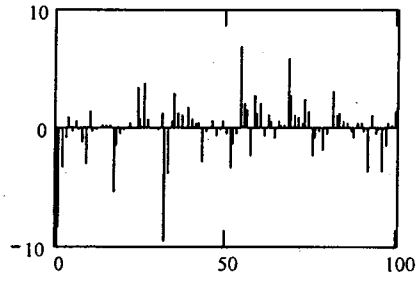
$$= \left(p_0 \frac{1}{1+\sigma^2 t^2} + p_1 \frac{1}{1+\sigma^2 (t+\rho s)^2}\right)\left(\frac{\theta}{1+\sigma^2 s^2} + \frac{1-\theta}{1+\sigma^2 s^2 (\frac{\rho^2 p_0}{1-p_2})}\right)$$

$$+ p_2 \frac{1}{1+\sigma^2 (t+\rho s)^2}.$$

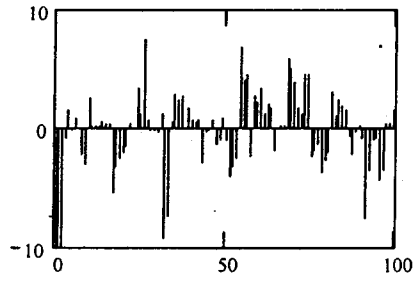
From the above expression, it may be noted that $\phi_{X_{n-1},X_n}(t,s) \neq \phi_{X_{n-1},X_n}(s,t)$. Hence the process is not time reversible.

The three parameters p_0, p_1 and ρ of the process may seem excessive for a first order autoregressive process, but simulation shows a wide range of behavior in sample

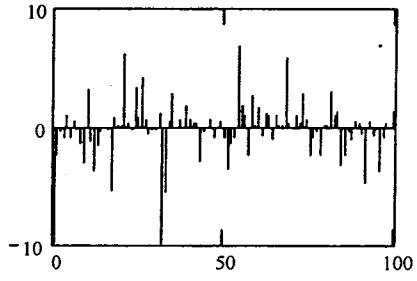
paths with different choices of parameters. The simulated sample path of the LGAR (1) process for various values of parameters is given in the following figures.



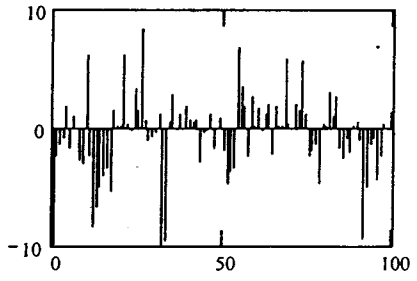
$p_o = 0.3, p_1 = 0.4, \rho = 0.4$



$p_o = 0.3, p_1 = 0.4, \rho = 0.8$



$p_o = 0.6, p_1 = 0.3, \rho = 0.4$



$p_o = 0.6, p_1 = 0.3, \rho = 0.8$

Figure 3.5.1

Simulated sample path of LGAR (1) process

3.5.2. Second order generalized autoregressive process

Now we define a second order general autoregressive model with following structure

$$X_n = \begin{cases} \varepsilon_n & \text{w.p. } p_0 \\ \rho_1 X_{n-1} + \varepsilon_n & \text{w.p. } p_1 \\ \rho_1 X_{n-1} & \text{w.p. } p_2 \\ \rho_2 X_{n-2} + \varepsilon_n & \text{w.p. } p_3 \\ \rho_2 X_{n-2} & \text{w.p. } p_4 \end{cases} . \qquad (3.5.6)$$

where $0 \le \rho_1, \rho_2 < 1$; $0 < p_i < 1$, $i = 0,1,2,3,4$ and $\sum_{i=0}^{4} p_i = 1$.

It is a difficult task to obtain the distribution of innovation sequence $\{\varepsilon_n\}$ so as to ensure that $\{X_n\}$ is stationary with Laplace marginal distribution. Hence we consider a particular case $\rho_1 = \rho_2 = \rho$, then ε_n is given by

$$\varepsilon_n = \begin{cases} L_n & \text{w.p. } \theta \\ \rho \left(\dfrac{p_0}{1 - p_2 - p_4} \right)^{1/2} L_n & \text{w.p. } 1-\theta \end{cases} \qquad (3.5.7)$$

where $\{L_n\}$ is a sequence of independent and identically distributed Laplace random variables with characteristic function (1.3.2) and $\theta = \dfrac{1-\rho^2}{(1-p_2-p_4) - p_0 \rho^2}$.

It may be noted that

(i) $\theta < 0$ if $\rho^2 > \dfrac{1-p_2-p_4}{p_0}$ and

(ii) $\theta > 1$ if $\rho^2 < \frac{p_2 + p_4}{1 - p_0}$.

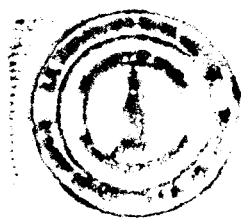
Therefore, whenever the values of the parameters do not satisfy the condition $\frac{p_2 + p_4}{1 - p_0} \le \rho^2 \le \frac{1 - p_2 - p_4}{p_0}$, random variable ε_n is a mixture of Laplace random variables with negative weights. Hence using the stationary autoregressive process (3.5.6) with $\rho_1 = \rho_2 = \rho$ and $X_0 \stackrel{d}{=} L(\sigma)$, one can obtain the mixture of Laplace random variables with negative weights.

3.5.3. First order asymmetric Laplace generalized autoregressive process

Let $\{X_n\}$ be a stationary sequence of asymmetric Laplace random variables given by the structure (3.5.1). Then the probability distribution for innovative sequence $\{\varepsilon_n\}$ is obtained by substituting $\phi_X(t) = \frac{1}{1 + \sigma^2 t^2 - i\mu t}$ in (3.5.3). The solution is much-complicated one, and to make the problem a simple one we consider the following autoregressive model developed by Mališić (1987).

Let $\{X_n\}$ be a sequence of random variables given by

$$X_n = \begin{cases} \rho\,\varepsilon_n & \text{w.p } p_0 \\ X_{n-1} + \rho\,\varepsilon_n & \text{w.p } p_1 \\ X_{n-1} & \text{w.p } p_2 \end{cases} \qquad (3.5.8)$$



where $0<p_0,p_1,p_2<1$ with $p_0+p_1+p_2=1$, $0<\rho\leq 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables.

Theorem 3.5.2.

Let $X_0 \stackrel{d}{=} AL(\mu,\sigma)$. Then the first order autoregressive equation given by (3.5.8) defines a stationary autoregressive process with asymmetric Laplace marginal distribution if and only if innovations are of the form

$$\varepsilon_n \stackrel{d}{=} AL\left(\frac{p_0}{(1-p_2)\rho}\mu,\ \sqrt{\frac{p_0}{(1-p_2)}}\frac{\sigma}{\rho}\right)$$

Proof:

From equation (3.5.8), using the characteristic functions we have

$$\phi_{X_n}(t)=p_0\,\phi_{\varepsilon_n}(\rho t)+p_1\,\phi_{X_{n-1}}(t)\,\phi_{\varepsilon_n}(\rho t)+p_2\,\phi_{X_{n-1}}(t). \qquad (3.5.9)$$

If we assume $\{X_n\}$ is stationary with asymmetric Laplace marginal distribution, then (3.5.9) implies

$$\phi_{\varepsilon_n}(\rho t)=\frac{\dfrac{1}{1+\sigma^2 t^2-i\mu t}-p_2\dfrac{1}{1+\sigma^2 t^2-i\mu t}}{p_0+p_1\dfrac{1}{1+\sigma^2 t^2-i\mu t}}$$

$$= \frac{1-p_2}{1-p_2+\sigma^2 p_0 t^2 - i\mu p_0 t}$$

$$= \frac{1}{1+\sigma^2 \frac{p_0}{1-p_2} t^2 - i\mu \frac{p_0}{1-p_2} t}.$$

Hence

$$\varepsilon_n \stackrel{d}{=} AL\left(\frac{p_0}{(1-p_2)\rho}\mu, \sqrt{\frac{p_0}{(1-p_2)}}\frac{\sigma}{\rho}\right). \tag{3.5.10}$$

Conversely, if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed asymmetric Laplace random variables given by (3.5.10) and $X_0 \stackrel{d}{=} AL(\mu,\sigma)$ then from (3.5.9), when $n=1$, we have

$$\phi_{X_1}(t) = p_0\, \phi_{\varepsilon_1}(\rho t) + p_1\, \phi_{X_0}(t)\, \phi_{\varepsilon_1}(\rho t) + p_2\, \phi_{X_0}(t)$$

$$= p_0 \frac{1}{1+\sigma^2 \frac{p_0}{1-p_2} t^2 - i\mu \frac{p_0}{1-p_2} t}$$

$$+ p_1 \frac{1}{1+\sigma^2\, t^2 - i\mu\, t} \frac{1}{1+\sigma^2 \frac{p_0}{1-p_2} t^2 - i\mu \frac{p_0}{1-p_2} t}$$

$$+ p_2 \frac{1}{1+\sigma^2\, t^2 - i\mu\, t}$$

$$= \frac{1}{1+\sigma^2 t^2 - i\mu t}.$$

Suppose $X_{n-1} \stackrel{d}{=} AL(\mu,\sigma)$, then it can be shown that $X_n \stackrel{d}{=} AL(\mu,\sigma)$.

Therefore, the process (3.5.8) is stationary with asymmetric Laplace marginal distribution if $X_0 \stackrel{d}{=} AL(\mu,\sigma)$.

Hence the theorem. □

The autocorrelation function $\rho(h)$ of the model (3.5.8) is obtained as follows:

Consider

$$E(X_n X_{n-1}) = E\big(E(X_n X_{n-1} / X_{n-1} = x)\big),$$

where

$$E(X_n X_{n-1} / X_{n-1} = x) = p_0 \, x \, \rho E(\varepsilon_n) + p_1 \big(E(X_{n-1} X_{n-1} / X_{n-1} = x) + x \, \rho \, E(\varepsilon_n)\big)$$

$$+ p_2 \, E(X_{n-1} X_{n-1} / X_{n-1} = x)$$

$$= (p_0 + p_1)\rho x \; E(\varepsilon_n) + (p_1 + p_2) x^2 \;,$$

$$= p_0 x \; \mu + (p_1 + p_2)x^2 \;, \text{ because } E(\varepsilon_n) = \frac{p_0}{(1-p_2)\rho}\mu$$

$$= p_0\mu^2 + (p_1 + p_2)\left(2\mu^2 + 2\sigma^2\right).$$

Hence

$$\mathrm{Cov}(X_n, X_{n-1}) = (p_1 + p_2)\left(\mu^2 + 2\sigma^2\right)$$

Therefore

$$\rho(1) = (p_1 + p_2).$$

In general, the autocorrelation function of the process (3.5.8) with asymmetric Laplace marginal distribution is given by $\rho(h) = (p_1 + p_2)^h$.

3.5.4. First order semi α-Laplace autoregressive process

Now we consider a stationary process using a generalized class of distributions, namely semi α-Laplace distributions, defined in Pillai (1985). Let $\{X_n\}$ be a stationary sequence with semi α-Laplace marginal distribution and with the structure (3.5.1). Then from (3.5.1) the question is whether there is a valid probability distribution for innovative sequence $\{\varepsilon_n\}$.

Theorem 3.5.3.

Let X_0 be a semi α-Laplace random variable with characteristic function given by (1.4.16) to (1.4.18). Then the first order autoregressive equation given by (3.5.1)

defines a stationary autoregressive process with semi α-Laplace marginal distribution if and only if innovation ε_n is of the form

$$\varepsilon_n = \begin{cases} SAL_n & \text{w.p.} \quad \theta \\ (\frac{p_0}{1-p_2})^{1/\alpha} \rho\, SAL_n & \text{w.p.} \quad 1-\theta, \end{cases} \tag{3.5.11}$$

where $\{SAL_n\}$ is a sequence of independent and identically distributed semi α-Laplace random variables with characteristic function (1.4.16) to (1.4.18) and $\theta = \dfrac{1-\rho^\alpha}{1-\rho^\alpha p_0 - p_2}$.

Proof:

If we assume $\{X_n\}$ is stationary with semi α-Laplace marginal distribution. Then (3.5.3) implies

$$\phi_{\varepsilon_n}(t) = \frac{\dfrac{1}{1+\psi(t)} - p_2 \dfrac{1}{1+\psi(\rho t)}}{p_0 + p_1 \dfrac{1}{1+\psi(\rho t)}}. \tag{3.5.12}$$

where $\psi(t)$ satisfies the functional equation $\psi(t) = \dfrac{1}{a}\psi(a^{\frac{1}{\alpha}} t)$.

Hence

$$\phi_{\varepsilon_n}(t)=\frac{1+\frac{\rho^{\alpha}-p_2}{1-p_2}\psi(t)}{(1+\psi(t))(1+\frac{p_0}{1-p_2}\rho^{\alpha}\psi(t))}$$

$$=\frac{\theta}{1+\psi(t)}+\frac{1-\theta}{1+\frac{p_0}{1-p_2}\rho^{\alpha}\psi(t)}, \text{ where } \theta=\frac{1-\rho^{\alpha}}{1-\rho^{\alpha}p_0-p_2}.$$

$$=\frac{\theta}{1+\psi(t)}+\frac{1-\theta}{1+\psi((\frac{p_0}{1-p_2})^{1/\alpha}\rho t)}.$$

That is, $\{\varepsilon_n\}$ is a sequence of mixture of semi α -Laplace random variables such that

$$\varepsilon_n=\begin{cases} SAL_n & \text{w.p.} \quad \theta \\ (\frac{p_0}{1-p_2})^{1/\alpha}\rho\, SAL_n & \text{w.p.} \quad 1-\theta, \end{cases} \tag{3.5.13}$$

where $\{SAL_n\}$ is a sequence of independent and identically distributed semi α -Laplace Laplace random variables with characteristic function (1.4.16) to (1.4.18).

Conversely, if $\{\varepsilon_n\}$ is a sequence of independent and identically distributed semi α -Laplace random variables given by (3.5.13) and X_0 is a semi α -Laplace random

variable then using mathematical inductive argument it is easily verified that $\{X_n\}$ is stationary with semi α-Laplace marginal distribution.

Hence the theorem. □

Using the Theorem 3.5.3, we can define first order semi α-Laplace autoregressive process as follows:

Let X_0 be a semi α-Laplace random variable with characteristic function (1.4.16) to (1.4.18) and $\{X_n\}$ be a sequence of random variables defined by

$$X_n = \begin{cases} \varepsilon_n & \text{w.p } p_0 \\ \rho X_{n-1} + \varepsilon_n & \text{w.p } p_1 \\ \rho X_{n-1} & \text{w.p } p_2 \end{cases}$$

where $0 < p_0, p_1, p_2 < 1$ with $p_0 + p_1 + p_2 = 1$, $0 \le \rho < 1$ and

$$\varepsilon_n = \begin{cases} SAL_n & \text{w.p. } \theta \\ (\frac{p_0}{1-p_2})^{1/\alpha} \rho\, SAL_n & \text{w.p. } 1-\theta, \end{cases}$$

where $\{SAL_n\}$ is a sequence of independent and identically distributed semi α-Laplace random variables and $\theta = \dfrac{1-\rho^\alpha}{1-\rho^\alpha p_0 - p_2}$.

Note that

(i) when h(t) is a constant in (1.4.19), the semi α-Laplace autoregressive process reduced to a α-Laplace process and

(ii) when $\alpha = 2$, and h(t) is a constant, the semi α-Laplace autoregressive process reduced to LGAR(1) process discussed in Section 3.5.1.

3.5.5. First order Laplace moving average process

Here we discuss the stationary sequence of random variables, which are formed from independent and identically distributed Laplace random variables according to the model

$$X_n = \begin{cases} \varepsilon_n & \text{w.p} \quad p_0 \\ \rho\varepsilon_{n-1} + \varepsilon_n & \text{w.p} \quad p_1 \\ \rho\varepsilon_{n-1} & \text{w.p} \quad p_2, \end{cases} \tag{3.5.14}$$

where $0 < p_0, p_1, p_2 < 1$ and p_2 are probabilities with $p_0 + p_1 + p_2 = 1$ and $0 \le \rho < 1$.

From the model (3.5.14), using characteristic function, we get

$$\phi_{X_n}(t) = p_0 \phi_{\varepsilon_n}(t) + p_1 \phi_{\varepsilon_{n-1}}(\rho t)\phi_{\varepsilon_n}(t) + p_2 \phi_{\varepsilon_{n-1}}(\rho t).$$

Since $\{\varepsilon_n\}$ is an independent and identically distributed sequence of Laplace random variables, under stationarity assumption, we get

$$\phi_X(t) = \frac{\theta}{1+\sigma^2 t^2} + \frac{1-\theta}{1+\rho^2\sigma^2 t^2} \tag{3.5.15}$$

where $\theta = \frac{1-\rho\, p_1 - p_2}{1-\rho}$.

Hence, $\{X_n\}$ is a sequence of convex mixture of Laplace random variables such that

$$X_n = \begin{cases} L_n & \text{w.p.} \quad \theta \\ \rho L_n & \text{w.p.} \quad 1-\theta, \end{cases} \tag{3.5.16}$$

where $\{L_n\}$ is a sequence of Laplace random variables with characteristic function(1.3.2).

Hence we can define first order Laplace moving average (LMA (1)) process as follows:

For n=1,2... define

$$X_n = \begin{cases} \varepsilon_n & \text{w.p} \quad p_0 \\ \rho\varepsilon_{n-1} + \varepsilon_n & \text{w.p} \quad p_1 \\ \rho\varepsilon_{n-1} & \text{w.p} \quad p_2 \end{cases}. \tag{3.5.17}$$

where $0 \le \rho < 1$, $0 < p_0, p_1, p_2 < 1$, $p_0 + p_1 + p_2 = 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed Laplace random variables.

If $\rho^2 > \frac{p_2}{1-p_0}$ or $\rho^2 > \frac{1-p_2}{p_0}$ then θ or $1-\theta$ will be negative and so the distribution of $\{X_n\}$ given by (3.5.16) is a mixture of Laplace random variables with negative weights. Hence, using the first order Laplace moving average process (3.5.17) it is possible to generate sequence of random variables that are mixtures of Laplace random variables with negative weights.

LAPLACE AUTOREGRESSIVE TIME SERIES MODELS

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Chapter-IV

A Bivariate Laplace Autoregressive Process

4.1. Introduction

In various applications it is frequently of interest to study time series models for bivariate data and so autoregressive models are extended to bivariate time series models by several researchers. Dewald *et al.* (1989) developed and studied a bivariate exponential autoregressive process, which is broader and simpler than the models described by Gaver and Lewis (1980) and Raftery (1982). Block *et al.* (1988) introduced additive first order autoregressive bivariate exponential and geometric processes and studied their properties. Ristić and Popović (2003) discussed a bivariate autoregressive process with uniform marginal distribution and examined the autocorrelation structure of the process. Since the bivariate time series observations that can be modeled using Laplace distribution appear in many areas, it is interesting to develop a bivariate autoregressive process with Laplace marginal distribution. Here we introduce and study a bivariate time series model with Laplace marginal distribution.

Dewald and Lewis (1985) introduced and studied time series model for Laplace variables, called NLAR (2) model, using second order random coefficient autoregressive process with following structure

$$X_n = \Theta_{n1} X_{n-1} + \Theta_{n2} X_{n-2} + \varepsilon_n, \qquad (4.1.1)$$

where joint distribution of $\{(\Theta_{n1}, \Theta_{n2})\}$ is given by

$$(\Theta_{n1}, \Theta_{n2}) = \begin{cases} (\beta_1, 0) & \text{w.p.} \quad p_1 \\ (0, \beta_2) & \text{w.p.} \quad p_2 \\ (0,0) & \text{w.p.} \quad 1-p_1-p_2, \end{cases} \tag{4.1.2}$$

and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables and $0 < \beta_1, \beta_2 \leq 1$ and $0 \leq p_1, p_2 \leq 1$.

They have shown that the innovation sequence $\{\varepsilon_n\}$ is a convex mixture of Laplace random variables. When $\beta_1 = \beta_2 = 1$, the model is very simple and the innovation sequence $\{\varepsilon_n\}$ is a sequence of scaled Laplace random variables. Such model is known as TLAR (2) process with two parameters .We use the TLAR (2) model as the basis for construction of our bivariate model.

4.2. The bivariate Laplace model

Let the two stationary processes $\{X_n\}$ and $\{Y_n\}$ be defined as

$$X_n = \Theta_{11}^{(n)} X_{n-1} + \Theta_{12}^{(n)} Y_{n-1} + \varepsilon_{n1} \tag{4.2.1}$$

$$Y_n = \Theta_{21}^{(n)} X_{n-1} + \Theta_{22}^{(n)} Y_{n-1} + \varepsilon_{n2} \tag{4.2.2}$$

where $\{(\Theta_{il}^{(n)}, \Theta_{i2}^{(n)})\}, i = 1,2$, are independent sequences of random vectors with the discrete probability distribution

$$\left(\Theta_{i1}^{(n)},\Theta_{i2}^{(n)}\right)=\begin{cases}(1,0) & \text{w.p.} \quad p_{i1}\\ (0,1) & \text{w.p.} \quad p_{i2}\\ (0,0) & \text{w.p.} \quad 1-p_{i1}-p_{i2}.\end{cases} \tag{4.2.3}$$

Let $\{X_n\}$ and $\{Y_n\}$ be independent sequence of Laplace random variables with characteristic function $\phi(t)=\frac{1}{1+\sigma^2 t^2}$. Assume the random vectors $\left(\Theta_{i1}^{(n)},\Theta_{i2}^{(n)}\right), i=1,2$ and $\{(\varepsilon_{n1},\varepsilon_{n2})\}$ are independent.

In terms of characteristic function, equation (4.2.1) can be written as

$$\phi_{X_n}(t)=p_{11}\,\phi_{X_{n-1}}(t)\,\phi_{\varepsilon_{n1}}(t)+p_{12}\,\phi_{X_{n-1}}(t)\,\phi_{\varepsilon_{n1}}(t)+(1-p_{11}-p_{12})\,\phi_{\varepsilon_{n1}}(t).$$

Hence

$$\phi_{\varepsilon_{n1}}(t)=\frac{\phi_{X_{n-1}}(t)}{p_{11}\,\phi_{X_{n-1}}(t)+p_{12}\,\phi_{X_{n-1}}(t)+(1-p_{11}-p_{12})}. \tag{4.2.4}$$

Since $\{X_n\}$ is stationary with Laplace marginal distribution, equation (4.2.4) gives

$$\phi_{\varepsilon_{n1}}(t)=\frac{\frac{1}{1+\sigma^2 t^2}}{p_{11}\,\frac{1}{1+\sigma^2 t^2}+p_{12}\,\frac{1}{1+\sigma^2 t^2}+(1-p_{11}-p_{12})}$$

$$= \frac{1}{1+(1-p_{11}-p_{12})\sigma^2 t^2}.$$

Similarly we can show that

$$\phi_{\varepsilon_{n2}}(t) = \frac{1}{1+(1-p_{21}-p_{22})\sigma^2 t^2}. \tag{4.2.5}$$

Hence $\{\varepsilon_{n1}\}$ and $\{\varepsilon_{n2}\}$ are independent sequences of independent and identically distributed random variables such that $\varepsilon_{ni} \stackrel{d}{=} \sqrt{(1-p_{i1}-p_{i2})}\, L_{ni}$, $i=1,2$, where $\{L_{ni}\}$ is a sequence of Laplace random variables with characteristic function $\frac{1}{1+\sigma^2 t^2}$.

So by (4.2.1) and (4.2.2) we can define a first order autoregressive time series model using Laplace variables. The equations (4.2.1) and (4.2.2) may be combined and written in the vector form

$$Z_n = \Theta_n Z_{n-1} + \varepsilon_n \tag{4.2.6}$$

where $Z_n = (X_n, Y_n)'$, $\Theta_n = \begin{pmatrix} \Theta_{11}^{(n)} & \Theta_{12}^{(n)} \\ \Theta_{21}^{(n)} & \Theta_{22}^{(n)} \end{pmatrix}$ and $\varepsilon_n = (\varepsilon_{n1}, \varepsilon_{n2})'$.

From the vector representation (4.2.6) it is evident that $\{Z_n\}$ is a Markov process. We call the process (4.2.6) as first order bivariate Laplace autoregressive (BLAR (1)) process.

4.3. Autocorrelation structure of the model

In this Section we consider the autocorrelation structure of the BLAR (1) process (4.2.6). The autocovariance matrix of Z_n at lag k is given by

$$\Gamma(k) = \mathrm{Cov}(Z_n, Z_{n-k})$$

$$= \begin{pmatrix} \gamma_{XX}(k) & \gamma_{XY}(k) \\ \gamma_{YX}(k) & \gamma_{YY}(k) \end{pmatrix} \tag{4.3.1}$$

where

$$\gamma_{XX}(k) = \mathrm{Cov}(X_n, X_{n-k}), \gamma_{YY}(k) = \mathrm{Cov}(Y_n, Y_{n-k}),$$

$$\gamma_{XY}(k) = \mathrm{Cov}(X_n, Y_{n-k}) \ , \ \gamma_{YX}(k) = \mathrm{Cov}(Y_n, X_{n-k}).$$

Theorem 4.3.1.

The autocovariance matrix $\Gamma(k)$ of BLAR (1) process (4.2.6) is given by

$$\Gamma(k) = M^k \Gamma(0), \tag{4.3.2}$$

where $M=\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, $\Gamma(0)=2\sigma^2\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$ and $\alpha=\dfrac{(p_{11}p_{21}+p_{12}p_{22})}{1-p_{11}p_{22}-p_{12}p_{21}}$.

Proof:

From the representation (4.2.6) of bivariate Laplace autoregressive process we have

$$\Gamma(k)=\mathrm{Cov}(Z_n, Z_{n-k})$$

$$=E(Z_n Z_{n-k})-E(Z_n)E(Z_{n-k})$$

$$=E(\Theta_n)\left(E(Z_{n-1} Z_{n-k})-E(Z_{n-1})E(Z_{n-k})\right)$$

$$=M\Gamma(k-1), \text{ where } M=E(\Theta_n)=\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.$$

Hence

$$\Gamma(k)=M^k\Gamma(0).$$

To find $\Gamma(0)$ of the BLAR (1) process consider the definition (4.2.1) and (4.2.2) of the process.

From (4.2.1) and (4.2.2) we have

$$X_n Y_n = \Theta_{11}^{(n)} \Theta_{21}^{(n)} X_{n-1}^2 + \Theta_{11}^{(n)} \Theta_{22}^{(n)} X_{n-1} Y_{n-1} + \Theta_{11}^{(n)} X_{n-1} \varepsilon_{n2}$$

$$+ \Theta_{12}^{(n)} \Theta_{22}^{(n)} Y_{n-1}^2 + \Theta_{12}^{(n)} \Theta_{21}^{(n)} X_{n-1} Y_{n-1} + \Theta_{12}^{(n)} Y_{n-1} \varepsilon_{n2} \qquad (4.3.3)$$

$$+ \Theta_{21}^{(n)} X_{n-1} \varepsilon_{n1} + \Theta_{22}^{(n)} Y_{n-1} \varepsilon_{n1} + \varepsilon_{n1} \varepsilon_{n2}.$$

Hence

$$E(X_n Y_n) = p_{11} p_{21} 2\sigma^2 + p_{11} p_{22} E(X_{n-1} Y_{n-1})$$

$$+ p_{12} p_{22} 2\sigma^2 + p_{12} p_{21} E(X_{n-1} Y_{n-1})$$

Since $\{X_n\}$ and $\{Y_n\}$ are stationary, we obtain

$$E(X_n Y_n)(1 - p_{11} p_{22} - p_{12} p_{21}) = (p_{11} p_{21} + p_{12} p_{22}) 2\sigma^2$$

Therefore

$$E(X_n Y_n) = \frac{(p_{11} p_{21} + p_{12} p_{22})}{(1 - p_{11} p_{22} - p_{12} p_{21})} 2\sigma^2. \qquad (4.3.4)$$

Hence

$$Cov(X_n, Y_n) = E(X_n Y_n) - E(X_n) E(Y_n)$$

$$= \frac{(p_{11} p_{21} + p_{12} p_{22})}{(1 - p_{11} p_{22} - p_{12} p_{21})} 2\sigma^2$$

$$= 2\sigma^2 \alpha, \text{ where } \alpha = \frac{(p_{11} p_{21} + p_{12} p_{22})}{1 - p_{11} p_{22} - p_{12} p_{21}}.$$

Hence

$$\Gamma(0) = 2\sigma^2 \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}. \tag{4.3.5}$$

Therefore, the autocovariance matrix of the BLAR (1) process is

$$\Gamma(k) = M^k \Gamma(0),$$

where $M = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, $\Gamma(0) = 2\sigma^2 \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$ and $\alpha = \dfrac{(p_{11}p_{21} + p_{12}p_{22})}{1 - p_{11}p_{22} - p_{12}p_{21}}$.

Hence the theorem.

Remark 4.3.1.

The autocorrelation matrix of BLAR (1) process is given by

$$R(k) = \mathrm{Corr}(Z_n, Z_{n-k})$$

$$= M^k R(0), \text{ where } R(0) = \begin{pmatrix} \rho_{XX}(0) & \rho_{XY}(0) \\ \rho_{YX}(0) & \rho_{YY}(0) \end{pmatrix}.$$

Since $\{X_n\}$ and $\{Y_n\}$ are sequence of Laplace random variables we have $E(X_n) = E(Y_n) = 0$ and $V(X_n) = V(Y_n) = 2\sigma^2$.

Hence

$$R(0) = \frac{\Gamma(0)}{2\sigma^2}. \tag{4.3.6}$$

Since $\alpha = \frac{(p_{11}p_{21} + p_{12}p_{22})}{1 - p_{11}p_{22} - p_{12}p_{21}} \in [0,1]$, $\Gamma(0)$ contains no negative terms in general and so the autocorrelation matrix (4.3.6) of bivariate Laplace autoregressive process is positive. Hence the BLAR (1) process has only positive correlation.

Negative correlation may be introduced by relaxing following assumptions that made while defining the BLAR (1) process

(i) $\{\varepsilon_{n1}\}$ and $\{\varepsilon_{n2}\}$ are independent sequence of Laplace variables and

(ii) $\{(\Theta_{11}^{(n)}, \Theta_{12}^{(n)})\}$ is an independent and identically distributed sequence of random vectors independent of $\{(\Theta_{21}^{(n)}, \Theta_{22}^{(n)})\}$ for all n.

Case (i): Assume $\{\varepsilon_{n1}\}$ and $\{\varepsilon_{n2}\}$ are correlated sequence.

If we assume $\{\varepsilon_{n1}\}$ and $\{\varepsilon_{n2}\}$ are correlated sequence with

$$\mathrm{Cov}(\varepsilon_{n1}, \varepsilon_{n2+k}) = \begin{cases} \sigma(\varepsilon_{n1}, \varepsilon_{n2}) & \text{when } k = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma(\varepsilon_{n1}, \varepsilon_{n2})$ denote the covariance between ε_{n1} and ε_{n2}.

From (4.3.3) by taking expectation, we get

$$E(X_n\ Y_n) = p_{11}p_{21}\,2\sigma^2 + p_{11}p_{22}\,E(X_{n-1}Y_{n-1})$$
$$+ p_{12}p_{22}\,2\sigma^2 + p_{12}p_{21}\,E(X_{n-1}Y_{n-1}) + \sigma(\varepsilon_{n1},\varepsilon_{n2})$$

Hence

$$Cov(X_n, Y_n) = E(X_n\ Y_n) - E(X_n)\,E(Y_n)$$

$$= \frac{2\sigma^2(p_{11}p_{21} + p_{12}\,p_{22}) + \sigma(\varepsilon_{n1},\varepsilon_{n2})}{1 - p_{11}\,p_{22} - p_{12}\,p_{21}}.$$

Therefore

$$\Gamma(0) = 2\sigma^2 \begin{pmatrix} 1 & \alpha_1 \\ \alpha_1 & 1 \end{pmatrix}, \tag{4.3.8}$$

where $\alpha_1 = \dfrac{(p_{11}\,p_{21} + p_{12}\ p_{22}) + \dfrac{1}{2\sigma^2}\sigma(\varepsilon_{n1},\varepsilon_{n2})}{1 - p_{11}\,p_{22} - p_{12}\,p_{21}}$.

Case (ii): Assume the rows of matrix Θ_n are not stochastically independent.

Let us assume $\Theta_{ij}^{(n)}$ and $\Theta_{lm}^{(n)}$ are correlated sequence such that

$$Cov(\Theta_{ij}^{(n)}, \Theta_{lm}^{(n+s)}) = \begin{cases} \sigma_{ij:lm} & \text{if } s = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma_{ij:lm}$ denote the covariance between $\Theta_{ij}^{(n)}$ and $\Theta_{lm}^{(n)}$.

Then using (4.3.3) it is simple to establish that

$$\mathrm{Cov}(X_n, Y_n) = E(X_n\, Y_n) - E(X_n)\, E(Y_n)$$

$$= \frac{2\sigma^2\left(p_{11}\, p_{21} + p_{12}\, p_{22} + \sigma_{11:21} + \sigma_{12:22}\right)}{1 - p_{11}\, p_{22} - p_{12}\, p_{21} - \sigma_{11:22} - \sigma_{12:21}}.$$

Hence

$$\Gamma(0) = 2\sigma^2 \begin{pmatrix} 1 & \alpha_2 \\ \alpha_2 & 1 \end{pmatrix}, \tag{4.3.9}$$

where $\alpha_2 = \dfrac{\left(p_{11}p_{21} + p_{12}\, p_{22} + \sigma_{11:21} + \sigma_{12:22}\right)}{1 - p_{11}\, p_{22} - p_{12}\, p_{21} - \sigma_{11:22} - \sigma_{12:21}}$.

Case (iii): Assume $\{\varepsilon_{n1}\}$ and $\{\varepsilon_{n2}\}$ are correlated sequence and rows of matrix Θ_n are not stochastically independent.

If there is dependence from both the innovation and cross-coupled attenuation then from (4.3.3) we obtain

$$\mathrm{Cov}(X_n, Y_n) = E(X_n\, Y_n) - E(X_n)\, E(Y_n)$$

$$= \frac{2\sigma^2\left(p_{11}p_{21} + p_{12}\, p_{22} + \sigma_{11:21} + \sigma_{12:22}\right) + \sigma(\varepsilon_{n1}, \varepsilon_{n2})}{1 - p_{11}\, p_{22} - p_{12}\, p_{21} - \sigma_{11:22} - \sigma_{12:21}}.$$

Hence

$$\Gamma(0) = 2\sigma^2 \begin{pmatrix} 1 & \alpha_3 \\ \alpha_3 & 1 \end{pmatrix} \tag{4.3.10}$$

$$\text{where } \alpha_3 = \frac{\left(p_{11}p_{21} + p_{12}p_{22}\sigma_{11:21} + \sigma_{12:22}\right) + \frac{1}{2\sigma^2}\sigma(\varepsilon_{n1}, \varepsilon_{n2})}{1 - p_{11}p_{22} - p_{12}p_{21} - \sigma_{11:22} - \sigma_{12:21}}.$$

It may be noted that the expressions (4.3.8), (4.3.9) and (4.3.10) may be negative for some values of $\sigma(\varepsilon_{n1}, \varepsilon_{n2})$ and $\sigma_{ij:lm}$. Hence the bivariate Laplace autoregressive processes under the above assumptions have the capability to exhibit negative dependence.

4.4. Autocorrelation structure of marginal process

Let $\{X_n\}$ be an ARMA(p,q) process given by (1.1.5) for which the polynomials $\rho(.)$ and $\theta(.)$ have no common zeroes and $\rho(z) \neq 0$ for all $|z| \leq 1$, then we have the representation

$$X_n = \sum_{j=0}^{\infty} \psi_j \varepsilon_{n-j}, \tag{4.4.1}$$

where the sequence $\{\psi_j\}$ is determined by $\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\rho(z)}$, $|z| \leq 1$.

If $\{\varepsilon_n\}$ is an independent and identically distributed sequence of random variables with mean zero and variance σ^2 then the autocovariance function of the ARMA(p,q) process represented by (4.4.1) satisfy the difference equation

$$\gamma(k)-\rho_1\gamma(k-1)-\cdots-\rho_p\gamma(k-p)=\sigma^2\sum_{k\le j\le q}\theta_j\psi_{j-k}\quad,0\le k<\max(p,q+1)\quad(4.4.2)$$

$$\gamma(k)-\rho_1\gamma(k-1)-\cdots-\rho_p\gamma(k-p)=0\quad,k\ge\max(p,q+1).\qquad(4.4.3)$$

Now we can establish that the autocovariance function of the marginal processes $\{X_n\}$ and $\{Y_n\}$ of the BLAR (1) process satisfy autocovariance structure of an ARMA(p,q) process with $p\le 2$ and $q\le 1$.

Theorem 4.4.1.

The autocorrelation function of the marginal process $\{X_n\}$ given by (4.2.1) of the BLAR (1) process satisfies the difference equation

$$\rho_{XX}(1)=p_{11}+p_{12}\,\rho_{YX}(0)\text{ and}$$

$$\rho_{XX}(k)-(p_{11}+p_{22})\,\rho_{Xx}(k-1)-(p_{12}\,p_{21}-p_{11}\,p_{22})\,\rho_{XX}(k-2)=0,k\ge 2,$$

where $\rho_{XX}(k)=\mathrm{Corr}(X_n,X_{n-k})$ and $\rho_{XY}(k)=\rho_{YX}(k)=\mathrm{Corr}(X_n,Y_{n-k})$.

Proof:

The autocovariance function of BLAR (1) process is $\Gamma(k)=M^k\Gamma(0)$.

When $k=1$, we have

$$\Gamma(1)=M\Gamma(0).$$

That is,

$$\begin{pmatrix} \gamma_{XX}(1) & \gamma_{XY}(1) \\ \gamma_{YX}(1) & \gamma_{YY}(1) \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} \gamma_{XX}(0) & \gamma_{XY}(0) \\ \gamma_{YX}(0) & \gamma_{YY}(0) \end{pmatrix}$$

Simplifying, we get

$$\gamma_{XX}(1) = p_{11}\gamma_{XX}(0) + p_{12}\gamma_{YX}(0)$$

Hence

$$\rho_{XX}(1) = p_{11} + p_{12}\,\rho_{YX}(0). \tag{4.4.4}$$

Since $M = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ is a square matrix, using Cayley–Hamilton Theorem, it follows that $M^2 - \phi_1 M - \phi_2 I = 0$.

This implies that

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}^2 = \phi_1 \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} + \phi_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Simplifying, we get

$$\phi_1 p_{11} + \phi_2 = p_{11}^2 + p_{12}\,p_{21} \text{ and } \phi_1 p_{22} + \phi_2 = p_{22}^2 + p_{21}\,p_{12}.$$

Solving the above equation, we get ϕ_1 and ϕ_2 in terms of the elements of M as

$$\phi_1 = p_{11} + p_{22} \text{ and } \phi_2 = p_{12}\, p_{21} - p_{11}\, p_{22} \tag{4.4.5}$$

Now we represent the autocovariance function $\Gamma(k)$ as

$$\Gamma(k) = M^2 \Gamma(k-2)$$

$$= (\phi_1 M + \phi_2 I)\, \Gamma(k-2)$$

$$= \phi_1 \Gamma(k-1) + \phi_2 \Gamma(k-2).$$

Therefore, for the BLAR (1) process, the autocovariance matrix satisfies the difference equation

$$\Gamma(k) - \phi_1 \Gamma(k-1) - \phi_2 \Gamma(k-2) = 0, k \geq 2. \tag{4.4.6}$$

The autocorrelation functions of the marginal processes are obtained by applying the difference equation (4.4.6) to each element of the matrix $\Gamma(k)$.

From (4.4.6), it is easily seen that the autocorrelation function of the marginal processes $\{X_n\}$ satisfies

$$\rho_{XX}(k) - \phi_1\, \rho_{XX}(k-1) - \phi_2\, \rho_{XX}(k-2) = 0, k \geq 2. \tag{4.4.7}$$

Substituting the values of ϕ_1 and ϕ_2 obtained in (4.4.5) in equation (4.4.7), we get

$$\rho_{XX}(k) - (p_{11} + p_{22})\, \rho_{XX}(k-1) - (p_{12}\, p_{21} - p_{11}\, p_{22})\, \rho_{XX}(k-2) = 0, k \geq 2. \tag{4.4.8}$$

From (4.4.4) and (4.4.8), the marginal process $\{X_n\}$ satisfies the difference equations

$$\rho_{XX}(1) = p_{11} + p_{12}\,\rho_{YX}(0)$$

and

$$\rho_{XX}(k) - (p_{11} + p_{22})\,\rho_{Xx}(k-1) - (p_{12}\,p_{21} - p_{11}\,p_{22})\,\rho_{XX}(k-2) = 0,\ k \geq 2.$$

Hence the theorem.

Similarly, the autocorrelation function of the marginal process $\{Y_n\}$ satisfies the difference equation

$$\rho_{YY}(1) = p_{22} + p_{12}\,\rho_{XY}(0) \tag{4.4.9}$$

and

$$\rho_{YY}(k) - (p_{11} + p_{22})\rho_{YY}(k-1) - (p_{12}\,p_{21} - p_{11}\,p_{22})\,\rho_{YY}(k-2) = 0,\ k \geq 2, \tag{4.4.10}$$

where $\rho_{YY}(k) = \mathrm{Corr}(Y_n, Y_{n-k})$.

Remark 4.4.1.

From the difference equations obtained above it can be noted that in general the marginal processes $\{X_n\}$ and $\{Y_n\}$ have the correlations structure of an ARMA(2,1) process. Their actual form will depend upon the values of probabilities p_{ij} ; $i, j = 1, 2$.

Now we verify the structure of the autocorrelation functions of the marginal processes for the following simple cases.

Case (i). Suppose $p_{12} = p_{21} = 0$.

In this case the marginal processes $\{X_n\}$ and $\{Y_n\}$ are independent with following structure

$$X_n = \begin{cases} \varepsilon_{n1} & \text{w.p. } 1-p_{11} \\ X_{n-1} + \varepsilon_{n1} & \text{w.p. } p_{11} \end{cases}$$

and

$$Y_n = \begin{cases} \varepsilon_{n2} & \text{w.p. } 1-p_{22} \\ Y_{n-1} + \varepsilon_{n2} & \text{w.p. } p_{22}. \end{cases}$$

If $\{X_n\}$ and $\{Y_n\}$ are stationary with Laplace marginal distribution then the innovation sequence $\{\varepsilon_{ni}; i = 1,2\}$ is a sequence of scaled Laplace random variables. From the equation (4.4.4), we have

$$\rho_{XX}(1) = p_{11}.$$

When $k = 2$, equation (4.4.8) gives

$$\rho_{XX}(2) = (p_{11} + p_{22})\ \rho_{XX}(1) - p_{11}\, p_{22}$$

$$= (p_{11} + p_{22})\ p_{11} - p_{11}\, p_{22}$$

$$= p_{11}^2.$$

In general, we can show that $\rho_{XX}(h) = p_{11}^h$.

Similarly the autocorrelation function of the marginal process $\{Y_n\}$ satisfies $\rho_{YY}(h) = p_{22}^h$.

Case (ii). Suppose $p_{11} = p_{22} = 0$.

In this case the marginal processes $\{X_n\}$ and $\{Y_n\}$ are coupled processes with following structure

$$X_n = \begin{cases} \varepsilon_{n1} & \text{w.p. } 1-p_{12} \\ Y_{n-1} + \varepsilon_{n1} & \text{w.p. } p_{12} \end{cases}$$

and

$$Y_n = \begin{cases} \varepsilon_{n2} & \text{w.p. } 1-p_{21} \\ X_{n-1} + \varepsilon_{n2} & \text{w.p. } p_{21}. \end{cases}$$

From the equation (4.4.4), we have

$$\rho_{XX}(1) = p_{12}\, \rho_{XY}(0).$$

From the equation (4.4.8), we have

$$\rho_{XX}(k) = p_{12}p_{21}\, \rho_{XX}(k-2)$$

Hence we obtain

$$\rho_{XX}(2h) = (p_{12}p_{21})^h, \text{ when } k = 2h$$

and

$$\rho_{XX}(2h+1) = (p_{12}p_{21})^{h} p_{12} \rho_{XY}(0) \text{ , when } k = 2h+1.$$

Similar expression will be obtained corresponding to the autocorrelation function of the marginal process $\{Y_n\}$.

Case (iii). Suppose $p_{11} = p_{21} = 0$.

The marginal processes corresponding to this case are given by

$$X_n = \begin{cases} \varepsilon_{n1} & \text{w.p. } 1-p_{12} \\ Y_{n-1} + \varepsilon_{n1} & \text{w.p. } p_{12} \end{cases}$$

and

$$Y_n = \begin{cases} \varepsilon_{n1} & \text{w.p. } 1-p_{22} \\ Y_{n-1} + \varepsilon_{n1} & \text{w.p. } p_{22}. \end{cases}$$

The autocorrelation function of the marginal process $\{X_n\}$ is obtained from equation (4.4.4) and (4.4.8) and given by

$$\rho_{XX}(h) = (p_{12} \rho_{XY}(0)) p_{22}^{h-1}, h \geq 1 .$$

Thus $\{X_n\}$ is a Laplace process with ARMA (1,1) correlation structure provided $\rho_{XY}(0) \neq 0$.

The autocorrelation function of the marginal process $\{Y_n\}$ is obtained from (4.4.9) and (4.4.10) and given by the expression

$$\rho_{YY}(h) = p_{22}{}^{h}, h \geq 0.$$

Hence $\{Y_n\}$ is a Laplace process with AR (1) correlation structure.

Case (iv). Suppose $p_{11} = p_{21} = p_{22} = 0$.

The marginal processes under this assumption reduced to the form

$$X_n = \begin{cases} \varepsilon_{n1} & \text{w.p. } 1-p_{12} \\ Y_{n-1} + \varepsilon_{n1} & \text{w.p. } p_{12} \end{cases}$$

and

$$Y_n = \varepsilon_{n2}.$$

If $\varepsilon_{n1} = \varepsilon_{n2}$ then $\{X_n\}$ is a Laplace process with correlation structure of MA (1) process and $\{Y_n\}$ is a sequence of independent and identically distributed Laplace random variables.

4.5. Joint distribution of (X_n, Y_n)

Bell and Smith (1986) investigated an autoregressive process of first order in which the innovation sequence has a given distribution function. Andel (1988)

considered this approach and discussed the AR (1) process with exponentially distributed innovation sequence. Now we make use of this approach to identify the joint distribution of (X_n, Y_n) of stationary processes $\{X_n\}$ and $\{Y_n\}$ if we assume innovation sequence $\{(\varepsilon_{n1}, \varepsilon_{n2})\}$ has bivariate Laplace distribution.

The joint distribution of (X_n, Y_n) is obtained by finding the joint characteristic function of bivariate model described by (4.2.1) and (4.2.2). The expression for the joint characteristic function $\phi_{X_n, Y_n}(t,s) = E(e^{itX_n + isY_n})$ for the general case is complicated and hence we consider a particular case of the model. We consider the case where

$$\Theta_{11}^{(n)} = \Theta_{22}^{(n)} = \begin{cases} 0 & \text{w.p. } 1-p \\ 1 & \text{w.p. } p \end{cases}$$

and

$$\Theta_{12}^{(n)} = \Theta_{21}^{(n)} = 0 \text{ for all } n.$$

Then, model (4.2.1) and (4.2.2) reduced to the form

$$X_n = \begin{cases} \varepsilon_{n1} & \text{w.p. } 1-p \\ X_{n-1} + \varepsilon_{n1} & \text{w.p. } p \end{cases}$$

and

$$Y_n = \begin{cases} \varepsilon_{n2} & \text{w.p. } 1-p \\ Y_{n-1} + \varepsilon_{n2} & \text{w.p. } p. \end{cases}$$

Now, the joint characteristic function (X_n, Y_n) is given by

$$\phi_{X_n,Y_n}(t,s)=\left[p\phi_{X_{n-1},Y_{n-1}}(t,s)+(1-p)\right]\phi_{\varepsilon_{n1},\varepsilon_{n2}}(t,s).$$

Under stationarity assumption, we get

$$\phi_{X,Y}(t,s)=\frac{(1-p)\,\phi_{\varepsilon_{n1},\varepsilon_{n2}}(t,s)}{1-p\phi_{\varepsilon_{n1},\varepsilon_{n2}}(t,s)}. \qquad (4.5.1)$$

Suppose $(\varepsilon_{n1},\varepsilon_{n2})$ be a bivariate symmetric Laplace random vector with mean vector **0** and variance-covariance matrix $\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}$. We shall use the notation $(\varepsilon_{n1},\varepsilon_{n2}) \overset{d}{=} \mathrm{BSL}(\sigma_1,\sigma_2,\rho)$ to describe membership in this family, and the characteristic function of $(\varepsilon_{n1},\varepsilon_{n2})$ is given by

$$\phi_{\varepsilon_{n1},\varepsilon_{n2}}(t,s)=\frac{1}{1+\frac{\sigma_1^2}{2}t^2+\rho\sigma_1\sigma_2\,ts+\frac{\sigma_2^2}{2}s^2}. \qquad (4.5.2)$$

where $\sigma_1 \geq 0, \sigma_2 \geq 0$, $\rho \in [0,1]$ (for more details see Kotz *et al.* (2001)).

Substituting (4.5.2) in (4.5.1) we get

$$\phi_{X,Y}(t,s)=\frac{1}{1+\frac{1}{2}\frac{\sigma_1^2}{1-p}t^2+\rho\frac{\sigma_1\sigma_2}{1-p}ts+\frac{1}{2}\frac{\sigma_2^2}{1-p}s^2}. \qquad (4.5.3)$$

Hence the joint distribution of (X_n, Y_n) is bivariate symmetric Laplace distribution with mean vector **0** and variance-covariance matrix $\frac{1}{1-p}\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}$.

That is,

$$(X_n, Y_n) \underline{d} \equiv \mathrm{BSL}\left(\frac{\sigma_1}{\sqrt{1-p}}, \frac{\sigma_2}{\sqrt{1-p}}, \rho\right).$$

4.6. The bivariate asymmetric Laplace model

Let $\{X_n\}$ and $\{Y_n\}$ be stationary autoregressive processes defined by equations (4.2.1), (4.2.2) and (4.2.3) and assume $\{X_n\}$ and $\{Y_n\}$ are independent sequence of asymmetric Laplace random variables with characteristic function $\phi(t)=\frac{1}{1+\sigma^2t^2-i\mu t}$.

Then from equation (4.2.4) we obtain

$$\phi_{\varepsilon_{nl}}(t)=\frac{\frac{1}{1+\sigma^2t^2-i\mu t}}{p_{11}\frac{1}{1+\sigma^2t^2-i\mu t}+p_{12}\frac{1}{1+\sigma^2t^2-i\mu t}+(1-p_{11}-p_{12})}$$

$$= \frac{1}{1+\sigma^2(1-p_{11}-p_{12})\,t^2 - i\mu(1-p_{11}-p_{12})t}.$$

Similarly

$$\phi_{\varepsilon_{n2}}(t) = \frac{1}{1+\sigma^2(1-p_{11}-p_{12})\,t^2 - i\mu(1-p_{11}-p_{12})t}.$$

Hence $\{\varepsilon_{n1}\}$ and $\{\varepsilon_{n2}\}$ are two independent sequences of independent and identically distributed asymmetric Laplace random variables such that

$$\varepsilon_{ni} \stackrel{d}{=} AL_{ni}\left(\mu(1-p_{i1}-p_{i2}), \sqrt{(1-p_{i1}-p_{i2})}\,\sigma\right),\ i=1,2, \qquad (4.6.1)$$

where $\{AL_{ni}\}$ is a sequence of asymmetric Laplace random variables with characteristic function $\frac{1}{1+\sigma^2 t^2 - i\mu t}$.

So by (4.2.1) and (4.2.2) we can define a first order autoregressive time series model using asymmetric Laplace variables. We call such process as bivariate asymmetric Laplace autoregressive (BALAR (1)) process.

Since $\{X_n\}$ and $\{Y_n\}$ are sequence of asymmetric Laplace random variables

$$E(X_n) = E(Y_n) = \mu \text{ and } E(X_n^2) = E(Y_n^2) = 2(\mu^2+\sigma^2).$$

Using equation (4.3.3) we obtain

$$E(X_n\ Y_n) = p_{11}p_{21}\,2(\mu^2+\sigma^2) + p_{11}p_{22}\,E(X_{n-1}Y_{n-1}) + p_{11}(1-p_{21}-p_{22})\mu^2$$
$$+ p_{12}p_{22}\,2(\mu^2+\sigma^2) + p_{12}p_{21}\,E(X_{n-1}Y_{n-1}) + p_{12}(1-p_{21}-p_{22})\mu^2$$
$$+ p_{21}(1-p_{11}-p_{12})\mu^2 + p_{22}(1-p_{11}-p_{12})\mu^2 + (1-p_{11}-p_{12})(1-p_{21}-p_{22})\mu^2.$$

Since $\{X_n\}$ and $\{Y_n\}$ are stationary processes we obtain

$$E(X_n\ Y_n)(1-p_{11}\,p_{22}-p_{12}\,p_{21}) = (p_{11}p_{21}+p_{12}p_{22})\,2(\mu^2+\sigma^2)$$
$$+\mu^2(1-p_{11}\,p_{21}-p_{12}\,p_{21}-p_{11}\,p_{22}-p_{12}\,p_{22}).$$

Hence

$$\mathrm{Cov}(X_n, Y_n) = E(X_n\ Y_n) - E(X_n)\,E(Y_n)$$

$$= \frac{(p_{11}\,p_{21}+p_{12}\,p_{22})}{(1-p_{11}\,p_{22}-p_{12}\,p_{21})}\,(\mu^2+2\sigma^2). \qquad (4.6.2)$$

Therefore, the autocovariance matrix of the bivariate asymmetric Laplace autoregressive process is $\Gamma(k) = M^k\,\Gamma(0)$, where $M = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$,

$\Gamma(0) = (\mu^2+2\sigma^2)\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$ and $\alpha = \dfrac{(p_{11}p_{21}+p_{12}\,p_{22})}{1-p_{11}\,p_{22}-p_{12}\,p_{21}}$.

The autocorrelation function of the process is always positive. Hence the BALAR (1) process is positively correlated. Negative correlated bivariate asymmetric Laplace autoregressive process may be constructed by the procedure discussed in the case of the BLAR (1) process.

LAPLACE AUTOREGRESSIVE TIME SERIES MODELS

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Chapter-V

Marginal Laplace and Linnik and Bivariate Semi α-Laplace Distributions and Processes

5.1. Introduction

We can observe that heavy tailed bivariate distributions with different tail behavior are useful for modeling bivariate data where the vectors are sums of a random number of small random shocks with heavy tails, and each component has a different tail index. Empirical analysis of some important bivariate data, especially in the fields of mathematical finance, biology, communication theory, environmental science etc. shows that bivariate observations are asymmetric and heavy tailed with different tail behavior. Kozubowski *et al.* (2005) considered a bivariate distribution related to Laplace and Linnik distribution, namely marginal Laplace and Linnik distribution, which can be applied for modeling bivariate data with this character. In this context they studied a general class of multivariate distributions known as operator geometric stable laws.

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random vectors in R^d and let N_p be a geometric random variable with mean $\frac{1}{p}$, and independent of X_n's. If there exists a weak limit of

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This Chapter is based on Kuttykrishnan and Jayakumar (2005, 2006).

$$S_p = a(p)\sum_{i=1}^{N_p}(X_i + b(p)), \tag{5.1.1}$$

when $p \to 0$, where $a(p)$ is a linear operator on R^d and $b(p) \in R^d$, then the class of limiting distributions is called operator geometric stable (OGS) laws. The OGS laws are reduced to geometric stable laws under scalar normalization and in this case we obtain skew Laplace distribution as limiting law when components have finite second moments. If the sum S_p given by (5.1.1) is defined with $N_p = n$, then the limiting distributions belongs to the class of operator stable laws. If $N_p = n$ and $a(p)$ is a scalar then the limiting distribution of (5.1.1) is a member of class of stable laws. Hence the class of OGS laws is a general class of distributions and developed by combining the concepts operator norming and geometric randomized sums.

Kozubowski *et al.* (2005) studied the class of OGS distributions and derived many important properties. They discussed the one to one correspondence between operator stable and OGS distributions and established that the characteristic function $\psi(t)$ of an OGS distribution has the form

$$\psi(t) = \frac{1}{1 - \log \varphi(t)}, t \in R^d, \tag{5.1.2}$$

where $\varphi(t)$ is the characteristic function of an operator stable distribution.

Let $X = (X_1, X_2)$ be a bivariate operator stable random vector with characteristic function

$$\varphi(t,s)=\exp(-\sigma^2 t^2-\eta^\alpha |s|^\alpha),(t,s)\in R^2,\alpha\in(0,2], \tag{5.1.3}$$

then the corresponding OGS random vector $Y=(Y_1,Y_2)$ has the characteristic function

$$\psi(t,s)=\frac{1}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha},(t,s)\in R^2,\alpha\in(0,2]. \tag{5.1.4}$$

From this representation, it is clear that the marginal distributions of $Y=(Y_1,Y_2)$ are Laplace and Linnik distributions with respective characteristic functions

$$\psi(t,0)=\frac{1}{1+\sigma^2 t^2} \tag{5.1.5}$$

and

$$\psi(0,s)=\frac{1}{1+\eta^\alpha |s|^\alpha}, \tag{5.1.6}$$

where $t\in R, s\in R$ and $\alpha\in(0,2]$.

Hence they referred to the distribution of bivariate vector $Y=(Y_1,Y_2)$ with characteristic function (5.1.4) as marginal Laplace and Linnik distribution with parameters α,σ and η.

If $Y=(Y_1,Y_2)$ follows marginal Laplace and Linnik distribution with parameters α,σ and η then we represent it as $Y \underline{d} MLL_\alpha(\sigma,\eta)$.

The applications of this class of distributions in mathematical finance and other related fields are discussed in Kozubowski *et al.* (2005). The applications of this bivariate distribution in time series modeling have not been explored so far. Here we take up this task and develop and study time series models using marginal Laplace and Linnik distributions.

5. 2. Some characterizations of marginal Laplace and Linnik distribution

Kozubowski *et al.* (2005) derived the representation of $Y \underline{\underline{d}} MLL_{\alpha}(\sigma,\eta)$ random vector as

$$Y \underline{\underline{d}} (Z^{1/2} X_1, Z^{1/\alpha} X_2)$$

where Z is unit exponential, X_1 and X_2 are normal and α-stable random variables with $\varphi(t,0) = \exp(-\sigma^2 t^2)$ and $\varphi(0,s) = \exp(-\eta^{\alpha} |s|^{\alpha})$ as respective characteristic functions. It may be noted that Z, X_1 and X_2 are mutually independent random variables. Using this representation, the distribution function $F(y_1,y_2)$ and density function $f(y_1,y_2)$ of $Y = (Y_1, Y_2)$ are given by

$$F(y_1,y_2) = \int_0^{\infty} \Phi(\frac{y_1}{\sqrt{2z}\,\sigma})\, \Omega_{\alpha}(\frac{y_2}{z^{1/\alpha}\eta})\, e^{-z} dz,\ (y_1,y_2) \in R^2, \qquad (5.2.1)$$

$$f(y_1,y_2) = \frac{1}{2\sigma\eta\sqrt{\pi}} \int_0^{\infty} z^{-(\frac{1}{\alpha}+\frac{1}{2})}\, e^{-z-\frac{y_1^2}{4\sigma^2 z}}\, P_{\alpha}(\frac{y_2}{z^{1/\alpha}\eta}) dz,\ (y_1,y_2) \neq (0,0). \qquad (5.2.2)$$

where Φ is the distribution function of standard normal random variable and Ω_α and P_α are the distribution and density functions of α-stable random variable with characteristic function $\exp(-|s|^\alpha)$.

The following theorem establishes the relation between $MLL_\alpha(\sigma,\eta)$ and bivariate operator stable distributions.

Theorem 5.2.1.

Let $\{(Y_{1i}, Y_{2i}), i \geq 1\}$ be a sequence of independent and identically distributed $MLL_\alpha(\sigma,\eta)$ random vectors. Then the random vector

$$(U_n, V_n) = \left((1/n)^{1/2} \sum_{i=1}^{n} Y_{1i},\ (1/n)^{1/\alpha} \sum_{i=1}^{n} Y_{2i} \right) \tag{5.2.3}$$

is asymptotically distributed as bivariate symmetric stable with independent components.

Proof:

Let $\pi_{(U_n,V_n)}(t,s)$ be the characteristic function of (U_n, V_n). Then from (5.2.3), we have

$$\pi_{(U_n,V_n)}(t,s) = E\left(\exp(i t (1/n)^{1/2} \sum_{i=1}^{n} Y_{1i} + i s (1/n)^{1/\alpha} \sum_{i=1}^{n} Y_{2i}) \right)$$

$$= \left[\pi_{(Y_{1i}, Y_{2i})} \left((1/n)^{1/2} t,\ (1/n)^{1/\alpha} s \right) \right]^n$$

$$= \left[\frac{1}{1 + \sigma^2 \frac{t^2}{n} + \eta^\alpha \frac{|s|^\alpha}{n}} \right]^n,$$

since $\left\{ (Y_{1i}, Y_{2i}), i \geq 1 \right\}$ is a sequence of independent and identically distributed $MLL_\alpha(\sigma, \eta)$ random vectors..

When $n \to \infty$, $\pi_{(U_n, V_n)}(t,s) = \exp\left\{ -(\sigma^2 t^2 + \eta^\alpha |s|^\alpha) \right\}$, which is the characteristic function of a bivariate symmetric stable distribution with independent components. Hence the theorem. □

Kozubowski *et al.* (2005) derived the property of stability of OGS distribution with respect to geometric summation and hence corresponding to $MLL_\alpha(\sigma, \eta)$ distribution we have the following theorem.

Theorem 5.2.2.

Let $\left\{ (Y_{1i}, Y_{2i}), i \geq 1 \right\}$ be a sequence of independent and identically distributed $MLL_\alpha(\sigma, \eta)$ random vectors and let $N_p, p \in (0,1)$ be a geometric random variable with mean $\frac{1}{p}$, independent of (Y_{1i}, Y_{2i})'s. Then

$$(p^{1/2}U_{N_p}, p^{1/\alpha}V_{N_p}) \stackrel{d}{=} (Y_{1i}, Y_{2i}) \tag{5.2.4}$$

for all p, where $U_{N_p} = \sum_{i=1}^{N_p} Y_{1i}$ and $V_{N_p} = \sum_{i=1}^{N_p} Y_{2i}$.

Proof:

Let $\pi(t,s)$ and $\psi(t,s)$ be the characteristic functions of $(p^{1/2}U_{N_p}, p^{1/\alpha}V_{N_p})$ and (Y_{1i}, Y_{2i}) respectively.

Then by definition of (U_{N_p}, V_{N_p}), we have

$$\pi(t,s) = E\left(\exp(i t\ p^{1/2} \sum_{i=1}^{N_p} Y_{1i} + i s p^{1/\alpha} \sum_{i=1}^{N_p} Y_{2i}) \right)$$

$$= \sum_{n=1}^{\infty} \left(E(\exp(i t p^{1/2} Y_{1i} + i s p^{1/\alpha} Y_{2i}) \right)^n \ p q^{n-1}$$

$$= \sum_{n=1}^{\infty} \left(\psi(p^{1/2} t, p^{1/\alpha} s) \right)^n p q^{n-1}.$$

Hence

$$\pi(t,s) = \frac{p\ \psi\ (p^{1/2} t,\ p^{1/\alpha} s)}{1-(1-p)\ \psi(p^{1/2} t, p^{1/\alpha} s)}. \tag{5.2.5}$$

Assume $(Y_{1i}, Y_{2i}) \stackrel{d}{=} MLL_{\alpha}(\sigma, \eta)$. Then (5.2.5) gives

$$\pi(t,s)=\frac{p\dfrac{1}{1+\sigma^2 p\,t^2+\eta^\alpha\, p\,|s|^\alpha}}{1-(1-p)\dfrac{1}{1+\sigma^2 p\,t^2+\eta^\alpha\, p\,|s|^\alpha}}$$

$$=\frac{1}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha}.$$

Hence

$$(\,p^{1/2}U_{N_p},p^{1/\alpha}V_{N_p}\,)\ \underline{\underline{d}}\ \mathrm{MLL}_\alpha(\sigma,\eta).$$

That is,

$$(\,p^{1/2}U_{N_p},p^{1/\alpha}V_{N_p}\,)\ \underline{\underline{d}}\ (Y_{1i},Y_{2i}).$$

Hence the theorem. □

Now we obtain a characterization of $\mathrm{MLL}_\alpha(\sigma,\eta)$ random vector using bivariate compounding.

Let (N_1,N_2) be a random vector with bivariate geometric distribution having survival function

$$P(N_1>n_1,N_2>n_2)=\begin{cases}p_{11}^{n_1}(\,p_{01}+p_{11}\,)^{n_2-n_1} & \text{if } n_1\le n_2\\ p_{11}^{n_2}(\,p_{10}+p_{11}\,)^{n_1-n_2} & \text{if } n_1\ge n_2,\end{cases}\qquad(5.2.6)$$

where $0<p_{01},p_{10},p_{11}<1\,;\,0<p_{01}+p_{10}+p_{11}\le 1\,;\,n_1,n_2=1,2,\ldots$ (see Block (1977)).

Theorem 5.2.3.

The distribution of (Y_{1i}, Y_{2i}) is $MLL_{\alpha}(\sigma,\eta)$ if and only if $(p_{01}^{1/2} U_{N_1}, p_{10}^{1/\alpha} U_{N_2})$ is distributed as marginal Laplace and Linnik distribution with independent marginals, where $U_{N_1} = \sum_{i=1}^{N_1} Y_{1i}$, $V_{N_2} = \sum_{i=1}^{N_2} Y_{2i}$, and (N_1, N_2) has bivariate geometric distribution with survival function (5.2.6) and $p_{01} + p_{10} + p_{11} = 1$.

Proof:

Let $\pi(t,s)$ and $\psi(t,s)$ be the characteristic functions of (U_{N_1}, V_{N_2}) and (Y_{1i}, Y_{2i}) respectively.

Paulson and Uppuluri (1972) obtained a characterization of the bivariate geometric distribution and obtained the characteristic function of (U_{N_1}, V_{N_2}) as

$$\pi(t,s) = \psi_1(t,0)\psi_2(0,s)\left[p_{00} + p_{01}\pi(0,s) + p_{10}\pi(t,0) + p_{11}\pi(t,s)\right].$$

When $p_{00} = 0$ and the marginals are independent, the characteristic function of (U_{N_1}, V_{N_2}) is given by

$$\pi(t,s) = \psi(t,s)\left[p_{01}\pi(0,s) + p_{10}\pi(t,0) + p_{11}\pi(t,s)\right] \qquad (5.2.7)$$

When $s = 0$,

$$\pi(t,0) = \psi(t,0)\left[p_{01} + p_{10}\,\pi(t,0) + p_{11}\,\pi(t,0)\right].$$

Hence

$$\pi(t,0) = \frac{(1-p_{10}-p_{11})\,\psi(t,0)}{1-(p_{10}+p_{11})\,\psi(t,0)} \tag{5.2.8}$$

Similarly

$$\pi(0,s) = \frac{(1-p_{01}-p_{11})\,\psi(0,s)}{1-(p_{01}+p_{11})\,\psi(0,s)}. \tag{5.2.9}$$

Let $(Y_{1i}, Y_{2i}) \underline{d} \equiv \mathrm{MLL}_{\alpha}(\sigma,\eta)$. Then $\psi(t,s) = \dfrac{1}{1+\sigma^2 t^2 + \eta^{\alpha}|s|^{\alpha}}$.

Substituting $\psi(t,0)$ in (5.2.8) we get

$$\pi(t,0) = \frac{(1-p_{10}-p_{11})\dfrac{1}{1+\sigma^2 t^2}}{1-(p_{10}+p_{11})\dfrac{1}{1+\sigma^2 t^2}}$$

$$= \frac{p_{01}}{p_{01}+\sigma^2 t^2}.$$

Similarly substituting $\psi(0,s)$ in (5.2.9), we get

$$\pi(0,s) = \frac{p_{10}}{p_{10}+\eta^{\alpha}|s|^{\alpha}}.$$

Hence $p_{01}^{1/2} U_{N_1} \stackrel{d}{=} Y_{1i}$ and $p_{10}^{1/\alpha} U_{N_2} \stackrel{d}{=} Y_{2i}$.

To prove Converse, substitute $\pi(t,s)$, $\pi(t,0)$ and $\pi(0,s)$ in (5.2.7).

This yields $\psi(t,s)$ as $\psi(t,s) = \dfrac{1}{1+\sigma^2 t^2 + \eta^\alpha |s|^\alpha}$.

Hence the theorem. □

5.3. A bivariate autoregressive marginal Laplace and Linnik process

In various applications, one frequently encounters the situations where bivariate data exhibit heavy tailed character with a different tail behavior. Now we develop time series models that can be used for modeling heavy tailed bivariate time series observations with a different tail behavior for each component. Since marginal Laplace and Linnik random vector have different tail behavior for each component and since the distribution is heavy tailed we can use bivariate time series model using marginal Laplace and Linnik distribution in such situations.

Now we develop a bivariate AR (1) process $\{(X_n, Y_n); n = 1,2,...\}$ given by the first order autoregressive structure

$$X_n = p^{1/2} X_{n-1} + \varepsilon_n \tag{5.3.1}$$

and

$$Y_n = p^{1/\alpha} Y_{n-1} + \eta_n, \tag{5.3.2}$$

where $0 < p < 1$ and (X_i, Y_i) is independent of (ε_n, η_n) for $i < n$.

Theorem 5.3.1.

The bivariate first order autoregressive process $\{(X_n, Y_n)\}$ given by the structure (5.3.1) and (5.3.2) is a stationary bivariate time series model with $MLL_\alpha(\sigma,\eta)$ marginal distribution if and only if $\{(\varepsilon_n, \eta_n), n \geq 1\}$ is a sequence of independent and identically distributed random vectors such that

$$(\varepsilon_n, \eta_n) = \begin{cases} (0,0) & \text{w.p.} \quad p \\ MLL_\alpha(\sigma,\eta) & \text{w.p.} \quad 1-p, \end{cases} \tag{5.3.3}$$

provided $(X_0, Y_0) \overset{d}{=} MLL_\alpha(\sigma,\eta)$.

Proof:

Let $\psi_{(X_n,Y_n)}(t,s)$ and $\psi_{(\varepsilon_n,\eta_n)}(t,s)$ be the characteristic function of (X_n, Y_n) and (ε_n, η_n) respectively.

Then from (5.3.1) and (5.3.2) we have

$$\psi_{(\varepsilon_n,\eta_n)}(t,s) = \frac{\psi_{(X_n,Y_n)}(t,s)}{\psi_{(X_{n-1},Y_{n-1})}(p^{1/2}t,\ p^{1/\alpha}s)} \ . \tag{5.3.4}$$

Let $\{(X_n, Y_n)\}$ be a stationary sequence with $MLL_\alpha(\sigma,\eta)$ marginal distribution.

Then from (5.3.4)

$$\psi_{(\varepsilon_n,\eta_n)}(t,s)=\frac{1+\sigma^2 p t^2+\eta^\alpha p|s|^\alpha}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha}$$

$$=p+(1-p)\frac{1}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha}. \qquad (5.3.5)$$

Hence

$$(\varepsilon_n,\eta_n)=\begin{cases}(0,0) & \text{w.p.} \quad p\\ MLL_\alpha(\sigma,\eta) & \text{w.p.} \quad 1-p\end{cases}.$$

Conversely, assume $\{(\varepsilon_n,\eta_n), n\geq 1\}$ be a sequence of independent and identically distributed bivariate random vectors given by (5.3.3) and let (X_0,Y_0) be a $MLL_\alpha(\sigma,\eta)$ random vector with characteristic function (5.1.4).

When $n=1$, in terms of characteristic function, (5.3.1) and (5.3.2) can be represented as

$$\psi_{(X_1,Y_1)}(t,s)=\psi_{(X_0,Y_0)}(p^{1/2}t,\ p^{1/\alpha}s)\,\psi_{(\varepsilon_1,\eta_1)}(t,s)$$

$$=\frac{1}{1+\sigma^2 p\, t^2+\eta^\alpha p\,|s|^\alpha}\left[p+(1-p)\frac{1}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha}\right],$$

using (5.3.5) and since (X_0,Y_0) is a $MLL_\alpha(\sigma,\eta)$ random vector.

On simplification, we get

$$\psi_{(X_1,Y_1)}(t,s)=\frac{1}{1+\sigma^2\ t^2+\eta^\alpha\ |s|^\alpha}.$$

Hence $(X_1,Y_1) \stackrel{d}{=} MLL_\alpha(\sigma,\eta)$.

So, using mathematical induction argument we can prove $(X_n,Y_n) \stackrel{d}{=} MLL_\alpha(\sigma,\eta)$ for all n.

Therefore, the first order autoregressive process $\{(X_n,Y_n)\}$ given by (5.3.1) and (5.3.2) is stationary with $MLL_\alpha(\sigma,\eta)$ marginal distribution.

Hence the theorem. □

The equation (5.3.1) and (5.3.2) can be represented in the vector form

$$Z_n = M Z_{n-1} + W_n, \qquad (5.3.6)$$

where $Z_n=(X_n,Y_n)'$, $M=\begin{bmatrix} p^{1/2} & 0 \\ 0 & p^{1/\alpha} \end{bmatrix}$, and $W_n=(\varepsilon_n,\eta_n)'$. We call the process (5.3.6) such that

$$(\varepsilon_n,\eta_n)=\begin{cases} (0,0) & \text{w.p.} \quad p \\ MLL_\alpha(\sigma,\eta) & \text{w.p.} \quad 1-p, \end{cases}$$

and $(X_0,Y_0) \stackrel{d}{=} MLL_\alpha(\sigma,\eta)$ as first order marginal Laplace and Linnik autoregressive (MLLAR(1)) process.

It can be noted that the bivariate process (5.3.1) and (5.3.2) can be represented as

$$X_n = (p^{1/2})^n X_0 + \sum_{k=0}^{n-1} (p^{1/2})^k \varepsilon_{n-k} \tag{5.3.7}$$

and

$$Y_n = (p^{1/\alpha})^n Y_0 + \sum_{k=0}^{n-1} (p^{1/\alpha})^k \eta_{n-k} . \tag{5.3.8}$$

Using characteristic function, (5.3.7) and (5.3.8) can be represented as

$$\psi_{(X_n,Y_n)}(t,s) = \psi_{(X_0,Y_0)}\left((p^{1/2})^n t,(p^{1/\alpha})^n s\right)\prod_{k=0}^{n-1}\psi_{(\varepsilon_{k+1},\eta_{k+1})}\left((p^{1/2})^k t,(p^{1/\alpha})^k s\right).$$

If (X_0, Y_0) is distributed arbitrary and $\{(\varepsilon_n, \eta_n), n \geq 1\}$ is a sequence of independent and identically distributed bivariate random variables given by (5.3.3), then from the above expression, we have

$$\psi_{(X_n,Y_n)}(t,s) = \psi_{(X_0,Y_0)}\left((p^{1/2})^n t,(p^{1/\alpha})^n s\right)\prod_{k=0}^{n-1}\left(\frac{1+\sigma^2 p^k t^2+\eta^\alpha p^k |s|^\alpha}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha}\right)$$

$$= \psi_{(X_0,Y_0)}\left((p^{1/2})^n t,(p^{1/\alpha})^n s\right)\frac{1+\sigma^2 p^{n-1} t^2+\eta^\alpha p^{n-1}|s|^\alpha}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha}.$$

When $n \to \infty$, $\psi_{(X_n,Y_n)}(t,s) = \dfrac{1}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha}$.

Hence, the first order autoregressive process given by (5.3.1) and (5.3.2) is asymptotically stationary with marginal Laplace and Linnik marginal distribution if (X_0, Y_0) is distributed arbitrary and $\{(\varepsilon_n, \eta_n), n \geq 1\}$ is a sequence of independent and identically distributed bivariate random variables given by (5.3.3).

We can note that the characteristic function (5.1.4) when $\alpha = 2$ is $\psi(t,s) = \frac{1}{1+\sigma^2 t^2 + \eta^2 s^2}$, which is the characteristic function of bivariate symmetric Laplace (BSL) distribution with mean vector **0** and variance-covariance matrix $\Sigma = \begin{pmatrix} 2\sigma^2 & 0 \\ 0 & 2\eta^2 \end{pmatrix}$ (see Kotz *et al.* (2001)).

In this case the bivariate autoregressive process defined by (5.3.1) and (5.3.2) with BSL marginal distribution have the solution

$$(\varepsilon_n, \eta_n) = \begin{cases} (0,0) & \text{w.p.} \quad p \\ (\delta_n, \gamma_n) & \text{w.p.} \quad 1-p, \end{cases}$$

where (δ_n, γ_n) is a BSL random vector with characteristic function $\psi(t,s) = \frac{1}{1+\sigma^2 t^2 + \eta^2 s^2}$.

For a BSL random vector, mean and variance-covariance matrix exists. So we can study the second order properties of bivariate process using the vector representation (5.3.6).

The auto covariance matrix $\Gamma(h)$ is given by

$$\Gamma(h) = \mathrm{Cov}(Z_n, Z_{n-h}),$$

where

$$\mathrm{Cov}(Z_n, Z_{n-h}) = E(Z_n Z_{n-h}) - E(Z_n)\,E(Z_{n-h})$$

$$= E((M Z_{n-1} + W_n) Z_{n-h}) - E(M Z_{n-1} + W_n)\,E(Z_{n-h})$$

$$= M\left(E(Z_{n-1} Z_{n-h}) - E(Z_{n-1})\,E(Z_{n-h})\right)$$

$$= M\Gamma(h-1).$$

By repeated use of the above equation, we obtain

$$\mathrm{Cov}(Z_n, Z_{n-h}) = M^h\,\Gamma(0).$$

Hence, autocovariance matrix $\Gamma(h)$ of the process (5.3.6) with BSL marginal distribution is given by

$$\Gamma(h) = M^h\,\Gamma(0),$$

where $\Gamma(0) = \begin{pmatrix} 2\sigma^2 & 0 \\ 0 & 2\eta^2 \end{pmatrix}$ and $M = \begin{bmatrix} p^{1/2} & 0 \\ 0 & p^{1/\alpha} \end{bmatrix}$.

5.4. A bivariate marginal asymmetric Laplace and asymmetric Linnik process

In practical situation we may come across the cases where empirical distribution of the components of bivariate vector appears to be asymmetric, with

steep peak and tails heavier than those allowed by normal distribution. In such a situation, a skewed form of bivariate distribution is well suited for modeling the data. Here we consider a class of bivariate distributions, which is a member of class of marginally geometric stable laws, with asymmetric Laplace and asymmetric Linnik distributions as marginals.

Definition 5.4.1.

Let $Y = (Y_1, Y_2)$ be a random vector with characteristic function

$$\psi(t,s) = \frac{1}{1+\sigma^2 t^2 + \eta^\alpha |s|^\alpha - i\mu t - i\nu s}, \tag{5.4.1}$$

where $\sigma, \eta \geq 0$, $\mu, \nu \in (-\infty, \infty)$, $\alpha \in (0,2]$. Then we say that $Y = (Y_1, Y_2)$ follows marginal asymmetric Laplace and asymmetric Linnik distribution, and denote it by $Y \underline{d} \text{MALAL}_\alpha(\mu, \nu, \sigma, \eta)$.

Remark 5.4.1.

Kotz *et al.* (2001) discussed a five-parameter family of bivariate distribution, namely bivariate asymmetric Laplace distribution ($\text{BAL}(\mu, \nu, \sigma, \eta, \rho)$), with characteristic function

$$\psi(t,s) = \frac{1}{1+\sigma^2 t^2 + 2\rho\sigma\eta ts + \eta^2 s^2 - i\mu t - i\nu s},$$

where $\sigma, \eta \geq 0$, $\mu, \nu \in (-\infty, \infty)$ and $\rho \in [0,1]$. When $\alpha = 2$, the characteristic function (5.4.1) reduces to the characteristic function of a bivariate asymmetric Laplace distribution $BAL(\mu, \nu, \sigma, \eta, \rho)$ with $\rho = 0$.

Now we define a first order bivariate autoregressive process with $MALAL_{\alpha}(\mu, \nu, \sigma, \eta)$ marginal distribution. If we define the process with structure (5.3.1) and (5.3.2), then it seems to be difficult to get the solution for the equation (5.3.4). Hence we are introducing a bivariate autoregressive process with structural relationship equivalent to the one-parameter TEAR (1) model discussed in Lawrance and Lewis (1981). This construction is based on the property of geometric infinite divisibility satisfied by the OGS laws.

Define an AR (1) process $\{(X_n, Y_n); n = 1, 2, ...\}$ given by the autoregressive structure

$$X_n = \begin{cases} \varepsilon_n & \text{w.p.} \quad p \\ X_{n-1} + \varepsilon_n & \text{w.p.} \quad 1-p \end{cases} \tag{5.4.2}$$

and

$$Y_n = \begin{cases} \eta_n & \text{w.p.} \quad p \\ Y_{n-1} + \eta_n & \text{w.p.} \quad 1-p, \end{cases} \tag{5.4.3}$$

where $0 < p < 1$ and (X_i, Y_i) is independent of (ε_n, η_n) for $i < n$.

Theorem 5.4.1.

Let $\{(\varepsilon_n, \eta_n), n \geq 1\}$ be a sequence of independent and identically distributed marginal asymmetric Laplace and asymmetric Linnik random vectors such that

$$(\varepsilon_n, \eta_n) \stackrel{d}{=} \text{MALAL}_\alpha\left(\mu p, \nu p, \sigma p^{1/2}, \eta p^{1/\alpha}\right) \tag{5.4.4}$$

and $(X_0, Y_0) \stackrel{d}{=} \text{MALAL}_\alpha(\mu, \nu, \sigma, \eta)$. Then the bivariate first order autoregressive process $\{(X_n, Y_n)\}$ given by (5.4.2) and (5.4.3) defines a stationary bivariate time series model with $\text{MALAL}_\alpha(\mu, \nu, \sigma, \eta)$ marginal distribution.

Proof:

Let $\psi_{(X_n, Y_n)}(t,s)$ and $\psi_{(\varepsilon_n, \eta_n)}(t,s)$ be the characteristic function of (X_n, Y_n) and (ε_n, η_n) respectively.

Then, from (5.4.2) and (5.4.3) we get

$$\psi_{(\varepsilon_n, \eta_n)}(t,s) = \frac{\psi_{(X_n, Y_n)}(t,s)}{p + (1-p)\,\psi_{(X_{n-1}, Y_{n-1})}(t,s)} . \tag{5.4.5}$$

If $\{(X_n, Y_n)\}$ is stationary process with $\text{MALAL}_\alpha(\mu, \nu, \sigma, \eta)$ marginal distribution, then from (5.4.5) we have

$$\psi_{(\varepsilon_n,\eta_n)}(t,s)=\frac{\dfrac{1}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha -i\mu t-ivs}}{p+(1-p)\dfrac{1}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha -i\mu t-ivs}}$$

$$=\frac{1}{1+\sigma^2 p\, t^2+\eta^\alpha p\,|s|^\alpha -i\mu pt-ivps}.$$

Hence

$$(\varepsilon_n,\eta_n) \overset{d}{=} \text{MALAL}_\alpha\left(\mu p, \nu p, \sigma p^{1/2}, \eta p^{1/\alpha}\right).$$

Conversely, assume $(X_0,Y_0) \overset{d}{=} \text{MALAL}_\alpha(\mu,\nu,\sigma,\eta)$ and $\{(\varepsilon_n,\eta_n), n\geq 1\}$ is a sequence of independent and identically distributed bivariate random variables given by (5.4.4).

When $n=1$, using equations (5.4.2) and (5.4.3), we can write

$$\psi_{(X_1,Y_1)}(t,s)=p\,\psi_{(\varepsilon_1,\eta_1)}(t,s)+(1-p)\,\psi_{(X_0,Y_0)}\,\psi_{(\varepsilon_1,\eta_1)}(t,s).$$

By substituting $\psi_{(X_0,Y_0)}(t,s)$ and $\psi_{(\varepsilon_1,\eta_1)}(t,s)$, we get

$$\psi_{(X_1,Y_1)}(t,s)=\frac{1}{1+\sigma^2 t^2+\eta^\alpha |s|^\alpha -i\mu t-ivs}.$$

Hence

$$(X_1,Y_1) \overset{d}{=} \text{MALAL}_\alpha(\mu,\nu,\sigma,\eta).$$

Therefore, using mathematical induction argument we can prove

$$(X_n, Y_n) \overset{d}{=} \text{MALAL}_\alpha(\mu, \nu, \sigma, \eta) \text{ for all } n.$$

Hence, the first order autoregressive process $\{(X_n, Y_n)\}$ given by (5.4.2) and (5.4.3) is stationary with $\text{MALAL}_\alpha(\mu, \nu, \sigma, \eta)$ marginal distribution.

Hence the theorem. □

We call the process (5.4.2) and (5.4.3) where

$$(\varepsilon_n, \eta_n) \overset{d}{=} \text{MALAL}_\alpha\left(\mu p, \nu p, \sigma p^{1/2}, \eta p^{1/\alpha}\right)$$

and $(X_0, Y_0) \overset{d}{=} \text{MALAL}_\alpha(\mu, \nu, \sigma, \eta)$ as first order marginal asymmetric Laplace and asymmetric Linnik autoregressive (MALALAR (1)) process. When $\alpha = 2$, the MALALAR (1) defines a first order autoregressive process with bivariate asymmetric Laplace distribution as marginal distribution.

Remark 5.4.2.

Since OGS laws are infinite divisible

$$\psi(t,s) = \left(\frac{1}{1+\sigma^2 t^2 + \eta^\alpha |s|^\alpha}\right)^\tau, \tag{5.4.6}$$

where $\sigma, \eta \geq 0$, $\alpha \in (0,2]$ and $\tau \geq 0$ is a legitimate characteristic function. We call

the bivariate random vector $Y=(Y_1,Y_2)$ with characteristic function (5.4.6) as generalized marginal Laplace and Linnik distribution ($GMLL_\alpha(\sigma,\eta,\tau)$).

The generalized marginal Laplace and Linnik distribution with characteristic function (5.4.6) can be regarded as a special case of a large class of distributions, namely operator υ-stable laws, introduced and studied by Kozubowski *et al.* (2003). It may be noted that generalized marginal Laplace and Linnik distribution is obtained when υ is a gamma distribution in the operator υ-stable laws.

Remark 5.4.3.

A bivariate stationary first order autoregressive model using $GMLL_\alpha(\sigma,\eta,\tau)$ marginal distribution can be constructed using the same argument of Theorem 5.3.1. The process is defined as follows:

Suppose $\{(\varepsilon_n,\eta_n), n\geq 1\}$ is distributed independently and identically as τ-fold convolution of random vectors (δ_n,γ_n), where

$$(\delta_n,\gamma_n)=\begin{cases}(0,0) & \text{w.p. } p\\ MLL_\alpha(\sigma,\eta) & \text{w.p. } 1-p\end{cases}$$

and let $(X_0,Y_0) \stackrel{d}{=} GMLL_\alpha(\sigma,\eta,\tau)$. Then the relation (5.3.1) and (5.3.2) defines a stationary bivariate first order autoregressive time series model with $GMLL_\alpha(\sigma,\eta,\tau)$ marginal distribution.

5.5. Bivariate semi α- Laplace distribution and process

In this Section we study a general class of bivariate distributions, which we shall call bivariate semi α-Laplace distribution, and develop a bivariate autoregressive process with bivariate semi α-Laplace distribution as the marginal distribution.

Definition 5.5.1.

A random vector (X, Y) with characteristic function $\phi(t, s)$ is said to follow bivariate semi α-Laplace distribution with exponent (α_1, α_2), and we write $(X, Y) \stackrel{d}{=} BS\alpha L(\alpha_1, \alpha_2)$, if its characteristic function is

$$\phi(t, s) = \frac{1}{1+\eta(t, s)}, \tag{5.5.1}$$

where $\eta(t, s)$ satisfies the functional equation

$$\eta(t,s) = \frac{1}{p}\eta\,(p^{1/\alpha_1}\,t, p^{1/\alpha_2}\,s), 0 < p < 1, 0 < \alpha_1, \alpha_2 \le 2. \tag{5.5.2}$$

A solution of the functional equation (5.5. 2) is

$$\eta(t, s) = |t|^{\alpha_1} h(t) + |s|^{\alpha_2} h(s) \tag{5.5.3}$$

where $h(t)$ and $h(s)$ are periodic functions in $\ln|t|$ and $\ln|s|$ with periods $\frac{2\pi\alpha_1}{-\ln p}$ and $\frac{2\pi\alpha_2}{-\ln p}$ respectively.

But this solution is not unique as in the univariate case given by the characteristic function (1.4.16) and (1.4.17). For example the function

$$\eta(t,s)=\left[\frac{1}{2}(t,s)\Sigma(t,s)'\right]^{\frac{\alpha}{2}} \tag{5.5.4}$$

where $\alpha\in(0,2]$ and Σ is any non-negative definite matrix satisfy the functional equation (5.5.2) with $\alpha_1=\alpha_2=\alpha$ and any $p\in(0,1]$.

We can note that this function does not have the form of the solution given by (5.5.3). In such case the characteristic function given by (5.5.1) and (5.5.2) is equal to the characteristic function of a bivariate Linnik distribution defined in Anderson (1992).

When $\Sigma=\begin{pmatrix}\sigma_1^{\,2} & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^{\,2}\end{pmatrix}$ in (5.5.4), the characteristic function (5.5.1) becomes

$$\phi(t,s)=\frac{1}{1+\frac{\sigma_1^{\,2}}{2}t^2+\rho\sigma_1\sigma_2\,t\,s+\frac{\sigma_2^{\,2}}{2}s^2}, \tag{5.5.5}$$

where $\sigma_1\geq 0,\sigma_2\geq 0,\ \rho\in[0,1]$, is the characteristic function of bivariate symmetric Laplace distribution with mean vector $\mathbf{0}$ and variance-covariance matrix Σ. If $\rho=0,\sigma_1=\sigma_2=1$ then the distribution is called standard bivariate Laplace distribution (for more details see Kotz *et al.* (2001)).

5.5.1. Some properties of bivariate semi α-Laplace distribution

A representation of the bivariate semi α-Laplace distribution is possible using the joint distribution function $F(x,y)$. The following theorem gives a characterization of bivariate semi α-Laplace distribution in terms of joint distribution functions.

Theorem 5.5.1.

Let $F(x,y)$ be the joint distribution function of the random vector (X,Y). Then $(X,Y) \underline{d} BS\alpha L(\alpha_1,\alpha_2)$ if and only if $F(x,y)$ satisfies the functional equation

$$F(x,y) = p\,F_1(x,y) + q\,F_2(x,y);\ 0<p,q<1;\ p+q=1, \qquad (5.5.5)$$

where $F_1(x,y)$ is the distribution function of the random vector $(p^{1/\alpha_1}X, p^{1/\alpha_2}Y)$ and $F_2(x,y)$ is the convolution of F and F_1.

Proof:

Let $\phi(t,s)$ be the characteristic function of (X,Y). Then equation (5.5.5) in terms of characteristic function is

$$\phi(t,s) = p\,\phi(p^{1/\alpha_1}t, p^{1/\alpha_2}s) + q\,\phi(t,s)\,\phi(p^{1/\alpha_1}t, p^{1/\alpha_2}s). \qquad (5.5.7)$$

Hence

$$\phi(t,s)=\frac{p\,\phi(p^{1/\alpha_1}t,p^{1/\alpha_2}s)}{1-q\,\phi(p^{1/\alpha_1}t,p^{1/\alpha_2}s)}. \tag{5.5.8}$$

It can be easily verified that $\phi(t,s)$ has no zeros and if we represent $\phi(t,s)=\frac{1}{1+\eta(t,s)}$ in (5.5.8), we get

$$\frac{1}{1+\eta(t,s)}=\frac{p\dfrac{1}{1+\eta(p^{1/\alpha_1}t,p^{1/\alpha_2}s)}}{1-(1-p)\dfrac{1}{1+\eta(p^{1/\alpha_1}t,p^{1/\alpha_2}s)}}$$

$$=\frac{1}{1+\frac{1}{p}\eta(p^{1/\alpha_1}t,p^{1/\alpha_2}s)}.$$

So

$$\eta(t,s)=\frac{1}{p}\eta(p^{1/\alpha_1}t,p^{1/\alpha_2}s).$$

Hence the characteristic function of (X,Y) is given by $\phi(t,s)=\frac{1}{1+\eta(t,s)}$, where $\eta(t,s)$ satisfies the functional equation (5.5.2).

Therefore

$$(X,Y)\ \underline{\underline{d}}\ BS\alpha L(\alpha_1,\alpha_2).$$

Conversely, if $(X,Y)\ \underline{\underline{d}}\ BS\alpha L(\alpha_1,\alpha_2)$, by retracing the steps one can easily prove that $F(x,y)$ satisfies the functional equation (5.5.5).

Hence the theorem. □

Definition 5.5.2.

A random vector (X, Y) with characteristic function $\phi(t,s)$ is said to follow bivariate semi stable distribution with exponent (α_1, α_2), and we write $(X,Y) \underline{d} \equiv BSS(\alpha_1, \alpha_2)$, if its characteristic function is

$$\phi(t, s) = e^{-\eta(t, s)}, \tag{5.5.9}$$

where $\eta(t, s)$ satisfies the functional equation (5.5.2).

The following result establishes the relation between bivariate semi α-Laplace and bivariate semi stable distributions.

Theorem 5.5.2.

Let $\{(X_j, Y_j), i \geq 1\}$, be a sequence of independent and identically distributed random vectors such that $(X_i, Y_i) \underline{d} \equiv BS\alpha L(\alpha_1, \alpha_2)$. Then

$$(U_n, V_n) = \left((1/n)^{1/\alpha_1} \sum_{i=1}^{n} X_i,\ (1/n)^{1/\alpha_2} \sum_{i=1}^{n} Y_i \right) \tag{5.5.10}$$

where $\alpha_1 > 0, \alpha_2 > 0; n > 1$, is asymptotically distributed as bivariate semi stable.

Proof:

The characteristic function of (U_n, V_n) is given by

$$\phi_{(U_n, V_n)}(t, s) = E\left(\exp(i\, t (1/n)^{1/\alpha_1} \sum_{i=1}^{n} X_i + i\, s (1/n)^{1/\alpha_2} \sum_{i=1}^{n} Y_i) \right) \tag{5.5.11}$$

Since $(X_i, Y_i) \underline{d} \equiv BS\alpha L(\alpha_1, \alpha_2)$ the characteristic function of (X_i, Y_i) is given by

$\phi_{(X_i,Y_i)}(t,s) = \frac{1}{1+\eta(t,s)}$ for all $i = 1,2,\ldots$, where $\eta(t, s)$ satisfies the functional equation (5.5.2).

Substituting this in (5.5.11), we have

$$\phi_{(U_n,V_n)}(t,s) = \left(\phi_{(X_i,Y_i)}((1/n)^{1/\alpha_1} t, (1/n)^{1/\alpha_2} s)\right)^n.$$

$$= \left[\frac{1}{1+\eta((1/n)^{1/\alpha_1} t, (1/n)^{1/\alpha_2} s)}\right]^n$$

$$= \left[\frac{1}{1+\frac{1}{n}\eta(t,s)}\right]^n$$

$$= e^{-\eta(t,s)}, \text{ as } n \to \infty.$$

Therefore, $\phi_{(U_n,V_n)}(t,s)$ is the characteristic function of semi stable distribution.

Hence the theorem. □

Now we give a characterization of bivariate semi α-Laplace distribution using geometric compounding. Let $U_N = \sum_{i=1}^{N} X_i$ and $V_N = \sum_{i=1}^{N} Y_i$, where N has a geometric distribution with mean $\frac{1}{p}$, independent of (X_i, Y_i)'s. Since the random

vector (U_N, V_N) frequently appears in many applied problems in various areas such as actuarial science, reliability studies, etc, the bivariate semi α-Laplace distribution have a wide variety of applications (see Gnedenko and Korolev (1996)).

Theorem 5.5.3.

Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independent and identically distributed random vectors and let N be a geometric random variable with probability mass function (1.4.5). Further, assume that N is independent of the (X_i, Y_i) where $i = 1, 2, \ldots$ If $U_N = \sum_{i=1}^{N} X_i$ and $V_N = \sum_{i=1}^{N} Y_i$ then the random vectors $(p^{1/\alpha_1} U_N, p^{1/\alpha_2} V_N)$ and (X_i, Y_i) are identically distributed if and only if (X_i, Y_i) follows bivariate semi α-Laplace distribution.

Proof:

Let $\phi(t, s)$ and $\psi(t, s)$ be the characteristic functions of (X_i, Y_i) and $(p^{1/\alpha_1} U_N, p^{1/\alpha_2} V_N)$ respectively.

Then

$$\psi(t,s) = E\left(e^{itp^{1/\alpha_1}\sum_{i=1}^{N} X_i + isp^{1/\alpha_2}\sum_{i=1}^{N} Y_i}\right)$$

$$= \sum_{k=1}^{n} \left(\phi_{X_i, Y_i}(p^{1/\alpha_1} t, p^{1/\alpha_2} s)\right)^k p(1-p)^{k-1}$$

Simplifying, we get

$$\psi(t,s)=\frac{p\,\phi(p^{1/\alpha_1}t,p^{1/\alpha_2}s)}{1-(1-p)\,\phi(p^{1/\alpha_1}t,p^{1/\alpha_2}s)}. \tag{5.5.12}$$

Suppose $(p^{1/\alpha_1}U_N,p^{1/\alpha_2}V_N)\underline{\underline{d}}(X_i,Y_i)$ then by (5.5.12) we have

$$\frac{p\,\phi(p^{1/\alpha_1}t,p^{1/\alpha_2}s)}{1-(1-p)\,\phi(p^{1/\alpha_1}t,p^{1/\alpha_2}s)}=\phi(t,s). \tag{5.5.13}$$

If we represent $\phi(t,s)=\frac{1}{1+\eta(t,s)}$ then (5.5.13) gives

$$\frac{1}{1+\frac{1}{p}\eta(p^{1/\alpha_2}t,p^{1/\alpha_2}s)}=\frac{1}{1+\eta(t,s)}.$$

Hence $\eta(t,s)$ satisfies the functional equation (5.5. 2).

So $(X_i,Y_i)\underline{\underline{d}}\,BS\alpha L(\alpha_1,\alpha_2)$.

Conversely, suppose $(X_i,Y_i)\underline{\underline{d}}\,BS\alpha L(\alpha_1,\alpha_2)$.

Using the representation $\phi(t,s)=\frac{1}{1+\eta(t,s)}$ in (5.5.12), the characteristic function of

$(p^{1/\alpha_1}U_N,\ p^{1/\alpha_2}V_N)$ is given by

$$\psi(t,s)=\frac{1}{1+\frac{1}{p}\eta(p^{1/\alpha_1}t,p^{1/\alpha_2}s)}$$

$$= \frac{1}{1+\eta(t,s)} \quad \text{, because } (X_i, Y_i) \underline{d} \equiv BS\alpha L(\alpha_1, \alpha_2).$$

$$= \phi(t,s)$$

So

$$(p^{1/\alpha_1} U_N, p^{1/\alpha_2} V_N) \underline{d} \equiv (X_i, Y_i).$$

Hence the theorem. □

Let us define

$$U_{N_{k-1}} = \sum_{i=1}^{N_{k-1}} X_i \text{ and } V_{N_{k-1}} = \sum_{i=1}^{N_{k-1}} Y_i$$

where $\{N_k; k \geq 1\}$ is a sequence of geometric random variables with parameters p_k, where $0 < p_k < 1$.

If $\phi_k(t,s) = E(e^{it U_{N_{k-1}} + is V_{N_{k-1}}})$ is the characteristic function of geometric (p_{k-1}) sum of independent and identically distributed random vectors with common characteristic function $\phi_{k-1}(t,s)$ where $k = 2,3,...$ then we get the following characterization of the bivariate semi α-Laplace distribution.

Theorem 5.5.4.

Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independent and identically distributed random vectors with common characteristic function $\phi(t,s)$. Define $\phi_1 = \phi$ and ϕ_k as the characteristic function of a geometric (p_{k-1}) sum of independent and

identically distributed random vectors with common characteristic function $\phi_{k-1}, k = 2,3,...$ Then

$$\phi_k\left((\prod_{j=1}^{k-1} p_j)^{1/\alpha_1} t, (\prod_{j=1}^{k-1} p_j)^{1/\alpha_2} s\right) = \phi(t,s)$$

if and only if (X_i, Y_i) follows a bivariate semi α-Laplace distribution.

Proof:

By definition, the characteristic function of $(U_{N_{k-1}}, V_{N_{k-1}})$ is

$$\phi_k(t,s) = E(e^{\,it\,U_{N_{k-1}} + \,is\,V_{N_{k-1}}})$$

$$= \sum_{m=1}^{n} (\phi_{k-1}(t,s))^m p_k (1-p_k)^{m-1}$$

$$\phi_k(t,s) = \frac{p_{k-1}\,\phi_{k-1}(t,s)}{1-(1-p_{k-1})\,\phi_{k-1}(t,s)}. \qquad (5.5.14)$$

Write $\phi_k(t,s) = \frac{1}{1+\eta_k(t,s)}$ in equation (5.5.14) we have

$$\frac{1}{1+\eta_k(t,s)} = \frac{1}{1+\frac{1}{p_{k-1}}\eta_{k-1}(t,s)}$$

Hence

$$\eta_k(t,s) = \frac{1}{p_{k-1}}\eta_{k-1}(t,s), k = 2,3,\ldots.$$

Using this relation recursively, we obtain

$$\eta_k(t,s)=\frac{1}{\prod_{j=1}^{k-1}p_{k-1}}\eta_1(t,s)\,. \tag{5.5.15}$$

Since $\phi_1=\phi$, $\eta_1=\eta$, the equation (5.5.15) becomes

$$\eta_k(t,s)=\frac{1}{\prod_{j=1}^{k-1}p_{j-1}}\eta(t,s) \tag{5.5.16}$$

Now $\phi_k\left((\prod_{j=1}^{k-1}p_j)^{1/\alpha_1}t,(\prod_{j=1}^{k-1}p_j)^{1/\alpha_2}s\right)=\phi(t,s)$ implies

$$\frac{1}{1+\frac{1}{\prod_{j=1}^{k-1}p_j}\eta\left((\prod_{j=1}^{k-1}p_j)^{1/\alpha_1}t,(\prod_{j=1}^{k-1}p_j)^{1/\alpha_2}s\right)}=\frac{1}{1+\eta(t,s)}$$

Hence $\eta(t,s)$ satisfies the functional equation (5.5.2).

So $(X_i,Y_i)\underline{\underline{d}}\,BS\alpha L(\alpha_1,\alpha_2)$.

Conversely, suppose $(X_i,Y_i)\underline{\underline{d}}\,BS\alpha L(\alpha_1,\alpha_2)$.

Using equation (5.5.16) we have

$$\phi_k\left((\prod_{j=1}^{k-1}p_j)^{1/\alpha_1}t,(\prod_{j=1}^{k-1}p_j)^{1/\alpha_2}s\right)=\frac{1}{1+\frac{1}{\prod_{j=1}^{k-1}p_j}\eta\left((\prod_{j=1}^{k-1}p_j)^{1/\alpha_1}t,(\prod_{j=1}^{k-1}p_j)^{1/\alpha_2}s\right)}$$

$$= \frac{1}{1+\eta(t,s)}, \text{ because } (X_i, Y_i) \underline{d} \underline{=} BS\alpha L(\alpha_1, \alpha_2)$$

$$= \phi(t, s).$$

Hence the theorem. □

5.5.2. A first order autoregressive bivariate semi α-Laplace process

We now present a bivariate AR (1) model as follows:

Let $\{(\varepsilon_n, \delta_n); n \geq 1\}$ be a sequence of independent and identically distributed random vectors and define a sequence of random vectors $\{(X_n, Y_n); n \geq 1\}$ as

$$X_n = \begin{cases} p^{1/\alpha_1} X_{n-1} & \text{w.p.} \quad p \\ p^{1/\alpha_1} X_{n-1} + \varepsilon_n & \text{w.p.} \quad 1-p \end{cases} \tag{5.5.17}$$

and

$$Y_n = \begin{cases} p^{1/\alpha_1} Y_{n-1} & \text{w.p.} \quad p \\ p^{1/\alpha_1} Y_{n-1} + \delta_n & \text{w.p.} \quad 1-p \end{cases} \tag{5.5.18}$$

where $0 < p < 1, 0 < \alpha_1, \alpha_2 \leq 2$.

Theorem 5.5.5.

Assuming that $(X_0, Y_0) \underline{d} \underline{=} (\varepsilon_1, \delta_1)$, then the process $\{(X_n, Y_n); n \geq 1\}$ defined by (5.5.17) and (5.5.18) is stationary if and only if $(\varepsilon_n, \delta_n) \underline{d} \underline{=} BS\alpha L(\alpha_1, \alpha_2)$.

Proof:

Let $\phi_{(X_n,Y_n)}(t,s)$ be the bivariate characteristic function of (X_n, Y_n). Then, (5.5.17) and (5.5.18) produces

$$\phi_{(X_n,Y_n)}(t,s) = p\,\phi_{(X_{n-1},Y_{n-1})}(p^{1/\alpha_1}t,\ p^{1/\alpha_2}s) + (1-p)\phi_{(X_{n-1},Y_{n-1})}(p^{1/\alpha_1}t, p^{1/\alpha_2}s)\phi_{(\varepsilon_n,\delta_n)}(t,s). \tag{5.5.19}$$

Assuming that the process $\{(X_n, Y_n); n \geq 1\}$ is stationary and $(X_0, Y_0) \underline{d} (\varepsilon_1, \delta_1)$, then equation (5.5.19) is same as

$$\phi(t,s) = p\ \phi(p^{1/\alpha_1}t, p^{1/\alpha_2}s) + (1-p)\ \phi(p^{1/\alpha_1}t,\ p^{1/\alpha_2}s)\ \phi(t,s),$$

where $\phi(t,s)$ is the characteristic function of stationary sequence $\{(X_n, Y_n); n \geq 1\}$.

Hence

$$\phi(t,s) = \frac{p\ \phi(\ p^{1/\alpha_1}t, p^{1/\alpha_2}s)}{1-(1-p)\ \phi(p^{1/\alpha_1}t, p^{1/\alpha_2}s)}. \tag{5.5.20}$$

If we write $\phi(t,s) = \dfrac{1}{1+\eta(t,s)}$ in (5.5.20), we get

$$\eta(t,s) = \frac{1}{p}\,\eta(p^{1/\alpha_1}t, p^{1/\alpha_2}s).$$

Hence the characteristic function $\phi(t,s)$ of stationary sequence $\{(X_n, Y_n); n \geq 1\}$ is given by

$$\phi(t,s) = \frac{1}{1+\eta(t,s)}, \text{ where } \eta(t,s) = \frac{1}{p}\eta\,(p^{1/\alpha_1}t, p^{1/\alpha_2}s).$$

That is, the bivariate AR (1) process given by (5.5.17) and (5.5.18) is stationary with bivariate semi α-Laplace marginal distribution.

Again if $(X_0, Y_0) \underline{d} (\varepsilon_1, \delta_1)$ and $(X_0, Y_0) \underline{d} BS\alpha L(\alpha_1, \alpha_2)$ then we can show that the process is stationary using the following inductive argument.

Suppose $(X_{n-1}, Y_{n-1}) \underline{d} BS\alpha L(\alpha_1, \alpha_2)$. Then from (5.5.19) we get

$$\phi_{(X_n,Y_n)}(t,s) = p\frac{1}{1+\eta(p^{1/\alpha_1}t, p^{1/\alpha_2}s)} + (1-p)\frac{1}{1+\eta(p^{1/\alpha_1}t, p^{1/\alpha_2}s)}\frac{1}{1+\eta(t,s)}$$

$$= \frac{1}{1+\eta(p^{1/\alpha_1}t, p^{1/\alpha_2}s)}\left[p + (1-p)\frac{1}{1+\eta(t,s)}\right]$$

$$= \frac{1}{1+p\,\eta(t,s)}\left[p + (1-p)\frac{1}{1+\eta(t,s)}\right], \text{ because } \eta(t,s) \text{ satisfies}$$

the functional equation (5.5.2).

Therefore

$$\phi_{(X_n,Y_n)}(t,s) = \frac{1}{1+\eta(t,s)}.$$

Hence

$$(X_n, Y_n) \stackrel{d}{=} BS\alpha L(\alpha_1, \alpha_2) \text{ for all } n \geq 1.$$

Hence the theorem. □

So we can define first order autoregressive bivariate semi α-Laplace process $\{(X_n, Y_n); n \geq 1\}$ as follows:

Let $(X_0, Y_0) \stackrel{d}{=} (\varepsilon_1, \delta_1)$ and for $n = 1, 2, 3, \ldots$ define

$$(X_n, Y_n) = \begin{cases} (p^{1/\alpha_1} X_{n-1}, p^{1/\alpha_2} Y_{n-1}) & \text{w.p. } p \\ (p^{1/\alpha_1} X_{n-1} + \varepsilon_n, p^{1/\alpha_2} Y_{n-1} + \delta_n) & \text{w.p. } 1-p \end{cases} \tag{5.5.21}$$

where $0 < \alpha_1, \alpha_2 \leq 2$, $0 < p < 1$ and $\{(\varepsilon_n, \delta_n), n \geq 1\}$ is a sequence of independent and identically distributed $BS\alpha L(\alpha_1, \alpha_2)$ random vectors.

5.5.3. Properties of first order autoregressive bivariate semi α-Laplace process

The first order autoregressive bivariate semi α-Laplace process given by the equation (5.5.21) can be represented in the vector form

$$Z_n = M Z_{n-1} + I_n W_n \tag{5.5.22}$$

where $Z_n = (X_n, Y_n)'$, $M = \begin{bmatrix} p^{1/\alpha_1} & 0 \\ 0 & p^{1/\alpha_2} \end{bmatrix}$, $I_n = \begin{bmatrix} U_{n_1} & 0 \\ 0 & U_{n_2} \end{bmatrix}$, $W_n = (\varepsilon_n, \delta_n)'$

and $\{U_{n_i}\}$ is a sequence of independent random variables with probability distribution

$$P(U_{n_i}=0)=1-p(U_{n_i}=1)=p \text{ , for } i=1,2.$$

For the bivariate semi α-Laplace distribution it is not possible to obtain the exact expression of its moments due to its implicit nature.

Consider a simple case, where $\alpha_1=\alpha_2=2$ and $\Sigma=\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}$ in (5.5.4). Then the characteristic function (5.5.1) is the characteristic function of bivariate symmetric Laplace distribution with mean vector **0** and variance-covariance matrix Σ and is given by the expression (5.5.5).

In such situation the auto covariance matrix of the first order autoregressive bivariate semi α-Laplace process (bivariate symmetric Laplace process) is given by

$$\Gamma(k)=\mathrm{Cov}(Z_n,Z_{n-k})$$

$$=M^k\Gamma(0), \text{ using the representation (5.5.22).}$$

$$=M^k\Sigma.$$

Hence the covariance structure of the bivariate symmetric Laplace process is obtained from $\Gamma(k)=M^k\Sigma$.

If (X_0,Y_0) has an arbitrary bivariate distribution and $\{(\varepsilon_n,\delta_n);n\geq 1\}$ is a sequence of independent and identically distributed $BS\alpha L(\alpha_1,\alpha_2)$ random vectors, then using equation (5.5.17) and (5.5.18) it can be easily verified that

$$\phi_{(X_n,Y_n)}(t,s)=\phi_{(X_0,Y_0)}(p^{n/\alpha_1}t,p^{n/\alpha_2}s)\prod_{i=0}^{n-1}\left(p+(1-p)\frac{1}{1+\eta(p^{i/\alpha_1}t,p^{i/\alpha_2}s)}\right)$$

$$=\phi_{(X_0,Y_0)}(p^{n/\alpha_1}t,p^{n/\alpha_2}s)\frac{1+p^n\eta(t,s)}{1+\eta(t,s)}.$$

Hence

$$\phi_{(X_n,Y_n)}(t,s)=\frac{1}{1+\eta(t,s)},\text{as}\quad n\to\infty,\ \text{where}\ \eta(t,s)\ \text{satisfies the}$$

functional equation (5.5.2).

Hence the process (5.5.21) is asymptotically stationary with bivariate semi α-Laplace marginal distribution if (X_0,Y_0) has an arbitrary bivariate distribution.

LAPLACE AUTOREGRESSIVE TIME SERIES MODELS

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Chapter-VI

Discrete Laplace distribution and Process

6.1. Introduction

Kemp (1997) studied a class of discrete distribution, namely discrete normal distribution, supported on the set of integers and analogue of the normal distribution. The probability mass function of the discrete normal random variable is obtained using the method

$$P(Y=k)=\frac{f(k)}{\sum_{j=-\infty}^{\infty} f(j)}, k=0,\pm1,\pm2,... \qquad (6.1.1)$$

where $f(x)$ is the probability density function of a normal random variable.

Corresponding to any continuous random variable X we can construct a discrete random variable Y on integers with probability mass function given by the equation (6.1.1). Hence, a discrete random variable corresponding to a classical Laplace random variable with probability density function (1.3.1) can be defined and the resulting discrete random variable Y has the probability mass function

$$P(Y=k)=\frac{\frac{1}{2\sigma}e^{-|k|/\sigma}}{\sum_{j=-\infty}^{\infty}\frac{1}{2\sigma}e^{-|j|/\sigma}},$$

On simplification, we get

$$P(Y=k)=\frac{1-p}{1+p}p^{|k|}, \text{ where } p=e^{-1/\sigma}.$$

Definition 6.1.1.

A random variable Y is said to follow discrete Laplace distribution with parameter $p \in (0,1)$ if its probability mass function is

$$f_p(k)=\frac{1-p}{1+p}\,p^{|k|}, 0<p<1, k=0,\pm1,\pm2,... \qquad (6.1.2)$$

and it is represented by $X \sim DL(p)$.

Inusah and Kozubowski (2006) studied various properties of the discrete Laplace distribution. Some properties of discrete Laplace distribution are

(1) Let $X \sim DL(p)$, then mean and variance are given by

$E(X)=0$ and $V(X)=\frac{2p}{(1-p)^2}$ respectively.

The absolute moment is equal to $E|X|=\frac{2p}{(1+p)}$.

The characteristic function is equal to $\phi(t)=\frac{(1-p)^2}{(1-e^{it}p)(1-e^{-it}p)}$.

(2) Among the discrete distributions on integers with non-vanishing probability density function and $E|X|=c>0$, the discrete Laplace distribution has the maximum entropy.

(3) The discrete Laplace distribution is infinitely divisible, geometrically infinitely divisible and stable with respect to geometric compounding.

We know that a Laplace random variable has the same distribution as the difference of two independent and identically distributed exponential random variables and an analogue property holds in the case of discrete Laplace distribution. Hence a discrete Laplace random variable has the same distribution of difference of two independent and identically distributed geometric random variables. This property leads to applications of the discrete Laplace distribution in analysis of uncertainty in hydro climatic systems. The hydro climatic episodes such as droughts, floods, warm spells and cold spells are commonly quantified in terms of their duration and magnitude. The durations of episodes above and below the reference level known as positive and negative episodes respectively are frequently modeled by geometric distribution. Hence, the difference of positive and negative episodes may be modeled using the discrete Laplace model (see Biondi *et al.* (2002) Inusah and Kozubowski (2006)).

6.2.First order discrete Laplace autoregressive process

Statistical data expressed in terms of counts taken sequentially in time and which are correlated arise in many contexts. Examples of this process are the number of patients in a hospital at a specific point of time or the number of persons waiting in a queue for certain moment. In each of these examples, an element of the process at time t can be either the survival of an element of the process at previous time or an arrival (innovation) sequence, which has a certain discrete distribution. The statistical data corresponding to such processes constitute a discrete time series. Recently much effort has been put in for the study of discrete time series. Al-Osh

and Alzaid (1987), Alzaid and Al-Osh (1990), Jin Guan and Yuan (1991) and Pillai and Jayakumar (1995) have developed discrete time series models with binomial, Poisson, geometric and discrete Mittag-Leffler marginal distributions. Now we develop an integer valued autoregressive time series model with discrete Laplace distribution as marginal distribution.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined by the autoregressive equation

$$X_n = \begin{cases} \varepsilon_n & \text{w.p. } \theta \\ X_{n-1} + \varepsilon_n & \text{w.p. } 1-\theta \end{cases}, \qquad (6.2.1)$$

where $0 < \theta < 1$ and $\{\varepsilon_n\}$ be a sequence of independent and identically distributed random variables such that X_m and ε_n are independent when $m < n$.

Theorem 6.2.1.

Let $\{\varepsilon_n\}$ be a sequence of independent and identically distributed discrete Laplace random variables such that $\varepsilon_n \sim DL(\delta)$, then the first order autoregressive process $\{X_n, n \geq 1\}$ given by (6.2.1) defines a stationary time series of DL(p) random variables, where $\delta = 1 - \frac{1-p}{2\theta p}\left(\sqrt{(1-p)^2 + 4\theta p} - 1 + p\right)$.

Proof:

Let the characteristic functions of $\{X_n\}$ and $\{\varepsilon_n\}$ are denoted by $\phi_{X_n}(t)$ and $\phi_{\varepsilon_n}(t)$ respectively, then from (6.2.1) we get

$$\phi_{X_n}(t) = \theta\, \phi_{\varepsilon_n}(t) + (1-\theta)\, \phi_{X_{n-1}}(t)\, \phi_{\varepsilon_n}(t). \tag{6.2.2}$$

Assume $\{X_n\}$ is stationary with discrete Laplace marginal distribution then from (6.2.2) we have

$$\phi_{\varepsilon_n}(t) = \frac{\phi_X(t)}{\theta + (1-\theta)\,\phi_X(t)}$$

$$= \frac{\dfrac{(1-p)^2}{(1-e^{it}p)(1-e^{-it}p)}}{\theta + (1-\theta)\dfrac{(1-p)^2}{(1-e^{it}p)(1-e^{-it}p)}}.$$

Hence

$$\phi_{\varepsilon_n}(t) = \frac{(1-p)^2}{\theta(1-e^{it}p)(1-e^{-it}p) + (1-\theta)(1-p)^2}. \tag{6.2.3}$$

Now we show that the characteristic function (6.2.3) is same as the characteristic function of DL(δ) random variable where δ is given by

$$\delta = 1 - \frac{1-p}{2\theta p}\left(\sqrt{(1-p)^2 + 4\theta p} - 1 + p\right).$$

Set (6.2.3) equal to $\frac{(1-\delta)^2}{(1-e^{it}\delta)(1-e^{-it}\delta)}$, which is the characteristic function of DL(δ) random variable, we have

$$\frac{(1-p)^2}{\theta(1-e^{it}p)(1-e^{-it}p)+(1-\theta)(1-p)^2}=\frac{(1-\delta)^2}{(1-e^{it}\delta)(1-e^{-it}\delta)}$$

This produces the equation

$$(1-p)^2(1-e^{it}\delta)(1-e^{-it}\delta)=(1-\delta)^2\theta(1-e^{it}p)(1-e^{-it}p)+(1-\theta)(1-p)^2(1-\delta)^2$$

and hold for each $t\in(-\infty,\infty)$.

The above equation is satisfied provided

$$p\theta(1-\delta)^2=\delta(1-p)^2$$

and

$$(1+p)^2(1-\delta)^2\theta=(1+\delta^2)(1-p)^2-(1-p)^2(1-\theta)(1-\delta)^2.$$

On simplification, we get that these two equations are equivalent to the quadratic equation in δ where $\delta\in(0,1)$ and given by

$$h(\delta)=\theta p\delta^2-\delta(2\theta p+(1-p)^2)+\theta p=0$$

where $p,\theta\in(0,1)$.

Since $h(0)=p\theta>0$ and $h(1)=-(1-p)^2<0$, there exists a unique solution of δ in the interval $(0,1)$.

Hence

$$\delta = \frac{(2\ \theta p + (1-p)^2) - \sqrt{(1-p)^2\left((1-p)^2 + 4\ \theta p\right)}}{2\ \theta p}$$

$$= 1 - \frac{1-p}{2\theta p}\left(\sqrt{(1-p)^2 + 4\theta p} - 1 + p\right).$$

Therefore, from the above discussion equation (6.2.3) can be written as

$$\phi_{\varepsilon_n}(t) = \frac{(1-\delta)^2}{(1-e^{it}\delta)(1-e^{-it}\delta)}.$$

So

$$\varepsilon_n \sim DL(\delta) \quad \text{where} \quad \delta = 1 - \frac{1-p}{2\theta p}\left(\sqrt{(1-p)^2 + 4\theta p} - 1 + p\right).$$

Hence the theorem.

The autocovariance function of the process (6.2.1) is given by

$$Cov(X_n, X_{n-h}) = \theta\, Cov(\varepsilon_n, X_{n-h}) + (1-\theta)\, Cov(X_{n-1} + \varepsilon_n, X_{n-h})$$

$$= (1-\theta) Cov(X_{n-1}, X_{n-h})$$

$$= (1-\theta)^h Var(X_{n-h}), \text{ by repeating the procedure.}$$

Hence

$$\rho(h) = (1-\theta)^h. \tag{6.2.4}$$

So the autocorrelation function is positive and decays exponentially and has the same form as the Yule-Walker equation of the first order autoregressive process.

The conditional expectation of the process $\{X_n\}$ is linear in x and given by the expression $E(X_n / X_{n-1} = x) = (1-\theta)\, x$.

Now $P(X_n > X_{n-1})$ of the process is obtained as follows:

By definition of the process

$$P(X_n > X_{n-1}) = \theta\, P(\varepsilon_n > X_{n-1}) + (1-\theta)\, P(\varepsilon_n > 0). \qquad (6.2.5)$$

Consider

$$P(\varepsilon_n > X_{n-1}) = \sum_{x=-\infty}^{\infty} P(\varepsilon_n > X_{n-1} / X_{n-1} = x)\, P(X_{n-1} = x)$$

$$= \sum_{x=-\infty}^{\infty} P(\varepsilon_n > x)\, P(X_{n-1} = x)$$

$$= \sum_{x=-\infty}^{\infty} \frac{1-\delta}{1+\delta}\delta^{|x|}\ \frac{1-p}{1+p}p^{|x|}$$

$$= \frac{1-\delta}{1+\delta}\frac{1-p}{1+p}\sum_{x=-\infty}^{\infty}(\delta p)^{|x|}$$

$$= \frac{1-\delta}{1+\delta}\frac{1-p}{1+p}\frac{1+\delta p}{1-\delta p}.$$

Also note that

$$P(\varepsilon_n > 0) = \sum_{x=1}^{\infty} P(\varepsilon_n = x)$$

$$= \sum_{x=1}^{\infty} \frac{1-\delta}{1+\delta}\delta^x = \frac{\delta}{1+\delta}.$$

Hence, using (6.2.5) we have

$$P(X_n > X_{n-1}) = \theta\frac{1-\delta}{1+\delta}\frac{1-p}{1+p}\frac{1+\delta p}{1-\delta p} + (1-\theta)\frac{\delta}{1+\delta}. \qquad (6.2.6)$$

The probability that X_n is greater than X_{n-1} for the discrete Laplace autoregressive model is a function of θ and p (note that δ is a function of p).

Now we can estimate the parameters θ and p from the equations (6.2.4) and (6.2.6) using the first order sample autocorrelation function $\hat{\rho}(1)$ and the relative frequency of the number of up-runs in the sample $x_0, x_1, ..., x_n$ of $n+1$ observations.

Suppose $\hat{\rho}(1)$ be the first order sample autocorrelation function. Then

$$\hat{\rho}(1) = \frac{\sum_{j=0}^{n-1}\left(x_{j+1} - \bar{x}\right)\left(x_j - \bar{x}\right)}{\sum_{j=0}^{n}\left(x_j - \bar{x}\right)^2}, \qquad (6.2.7)$$

where $\bar{x} = \frac{1}{n+1}\sum_{j=0}^{n} x_j$.

Let us define

$$I(X_i > X_{i-1}) = \begin{cases} 1 & \text{if } X_i > X_{i-1} \\ 0 & \text{otherwise.} \end{cases}$$

Then the estimate of $P(X_n > X_{n-1})$, denoted by $\hat{P}$ is given by

$$\hat{P} = \frac{1}{n}\sum_{i=1}^{n} I(X_i > X_{i-1}). \tag{6.2.8}$$

Hence, the estimate of θ is $\hat{\theta} = 1 - \hat{\rho}(1)$ and the estimate $\hat{p}$ of p is obtained by substituting $\hat{P}$ for $P(X_n > X_{n-1})$ and $\hat{\theta}$ for θ in (6.2.6) where the values of $\hat{\rho}(1)$ and $\hat{P}$ are available from (6.2.7) and (6.2.8) respectively.

6.3. First order skewed discrete Laplace process

A discrete random variable Y on set of integers corresponding to the asymmetric Laplace distribution with probability density function (1.4.13) can be defined using the equation (6.1.1). The distribution of such random variable is known as skewed discrete Laplace distribution. Kozubowski and Inusah (2006) studied skewed discrete Laplace distributions.

Definition 6.3.1.

A random variable Y is said to follow skewed discrete Laplace distribution with parameter $p_1 \in (0,1)$ and $p_2 \in (0,1)$ if its probability mass function is

$$f_{p_1,p_2}(k)=\frac{(1-p_1)(1-p_2)}{1-p_1p_2}\begin{cases} p_1^{\,k} & \text{if } k=0,1,2,\ldots \\ p_2^{\,-k} & \text{if } k=-1,-2,\ldots \end{cases} \tag{6.3.1}$$

and it is represented by $Y \sim SDL(p_1,p_2)$.

It may be noted that when $p_1=p_2=p$, we obtain symmetric discrete Laplace distribution with probability mass function (6.1.2).

Kozubowski and Inusah (2006) studied various properties including unimodality, infinite divisibility, geometric infinitely divisibility and maximum entropy property and obtained expressions for mean, variance, characteristic function etc. of the skewed discrete Laplace distribution.

The characteristic function of $X \sim SDL(p_1,p_2)$ is given by

$$\phi(t)=\frac{(1-p_1)(1-p_2)}{(1-e^{it}\,p_1)(1-e^{-it}\,p_2)}. \tag{6.3.2}$$

The mean and variance of $Y \sim SDL(p_1,p_2)$, may be determined from (6.3.2) or directly using (6.3.1). The mean and variance are obtained as

$$E(X)=\frac{p_1}{1-p_1}-\frac{p_2}{1-p_2}$$

and

$$V(X)=\frac{1}{(1-p_1)^2(1-p_2)^2}\left[\frac{p_2(1-p_1)^3(1+p_2)+p_1(1-p_2)^3(1+p_1)}{1-p_1p_2}-(p_1-p_2)^2\right],$$

respectively.

Similar to the representation of a skewed Laplace random variable as the difference of two exponential random variables with different parameters an analogue result for the skewed discrete Laplace random variable is possible. Hence a skewed discrete Laplace random variable can be considered as the difference of two independent geometric random variables. Similar to discrete Laplace distribution skewed discrete Laplace distribution can be used to model in the analysis of uncertainty in the hydro climatic episodes.

Now we develop a stationary time series model using skewed discrete Laplace random variable.

Theorem 6.3.1.

Let $\{\varepsilon_n\}$ be a sequence of independent and identically distributed skewed discrete Laplace random variables such that $\varepsilon_n \sim \text{SDL}(\delta_1, \delta_2)$, then the first order autoregressive process $\{X_n, n \geq 1\}$ given by (6.2.1) defines a stationary time series of $\text{SDL}(p_1, p_2)$ random variables, where

$$\delta_1 = \frac{2 p_1 \theta}{(p_1 + p_2)\theta + (1-p_1)(1-p_2) + \sqrt{\left((p_1 + p_2)\theta + (1-p_1)(1-p_2)\right)^2 - 4p_1 p_2 \theta^2}} \quad (6.3.3)$$

and

$$\delta_2 = \frac{p_2 \delta_1}{p_1}. \quad (6.3.4)$$

Proof:

Assume $\{X_n\}$ is stationary with skewed discrete Laplace marginal distribution then from (6.2.2) we have

$$\phi_{\varepsilon_n}(t) = \frac{\dfrac{(1-p_1)(1-p_2)}{(1-e^{it}p_1)(1-e^{-it}p_2)}}{\theta + (1-\theta)\dfrac{(1-p_1)(1-p_2)}{(1-e^{it}p_1)(1-e^{-it}p_2)}}$$

$$= \frac{(1-p_1)(1-p_2)}{\theta(1-e^{it}p_1)(1-e^{-it}p_2) + (1-\theta)(1-p_1)(1-p_2)}.$$

Hence

$$\phi_{\varepsilon_n}(t) = \frac{(1-p_1)(1-p_2)}{\theta(1-e^{it}p_1)(1-e^{-it}p_2) + (1-\theta)(1-p_1)(1-p_2)}. \qquad (6.3.5)$$

Now we show that the characteristic function $\phi_{\varepsilon_n}(t)$ is same as the characteristic function of SDL(δ_1,δ_2) random variable where

$$\delta_1 = \frac{2p_1\theta}{(p_1+p_2)\theta + (1-p_1)(1-p_2) + \sqrt{\left((p_1+p_2)\theta + (1-p_1)(1-p_2)\right)^2 - 4p_1p_2\theta^2}} \text{ and}$$

$$\delta_2 = \frac{p_2\delta_1}{p_1}.$$

Set $\phi_{\varepsilon_n}(t)$ equal to $\frac{(1-\delta_1)(1-\delta_2)}{(1-e^{it}\delta_1)(1-e^{-it}\delta_2)}$, which is the characteristic function of SDL(δ_1,δ_2) random variable, we have

$$\frac{(1-p_1)(1-p_2)}{\theta(1-e^{it}p_1)(1-e^{-it}p_2)+(1-\theta)(1-p_1)(1-p_2)}=\frac{(1-\delta_1)(1-\delta_2)}{(1-e^{it}\delta_1)(1-e^{-it}\delta_2)}$$

This produces the equation

$$(1-p_1)(1-p_2)(1-e^{it}\delta_1)(1-e^{-it}\delta_2)=(1-\delta_1)(1-\delta_2)\theta(1-e^{it}p_1)(1-e^{-it}p_2)$$
$$+(1-\theta)(1-p_1)(1-p_2)(1-\delta_1)(1-\delta_2)$$

and hold for each $t\in(-\infty,\infty)$.

The above equation is satisfied provided

$$p_2(1-\delta_1)(1-\delta_2)\theta=\delta_2(1-p_1)(1-p_2)$$

and

$$p_1(1-\delta_1)(1-\delta_2)\theta=\delta_1(1-p_1)(1-p_2).$$

Dividing the corresponding sides of the above equation leads to $\frac{p_2}{p_1}=\frac{\delta_2}{\delta_1}$.

Substituting $\delta_2=\frac{p_2\,\delta_1}{p_1}$ in the first equation resulted into a quadratic equation of the form

$$h(\delta_1) = p_1 \delta_1^{\,2}\theta - \delta_1\left[(p_1+p_2)\theta + (1-p_1)(1-p_2)\right] + p_1\theta = 0$$

Since $h(0) = p_1\theta > 0$ and $h(1) = -(1-p_1)(1-p_2) < 0$, there exists a unique solution of δ_1 in the interval $(0,1)$.

Hence

$$\delta_1 = \frac{(p_1+p_2)\,\theta + (1-p_1)(1-p_2) - \sqrt{\left((p_1+p_2)\theta + (1-p_1)(1-p_2)\right)^2 - 4p_1p_2\theta^2}}{2\ \theta\, p_2}$$

$$= \frac{2\,p_1\,\theta}{(p_1+p_2)\,\theta + (1-p_1)\,(1-p_2) + \sqrt{\left((p_1+p_2)\,\theta + (1-p_1)\,(1-p_2)\right)^2 - 4p_1\,p_2\theta^2}}$$

and

$$\delta_2 = \frac{p_2\,\delta_1}{p_1}.$$

Therefore, from the above discussion equation (6.3.5) can be written as

$$\phi_{\varepsilon_n}(t) = \frac{(1-\delta_1)\,(1-\delta_2)}{(1-e^{it}\delta_1)\,(1-e^{-it}\delta_2)}.$$

Hence

$$\varepsilon_n \sim \mathrm{SDL}(\delta_1, \delta_2),$$

where δ_1 and δ_2 are given by (6.3.3) and (6.3.4) respectively.

Hence the theorem.

From the definition (6.2.1) of the stationary first order autoregressive process with skewed discrete Laplace marginal distribution we can show that the autocorrelation function is

$$\rho(h) = (1-\theta)^h$$

and the conditional expectation of the process $\{X_n\}$ is given by the expression

$$E(X_n / X_{n-1} = x) = (1-\theta)\, x + \frac{\delta_1 - \delta_2}{(1-\delta_1)(1-\delta_2)}.$$

LAPLACE AUTOREGRESSIVE TIME SERIES MODELS

Thesis Submitted to the University of Calicut for the degree of

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in Statistics

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Chapter-VII

Applications

7.1. Introduction

Several models have been used for modeling time series observations. A mathematical model representing the set of observations consists of a certain mathematical form and a set of parameters. Hence fitting an appropriate model to an observed series involves two interrelated problems, namely determining the order and estimating the parameters of the model. A systematic approach to modeling time series observations involves following phases.

(i) Select the type of model among the various linear time series models.

(ii) Determine the order of the model.

(iii) Use proper estimation method to estimate the parameters of the model.

(iv) Testing the goodness of fit of the model.

The overall time series modeling is an iterative process with the feedback and interaction between the above-referred phases.

The selection of the type of the linear model generally depends on the statistical characteristics like autocorrelation and partial autocorrelation functions of the given time series. Let $\{x_0, x_1, .., x_n\}$ be the given time series values, then the sample autocorrelation function $\hat{\rho}(h)$ of order h is given by

$$\hat{\rho}(h) = \frac{\sum_{j=0}^{n-h}\left(x_{j+h} - \bar{x}\right)\left(x_j - \bar{x}\right)}{\sum_{j=0}^{n}\left(x_j - \bar{x}\right)^2}, \qquad (7.1.1)$$

where $\bar{x} = \frac{1}{n+1}\sum_{j=0}^{n} x_j$ and $h < n$.

The partial autocorrelation function at lag h denoted by $\alpha(h) = \phi_{hh}, h \geq 1$ of the stationary process $\{X_n\}$ is uniquely determined by the equation

$$\begin{bmatrix} \rho(0) & \rho(1) & \dots & \rho(h-1) \\ \rho(1) & \rho(0) & \dots & \rho(h-2) \\ \vdots & \vdots & \vdots & \vdots \\ \rho(h-1) & \rho(h-2) & \dots & \rho(0) \end{bmatrix} \begin{bmatrix} \phi_{h1} \\ \phi_{h2} \\ \vdots \\ \phi_{hh} \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(h) \end{bmatrix}. \qquad (7.1.2)$$

Hence the sample partial autocorrelation $\hat{\alpha}(h)$ at lag h of $\{x_0, x_1, .., x_n\}$, provided $x_i \neq x_j$ for some i and j, is given by $\hat{\alpha}(h) = \hat{\phi}_{hh}, 1 \leq h < n$, where $\hat{\phi}_{hh}$ is uniquely determined by (7.1.2) with each $\rho(h)$ replaced by the corresponding sample autocorrelation $\hat{\rho}(h)$ given by (7.1.1).

Box and Jenkins (1970) discussed the model identification procedure using autocorrelation and partial autocorrelation functions. If the data exhibits stationarity character and have rapidly decreasing autocorrelation function we choose an autoregressive process with order p to model the data. To determining the order of

the process we choose the partial autocorrelation function . It can be seen that for an AR (p) process the partial autocorrelations ϕ_{mm} are equal to zero when $m > p$. In particular the estimate of p is obtained by finding the smallest value of r such that the sample partial autocorrelation function $\hat{\phi}_{mm}$ satisfies $|\hat{\phi}_{mm}| < \frac{1.96}{\sqrt{n}}$ for $m > r$, where n is the number of observations. If the sample autocorrelation and partial autocorrelation functions of observed series is in tune with the theoretical pattern of the known model, we use such model to model the given series of observations.

For an AR (1) process the first order autocorrelation function decays exponentially, while the partial autocorrelation function cuts off after the first lag. If the autocorrelation function and partial autocorrelation function of the sample are consistent with that of an AR (1) model we can identify the model as AR (1). To select the proper marginal distribution to fit the observed data, statistical tests like chi-square and Kolmogrov- Smirnov tests are used to test the goodness of fit. The same problem can also be addressed using the methods percentage-percentage (P-P) probability plots / quantile- quantile (Q-Q) probability plots (for more discussions see Sim (1994)).

When a satisfactory model is found, the next stage is extrapolating past behavior into the future and hence forecast future values of the observed series. Given the information set $\{x_0, x_1, ..., x_n\}$, the variable $X_{n+m}, m = 1, 2, ...$ can be predicted. The classical statistical theory tells us that the minimum mean square error point forecast of $X_{n+m}, m = 1, 2, ...$, given the information $\{x_0, x_1, ..., x_n\}$ is

$E(X_{n+m}/x_0, x_1, ..., x_n)$. Hence for the Markov process the point forecasts are obtained by finding $E(X_{n+m}/x_n)$ where $m = 1, 2, ...$

7.2. An Application

The class of asymmetric Laplace distributions is well suited for modeling phenomena where the variable of interest results from a large random number of independent observations, while the empirical distribution appears to be asymmetric, with steep peak and tails heavier than those allowed by normal distribution. Usually the empirical distributions of data sets related to currency exchange rate, interest rate, stock price changes, industrial production rate etc. often do not exhibit the character of normal law but with asymmetric, steep peak and heavy tailed character. Kozubowski and Podgórski (1999, 2000) used asymmetric Laplace distributions for modeling interest rate and currency exchange rate. Although the theory and applications of asymmetric Laplace distributions is well developed in recent years, applications in time series modeling is not much developed. Here we are discussing an application of asymmetric Laplace distribution in the field of time series modeling.

As an application, ALAR (1) model is fitted to the monthly industrial production index of the USA. The historical data consists of 1020 values of index of industrial production of the USA (data is taken from the web site www.economagic.com) from January 1921 to December 2005. Time series plot of

the data $\{Y_n\}$ is given in Figure 7.2.1. From the figure we can notice an upward trend in the index of industrial production.

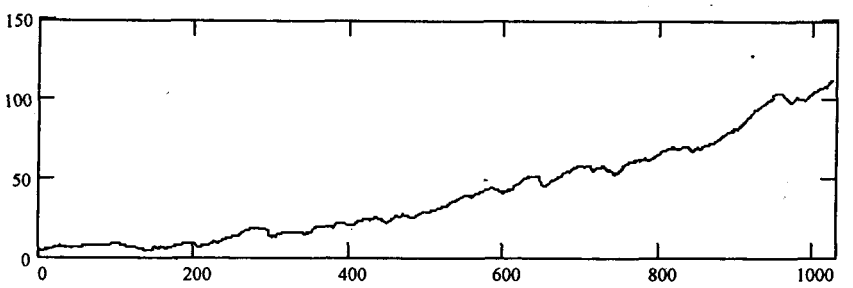

Figure 7.2.1

Time series plot of index of industrial production

The given time series $\{Y_n\}$ is non-stationary. Now we generate a new series by the transformation $\ln Y_n$. Time series plot of $\{\ln Y_n\}$ is given in the Figure 7.2.2.

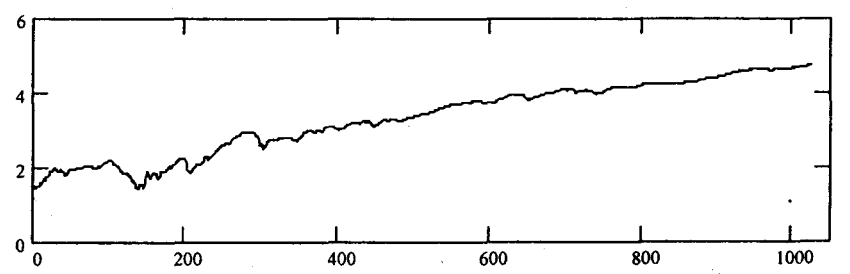

Figure 7.2.2

Time series plot of logarithm of index of industrial production

The non-stationary historical series $\{\ln Y_n\}$ is made stationary by taking the first order difference of the historical data. Let us define $X_i = \ln Y_i - \ln Y_{i-1}, i = 1, 2, \ldots$ Time series plot of series $\{X_n\}$ is given in

Figure 7.2.3. Now it is seen that the resulting time series $\{X_n\}$ can be modeled using a stationary process. From the time series plot of the stationary series $\{X_n\}$ we can notice large fluctuations at the initial stage and as time increases fluctuations become small. Also a general asymmetry and sharp peaks in the fluctuations is exhibited.

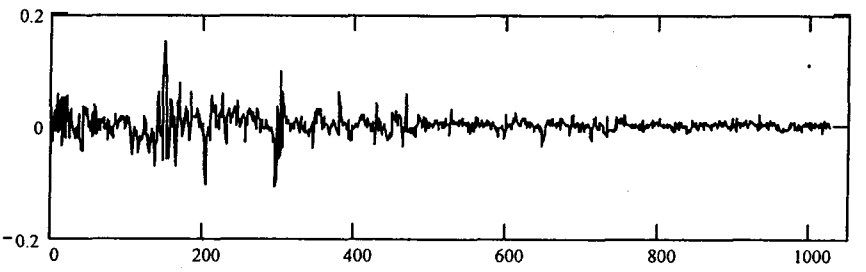

Figure 7.2.3

Time series plot of first order difference of logarithm of index of industrial production

The sample autocorrelation function (ACF) and partial autocorrelation (PACF) values of the same corresponding to different lags of the stationary series are given in the Table 7.2.1.

Lag	1	2	3	4	5	6	7	8
ACF	0.508	0.213	0.084	-0.003	-0.02	-0.03	0.08	0.115
PACF	0.508	-0.061	-0.001	-0.049	0.007	-0.028	0.164	0.110

Table 7.2.1

Sample autocorrelation and partial autocorrelation function

From the above table we can note that, sample autocorrelation values are exponentially decreasing and lag-2 sample partial autocorrelation function $\hat{\phi}_{22} = -0.061$ falls within the limits $\pm\frac{1.96}{\sqrt{n}}$. Hence the autocorrelation and partial autocorrelation of the sample data are consistent with that of a first order autoregressive process. So we prefer to model the data using a first order autoregressive model.

Next we shall examine whether the data $\{X_n\}$ can be modeled using an asymmetric Laplace distribution. The minimum value of the data is –0.11 and maximum value is 0.153. A frequency table is prepared by taking equal class intervals starting from –0.11 and ending at 0.153 and the histogram of first order difference of the logarithm of index of industrial production is presented in Figure 7.2.4.

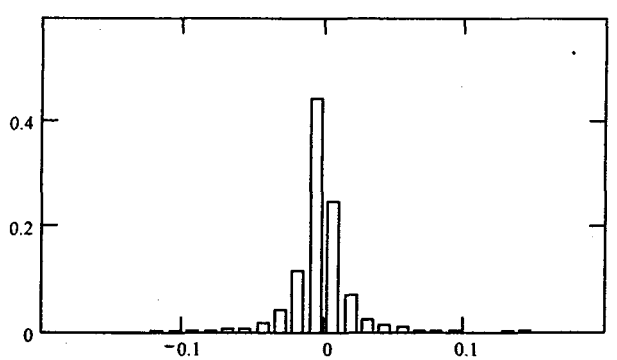

Figure 7.2.4

Histogram of first order difference of logarithm of index of industrial production data

The histogram of the series demonstrates that the historical data corresponding to the first order difference of logarithm of index of industrial production has peaked with asymmetric character. Now we estimate the model and distribution parameters using the method described in Section 2.2.2 of Chapter-II.

The estimates values of ρ, μ and σ are $\hat{\rho} = 0.509$, $\hat{\mu} = 0.003$ and $\hat{\sigma} = 0.013$ respectively.

Hence the estimate $\hat{\kappa}$ of κ is given by $\hat{\kappa} = \dfrac{2}{\frac{\hat{\mu}}{\hat{\sigma}} + \sqrt{4 + (\frac{\hat{\mu}}{\hat{\sigma}})^2}} = 0.853$.

Using these estimate values simulate a sequence $\{X_n\}$ of ALAR (1) process of the form (2.2.3).

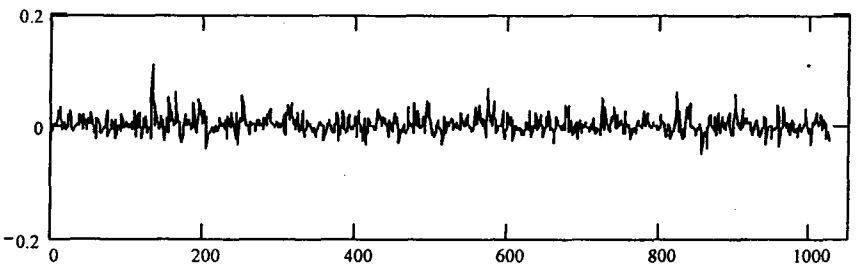

Figure 7.2.5

Time series plot of simulated ALAR (1) sequence

In order to assess the adequacy of fit, we use the technique of graphical method. The histogram of the sequence $\{X_n\}$ with the best-fitted sequence of ALAR (1) process super imposed on it is given in the Figure 7.2.6a. The P-P plot of the same is given in the Figure 7.2.6 b. Using the historical data $\{X_n\}$ we draw the

cumulative frequency curve with the cumulative frequency curve of simulated ALAR (1) process embedded on it. The same is given in the Figure 7.2.6 c. The plots of data quantile versus asymmetric Laplace quantile corresponding to the estimated parameters are given the Figure 7.2.6 d.

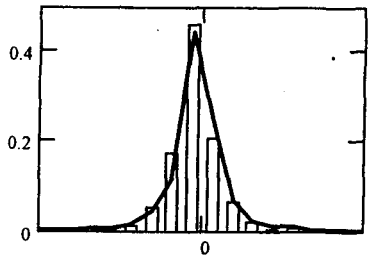

Figure 7.2.6a

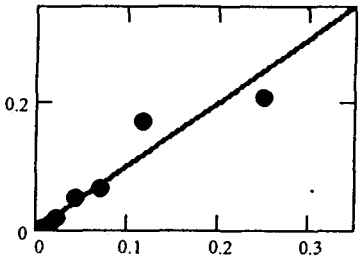

Figure 7.2.6b

Histogram and Q-Q plot of ALAR (1) sequence

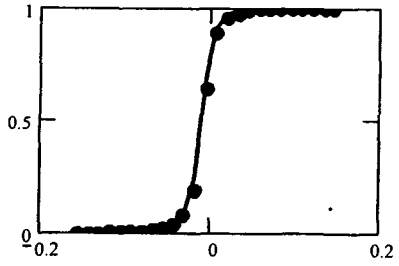

Figure 7.2.6c

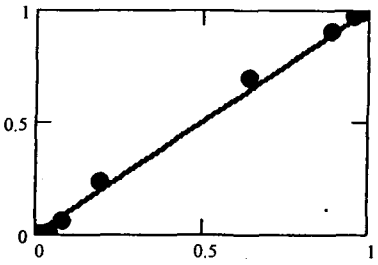

Figure 7.2.6d

Cumulative frequency curve and P-P plot of ALAR (1) sequence

The histogram of the given historical data overlaid with the corresponding estimated ALAR (1) sequence shows that the data are in close agreement with the ALAR (1) model. Also the P-P plot and Q-Q plot are very close to a straight line.

Hence we conclude that the fitted model, ALAR (1) model, is adequate for modeling the given time series observations $\{X_n\}$.

Next we shall verify whether the fitted model generates the historical sequence. The historical and simulated sequence of observations is presented in Figure 7.2.7 (the solid-line represents simulated sequence of observations and dot-line represents historical data). From this figure we can see that the historical data bear a close resemblance to the simulated data. It is observed that the historical and the simulated series agree in all the 100 repetitions we have made. Thus we can conclude that ALAR (1) process is appropriate for modeling the given set of time series data.

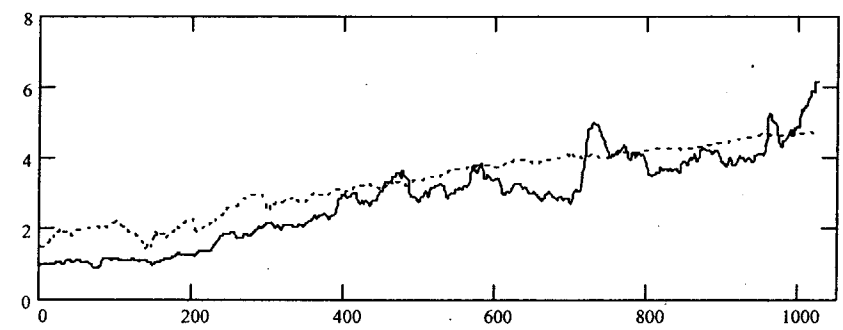

Figure 7.2.7

Time series plot of simulated (solid line) and historical data (dot line) of ALAR (1) sequence

For the ALAR (1) process we know $E(X_n / X_{n-1} = x) = \rho x + (1-\rho)\mu$. Now we forecast observations using the same technique adopted in simulation and the forecast is done by taking X_0 as the last observation in the historical data. The

generated 12 observations of X_i are calculated and hence we can forecast index of industrial production in the USA for the year 2006. The results are given in the following table.

t	Y_t
January 2006	112.098
February 2006	112.099
March 2006	112.102
April 2006	112.104
May 2006	112.107
June 2006	112.110
July 2006	112.113
August 2006	112.117
September 2006	112.120
October 2006	112.123
November 2006	112.126
December 2006	112.129

Table 7.2.1

Forecast of index of industrial production in USA for the year 2006

LAPLACE AUTOREGRESSIVE TIME SERIES MODELS

Thesis Submitted to the University of Calicut for the degree of

DOCTOR OF PHILOSOPHY

in Statistics

By

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