D 31465 **(Pages : 3)** Name.................................

Reg. No.................................

THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 3C 12—FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Consider a measurable subset E = $[a, b]$ of $\mathbb{R}$ and $1 \leq p < \infty$. Show that the set of all step functions on E is dense in $L^p([a,b])$.

2. Let $\| \ \|$ and $\| \ \|'$ be two norms a linear space X over $\mathbb{K}$. Show that $\| \ \|$ is equivalent to $\| \ \|'$ if and only if there are $\alpha, \beta > 0$ such that $\beta\|x\| \leq \|x\|' \leq \alpha\|x\|$ for every $x \in X$.

3. Let $X = \mathbb{R}^2$ with $\| \ \|_\infty$. Find the norm of the linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x(1), x(2)) = (x(1) + x(2), x(1) - x(2))$.

4. Let E be a non-empty convex subset of a normed space X over $\mathbb{K}$. If $a \in X$ but $a \notin \overline{E}$, then prove that there is $f \in X'$ and $t \in \mathbb{R}$ such that $\operatorname{Re} f(x) \leq t < \operatorname{Re} f(a)$ for all $x \in \overline{E}$.

5. Let $X = X_1 \times X_2 \times \ldots \times X_m$ is a Banach space. Prove that each $X_1, X_2, \ldots, X_m$ are Banach spaces.

6. Let X be a normed space over $\mathbb{K}$. If $0 \neq a \in X$, show that there is some $f \in X'$ such that $f(a) = \|a\|$ and $\|f\| = 1$.

7. Define a projection map on a linear space X. Show that if P is a projection so is I - P, where I is the identity operator.

8. State Bounded Inverse Theorem. Show by an example that completeness of the spaces cannot be omitted.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Show that the metric space $L^{\infty}([a,b])$ is not separable.

10. Give an example to show that $\mathbb{K}^n$ with the norms $\| \|_1$ and $\| \|_\infty$ is not strictly convex. Also show that $\mathbb{K}^n$ is strictly convex with the norm $\| \|_2$.

11. Let X and Y be normed spaces and F : X $\to$ Y be linear. Prove that F is continuous if and only if for every Cauchy sequence (x_n) in X, the sequence $(F(x_n))$ is Cauchy in Y.

12. Consider $Z = \{(x(1), x(2)) \in X : x(1) = x(2)\}$, and define $g \in Z'$ by $g(x(1), x(2)) = x(1)$. Find any two Hahn Banach extensions of g.

13. Let X normed space and Y be a Banach space. Consider a dense subspace X_o of X and $F_o \in BL(X_o, Y)$. Prove that there is a unique $F \in BL(X,Y)$ such that $F/X_o = F_o$.

14. Let X and Y be normed spaces. If Z is a closed subspace of X, then prove that the quotient map Q from X to X/Z is continuous and open.

(4 × 4 = 16 marks)

Part C

Answer **either** *A* **or** *B of each question.*
Each question carries 12 marks.

15. A (a) State and prove Baire's theorem for metric space.

(b) For $1 \le p < \infty$, show that the metric space l^p is separable.

B (a) Let Y be a finite dimensional subspace of a normed space X. Prove that Y is complete and hence closed in X.

(b) Let E be a measurable subset of $\mathbb{R}$. For $1 \le p < \infty$, show that the set of all simple measurable functions on E which are zero outside subsets of finite measure is dense in $L^p(E)$.

16. A (a) Let X and Y be normed spaces and F : X $\to$ Y be linear map such that the range R(F) of F is finite dimensional. Show that F is continuous if and only if the zero space Z(F) of F is closed in X.

(b) There exists a discontinuous linear map on an infinite dimensional normed linear space X. Prove or disprove.

B (a) Let $M = (K_{ij})$ be an infinite matrix with scalar entries such that $\sup\left\{\sum_{j=1}^{\infty} |K_{ij}| : i = 1,2,3\ldots\right\} < \infty$. For $x \in l^\infty$, let $M(x) \in l^\infty$ be defined by $(Mx)(i) = \sum_{j=1}^{\infty} K_{ij} x(j)$. Show that M is a continuous linear map.

(b) Let X be a normed space over $\mathbb{K}$, E be a non-empty open convex subset of X and Y be a subspace of X such that $E \cap Y = \varnothing$. Prove that there exists an $f \in X'$ such that $f(x) = 0$ for every $x \in Y$ but $\operatorname{Re} f(x) \neq 0$ for every $x \in E$.

17. A (a) Let X and Y be normed space and $X \neq \{0\}$. Prove that BL (X, Y) is a Banach space in the operator norm if and only if Y is a Banach space.

(b) State and prove Hahn Banach extension theorem.

B (a) Let X be normed space and Y be a closed subspace of X. Prove that X is a Banach space if and only if Y and X/Y are Banach spaces in the induced norm and quotient norm, respectively.

(b) If Y is a proper dense subspace of a Banach space X, then prove that Y is not a Banach space in the induced norm.

18. A State and prove closed graph theorem.

B (a)Let X and Y be Banach spaces and F : X → Y be a linear map which is closed and surjective. Prove that F is continuous and open.

(b) Give an example to show that open mapping theorem may not hold if the normed spaces X and/or Y are not complete.

(4 × 12 = 48 marks)

D 31464 **(Pages : 4)** **Name.......................................**

Reg. No...................................

THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 3C 11—COMPLEX ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Find the image of $\{z : \operatorname{Re} z < 0, |\operatorname{Im} z| < \pi\}$ under the exponential function.

2. Prove that if S is a Möbius transformation, then S is a composition of translations, dilations and inversion.

3. Evaluate $\int_\gamma \frac{\sin z}{z^3}\, dz, \gamma(t) = e^{it}, 0 \le t \le 2\pi.$

4. Find the power series expansion of $\log z$ about $z = i$.

5. Evaluate $\int_{|z|=3} \frac{2z-3}{z^2-3z+2}\, dz.$

6. Find $\int_\gamma \left(\frac{e^z}{z-1}\right)^n dz$ for all positive integers n, where $\gamma(t) = 1 + e^{it}$ for $0 \le t \le 2\pi$.

7. Find the poles of $f(z) = \frac{z-2}{z^2+z-2}$ and find the residue at a pole.

8. Prove that an entire function has a removable singularity at infinity if and only if it is a constant.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let G be some open disk. If $u : G \rightarrow R$ is a harmonic function, then prove that u has a harmonic conjugate.

10. State and prove Liouville's theorem.

11. Let f be analytic in the disk $B(a; R)$ and suppose that γ is closed rectifiable curve in $B(a; R)$. Then prove that $\int_{\gamma} f = 0$.

12. Find all possible values of $\int_{\gamma} \frac{dz}{1+z^2}$ where γ is any closed rectifiable curve in C not passing through $\pm i$.

13. Let f be analytic in a region G and a is a point in G with $|f(a)| \geq |f(z)|$ for all z in G. Then prove that f must be a constant function.

14. Let $f(z) = \frac{1}{z(z-2)}$; give the Laurent expansion of $f(z)$ in each of the annuli :

    (a) $ann(0; 1, 2)$; and

    (b) $ann(0; 2, \infty)$.

(4 × 4 = 16 marks)

Part C

Answer **either** *A* **or** *B of each of the following questions.*
Each question carries 12 marks.

UNIT I

15. A. (i) For the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, define the number R,

$0 \le R \le \infty$ by $\frac{1}{R} = \lim\sup |a_n|^{1/n}$. Then prove that :

a) If $|z-a| < R$, the series converges absolutely ; and

b) If $|z-a| > R$, the series diverges.

(ii) If $\sum_{n=0}^{\infty} a_n (z-a)^n$ is a power series with radius of convergence R, then prove that

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right|.$$

B (i) If z_1, z_2, z_3, z_4 are distinct points in C_∞, then prove that (z_1, z_2, z_3, z_4) is a real number if and only if all four points lie on a circle..

(ii) Prove that a Möbius transformation takes circles onto circles.

UNIT II

16. A (i) Let f be analytic in $B(a;R)$ Then prove that

$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a| < R$ where $a_n = \frac{1}{n!} f^{(n)}(a)$ and this series has radius of convergence $\ge R$.

(ii) State and prove Cauchy's estimate.

Turn over

B (i) If $\gamma : [0, 1] \to C$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then prove that $\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}$ is an integer.

(ii) Let γ be a closed rectifiable curve in C. Then prove that $n(\gamma ; a)$ is a constant for a belonging to a component of $G = C - \{\gamma\}$.

UNIT III

17. A (i) State and prove *Goursat's* theorem .

(ii) If G is simply connected and $f : G \to C$ is analytic in G, then prove that *f* has a primitive in G.

B (i) Let G be a region and suppose that *f* is a non-constant analytic function on G. Then prove that for any open set U in G, $f(U)$ is open.

(ii) If $f : G \to C$ is an analytic function and γ is a closed rectifiable curve in G such that $\gamma \sim 0$. Then prove that $\int_{\gamma} f = 0$.

UNIT IV

18. A (i) State and prove residue theorem.

(ii) Show that $\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} dx = \frac{\pi}{\sqrt{2}}$.

B (i) Prove that if *f* has an isolated singularity at *a* then the point $z = a$ is a removable singularity if and only if $\lim_{z \to a} (z - a) f(z) = 0$.

(ii) Prove that if *f* has an essential singularity at $z = a$ then for every $\delta > 0$, $\{f[ann(a ; 0, \delta)]\}^{-} = C$.

(4 × 12 = 48 marks)