SOME BIVARIATE DISTRIBUTIONS GENERATED THROUGH COMPOUNDING

Thesis submitted to the University of Calicut for the degree of

DOCTOR OF PHILOSOPHY

in Statistics

By

DAVIS ANTONY MUNDASSERY

Under the Supervision of

Dr. K. JAYAKUMAR

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALICUT
KERALA– 673 635
INDIA

August 2007

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17, August 2007

CERTIFICATE

This is to certify that work reported in this thesis entitled "**SOME BIVARIATE DISTRIBUTIONS GENERATED THROUGH COMPOUNDING**" submitted to the University of Calicut for the award of degree of Doctor of Philosophy in Statistics is a bonafide research work carried out by **Mr. Davis Antony Mundassery** under my supervision and guidance in the Department of Statistics, University of Calicut. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

Dr. K. Jayakumar

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DECLARATION

I hereby declare that the matter embodied in this thesis entitled "**SOME BIVARIATE DISTRIBUTIONS GENERATED THROUGH COMPOUNDING**" is the result of investigations carried out by me in the Department of Statistics, University of Calicut under the supervision of **Dr. K. Jayakumar**, Lecturer Senior Scale, Department of Statistics, University of Calicut. This thesis contains no material which can be accepted for award of any degree or diploma in any University or Institute and to the best of my knowledge and belief, it contains no material previously published by any other person, except where the due references are made in the text of the thesis.

Calicut University Campus,
17-8-2007.

DAVIS ANTONY MUNDASSERY

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Chapter 1

Introduction

1.1 Introduction

Probability is a common word to everyone now-a-days, which is associated with uncertain situations. The origin of this concept was in 16^{th} century and it was as a result of the problems associated with the theory of gambling. However, this association with gambling theory contributed slow and sporadic growth to probability theory due to inadequate mathematical support. The measure theoretic approach to probability in the beginning of the last century provided a logically consistent foundation for probability theory and consequently imparted a rapid and systematic development to the same. Today, probability theory extends its influence and applications to many spheres.

One important aspect of probability theory is to provide probability distributions to random variables defined over those uncertain situations. The development incurred in probability theory enhanced the branch of probability distributions also, to higher altitudes. It comprises of many commonly occurring probability distributions like binomial, Poisson, geometric, hypergeometric, normal, exponential, gamma, Weibull, etc. Exponential distribution is an important probability distribution due to its memoryless property. This property makes it a powerful device in reliability contexts, especially in life testing. The constant hazard rate is one of its characterizing properties. Weibull and gamma distributions have tremendous applications in situations where the failure rate is monotone.

However, there are numerous situations where more heavy tailed distributions than exponential are essential in modeling certain data especially in finance, economics, etc. The most commonly observed properties of financial data sets are their heavy tails and high peaks. Pillai (1990) introduced Mittag-Leffler distribution as a generalization to exponential distribution. The data consisting of daily stock returns, commodity prices, foreign currency exchange rates and other financial data can be adequately modeled using Mittag-Leffler distribution. Kozubowski and Rachev (1994) well established the applications of Mittag-Leffler distribution in financial modeling. Jayakumar (2003) developed the first order autoregressive processes with Mittag-Leffler marginals to study the rate of flow of water in Kallada river, Kerala, India. Weron and Kotulski (1996) used Mittag-Leffler distribution to describe the Cole-Cole relaxation phenomena in Physics. The applications of Mittag-Leffler distribution in various socio-economic problems motivated to study its extension to higher dimensions.

The probability distribution of sums of independently and identically distributed random variables was a subject of intensive investigation during the development of distribution theory. The central limit problem dealing with limit distribution of sums of independently and identically distributed random variables emphasizes the importance of normal law as the limiting distribution. However, the limitations of normal law and its inadequacy to many natural phenomena necessitated the research along these lines on other distributions also.

In compounding, we consider the probability distributions of random sums of identically and independently distributed random variables. Gnedenko and Korolev (1996) gave a number of contexts where we usually encounter the distributions of random sums, especially geometric summation. Milne and Yeo (1989) considered the closure property of exponential distribution under geometric summation and offered many practical interpretations of this property. Mittag-Leffler distribution is also closed under geometric summation. Therefore it has excellent applications in situations when a geometric sum and corresponding summands have same distribution. For example, suppose that an investor has invested a fixed amount of money in 'n' assets. Let $X_1, X_2, ..., X_n$ are the rates of returns which are assumed to be independently and identically distributed according to Mittag-Leffler. Then the return X is considered as sum of smaller changes X_i, taken over a period of random time $N(p)$.

The random variable $N(p)$ has geometric distribution with mean $\frac{1}{p}$. Kozubowski and Rachev (1994) used the distribution of geometric sums to model foreign currency exchange data. The distribution of geometric sums has wide applications in reliability contexts also. In this work, we introduce bivariate forms of Mittag-Leffler distribution, its generalizations and their discrete counterparts using compounding techniques. Mundassery and Jayakumar (2007a) introduced a bivariate Mittag-Leffler distribution and studied its properties. A discrete analogue of bivariate Mittag-Leffler distribution is obtained in Mundassery and Jayakumar (2006).

The organization of the thesis is as follows: In the present Chapter, we make a quick review on various materials needed for the discussion in the succeeding Chapters.

In Chapter 2, we introduce a bivariate form of the Mittag-Leffler distribution. Its distribution function, density function, product moments, etc are obtained. Various characterizations of the bivariate Mittag-Leffler are obtained. The parameters of bivariate Mittag-Leffler distribution are estimated using log moments. First order autoregressive models with bivariate Mittag-Leffler marginals are obtained. Bivariate Mittag-Leffler forms of important bivariate exponential distributions like Marshall-Olkin's bivariate exponential, Hawkes' bivariate exponential and Paulson's bivariate exponential are introduced.

As a generalization of the bivariate Mittag-Leffler distribution, bivariate quasi factorial gamma distribution is introduced in Chapter 3. Various distributional properties of bivariate quasi factorial gamma distribution are studied. Characterizations of bivariate quasi factorial gamma are obtained using negative binomial compounding. First order stationary autoregressive processes with bivariate quasi factorial gamma marginals are developed. As a special case of bivariate quasi factorial gamma distribution, Moran's bivariate gamma distribution is studied. Bivariate semi quasi factorial gamma distribution is introduced and studied as a generalization of bivariate quasi factorial gamma distribution.

A discrete analogue of the bivariate Mittag-Leffler distribution, namely bivariate discrete Mittag-Leffler distribution is introduced in Chapter 4. The properties of bivariate discrete Mittag-Leffler distribution including joint probabilities, factorial moment generating function, attraction towards discrete stable law, etc are studied. We obtain characterizations of bivariate discrete Mittag-Leffler using geometric compounding. The estimates of the parameters are obtained. First order autoregressive models with bivariate discrete Mittag-Leffler marginals are developed. Discrete analogues of bivariate Mittag-Leffler that will generalize Marshall-Olkin's bivariate exponential and Hawkes' bivariate exponential distributions are introduced. In this context, we obtain bivariate geometric distributions as special cases of bivariate discrete Mittag-Leffler distributions.

We introduce a bivariate form of the discrete Linnik distribution and study its distributional properties in Chapter 5. Characterizations of bivariate discrete Linnik distribution are obtained using the negative binomial sums of independently and identically distributed bivariate discrete Mittag-Leffler random variables. The first order autoregressive models with bivariate discrete Linnik marginals are obtained.

We introduce the tailed forms of the bivariate Mittag-Leffler distribution and bivariate discrete Mittag-Leffler distributions in Chapter 6. The first order autoregressive models with marginals following bivariate tailed Mittag-Leffler and bivariate tailed discrete Mittag-Leffler distributions are developed. The tailed form of Moran's bivariate exponential distribution is studied as a special case of bivariate tailed Mittag-Leffler distribution. The summary and conclusion of the thesis is presented in Chapter 7.

1.2 Infinite Divisibility and Geometric Infinite Divisibility

A random variable X is said to be infinitely divisible if for every positive integer n, X can be written as

$$X \stackrel{d}{=} X_{n,1} + X_{n,2} + \ldots + X_{n,n}$$

where $X_{n,1}, X_{n,2}, \ldots, X_{n,n}$ are independently and identically distributed random variables. Infinite divisibility of X is in fact a property of the distribution of X. Therefore the probability distribution and distribution function of an infinitely divisible random variable will also be called as infinitely divisible. Thus a distribution function $F(x)$ is said to be infinitely divisible if for every positive integer n, there exists a distribution function $F_n(x)$ such that

$$F(x) = \overbrace{F_n(x) * F_n(x) * \ldots * F_n(x)}^{n\ times}.$$

This implies that $F(x)$ is the n-fold convolution of $F_n(x)$. Equivalently, a characteristic function $\psi(t)$ of a random variable X is said to be infinitely divisible if for every positive integer n, there exists a characteristic function $\psi_n(t)$, such that $\psi(t) = (\psi_n(t))^n$. Here, $\psi_n(t)$ represents the common characteristic function of the set of independently and identically distributed random variables $X_{n.j}$, $j = 1, 2, \ldots n$. Gamma and Mittag-Leffler distributions are infinitely divisible while uniform distribution over [-1, 1] is not. The class of infinitely divisible distributions plays an important role in the study of decomposition of probability distributions. A discussion on infinitely divisible distributions in relation to central limit problem can be found in Feller (1971). Bondesson et al. (1996) studied the infinite divisibility of the integer parts of positive real valued random variables. A detailed discussion on properties of infinitely divisible distributions can be found in Steutel and van Harn (2004).

Klebanov et al. (1984) introduced the concept of geometric infinite divisibility. A random variable X is geometrically infinitely divisible if

$$X \stackrel{d}{=} \sum_{j=1}^{N(p)} X_j^{(p)}$$

where $N(p)$ has geometric distribution such that $P(N(p) = k) = p(1 - p)^{k-1}$,

$k = 1,2,3,...;\ \ p \in (0,1);\ X_j^{(p)}, j = 1,2,...$ are independently and identically distributed random variables and also independent of $N(p)$. Equivalently, in terms of characteristic function, X is geometrically infinitely divisible if

$$\psi(t) = \frac{p\ \psi_p(t)}{1-(1-p)\ \psi_p(t)}$$

where $\psi(t)$ and $\psi_p(t)$ are the characteristic functions of X and $X_j^{(p)}$ respectively. Klebanov et al. (1984) obtained that a characteristic function $\psi(t)$ is geometrically infinitely divisible if and only if $e^{1-\frac{1}{\phi(t)}}$ is infinitely divisible. Sandhya and Pillai (1999) showed that the class of geometrically infinitely divisible distributions is a proper of subclass of infinitely divisible distributions. Aly and Bouzar (2000) and Mohan et al. (1993) obtained characterizations geometrically infinitely divisible distributions. Pillai and Sandhya (1990) showed that the class of distribution functions having complete monotone derivative is a proper subclass of geometrically infinitely divisible distributions. The applications of geometric infinite divisibility in autoregressive time series modeling are discussed in Jose and Pillai (1995).

1.3 Mittag-Leffler and Semi Mittag-Leffler Distributions

The function $E_\alpha(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(1+\alpha k)}$, $u \in (0,\infty)$ was first introduced by Mittag-Leffler in 1903 (see Erdelyi et al. (1955)). In Feller (1971), the Laplace transform of $E_\alpha(-x^\alpha)$ with $0 < \alpha \leq 1$, is shown to be $\frac{\lambda^{\alpha-1}}{1+\lambda^\alpha}$, $\lambda \geq 0$. But $E_\alpha(-x^\alpha)$ is not a probability distribution. Pillai (1990) showed that $F_\alpha(x) = 1 -\ E_\alpha(-x^\alpha)$

is a distribution function. Hence he named $F_\alpha(x)$ as Mittag-Leffler distribution. We have

$$F_\alpha(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{\alpha k}}{\Gamma(1+\alpha k)}, \quad 0 < \alpha \leq 1, \quad x \geq 0 \tag{1.1}$$

and the corresponding density function is

$$f_\alpha(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{\alpha k - 1}}{\Gamma(\alpha k)}.$$

The plots of $f_\alpha(x)$ is presented in Fig. 1.1. The Laplace of transform (1.1) is

$$\phi(\lambda) = \frac{1}{1+\lambda^\alpha}, \lambda \geq 0. \tag{1.2}$$

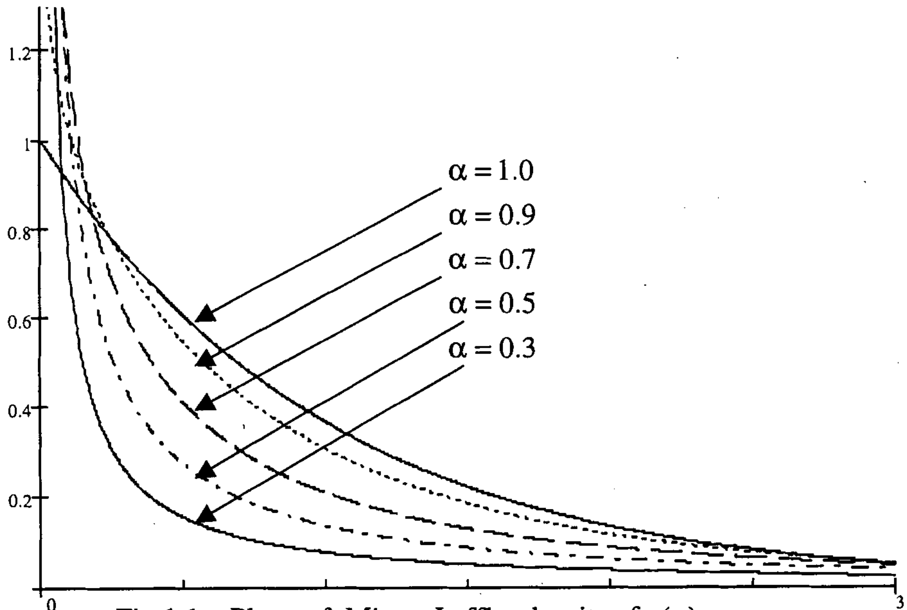

Fig.1.1 Plots of Mittag-Leffler density, $f_\alpha(x)$.

Mittag-Leffler distribution has been received much attention by many researchers recently (see Jayakumar and Suresh (2003), Kozubowski (1999, 2000a, b), Lin (1998, 2001) and the references therein). Note that when $\alpha = 1$, (1.2) reduces to the Laplace transform of exponential distribution. Mittag-Leffler distribution is geometrically infinitely divisible, self decomposable and is normally attracted to stable law. Lin (1998) proved that Mittag-Leffler distribution belongs to the class of distributions with completely monotone derivative. Pillai and Anil (1996) characterized Mittag-Leffler distribution using the integrated Cauchy functional equation. Based on Mohan et al. (1993), the Mittag-Leffler distribution can be called as positive geometric right stable law. Kozubowski (2000a) showed that the Mittag-Leffler random variable can be represented as $X = W^{\frac{1}{\alpha}}Z$ where Z and W are independent, Z is standard exponential and W follows positive stable distribution with density function

$$f_W(x) = \frac{sin\alpha\pi}{\alpha\pi(x^2 + 2xcos\alpha\pi + 1)}, \; x > 0. \tag{1.3}$$

Parameters of Mittag-Leffler distribution are estimated using fractional moments in Kozubowski (2001). Jayakumar and Pillai (1996) obtained certain characterizations of Mittag-Leffler distribution and showed that within the class of infinitely divisible laws with positive support, the Mittag-Leffler distribution $F_\alpha(x)$ is the unique distribution function $F(x)$ satisfying the relation

$$\phi(\lambda) = e^{\int\limits_0^\infty \frac{e^{-\lambda x}-1}{x}\alpha(1-F(x))dx}$$

where $\phi(\lambda)$ is the Laplace transform of $F(x)$. Pillai and Jayakumar (1994) derived the innovation distribution of the p^{th} order autoregressive process with Mittag-Leffler marginals and using it specialized class L distributions are introduced. The Mittag-Leffler distribution has potential applications in various fields (see Jayakumar (2003), Kozubowski and Rachev (1994) and Weron and Kotulski (1996)).

A random variable X with positive support is said to follow semi Mittag-Leffler distribution with exponent α, $0 < \alpha \leq 1$ if its Laplace transform is

$$\phi(\lambda) = \frac{1}{1 + \xi(\lambda)} \tag{1.4}$$

where $\xi(\lambda)$ satisfies the functional equation

$$\xi(\lambda) = \frac{1}{p}\xi(p^{\frac{1}{\alpha}}\lambda) \tag{1.5}$$

for some p, such that $0 < p \leq 1$. The solution of the functional equation (1.5) is $\xi(\lambda) = \lambda^{\alpha}h(\lambda)$ where $h(\lambda)$ is periodic in $\ln\lambda$ with period $\dfrac{-2\pi\alpha}{\ln p}$ (see Kagan et al. (1973)). When $h(\lambda) = 1$, we get the Mittag-Leffler distribution. Jayakumar and Pillai (1993) developed first order stationary autoregressive process with marginals as semi Mittag-Leffler distribution and studied its properties. Bunge (1996) explained the use of semi Mittag-Leffler distribution in the study of random stability.

1.4 Discrete Mittag-Leffler and Discrete Semi Mittag-Leffler Distributions

A discrete analogue of Mittag-Leffler distribution was obtained in Pillai and Jayakumar (1995). A random variable X on {0,1,2,...} is said to follow discrete Mittag-Leffler distribution if its probability generating function (p.g.f.) is

$$P(s) = \frac{1}{1 + c(1-s)^{\alpha}}, \quad 0 < \alpha \leq 1, \quad c > 0, \quad | s | \leq 1. \tag{1.6}$$

The discrete Mittag-Leffler distribution can be viewed as the distribution of geometric sum of independently and identically distributed Sibuya random variables. In a sequence of independent Bernoulli trials, let $\frac{\alpha}{k}$ be the probability of success in k^{th} trial. Then the number of trials required to obtain the first success has Sibuya distribution (see Devroye (1993)).

Discrete Mittag-Leffler distribution is a generalization of geometric distribution, since in (1.6) when $\alpha = 1$, we get geometric. Pillai and Jayakumar (1995) obtained some distributional properties of the discrete Mittag-Leffler. It is geometrically infinitely divisible, belongs to discrete class L and is normally attracted to stable law. Pillai and Jayakumar (1995) developed autoregressive models with marginals as discrete Mittag-Leffler distribution. Bouzar (2002) gave mixture representations of discrete Mittag-Leffler distribution. Jayakumar and Sreenivas (2003) noted that if X has Mittag-Leffler distribution and $N_c(.)$ is a unit Poisson process with parameter c, independent of X then $Y = N_c(X)$ has discrete Mittag-Leffler distribution.

A random variable X on $\{0,1,2,...\}$ has discrete semi Mittag-Leffler distribution if its p.g.f. is

$$P(s) = \frac{1}{1+\xi(1-s)}, \tag{1.7}$$

where $\xi(1-s) = \frac{1}{p}\xi(p^{\frac{1}{\alpha}}(1-s))$ for some p, $0 < p \leq 1$. The discrete semi Mittag-Leffler distribution was introduced by Jayakumar (1995a) as the discrete analogue of semi Mittag-Leffler distribution.

1.5 Quasi Factorial Gamma and Semi Quasi Factorial Gamma Distributions

Pakes (1995) introduced the positive Linnik distribution. A non negative random variable X is said to follow positive Linnik distribution if its Laplace transform is

$$\phi(\lambda) = \left(\frac{1}{1+\lambda^{\alpha}}\right)^{v}, \quad 0 < \alpha \leq 1,\ v > 0,\ \lambda \geq 0. \tag{1.8}$$

Jayakumar and Gadag (1999) studied this Laplace transform as a generalization to the Mittag-Leffler distribution and called the corresponding distribution as quasi factorial gamma. A non negative random variable X is said to follow quasi factorial gamma distribution if it has the distribution function

$$F_{\alpha,v}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(k+v)x^{\alpha(k+v)}}{\Gamma(v)k!\Gamma(1+\alpha(k+v))}, \quad x \geq 0,\ v > 0,\ 0 < \alpha \leq 1. \tag{1.9}$$

The density plot of quasi factorial gamma distribution is presented in Fig.1.2. This family of distributions accommodates many important distributions such as gamma, Mittag-Leffler, exponential etc (see Fig 1.3). Jayakumar and Gadag (1999) studied various distributional properties of quasi factorial gamma. They have also developed first order stationary autoregressive process with marginals as quasi factorial gamma distribution. Jayakumar and Gadag (1999) introduced semi quasi factorial gamma distribution as a generalization to the quasi factorial gamma distribution. A random variable X on $(0, \infty)$ has semi quasi factorial gamma distribution if its Laplace transform

$$\phi(\lambda) = \left(\frac{1}{1+\xi(\lambda)}\right)^{v}.$$

where $\xi(\lambda)$ satisfies the functional equation in (1.5).

1.6 Discrete Linnik distribution

Christoph and Schreiber (1998a) studied the discrete analogue of the positive Linnik distribution in (1.8). A non negative integer valued random variable is said to be discrete Linnik distributed with exponent $\alpha \in (0, 1]$ and scale parameter c if it has p.g.f.

$$P(s) = \begin{cases} \left(\frac{1}{1+c(1-s)^{\alpha}}\right)^{v} & \text{for } 0 < v < \infty \\ e^{-c(1-s)^{\alpha}} & \text{for } v = \infty \end{cases} \tag{1.10}$$

When $v = 1$, it coincides with the discrete Mittag-Leffler distribution in (1.6). We get the p.g.f. of negative binomial distribution when $\alpha = 1$.

Probabilities of the discrete Linnik distribution, some properties of the probabilities and characterizations via survival function are investigated in Christoph and

Schreiber (1998a). Bouzar (2002) obtained representations for discrete Linnik distribution using Poisson mixtures and established the infinite divisibility of the distribution. They have also obtained the mixture representation of discrete Linnik laws by way of stable laws. Christoph and Schreiber (1998c) proved that discrete Linnik distribution belongs to the domain of discrete attraction of a discrete stable law as well as to the domain of attraction of non negative strictly stable law and obtained the rate of convergence in both cases.

1.7 Stable and Semi Stable Distributions

The class of stable distributions was developed in 1920's during the investigations of the behavior of sums of independently and identically distributed random variables. A random variable X is said to follow stable distribution in broad sense, if for any positive constants a and b, X satisfies

$$aX_1 + bX_2 \stackrel{d}{=} cX + d, \; c > 0, \; d \in R$$

where X_1 and X_2 are independent copies of X. Equivalently in terms of characteristic function

$$\psi(at).\psi(bt) = \psi(ct)e^{idt}.$$

It is a strictly stable distribution when $d = 0$. The most often used parametrization of stable distribution is that discussed in Samorodnitski and Taqqu (1994). A random variable X is said to follow stable distribution if it has characteristic function

$$\psi(t) = \begin{cases} e^{iat-c|t|^{\alpha}(1-i\beta\,\mathrm{sign}(t)\,\tan\frac{\pi\alpha}{2})} & \text{if } \alpha \neq 1 \\ e^{iat-c|t|(1+i\beta\frac{2}{\pi}\mathrm{sign}(t)\ln|t|)} & \text{if } \alpha = 1 \end{cases} . \tag{1.11}$$

Here, α is known as the exponent of stable distribution and $0 < \alpha \leq 2$, c is the scale parameter, $c > 0$, a is the location parameter, $-\infty < a < \infty$ and β is the symmetry parameter, $-1 \leq \beta \leq 1$. When $\beta = 0$, we get symmetric stable distribution.

$$\mathrm{sign}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}$$

When $\alpha = 2$, $\beta = 0, c = \dfrac{\sigma^2}{2}$ and $b = \mu$, then $\psi(t)$ is the characteristic function of normal distribution, $N(\mu, \sigma^2)$. For $\alpha = 1$, $\beta = 0, c = \gamma$ and $b = \delta$ $\psi(t)$ represents the characteristic function of Cauchy (γ, δ) distribution. Note that Laplace distribution is not stable.

Nolan (2005a) discussed various distributional properties of univariate and multivariate stable distributions and examples of stable laws arising in different problems. Estimation of various parameters of stable distribution including moment estimators are discussed in Press (1972a). For representation, estimation and other properties of stable distributions, see Laha and Rohatgi (1979) and Nolan (2001). Note that the Laplace transform of positive stable distribution is

$$\phi(\lambda) = e^{-c\,\lambda^{\alpha}}, \quad c > 0, \quad 0 < \alpha \leq 1. \tag{1.12}$$

The density function corresponding to (1.12) is stated in (1.3).

Stable distributions have applications in many fields. This is because, stable distributions provide approximations for sums of independently and identically distributed random variables that are heavy tailed. So they are seemed to be appropriate in modeling skewed data. In economics, statistical physics, telecommunications etc, we meet such data sets. Another important reason for focussing on stable distribution is its outstanding feature of having domains of attraction. That is, they are the only possible limits of sums of independently and identically distributed random variables (see Kozubowski and Rachev (1994)).

The multivariate forms of stable distributions are provided in Nolan (1998). Nolan (2005b) studied multivariate stable densities and distribution functions in the elliptical case. A detailed discussion on multivariate stable distribution can be found in Press (1972b, c) and Samorodnitski and Taqqu (1994).

A random variable X is said to follow positive semi stable distribution if its Laplace transform is $\phi(\lambda) = e^{-\xi(\lambda)}$ where $\xi(\lambda)$ is as given in (1.5). The properties of semi stable distributions are discussed in Pillai (1971).

1.8 Discrete Stable and Discrete Semi Stable Distributions

A discrete analogue of stable distributions is obtained in Steutel and van Harn (1979). A random variable X with support on on non negative integers, is said to

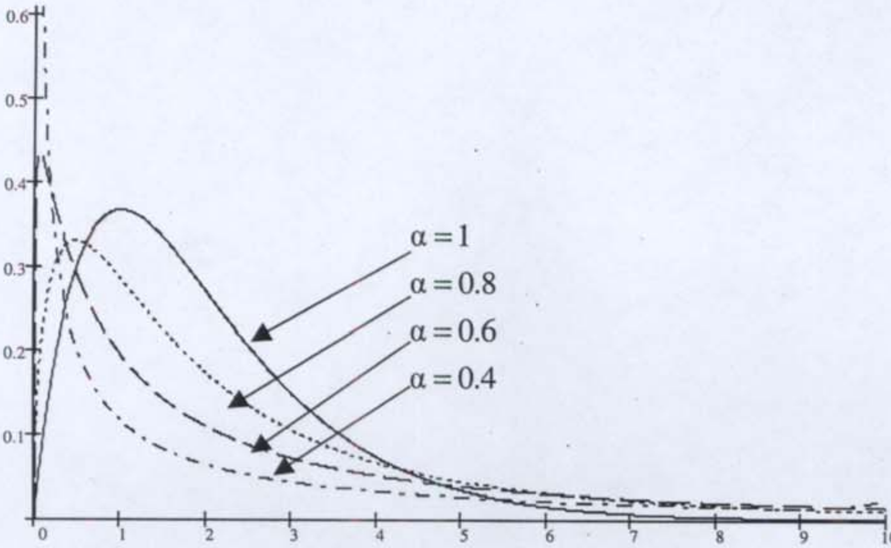

Fig.1.2 Density Plot of Quasi Factorial Gamma ($\upsilon = 2$)

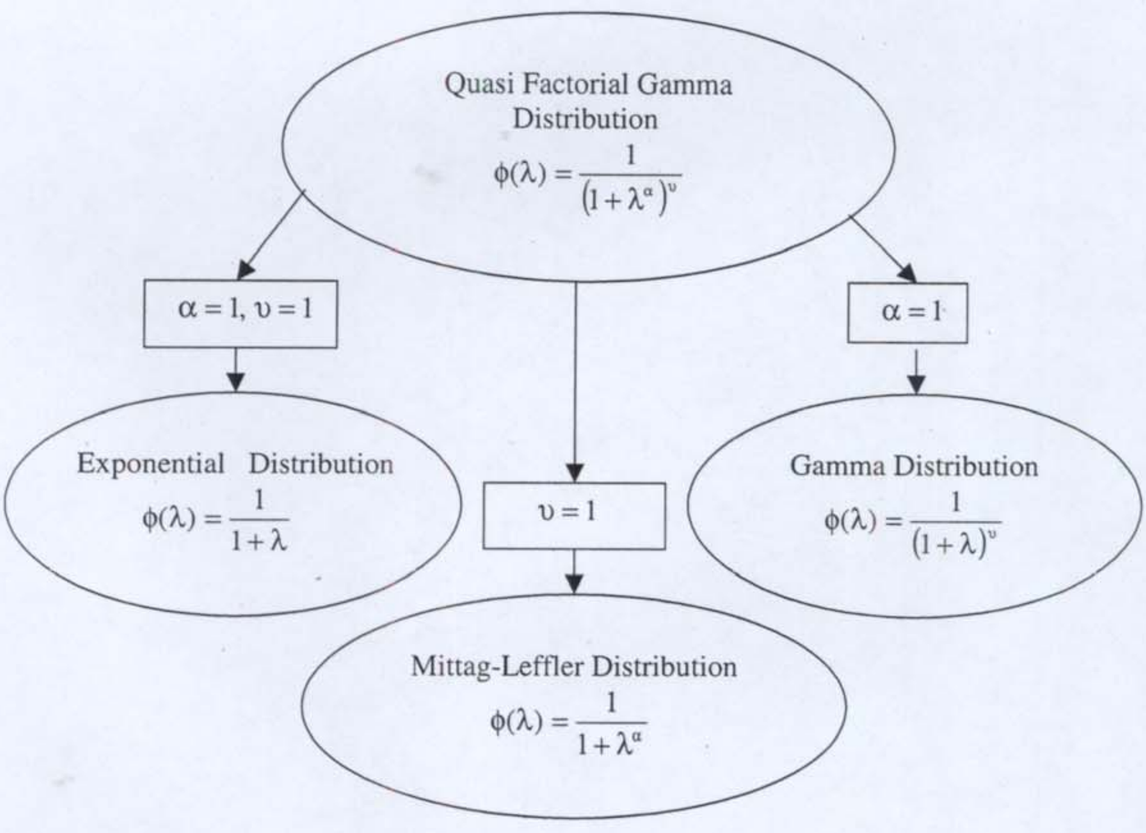

Fig.1.3 Relations between Quasi Factorial Gamma and other distributions

follow discrete stable distribution with exponent $\gamma(>0)$ if it satisfies the relation

$$X \stackrel{d}{=} \alpha \oplus X_1 + (1-\alpha^\gamma)^{\frac{1}{\gamma}} \oplus X_2, \quad 0 < \alpha \leq 1$$

where X_1 and X_2 are independent copies of X.

Jayakumar (1995a) considerd the operator '$\oplus$' as follows: If X has p.g.f. $P(s)$, then $\alpha \oplus X$ is defined (in distribution) by its p.g.f. $P(1-\alpha+\alpha s)$ or by $\alpha \oplus X = \sum_{j=1}^{X} N_j$ where $P(N_j = 1) = 1 - P(N_j = 0) = \alpha$, all random variables being independent.

Therefore, a random variable X is said to follow discrete stable distribution if its p.g.f. satisfies

$$P(s) = P(1-\alpha(1-s))P(1-(1-\alpha^\gamma)^{\frac{1}{\gamma}}(1-s)), \quad |\, s \,| \leq 1. \tag{1.13}$$

Stuetel and van Harn (1979) established that the p.g.f. in (1.13) exists only for $0 < \gamma \leq 1$ and a random variable X defined on $\{0, 1, 2, ..\}$ has discrete stable distribution if and only if its p.g.f. is

$$P(s) = e^{-c(1-s)^\gamma}, \quad c > 0 \tag{1.14}$$

In case $\gamma = 1$, (1.14) represents the p.g.f. of a Poisson random variable with mean c. Christoph and Schreiber (1998b) obtained the expression for probability distribution of X when $0 < \gamma \leq 1$.

$$P(X = k) = (-1)^k e^{-c} \sum_{m=0}^{k} \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \begin{pmatrix} \gamma j \\ k \end{pmatrix} (-1)^j \frac{c^m}{m!}, \qquad k = 0, 1, 2, ...$$

Let $X_1,\ X_2, ...$ be a sequence of independently and identically distributed Sibuya random variables and Y be a Poisson random variable with mean c, independent of $X_1,\ X_2, ...$ Then Devroye (1993) obtained that $X = X_1 +\ X_2 + ... + X_n$ has discrete stable distribution.

Jayakumar and Sreenivas (2003) obtained mixtures of discrete stable distribution as a generalization to Poisson mixtures and studied its properties. Christoph and Schreiber (1998c) showed that a discrete stable random variable belongs to the domain of attraction of a positive stable random variable and obtained the rate of convergence. For other various distributional properties of discrete stable distribution, see Christoph and Schreiber (1998b) and Remillard and Theodorescu (2000).

The concept of discrete semi stability was introduced in Bouzar (2004). A non degenerate random variable on $\{0, 1, 2, ...\}$ is said to have discrete semi stable distribution with exponent $\gamma(> 0)$ and order $\alpha \in (0, 1]$ if its p.g.f. $P(s)$ satisfies

$$\ln P(1 - \alpha + \alpha s) = a^{\gamma} \ln P(s), \quad | s | \leq 1$$

In other words, a distribution with support on $\{0, 1, 2, ...\}$ is said to be discrete semi stable if its p.g.f. is

$$P(s) = e^{-\xi(1-s)},$$

where the function $\xi(.)$ is as stated in (1.7).

Discrete semi stable distributions exist only for $0 < \gamma \leq 1$. Bouzar (2004) proved that discrete semi stable distributions are infinitely divisible and obtained its characterizations. Jayakumar and Sreenivas (2003) obtained discrete semi stable distribution mixtures as a generalization to discrete stable mixtures. They found that a random variable X on $\{0, 1, 2, ...\}$ is discrete semi stable mixture with mixing distribution function $F(.)$ defined over $(0, \infty)$ if the p.g.f. of X can be expressed as

$$P(s) = \int_0^\infty e^{-x\xi(1-s)} dF(x)$$

where $\xi(.)$ satisfies the functional equation given in (1.7). Bouzar and Jayakumar (2007) developed stationary integer valued first order autoregressive process with discrete semi stable marginals.

1.9 Geometric Stable Distributions

Geometric stable distributions are the weak limits of random sums of independently and identically distributed random variables. A random variable Y is said to be geometrically stable if there exists a sequence of independently and identically distributed random variables $X_1, X_2, ...$ such that for constants, $a = a(p) > 0$ and $b = b(p) \in R$, the relation

$$a(p) \sum_{i=1}^{N(p)} Xi + b(p) \xrightarrow{d} Y \quad \text{as} \quad p \longrightarrow 0$$

holds. $N(p)$ is a geometric random variable, independent of X_i such that $P(N(p) = n) = p(1-p)^{n-1}, \quad 0 < p < 1, \quad n = 1, 2, 3, ...$ The case $b(p) = 0$ gives strictly geometric stable distribution.

Kozubowski and Rachev (1999a) discussed various parameterizations of geometric stable distributions. Among these the most standard representation has the characteristic function

$$G(t) = \left[1 + c|\ t\ |^{\alpha}\omega(t,\ \alpha,\ \beta) - iat\right]^{-1}$$

where

$$\omega(t,\ \alpha,\ \beta) = \begin{cases} 1 - i\beta\,\mathrm{sign}(t)\ \tan\frac{\pi\alpha}{2} & \text{if } \alpha \neq 1 \\ 1 + i\beta\frac{2}{\pi}\mathrm{sign}(t)\ln|t| & \text{if } \alpha = 1 \end{cases}.$$

α is the exponent of the distribution, $0 < \alpha \leq 2$; c is the scale parameter, $c > 0$; a is the location parameter, $-\infty < a < \infty$ and β is the symmetry parameter, $-1 \leq \beta \leq 1$.

Mittnik and Rachev (1991) obtained the following characterization of geometric stable distribution. A random variable Y is geometric stable if and only if its characteristic function $\psi(t)$ has the form

$$G(t) = \frac{1}{1 - \ln\psi(t)}$$

where $\psi(t)$ is the characteristic function of stable distribution given in (1.11). Kozubowski (2000a) showed that every strictly geometric stable random variable can be represented as product of an exponentially distributed random variable and an independent random variable with stable distribution. The estimators of the parameters of geometric stable distribution and their properties can be found in Kozubowski (1999). The properties of geometric stable distribution like representation, computer simulation etc, are discussed in Kozubowski (1994, 2000b).

Geometric stable distributions approximate random sums of identically and independently distributed random variables where the number of summands has geometric distribution. Such situations arise in a variety of applied problems in risk analysis, biology, economics, queueing theory and reliability. Consequently geometric stable distributions have potential applications in these areas. Moreover, geometric stable laws have an important role in modeling heavy tailed data (see Kozubowski and Rachev (1994) and Mittnik and Rachev (1991, 1993)). Multivariate extensions of geometric stable distributions are studied in Kozubowski et al. (2005), Kozubowski and Panorska (1999a) and Kozubowski and Rachev (1999b).

1.10 Bivariate Geometric Distribution

We consider the following bivariate geometric distribution. A random vector (N_1, N_2) is said to follow bivariate geometric distribution if it has the survival function

$$P(N_1 > n_1, N_2 > n_2) = \begin{cases} p_{11}^{n_1}(p_{01} + p_{11})^{n_2 - n_1} & \text{if} \quad n_1 \leq n_2 \\ p_{11}^{n_2}(p_{10} + p_{11})^{n_1 - n_2} & \text{if} \quad n_2 \leq n_1 \end{cases} \tag{1.15}$$

where $p_{00} + p_{01} + p_{10} + p_{11} = 1$, $p_{10} + p_{11} < 1$, $p_{01} + p_{11} < 1$, n_1, $n_2 = 1, 2, 3, \ldots$. A practical interpretation where this bivariate geometric distribution becomes appropriate can be found in reliability context. Consider a system consisting of two components in which both are subjected to shocks causing failure. Suppose that with probability p_{11}, both components survive after shocks, p_{10}, the first survives and second does not, with probability p_{01}, the first component fails and the second survives, and with probability p_{00}, both components fail. Let (N_1, N_2) represent the number of shocks caused to failure of the components. Then (N_1, N_2) admits the probability distribution given in (1.15).

The p.g.f. of (1.15) is

$$P(s_1, s_2) = \frac{s_1 s_2}{1 - p_{11} s_1 s_2} \left(p_{00} + \frac{p_{01}(p_{00} + p_{10}) s_2}{1 - (p_{01} + p_{11}) s_2} + \frac{p_{10}(p_{00} + p_{01}) s_1}{1 - (p_{10} + p_{11}) s_1} \right). \tag{1.16}$$

Block (1977) have obtained many bivariate exponential and geometric distributions using the bivariate geometric sums of independently and identically distributed random vectors. Balakrishna and Nair (1996) obtained characterizations of Moran's bivariate exponential using the distribution of sums of independently and identically distributed random vectors when the number of summands follow the bivariate geometric distribution in (1.15).

1.11 Moran's Bivariate Exponential Distribution

Due to the potential applications of exponential distribution, it is natural to consider its extensions to higher dimensions. However, unlike normal distribution, no unique extension of exponential distribution is available. A survey on various bivariate exponential distribution was found in Kotz et al. (2000). In the present study special attention is paid to Moran's bivariate exponential distribution and its generalizations. Moran (1967) introduced a bivariate exponential distribution which was later popularized by Downton (1970) as a model to describe the failure time of a system having two components. The joint density function of Moran's bivariate exponential distribution is

$$f(x,y) = \frac{\mu_1\mu_2}{1-\theta} I_0\left(\frac{2\sqrt{(\mu_1\mu_2\theta xy)}}{1-\theta}\right) exp - \left(\frac{\mu_1 x + \mu_2 y}{1-\theta}\right),$$

where $\mu_1, \mu_2 > 0$; $x, y > 0$; $0 \le \theta \le 1$ and $I_0(z) = \sum_{j=0}^{\infty}\left(\frac{z}{2j!}\right)^{2j}$ is the modified Bessel function of the first kind of order zero. We denote the Moran's bivariate exponential distribution by MBE (μ_1, μ_2, θ). The Laplace transform of MBE (μ_1, μ_2, θ) distribution is

$$\phi(\lambda_1, \lambda_2) = \frac{\mu_1\mu_2}{(\mu_1+\lambda_1)(\mu_2+\lambda_2) - \theta\lambda_1\lambda_2}. \tag{1.17}$$

$I_0(.)$ is the modified Bessel function of the first kind of order zero. Downton (1970) presented an interpretation of this bivariate exponential distribution which is applicable in reliability context. Consider a system in which the two components are subjected to non fatal shocks occurring according to independent Poisson process with parameters δ_1 and δ_2. Let X_{ij}, $i = 1, 2$, $j = 1, 2, 3, ...$ denote the inter arrival time of the i^{th} process. Assume that i^{th} component fails after N_i shocks where N_i follows geometric distribution. The time to failure of the components are given by

$$(X, Y) = \left(\sum_{j=1}^{N_1} X_{1j}, \sum_{j=1}^{N_2} X_{2j}\right). \tag{1.18}$$

Suppose that $(N_1,\ N_2)$ has bivariate geometric distribution with p.g.f.

$$P(s_1,\ s_2) = \frac{s_1\ s_2}{1 + d_1 + d_2 + d_3 - d_1 s_1 - d_2 d_2 - d_3 s_1 s_2}$$

where d_1, d_3 and d_3 are non negative. Then Downton (1970) showed that the $(X,\ Y)$ in (1.18) follows MBE (μ_1, μ_2, θ) and its Laplace transform is in (1.17) where

$$\mu_1 = \frac{\delta_1}{1+d_1+d_3}, \quad \mu_2 = \frac{\delta_2}{1+d_2+d_3} \text{ and } \theta = \frac{d_1 d_2 + d_2 d_3 + d_1 d_3 + d_3 + d_3^2}{(1+d_1+d_3)(1+d_2+d_3)}.$$

As a generalization to MBE (μ_1, μ_2, θ) distribution, Moran (1967) obtained bivariate gamma distribution with Laplace transform

$$\phi(\lambda_1, \lambda_2) = \left(\frac{\mu_1 \mu_2}{(\mu_1 + \lambda_1)(\mu_2 + \lambda_2) - \theta\lambda_1\lambda_2} \right)^v, \quad v > 0. \tag{1.19}$$

It is denoted by MBG $(\mu_1, \mu_2, \theta, v)$. The joint density function corresponding to this Laplace transform and its properties are found in Kotz et al. (2000).

1.12 Random Summation

The probability distributions of random sums of independently and identically distributed random variables, especially the geometric sum, have been extensively studied by many researchers during the last decade. For example, Gnedenko and Korolev (1996), Klebanov et al. (1984), Kozubowski and Panorska (1999b), Kozubowski and Rachev (1999a) and Rolski et al. (1997). The problem of geometric summation can be described as follows: Consider a sequence $\{X_i, i \geq 1\}$ of independently and identically distributed random variables. Define

$$S_N = X_1 + X_2 + ... + X_N$$

where N follows geometric distribution. When cS_N and X_i are identically distributed for some $c > 0$, we say that distribution of X_i is stable under geometric summation. The applicability of the result that geometric sum of exponential random variables is exponential, is investigated in Milne and Yeo (1989). Chufang (1997) characterized the Marshall-Olkin type distributions using bivariate geometric summation. Klebanov and Rachev (1996) studied the distribution of random sums and their applications when the number of summands is infinitely divisible laws. Lin and Stoyanov (2002) studied the moment problem for the distribution of geometric sums. Cai and Kalashnikov (2000) extended the new worse than property (NWU) of geometric sums to the class of random sums distribution. Later, Li et al. (2006) investigated the negative ageing property of random sums and obtained that the property is solely determined by the negative ageing property of the distribution of N.

A generalization of geometric sums is obtained in Milne and Yeo (1989) by considering negative binomial sums and found that the random sum has gamma distribution when N follows negative binomial distribution and summands are exponential. Prokhorov and Ushakov (2001) studied the conditions of reconstructing the distribution of independently and identically distributed random variables when the distribution of the random sum is given.

1.13 Autoregressive Processes

Autoregressive processes of appropriate orders are used for modeling time series data. The general form of p^{th} order autoregressive process with parameters $a_1, a_2, ..., a_p$ is

$$X_n = a_1 X_{n-1} + a_2 X_{n-2} + ... + a_p X_{n-p} + \varepsilon_n$$

where $\{\epsilon_n, n \geq 1\}$, called the innovations, is a sequence of independently and identically distributed random variables and independent of $X_{n-1}, ..., X_{n-p}$. Even though

Guassian autoregressive models dominated in the development of time series modeling, autoregressive processes with non Guassian marginal distributions are a fast growing area of investigation in recent years due to its wide applications in many naturally arising time series models (see Gaver and Lewis (1980) and Lawrance and Lewis (1980, 1981)). Exploiting the class L property of exponential distribution, Gaver and Lewis (1980) developed a first order exponential autoregressive (EAR(1)) model. The EAR(1) process has the structure

$$X_n = \begin{cases} \rho X_{n-1} & \text{with probability } \rho \\ \rho X_{n-1} + \epsilon_n & \text{with probability } 1-\rho \end{cases} \tag{1.20}$$

where $0 \leq \rho < 1$. $\{\epsilon_n, n \geq 1\}$ is a sequence of independent exponential random variables such that $X_0 \stackrel{d}{=} \epsilon_1$.

Further, Jayakumar and Pillai (1993) developed a first order Mittag-Leffler autoregressive (MLAR(1)) process.

Construction of stationary autoregressive models with marginals have bivariate distributions is an emerging area of research recently. Block et al. (1988) developed a first order autoregressive additive process with bivariate geometric marginals and studied its properties. Dewald et al. (1989) developed an additive first order autoregressive bivariate exponential process. Balakrishna and Jayakumar (1996) obtained an extension of the first order autoregressive exponential minification process of Tavares (1980) to bivariate case and developed Gumbel's bivariate exponential autoregressive process. Ristic and Popovic (2003) introduced a first order stationary autoregressive process (BUAR(1)) having bivariate uniform marginals over (0,1) and obtained the estimates of the parameters of the process. Balakrishna and Jayakumar (1997) developed a bivariate minification process using bivariate semi Pareto distribution and studied its properties. A class of stationary bivariate minification process is obtained in Ristic (2006) and established that process has the uniformly mixing property.

1.14 Tailed Distributions

In tailed distributions, tail of a nonnegative random variable refers to the positive part of the sample space excluding the point zero. The random variable X is treated as a mixture of an atom at zero with probability σ and the whole positive part with probability $1-\sigma$. We encounter such situations in life testing experiments where an item fails instantaneously and hence the observed lifetime becomes zero. During the dry days of season, the volume of water in a reservoir may be zero. In modeling the daily data of river flow, there may be days without flow. That is, the rivers will be dry during certain days of an year. In epidemiology, if X is the percentage of progress of an infectious disease, the incubation period ($X = 0$) has non zero probability. Similarly, in clinical trials, it happens that initially a medicine has no response with certain probability and on a later stage, there is some response, then the length of the response is described by certain probability distribution. Such situations are common in agriculture and inventory also. As an example of tailed distribution, consider exponential tailed distribution. A random variable X is said to follow exponential tailed distribution if $P(X = 0) = \sigma$ and $P(X > x) = (1-\sigma)e^{-\mu x}$, $x > 0$, $\mu > 0$ and $0 \leq \sigma < 1$. The Laplace transform of exponential tailed distribution is

$$\begin{aligned} \phi(\lambda) &= \sigma + (1-\sigma)\frac{\mu}{\mu+\lambda} \\ &= \frac{\mu+\sigma\lambda}{\mu+\lambda}. \end{aligned}$$

Kemp (2004) considered a geometric distribution in a modified form. A random variable X is said to follow ' zero modified ' geometric distribution if its probability distribution is

$$P(X = x) = \begin{cases} \sigma + (1-\sigma)p \quad \text{when } x = 0 \\ (1-\sigma)p(1-p)^x, \quad x = 1, 2, 3, ..., \quad 0 < p < 1, \quad 0 \leq \sigma < 1. \end{cases} \tag{1.21}$$

The p.g.f. of (1.21) is

$$P(s) = \sigma + (1-\sigma)\frac{p}{1-(1-p)s}.$$

SOME BIVARIATE DISTRIBUTIONS GENERATED THROUGH COMPOUNDING

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Chapter 2

Bivariate Mittag-Leffler Distribution

2.1 Introduction

Mittag-Leffler distribution has been studied extensively by many authors in the past decade (see Jayakumar and Pillai (1993), Kozubowski (1994, 1999), Lin (1998, 2001), Pillai (1990) and Weron and Kotulski (1996)). Pillai (1990) established that the Mittag-Leffler distribution is geometrically infinitely divisible. Various distributional properties of Mittag-Leffler are discussed in Jayakumar and Suresh (2003) and Lin (1998). Kozubowski (2001) estimated the parameters of Mittag-Leffler distribution using fractional moments. Even though a lot of investigations on Mittag-Leffler distribution were carried out, studies in the direction of its extensions to higher dimensions are not yet explored.

Mundassery and Jayakumar (2007a) introduced a bivariate Mittag-Leffler distribution.

Definition 2.1. A non negative random vector (X, Y) is said to follow bivariate Mittag-Leffler distribution with parameters $\mu_1, \mu_2, \alpha_1, \alpha_2$ and θ, denoted by BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$, if its Laplace transform is

$$\phi(\lambda_1, \lambda_2) = \frac{1}{(1+\mu_1\lambda_1^{\alpha_1})(1+\mu_2\lambda_2^{\alpha_2}) - \theta\mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}}, \tag{2.1}$$

$$\lambda_1, \lambda_2 \geq 0;\ \ 0 < \alpha_1, \alpha_2 \leq 1;\ \mu_1,\ \mu_2 > 0;\ 0 \leq \theta \leq 1.$$

Note that

$$\phi(\lambda_1, 0) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{1}{1+\mu_2\lambda_2^{\alpha_2}}.$$

When $\theta = 1$,

$$\phi(\lambda_1, \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}}. \tag{2.2}$$

and θ =0 implies that X and Y are independent. When $\alpha_1 = \alpha_2 = 1$, BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ distribution gives a generalization of the MBE (μ_1, μ_2, θ) distribution discussed in (1.17).

Now, we define a bivariate positive stable distribution.

Definition 2.2. A non negative random vector (W_1, W_2) is said to follow bivariate positive stable distribution if its Laplace transform is

$$\varphi(\lambda_1, \lambda_2) = e^{-\mu_1\lambda_1^{\alpha_1} - \mu_2\lambda_2^{\alpha_2} - r\mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}}, \tag{2.3}$$

$$0 < \alpha_1, \alpha_2 \leq 1,\ \mu_1,\ \mu_2 > 0,\ 0 \leq r \leq 1.$$

When W_1 and W_2 are independent ($r = 0$),

$$\varphi(\lambda_1, \lambda_2) = e^{-\mu_1 \lambda_1^{\alpha_1} - \mu_2 \lambda_2^{\alpha_2}}. \tag{2.4}$$

In Section 2, we discuss various distributional properties of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$. The characterizations of the distribution using geometric compounding are obtained in Section 3, while in Section 4, we have the characterizations using bivariate geometric compounding. Estimates of the parameters of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ are obtained in Section 5. Autoregressive processes with BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ marginals are developed in Section 6. Bivariate Mittag-Leffler distributions that generalize Marshall-Olkin's bivariate exponential, Hawkes' bivariate exponential and Paulson's bivariate exponential are introduced in Section 7. A bivariate semi Mittag-Leffler distribution is introduced and studied in Section 8.

2.2 Distributional Properties

Kozubowski et al. (2005) established the one to one correspondence between the operator stable and operator geometric stable distributions as

$$G(t) = \frac{1}{1 - \ln \psi(t)}$$

where $G(t)$ and $\psi(t)$ are the characteristic functions of the operator geometric stable and operator stable distributions respectively. In the light of this , we note that BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ distribution is the geometric stable form of the bivariate positive stable distribution with Laplace transform given in (2.3).

Therefore,

$$\phi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2} + r\mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}}.$$

The following theorem gives a mixture representation of BML $(\mu_1,\ \mu_2,\ \alpha_1, \alpha_2, 1)$ distribution.

Theorem 2.1. *Let (W_1, W_2) have positive stable distribution with Laplace transform in (2.4) and Z, independent of (W_1, W_2), have standard exponential distribution. Then $(X, Y) = (Z^{\frac{1}{\alpha_1}}W_1,\ Z^{\frac{1}{\alpha_2}}W_2)$ has BML $(\mu_1,\ \mu_2,\ \alpha_1, \alpha_2, 1)$ distribution.*

Proof. The Laplace transform of $(X,\ Y)$ is

$$\begin{aligned}
\phi(\lambda_1, \lambda_2) &= \int_0^\infty E(e^{-\lambda_1 Z^{\frac{1}{\alpha_1}}W_1 - \lambda_2 Z^{\frac{1}{\alpha_2}}W_2}/Z) f_Z(z)dz \\
&= \int_0^\infty E(e^{-\lambda_1 z^{\frac{1}{\alpha_1}}W_1 - \lambda_2 z^{\frac{1}{\alpha_2}}W_2})e^{-z}dz \\
&= \int_0^\infty e^{-z(\mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2})}e^{-z}dz \\
&= \frac{1}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}}.
\end{aligned}$$

□

Theorem 2.1 implies that a random vector (X, Y) with BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution admits the representation

$$(X, Y) \overset{d}{=} (Z^{\frac{1}{\alpha_1}}W_1,\ Z^{\frac{1}{\alpha_2}}W_2). \tag{2.5}$$

Now, we obtain the distribution function of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$.

From the representation given in (2.5), we have

$$\begin{aligned} P(X \leq x, Y \leq y) &= P(Z^{\frac{1}{\alpha_1}} W_1 \leq x, Z^{\frac{1}{\alpha_2}} W_2 \leq y) \\ &= \int_0^\infty P(Z^{\frac{1}{\alpha_1}} W_1 \leq x, Z^{\frac{1}{\alpha_2}} W_2 \leq y/Z) f_Z(z) dz \\ &= \int_0^\infty F_{W_1,W_2}(\frac{x}{z^{\frac{1}{\alpha_1}}}, \frac{y}{z^{\frac{1}{\alpha_2}}}) e^{-z} dz \end{aligned}$$

where $F_{W_1,W_2}(.\,,\,.)$ represents bivariate positive stable distribution function with Laplace transform in (2.3).

When W_1 and W_2 are independent ($r = 0$),

$$P(X \leq x, Y \leq y) = \int_0^\infty S_{\alpha_1}(\frac{x}{z^{\frac{1}{\alpha_1}} \mu_1}) S_{\alpha_2}(\frac{y}{z^{\frac{1}{\alpha_2}} \mu_2}) e^{-z} dz$$

where $S_{\alpha_i}(.), (i = 1, 2.)$ represents the distribution function of a standard positive stable random variable Laplace transform, $\phi(\lambda) = e^{-\lambda^\alpha}$.

The joint density function of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ distribution is

$$h_{X,Y}(x, y) = \int_0^\infty f_{W_1,W_2}(\frac{x}{z^{\frac{1}{\alpha_1}}}, \frac{y}{z^{\frac{1}{\alpha_2}}}) e^{-z} dz$$

where $f_{W_1,W_2}(.\,,\,.)$ is the joint density function of bivariate positive stable distribution. When W_1 and W_2 are independent,

$$h_{X,Y}(x, y) = \int_0^\infty D_{\alpha_1}(\frac{x}{z^{\frac{1}{\alpha_1}} \mu_1}) D_{\alpha_2}(\frac{y}{z^{\frac{1}{\alpha_2}} \mu_2}) e^{-z} dz$$

where $D_{\alpha_i}(\cdot)$, (i=1,2.), represents the density function of positive stable random variable stated in (2.2).

We note that the Laplace transform in (2.2) is the geometric version of the bivariate positive stable distribution given in (2.4). Therefore, BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ is geometrically infinitely divisible (see Kozubowski (2005)).

The product moments of the distribution, $E(X^{\delta_1}Y^{\delta_2})$, exists if and only if $0 < \delta_1 < \alpha_1,\ 0 < \delta_2 < \alpha_2$ and

$$E(X^{\delta_1}Y^{\delta_2}) = \frac{\Gamma(\frac{\delta_1}{\alpha_1} + \frac{\delta_2}{\alpha_2} + 1)\Gamma(1 - \frac{\delta_1}{\alpha_1})\Gamma(1 + \frac{\delta_1}{\alpha_1})\Gamma(1 - \frac{\delta_2}{\alpha_2})\Gamma(1 + \frac{\delta_2}{\alpha_2})}{\Gamma(1-\delta_1)\Gamma(1-\delta_2)}.$$

In (2.5), assume that Z, W_1 and W_2 are independent. Then

$$E(X^{\delta_1}Y^{\delta_2}) = E(Z^{\frac{\delta_1}{\alpha_1} + \frac{\delta_2}{\alpha_2}})E(W_1^{\delta_1}W_2^{\delta_2}). \tag{2.6}$$

But

$$E(Z^{\frac{\delta_1}{\alpha_1} + \frac{\delta_2}{\alpha_2}}) = \Gamma(\frac{\delta_1}{\alpha_1} + \frac{\delta_2}{\alpha_2} + 1).$$

$$E(W_i^{\delta_i}) = \frac{\Gamma(1 - \frac{\delta_i}{\alpha_i})\Gamma(1 + \frac{\delta_i}{\alpha_i})}{\Gamma(1-\delta_i)} \quad \text{for } i = 1, 2.$$

Substituting in (2.6), we get the required result.

The following theorem shows the attraction of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution towards the bivariate positive stable distribution in (2.4).

Theorem 2.2. *Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of random vectors which are independently and identically distributed as BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$. Define*

$$U_n = n^{\frac{-1}{\alpha_1}}(X_1 + X_2 + ... + X_n) \quad \text{and} \quad V_n = n^{\frac{-1}{\alpha_2}}(Y_1 + Y_2 + ... + Y_n).$$

Then (U_n, V_n) is asymptotically distributed as bivariate positive stable law with Laplace transform in (2.4).

Proof. Suppose that $(X_i, Y_i), i \geq 1$ are distributed as BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$.

Therefore,

$$\phi(\lambda_1,\ \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

The Laplace transform of (U_n, V_n) is

$$\begin{aligned}
\varphi_{U_n,V_n}(\lambda_1,\lambda_2) &= E(e^{-\lambda_1 n^{\frac{-1}{\alpha_1}}(X_1+X_2+\ldots+X_n)-\lambda_2 n^{\frac{-1}{\alpha_2}}(Y_1+Y_2+\ldots+Y_n)}) \\
&= \left[E(e^{-\lambda_1 n^{\frac{-1}{\alpha_1}}X_i-\lambda_2 n^{\frac{-1}{\alpha_2}}Y_i})\right]^n \\
&= \left[\frac{1}{1+\frac{1}{n}(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2})}\right]^n.
\end{aligned}$$

When $n \longrightarrow \infty$, we get

$$\varphi_{U_n,V_n}(\lambda_1,\lambda_2) \longrightarrow e^{-\mu_1\lambda_1^{\alpha_1}-\mu_2\lambda_2^{\alpha_2}}.$$

□

2.3 Characterization of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ through Geometric Compounding

Let $\{(X_i,\ Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors with Laplace transform $\psi(\lambda_1,\ \lambda_2)$. Define

$$U_N = \sum_{i=1}^{N} X_i \qquad V_N = \sum_{i=1}^{N} Y_i \tag{2.7}$$

where N follows geometric distribution such that

$$P(N=n) = p(1-p)^{n-1},\ \ n = 1, 2, 3, \ldots, 0 < p < 1 \tag{2.8}$$

and is independent of $(X_i,\ Y_i), i \geq 1$. Then the Laplace transform of (U_N, V_N) is

$$\begin{aligned}
\phi(\lambda_1,\ \lambda_2) &= E(e^{-\lambda_1 U_N - \lambda_2 V_N}) \\
&= \sum_{n=1}^{\infty} E(e^{-\lambda_1(X_1+X_2+\ldots+X_N)-\lambda_2(Y_1+Y_2+\ldots+Y_N)}/N = n)P(N = n) \\
&= \sum_{n=1}^{\infty} E(e^{-\lambda_1(X_1+X_2+\ldots+X_n)-\lambda_2(Y_1+Y_2+\ldots+Y_n)})P(N = n) \\
&= \sum_{n=1}^{\infty} \left[E(e^{-\lambda_1 X_i - \lambda_2 Y_i})\right]^n p(1-p)^{n-1} \\
&= \frac{p\psi(\lambda_1,\ \lambda_2)}{1-(1-p)\psi(\lambda_1,\ \lambda_2)}. \qquad (2.9)
\end{aligned}$$

Now, we obtain a characterization of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution.

Theorem 2.3. *Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors and N, independent of $(X_i, Y_i), i \geq 1$, follows geometric distribution in (2.8). Suppose that U_N and V_N are as defined in (2.7). Then $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ is distributed as BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ if and only if $(X_i, Y_i), i \geq 1$ follow BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution.*

Proof. From (2.9), the Laplace transform of $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ is

$$\phi(\lambda_1, \lambda_2) = \frac{p\psi(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)}{1-(1-p)\psi(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)} \qquad (2.10)$$

where $\psi(\lambda_1, \lambda_2)$ represents the Laplace transform of $(X_i, Y_i), i \geq 1$.

Taking

$$\psi(\lambda_1,\ \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

Substituting $\psi(\lambda_1, \lambda_2)$ in (2.10) and simplifying, we get

$$\phi(\lambda_1,\ \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

Conversely, assume that $(p^{\frac{1}{\alpha_1}}U_N, p^{\frac{1}{\alpha_2}}V_N)$ has BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution. From (2.10),

$$\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}} = \frac{p\psi(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)}{1-(1-p)\psi(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)}.$$

Solving, we get

$$\psi(\lambda_1,\ \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

□

A characterization of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution is obtained using repeated geometric compounding.

Theorem 2.4. *Let (U_{N_k}, V_{N_k}) be defined as*

$$U_{N_k} = p_{k-1}^{\frac{1}{\alpha_1}} \sum_{i=1}^{N_{k-1}} X_i \qquad V_{N_k} = p_{k-1}^{\frac{1}{\alpha_2}} \sum_{i=1}^{N_{k-1}} Y_i$$

where N_{k-1}, independent of $(X_i, Y_i), i \geq 1$, follows geometric distribution such that

$$P(N_{k-1}=n) = (1-p_{k-1})^{n-1}p_{k-1},\ 0<p_{k-1}<1,\quad n=1,2,3,...$$

$\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed random vectors with distribution function $F_{k-1}(.,.)$ and Laplace transform $\phi_{k-1}(\lambda_1, \lambda_2)$ k=2,3,... To start with, take $F_1(.,.) = F(.,.)$ and the corresponding Laplace transform as $\phi(\lambda_1, \lambda_2)$. Then (U_{N_k}, V_{N_k}) is distributed as BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ if and only if $(X_i, Y_i), i \geq 1$ are distributed as BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$.

Proof. From (2.10), the Laplace transform of (U_{N_k}, V_{N_k}) is

$$\phi_k(\lambda_1, \lambda_2) = \frac{p_{k-1}\phi_{k-1}(p_{k-1}^{\frac{1}{\alpha_1}}\lambda_1, p_{k-1}^{\frac{1}{\alpha_2}}\lambda_2)}{1-(1-p_{k-1})\phi_{k-1}(p_{k-1}^{\frac{1}{\alpha_1}}\lambda_1, p_{k-1}^{\frac{1}{\alpha_2}}\lambda_2)}. \tag{2.11}$$

Therefore,

$$\phi_2(\lambda_1,\lambda_2) = \frac{p_1\phi(p_1^{\frac{1}{\alpha_1}}\lambda_1, p_1^{\frac{1}{\alpha_2}}\lambda_2)}{1-(1-p_1)\phi(p_1^{\frac{1}{\alpha_1}}\lambda_1, p_1^{\frac{1}{\alpha_2}}\lambda_2)},$$

since $F_1(.,.) = F(.,.)$, $\quad \phi_1(\lambda_1,\lambda_2) = \phi(\lambda_1,\lambda_2)$.

Applying recursively (2.11),

$$\phi_k(\lambda_1,\lambda_2) = \frac{\prod_{i=1}^{k-1} p_i\, \phi\left[\prod_{i=1}^{k-1} p_i^{\frac{1}{\alpha_1}}\lambda_1,\ \prod_{i=1}^{k-1} p_i^{\frac{1}{\alpha_2}}\lambda_2\right]}{1-\phi\left[\prod_{i=1}^{k-1} p_i^{\frac{1}{\alpha_1}}\lambda_1,\ \prod_{i=1}^{k-1} p_i^{\frac{1}{\alpha_2}}\lambda_2\right] + \prod_{i=1}^{k-1} p_i\phi\left[\prod_{i=1}^{k-1} p_i^{\frac{1}{\alpha_1}}\lambda_1,\ \prod_{i=1}^{k-1} p_i^{\frac{1}{\alpha_2}}\lambda_2\right]}. \tag{2.12}$$

Suppose that $(X_i, Y_i), i \geq 1$ are distributed as BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$.

Therefore,

$$\phi(\lambda_1,\ \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

Substituting $\phi(\lambda_1,\ \lambda_2)$ in (2.12) and simplifying, we get

$$\phi_k(\lambda_1,\lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

Conversely, assuming that (U_{N_k}, V_{N_k}) is distributed as BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$. Substituting $\phi_k(\lambda_1,\ \lambda_2)$ in (2.12) and simplifying, we get

$$\phi(\lambda_1,\ \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

$\square$

Now we introduce BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ distribution by considering the geometric sum of a set of independently and identically distributed random variables. Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors such that the components X_i and Y_i are independently distributed as Mittag-Leffler.

Therefore,

$$\psi_{X_i}(\lambda_1,\ 0) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}}, \quad \psi_{Y_i}(0,\ \lambda_2) = \frac{1}{1+\mu_2\lambda_2^{\alpha_2}} \text{ for } \quad i = 1, 2, 3, ...$$

The joint Laplace transform of $(X_i, Y_i), i \geq 1$ is

$$\psi(\lambda_1, \lambda_2) = \frac{1}{(1+\mu_1\lambda_1^{\alpha_1})(1+\mu_2\lambda_2^{\alpha_2})}. \tag{2.13}$$

From (2.10), the Laplace transform of $(p^{\frac{1}{\alpha_1}}U_N, p^{\frac{1}{\alpha_2}}V_N)$ is

$$\begin{aligned}
\phi(\lambda_1, \lambda_2) &= \frac{p}{(1+p\mu_1\lambda_1^{\alpha_1})(1+p\mu_2\lambda_2^{\alpha_2})-1+p} \\
&= \frac{p}{1+p\mu_1\lambda_1^{\alpha_1}+p\mu_2\lambda_2^{\alpha_2}+p^2\mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}-1+p} \\
&= \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}+p\mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}}.
\end{aligned}$$

Comparing with (2.1), $\theta = 1 - p$.

Therefore, $(p^{\frac{1}{\alpha_1}}U_N, p^{\frac{1}{\alpha_2}}V_N)$ has BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1-p)$ distribution.

On the other hand suppose that $(p^{\frac{1}{\alpha_1}}U_N, p^{\frac{1}{\alpha_2}}V_N)$ follows BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1-p)$ distribution. From (2.10),

$$\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}+p\mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}} = \frac{p\psi(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)}{1-(1-p)\psi(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)}.$$

Solving, we obtain

$$\psi(\lambda_1, \lambda_2) = \frac{1}{(1+\mu_1\lambda_1^{\alpha_1})(1+\mu_2\lambda_2^{\alpha_2})}.$$

Hence we have the following theorem.

Theorem 2.5. *Consider a sequence of independently and identically distributed random vectors* $\{(X_i, Y_i), i \geq 1\}$ *and* N *has geometric distribution given in (2.8). Assume that* N *is independent of* $(X_i, Y_i), i \geq 1$. *Then* $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ *has BML* $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1-p)$ *distribution if and only if* X_i *and* Y_i *are independently distributed as Mittag-Leffler .*

Hence Theorem 2.5 enables to generate BML $(\mu_1,\ \mu_2,\ \alpha_1, \alpha_2, \theta)$ distribution as a geometric compound of random vectors $(X_i, Y_i), i \geq 1$ such that X_i and $Y_i, i \geq 1$ are independent Mittag-Leffler random variables.

The following theorem gives a characterization of geometric distribution.

Theorem 2.6. *Suppose that* $\{(X_i, Y_i), i \geq 1\}$ *is a sequence of independently and identically distributed BML* $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ *random vectors. Then* $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ *and* $(X_i, Y_i), i \geq 1$ *are identically distributed if and only if* N *is geometric.*

Proof. The proof of the 'if' part is omitted as it is presented in Theorem 2.3.

To prove the converse, assume that $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ and $(X_i, Y_i), i \geq 1$ are identically distributed as BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$. Without loss of generality, we take $\mu_1 = \mu_2 = 1$. The Laplace transform of $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ is given by

$$\phi(\lambda_1, \lambda_2) = \sum_{n=1}^{\infty} \left(\psi(p^{\frac{1}{\alpha_1}} \lambda_1, p^{\frac{1}{\alpha_2}} \lambda_2) \right)^n P(N = n).$$

where $\psi(\lambda_1, \lambda_2)$ represents the Laplace transform of $(X_i, Y_i), i \geq 1$. Therefore, from the assumption it follows,

$$\sum_{n=1}^{\infty}\left(\frac{1}{1+p\lambda_1^{\alpha_1}+p\lambda_2^{\alpha_2}}\right)^n P(N=n)=\frac{1}{1+\lambda_1^{\alpha_1}+\lambda_2^{\alpha_2}}.$$

Expanding both sides,

$$\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j(j+n-1)!(\lambda_1^{\alpha_1}+\lambda_2^{\alpha_2})^j p^j}{j!(n-1)!}P(N=n)=\sum_{j=0}^{\infty}(-1)^j(\lambda_1^{\alpha_1}+\lambda_2^{\alpha_2})^j.$$

Comparing the coefficients of $(\lambda_1^{\alpha_1}+\lambda_2^{\alpha_2})^j$,

$$\sum_{n=1}^{\infty}\frac{n(n+1)(n+2)...(n+j-1)p^j P(N=n)}{j!}=1, for\ j\ =1,2,3,...$$

Therefore,

$$E(N)=\frac{1}{p},\quad E(N(N+1))=\frac{2}{p^2}\ \text{ and so on.}$$

Consider

$$\begin{aligned}
E(1-t)^{-N} &= 1+\frac{t}{1!}E(N)+\frac{t^2}{2!}E(N(N+1))+\frac{t^3}{3!}E(N(N+1)(N+2))+...\\
&= \frac{p}{p-t}\\
&= \frac{p}{1-p}\sum_{n=1}^{\infty}\left(\frac{1-p}{1-t}\right)^n\\
&= p\sum_{n=1}^{\infty}(1-t)^{-n}(1-p)^{n-1}.
\end{aligned}$$

Also

$$E(1-t)^{-N}=\sum_{n=1}^{\infty}(1-t)^{-n}P(N=n).$$

Therefore,

$$\sum_{n=1}^{\infty}(1-t)^{-n}P(N=n)=p\sum_{n=1}^{\infty}(1-t)^{-n}(1-p)^{n-1}.$$

Comparing we get,

$$P(N = n) = (1 - p)^{n-1} p, \quad n = 1, 2, 3, ...$$

□

2.4 Characterization of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ through Bivariate Geometric Compounding

Block (1977) discussed the probability distributions of random sums of independently and identically distributed random vectors when the number of summands follow the bivariate geometric distribution in (1.15).

Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors with Laplace transform $\psi(\lambda_1, \lambda_2)$. Define

$$U_{N_1} = \sum_{i=1}^{N_1} X_i \text{ and } V_{N_2} = \sum_{i=1}^{N_2} Y_i \tag{2.14}$$

where (N_1, N_2) has the bivariate geometric distribution with p.g.f. in (1.16). Block (1977) has given the expression for the Laplace transform of (U_{N_1}, V_{N_2}).

$$\phi(\lambda_1, \lambda_2) = \psi(\lambda_1, \lambda_2)(p_{00} + p_{10}\phi(\lambda_1, 0) + p_{01}\phi(0, \lambda_2) + p_{11}\phi(\lambda_1, \lambda_2)). \tag{2.15}$$

Therefore,

$$\phi(\lambda_1, \lambda_2) = \frac{\psi(\lambda_1, \lambda_2)}{1 - p_{11}\psi(\lambda_1, \lambda_2)} (p_{00} + p_{10}\phi(\lambda_1, 0) + p_{01}\phi(0, \lambda_2)). \tag{2.16}$$

Also, from (2.15)

$$\phi(\lambda_1,0)=\frac{(p_{00}+p_{01})\psi(\lambda_1,0)}{1-(p_{11}+p_{10})\psi(\lambda_1,0)} \text{ and } \phi(0,\lambda_2)=\frac{(p_{00}+p_{10})\psi(0,\lambda_2)}{1-(p_{11}+p_{01})\psi(0,\lambda_2)}. \quad (2.17)$$

By choosing appropriately $\psi(\lambda_1,\ \lambda_2)$, p_{00}, p_{10}, p_{01} and p_{11} we obtain characterizations of BML $(\mu_1,\mu_2,\alpha_1,\alpha_2,\theta)$ distribution and also generate some other forms of bivariate Mittag-Leffler distribution.

Theorem 2.7. *Consider a sequence $\{(X_i,Y_i), i\geq 1\}$ of independently and identically distributed random vectors. Let (N_1,N_2) be independent of $(X_i,Y_i), i\geq 1$ and have the bivariate geometric distribution with p.g.f. in (1.16) such that $p_{00}=0$ and $p_{10}+p_{01}+p_{11}=1$. Let $U_{N_1}=\sum_{i=1}^{N_1} X_i$ and $V_{N_2}=\sum_{i=1}^{N_2} Y_i$. Then $(p_{01}^{\frac{1}{\alpha_1}}U_{N_1}, p_{10}^{\frac{1}{\alpha_2}}V_{N_2})$ follows bivariate Mittag-Leffler distribution with independent marginals where (U_{N_1},V_{N_2}) are as stated in (2.14) if and only if $(X_i,Y_i), i\geq 1$ have BML $(\mu_1,\mu_2,\alpha_1,\alpha_2,1)$ distribution.*

Proof. From (2.15), the Laplace transform of $(p_{01}^{\frac{1}{\alpha_1}}U_{N_1}, p_{10}^{\frac{1}{\alpha_2}}V_{N_2})$ is

$$\phi(\lambda_1,\ \lambda_2)=\psi(p_{01}^{\frac{1}{\alpha_1}}\lambda_1,\ p_{10}^{\frac{1}{\alpha_2}}\lambda_2)(p_{00}+p_{10}\phi(\lambda_1,\ 0)+p_{01}\phi(0,\ \lambda_2)+p_{11}\phi(\lambda_1,\ \lambda_2)). \quad (2.18)$$

When $p_{00}=0$ and $p_{10}+p_{01}+p_{11}=1$,

$$\phi(\lambda_1,\lambda_2)=\frac{\psi(p_{01}^{\frac{1}{\alpha_1}}\lambda_1, p_{10}^{\frac{1}{\alpha_2}}\lambda_2)}{1-p_{11}\psi(p_{01}^{\frac{1}{\alpha_1}}\lambda_1, p_{10}^{\frac{1}{\alpha_2}}\lambda_2)}\left(p_{10}\phi(\lambda_1,0)+p_{01}\phi(0,\lambda_2)\right). \quad (2.19)$$

Also, from (2.18)

$$\phi(\lambda_1,0)=\frac{p_{01}\psi(p_{01}^{\frac{1}{\alpha_1}}\lambda_1,0)}{1-(1-p_{01})\psi(p_{01}^{\frac{1}{\alpha_1}}\lambda_1,0)} \text{ and } \phi(0,\lambda_2)=\frac{p_{10}\psi(0,p_{10}^{\frac{1}{\alpha_2}}\lambda_2)}{1-(1-p_{10})\psi(0,p_{10}^{\frac{1}{\alpha_2}}\lambda_2)}. \quad (2.20)$$

If $(X_i, Y_i), i \geq 1$ have BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution, then from (2.20), we get

$$\phi(\lambda_1, 0) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{1}{1+\mu_2\lambda_2^{\alpha_2}}.$$

Substituting $\phi(\lambda_1, 0)$ and $\phi(0, \lambda_2)$ in (2.19) and simplifying

$$\phi(\lambda_1, \lambda_2) = \frac{1}{(1+\mu_1\lambda_1^{\alpha_1})(1+\mu_2\lambda_2^{\alpha_2})}.$$

Conversely, suppose that $(p_{01}^{\frac{1}{\alpha_1}} U_{N_1}, p_{10}^{\frac{1}{\alpha_2}} V_{N_2})$ follows bivariate Mittag-Leffler distribution with independent marginals . Substituting the Laplace transform of $(p_{01}^{\frac{1}{\alpha_1}} U_{N_1}, p_{10}^{\frac{1}{\alpha_2}} V_{N_2})$ in (2.19),

$$\frac{1}{(1+\mu_1\lambda_1^{\alpha_1})(1+\mu_2\lambda_2^{\alpha_2})} = \frac{\psi(p_{01}^{\frac{1}{\alpha_1}}\lambda_1, p_{10}^{\frac{1}{\alpha_2}}\lambda_2)}{1-p_{11}\psi(p_{01}^{\frac{1}{\alpha_1}}\lambda_1, p_{10}^{\frac{1}{\alpha_2}}\lambda_2)} \left(p_{10}\frac{1}{1+\mu_1\lambda_1^{\alpha_1}} + p_{01}\frac{1}{1+\mu_2\lambda_2^{\alpha_2}}\right).$$

Therefore,

$$1-p_{11}\psi(p_{01}^{\frac{1}{\alpha_1}}\lambda_1, p_{10}^{\frac{1}{\alpha_2}}\lambda_2) = \psi(p_{01}^{\frac{1}{\alpha_1}}\lambda_1, p_{10}^{\frac{1}{\alpha_2}}\lambda_2)\left[p_{10}(1+\mu_2\lambda_2^{\alpha_2}) + p_{01}(1+\mu_1\lambda_1^{\alpha_1})\right].$$

On simplification, we get

$$\psi(\lambda_1, \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

□

Theorem 2.8. *Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed Mittag-Leffler random vectors with Laplace transform $\psi(\lambda_1, \lambda_2)$. Suppose that (N_1, N_2) is independent of $(X_i, Y_i), i \geq 1$ and follows bivariate geometric distribution with p.g.f. given in (1.16). Let $p_{00} = 0$, $p_{10}+p_{01}+p_{11} = 1$. Then $((1-p_{11})^{\frac{1}{\alpha_1}} U_{N_1}, (1-p_{11})^{\frac{1}{\alpha_2}} V_{N_2})$ has Laplace transform*

$$\phi(\lambda_1, \lambda_2) = \frac{1}{(1+(1-p_{11})\lambda_1^{\alpha_1})(1+(1-p_{11})\lambda_2^{\alpha_2})}$$

if and only if $\psi(\lambda_1, \lambda_2) = \dfrac{1}{1 + p_{01}\lambda_1^{\alpha_1} + p_{10}\lambda_2^{\alpha_2}}$

Proof. Suppose that

$$\psi(\lambda_1, \lambda_2) = \frac{1}{1 + p_{01}\lambda_1^{\alpha_1} + p_{10}\lambda_2^{\alpha_2}}.$$

Using (2.17),

$$\phi(\lambda_1, 0) = \frac{1}{1 + (1 - p_{11})\lambda_1^{\alpha_1}} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{1}{1 + (1 - p_{11})\lambda_2^{\alpha_2}}.$$

From (2.16) the Laplace transform of $((1 - p_{11})^{\frac{1}{\alpha_1}} U_{N_1}, (1 - p_{11})^{\frac{1}{\alpha_2}} V_{N_2})$ is

$$\phi(\lambda_1, \lambda_2) = \frac{\psi((1 - p_{11})^{\frac{1}{\alpha_1}}\lambda_1, (1 - p_{11})^{\frac{1}{\alpha_2}}\lambda_2)}{1 - p_{11}\psi((1 - p_{11})^{\frac{1}{\alpha_1}}\lambda_1, (1 - p_{11})^{\frac{1}{\alpha_2}}\lambda_2)} \left(p_{10}\phi(\lambda_1, 0) + p_{01}\phi(0, \lambda_2)\right). \tag{2.21}$$

Substituting $\phi(\lambda_1, 0)$ and $\phi(0, \lambda_2)$ in (2.21) and simplifying, we get

$$\phi(\lambda_1, \lambda_2) = \frac{1}{(1 + (1 - p_{11})\lambda_1^{\alpha_1})(1 + (1 - p_{11})\lambda_2^{\alpha_2})}.$$

In order to prove the converse, substituting $\phi(\lambda_1, \lambda_2)$, $\phi(\lambda_1, 0)$ and $\phi(0, \lambda_2)$ in (2.21). On simplification, we get

$$\psi(\lambda_1, \lambda_2) = \frac{1}{1 + p_{01}\lambda_1^{\alpha_1} + p_{10}\lambda_2^{\alpha_2}}.$$

□

Using the Laplace transform of bivariate compounding mentioned in (2.15), now we obtain BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ as the random sum distribution of independently and identically distributed random vectors in which the components have independent Mittag-Leffler.

Theorem 2.9. *Suppose that $\{(X_i, Y_i), i \geq 1\}$ are independently and identically distributed random vectors with Laplace transform*

$$\psi(\lambda_1, \lambda_2) = \left(1 + \frac{\mu_1 \lambda_1^{\alpha_1}}{1+m}\right)^{-1} \left(1 + \frac{\mu_2 \lambda_2^{\alpha_2}}{1+m}\right)^{-1}, 0 < \alpha_1, \alpha_2 \leq 1. \qquad (2.22)$$

Take $p_{00} = (1+m)^{-1}$, $p_{10} = p_{01} = 0$ and $p_{11} = m(1+m)^{-1}$, $\mu_1, \mu_2 > 0$, $m > 0$. Then the random vector (U_{N_1}, V_{N_2}) has BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \frac{m}{1+m})$ distribution if and only if $(X_i, Y_i), i \geq 1$ have the Laplace transform in (2.22).

Proof. Suppose that $(X_i, Y_i), i \geq 1$ have the Laplace transform in (2.22). From (2.15), the Laplace transform of (U_{N_1}, V_{N_2}) is

$$\begin{aligned}
\phi(\lambda_1, \lambda_2) &= \left(1 + \frac{\mu_1 \lambda_1^{\alpha_1}}{1+m}\right)^{-1} \left(1 + \frac{\mu_2 \lambda_2^{\alpha_2}}{1+m}\right)^{-1} \left((1+m)^{-1} + m(1+m)^{-1}\phi(\lambda_1, \lambda_2)\right) \\
&= \frac{(1+m)^{-1}}{\left(1 + \frac{\mu_1 \lambda_1^{\alpha_1}}{1+m}\right)\left(1 + \frac{\mu_2 \lambda_2^{\alpha_2}}{1+m}\right)}(1 + m\phi(\lambda_1, \lambda_2)) \\
&= \frac{1}{(1+m)\left(1 + \frac{\mu_1 \lambda_1^{\alpha_1}}{1+m}\right)\left(1 + \frac{\mu_2 \lambda_2^{\alpha_2}}{1+m}\right) - m} \\
&= \frac{1+m}{((1+m) + \mu_1 \lambda_1^{\alpha_1})\ ((1+m) + \mu_2 \lambda_2^{\alpha_2}) - m(1+m)} \\
&= \frac{1}{1 + \mu_1 \lambda_1^{\alpha_1} + \mu_2 \lambda_2^{\alpha_2} + \frac{\mu_1 \mu_2 \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}}{1+m}} \\
&= \frac{1}{(1 + \mu_1 \lambda_1^{\alpha_1})(1 + \mu_2 \lambda_2^{\alpha_2}) - \frac{m}{1+m}\mu_1 \mu_2 \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}}.
\end{aligned}$$

Comparing with (2.1), we get (U_{N_1}, V_{N_2}) has BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \frac{m}{1+m})$ distribution.

Conversely, let (U_{N_1}, V_{N_2}) have BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \frac{m}{1+m})$ distribution. From (2.15), we get

$$\psi(\lambda_1, \lambda_2) = \frac{(1+m)\phi(\lambda_1, \lambda_2)}{1 + m\phi(\lambda_1, \lambda_2)}.$$

Substituting $\phi(\lambda_1, \lambda_2)$ and simplifying, we get (2.22). □

2.5 Estimation of Parameters

In this Section, we obtain the log moment estimators of the parameters of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution. Kozubowski (2000a) showed that a random variable X following Mittag-Leffler distribution with parameter α can be represented as $ZW^{1/\alpha}$ where Z has standard exponential distribution and W follows positive stable distribution given in (1.3). Suppose that a random vector (X, Y) has BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution. Then (X, Y) admits the relation

$$(X, Y) \stackrel{d}{=} (Z(\mu_1 W_1)^{\frac{1}{\alpha_1}}, Z(\mu_2 W_2)^{\frac{1}{\alpha_2}})$$

where W_1 and W_2 are independently distributed stable random variables and also independent of Z which follows standard exponential distribution. Now let us consider the estimation of the parameter α_1. We have

$$X \stackrel{d}{=} Z(\mu_1 W_1)^{\frac{1}{\alpha_1}}.$$

Then it follows that the random variables $\ln X$ and $\ln Z + \dfrac{1}{\alpha_1}\ln(\mu_1 W_1)$ have same distributions.

Therefore,

$$\begin{aligned}(\ln X)^2 \stackrel{d}{=} (\ln Z)^2 &+ \frac{1}{\alpha_1^2}(\ln \mu_1)^2 + \frac{1}{\alpha_1^2}(\ln W_1)^2 \\ &+ \frac{2}{\alpha_1}\ln Z \ln \mu_1 + \frac{2}{\alpha_1}\ln Z \ln W_1 + \frac{2}{\alpha_1^2}\ln W_1 \ln \mu_1. \qquad (2.23)\end{aligned}$$

Taking expectations on both sides of (2.23)

$$\begin{aligned} E(\ln X)^2 &= E(\ln Z)^2 + \frac{1}{\alpha_1^2}(\ln \mu_1)^2 + \frac{1}{\alpha_1^2}E(\ln W_1)^2 \\ &+\frac{2}{\alpha_1}\ln \mu_1 E(\ln Z) + \frac{2}{\alpha_1}E(\ln Z)E(\ln W_1) \\ &+\frac{2}{\alpha_1^2}\ln \mu_1 E(\ln W_1). \end{aligned} \tag{2.24}$$

Let us consider first the moments of the random variable $\ln Z$. Define the function $h_k(m)$ as

$$h_k(m) = \int_0^\infty (\ln x)^k e^{-mx}dx, \; k = 1, 2.$$

From Gradshteyn and Ryzhik (1996), we have

$$h_1(m) = -\frac{1}{m}(\gamma + \ln m), \; Re(m) > 0$$

where γ is the Euler's constant.

Thus the first moment of the random variable $\ln Z$ is

$$E(\ln Z) = h_1(1) = -\gamma.$$

Consider now the second moment of the random variable $\ln Z$. From Gradshteyn and Ryzhik (1996), we have

$$h_2(m) = \frac{1}{m}\left(\frac{\pi^2}{6} + (\gamma + \ln m)^2\right), \; Re(m) > 0.$$

It follows that the second moment $E(\ln Z)^2$ is

$$E(\ln Z)^2 = h_2(1) = \gamma^2 + \frac{\pi^2}{6}.$$

Now, we will consider the first and the second moment of the random variable $\ln W_1$.

Define the function $g_1(t)$ as

$$g_1(t) = \int_0^\infty \frac{\ln x dx}{x^2 - 2x\cos t + 1}, \quad 0 < t < 2\pi.$$

Then

$$\int_0^\infty \frac{\ln x dx}{x^2 - 2x\cos t + 1} = \int_0^1 \frac{\ln x dx}{x^2 - 2x\cos t + 1} + \int_1^\infty \frac{\ln x dx}{x^2 - 2x\cos t + 1}.$$

Consider the second integral on the right side. The change of variables $x = 1/y$ gives

$$\int_1^\infty \frac{\ln x dx}{x^2 - 2x\cos t + 1} = -\int_0^1 \frac{\ln y dy}{y^2 - 2y\cos t + 1}.$$

Thus we obtain that

$$g_1(t) = \int_0^1 \frac{\ln x dx}{x^2 - 2x\cos t + 1} - \int_0^1 \frac{\ln y dy}{y^2 - 2y\cos t + 1} = 0.$$

The first moment of the random variable $\ln W_1$ is

$$E(\ln W_1) = \frac{\sin \pi\alpha}{\pi\alpha} \int_0^\infty \frac{\ln x dx}{x^2 + 2x\cos(\pi\alpha) + 1} = \frac{\sin \pi\alpha}{\pi\alpha} g_1(\pi + \alpha\pi) = 0.$$

Finally, consider the second moment of the random variable $\ln W_2$. Define the function $g_2(t)$ as

$$g_2(t) = \int_0^\infty \frac{(\ln x)^2 dx}{x^2 - 2x\cos t + 1}.$$

Taking the same argument as in the case of the function $g_1(t)$, we obtain that

$$\begin{aligned} g_2(t) &= \int_0^1 \frac{(\ln x)^2 dx}{x^2 - 2x\cos t + 1} + \int_1^\infty \frac{(\ln x)^2 dx}{x^2 - 2x\cos t + 1} \\ &= \int_0^1 \frac{(\ln x)^2 dx}{x^2 - 2x\cos t + 1} + \int_0^1 \frac{(\ln y)^2 dy}{y^2 - 2y\cos t + 1} \\ &= 2\int_0^1 \frac{(\ln x)^2 dx}{x^2 - 2x\cos t + 1}. \end{aligned}$$

Now, using Prudnikov et al. (1981), we obtain

$$g_2(t) = \frac{t(t-2\pi)(t-\pi)}{3\sin t}.$$

Then the second moment of the random variable $\ln W_1$ is

$$\begin{aligned} E(\ln W_1)^2 &= \frac{\sin\pi\alpha}{\pi\alpha}\int_0^\infty \frac{(\ln x)^2 dx}{x^2+2x\cos(\pi\alpha)+1}. \\ &= \frac{\sin\pi\alpha}{\pi\alpha}\cdot g_2(\pi+\alpha\pi) \\ &= \frac{\pi^2(1-\alpha^2)}{3}. \end{aligned}$$

Using the above results, from(2.24), the second moment of $(\ln X)^2$ as

$$\begin{aligned} E(\ln X)^2 &= \gamma^2+\frac{\pi^2}{6}+\frac{\pi^2(1-\alpha_1^2)}{3\alpha_1^2}+\frac{1}{\alpha_1^2}(\ln\mu_1)^2-\frac{2}{\alpha_1}\gamma\ln\mu_1 \\ &= \gamma^2-\frac{\pi^2}{6}+\frac{\pi^2}{3\alpha_1^2}+\frac{1}{\alpha_1^2}(\ln\mu_1)^2-\frac{2}{\alpha_1}\gamma\ln\mu_1 \end{aligned}$$

Replacing $E(\ln X)^2$ by the corresponding sample equivalent we get a quadratic equation in α_1,

$$\left(\gamma^2-\frac{\pi^2}{6}-\frac{1}{n}\sum_{i=1}^{n}(\ln X_i)^2\right)\alpha_1^2-2\gamma\alpha_1\ln\mu_1+(\ln\mu_1)^2+\frac{\pi^2}{3}.$$

Then as an estimate of α_1, we have

$$\widehat{\alpha}_1 = \frac{\gamma\ln\mu_1-\sqrt{\gamma^2(\ln\mu_1)^2-\left(\gamma^2-\frac{\pi^2}{6}-\frac{1}{n}\sum\limits_{i=1}^{n}(\ln X_i)^2\right)\left((\ln\mu_1)^2+\frac{\pi^2}{3}\right)}}{\gamma^2-\frac{\pi^2}{6}-\frac{1}{n}\sum\limits_{i=1}^{n}(\ln X_i)^2}.$$

The first moment of $\ln X$ is

$$E(\ln X) = -\gamma+\frac{1}{\alpha_1}\ln\mu_1.$$

Therefore, the estimate of μ_1 is

$$\widehat{\mu}_1 = e^{\widehat{\alpha}_1\left(\gamma+\frac{1}{n}\sum_{i=1}^{n}(\ln X_i)\right)}.$$

Similarly,

$$\widehat{\alpha}_2 = \frac{\gamma\ln\mu_2 - \sqrt{\gamma^2(\ln\mu_2)^2 - \left(\gamma^2 - \frac{\pi^2}{6} - \frac{1}{n}\sum_{i=1}^{n}(\ln Y_i)^2\right)\left((\ln\mu_2)^2 + \frac{\pi^2}{3})\right)}}{\gamma^2 - \frac{\pi^2}{6} - \frac{1}{n}\sum_{i=1}^{n}(\ln Y_i)^2}$$

and

$$\widehat{\mu}_2 = e^{\widehat{\alpha}_2\left(\gamma+\frac{1}{n}\sum_{i=1}^{n}(\ln Y_i)\right)}.$$

As an illustation, we estimate the unknown parameters using Monte Carlo method. For different values of the parameters α_1, α_2, μ_1 and μ_2, we simulate 10 sequences of 1000, 5000, 10000 observations following BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution. In Table 2.1, we present the averages and standard deviations of these estimators.

2.6 Autoregressive Processes with BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ Marginals

Gaver and Lewis (1980) developed exponential autoregressive process (EAR(1)) as the solution of the first order autoregressive equation

$$X_n = \rho X_{n-1} + \epsilon_n.$$

Table 2.1. The estimators of the parameters for different values of $\alpha_1, \alpha_2, \mu_1$ and μ_2.

n	$\alpha_1 = 0.2$	$\mu_1 = 2$	$\alpha_2 = 0.4$	$\mu_2 = 4$	$\alpha_1 = .4$	$\mu_1 = 15$
1000	0.2000	2.0271	0.4005	4.0141	0.3911	14.4661
	(0.0039)	(0.1027)	(0.0108)	(0.3228)	(4.8961)	(4.1313)
5000	0.2001	2.0024	0.4021	4.0713	0.3893	14.2291
	(0.0066)	(0.5242)	(0.0085)	(0.0205)	(0.0323)	(4.2531)
10000	0.1998	2.0041	0.4001	4.0009	0.3876	14.2775
	(0.0141)	(0.7456)	(0.1742)	(0.0088)	(0.3621)	(5.0321)
	$\alpha_1 = 0.6$	$\mu_1 = 4$	$\alpha_2 = 0.4$	$\mu_2 = 15$	$\alpha_1 = 0.4$	$\mu_1 = 25$
1000	0.6071	4.0161	0.3911	14.4661	0.3751	21.2812
	(0.0142)	(0.0199)	(0.0321)	(4.8961)	(0.0374)	(10.5741)
5000	0.5971	3.9772	0.3899	14.2293	0.3591	20.0881
	(0.0154)	(0.2523)	(0.0323)	(4.2471)	(0.0508)	(13.8091)
10000	0.6021	4.0012	0.3874	14.2775	0.3712	20.6721
	(0.0116)	(0.1419)	(0.0352)	(5.0321)	(0.0361)	(8.7241)
	$\alpha_1 = 0.2$	$\mu_1 = 12$	$\alpha_2 = 0.6$	$\mu_2 = 6$	$\alpha_1 = 0.6$	$\mu_1 = 12$
1000	0.1951	11.6382	0.6091	6.1207	0.6012	12.2212
	(0.0132)	(2.4531)	(0.0278)	(0.5421)	(0.0541)	(3.4431)
5000	0.1942	11.8014	0.5981	6.0061	0.5911	11.8123
	(0.0136)	(3.1876)	(0.0247)	(0.5532)	(0.0471)	(3.0312)
10000	0.1922	11.5876	0.6052	6.0851	0.5961	12.0881
	(0.0142)	(2.7423)	(0.0221)	(0.5421)	(0.0492)	(3.4213)
	$\alpha_1 = 0.6$	$\mu_1 = 15$	$\alpha_2 = 0.6$	$\mu_2 = 20$	$\alpha_2 = 0.6$	$\mu_2 = 25$
1000	0.5922	14.9612	0.5795	19.0121	0.5662	22.5121
	(0.0598)	(5.4567)	(0.0653)	(9.0564)	(0.0681)	(12.6615)
5000	0.5821	14.3987	0.5691	19.2131	0.5581	21.5112
	(0.0532)	(4.7867)	(0.0592)	(7.9412)	(0.0627)	(11.1321)
10000	0.5887	14.7633	0.5749	18.7131	0.5631	21.7231
	(0.0554)	(5.3554)	(0.0592)	(8.7823)	(0.0581)	(10.2567)
	$\alpha_1 = 0.8$	$\mu_1 = 2$	$\alpha_2 = 0.8$	$\mu_2 = 4$	$\alpha_2 = 0.8$	$\mu_1 = 9$
1000	0.7926	1.9786	0.7971	3.9581	0.8032	9.1321
	(0.0123)	(0.0756)	(0.0161)	(0.1551)	(0.0553)	(1.5993)
5000	0.7943	1.9657	0.7954	3.9234	0.8002	8.9643
	(0.0153)	(0.0478)	(0.0127)	(0.1132)	(0.0483)	(1.2632)
10000	0.7911	1.9517	0.8051	4.0581	0.8107	9.3725
	(0.0312)	(0.0725)	(0.0068)	(0.0871)	(0.0453)	(1.4076)
	$\alpha_1 = 0.8$	$\mu_1 = 9$	$\alpha_2 = 0.8$	$\mu_2 = 12$	$\alpha_2 = 0.8$	$\mu_2 = 25$
1000	0.8029	9.1342	0.7961	12.1324	0.7564	22.8571
	(0.8032)	(1.5910)	(0.0698)	(3.3469)	(0.0915)	(13.1954)
5000	0.8003	8.9744	0.7945	11.8692	0.7547	22.2315
	(0.0487)	(1.2443)	(0.0632)	(2.7856)	(0.0872)	(11.7345)
10000	0.8109	9.3732	0.8034	12.4121	0.7609	23.1074
	(0.0481)	(1.4341)	(0.0648)	(3.0183)	(0.0841)	(12.2243)

where $0 \leq \rho < 1$, $\{\epsilon_n, n \geq 1\}$ is a sequence of independently and identically distributed exponential random variables and X_n's have marginally exponential distribution. Later, Jayakumar and Pillai (1993) developed first order Mittag-Leffler autoregressive process (MLAR(1)) as a generalization of the EAR(1) process. The MLAR(1) process has structure,

$$X_n = \begin{cases} \rho X_{n-1} & \text{with probability } \rho^\alpha \\ \rho X_{n-1} + \epsilon_n & \text{with probability } 1 - \rho^\alpha \end{cases}$$

where $0 < \alpha \leq 1$. Mundassery and Jayakumar (2007b) introduced a bivariate first order autoregressive process with BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ marginals.

Theorem 2.10. *Consider a first order autoregressive process $\{(X_n, Y_n),\ n \geq 1\}$ with structure*

$$(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1) \quad \textit{and} \quad \textit{for } n = 1,2,3,\ldots$$

$$(X_n, Y_n) = \begin{cases} (\rho^{\frac{1}{\alpha_1}} X_{n-1}, \rho^{\frac{1}{\alpha_2}} Y_{n-1}) & \textit{with probability } \rho \\ (\rho^{\frac{1}{\alpha_1}} X_{n-1} + \epsilon_n, \rho^{\frac{1}{\alpha_2}} Y_{n-1} + \psi_n) & \textit{with probability } 1 - \rho \end{cases} \tag{2.25}$$

where $0 \leq \rho < 1$, $0 < \alpha_1, \alpha_2 \leq 1$ and $\{(\epsilon_n, \psi_n), n \geq 1\}$ is a sequence of independently and identically distributed random vectors. Then $\{(X_n, Y_n), n \geq 1\}$ represents a stationary first order autoregressive process with BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ marginals if and only if $(\epsilon_n, \psi_n), n \geq 1$, are distributed according to BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$.

Proof. The Laplace transform of the process in (2.25) is

$$\phi_{X_n,Y_n}(\lambda_1, \lambda_2)$$
$$= \rho\phi_{X_{n-1},Y_{n-1}}(\rho^{\frac{1}{\alpha_1}}\lambda_1, \rho^{\frac{1}{\alpha_2}}\lambda_2) + (1-\rho)\phi_{X_{n-1},Y_{n-1}}(\rho^{\frac{1}{\alpha_1}}\lambda_1, \rho^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon_n,\psi_n}(\lambda_1, \lambda_2). \tag{2.26}$$

When the process is stationary with BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ marginals, from (2.26) we have

$$\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}} = \frac{\rho}{1+\rho\mu_1\lambda_1^{\alpha_1}+\rho\mu_2\lambda_2^{\alpha_2}} + \frac{1-\rho}{1+\rho\mu_1\lambda_1^{\alpha_1}+\rho\mu_2\lambda_2^{\alpha_2}}\phi_{\epsilon,\psi}(\lambda_1,\lambda_2).$$

On simplification, we get

$$\phi_{\epsilon,\psi}(\lambda_1,\lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

Conversely, suppose that (ϵ_n, ψ_n), $n \geq 1$ are distributed according to BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ and $(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1)$. Choose n=1. From (2.26), we have

$$\phi_{X_1,Y_1}(\lambda_1,\lambda_2) = \rho\phi_{X_0,Y_0}(\rho^{\frac{1}{\alpha_1}}\lambda_1, \rho^{\frac{1}{\alpha_2}}\lambda_2) + (1-\rho)\phi_{X_0,Y_0}(\rho^{\frac{1}{\alpha_1}}\lambda_1, \rho^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon_1,\psi_1}(\lambda_1,\lambda_2).$$

Substituting the Laplace transform of (ϵ_1, ψ_1)

$$\begin{aligned}\phi_{X_1,Y_1}(\lambda_1,\lambda_2) &= \frac{\rho}{1+\rho\mu_1\lambda_1^{\alpha_1}+\rho\mu_2\lambda_2^{\alpha_2}} + \frac{1-\rho}{1+\rho\mu_1\lambda_1^{\alpha_1}+\rho\mu_2\lambda_2^{\alpha_2}}\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}} \\ &= \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.\end{aligned}$$

Hence by mathematical induction, the process is stationary with BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ marginals. $\square$

Now we obtain a generalization of the first order autoregressive process developed in (2.25). Consider an autoregressive process with structure:

$$(X_n, Y_n) = \begin{cases} (\epsilon_n, \psi_n), & \text{with probability } 1-p \\ (p^{\frac{1}{\alpha_1}}X_{n-1}, p^{\frac{1}{\alpha_2}}Y_{n-1}), & \text{with probability } p(1-q) \\ (p^{\frac{1}{\alpha_1}}X_{n-1}+\epsilon_n, p^{\frac{1}{\alpha_2}}Y_{n-1}+\psi_n), & \text{with probability } (1-p)(1-q) \end{cases} \tag{2.27}$$

where $\{(\epsilon_n, \psi_n), n \geq 1\}$ is a sequence of independently and identically distributed random vectors, (X_{n-1}, Y_{n-1}) and (ϵ_n, ψ_n) are independent random vectors and $0 < p < 1$, $q = 1 - p$ and $0 < \alpha_1, \alpha_2 \leq 1$. Note that for $q = 0$, we get the first order autoregressive process discussed in (2.25).

The following theorem gives a necessary and sufficient condition for the stationarity of the process in (2.27).

Theorem 2.11. *Let* $(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1)$. *The process* $\{(X_n, Y_n), n \geq 1\}$ *defined in (2.27) is stationary with* $BML(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ *marginals if and only if* $\{(\epsilon_n, \psi_n), n \geq 1\}$ *is a sequence of independently and identically distributed random vectors according to* $BML(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ *distribution.*

Proof. The Laplace transform of (2.27) is

$$\begin{aligned}
\phi_{X_n,Y_n}(\lambda_1, \lambda_2) &= q\phi_{\epsilon_n,\psi_n}(\lambda_1, \lambda_2) \\
&\quad + p(1-q)\phi_{X_{n-1},Y_{n-1}}(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2) \\
&\quad + (1-p)(1-q)\phi_{X_{n-1},Y_{n-1}}(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon_n,\psi_n}(\lambda_1, \lambda_2). \quad (2.28)
\end{aligned}$$

When the process is stationary, we have

$$\begin{aligned}
\phi_{X,Y}(\lambda_1, \lambda_2) = q\phi_{\epsilon,\psi}(\lambda_1, \lambda_2) &+ p(1-q)\phi_{X,Y}(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2) \\
&+ (1-p)(1-q)\phi_{X,Y}(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon,\psi}(\lambda_1, \lambda_2).
\end{aligned}$$

Assume that $\phi_{X,Y}(\lambda_1, \lambda_2)$ corresponds to BML$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution. Substituting and simplifying, we get

$$\phi_{\epsilon,\psi}(\lambda_1. \lambda_2) = \frac{1}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}}.$$

Proof of the converse is obtained by mathematical induction. Suppose that $(\epsilon_n, \psi_n), n \geq 1$ follow BML$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution and $(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1)$. Put $n = 1$ in (2.28) we obtain

$$\begin{aligned}\phi_{X_1,Y_1}(\lambda_1, \lambda_2) = q\phi_{\epsilon_1,\psi_1}(\lambda_1, \lambda_2) &+ p(1-q)\phi_{X_0,Y_0}(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2) \\ &+ (1-p)(1-q)\phi_{X_0,Y_0}(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon_1,\psi_1}(\lambda_1, \lambda_2).\end{aligned}$$

Under the assumption,

$$\begin{aligned}\phi_{X_1,Y_1}(\lambda_1, \lambda_2) = \frac{q}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}} &+ \frac{p(1-q)}{1+p(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2})} \\ &+ \frac{(1-p)(1-q)}{1+p(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2})}\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.\end{aligned}$$

On simplification, we get

$$\phi_{X_1,Y_1}(\lambda_1, \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

Hence by mathematical induction we get that the process is stationary with BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ marginals. □

As a remark, we obtain a bivariate first order autoregressive process with MBE $(\mu_1, \mu_2, 1)$ marginals .

Remark 2.1. Consider a first order autoregressive process $(X_n, Y_n), n \geq 1$ with following structure:

$$(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1) \text{ and for } n = 1, 2, 3, \ldots$$

$$(X_n, Y_n) = \begin{cases} (\epsilon_n, \psi_n), & \text{with probability } q \\ (pX_{n-1}, pY_{n-1}), & \text{with probability } (1-q)p \\ (pX_{n-1}+\epsilon_n, pY_{n-1}+\psi_n), & \text{with probability } (1-p)(1-q) \end{cases}$$

Then the process $\{(X_n, Y_n), n \geq 1\}$ is stationary with MBE $(\mu_1, \mu_2, 1)$ marginals if and only if the innovations $(\epsilon_n, \psi_n), n \geq 1$ follow MBE$(\mu_1, \mu_2, 1)$.

Proof of the Remark 2.1 is omitted since we can easily deduce from Theorem 2.11.

A first order autoregressive process $\{X_n, n \geq 1\}$, called TEAR(1), is introduced in Lawrance and Lewis (1980). Its structure is

$$X_n = \begin{cases} \rho\epsilon_n & \text{with} \quad \text{probability } \rho \\ X_{n-1} + \rho\epsilon_n & \text{with} \quad \text{probability } 1-\rho \end{cases} \tag{2.29}$$

where $\{\epsilon_n, n \geq 1\}$ is a sequence of independently and identically distributed random variables and $0 \leq \rho < 1$. Using this model, in the following theorem we develop a first order autoregressive process having BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ marginals.

Theorem 2.12. *Let a first order autoregressive process $\{(X_n, Y_n),\ n \geq 1\}$ have the structure*

$$(X_n, Y_n) = \begin{cases} (\rho^{\frac{1}{\alpha_1}}\epsilon_n, \rho^{\frac{1}{\alpha_2}}\psi_n) & \textit{with probability } \rho \\ (X_{n-1} + \rho^{\frac{1}{\alpha_1}}\epsilon_n, Y_{n-1} + \rho^{\frac{1}{\alpha_2}}\psi_n) & \textit{with probability } 1-\rho \end{cases} \tag{2.30}$$

where $\{(\epsilon_n, \psi_n), n \geq 1\}$ is a sequence of independently and identically distributed random vectors. Then $\{(X_n, Y_n), n \geq 1\}$ is a first order stationary autoregressive process with BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ marginals if and only if (ϵ_n, ψ_n), $n \geq 1$ are distributed according to BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$, provided $(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1)$.

Proof. The Laplace transform of (2.30) is

$$\phi_{X_n,Y_n}(\lambda_1,\lambda_2)=\rho\phi_{\epsilon_n,\psi_n}(\rho^{\frac{1}{\alpha_1}}\lambda_1,\rho^{\frac{1}{\alpha_2}}\lambda_2)+(1-\rho)\phi_{X_{n-1},Y_{n-1}}(\lambda_1,\lambda_2)\phi_{\epsilon_n,\psi_n}(\rho^{\frac{1}{\alpha_1}}\lambda_1,\rho^{\frac{1}{\alpha_2}}\lambda_2). \tag{2.31}$$

When the process is stationary,

$$\phi_{X,Y}(\lambda_1,\lambda_2)=\rho\phi_{\epsilon,\psi}(\rho^{\frac{1}{\alpha_1}}\lambda_1,\rho^{\frac{1}{\alpha_2}}\lambda_2)+(1-\rho)\phi_{X,Y}(\lambda_1,\lambda_2)\phi_{\epsilon,\psi}(\rho^{\frac{1}{\alpha_1}}\lambda_1,\rho^{\frac{1}{\alpha_2}}\lambda_2).$$

Suppose that the process has BML $(\mu_1,\mu_2,\alpha_1,\alpha_2,1)$ marginals. Then, we get

$$\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}=\rho\phi_{\epsilon,\psi}(\rho^{\frac{1}{\alpha_1}}\lambda_1,\rho^{\frac{1}{\alpha_2}}\lambda_2)+\frac{1-\rho}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}\phi_{\epsilon,\psi}(\rho^{\frac{1}{\alpha_1}}\lambda_1,\rho^{\frac{1}{\alpha_2}}\lambda_2).$$

On simplification, we get

$$\phi_{\epsilon,\psi}(\lambda_1,\lambda_2)=\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}$$

To prove the converse we use induction method. Suppose that $(\epsilon_n,\psi_n), n\geq 1$ are distributed according to BML $(\mu_1,\mu_2,\alpha_1,\alpha_2,1)$ and $(X_0,Y_0)\stackrel{d}{=}(\epsilon_1,\psi_1)$.
Put n=1 in (2.31), we get

$$\phi_{X_1,Y_1}(\lambda_1,\lambda_2)=\rho\phi_{\epsilon_1,\psi_1}(\rho^{\frac{1}{\alpha_1}}\lambda_1,\rho^{\frac{1}{\alpha_2}}\lambda_2)+(1-\rho)\phi_{X_0,Y_0}(\lambda_1,\lambda_2)\phi_{\epsilon_1,\psi_1}(\rho^{\frac{1}{\alpha_1}}\lambda_1,\rho^{\frac{1}{\alpha_2}}\lambda_2).$$

Substituting the Laplace transform of (ϵ_1,ψ_1)

$$\begin{aligned}\phi_{X_1,Y_1}(\lambda_1,\lambda_2) &= \frac{\rho}{1+\rho\mu_1\lambda_1^{\alpha_1}+\rho\mu_2\lambda_2^{\alpha_2}}+\frac{1-\rho}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}\frac{1}{1+\rho\mu_1\lambda_1^{\alpha_1}+\rho\mu_2\lambda_2^{\alpha_2}}\\ &= \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.\end{aligned}$$

Hence by mathematical induction, we get the process $\{(X_n,Y_n),\ n\geq\ 1\}$ is stationary with BML $(\mu_1,\mu_2,\alpha_1,\alpha_2,1)$ marginals. □

Lawrance and Lewis (1981) developed a first order new exponential autoregressive process (NEAR(1)) that will generalize both the EAR(1)and TEAR(1) processes. The structure of the NEAR(1) process is

$$X_n = \epsilon_n + \begin{cases} \beta X_{n-1} & \text{with probability } \rho \\ 0 & \text{with probability } 1-\rho \end{cases}$$

where $0 \leq \beta \leq 1$ and $0 \leq \rho \leq 1$. $\{\epsilon_n, n \geq 1\}$ is a sequence of independently and identically distributed random variables and could be generated as follows:

$$\epsilon_n = \begin{cases} E_n & \text{with probability } \frac{1-\beta}{1-(1-\rho)\beta} \\ (1-\rho)\beta E_n & \text{with probability } \frac{\rho\beta}{1-(1-\rho)\beta} \end{cases}$$

Note that for ρ=1, we get the EAR(1) process given in (1.20) and for $\beta = 1$, we have the TEAR(1) process given in (2.29).

Now, consider a first order autoregressive process with following structure.

$$(X_n, Y_n) = (\epsilon_n, \psi_n) + \begin{cases} (\beta^{\frac{1}{\alpha_1}} X_{n-1}, \beta^{\frac{1}{\alpha_2}} Y_{n-1}) & \text{with probability } \rho \\ 0 & \text{with probability } 1-\rho \end{cases}$$

where $\{(\epsilon_n, \psi_n), n \geq 1\}$ is a sequence of random vectors distributed according to BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$. $\{\epsilon_n, \psi_n, n \geq 1\}$ is defined as follows:

$$(\epsilon_n, \psi_n) = \begin{cases} (E_n. F_n) & \text{with probability } \frac{1-\beta}{1-(1-\rho)\beta} \\ ((1-\rho)^{\frac{1}{\alpha_1}}\beta E_n, (1-\rho)^{\frac{1}{\alpha_2}}\beta F_n) & \text{with probability } \frac{\rho\beta}{1-(1-\rho)\beta} \end{cases}.$$

When ρ=1 we get bivariate Mittag-Leffler autoregressive process developed in (2.25) and while β=1, it is the bivariate Mittag-Leffler autoregressive process given in (2.30).

2.7 Bivariate Mittag-Leffler Distributions Generated through Bivariate Geometric Compounding

In this Section, we introduce the bivariate Mittag-Leffler forms of some important bivariate exponential distributions (see Jayakumar and Mundassery (2006)). Marshall-Olkin (1967) obtained a bivariate exponential distribution which can be treated as a shock model. Consider a two component system which are subjected to fatal shocks. These shocks follow independent Poisson process with parameters δ_1, δ_2 and δ_{12} according as the shocks applied to component 1 only, component 2 only or both components respectively. Then the joint survival function of the life times of the components denoted by (X_1, X_2) is

$$\overline{F}(x_1, x_2) = e^{-\delta_1 x_1 - \delta_2 x_2 - \delta_{12} max(x_1, x_2)} \quad x_1,\ x_2 > 0, \quad \delta_1, \delta_2, \delta_{12}\ > 0.$$

The Laplace transform of Marshall-Olkin's bivariate exponential distribution is

$$\phi(\lambda_1,\ \lambda_2) = \frac{(\delta + \lambda_1 + \lambda_2)(\delta_1 + \delta_{12})(\delta_2 + \delta_{12}) + \lambda_1 \lambda_2 \delta_{12}}{(\delta + \lambda_1 + \lambda_2)(\delta_1 + \delta_{12} + \lambda_1)(\delta_2 + \delta_{12} + \lambda_2)} \tag{2.32}$$

where $\delta = \delta_1 + \delta_2 + \delta_{12}$.

We obtain a generalization of Marshall-Olkin's bivariate exponential distribution.

Theorem 2.13. *Consider a sequence* $\{(X_i, Y_i), i \geq 1\}$ *of independently and identically distributed BML* $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ *random vectors. Let* $U_{N_1} = \sum_{i=1}^{N_1} X_i$ *and* $V_{N_2} = \sum_{i=1}^{N_2} Y_i$ *where* (N_1, N_2) *has bivariate geometric distribution in (1.15) and independent of* $(X_i, Y_i), i \geq 1$. *Choose* $p_{00} = \delta_{12},\ p_{10} = \delta_2,\ p_{01} = \delta_1$ *and* $p_{11} = 1 - \delta$ *where*

$\delta = \delta_1 + \delta_2 + \delta_{12}$. *Then the distribution of* (U_{N_1}, V_{N_2}) *is the bivariate Mittag-Leffler generalization of the Marshall-Olkin's bivariate exponential distribution* .

Proof. Assume that $(X_i, Y_i), i \geq 1$ have the Laplace transform

$$\psi(\lambda_1, \lambda_2) = \frac{1}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}}.$$

Substituting p_{00}, p_{10}, p_{01}, p_{11} and $\psi(\lambda_1, \lambda_2)$ in (2.17), we get

$$\phi(\lambda_1, 0) = \frac{\delta_{12} + \delta_1}{\delta_{12} + \delta_1 + \mu_1\lambda_1^{\alpha_1}} \text{ and } \phi(0, \lambda_2) = \frac{\delta_{12} + \delta_2}{\delta_{12} + \delta_2 + \mu_2\lambda_2^{\alpha_2}}.$$

The Laplace transform of (U_{N_1}, V_{N_2}) is obtained by substituting $\phi(\lambda_1, 0)$ and $\phi(0, \lambda_2)$ in (2.16),

$$\phi(\lambda_1, \lambda_2) = \frac{1}{\delta + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}} \left(\delta_{12} + \frac{\delta_2(\delta_{12} + \delta_1)}{\delta_{12} + \delta_1 + \mu_1\lambda_1^{\alpha_1}} + \frac{\delta_1(\delta_{12} + \delta_2)}{\delta_{12} + \delta_2 + \mu_2\lambda_2^{\alpha_2}} \right).$$

On simplification, we get

$$\phi(\lambda_1, \lambda_2) = \frac{(\delta + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2})(\delta_1 + \delta_{12})(\delta_2 + \delta_{12}) + \mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\delta_{12}}{(\delta + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2})(\delta_1 + \delta_{12} + \mu_1\lambda_1^{\alpha_1})(\delta_2 + \delta_{12} + \mu_2\lambda_2^{\alpha_2})}. \tag{2.33}$$

When $\alpha_1 = \alpha_2 = 1$, $\phi(\lambda_1, \lambda_2)$ coincides with the Laplace transform of Marshall-Olkin's bivariate exponential distribution in (2.32). □

Hawkes (1972) obtained a bivariate exponential distribution which describes the failure time of a system having two components. Suppose that the two components are subjected to non fatal shocks occurring in independent Poisson fashion with parameters δ_1 and δ_2 $(\delta_1, \delta_2 > 0)$. Let the number of shocks needed to cause failure of these components have the bivariate geometric distribution with p.g.f. $P(s_1, s_2)$.

Then the waiting time for failure of the components, denoted by (X,Y), is given by the random sum

$$(X,Y)=\left(\sum_{j=1}^{N_1}X_{1j},\ \sum_{j=1}^{N_2}X_{2j}\right) \tag{2.34}$$

where $X_{i,j}$, j=1,2,3,... denote the inter arrival time of the i^{th} process, i=1.2. Then (X,Y) has the Laplace transform

$$\phi(\lambda_1,\lambda_2)=P\left(\frac{\delta_1}{\delta_1+\lambda_1},\frac{\delta_2}{\delta_2+\lambda_2}\right).$$

Hawkes (1972) considered the p.g.f.

$$P(s_1,s_2)=\frac{s_1s_2}{1-p_{00}s_1s_2}\left(p_{11}+\frac{p_{10}P_2s_2}{1-Q_2s_2}+\frac{p_{01}P_1s_1}{1-Q_1s_1}\right) \tag{2.35}$$

where $P_1=p_{11}+p_{10}$, $P_2=p_{11}+p_{01}$, $Q_1=p_{00}+p_{01}$. $Q_2=p_{00}+p_{10}$ and obtained a bivariate exponential distribution as bivariate geometric sum of independently and identically distributed random vectors. The Laplace transform of Hawkes' bivariate exponential distribution is

$$\phi(\lambda_1,\lambda_2=\frac{m_1m_2}{(m_1+\lambda_1)(m_2+\lambda_2)}\left(1+\frac{[p_{00}-(1-P_1)(1-P_2)]\lambda_1\lambda_2}{(m_1+P_1\lambda_1)(m_2+P_2\lambda_2)-p_{00}m_1m_2}\right), \tag{2.36}$$

$$m_i=\delta_iP_i,\ i=1,2.$$

In the following theorem we derive the bivariate Mittag-Leffler form of (2.36).

Theorem 2.14. *Let $\{(X_i,Y_i),i\geq 1\}$ be a sequence of independently and identically distributed random vectors such that each component has Mittag-Leffler distribution with Laplace transforms $\psi_{X_i}(\lambda_1)=\frac{\delta_1}{\delta_1+\lambda_1^{\alpha_1}}$ and $\psi_{Y_i}(\lambda_2)=\frac{\delta_2}{\delta_2+\lambda_2^{\alpha_2}}$, $\delta_1,\delta_2>0$, i=1,2,... respectively. Then the distribution of (X,Y) defined in (2.34) gives a generalization to (2.36) when (N_1,N_2) has the bivariate geometric distribution with p.g.f. in (2.35).*

Proof. Suppose that $(X_i, Y_i), i \geq 1$ are independently and identically distributed such that

$$\psi_{X_i}(\lambda_1) = \frac{\delta_1}{\delta_1 + \lambda_1^{\alpha_1}} \quad \text{and} \quad \psi_{Y_i}(\lambda_2) = \frac{\delta_2}{\delta_2 + \lambda_2^{\alpha_2}},\ i = 1, 2, ...$$

Let the joint Laplace transform of $(X_i, Y_i), i \geq 1$ be $\psi(\lambda_1, \lambda_2)$. From (2.35) the Laplace transform of (X, Y) is

$$\phi(\lambda_1, \lambda_2) = \frac{\psi(\lambda_1, \lambda_2)}{1 - p_{00}\psi(\lambda_1, \lambda_2)} \left(p_{11} + \frac{p_{10}P_2\psi_{Y_i}(\lambda_2)}{1 - Q_2\psi_{Y_i}(\lambda_2)} + \frac{p_{01}P_1\psi_{X_i}(\lambda_1)}{1 - Q_1\psi_{X_i}(\lambda_1)} \right).$$

Substituting $\psi_{X_i}(\lambda_1)$, $\psi_{Y_i}(\lambda_2)$ and $\psi(\lambda_1, \lambda_2)$

$$\begin{aligned} \phi(\lambda_1, \lambda_2) &= \frac{\delta_1\delta_2}{(\delta_1 + \lambda_1^{\alpha_1})(\delta_2 + \lambda_2^{\alpha_2}) - \delta_1\delta_2 p_{00}} \left(p_{11} + \frac{p_{10}P_2\delta_2}{P_2\delta_2 + \lambda_2^{\alpha_2}} + \frac{p_{01}P_1\delta_1}{P_1\delta_1 + \lambda_1^{\alpha_1}} \right) \\ &= \frac{m_1 m_2}{(m_1 + P_1\lambda_1^{\alpha_1})(m_2 + P_2\lambda_2^{\alpha_2}) - m_1 m_2 p_{00}} \left(p_{11} + \frac{p_{10}m_2}{m_2 + \lambda_2^{\alpha_2}} + \frac{p_{01}m_1}{m_1 + \lambda_1^{\alpha_1}} \right) \end{aligned}$$

where $m_i = \delta_i P_i$, i=1,2.

$$= \frac{m_1 m_2}{(m_1 + \lambda_1^{\alpha_1})(m_2 + \lambda_2^{\alpha_2})} \left(\frac{p_{11}(m_1 + \lambda_1^{\alpha_1})(m_2 + \lambda_2^{\alpha_2}) + p_{10}m_2(m_1 + \lambda_1^{\alpha_1}) + p_{01}m_1(m_2 + \lambda_2^{\alpha_2})}{(m_1 + P_1\lambda_1^{\alpha_1})(m_2 + P_2\lambda_2^{\alpha_2}) - m_1 m_2 p_{00}} \right).$$

On simplification,

$$\phi(\lambda_1, \lambda_2) = \frac{m_1 m_2}{(m_1 + \lambda_1^{\alpha_1})(m_2 + \lambda_2^{\alpha_2})} \left(1 + \frac{[p_{00} - (1 - P_1)(1 - P_2)]\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}}{(m_1 + P_1\lambda_1^{\alpha_1})(m_2 + P_2\lambda_2^{\alpha_2}) - p_{00}m_1 m_2} \right). \tag{2.37}$$

Hence when $\alpha_1 = \alpha_2 = 1$ we get (2.36) $\square$

Paulson (1973) obtained a bivariate exponential distribution using the bivariate geometric compounding given in (2.15). Choosing $p_{00} = a$, $p_{10} = b$, $p_{01} = c$, $p_{11} = d$

and

$$\psi(\lambda_1, \lambda_2) = \frac{1}{(1 + \theta_1\lambda_1)(1 + \theta_2\lambda_2)},$$

$\theta_1, \theta_2 > 0,\ \ 0 < a,\ b,\ c,\ d < 1, a + b + c + d = 1, b + d\ < 1,\ \ c + d\ < 1.$

From (2.15),

$$\phi(\lambda_1, 0) = \frac{1}{1 + \delta_1\lambda_1} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{1}{1 + \delta_2\lambda_2}$$

where $\delta_1 = \dfrac{\theta_1}{a + c}$ and $\delta_2 = \dfrac{\theta_2}{a + b}$. This implies that marginal distributions have exponential distribution. Hence $\phi(\lambda_1, \lambda_2)$ represents the Laplace transform of a bivariate exponential distribution. We obtain a bivariate Mittag-Leffler distribution that generalizes the Paulson's (1973) bivariate exponential distribution.

Choose p_{00}, p_{10}, p_{01}, and p_{11} as before and

$$\psi(\lambda_1, \lambda_2) = \frac{1}{(1 + \mu_1\lambda_1^{\alpha_1})(1 + \mu_2\lambda_2^{\alpha_2})}.$$

From (2.15), we get

$$\phi(\lambda_1, \lambda_2) = \frac{1}{(1 + \mu_1\lambda_1^{\alpha_1})(1 + \mu_2\lambda_2^{\alpha_2})}\left(a + \frac{b(a + c)}{a + c + \mu_1\lambda_1^{\alpha_1}} + \frac{c(b + d)}{b + d + \mu_2\lambda_2^{\alpha_2}} + d\phi(\lambda_1, \lambda_2)\right).$$

Hence

$$\phi(\lambda_1, 0) = \frac{1}{1 + \delta_1\lambda_1^{\alpha_1}} \quad \text{and} \quad \phi(0, \lambda_2) = \frac{1}{1 + \delta_2\lambda_2^{\alpha_2}}$$

where $\delta_1 = \dfrac{\mu_1}{a + c}$ and $\delta_2 = \dfrac{\mu_2}{a + b}$.

2.8 Bivariate Semi Mittag-Leffler Distribution

In the following definition we introduce a bivariate semi Mittag-Leffler distribution (see Mundassery and Jayakumar (2007c)).

Definition 2.3. A non negative random vector (X, Y) is said to follow bivariate semi Mittag-Leffler distribution, denoted by BSML (α_1, α_2, p), if its Laplace transform is

$$\phi(\lambda_1, \lambda_2) = \frac{1}{1+\xi(\lambda_1, \lambda_2)} \tag{2.38}$$

where $\xi(\lambda_1, \lambda_2)$ satisfies the functional equation

$$\xi(\lambda_1, \lambda_2) = \frac{1}{p}\xi(p^{\frac{1}{\alpha_1}}\lambda_1,\ p^{\frac{1}{\alpha_2}}\lambda_2),\ 0 < p < 1,\ 0 < \alpha_1, \alpha_2 \leq 1.$$

A solution of this functional equation is given by

$$\xi(\lambda_1, \lambda_2) = \lambda_1^{\alpha_1} h_1(\lambda_1) + \lambda_2^{\alpha_2} h_2(\lambda_2). \tag{2.39}$$

When $h_i(\lambda_i) = 1$ for $i = 1, 2$, we get BML $(1, 1, \alpha_1, \alpha_2, 1)$ distribution.

Using the geometric compounding stated in (2.9), we now obtain a characterization of BSML (α_1, α_2, p).

Theorem 2.15. *Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors. Define (U_N, V_N) as $U_N = \sum_{i=1}^{N} X_i$ and $V_N = \sum_{i=1}^{N} Y_i$ where N is independent of $(X_i, Y_i), i \geq 1$ and has the geometric distribution such that $P(N = n) = (1-p)^{n-1}p,\ \ 0 < p < 1,\ \ n = 1,2,3,\ldots$ Then $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ is distributed as BSML (α_1, α_2, p) if and only if $(X_i, Y_i), i \geq 1$ follow BSML (α_1, α_2, p) distribution.*

Proof. Suppose that $(X_i, Y_i), i \geq 1$ follows BSML (α_1, α_2, p). Substituting its Laplace transform $\psi(\lambda_1,\ \lambda_2)$, in (2.10) and on simplification the Laplace transform of $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ becomes

$$\phi(\lambda_1, \lambda_2) = \frac{1}{1 + \xi(\lambda_1, \lambda_2)}.$$

Conversely, suppose that $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ follows BSML (α_1, α_2, p) distribution. From (2.10),

$$\frac{1}{1 + \xi(\lambda_1, \lambda_2)} = \frac{p\psi(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)}{1 - (1-p)\psi(p^{\frac{1}{\alpha_1}}\lambda, p^{\frac{1}{\alpha_2}}\lambda_2)}.$$

Solving, we get

$$\psi(\lambda_1,\ \lambda_2) =) = \frac{1}{1 + \xi(\lambda_1, \lambda_2)}.$$

□

Now by repeated geometric compounding, we have the characterization of BSML (α_1, α_2, p).

Theorem 2.16. *Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors with distribution function $F(x, y)$ and Laplace transform $\phi(\lambda_1, \lambda_2)$. Suppose that N_{k-1} follows geometric distribution such that*

$$P(N_{k-1} = n) = (1 - p_{k-1})^{n-1} p_{k-1},\ 0 < p_{k-1} < 1,\quad n = 1, 2, 3, ...$$

and N_{k-1} is independent of $(X_i, Y_i), i \geq 1$. Define (U_{N_k}, V_{N_k}) as

$$U_{N_k} = p_{k-1}^{\frac{1}{\alpha_1}} \sum_{i=1}^{N_{k-1}} X_i \qquad \textit{and} \qquad V_{N_k} = p_{k-1}^{\frac{1}{\alpha_2}} \sum_{i=1}^{N_{k-1}} Y_i$$

where $(X_i, Y_i), i \geq 1$ *are the summands of the* $(k-1)^{th}$ *stage of compounding and independently and identically distributed according to* $F_{k-1}(.,.)$, $k = 2, 3, ...$ *For the initial stage, choose* $F_1(.,.) = F(.,.)$. *Then* $F_k(.,.)$, *the distribution of* (U_{N_k}, V_{N_k}), *and* $F(.,.)$ *are BSML* (α_1, α_2, p).

Proof of the theorem easily follows by the recursive application of (2.11).

Now, we obtain the bivariate extension of the semi Mittag-Leffler process developed in Jayakumar and Pillai (1993).

Theorem 2.17. *Let* $\{(X_n, Y_n),\ n \geq 1\}$ *constitute a first order autoregressive process with structure*

$$(X_n, Y_n) = \begin{cases} (\rho^{\frac{1}{\alpha_1}} X_{n-1}, \rho^{\frac{1}{\alpha_2}} Y_{n-1}) & \textit{with probability } \rho \\ (\rho^{\frac{1}{\alpha_1}} X_{n-1} + \epsilon_n, \rho^{\frac{1}{\alpha_2}} Y_{n-1} + \psi_n) & \textit{with probability } 1-\rho \end{cases} \tag{2.40}$$

where $\{(\epsilon_n, \psi_n), n \geq 1\}$ *is a sequence of independently and identically distributed random vectors. Then* $\{(X_n, Y_n), n \geq 1\}$ *defines a stationary first order autoregressive process with BSML* $(\alpha_1, \alpha_2, \rho)$ *marginals if and only if* (ϵ_n, ψ_n), $n \geq 1$ *are distributed as BSML* $(\alpha_1, \alpha_2, \rho)$, *provided* $(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1)$.

Proof of the theorem can be obtained by proceeding with arguments similar to that of Theorem 2.10.

Now, we develop a first order stationary autoregressive process with BSML (α_1, α_2, p) marginals that generalizes the process stated in (2.40).

Theorem 2.18. *Suppose that a bivariate first order autoregressive process* $(X_n, Y_n), n \geq 1$ *have the following structure:*

$$(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1) \text{ and for } n = 1, 2, 3, ...$$

$$(X_n, Y_n) = \begin{cases} (\epsilon_n, \psi_n), & \text{with probability } q \\ (p^{\frac{1}{\alpha_1}} X_{n-1}, p^{\frac{1}{\alpha_2}} Y_{n-1}), & \text{with probability } (1-q)p \\ (p^{\frac{1}{\alpha_1}} X_{n-1} + \epsilon_n, p^{\frac{1}{\alpha_2}} Y_{n-1} + \psi_n), & \text{with probability } (1-p)(1-q) \end{cases} \quad (2.41)$$

Then the process given in (2.41), is stationary with marginals BSML (α_1, α_2, p) *distribution if and only if* $\{(\epsilon_n, \psi_n), n \geq 1\}$ *is a sequence of independently and identically distributed random vectors according to BSML* (α_1, α_2, p) .

Proof of the theorem is omitted as it is obvious.

In the following theorem we develop a first order autoregressive process with BSML $(\alpha_1, \alpha_2, \rho)$ marginals, along the lines of the TEAR(1) process that discussed in Lawrance and Lewis (1980).

Theorem 2.19. *Consider a first order autoregressive process* $\{(X_n, Y_n), n \geq 1\}$ *with structure*

$$(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1)$$

$$(X_n, Y_n) = \begin{cases} (\rho^{\frac{1}{\alpha_1}} \epsilon_n, \rho^{\frac{1}{\alpha_2}} \psi_n) & \text{with probability } \rho \\ (X_{n-1} + \rho^{\frac{1}{\alpha_1}} \epsilon_n, Y_{n-1} + \rho^{\frac{1}{\alpha_2}} \psi_n) & \text{with probability } 1 - \rho \end{cases}$$

where $\{(\epsilon_n, \psi_n), n \geq 1\}$ *is a sequence of independently and identically distributed random vectors. Then* $\{(X_n, Y_n), n \geq 1\}$ *defines a stationary first order autoregressive* *process with BSML* $(\alpha_1, \alpha_2, \rho)$ *marginals if and only if* $\{(\epsilon_n, \psi_n),\ n \geq 1\}$ *are distributed as BSML* $(\alpha_1, \alpha_2, \rho)$.

Proof of the theorem follows easily.

A first order autoregressive process that will generalize the processes, mentioned in Theorem 2.17 and Theorem 2.19, could be developed along the lines of the NEAR(1) process discussed in Lawrance and Lewis (1981).

Chapter 3

A Generalization of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$

3.1 Introduction

As a generalization to Mittag-Leffler distribution, Jayakumar and Gadag (1999) introduced quasi factorial gamma distribution in (1.9). This family of distributions includes important distributions like gamma, Mittag-Leffler, exponential, etc. In this chapter we introduce bivariate extension of quasi factorial gamma distribution and study its properties. We will also obtain bivariate quasi factorial gamma as the distribution of sum of independently and identically distributed random vectors when the number of summands is treated as random.

We introduce a bivariate quasi factorial gamma distribution in the following definition.

Definition 3.1. A non negative random vector (X, Y) is said to follow bivariate quasi factorial gamma distribution if its Laplace transform is

$$\phi(\lambda_1, \lambda_2) = \left(\frac{1}{(1+\mu_1\lambda_1^{\alpha_1})(1+\mu_2\lambda_2^{\alpha_2}) - \theta\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}}\right)^v, \tag{3.1}$$

$$\lambda_1, \lambda_2 \geq 0, \quad \mu_1, \mu_2 > 0, \quad v > 0, \quad 0 \leq \theta \leq 1, \quad 0 < \alpha_1, \alpha_2 \leq 1.$$

It is denoted by BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta, v)$. Note that each of the marginal probability distribution is quasi factorial gamma. When $\theta = 0$, X and Y are independently distributed. When X and Y are perfectly correlated, the distribution is BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ and its Laplace transform is

$$\phi(\lambda_1, \lambda_2) = \left(\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}\right)^v \tag{3.2}$$

BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta, v)$ includes many important bivariate distributions. It generalizes the MBE (μ_1, μ_2, θ) distribution discussed in (1.17) and MBG $(\mu_1, \mu_2, \theta, v)$ distribution given in (1.19). In Table 3.1, we present the distributions that are generalized by BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta, v)$. Jayakumar and Mundassery (2007) studied MBG $(\mu_1, \mu_2, \theta, v)$ distribution as a special case of the BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta, v)$. The distributional properties of BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta, v)$ are studied in Section 2. In Section 3, we obtain characterization of BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta, v)$ distribution using negative binomial compounding. In Section 4, we develop autoregressive models having marginals BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ distribution. The bivariate semi quasi factorial gamma distribution is introduced in Section 5.

No	Parameters	Laplace transform	Probability distribution
1	$v=1$	$\frac{1}{(1+\mu_1\lambda_1^{\alpha_1})(1+\mu_2\lambda_2^{\alpha_2})-\theta\mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}}$	BML $(\mu_1,\mu_2,\alpha_1,\alpha_2,\theta)$.
2	$\alpha_1=1$ $\alpha_2=1$	$\left(\frac{1}{(1+\mu_1\lambda_1)(1+\mu_2\lambda_2)-\theta\mu_1\mu_2\lambda_1\lambda_2}\right)^v$	MBG (μ_1,μ_2,θ,v).
3	$\alpha_1=1$ $\alpha_2=1$ $v=1$	$\frac{1}{(1+\mu_1\lambda_1)(1+\mu_2\lambda_2)-\theta\mu_1\mu_2\lambda_1\lambda_2}$	MBE (μ_1,μ_2,θ).

Table 3.1.

3.2 Distributional Properties

The following theorem gives mixture representation of BQFG$(\mu_1,\mu_2,\alpha_1,\alpha_2,1,v)$.

Theorem 3.1. *Let G be a random variable following gamma distribution with parameters 1 and $v,(v>0)$. Suppose that W_1 and W_2 follow independently positive stable distribution in (2.4) and also independent of G. Then $(G^{\frac{1}{\alpha_1}}W_1, G^{\frac{1}{\alpha_2}}W_2)$ follows $BQFG(\mu_1,\mu_2,\alpha_1,\alpha_2,1,v)$ distribution.*

Proof. The Laplace transform of $(G^{\frac{1}{\alpha_1}}W_1,\ G^{\frac{1}{\alpha_2}}W_2)$ is

$$\begin{aligned}\phi(\lambda_1,\lambda_2) &= E(e^{-\lambda_1 G^{\frac{1}{\alpha_1}}W_1-\lambda_2 G^{\frac{1}{\alpha_2}}W_2})\\ &= E\left(E(e^{-\lambda_1 G^{\frac{1}{\alpha_1}}W_1-\lambda_2 G^{\frac{1}{\alpha_2}}W_2}/G)\right)\\ &= \int_0^\infty E(e^{-\lambda_1 g^{\frac{1}{\alpha_1}}W_1-\lambda_2 g^{\frac{1}{\alpha_2}}W_2})\frac{e^{-g}g^{v-1}dg}{\Gamma(v)}\end{aligned}$$

$$= \frac{1}{\Gamma(v)} \int_0^\infty e^{-g\mu_1\lambda_1^{\alpha_1} - g\mu_2\lambda_2^{\alpha_2}} e^{-g} g^{v-1} dg$$
$$= \left(\frac{1}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}} \right)^v .$$

□

Theorem 3.1 shows that a random vector (X, Y) with BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ distribution can be expressed as a mixture of gamma distribution and bivariate positive stable distribution with independent marginals. Hence (X, Y) satisfies the representation.

$$(X, Y) \stackrel{d}{=} (G^{\frac{1}{\alpha_1}} W_1,\ G^{\frac{1}{\alpha_2}} W_2). \tag{3.3}$$

Now, we obtain the distribution function of BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$.

$$\begin{aligned} P(X \leq x, Y \leq y) &= P(G^{\frac{1}{\alpha_1}} W_1 \leq x, G^{\frac{1}{\alpha_2}} W_2 \leq y) \\ &= \int_0^\infty P(G^{\frac{1}{\alpha_1}} W_1 \leq x, G^{\frac{1}{\alpha_2}} W_2 \leq y/G) f_G(g) dg \\ &= \int_0^\infty P(W_1 \leq \frac{x}{g^{\frac{1}{\alpha_1}}}, W_2 \leq \frac{y}{g^{\frac{1}{\alpha_2}}}) \frac{e^{-g} g^{v-1} dg}{\Gamma(v)} \\ &= \int_0^\infty F_{W_1,W_2}(\frac{x}{g^{\frac{1}{\alpha_1}}}, \frac{y}{g^{\frac{1}{\alpha_2}}}) \frac{e^{-g} g^{v-1} dg}{\Gamma(v)}. \end{aligned}$$

where $F_{W_1,W_2}(.\,, .)$ represents the distribution function of bivariate positive stable distribution with Laplace transform in (2.3).

When W_1 and W_2 are independent ($r = 0$)

$$P(X \leq x, Y \leq y) = \int_0^\infty S_{\alpha_1}(\frac{x}{z^{\frac{1}{\alpha_1}} \mu_1}) S_{\alpha_2}(\frac{y}{z^{\frac{1}{\alpha_2}} \mu_2}) \frac{e^{-g} g^{v-1} dg}{\Gamma(v)}$$

where $S_{\alpha_i}(.)$ represents the distribution function of standard positive stable random variable with Laplace transform $\phi(\lambda) = e^{-\lambda^\alpha}$.

The joint density function of BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta, v)$ is

$$h_{X,Y}(x,y) = \int_0^\infty f_{W_1,W_2}(\frac{x}{z^{\frac{1}{\alpha_1}}}, \frac{y}{z^{\frac{1}{\alpha_2}}})\frac{e^{-g}g^{v-1}dg}{\Gamma(v)}$$

where $f_{W_1,W_2}(.,.)$ is the joint density function of bivariate positive stable distribution.

When W_1 and W_2 are independent, we get

$$h_{X,Y}(x,y) = \int_0^\infty D_{\alpha_1}(\frac{x}{z^{\frac{1}{\alpha_1}}\mu_1})D_{\alpha_2}(\frac{y}{z^{\frac{1}{\alpha_2}}\mu_2})\frac{e^{-g}g^{v-1}dg}{\Gamma(v)}$$

where $D_{\alpha_i}(.)$ represents the density function of standard positive stable random variable given in (1.3).

We find the product moments of BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$.

Assume that in (3.3) G, W_1 and W_2 are independent. Then

$$E(X^{\delta_1}Y^{\delta_2}) = E(G^{\frac{\delta_1}{\alpha_1}+\frac{\delta_2}{\alpha_2}})E(W_1^{\delta_1}W_2^{\delta_2}). \tag{3.4}$$

But

$$E(G^{\frac{\delta_1}{\alpha_1}+\frac{\delta_2}{\alpha_2}}) = \frac{\Gamma(\frac{\delta_1}{\alpha_1}+\frac{\delta_2}{\alpha_2}+v)}{\Gamma(v)}.$$

$$E(W_i^{\delta_i}) = \frac{\Gamma(1-\frac{\delta_i}{\alpha_i})\Gamma(1+\frac{\delta_i}{\alpha_i})}{\Gamma(1-\delta_i)},$$

$$0 < \delta_i < \alpha_i, \quad for \quad i = 1, 2.$$

Substituting in (3.4), we get

$$E(X^{\delta_1}Y^{\delta_2}) = \frac{\Gamma(\frac{\delta_1}{\alpha_1}+\frac{\delta_2}{\alpha_2}+v)\Gamma(1-\frac{\delta_1}{\alpha_1})\Gamma(1+\frac{\delta_1}{\alpha_1})\Gamma(1-\frac{\delta_2}{\alpha_2})\Gamma(1+\frac{\delta_2}{\alpha_2})}{\Gamma(v)\Gamma(1-\delta_1)\Gamma(1-\delta_2)}.$$

Now we show that BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ distribution is attracted towards the bivariate positive stable distribution.

Theorem 3.2. *Suppose that $\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ random vectors. Define*

$$U_n = n^{\frac{-1}{\alpha_1}}(X_1 + X_2 + ... + X_n) \quad and \quad V_n = n^{\frac{-1}{\alpha_2}}(Y_1 + Y_2 + ... + Y_n).$$

Then (U_n, V_n) is asymptotically distributed as bivariate positive stable law with Laplace transform in (2.4).

Proof. The Laplace transform of (U_n, V_n) is

$$\begin{aligned}\varphi_{U_n,V_n}(\lambda_1, \lambda_2) &= E(e^{-\lambda_1 n^{\frac{-1}{\alpha_1}}(X_1+X_2+...+X_n)-\lambda_2 n^{\frac{-1}{\alpha_2}}(Y_1+Y_2+...+Y_n)}) \\ &= \left[E(e^{-\lambda_1 n^{\frac{-1}{\alpha_1}}X_i-\lambda_2 n^{\frac{-1}{\alpha_2}}Y_i})\right]^n.\end{aligned}$$

Taking the Laplace transform of $(X_i, Y_i), i \geq 1$ as

$$\phi(\lambda_1,\ \lambda_2) = \left(\frac{1}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}}\right)^v.$$

Hence

$$\varphi_{U_n,V_n}(\lambda_1, \lambda_2) = \left[\frac{1}{1 + \frac{1}{n}(\mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2})}\right]^{nv}.$$

As $n \to \infty$, we get

$$\varphi_{U_n,V_n}(\lambda_1, \lambda_2) = e^{-\mu_1 v\lambda_1^{\alpha_1} - \mu_2 v\lambda_2^{\alpha_2}}.$$

$\square$

3.3 Characterization of BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta, \upsilon)$ through Negative Binomial Compounding

We obtain the characterization of BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta, \upsilon)$ distribution using sums of independently and identically distributed random variables when the number of summands N has negative binomial distribution.

A random variable N is said to follow negative binomial distribution if

$$P(N = n) = \begin{pmatrix} n-1 \\ \upsilon - 1 \end{pmatrix} p^{\upsilon}(1-p)^{n-\upsilon}, \tag{3.5}$$

$$\upsilon = 1, 2, 3, \ldots; n = \upsilon, \upsilon+1, \upsilon+2, \ldots; \quad 0 < p < 1,$$

The p.g.f. of N is

$$P(s) = \left(\frac{ps}{1-(1-p)s}\right)^{\upsilon}. \tag{3.6}$$

Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors with Laplace transform $\psi(\lambda_1, \lambda_2)$. Let $U_N = \sum\limits_{i=1}^{N} X_i$ and $V_N = \sum\limits_{i=1}^{N} Y_i$ where N follows negative binomial distribution in (3.5). Then Laplace transform of (U_N, V_N) is

$$\begin{aligned}
\phi(\lambda_1, \lambda_2) &= E(e^{-\lambda_1 U_N - \lambda_2 V_N}) \\
&= \sum_{n=1}^{\infty} E(e^{-\lambda_1(X_1+X_2+\ldots+X_N)-\lambda_2(Y_1+Y_2+\ldots+Y_N)}/N = n)P(N = n) \\
&= \sum_{n=1}^{\infty} E(e^{-\lambda_1(X_1+X_2+\ldots+X_n)-\lambda_2(Y_1+Y_2+\ldots+Y_n)})P(N = n)
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \left[E(e^{-\lambda_1 X_i - \lambda_2 Y_i})\right]^n \begin{pmatrix} n-1 \\ v-1 \end{pmatrix} p^v (1-p)^{n-v}$$

$$= \left(\frac{p\psi(\lambda_1,\ \lambda_2)}{1-(1-p)\psi(\lambda_1,\ \lambda_2)}\right)^v. \tag{3.7}$$

In the following theorem, we obtain a characterization of BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$.

Theorem 3.3. *Consider a sequence $\{(X_i, Y_i),\ i \geq 1\}$ of independently and identically distributed random vectors. Let N follow the negative binomial distribution in (3.5) and independent of $(X_i, Y_i), i \geq 1$. Then, $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ has BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ distribution if and only if $(X_i, Y_i), i \geq 1$ are distributed according to BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$.*

Proof. Assume that $(X_i, Y_i), i \geq 1$ have BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution with Laplace transform

$$\psi(\lambda_1,\ \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

The Laplace transform of $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ is

$$\phi(\lambda_1, \lambda_2) = E(e^{-(\lambda_1 p^{\frac{1}{\alpha_1}} U_N + \lambda_2 p^{\frac{1}{\alpha_2}} V_N)}).$$

From (3.7),

$$\phi(\lambda_1, \lambda_2) = \left(\frac{p\psi(p^{\frac{1}{\alpha_1}}\lambda_1,\ p^{\frac{1}{\alpha_2}}\lambda_2)}{1-(1-p)\psi(p^{\frac{1}{\alpha_1}}\lambda_1,\ p^{\frac{1}{\alpha_2}}\lambda_2)}\right)^v. \tag{3.8}$$

Substituting the Laplace transform of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ in (3.8) we get

$$\phi(\lambda_1, \lambda_2) = \left(\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}\right)^v.$$

Conversely, suppose that $(p^{\frac{1}{\alpha_1}}U_N, p^{\frac{1}{\alpha_2}}V_N)$ follows BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ distribution. Substituting its Laplace transform in (3.8)

$$\left(\frac{1}{1+\mu_1 p\lambda_1^{\alpha_1}+\mu_2 p\lambda_2^{\alpha_2}}\right)^v = \left(\frac{p\psi(p^{\frac{1}{\alpha_1}}\lambda_1,\ p^{\frac{1}{\alpha_2}}\lambda_2)}{1-(1-p)\psi(p^{\frac{1}{\alpha_1}}\lambda_1,\ p^{\frac{1}{\alpha_2}}\lambda_2)}\right)^v.$$

On simplification,

$$\psi(\lambda_1, \lambda_2) = \frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}.$$

□

Along the lines of Theorem 3.3, a characterization of MBG $(\mu_1, \mu_2, 1, v)$ distribution is obtained (see Jayakumar and Mundassery (2007)).

Remark 3.1. Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors and N, independent of $(X_i, Y_i), i \geq 1$ follow negative binomial distribution in (3.5). Then (pU_N, pV_N) follows MBG $(\mu_1, \mu_2, 1, v)$ distribution if and only if $(X_i, Y_i), i \geq 1$ have MBE $(\mu_1, \mu_2, 1)$ distribution.

Proof of the Remark 3.1 can be obtained by replacing BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ in Theorem 3.3 by MBE $(\mu_1, \mu_2, 1)$ distribution.

Now, we obtain a characterization of BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ by repeated negative binomial compounding.

Theorem 3.4. *Consider the random vector (U_{N_k}, V_{N_k}) defined as*

$$U_{N_k} = p_{k-1}^{\frac{1}{\alpha_1}} \sum_{i=1}^{N_{k-1}} X_i \qquad V_{N_k} = p_{k-1}^{\frac{1}{\alpha_2}} \sum_{i=1}^{N_{k-1}} Y_i$$

where N_{k-1} *follows the negative binomial distribution given in (3.5) with parameters* v *and* p_{k-1}. $\{(X_i, Y_i), i \geq 1\}$, *independent of* N_{k-1}, *is a sequence of independently and identically distributed random vectors with distribution function* $F_{k-1}(.,.)$ *and with Laplace transform* $\phi_{k-1}(\lambda_1, \lambda_2)$, *k=2,3,...* *Assume that* $F_1(.,.) = F(.,.)$ *and the corresponding Laplace transform is* $\phi(\lambda_1, \lambda_2)$. *Then* (U_{N_k}, V_{N_k}) *is distributed as BQFG* $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ *if and only if* $\{(X_i, Y_i), i \geq 1\}$ *are distributed as BML* $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$.

Proof of the theorem is omitted as it is analogous to the proof of Theorem 2.4.

Now we obtain BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta, v)$ as the distribution of negative binomial sum of independently and identically distributed random vectors in which the components are Mittag-Leffler random variables.

Theorem 3.5. *Let* $\{(X_i, Y_i), i \geq 1\}$ *be a sequence of independently and identically distributed random vectors and* N *have negative binomial distribution given in (3.5). Assume that* N *is independent of* $(X_i, Y_i), i \geq 1$. *Then* $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ *follows BQFG* $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1-p, v)$ *distribution if and only if the components* $X_i's$ *and* $Y_i's$ *are independently distributed as Mittag-Leffler.*

Proof. When $X_i's$ and $Y_i's$ are independently distributed according to Mittag-Leffler, the joint Laplace transform of $(X_i, Y_i), i \geq 1$ is

$$\psi(\lambda_1, \lambda_2) = \frac{1}{(1 + \mu_1 \lambda_1^{\alpha_1})(1 + \mu_2 \lambda_2^{\alpha_2})}.$$

From (3.8), the Laplace transform of $(p^{\frac{1}{\alpha_1}}U_N, p^{\frac{1}{\alpha_2}}V_N)$ is

$$\begin{aligned}\phi(\lambda_1,\lambda_2) &= \left(\frac{p}{(1+p\mu_1\lambda_1^{\alpha_1})(1+p\mu_2\lambda_2^{\alpha_2})-1+p}\right)^v \\ &= \left(\frac{p}{1+p\mu_1\lambda_1^{\alpha_1}+p\mu_2\lambda_2^{\alpha_2}+p^2\mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}-1+p}\right)^v \\ &= \left(\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}+p\mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}}\right)^v.\end{aligned}$$

Comparing with (3.1), we get $\theta = 1-p$.

To prove the converse, substituting the Laplace transform of $(p^{\frac{1}{\alpha_1}}U_N, p^{\frac{1}{\alpha_2}}V_N)$ in (3.8), we get

$$\left(\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}+p\mu_1\mu_2\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}}\right)^v = \left(\frac{p\psi(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)}{1-(1-p)\psi(p^{\frac{1}{\alpha_1}}\lambda_1, p^{\frac{1}{\alpha_2}}\lambda_2)}\right)^v.$$

Solving, we obtain that $X_i's$ and $Y_i's$ are independently distributed as Mittag-Leffler.

□

Using similar arguments we can also generate MBG $(\mu_1, \mu_2, \theta, v)$ distribution. Here, we consider the distribution of sums of independently and identically distributed random vectors in which the components follow independently exponential distribution with means μ_1 and μ_2, while the number of summands have negative binomial distribution.

Remark 3.2. Consider a sequence of independently and identically distributed random vectors $\{(X_i, Y_i), i \geq 1\}$, and N, which is independent of $(X_i, Y_i), i \geq 1$ with negative binomial distribution given in (3.5). Then (pU_N, pV_N) follows MBG $(\mu_1, \mu_2, 1-p, v)$ if and only if the components of $(X_i, Y_i), i \geq 1$ follow independently exponential distribution with means μ_1 and μ_2.

Proof of Remark 3.2 follows easily.

A characterization of the negative binomial distribution is obtained in the following theorem.

Theorem 3.6. *Suppose that $\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed random vectors according to BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$. Then for $0 < p < 1, (p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ has BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ distribution if and only if N follows negative binomial distribution discussed in (3.5).*

Proof. The necessary part of the theorem is already discussed in Theorem 3.3. To prove the sufficiency part, take $\mu_1 = \mu_2 = 1$. Assume that $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ follows BQFG$(1, 1, \alpha_1, \alpha_2, 1, v)$ distribution.

Therefore,

$$\phi(\lambda_1, \lambda_2) = \left(\frac{1}{1 + \lambda_1^{\alpha_1} + \lambda_2^{\alpha_2}}\right)^v.$$

By definition, the Laplace transform of $(p^{\frac{1}{\alpha_1}} U_N, p^{\frac{1}{\alpha_2}} V_N)$ is

$$\phi(\lambda_1, \lambda_2) = \sum_{n=v}^{\infty} \psi(p^{\frac{1}{\alpha_1}} \lambda_1, p^{\frac{1}{\alpha_2}} \lambda_2)^n P(N = n)$$

where $\psi(\lambda_1, \lambda_2)$ represents the Laplace transform of $(X_i, Y_i), i \geq 1$. We have,

$$\sum_{n=v}^{\infty} \left(\frac{1}{1 + p\lambda_1^{\alpha_1} + p\lambda_2^{\alpha_2}}\right)^n P(N = n) = \left(\frac{1}{1 + \lambda_1^{\alpha_1} + \lambda_2^{\alpha_2}}\right)^v.$$

Expanding both sides and comparing the coefficients of $(\lambda_1^{\alpha_1} + \lambda_2^{\alpha_2})^j$ for j = 1,2,3,...

$$\sum_{n=v}^{\infty} n(n+1)(n+2)(n+3)...(n+j-1) P(N = n) = \frac{v(v+1)(v+2)(v+3)...(v+j-1)}{p^j}.$$

Therefore,

$$E(N)=\frac{v}{p},\quad E\left(N(N+1)\right)=\frac{v(v+1)}{p^2}\quad \text{and so on.}$$

Consider

$$\begin{aligned}
E(1-t)^{-N} &= 1+\frac{t}{1!}E(N)+\frac{t^2}{2!}E\left(N(N+1)\right)+\frac{t^3}{3!}E\left(N(N+1)(N+2)\right)+...\\
&= \left(\frac{p}{p-t}\right)^v\\
&= \left(\frac{p}{1-p}\right)^v\sum_{n=v}^{\infty}\begin{pmatrix} n-1\\ v-1\end{pmatrix}\left(\frac{1-p}{1-t}\right)^n\\
&= p^v\sum_{n=v}^{\infty}\begin{pmatrix} n-1\\ v-1\end{pmatrix}(1-t)^{-n}(1-p)^{n-v}.
\end{aligned}$$

But

$$E(1-t)^{-N}=\sum_{n=v}^{\infty}(1-t)^{-n}P(N=n).$$

Therefore,

$$\sum_{n=v}^{\infty}(1-t)^{-n}P(N=n)=p^v\sum_{n=v}^{\infty}\begin{pmatrix} n-1\\ v-1\end{pmatrix}(1-t)^{-n}(1-p)^{n-v}.$$

Comparing both sides, we get

$$P(N=n)=\begin{pmatrix} n-1\\ v-1\end{pmatrix}p^v(1-p)^{n-v},\quad n;v,v+1,v+2,....$$

□

It is to be noted that, we can obtain a characterization of negative binomial distribution when $\alpha_1=\alpha_2=1$. Here we assume that $(X_i,Y_i), i\geq 1$ have MBE $(1,1,1)$ distribution and (pU_N, pV_N) follows MBG$(1,1,1,v)$ distribution.

3.4 Autoregressive Processes with BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ Marginals

Consider the first order random coefficient autoregressive model with structure

$$X_n = U_n X_{n-1} + \epsilon_n, n = 1, 2, 3, \ldots.$$

where $\{\epsilon_n, n \geq 1\}$ is a sequence of independently and identically distributed random variables. Assume that $U_n, n \geq 1$ have distribution function $F(u) = u^{\alpha v}$, $0 < \alpha \leq 1$, $v > 0$ and $0 < u < 1$. Using this structure, Jayakumar and Gadag (1999) developed first order autoregressive process with marginals follow quasi factorial gamma distribution. In the following theorem we obtain a first order autoregressive process with BQFG $(\mu_1, \mu_2, \alpha, \alpha, 1, v+1)$ distribution as marginals. We take, $\mu_1 = \mu_2 = 1$ for easier simplification.

Theorem 3.7. *Let a first order autoregressive process $\{(X_n, Y_n), n \geq 1\}$ have the structure*

$$(X_n, Y_n) = (T_n X_{n-1} + \epsilon_n, T_n Y_{n-1} + \psi_n), \ for\ n = 1, 2, 3, \ldots \tag{3.9}$$

where (ϵ_n, ψ_n) is a sequence of independently and identically distributed random vectors. Let T_n be another sequence of random variables, independent of $(\epsilon_n, \psi_n), n \geq 1$ and with distribution function $F(t) = t^{\alpha v}$, $0 < \alpha \leq 1$, $v > 0$, $0 < t < 1$. Then $(X_n, Y_n), n \geq 1$ defines a stationary process with marginals $BQFG(1, 1, \alpha, \alpha, 1, v+1)$ distribution if and only if $(\epsilon_n, \psi_n), n \geq 1$ are distributed as $BML(1, 1, \alpha, \alpha, 1)$.

Proof. The Laplace transform of the model (3.9) is

$$\begin{aligned}\phi_{X_n,Y_n}(\lambda_1,\lambda_2) &= E(e^{-\lambda_1(T_nX_{n-1}+\epsilon_n)-\lambda_2(T_nY_{n-1}+\psi_n)}) \\ &= \phi_{\epsilon_n,\psi_n}(\lambda_1,\lambda_2)\int_0^1 \phi_{X_{n-1},Y_{n-1}}(\lambda_1 t,\lambda_2 t)\alpha v t^{\alpha v-1}dt.\end{aligned}$$

When the process is stationary,

$$\phi_{X,Y}(\lambda_1,\lambda_2)=\alpha v\phi_{\epsilon,\psi}(\lambda_1,\lambda_2)\int_0^1 \phi_{X,Y}(\lambda_1 t,\lambda_2 t)\,t^{\alpha v-1}dt. \tag{3.10}$$

Take $\lambda_j=\gamma_j^{\frac{1}{\alpha}}\lambda$ for j = 1,2. Therefore (3.10) becomes,

$$\phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)=\alpha v\phi_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)\int_0^1 \phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda t\right)\,t^{\alpha v-1}dt.$$

If $\lambda t=r$, then

$$\lambda^{\alpha v}\phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)=\phi_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)\,\alpha v\int_0^\lambda \phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})r\right)\,r^{\alpha v-1}dr. \tag{3.11}$$

Differentiating both sides of (3.11) with respect to λ,

$$\begin{aligned}&\lambda^{\alpha v}\phi'_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)(\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})+\alpha v\lambda^{\alpha v-1}\phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)= \\ &\qquad\alpha v\left(\phi_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)\phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)\lambda^{\alpha v-1}\right. \\ &\qquad\left.+\phi'_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)(\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\int_0^\lambda \phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})r\right)\,r^{\alpha v-1}dr\right).\end{aligned}$$

Substituting $\int_0^\lambda \phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})r\right)\,r^{\alpha v-1}dr$ from (3.11) and then dividing both sides by $\phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)$,

$$\frac{\phi'_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)(\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})}{\phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)}$$

$$= -\alpha\upsilon\left(\frac{1-\phi_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)}{\lambda}\right) + \frac{\phi'_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)(\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})}{\phi_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)}. \tag{3.12}$$

Writing (3.12) as $\frac{dln(.)}{d\lambda}$ and simplifying,

$$\phi_{X,Y}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right) = \phi_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right) e^{-\alpha\upsilon\int_0^{\lambda}\frac{1-\phi_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})y\right)}{y}dy}. \tag{3.13}$$

Assume that (ϵ,ψ) have BML$(1,1,\alpha,\alpha,1)$. We have

$$\begin{aligned}\phi_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right) &= \frac{1}{1+(\gamma_1,\gamma_2)\lambda^{\alpha}}\\ &= \frac{1}{1+\lambda_1^{\alpha}+\lambda_2^{\alpha}}.\end{aligned}$$

Substituting $\phi_{\epsilon,\psi}\left((\gamma_1^{\frac{1}{\alpha}},\gamma_2^{\frac{1}{\alpha}})\lambda\right)$ in (3.13) and simplifying, we get

$$\phi_{X,Y}(\lambda_1,\lambda_2) = \left(\frac{1}{1+\lambda_1^{\alpha}+\lambda_2^{\alpha}}\right)^{\upsilon+1}.$$

Conversely, suppose that $(X_n,Y_n), n\geq 1$ have BQFG$(1,1,\alpha,\alpha.1,\upsilon+1)$ distribution. Then from (3.10),

$$\begin{aligned}\frac{1}{\phi_{\epsilon,\psi}(\lambda_1,\lambda_2)} &= \frac{1}{\phi_{X,Y}(\lambda_1,\lambda_2)}\int_0^1 \phi_{X,Y}(\lambda_1 t,\lambda_2 t)\,\alpha\upsilon t^{\alpha\upsilon-1}dt\\ &= \frac{1}{\phi_{X,Y}(\lambda_1,\lambda_2)}\int_0^1 \frac{1}{(1+(\gamma_1,\gamma_2)\lambda^{\alpha}t^{\alpha})^{\upsilon+1}}\,\alpha\upsilon t^{\alpha\upsilon-1}dt\\ &= \frac{\upsilon}{\phi_{X,Y}(\lambda_1,\lambda_2)}\int_0^1 \frac{w^{\upsilon-1}}{(1+(\gamma_1,\gamma_2)\lambda^{\alpha}w)^{\upsilon+1}}dw\end{aligned}$$

by putting $t^{\alpha}=w$.

$$\frac{1}{\phi_{\epsilon,\psi}(\lambda_1,\lambda_2)} = \frac{v}{\phi_{X,Y}(\lambda_1,\lambda_2)(\gamma_1,\gamma_2)^v\lambda^{\alpha v}} \int_0^{(\gamma_1,\gamma_2)\lambda^\alpha} \frac{z^{v-1}}{(1+z)^{v+1}}dz.$$

where $z = (\gamma_1, \gamma_2)\lambda^\alpha w$.

Substituting $s = \dfrac{z}{1+z}$ and simplifying , we get

$$\phi_{\epsilon,\psi}(\lambda_1,\lambda_2) = \frac{1}{1+\lambda_1^\alpha+\lambda_2^\alpha}.$$

□

Gaver and Lewis (1980) considered the first order stationary autoregressive equation $X_n = \rho X_{n-1} + \epsilon_n$ where $0 \leq \rho < 1$ and obtained solution for marginals as gamma distribution with parameters m and n. The innovation distribution in this case has the Laplace transform

$$\phi_\epsilon(\lambda) = \left(\rho + (1-\rho)\frac{m}{m+\lambda}\right)^n.$$

Here, we obtain a first order autoregressive process with BQFG $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ marginals.

Theorem 3.8. *Consider a first order autoregressive process with structure*

$$(X_n, Y_n) = (\rho X_{n-1} + \epsilon_n, \rho Y_{n-1} + \psi_n), \quad 0 \leq \rho < 1 \tag{3.14}$$

where $(\epsilon_n, \psi_n), n \geq 1$ is a sequence of independently and identically distributed random vectors . Assume that (X_0, Y_0) has BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ distribution. Then the process given by (3.14) is stationary with BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ marginals if and only if the innovation random vectors $(\epsilon_n, \psi_n), n \geq 1$ have Laplace transform

$$\phi_{\epsilon.\psi}(\lambda_1, \lambda_2) = \left(\rho + \frac{1-\rho}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}\right)^v.$$

Proof. The Laplace transform of (3.14), is

$$\phi_{X_n,Y_n}(\lambda_1, \lambda_2) = \phi_{X_{n-1},Y_{n-1}}(\rho^{\frac{1}{\alpha_1}}\lambda_1, \rho^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon_n,\psi_n}(\lambda_1, \lambda_2). \tag{3.15}$$

Suppose that the process is stationary with BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ marginals. Then from (3.15) we have,

$$\left(\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}\right)^v = \left(\frac{1}{1+\rho\mu_1\lambda_1^{\alpha_1}+\rho\mu_2\lambda_2^{\alpha_2}}\right)^v \phi_{\epsilon.\psi}(\lambda_1, \lambda_2).$$

Solving, we get

$$\phi_{\epsilon.\psi}(\lambda_1, \lambda_2) = \left(\rho + \frac{1-\rho}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}\right)^v. \tag{3.16}$$

Converse of the proof is obtained by mathematical induction. Assume that $(\epsilon_n, \psi_n, n \geq 1)$ have the Laplace transform in (3.16). When n= 1, from (3.15), we get

$$\phi_{X_1,Y_1}(\lambda_1, \lambda_2) = \phi_{X_0,Y_0}(\rho^{\frac{1}{\alpha_1}}\lambda_1, \rho^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon_1,\psi_1}(\lambda_1, \lambda_2).$$

Under the assumption, we have

$$\begin{aligned}\phi_{X_1,Y_1}(\lambda_1, \lambda_2) &= \left(\frac{1}{1+\rho\mu_1\lambda_1^{\alpha_1}+\rho\mu_2\lambda_2^{\alpha_2}}\right)^v \left(\rho + \frac{1-\rho}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}\right)^v \\ &= \left(\frac{1}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}\right)^v.\end{aligned}$$

By mathematical induction, it follows that the process $\{(X_n, Y_n), n \geq 1\}$ is stationary with BQFG$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ marginals. □

3.5 Bivariate Semi Quasi Factorial Gamma Distribution

As a generalization to the bivariate quasi factorial gamma distribution introduced in (3.1), we obtain bivariate semi quasi factorial gamma distribution.

Definition 3.2. A non negative random vector (X, Y) is said to follow a bivariate semi quasi factorial gamma distribution if its Laplace transform is

$$\phi(\lambda_1, \lambda_2) = \frac{1}{(1 + \xi(\lambda_1,\ \lambda_2))^v} \tag{3.17}$$

where

$$\xi(\lambda_1,\ \lambda_2) = \frac{1}{p}\xi(p^{\frac{1}{\alpha_1}}\lambda_1,\ p^{\frac{1}{\alpha_2}}\lambda_2),\ 0 < p < 1,\ 0 < \alpha_1, \alpha_2 \leq 1.$$

A solution of this functional equation is in (2.39). We denote this distribution by BSQFG$(\alpha_1, \alpha_2, p, v)$. When v=1 we get the bivariate semi Mittag-Leffler distribution in (2.38). Putting $\xi(\lambda_1,\ \lambda_2) = \lambda_1^{\alpha_1} + \lambda_2^{\alpha_2}$ in (3.17), we get BQFG$(1, 1, \alpha_1, \alpha_2, 1, v)$ distribution. The following theorem gives a characterization of BSQFG$(\alpha_1, \alpha_2, p, v)$ distribution.

Theorem 3.9. *Let $\{(X_i, Y_i),\ i \geq 1\}$ be a sequence of independently and identically distributed random vectors and N, independent of $(X_i, Y_i),\ i \geq 1$ follows the negative binomial distribution in (3.5). Then $(p^{\frac{1}{\alpha_1}}U_N, p^{\frac{1}{\alpha_2}}V_N)$ has the BSQFG $(\alpha_1, \alpha_2, p, v)$ distribution if and only if $(X_i, Y_i), i \geq 1$ follow BSML (α_1, α_2, p) distribution in (2.38).*

Proof of this theorem is omitted as it is analogous to the proof of Theorem 3.3.

SOME BIVARIATE DISTRIBUTIONS GENERATED THROUGH COMPOUNDING

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Chapter 4

Bivariate Discrete Mittag-Leffler Distribution

4.1 Introduction

Pillai and Jayakumar (1995) obtained a discrete counterpart of Mittag-Leffler distribution as geometric sum of Sibuya random variables. Suppose that $\frac{\alpha}{k}$ is the probability of success in k^{th} trial of a sequence of independent Bernoulli trials, $0 < \alpha < 1$, k= 1,2,3,... Let S represent the number of trials before the first success. Then the distribution of S, called Sibuya distribution with parameter α, is

$$\begin{aligned} P(S=0) &= 0, \text{ and} \\ P(S=k) &= (1-\alpha)(1-\frac{\alpha}{2})(1-\frac{\alpha}{3})...(1-\frac{\alpha}{k-1})\frac{\alpha}{k} \end{aligned}$$

$$= (-1)^{k+1}\begin{pmatrix}\alpha\\ k\end{pmatrix}, \quad \text{for } k = 1, 2, 3, \ldots$$

Note that Sibuya distribution with parameter one is degenerate and $P(S = 1) = 1$.

The p.g.f. of the Sibuya distribution is

$$P(s) = 1 - (1-s)^{\alpha}, |\, s\,| \leq 1. \tag{4.1}$$

Discrete Mittag-Leffler distribution in (1.6) is obtained as the geometric sum of independently and identically distributed Sibuya random variables.

Mundassery and Jayakumar (2006) introduced a bivariate discrete Mittag-Leffler distribution and studied its properties.

Definition 4.1. A non negative integer valued random vector (X, Y) is said to follow bivariate discrete Mittag-Leffler distribution if it has p.g.f.

$$P(s_1, s_2) = \frac{1}{(1 + c_1(1-s_1)^{\alpha_1})(1 + c_2(1-s_2)^{\alpha_2}) - \theta c_1 c_2 (1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}, \tag{4.2}$$

$$|s_1| \leq 1,\ |s_2| \leq 1,\ c_1,\ c_2 > 0,\ 0 \leq \theta \leq 1,\ 0 < \alpha_1,\ \alpha_2\ \leq 1.$$

It is denoted by BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$. Note that each of the components of (X, Y) has discrete Mittag-Leffler distribution.

$$P(s_1, 1) = \frac{1}{1 + c_1(1-s_1)^{\alpha_1}} \quad \text{and} \quad P(1, s_2) = \frac{1}{1 + c_2(1-s_2)^{\alpha_2}}.$$

When $\theta = 1$,

$$P(s_1, s_2) = \frac{1}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2}}. \tag{4.3}$$

When X and Y are independent $(\theta = 0)$,

$$P(s_1, s_2) = \frac{1}{(1 + c_1(1-s_1)^{\alpha_1})(1 + c_2(1-s_2)^{\alpha_2})}.$$

When $\alpha_1 = \alpha_2 = 1$ in (4.2), we get a bivariate geometric distribution $(BGD(c_1, c_2, \theta))$ with p.g.f.

$$P(s_1, s_2) = \frac{1}{(1 + c_1(1 - s_1))(1 + c_2(1 - s_2)) - \theta c_1 c_2 (1 - s_1)(1 - s_2)}. \tag{4.4}$$

Phatak and Sreehari (1981) considered a bivariate geometric distribution which could be interpreted as a shock model. Assume that two components of a system are affected by shocks; with probability p_1, the first component survives, with probability p_2, the second component survives and with probability p_0, both components fail. Let N_1 and N_2 denote the number of shocks to the first and second components respectively before the first failure of the system. Then (N_1, N_2) has the following joint probability distribution,

$$P(N_1 = n_1, N_2 = n_2) = \binom{n_1 + n_2}{n_1} p_1^{n_1} p_2^{n_2} p_0, \tag{4.5}$$

$$p_0 + p_1 + p_2 = 1\ ,\ n_1, n_2 = 0, 1, 2, 3, \ldots$$

The p.g.f. of (4.5) is

$$\frac{1}{1 + \frac{p_1}{p_0}(1 - s_1) + \frac{p_2}{p_0}(1 - s_2)}. \tag{4.6}$$

We note that (N_1, N_2) has BGD $(c_1, c_2, 1)$ when $c_1 = \dfrac{p_1}{p_0}$ and $c_2 = \dfrac{p_2}{p_0}$.

In Section 2, we study the distributional properties of BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$. Various characterizations of the BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$ distribution are obtained in Sections 3 and 4. The parameters of the distribution are estimated in Section 5. In Section 6, we develop first order stationary autoregressive process with bivariate discrete Mittag-Leffler marginals . We introduce some other forms of bivariate discrete Mittag-Leffler distribution in Sections 7 and 8.

4.2 Distributional Properties

Mundassery and Jayakumar (2006) obtained the joint probabilities of a random vector (X, Y) following BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$. By definition, p.g.f. of (X, Y) is

$$P(s_1, s_2) = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} s_1^x\, s_2^y\, p_{x,y}$$

where $p_{x,y} = P(X = x, Y = y)$. Here,

$$P(s_1, s_2) = \frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}.$$

Therefore, $p_{x,y}$ can be obtained by expanding the following equation.

$$\left(\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} s_1^x s_2^y p_{x,y}\right)\left(1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}\right) = 1.$$

That is,

$$\begin{aligned}\big(p_{0,0} + p_{0,1}s_2 + p_{0,2}s_2^2 + p_{0,3}s_2^3 + \ldots &+ p_{1,0}s_1 + p_{1,1}s_1s_2 + p_{1,2}s_1s_2^2 + \ldots \\ &+ p_{2,0}s_1^2 + p_{2,1}s_1^2s_2 + p_{2,2}s_1^2s_2^2 + \ldots\big)\end{aligned}$$

$$\left(1 + c_1\left[1 - \begin{pmatrix}\alpha_1\\1\end{pmatrix}s_1 + \begin{pmatrix}\alpha_1\\2\end{pmatrix}s_1^2 - \begin{pmatrix}\alpha_1\\3\end{pmatrix}s_1^3 + \ldots\right] + \right.$$

$$\left. c_2\left[1 - \begin{pmatrix}\alpha_2\\1\end{pmatrix}s_2 + \begin{pmatrix}\alpha_2\\2\end{pmatrix}s_2^2 - \begin{pmatrix}\alpha_2\\3\end{pmatrix}s_2^3 + \ldots\right]\right) = 1.$$

Comparing the powers of $s_1 s_2$, we get

$$p_{0,0} = \frac{1}{1+c_1+c_2},\ p_{0,1} = p_{0,0}^2 c_2\alpha_2,\ p_{0,2} = p_{0,1}p_{0,0}c_2\alpha_2 + \frac{p_{0,0}^2 c_2\alpha_2(1-\alpha_2)}{1.2},$$

$$p_{1,0} = p_{0,0}^2 c_1\alpha_1,\ p_{2,0} = p_{1,0}p_{0,0}c_1\alpha_1 + \frac{p_{0,0}^2 c_1\alpha_1(1-\alpha_1)}{1.2} \text{ and so on.}$$

In general,

$$p_{i,j} = p_1\sum_{r=0}^{i-1}(-1)^{i-1-r}p_{r,j}\begin{pmatrix}\alpha_1\\ i-r\end{pmatrix} + p_2\sum_{r=0}^{j-1}(-1)^{j-1-r}p_{i,r}\begin{pmatrix}\alpha_2\\ j-r\end{pmatrix}$$

with the understanding $p_{i,j} = 0$ if i *or* $j < 0$.

The factorial moment generating function $G(s_1, s_2)$ of any random vector (X, Y) with p.g.f. $P(s_1, s_2)$ is $G(s_1, s_2) = P(s_1 + 1, s_2 + 1)$. Therefore in the case of BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$, we have

$$G(s_1, s_2) = \frac{1}{1 + c_1(-s_1)^{\alpha_1} + c_2(-s_2)^{\alpha_2}}.$$

We show that BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ is normally attracted to the bivariate positive stable law. Consider the Laplace transform corresponding to (4.3),

$$\phi(\lambda_1, \lambda_2) = \frac{1}{1 + c_1(1-e^{-\lambda_1})^{\alpha_1} + c_2(1-e^{-\lambda_2})^{\alpha_2}}. \tag{4.7}$$

Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors with Laplace transform in (4.7). Define

$$\begin{aligned} U_n &= n^{-\frac{1}{\alpha_1}}(X_1 + X_2 + \ldots + X_n) \quad \text{and} \\ V_n &= n^{-\frac{1}{\alpha_2}}(Y_1 + Y_2 + \ldots + Y_n). \end{aligned}$$

Then (U_n, V_n) has Laplace transform

$$\begin{aligned}\varphi_{U_n,V_n}(\lambda_1,\lambda_2) &= E(e^{-(\lambda_1 U_n+\lambda_2 V_n)})\\ &= \left(\phi(\lambda_1 n^{-\frac{1}{\alpha_1}}, \lambda_2 n^{-\frac{1}{\alpha_2}})\right)^n\\ &= \left(\frac{1}{1+c_1(1-e^{-\lambda_1 n^{-\frac{1}{\alpha_1}}})^{\alpha_1}+c_2(1-e^{-\lambda_2 n^{-\frac{1}{\alpha_2}}})^{\alpha_2}}\right)^n.\end{aligned}$$

But

$$\begin{aligned}(1-e^{-\lambda_1 n^{-\frac{1}{\alpha_1}}})^{\alpha_1} &= \frac{\lambda_1^{\alpha_1}}{n}(1+o(1/n)) \quad \text{and}\\ (1-e^{-\lambda_2 n^{-\frac{1}{\alpha_2}}})^{\alpha_2} &= \frac{\lambda_2^{\alpha_2}}{n}(1+o(1/n)).\end{aligned}$$

When $n \to \infty$, we get $\varphi_{U_n,V_n}(\lambda_1,\lambda_2) \to e^{-c_1\lambda_1^{\alpha_1}-c_2\lambda_2^{\alpha_2}}$, the Laplace transform of bivariate positive stable distribution in (2.4) .

We obtain bivariate Mittag-Leffler distribution discussed in (2.1) as limit of a sequence of bivariate discrete Mittag-Leffler random variables.

Suppose that a random vector (X, Y) has the Laplace transform in (4.7). Replace c_1 and c_2 by $c_1 n^{\alpha_1}$ and $c_2 n^{\alpha_2}$ respectively. Then

$$\phi(\lambda_1,\lambda_2) = \frac{1}{1+c_1 n^{\alpha_1}(1-e^{-\lambda_1})^{\alpha_1}+c_2 n^{\alpha_2}(1-e^{-\lambda_2})^{\alpha_2}}.$$

Therefore, Laplace transform of $\left(\frac{X}{n}, \frac{Y}{n}\right)$ will be

$$\phi_n(\lambda_1,\lambda_2) = \frac{1}{1+c_1 n^{\alpha_1}(1-e^{\frac{-\lambda_1}{n}})^{\alpha_1}+c_2 n^{\alpha_2}(1-e^{\frac{-\lambda_2}{n}})^{\alpha_2}}.$$

As $n \to \infty$,

$$\phi_n(\lambda_1,\lambda_2) \longrightarrow \frac{1}{1+c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2}}.$$

In the following definition, we have a bivariate discrete stable distribution.

Definition 4.2. A non negative integer valued random vector (X, Y) is said to follow bivariate discrete stable distribution if its p.g.f. is

$$P(s_1, s_2) = e^{-c_1(1-s_1)^{\alpha_1} - c_2(1-s_2)^{\alpha_2} - rc_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}, \tag{4.8}$$
$$0 < \alpha_1, \alpha_2 \leq 1,\ c_1,\ c_2 > 0,\ 0 \leq r \leq 1.$$

When X and Y are independent $(r = 0)$, we have

$$P(s_1, s_2) = e^{-c_1(1-s_1)^{\alpha_1} - c_2(1-s_2)^{\alpha_2}}. \tag{4.9}$$

Now, we consider the operator '$\oplus$', defined in Jayakumar (1995a), in the bivariate set up. Let (X, Y) have p.g.f. $P(s_1, s_2)$, then $(p \oplus X, p \oplus Y)$ is defined (in distribution) by the p.g.f. $P(1 - p + ps_1, 1 - p + ps_2)$.

We show that BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ is the attracted towards the bivariate discrete stable law.

Theorem 4.1. *Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independent random vectors and identically distributed according to BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$. Let*

$$U_n = n^{\frac{-1}{\alpha_1}} \oplus (X_1 + X_2 + ... + X_n) \quad and \quad V_n = n^{\frac{-1}{\alpha_2}} \oplus (Y_1 + Y_2 + ... + Y_n).$$

Then (U_n, V_n) is asymptotically distributed according to bivariate discrete stable law with p.g.f. in (4.9).

Proof. Let $(X_i, Y_i), i \geq 1$ have p.g.f.

$$P(s_1, s_2) = \frac{1}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2}}.$$

The p.g.f. of (U_n, V_n) is

$$\begin{aligned} P_{U_n,V_n}(s_1,s_2) &= E(s_1^{n^{\frac{-1}{\alpha_1}}\oplus(X_1+X_2+...+X_n)} s_2^{n^{\frac{-1}{\alpha_2}}\oplus(Y_1+Y_2+...+Y_n)}) \\ &= \left[E(s_1^{n^{\frac{-1}{\alpha_1}}\oplus X_i} s_2^{n^{\frac{-1}{\alpha_2}}\oplus Y_i})\right]^n \\ &= \frac{1}{\left[1+c_1(1-(1-n^{\frac{-1}{\alpha_1}}+n^{\frac{-1}{\alpha_1}}s_1))^{\alpha_1}+c_2(1-(1-n^{\frac{-1}{\alpha_2}}+n^{\frac{-1}{\alpha_2}}s_2))^{\alpha_2}\right]^n}. \\ &= \frac{1}{\left[1+\frac{c_1}{n}(1-s_1)^{\alpha_1}+\frac{c_2}{n}(1-s_2)^{\alpha_2}\right]^n}. \end{aligned}$$

When $n \to \infty$, we get

$$P_{U_n,V_n}(s_1,s_2) = e^{-c_1(1-s_1)^{\alpha_1}-c_2(1-s_2)^{\alpha_2}}.$$

□

Now, we express BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ as a mixture of bivariate discrete stable distribution and exponential distribution.

Suppose that the joint distribution of the random vector (S, T) is bivariate discrete stable in (4.9) with exponents α_1 and α_2 and parameters $c_1 = c_2 = W$. Hence its p.g.f. is

$$\varphi(s_1, s_2) = e^{-W(1-s_1)^{\alpha_1}-W(1-s_2)^{\alpha_2}}, 0 < \alpha_1, \alpha_2 \leq 1.$$

Also assume that W is a random variable following exponential distribution with mean $= \dfrac{1}{\beta}$. Then the p.g.f. of the unconditional distribution of (S, T) is

$$\begin{aligned} P(s_1,s_2) &= \int_0^\infty e^{-w((1-s_1)^{\alpha_1}+(1-s_2)^{\alpha_2})}\beta e^{-\beta w}dw \\ &= \frac{1}{1+\frac{1}{\beta}((1-s_1)^{\alpha_1}+(1-s_2)^{\alpha_2})}. \quad \text{Here we take } c_1 = c_2 = \frac{1}{\beta} \end{aligned}$$

4.3 Characterization of BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$ through geometric compounding

Characterizations of BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$ distribution are obtained using geometric sums of independently and identically distributed random vectors. Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors with p.g.f. $Q(s_1, s_2)$. Let $U_N = \sum_{i=1}^{N} X_i$ and $V_N = \sum_{i=1}^{N} Y_i$ where N follows geometric distribution such that $P(N = n) = (1-p)^{n-1}p,\ \ 0 < p < 1$ and n; 1,2,3,... Assume that N is independent of $(X_i, Y_i), i \geq 1$. Then p.g.f. of (U_N, V_N) is

$$\begin{aligned}
P(s_1, s_2) &= E(s_1^{U_N} s_2^{V_N}) \\
&= \sum_{n=1}^{\infty} E((s_1^{U_N} s_2^{V_N})/N = n)P(N = n) \\
&= \sum_{n=1}^{\infty} E(s_1^{X_1+X_2+\ldots+X_n} s_2^{Y_1+Y_2+\ldots+Y_n})P(N = n) \\
&= \sum_{n=1}^{\infty} (Q(s_1, s_2))^n p(1-p)^{n-1} \\
&= \frac{pQ(s_1, s_2)}{1-(1-p)Q(s_1, s_2)}. \qquad (4.10)
\end{aligned}$$

The following theorem gives a characterization of BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution.

Theorem 4.2. *Consider a sequence of independently and identically distributed random vectors $\{(X_i, Y_i), i \geq 1\}$ and N with geometric distribution such that $P(N = n) = p(1-p)^{n-1}$, n; 1,2,3... Assume that N is independent of $(X_i, Y_i), i \geq 1$. Then $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ and $(X_i, Y_i), i \geq 1$ are identically distributed if and only if $(X_i, Y_i), i \geq 1$ have BDML$(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution.*

Proof. Assume that $(X_i, Y_i), i \geq 1$ have BDML$(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution with p.g.f. $Q(s_1, s_2)$. That is,

$$Q(s_1, s_2) = \frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}.$$

From (4.10) the p.g.f. of $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ is

$$\begin{aligned} P(s_1, s_2) &= \sum_{n=1}^{\infty} \left(Q(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1, 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2)\right)^n P(N = n) \\ &= \frac{pQ(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1, 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2)}{1 - (1 - p)Q(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1, 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2)}. \end{aligned} \tag{4.11}$$

Substituting $Q(s_1, s_2)$ in (4.11) and simplifying, we get

$$P(s_1, s_2) = \frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}. \tag{4.12}$$

Conversely, assume that $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ has BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution. Substituting (4.12) in (4.11).

$$\frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}} = \frac{pQ(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1, 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2)}{1 - (1 - p)Q(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1, 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2)}.$$

Solving, we get

$$Q(s_1, s_2) = \frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}.$$

$\square$

In the following remark, we obtain a characterization of BGD $(c_1, c_2, 1)$ distribution.

Remark 4.1. Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors and N have geometric distribution as stated in Theorem

4.2. Then $(p \oplus U_N, p \oplus V_N)$ and $(X_i, Y_i), i \geq 1$ are identically distributed if and only if $(X_i, Y_i), i \geq 1$ are BGD$(c_1, c_2, 1)$ random vectors.

Proof of the Remark 4.1 is obvious.

In the next theorem, we obtain BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$ as the distribution of geometric sum of independently and identically distributed random vectors.

Theorem 4.3. *Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors and N be geometric, independent of $(X_i, Y_i), i \geq 1$. Then $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ follows BDML $(c_1, c_2, \alpha_1, \alpha_2, 1-p)$ distribution if and only if X_i and Y_i are independently distributed discrete Mittag-Leffler random variables with parameters α_1, c_1 and α_2, c_2 respectively.*

Proof. Suppose that X_i and Y_i, i=1,2,... are independently distributed according to discrete Mittag-Leffler distributions with parameters α_1, c_1 and α_2, c_2 respectively. Then the joint p.g.f. of $(X_i, Y_i), i \geq 1$ is

$$Q(s_1, s_2) = \frac{1}{(1 + c_1(1 - s_1^{\alpha_1}))(1 + c_2(1 - s_2^{\alpha_2}))}.$$

From (4.11), the p.g.f. of $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ is

$P(s_1, s_2)$

$$= \frac{p\left[(1 + c_1(1 - (1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1))^{\alpha_1})(1 + c_2(1 - (1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2))^{\alpha_2})\right]^{-1}}{1 - (1-p)\left[(1 + c_1(1 - (1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1))^{\alpha_1})(1 + c_2(1 - (1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2))^{\alpha_2})\right]^{-1}} \tag{4.13}$$

$$= \frac{p}{c_1 p (1 - s_1)^{\alpha_1} + c_2 p (1 - s_2)^{\alpha_2} + c_1 c_2 p^2 (1 - s_1)^{\alpha_1} (1 - s_2)^{\alpha_2} + p}$$

$$= \frac{1}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2} + c_1c_2p(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}.$$

Comparing with (4.2), we get $1-\theta = p$. Hence $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ follows BDML $(c_1, c_2, \alpha_1, \alpha_2, 1-p)$ distribution.

Conversely assume that $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ follows BDML $(c_1, c_2, \alpha_1, \alpha_2, 1-p)$ distribution. Substituting its p.g.f. in (4.11) and on simplification, we get

$$Q(s_1, s_2) = \frac{1}{(1 + c_1(1-s_1^{\alpha_1}))(1 + c_2(1-s_2^{\alpha_2}))}.$$

□

Now we obtain BGD$(c_1, c_2, 1-p)$ as the geometric compound of independently and identically distributed random vectors.

Remark 4.2. Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors and also independent of N, that following geometric distribution. Then $(p \oplus U_N, p \oplus V_N)$ follows BGD$(c_1, c_2, 1-p)$ if and only $(X_i, Y_i), i \geq 1$ are independently and identically distributed according to BGD$(c_1, c_2, 0)$.

We omit proof the Remark 4.2 as it is obvious.

Now we obtain a characterization of the geometric distribution.

Theorem 4.4. *Suppose that $\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed random vectors according to BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$. Then for $0 < p < 1$, $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \ominus V_N)$ and $(X_i, Y_i), i \geq 1$ are identically distributed if and only if N follows geometric distribution.*

Proof of the theorem is analogous to that of Theorem 2.6.

In a similar manner, we can obtain a characterization of geometric distribution using the random sum of random vectors following BGD$(c_1, c_2, 1)$.

Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors. Then for $0 < p < 1$, $(p \oplus U_N, p \oplus V_N)$ and $(X_i, Y_i), i \geq 1$ are identically distributed if and only if N is geometric.

4.4 Characterization of BDML $(c_1,\ c_2,\ \alpha_1,\ \alpha_2,\ \theta)$ through Bivariate Geometric Compounding

In this Section, we obtain characterization of BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$ using bivariate geometric compounding. Suppose that $\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed random vectors with p.g.f. $Q(s_1, s_2)$. Let $U_{N_1} = \sum_{i=1}^{N_1} X_i$ and $V_{N_2} = \sum_{i=1}^{N_2} Y_i$ where (N_1, N_2) has the bivariate geometric distribution in (1.15). Here we consider the discrete version of (2.15) to obtain characterization of BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$ distribution. Therefore (U_{N_1}, V_{N_2}) has p.g.f.

$$P(s_1, s_2) = Q(s_1, s_2)(p_{00} + p_{10}P(s_1,\ 1) + p_{01}P(1,\ s_2) + p_{11}P(s_1,\ s_2)). \tag{4.14}$$

From (4.14), we have

$$P(s_1, s_2) = \frac{Q(s_1, s_2)}{1 - p_{11}P(s_1, s_2)} \left(p_{00} + p_{10}P(s_1, 1) + p_{01}P(1, s_2)\right). \tag{4.15}$$

Therefore,

$$P(s_1,1)=\frac{(p_{00}+p_{01})Q(s_1,1)}{1-(p_{11}+p_{10})Q(s_1,1)} \text{ and } P(1,s_2)=\frac{(p_{00}+p_{10})Q(1,s_2)}{1-(p_{11}+p_{01})Q(1,s_2)}. \quad (4.16)$$

The following theorem gives a characterization of BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution.

Theorem 4.5. *Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors and (N_1, N_2) have the bivariate geometric distribution in (1.15) such that $p_{00} = 0$, $p_{10} + p_{01} + p_{11} = 1$. Also suppose that (N_1, N_2) is independent of $(X_i, Y_i), i \geq 1$. Then $(p_{01}^{\frac{1}{\alpha_1}} \oplus U_{N_1}, p_{10}^{\frac{1}{\alpha_2}} \oplus V_{N_2})$ has bivariate discrete Mittag-Leffler distribution with independent marginals if and only if $(X_i, Y_i), i \geq 1$ have BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution.*

Proof. Suppose that $(X_i, Y_i), i \geq 1$ have BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution with p.g.f. $Q(s_1, s_2)$. From (4.15), the p.g.f. of $(p_{01}^{\frac{1}{\alpha_1}} \oplus U_{N_1}, p_{10}^{\frac{1}{\alpha_2}} \oplus V_{N_2})$ is

$$P(s_1,s_2)=\frac{Q(1-p_{01}^{\frac{1}{\alpha_1}}+p_{01}^{\frac{1}{\alpha_1}}s_1, 1-p_{10}^{\frac{1}{\alpha_2}}+p_{10}^{\frac{1}{\alpha_2}}s_2)}{1-p_{11}Q(1-p_{01}^{\frac{1}{\alpha_1}}+p_{01}^{\frac{1}{\alpha_1}}s_1, 1-p_{10}^{\frac{1}{\alpha_2}}+p_{10}^{\frac{1}{\alpha_2}}s_2)}\left(p_{10}P(s_1,1)+p_{01}P(1,s_2)\right). \quad (4.17)$$

Moreover, from (4.16)

$$\begin{aligned} P(s_1,1) &= \frac{p_{01}Q(1-p_{01}^{\frac{1}{\alpha_1}}+p_{01}^{\frac{1}{\alpha_1}}s_1,1)}{1-(1-p_{01})Q(1-p_{01}^{\frac{1}{\alpha_1}}+p_{01}^{\frac{1}{\alpha_1}}s_1,1)} \quad \text{and} \\ P(1,s_2) &= \frac{p_{10}Q(1,1-p_{10}^{\frac{1}{\alpha_2}}+p_{10}^{\frac{1}{\alpha_2}}s_2)}{1-(1-p_{10})Q(1,1-p_{10}^{\frac{1}{\alpha_2}}+p_{10}^{\frac{1}{\alpha_2}}s_2)}. \end{aligned} \quad (4.18)$$

Then from (4.17), we get

$$P(s_1,s_2)=\frac{1}{(1+c_1(1-s_1)^{\alpha_1})(1+c_2(1-s_2)^{\alpha_2})}.$$

To prove the converse, substituting $P(s_1, s_2)$ in (4.17), we have

$$\frac{1}{(1+c_1(1-s_1^{\alpha_1}))(1+c_2(1-s_2^{\alpha_2}))} = \frac{Q(1-p_{01}^{\frac{1}{\alpha_1}}+p_{01}^{\frac{1}{\alpha_1}}s_1, 1-p_{10}^{\frac{1}{\alpha_2}}+p_{10}^{\frac{1}{\alpha_2}}s_2)}{1-p_{11}Q(1-p_{01}^{\frac{1}{\alpha_1}}+p_{01}^{\frac{1}{\alpha_1}}s_1, 1-p_{10}^{\frac{1}{\alpha_2}}+p_{10}^{\frac{1}{\alpha_2}}s_2)} \left(p_{10}\frac{1}{1+c_1(1-s_1)^{\alpha_1}}+p_{01}\frac{1}{1+c_2(1-s_2)^{\alpha_2}}\right).$$

On simplification, we get

$$Q(s_1, s_2) = \frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}.$$

□

We obtain a characterization of BGD $(c_1, c_2, 1)$ using bivariate geometric compounding in the following remark.

Remark 4.3. Suppose that $\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed random vectors and (N_1, N_2) follows bivariate geometric distribution with p.g.f. in (1.15) and p_{00}, p_{10}, p_{01} and p_{11} are as stated in Theorem 4.5. Then $(p_{01} \oplus U_{N_1}, p_{10} \oplus V_{N_2})$ have bivariate geometric distribution with independent marginals if and only if $(X_i, Y_i), i \geq 1$ have BGD $(c_1, c_2, 1)$.

Theorem 4.6. *Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors according to bivariate discrete Mittag-Leffler distribution with p.g.f. $Q(s_1, s_2)$. Let (N_1, N_2) be independent of $(X_i, Y_i), i \geq 1$ and have bivariate geometric distribution such that p_{00}, p_{10}, p_{01} and p_{11} are as stated in Theorem 4.5. Then $((1-p_{11})^{\frac{1}{\alpha_1}} \oplus U_{N_1}, (1-p_{11})^{\frac{1}{\alpha_2}} \oplus V_{N_2})$ has p.g.f.*

$$P(s_1, s_2) = \frac{1}{(1+(1-p_{11})s_1^{\alpha_1})(1+(1-p_{11})s_2^{\alpha_2})}.$$

if and only if $Q(s_1, s_2) = \dfrac{1}{1+p_{01}s_1^{\alpha_1}+p_{10}s_2^{\alpha_2}}$.

Proof. Assume that $(X_i, Y_i), i \geq 1$ have p.g.f.

$$Q(s_1, s_2) = \frac{1}{1 + p_{01}s_1^{\alpha_1} + p_{10}s_2^{\alpha_2}}.$$

From (4.18), we get

$$P(s_1, 1) = \frac{1}{1 + (1 - p_{11})s_1^{\alpha_1}} \quad \text{and} \quad P(1, s_2) = \frac{1}{1 + (1 - p_{11})s_2^{\alpha_2}}.$$

Substituting in (4.17) and simplifying

$$P(s_1, s_2) = \frac{1}{(1 + (1 - p_{11})s_1^{\alpha_1})(1 + (1 - p_{11})s_2^{\alpha_2})}.$$

Proof of the converse easily follows by substituting $P(s_1, s_2)$, $P(s_1, 1)$ and $P(1, s_2)$ in (4.17) and then simplifying. □

In these lines, we have a characterization of bivariate geometric distribution.

Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors following bivariate geometric distribution with p.g.f. $Q(s_1, s_2)$ and (N_1, N_2) with the bivariate geometric distribution as stated in Theorem 4.5. Then $((1 - p_{11}) \oplus U_{N_1}, (1 - p_{11}) \oplus V_{N_2})$ has p.g.f.

$$P(s_1, s_2) = \frac{1}{(1 + (1 - p_{11})s_1)(1 + (1 - p_{11})s_2)}.$$

if and only if

$$Q(s_1, s_2) = \frac{1}{1 + p_{01}s_1 + p_{10}s_2}.$$

Now, we obtain BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$ distribution as the geometric sum of identically and independently distributed random vectors.

Theorem 4.7. *Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors with p.g.f.*

$$Q(s_1, s_2) = \left(1 + \frac{c_1(1-s_1)^{\alpha_1}}{1+m}\right)^{-1}\left(1 + \frac{c_2(1-s_2)^{\alpha_2}}{1+m}\right)^{-1}, 0 < \alpha_1 . \alpha_2 \leq 1 \qquad (4.19)$$

and (N_1, N_2) follow the bivariate geometric distribution given in (1.15) such that $p_{00} = (1+m)^{-1}$, $p_{10} = p_{01} = 0$ and $p_{11} = m(1+m)^{-1}$, $m > 0$. Then (U_{N_1}, V_{N_2}) follows BDML $(c_1, c_2, \alpha_1, \alpha_2, \frac{m}{1+m})$ distribution if and only if $Q(s_1, s_2)$ is as in (4.19)

Proof. Let $\{(X_i, Y_i), i \geq 1\}$ have the p.g.f. in (4.19). From (4.14) the p.g.f. of (U_{N_1}, V_{N_2}) is

$$\begin{aligned} P(s_1, s_2) &= \left(1 + \frac{c_1(1-s_1)^{\alpha_1}}{1+m}\right)^{-1} \qquad (4.20) \\ &\quad \left(1 + \frac{c_2(1-s_2)^{\alpha_2}}{1+m}\right)^{-1}\left((1+m)^{-1} + m(1+m)^{-1}P(s_1, s_2)\right) \\ &= \frac{(1+m)^{-1}}{\left(1 + \frac{c_1(1-s_1)^{\alpha_1}}{1+m}\right)\left(1 + \frac{c_2(1-s_2)^{\alpha_2}}{1+m}\right)}(1 + mP(s_1, s_2)) \\ &= \frac{1}{(1+m)\left(1 + \frac{c_1(1-s_1)^{\alpha_1}}{1+m}\right)\left(1 + \frac{c_2(1-s_2)^{\alpha_2}}{1+m}\right) - m} \\ &= \frac{1+m}{((1+m) + c_1(1-s_1)^{\alpha_1})\ ((1+m) + c_2(1-s_2)^{\alpha_2}) - m(1+m)} \\ &= \frac{1}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2} + \frac{c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}{1+m}} \\ &= \frac{1}{(1 + c_1(1-s_1)^{\alpha_1})(1 + c_2(1-s_2)^{\alpha_2}) - \frac{m}{1+m}c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}. \end{aligned}$$

Comparing with (4.2), we get that (U_{N_1}, V_{N_2}) follows BDML $(c_1, c_2, \alpha_1, \alpha_2, \frac{m}{1+m})$ distribution.

To prove the converse, assume that (U_{N_1}, V_{N_2}) has BDML $(c_1, c_2, \alpha_1, \alpha_2, \frac{m}{1+m})$ distribution. From (4.14), we have

$$Q(s_1, s_2) = \frac{(1+m)P(s_1, s_2)}{1 + mP(s_1, s_2)}.$$

Substituting $P(s_1, s_2)$ and simplifying, we get the sufficiency part. □

Theorem 4.7 shows that BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$ distribution can be obtained as the bivariate geometric compound of independently and identically distributed random vectors following BDML $(c_1, c_2, \alpha_1, \alpha_2, 0)$.

In the following remark, we can obtain BGD (c_1, c_2, θ) as the distribution of bivariate geometric compound of independently and identically distributed random vectors.

Remark 4.4. Suppose that $\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed random vectors with p.g.f.

$$Q(s_1, s_2) = \left(1 + \frac{c_1(1-s_1)}{1+m}\right)^{-1} \left(1 + \frac{c_2(1-s_2)}{1+m}\right)^{-1} \tag{4.21}$$

and (N_1, N_2) is as stated in Theorem 4.7. Then (U_{N_1}, V_{N_2}) follows BGD$(c_1, c_2, \frac{m}{1+m})$ if and only if $Q(s_1, s_2)$ is as given in (4.21) .

Proof of Remark 4.4 follows easily.

4.5 Estimation of Parameters

Let us consider the estimation of the unknown parameters c_1, c_2, α_1 and α_2, The parameters can be estimated following Remillard and Theodorescu (2000) as

$$\hat{\alpha}_1 = \sum_{j=1}^{q} L_{nX}(t_j)b_j, \qquad \hat{\alpha}_2 = \sum_{j=1}^{q} L_{nY}(t_j)b_j,$$
$$\hat{c}_1 = \exp\left\{\frac{1}{q}\sum_{j=1}^{q} L_{nX}(t_j) - \beta\hat{\alpha}_1\right\}, \quad \hat{c}_2 = \exp\left\{\frac{1}{q}\sum_{j=1}^{q} L_{nY}(t_j) - \beta\hat{\alpha}_2\right\},$$

where

$$L_{nX}(t) = \begin{cases} \ln\left(\frac{1}{g_{nX}(t)} - 1\right), & g_{nX}(t) \in (0,1) \\ 1, & \text{otherwise} \end{cases}$$

$$L_{nY}(t) = \begin{cases} \ln\left(\frac{1}{g_{nY}(t)} - 1\right), & g_{nY}(t) \in (0,1) \\ 1, & \text{otherwise,} \end{cases}$$

where g_{nX} and g_{nY} are the empirical probability generating function for X and Y, respectively, $\beta = \frac{1}{q}\sum_{j=1}^{q} \ln(1 - t_j)$ and

$$b_i = \frac{\ln(1-t_i) - \beta}{\sum_{j=1}^{q}(\ln(1-t_j) - \beta)^2}, \quad i = 1, 2, ...q.$$

We estimate the unknown parameters using Monte Carlo method. For different values of the parameters c_1, c_2, α_1, α_2, we simulate 10 sequences of 1000, 5000 and 10000 observations. In estimation, we use the values $t_1 = 0.1$ and $t_2 = 0.2$. In Table 4.1, we illustrate the averages of these estimators with standard deviations in brackets.

Table 4.1. The estimators of the parameters for different values of c_1, c_2, α_1 and α_2.

n	$c_1 = 0.2$	$c_2 = 0.2$	$\alpha_1 = 0.2$	$\alpha_2 = 0.2$
1000	0.20020	0.20316	0.20319	0.20103
	(0.02016)	(0.02155)	(0.03082)	(0.03032)
5000	0.19948	0.20015	0.20138	0.19855
	(0.00929)	(0.00961)	(0.01579)	(0.01303)
10000	0.19969	0.19988	0.20098	0.19929
	(0.00621)	(0.00626)	(0.01015)	(0.00933)
	$c_1 = 0.7$	$c_2 = 0.9$	$\alpha_1 = 0.5$	$\alpha_2 = 0.5$
1000	0.70858	0.91300	0.49447	0.50478
	(0.07149)	(0.09982)	(0.04004)	(0.04452)
5000	0.70135	0.89726	0.49868	0.49668
	(0.03173)	(0.04048)	(0.01841)	(0.01713)
10000	0.69972	0.90124	0.49778	0.49764
	(0.02325)	(0.02708)	(0.01291)	(0.01302)
	$c_1 = 2$	$c_2 = 3$	$\alpha_1 = 0.9$	$\alpha_2 = 0.2$
1000	2.07103	3.07039	0.91617	0.20467
	(0.23953)	(0.45107)	(0.08257)	(0.03533)
5000	2.01874	3.02542	0.90379	0.20064
	(0.10346)	(0.17818)	(0.03612)	(0.01534)
10000	2.00939	3.00945	0.90281	0.20045
	(0.07770)	(0.12292)	(0.02847)	(0.01053)
	$c_1 = 0.5$	$c_2 = 0.3$	$\alpha_1 = 0.4$	$\alpha_2 = 0.2$
1000	0.48957	0.30236	0.39545	0.20397
	(0.03705)	(0.02561)	(0.03373)	(0.02845)
5000	0.49898	0.30208	0.39733	0.20006
	(0.01848)	(0.01244)	(0.01601)	(0.01197)
10000	0.49960	0.30238	0.39842	0.20083
	(0.01172)	(0.00867)	(0.01077)	(0.00907)
	$c_1 = 1.5$	$c_2 = 0.8$	$\alpha_1 = 0.8$	$\alpha_2 = 0.5$
1000	5.07397	1.97708	0.30091	0.39910
	(0.62618)	(0.18638)	(0.04609)	(0.03714)
5000	5.04043	2.00126	0.29994	0.39965
	(0.29254)	(0.09229)	(0.02093)	(0.01757)
10000	5.01553	2.0038	0.29963	0.39951
	(0.19998)	(0.06703)	(0.01311)	(0.01090)

4.6 Autoregressive Processes with BDML$(c_1, c_2, \alpha_1, \alpha_2, 1)$ Marginals

There are many situations in which discrete variate time series arise often as, counts of events, objects or individuals in consecutive intervals. For example, number of road accidents that occurred on national highways of a country on a day, number of customers waiting in a ticket booking counter that recorded in every one hour duration, number of calls received in a fire rescue center in a week, etc. Moreover such data can also be obtained by discretization of continuous variate time series.

Stuetel and van Harn (1979) introduced the concept of discrete self decomposability.

Definition 4.3. An integer valued random variable X on $\{0, 1, 2, ...\}$ is called discrete self decomposable if its p.g.f. satisfies the relation

$$P(s) = P(1 - \alpha + \alpha s)P_\alpha(s), |\, s \,| \leq 1, \alpha \in (0, 1)$$

where $P_\alpha(s)$ is a p.g.f. This relation can be expressed as $X \stackrel{d}{=} \alpha \oplus X' + X_\alpha$ where $\alpha \oplus X'$ and X_α are independent and $X' \stackrel{d}{=} X$. Using this concept of discrete self decomposability, Al-Osh and Alzaid (1987) introduced first order integer valued autoregressive process. For a p^{th} order integer valued autoregressive time series model, see Jayakumar (1995b). Pillai and Jayakumar (1995) developed the first order autoregressive DML process as a generalization of first order autoregressive geometric process of McKenzie (1986).

Now, we develop a first order autoregressive process with BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ marginals.

Theorem 4.8. *Consider a first order autoregressive process with structure*

$$
\begin{aligned}
&(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1) \quad \text{and for} \quad n = 1, 2, 3, \ldots \\
&(X_n, Y_n) = \begin{cases} (\rho^{\frac{1}{\alpha_1}} \oplus X_{n-1}, \rho^{\frac{1}{\alpha_2}} \oplus Y_{n-1}) & \text{with probability } \rho \\ (\rho^{\frac{1}{\alpha_1}} \oplus X_{n-1} + \epsilon_n, \rho^{\frac{1}{\alpha_2}} \oplus Y_{n-1} + \psi_n) & \text{with probability } 1 - \rho \end{cases}
\end{aligned} \tag{4.22}
$$

where $\{(\epsilon_n, \psi_n), n \geq 1\}$ *is a sequence independently and identically distributed random vectors. Then* $\{(X_n, Y_n), n \geq 1\}$ *defines a stationary first order autoregressive process with BDML* $(c_1, c_2, \alpha_1, \alpha_2, 1)$ *marginals if and only if* $(\epsilon_n, \psi_n), n \geq 1$ *are distributed as BDML* $(c_1, c_2, \alpha_1, \alpha_2, 1)$.

Proof. The p.g.f. of the process in (4.22) is

$$
\begin{aligned}
P_{X_n,Y_n}(s_1, s_2) &= \rho P_{X_{n-1},Y_{n-1}}(1 - \rho^{\frac{1}{\alpha_1}} + \rho^{\frac{1}{\alpha_1}} s_1,\ 1 - \rho^{\frac{1}{\alpha_2}} + \rho^{\frac{1}{\alpha_2}} s_2) \\
&\quad + (1-\rho)_{X_{n-1},Y_{n-1}}(1 - \rho^{\frac{1}{\alpha_1}} + \rho^{\frac{1}{\alpha_1}} s_1,\ 1 - \rho^{\frac{1}{\alpha_2}} + \rho^{\frac{1}{\alpha_2}} s_2) P_{\epsilon_n,\psi_n}(s_1, s_2).
\end{aligned} \tag{4.23}
$$

When the process is stationary, (4.23) becomes

$$
\begin{aligned}
P_{X,Y}(s_1, s_2) &= \rho P_{X,Y}(1 - \rho^{\frac{1}{\alpha_1}} + \rho^{\frac{1}{\alpha_1}} s_1,\ 1 - \rho^{\frac{1}{\alpha_2}} + \rho^{\frac{1}{\alpha_2}} s_2) \\
&\quad + (1-\rho) P_{X,Y}(1 - \rho^{\frac{1}{\alpha_1}} + \rho^{\frac{1}{\alpha_1}} s_1,\ 1 - \rho^{\frac{1}{\alpha_2}} + \rho^{\frac{1}{\alpha_2}} s_2) P_{\epsilon,\psi}(s_1, s_2).
\end{aligned}
$$

Taking that, $(X_n, Y_n), n \geq 1$ follow BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution. Then the p.g.f. of (ϵ_n, ψ_n) is

$$P_{\epsilon_n,\psi_n}(s_1, s_2) = \frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}.$$

Proof of the converse is obtained by mathematical induction. Let $n = 1$. Assume that $(\epsilon_n, \psi_n), n \geq 1$ follow BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution. From (4.23) we have,

$$\begin{aligned} P_{X_1,Y_1}(s_1, s_2) &= \frac{\rho}{1 + c_1\rho(1 - s_1)^{\alpha_1} + c_2\rho(1 - s_2)^{\alpha_2}} + \\ & \frac{1 - \rho}{1 + c_1\rho(1 - s_1)^{\alpha_1} + c_2\rho(1 - s_2)^{\alpha_2}} \frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}. \end{aligned}$$

On simplification, we get

$$P_{X_1,Y_1}(s_1, s_2) = \frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}.$$

Therefore, by the mathematical induction, we get that the process $\{(X_n, Y_n), n \geq 1\}$ is stationary with BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ marginals. $\square$

We develop a bivariate first order autoregressive process with BGD $(c_1, c_2, 1)$ marginals.

Remark 4.5. Let a first order autoregressive process $\{(X_n, Y_n), n \geq 1\}$ have structure

$$(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1) \text{ and for } n = 1, 2, 3, ...$$
$$(X_n, Y_n) = \begin{cases} (\rho \oplus X_{n-1}, \rho \oplus Y_{n-1}) & \text{with probability } \rho \\ (\rho \oplus X_{n-1} + \epsilon_n, \rho \oplus Y_{n-1} + \psi_n) & \text{with probability } 1 - \rho \end{cases}$$

where $\{(\epsilon_n, \psi_n), n \geq 1\}$ is a sequence independently and identically distributed random vectors. Then the process $\{(X_n, Y_n), n \geq 1\}$ is stationary with BGD $(c_1, c_2, 1)$ marginals if and only if $\{(\epsilon_n, \psi_n), n \geq 1\}$ are distributed according to BGD $(c_1, c_2, 1)$.

Proof of the remark follows easily.

A generalization to the autoregressive process considered in (4.22) is obtained in Theorem 4.9.

Theorem 4.9. *Let $\{(X_n, Y_n), n \geq 1\}$ form a first order autoregressive process with structure as given below:*

$$(X_n, Y_n) = \begin{cases} (\epsilon_n, \psi_n), & \textit{with probability } q \\ (p^{\frac{1}{\alpha_1}} \oplus X_{n-1}, p^{\frac{1}{\alpha_2}} \ominus Y_{n-1}), & \textit{with probability } (1-q)p \\ (p^{\frac{1}{\alpha_1}} \oplus X_{n-1} + \epsilon_n, p^{\frac{1}{\alpha_2}} \oplus Y_{n-1} + \psi_n), & \textit{with probability } (1-p)(1-q). \end{cases} \tag{4.24}$$

Then the process is stationary with BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ marginals if and only if $\{(\epsilon_n, \psi_n), n \geq 1\}$ is a sequence of independently and identically distributed random vectors with BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution provided $(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1)$.

Proof. Assume that $(X_n, Y_n), n \geq 1$ are distributed according to BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$. The p.g.f. of (4.24) is

$$\begin{aligned} P_{X_n,Y_n}(s_1, s_2) &= qP_{\epsilon_n,\psi_n}(s_1, s_2) \\ &\quad +p(1-q)P_{X_{n-1},Y_{n-1}}(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1, 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2) \qquad (4.25) \\ &\quad +(1-p)(1-q)P_{X_{n-1},Y_{n-1}}(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1, 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2) \\ &\quad\quad P_{\epsilon_n,\psi_n}(s_1, s_2). \end{aligned}$$

Under the condition of stationary, we have

$$\begin{aligned}
P_{X,Y}(s_1,s_2) &= qP_{\epsilon,\psi}(s_1,s_2) \\
&\quad + p(1-q)P_{X,Y}(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1, 1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2) \\
&\quad + (1-p)(1-q)P_{X,Y}(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1, 1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2)P_{\epsilon,\psi}(s_1,s_2).
\end{aligned}$$

Substituting $P_{X,Y}(s_1,s_2)$ in the above expression and simplifying, we get

$$P_{\epsilon,\psi}(s_1,s_2) = \frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}.$$

Conversely assume that $(\epsilon_n,\psi_n), n \geq 1$ follow BDML$(c_1,c_2,\alpha_1,\alpha_2,1)$ distribution. When $n=1$, from (4.25) we have

$$\begin{aligned}
P_{X_1,Y_1}(s_1,s_2) &= qP_{\epsilon_1,\psi_1}(s_1,s_2) \\
&\quad + p(1-q)P_{X_0,Y_0}(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1, 1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2) \\
&\quad + (1-p)(1-q)P_{X_0,Y_0}(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1, 1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2)P_{\epsilon_1,\psi_1}(s_1,s_2).
\end{aligned}$$

Under the assumption, we get

$$P_{X_1,Y_1}(s_1,s_2) = \frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}.$$

Hence by mathematical induction, we obtain that the process is stationary with marginals follow BDML $(c_1,c_2,\alpha_1,\alpha_2,1)$ distribution. □

We can develop a similar bivariate autoregressive process with BGD $(c_1,c_2,1)$ marginals, by putting $\alpha_1=\alpha_2=1$ in Theorem 4.9.

A first order autoregressive model called, TEAR(1), is introduced in Lawrance and Lewis (1980). In the following theorem we develop a first order autoregressive process having BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ marginals.

Theorem 4.10. *Consider a first order autoregressive process with structure:*

$$(X_n, Y_n) = \begin{cases} (\rho^{\frac{1}{\alpha_1}} \oplus \epsilon_n, \rho^{\frac{1}{\alpha_2}} \oplus \psi_n) & \text{with probability } \rho \\ (X_{n-1} + \rho^{\frac{1}{\alpha_1}} \oplus \epsilon_n, Y_{n-1} + \rho^{\frac{1}{\alpha_2}} \oplus \psi_n) & \text{with probability } 1-\rho \end{cases} \tag{4.26}$$

where $(\epsilon_n, \psi_n), n \geq 1$ are independently and identically distributed random vectors. Then the process $\{(X_n, Y_n), n \geq 1\}$ is stationary with BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ marginals if and only if $(\epsilon_n, \psi_n), n \geq 1$ are distributed according to BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ and $(X_0, Y_0) \stackrel{d}{=} (\epsilon_1, \psi_1)$.

Proof. The p.g.f. of (4.26) is

$$\begin{aligned} P_{X_n,Y_n}(s_1, s_2) &= \rho P_{\epsilon_n,\psi_n}(1 - \rho^{\frac{1}{\alpha_1}} + \rho^{\frac{1}{\alpha_1}} s_1, 1 - \rho^{\frac{1}{\alpha_2}} + \rho^{\frac{1}{\alpha_2}} s_2) \\ &\quad +(1-\rho) P_{X_{n-1},Y_{n-1}}(s_1, s_2) P_{\epsilon_n,\psi_n}(1 - \rho^{\frac{1}{\alpha_1}} + \rho^{\frac{1}{\alpha_1}} s_1, 1 - \rho^{\frac{1}{\alpha_2}} + \rho^{\frac{1}{\alpha_2}} s_2). \end{aligned} \tag{4.27}$$

Let the process be stationary with BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ marginals. Substituting $P_{X,Y}(s_1, s_2)$ in (4.27) and simplifying, we get

$$P_{\epsilon,\psi}(s_1, s_2) = \frac{1}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2}}.$$

The converse of the theorem is proved by mathematical induction. Suppose that $\{(\epsilon_n, \psi_n), n \geq 1\}$ are distributed according to BDML$(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution.

Putting $n = 1$ in (4.27), we get

$$\begin{aligned}
P_{X_1,Y_1}(s_1, s_2) &= \rho P_{\epsilon_1,\psi_1}(1-\rho^{\frac{1}{\alpha_1}}+\rho^{\frac{1}{\alpha_1}}s_1, 1-\rho^{\frac{1}{\alpha_2}}+\rho^{\frac{1}{\alpha_2}}s_2) \\
&\quad +(1-\rho)P_{X_0,Y_0}(s_1,s_2)P_{\epsilon_1,\psi_1}(1-\rho^{\frac{1}{\alpha_1}}+\rho^{\frac{1}{\alpha_1}}s_1, 1-\rho^{\frac{1}{\alpha_2}}+\rho^{\frac{1}{\alpha_2}}s_2). \\
&= \rho\frac{1}{1+\rho c_1(1-s_1)^{\alpha_1}+\rho c_2(1-s_2)^{\alpha_2}} \\
&\quad +1-\rho\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}} \\
&\qquad\qquad \frac{1}{1+\rho c_1(1-s_1)^{\alpha_1}+\rho c_2(1-s_2)^{\alpha_2}}.
\end{aligned}$$

On simplification, we get

$$P_{X_1,Y_1}(s_1, s_2) = \frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}.$$

Hence by mathematical induction, we have that the process $\{(X_n, Y_n), n \geq 1\}$ is stationary with marginals follow BDML$(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution. □

By putting $\alpha_1 = \alpha_2 = 1$ in Theorem 4.10 we obtain a first order autoregressive process with BGD $(c_1, c_2, 1)$ marginals.

4.7 Bivariate Discrete Mittag-Leffler Distributions Generated through Bivariate Geometric Compounding

Jayakumar and Mundassery (2006) obtained the discrete analogues of bivariate Mittag-Leffler distribution introduced in Section 2.7. In the following theorem, we introduce discrete analogue of the bivariate Mittag-Leffler distribution that generalizes

Marshall-Olkin's bivariate exponential distribution.

Theorem 4.11. *Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors following BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$. Let $U_{N_1} = \sum_{i=1}^{N_1} X_i$ and $V_{N_2} = \sum_{i=1}^{N_2} Y_i$ where (N_1, N_2) has bivariate geometric distribution in (1.15) and $p_{00} = \delta_{12}$, $p_{10} = \delta_2$, $p_{01} = \delta_1$ and $p_{11} = 1 - \delta$ where $\delta = \delta_1 + \delta_2 + \delta_{12}$. Assume that (N_1, N_2) is independent of $(X_i, Y_i), i \geq 1$. Then the distribution of (U_{N_1}, V_{N_2}) is the discrete analogue of bivariate Mittag-Leffler distribution discussed in (2.33) .*

Proof. Suppose that $\{(X_i, Y_i), i \geq 1\}$ have p.g.f.

$$Q(s_1, s_2) = \frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}, \quad 0 < \alpha_1, \alpha_2 \leq 1.$$

From (4.15) and (4.16)

$$P(s_1, s_2) = \frac{1}{\delta + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}} \left(\delta_{12} + \frac{\delta_2(\delta_{12} + \delta_1)}{\delta_{12} + \delta_1 + c_1(1 - s_1)^{\alpha_1}} + \frac{\delta_1(\delta_{12} + \delta_2)}{\delta_{12} + \delta_2 + c_2(1 - s_2)^{\alpha_2}}\right).$$

On simplification, we get

$$P(s_1,\ s_2) = \frac{(\delta + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2})(\delta_1 + \delta_{12})(\delta_2 + \delta_{12}) + c_1 c_2 (1 - s_1)^{\alpha_1}(1 - s_2)^{\alpha_2}\delta_{12}}{(\delta + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2})(\delta_1 + \delta_{12} + c_1(1 - s_1)^{\alpha_1})(\delta_2 + \delta_{12} + c_2(1 - s_2)^{\alpha_2})}.$$

□

In the following remark, we obtain the discrete analogue of Marshall-Olkin's bivariate exponential distribution.

Remark 4.6. Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors according to BGD $(c_1, c_2, 1)$ and independent of (N_1, N_2) where (N_1, N_2) has bivariate geometric distribution given in (1.15). Choose p_{00}, p_{10}, p_{01} and p_{11} as in Theorem 4.11. Then the distribution of (U_{N_1}, V_{N_2}) is the discrete analogue of Marshall-Olkin's bivariate exponential distribution discussed in (2.32). The p.g.f. of (U_{N_1}, V_{N_2}) is

$$P(s_1, (s_2) = \frac{(\delta + c_1(1 - s_1) + c_2(1 - s_2))(\delta_1 + \delta_{12})(\delta_2 + \delta_{12}) + c_1 c_2(1 - s_1)(1 - s_2)\delta_{12}}{(\delta + c_1(1 - s_1) + c_2(1 - s_2))(\delta_1 + \delta_{12} + c_1(1 - s_1))(\delta_2 + \delta_{12} + c_2(1 - s_2))}.$$

Now we introduce the discrete version of the bivariate Mittag-Leffler distribution that generalizes the Hawkes' (1972) exponential distribution.

Theorem 4.12. *Suppose that the components of a sequence of independently and identically distributed random vectors* $\{(X_i, Y_i), i \geq 1\}$ *have discrete Mittag-Leffler distribution with p.g.f.s* $Q_{X_i}(s_1) = \dfrac{\delta_1}{\delta_1 + (1 - s_1)^{\alpha_1}}$ *and* $Q_{Y_i}(s_2) = \dfrac{\delta_2}{\delta_2 + (1 - s_2)^{\alpha_2}}$, *i=1,2,.... respectively and* $\delta_1, \delta_2 > 0$. *Then the distribution of* (U_{N_1}, V_{N_2}) *is discrete analogue of the bivariate Mittag-Leffler distribution in (2.37) where* (N_1, N_2) *has the p.g.f. stated in (2.35).*

Proof. Assume that $\{(X_i, Y_i), i \geq 1\}$ are independently and identically distributed random vectors such that the components have independent discrete Mittag-Leffler distribution. Therefore the joint p.g.f. is

$$Q(s_1, s_2) = \frac{\delta_1 \delta_2}{(\delta_1 + (1 - s_1)^{\alpha_1})(\delta_2 + (1 - s_2)^{\alpha_2})}.$$

From (2.35) the p.g.f. of (U_{N_1}, V_{N_2}) is

$$P(s_1, s_2) = \frac{Q(s_1, s_2)}{1 - p_{00}Q(s_1, s_2)} \left(p_{11} + \frac{p_{10}P_2 Q_{Y_i}(s_2)}{1 - Q_2 Q_{Y_i}(s_2)} + \frac{p_{01}P_1 Q_{X_i}(s_1)}{1 - Q_1 Q_{X_i}(s_1)} \right).$$

Substituting $Q_{X_i}(s_1), Q_{Y_i}(s_2)$ and $Q(s_1, s_2)$

$$\begin{aligned} P(s_1, s_2) &= \frac{\delta_1 \delta_2}{(\delta_1 + (1-s_1)^{\alpha_1})(\delta_2 + (1-s_2)^{\alpha_2}) - \delta_1\delta_2 p_{00}} \\ &\qquad \left(p_{11} + \frac{p_{10}P_2\delta_2}{P_2\delta_2 + (1-s_2)^{\alpha_2}} + \frac{p_{01}P_1\delta_1}{P_1\delta_1 + (1-s_1)^{\alpha_1}} \right) \\ &= \frac{m_1 m_2}{(m_1 + P_1(1-s_1)^{\alpha_1})(m_2 + P_2(1-s_2)^{\alpha_2}) - m_1 m_2 p_{00}} \\ &\qquad \left(p_{11} + \frac{p_{10}m_2}{m_2 + (1-s_2)^{\alpha_2}} + \frac{p_{01}m_1}{m_1 + (1-s_1)^{\alpha_1}} \right) \end{aligned}$$

where $m_i = \delta_i P_i$, i=1,2.

$$P(s_1, s_2) = \frac{m_1 m_2}{(m_1 + (1-s_1)^{\alpha_1})(m_2 + (1-s_2)^{\alpha_2})}$$

$$\left(\frac{p_{11}(m_1 + (1-s_1)^{\alpha_1})(m_2 + (1-s_2)^{\alpha_2}) + p_{10}m_2(m_1 + (1-s_1)^{\alpha_1}) + p_{01}m_1(m_2 + (1-s_2)^{\alpha_2})}{(m_1 + P_1(1-s_1)^{\alpha_1})(m_2 + P_2(1-s_2)^{\alpha_2}) - m_1 m_2 p_{00}} \right).$$

On simplification, we get

$$\begin{aligned} P(s_1, s_2) &= \frac{m_1 m_2}{(m_1 + (1-s_1)^{\alpha_1})(m_2 + (1-s_2)^{\alpha_2})} \\ &\qquad \left(1 + \frac{[p_{00} - (1-P_1)(1-P_2)](1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}{(m_1 + P_1(1-s_1)^{\alpha_1})(m_2 + P_2(1-s_2)^{\alpha_2}) - p_{00}m_1 m_2} \right). \end{aligned}$$

□

We obtain a bivariate geometric distribution, as the discrete analogue of the bivariate exponential distribution considered in Hawkes (1972). For this, taking $Q_{X_i}(s_1)$ and $Q_{Y_i}(s_2)$ are as follows:

$$Q_{X_i}(s_1) = \frac{\delta_1}{\delta_1 + (1-s_1)} \quad \text{and} \quad Q_{Y_i}(s_2) = \frac{\delta_2}{\delta_2 + (1-s_2)} \qquad i = 1, 2, ...$$

Then the p.g.f. of (U_{N_1}, V_{N_2}) is

$$\frac{m_1 m_2}{(m_1 + (1-s_1))(m_2 + (1-s_2))}\left(1 + \frac{[p_{00} - (1-P_1)(1-P_2)](1-s_1)(1-s_2)}{(m_1 + P_1(1-s_1))(m_2 + P_2(1-s_2)) - p_{00} m_1 m_2}\right).$$

4.8 Bivariate Discrete Mittag-Leffler Distributions Through Translated form of Block's Bivariate Geometric

Block (1977) considered the bivariate geometric compound of a set of independently and identically distributed random vectors. In that context, he considered the random vector (N_1, N_2) that follows bivariate geometric distribution such that each of the components realizes values from 'one' onwards (see (1.15)). Here we consider a translated model of this bivariate geometric distribution so that the random variable takes values from 'zero' onwards. The survival function of this translated form of Block's bivariate geometric distribution is

$$\overline{F}(n_1, n_2) = P(N_1 \geq n_1,\ N_2 \geq n_2) = \begin{cases} p_{11}^{n_1}(p_{01} + p_{11})^{n_2 - n_1} & \text{if } n_1 \leq n_2 \\ p_{11}^{n_2}(p_{10} + p_{11})^{n_1 - n_2} & \text{if } n_2 \leq n_1 \end{cases} \tag{4.28}$$

where $n_1,\ n_2 = 0, 1, 2, 3, \dots$, $p_{00} + p_{01} + p_{10} + p_{11} = 1$, $p_{10} + p_{11} < 1$, and $p_{01} + p_{11} < 1$. The joint probability distribution is

$$P(N_1 = n_1,\ N_2 = n_2) = \begin{cases} p_{01} p_{11}^{n_1}(p_{00} + p_{10})(p_{01} + p_{11})^{n_2 - n_1 - 1} & \text{if } n_1 < n_2 \\ p_{10} p_{11}^{n_2}(p_{00} + p_{01})(p_{10} + p_{11})^{n_1 - n_2 - 1} & \text{if } n_2 < n_1 \\ p_{11}^{n} p_{00} & \text{if } n_1 = n_2 = n. \end{cases}$$

The p.g.f. of (4.28) is

$$P(s_1, s_2) = \frac{1}{1 - p_{11}s_1s_2}\left(p_{00} + \frac{p_{10}(p_{00} + p_{01})s_1}{1 - (p_{11} + p_{10})s_1} + \frac{p_{01}(p_{00} + p_{10})s_2}{1 - (p_{11} + p_{01})s_2}\right).$$

That is,

$$P(s_1, s_2) = p_{00} + p_{10}P(s_1, 1)s_1 + p_{01}P(1, s_2)s_2 + p_{11}P(s_1, s_2)s_1s_2. \tag{4.29}$$

Note that (4.29) is a special case of the characteristic function equation

$$\phi(t_1, t_2) = p_{00} + p_{10}\phi(t_1, 0)\psi(t_1, 0) + p_{01}\phi(0, t_2)\psi(0, t_2) + p_{11}\phi(t_1, t_2)\psi(t_1, t_2). \tag{4.30}$$

Therefore, along the lines of Block (1977), $\phi(t_1, t_2)$ represents the characteristic function of (U_{N_1}, V_{N_2}) where $U_{N_1} = \sum_{i=1}^{N_1} X_i$ and $V_{N_2} = \sum_{i=1}^{N_2} Y_i$. $\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed random vectors with characteristic function $\psi(t_1, t_2)$. Moreover (N_1, N_2) is independent of $\{(X_i, Y_i), i \geq 1\}$ and follows the bivariate geometric distribution given in (4.28). A discrete analogue of (4.30) is expressed below in terms of p.g.f.

$$P(s_1, s_2) = p_{00} + p_{10}P(s_1, 1)Q(s_1, 1) + p_{01}P(1, s_2)Q(1, s_2) + p_{11}P(s_1, s_2)Q(s_1, s_2). \tag{4.31}$$

where $P(s_1, s_2)$ is the p.g.f. of (U_{N_1}, V_{N_2}) and $Q(s_1, s_2)$ is that of $(X_i, Y_i), i \geq 1$.

From (4.31) we get,

$$P(s_1, 1) = \frac{p_{00} + p_{01}}{1 - (p_{11} + p_{10})Q(s_1, 1)} \quad \text{and} \quad P(1, s_2) = \frac{p_{00} + p_{10}}{1 - (p_{11} + p_{01})Q(1, s_2)}. \tag{4.32}$$

Mundassery and Jayakumar (2006) introduced bivariate discrete Mittag-Leffler distribution as the probability distribution of random sums of independently and identically distributed random vectors when the number of summands have the bivariate geometric distribution given in (4.28).

Now, we define a bivariate Sibuya distribution.

Definition 4.4. A non negative integer valued random vector (X, Y) is said to follow bivariate Sibuya distribution if its p.g.f. is

$$Q(s_1, s_2) = 1 - (1 - s_1)^{\alpha_1} - (1 - s_2)^{\alpha_2}, \quad 0 < \alpha_1, \alpha_2 \leq 1. \tag{4.33}$$

Note that the marginal p.g.f.s are those of the Sibuya distribution discussed in (4.1).

Theorem 4.13. *Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors. Let (N_1, N_2) have the bivariate geometric distribution in (4.28) and independent of $\{(X_i, Y_i), i \geq 1\}$. Choose $p_{00} = p_1 p_2$, $p_{10} = p_2(1 - p_1)$, $p_{01} = p_1(1 - p_2)$ and $p_{11} = (1 - p_1)(1 - p_2)$ so that N_1 and N_2 are independent. Then (U_{N_1}, V_{N_2}) has bivariate discrete Mittag-Leffler distribution with independent marginals if and only if $(X_i, Y_i), i \geq 1$ have p.g.f.*

$$Q(s_1, s_2) = \left(1 - (1 - s_1)^{\alpha_1}\right)\left(1 - (1 - s_2)^{\alpha_2}\right), 0 < \alpha_1, \alpha_2 \leq 1. \tag{4.34}$$

Proof. Assume that $(X_i, Y_i), i \geq 1$ have the p.g.f. in (4.34). From (4.31), the p.g.f. of (U_{N_1}, V_{N_2}) is

$$\begin{aligned} P(s_1, s_2) = p_1 p_2 &+ p_2(1 - p_1)P(s_1, 1)(1 - (1 - s_1)^{\alpha_1}) \\ &+ p_1(1 - p_2)P(1, s_2)(1 - (1 - s_2)^{\alpha_2}) \\ &+ (1 - p_1)(1 - p_2)P(s_1, s_2)(1 - (1 - s_1)^{\alpha_1})(1 - (1 - s_2)^{\alpha_2}). \end{aligned}$$

Solving, we get

$$P(s_1, 1) = \frac{1}{1 + \frac{(1-p_1)}{p_1}(1 - s_1)^{\alpha_1}} \quad \text{and} \quad P(1,\ s_2) = \frac{1}{1 + \frac{(1-p_2)}{p_2}(1 - s_2)^{\alpha_2}}.$$

Substituting $P(s_1, 1)$, $P(1, s_2)$ in (4.31) and simplifying, we get

$$P(s_1, s_2) = \frac{1}{\left(1 + \frac{1-p_1}{p_1}(1 - s_1)^{\alpha_1}\right)\left(1 + \frac{1-p_2}{p_2}(1 - s_2)^{\alpha_2}\right)}. \tag{4.35}$$

Conversely, assume that (U_{N_1}, V_{N_2}) has bivariate discrete Mittag-Leffler distribution with p.g.f. in (4.35). Using this in (4.32) obtain $Q(s_1, 1)$ and $Q(1, s_2)$. Then from (4.31) we get,

$$Q(s_1, s_2) = (1 - (1 - s_1)^{\alpha_1})(1 - (1 - s_2)^{\alpha_2}).$$

□

In Theorem 4.13, if we take that the sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors have bivariate Sibuya distribution in (4.33), then (U_{N_1}, V_{N_2}) has bivariate discrete Mittag-Leffler distribution with p.g.f.

$$P(s_1, s_2) = \frac{1 - (1 - p_1)(1 - p_2)(1 - (1 - s_1)^{\alpha_1})(1 - (1 - s_2)^{\alpha_2})}{1 - (1 - p_1)(1 - p_2)\left[1 - (1 - s_1)^{\alpha_1} - (1 - s_2)^{\alpha_2}\right]} \tag{4.36}$$
$$\frac{1}{\left(1 + \frac{1-p_1}{p_1}(1 - s_1)^{\alpha_1}\right)\left(1 + \frac{1-p_2}{p_2}(1 - s_2)^{\alpha_2}\right)}.$$

In order to prove this, assume that $Q(s_1, s_2)$ is as in (4.33). Substituting $Q(s_1, s_2)$ in (4.32) to obtain the expressions of $P(s_1, 1)$ and $P(1, s_2)$. We get (4.36) by substituting $P(s_1, 1)$ and $P(1, s_2)$ in (4.31) and then on simplification.

Suppose that in the bivariate geometric distribution considered (4.28), $p_{00} = 0$ and $p_{11} = p_0$, $p_{01} = p_1$ and $p_{10} = p_2$. Then (4.29) becomes

$$P(s_1, s_2) = p_1 P(s_1, 1)s_1 + p_2 P(1, s_2)s_2 + p_0 P(s_1, s_2)s_1 s_2. \tag{4.37}$$

Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors with p.g.f. $Q(s_1, s_2)$ and (N_1, N_2), independent of $(X_i, Y_i), i \geq 1$, have p.g.f. in (4.37). Then along the lines of Block (1977), (U_{N_1}, V_{N_2}) has the p.g.f.

$$P(s_1, s_2) = \frac{p_1 p_2}{1 - p_0 Q(s_1, s_2)} \left(\frac{Q(s_1, 1) + Q(1, s_2) - Q(s_1, 1)Q(1, s_2)(2 - p_1 - p_2)}{(1 - (1 - p_1)Q(s_1, 1))(1 - (1 - p_2)Q(1, s_2))} \right). \tag{4.38}$$

From (4.38), we get

$$P(s_1, 1) = \frac{p_1}{1 - (1 - p_1)Q(s_1, 1)} \quad \text{and} \quad P(1, s_2) = \frac{p_2}{1 - (1 - p_2)Q(1, s_2)}. \tag{4.39}$$

This bivariate geometric compounding was considered in Arnold (1975) to obtain the characterization of bivariate exponential distribution. Using (4.38), now we generate a bivariate discrete Mittag-Leffler distribution.

Theorem 4.14. *Suppose that $\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed random vectors and (N_1, N_2), independent of $(X_i, Y_i), i \geq 1$ follows bivariate geometric distribution with p.g.f. in (4.37) . Then (U_{N_1}, V_{N_2}) follows bivariate discrete Mittag-Leffler distribution if and only if $(X_i, Y_i), i \geq 1$ have the bivariate Sibuya distribution in (4.33).*

Proof. Let $(X_i, Y_i), i \geq 1$ follow bivariate Sibuya distribution in (4.33). Substituting

$Q(s_1, 1)$ and $Q(1, s_2)$ in (4.39), we get

$$P(s_1, 1) = \frac{1}{1 + \frac{1-p_1}{p_1}(1 - s_1)^{\alpha_1}} \quad \text{and} \quad P(1, s_2) = \frac{1}{1 + \frac{1-p_2}{p_2}(1 - s_2)^{\alpha_2}}.$$

Also from (4.38), the p.g.f. of (U_{N_1}, V_{N_2}) is

$$P(s_1, s_2) = \frac{1 - (1 - s_1)^{\alpha_1}(1 - s_2)^{\alpha_2} - p_0(1 - (1 - s_1)^{\alpha_1})(1 - (1 - s_2)^{\alpha_2})}{(1 - p_0(1 - (1 - s_1)^{\alpha_1} - (1 - s_2)^{\alpha_2}))\left(1 + \frac{1-p_1}{p_1}(1 - s_1)^{\alpha_1}\right)\left(1 + \frac{1-p_2}{p_2}(1 - s_2)^{\alpha_2}\right)}. \tag{4.40}$$

Conversely assume that (U_{N_1}, V_{N_2}) has the p.g.f. in (4.40). Substitute $P(s_1, 1)$ and $P(1, s_2)$ in (4.39). We obtain

$$Q(s_1, 1) = 1 - (1 - s_1)^{\alpha_1} \qquad Q(1, s_2,) = 1 - (1 - s_2)^{\alpha_2}.$$

Further substituting $Q(s_1, 1)$ and $Q(1, s_2)$ in (4.38) and on simplification, we get (4.33). □

Suppose that in Theorem 4.14, the sequence of random vectors $\{(X_i, Y_i), i \geq 1\}$ have bivariate Sibuya distribution with p.g.f. in (4.34). Let (N_1, N_2) follow bivariate geometric distribution with p.g.f. in (4.37) and independent of $(X_i, Y_i), i \geq 1$. Then the p.g.f. (U_{N_1}, V_{N_2}) is as follows:

$$P(s_1, s_2) = \frac{1 - (1 - s_1)^{\alpha_1}(1 - s_2)^{\alpha_2} - p_0(1 - (1 - s_1)^{\alpha_1})(1 - (1 - s_2)^{\alpha_2})}{(1 - p_0(1 - (1 - s_1)^{\alpha_1})(1 - (1 - s_2)^{\alpha_2}))\left(1 + \frac{1-p_1}{p_1}(1 - s_1)^{\alpha_1}\right)\left(1 + \frac{1-p_2}{p_2}(1 - s_2)^{\alpha_2}\right)}.$$

SOME BIVARIATE DISTRIBUTIONS GENERATED THROUGH COMPOUNDING

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Chapter 5

A Generalization of BDML$(c_1, c_2, \alpha_1, \alpha_2, \theta)$

5.1 Introduction

As discussed in Chapter 1, a discrete analogue of the quasi factorial gamma distribution called discrete Linnik distribution is investigated in Christoph and Schreiber (1998a). In this Chapter, we introduce a bivariate form of discrete Linnik distribution and study its properties.

We define a bivariate discrete Linnik distribution as follows:

Definition 5.1. A non negative integer valued random vector (X, Y) is said to follow bivariate discrete Linnik distribution if it has p.g.f.

$$P(s_1, s_2) = \left(\frac{1}{(1 + c_1(1 - s_1)^{\alpha_1})(1 + c_2(1 - s_2)^{\alpha_2}) - \theta c_1 c_2 (1 - s_1)^{\alpha_1}(1 - s_2)^{\alpha_2}}\right)^{v}, \tag{5.1}$$

$$c_1,\ c_2 > 0,\ v > 0,\ 0 < \alpha_1,\ \alpha_2\ \leq 1,\ 0 \leq \theta \leq 1,\ |s_1| \leq 1,\ |s_2| \leq 1.$$

We represent the distribution with the above p.g.f. by BDL $(c_1, c_2, \alpha_1, \alpha_2, \theta, v)$. It is to be noted that BDL $(c_1, c_2, \alpha_1, \alpha_2, \theta, v)$ generalizes many important distributions. When $\alpha_1 = \alpha_2 = 1$, we get

$$P(s_1, s_2) = \left(\frac{1}{(1 + c_1(1 - s_1))(1 + c_2(1 - s_2)) - \theta c_1 c_2(1 - s_1)(1 - s_2)}\right)^v. \tag{5.2}$$

(5.2) represents the p.g.f. of a random vector with bivariate negative binomial distribution and we denote the respective distribution by BNBD (c_1, c_2, θ, v). When $v = 1$, (5.1) coincides with the p.g.f. of BDML $(c_1, c_2, \alpha_1, \alpha_2, \theta)$ distribution obtained in (4.2). Moreover, when $\alpha_1 = \alpha_2 = v = 1$, $P(s_1, s_2)$ represents the p.g.f. of BGD (c_1, c_2, θ).

From (5.1), it is clear that p.g.f.s of the components of (X, Y) are that of univariate discrete Linnik distribution given in (1.10).

When $\theta = 1$, (5.1) becomes

$$P(s_1, s_2) = \left(\frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}\right)^v. \tag{5.3}$$

Distributional properties of BDL $(c_1, c_2, \alpha_1, \alpha_2, \theta, v)$ are studied in Section 2. In Section 3, we obtain characterizations BDL $(c_1, c_2, \alpha_1, \alpha_2, \theta, v)$ using negative binomial compounding. Autoregressive models with BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ marginals are developed in Section 4.

5.2 Distributional Properties

We show that a random vector (X, Y) following BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ is normally attracted to the bivariate positive stable law.

Suppose that (X, Y) follow BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ distribution with p.g.f. in (5.3). Then the Laplace transform corresponding to (5.3) is

$$\phi(\lambda_1, \lambda_2) = \left(\frac{1}{1 + c_1(1 - e^{-\lambda_1})^{\alpha_1} + c_2(1 - e^{-\lambda_2})^{\alpha_2}}\right)^v. \tag{5.4}$$

Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors with Laplace transform in (5.4). Define

$$\begin{aligned} U_n &= n^{-\frac{1}{\alpha_1}}(X_1 + X_2 + ... + X_n) \quad \text{and} \\ V_n &= n^{-\frac{1}{\alpha_2}}(Y_1 + Y_2 + ... + Y_n). \end{aligned}$$

Then (U_n, V_n) has Laplace transform

$$\begin{aligned} \varphi_{U_n,V_n}(\lambda_1, \lambda_2) &= E(e^{-(\lambda_1 U_n + \lambda_2 V_n)}) \\ &= \left(\phi(\lambda_1 n^{-\frac{1}{\alpha_1}}, \lambda_2 n^{-\frac{1}{\alpha_2}})\right)^n \\ &= \left(\frac{1}{1 + c_1(1 - e^{-\lambda_1 n^{-\frac{1}{\alpha_1}}})^{\alpha_1} + c_2(1 - e^{-\lambda_2 n^{-\frac{1}{\alpha_2}}})^{\alpha_2}}\right)^{nv}. \end{aligned}$$

Also we have

$$(1 - e^{-\lambda_1 n^{-\frac{1}{\alpha_1}}})^{\alpha_1} = \frac{\lambda_1^{\alpha_1}}{n}(1 + o(1/n)) \quad \text{and} \quad (1 - e^{-\lambda_2 n^{-\frac{1}{\alpha_2}}})^{\alpha_2} = \frac{\lambda_2^{\alpha_2}}{n}(1 + o(1/n)).$$

When $n \to \infty$, we get, $\varphi_{U_n,V_n}(\lambda_1, \lambda_2) \to e^{-c_1 v \lambda_1^{\alpha_1} - c_2 v \lambda_2^{\alpha_2}}$.

Hence BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ distribution is normally attracted to bivariate positive stable law given in (2.4).

We can also obtain bivariate quasi factorial gamma distribution discussed in (3.2) as limit of a sequence of random vectors following bivariate discrete Linnik distribution. For this, consider a random vector (X, Y) with Laplace transform in (5.4). Replace c_1 and c_2 by $c_1 n^{\alpha_1}$ and $c_2 n^{\alpha_2}$ respectively. Then the Laplace transform of $\left(\frac{X}{n}, \frac{Y}{n}\right)$ will be

$$\phi_n(\lambda_1, \lambda_2) = \left(\frac{1}{1 + c_1 n^{\alpha_1}(1 - e^{\frac{-\lambda_1}{n}})^{\alpha_1} + c_2 n^{\alpha_2}(1 - e^{\frac{-\lambda_2}{n}})^{\alpha_2}}\right)^v.$$

When $n \longrightarrow \infty$, we get

$$\phi_n(\lambda_1, \lambda_2) \longrightarrow \left(\frac{1}{1 + c_1 \lambda_1^{\alpha_1} + c_2 \lambda_2^{\alpha_2}}\right)^v.$$

The following theorem shows the attraction of the BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ towards bivariate discrete stable law.

Theorem 5.1. *Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independent random vectors and identically distributed according to BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$. Define*

$$U_n = n^{\frac{-1}{\alpha_1}} \oplus (X_1 + X_2 + ... + X_n) \quad and \quad V_n = n^{\frac{-1}{\alpha_2}} \oplus (Y_1 + Y_2 + ... + Y_n).$$

Then (U_n, V_n) is asymptotically distributed according to bivariate discrete stable law with p.g.f. in (4.9)

Proof. Let $(X_i, Y_i), i \geq 1$ are distributed as BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$.

$$P(s_1, s_2) = \left(\frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}\right)^v.$$

The p.g.f. of (U_n, V_n) is

$$
\begin{aligned}
P_{U_n,V_n}(s_1,s_2) &= E(s_1^{n^{\frac{-1}{\alpha_1}} \oplus (X_1+X_2+\ldots+X_n)} s_2^{n^{\frac{-1}{\alpha_2}} \oplus (Y_1+Y_2+\ldots+Y_n)}) \\
&= \left[E(s_1^{n^{\frac{-1}{\alpha_1}} \oplus X_i} s_2^{n^{\frac{-1}{\alpha_2}} \oplus Y_i})\right]^n \\
&= \left(\frac{1}{1+c_1(1-(1-n^{\frac{-1}{\alpha_1}}+n^{\frac{-1}{\alpha_1}}s_1))^{\alpha_1}+c_2(1-(1-n^{\frac{-1}{\alpha_2}}+n^{\frac{-1}{\alpha_2}}s_2))^{\alpha_2}}\right)^{nv} . \\
&= \left(\frac{1}{1+\frac{c_1}{n}(1-s_1)^{\alpha_1}+\frac{c_2}{n}(1-s_2)^{\alpha_2}}\right)^{nv} .
\end{aligned}
$$

As $n \longrightarrow \infty$, we get

$$
P_{U_n,V_n}(s_1,s_2) \longrightarrow e^{-c_1 v(1-s_1)^{\alpha_1} - c_2 v(1-s_2)^{\alpha_2}} .
$$

□

We obtain BDL $(c_1, c_2, \alpha_1, \alpha_2, \theta, v)$ as a mixture of bivariate discrete stable distribution and gamma distribution with parameters β and v.

Let the joint distribution of the random vector (S, T) be bivariate discrete stable having p.g.f. in (4.9), exponents α_1, and α_2 and parameters $c_1 = c_2 = W$. Suppose that W follows gamma distribution with parameters β and v. Consider the unconditional distribution of (S, T). Its p.g.f. is

$$
\begin{aligned}
P(s_1,s_2) &= \int_0^\infty e^{-w((1-s_1)^{\alpha_1}+(1-s_2)^{\alpha_2})} \frac{\beta^v e^{-\beta w} w^{v-1}}{\Gamma(v)} dw \\
&= \left(\frac{\beta}{\beta+(1-s_1)^{\alpha_1}+(1-s_2)^{\alpha_2}}\right)^v \\
&= \left(\frac{1}{1+\frac{1}{\beta}((1-s_1)^{\alpha_1}+(1-s_2)^{\alpha_2})}\right)^v .
\end{aligned}
$$

Hence (S, T) follows BDL $(c_1, c_2, \alpha_1, \alpha_2, \theta, 1)$ distribution such that $c_1 = c_2 = \dfrac{1}{\beta}$.

5.3 Characterization of BDL $(c_1, c_2, \alpha_1, \alpha_2, \theta, \upsilon)$ through Negative Binomial Compounding

In order to obtain characterizations of BDL $(c_1, c_2, \alpha_1, \alpha_2, \theta, \upsilon)$ distribution, we consider the negative binomial sums of independently and identically distributed random vectors. Let $U_N = \sum_{i=1}^{N} X_i, V_N = \sum_{i=1}^{N} Y_i$ where $\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed random vectors with p.g.f. $Q(s_1, s_2)$ and N follows negative binomial distribution in (3.5). Then the p.g.f. of (U_N, V_N) is

$$\begin{aligned}
P(s_1, s_2) &= E(s_1^{U_N} s_2^{V_N}) \\
&= \sum_{n=1}^{\infty} E(s_1^{X_1+X_2+\ldots+X_N} s_2^{Y_1+Y_2+\ldots+Y_N} / N = n) P(N = n) \\
&= \sum_{n=1}^{\infty} E(s_1^{X_1+X_2+\ldots+X_n} s_2^{Y_1+Y_2+\ldots+Y_n}) P(N = n) \\
&= \sum_{n=1}^{\infty} \left[E(s_1^{X_i} s_2^{Y_i})\right]^n \begin{pmatrix} n-1 \\ \upsilon - 1 \end{pmatrix} p^{\upsilon}(1-p)^{n-\upsilon}, \\
&= \left(\frac{p\, Q(s_1,\, s_2)}{1-(1-p)Q(s_1,\, s_2)}\right)^{\upsilon}. \qquad (5.5)
\end{aligned}$$

Using this compounding a characterization of BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ distribution is obtained in the following theorem.

Theorem 5.2. *Let $\{(X_i, Y_i),\ i \geq 1\}$ be a sequence of independently and identically distributed random vectors and N, independent of $(X_i, Y_i), i \geq 1$, be a random variable with negative binomial distribution stated in (3.5). Then, $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ has BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ distribution if and only if $(X_i, Y_i), i \geq 1$ follow BDML $(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution.*

Proof. Suppose that $(X_i, Y_i), i \geq 1$ follow BDML$(c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution with p.g.f. $Q(s_1, s_2)$. Then p.g.f. of $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ is

$$P(s_1, s_2) = E(s_1^{p^{\frac{1}{\alpha_1}} \oplus U_N} s_2^{p^{\frac{1}{\alpha_2}} \oplus V_N}).$$

From (5.5),

$$P(s_1, s_2) = \left(\frac{pQ(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1,\ 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2)}{1 - (1-p)Q(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1,\ 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2)} \right)^v. \tag{5.6}$$

In (5.6) substituting the p.g.f. of $(X_i, Y_i), i \geq 1$, we get

$$P(s_1, s_2) = \left(\frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}} \right)^v.$$

In order to prove the converse, suppose that $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ follows BDL$(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ distribution. Substituting its p.g.f. in (5.6)

$$\left(\frac{1}{1 + c_1 p(1 - s_1)^{\alpha_1} + c_2 p(1 - s_2)^{\alpha_2}} \right)^v = \left(\frac{pQ(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1,\ 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2)}{1 - (1-p)Q(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1,\ 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2)} \right)^v.$$

On simplification, we get

$$Q(s_1, s_2) = \frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}}.$$

□

Jayakumar and Mundassery (2007) obtained a characterization of the BNBD $(c_1, c_2, 1, v)$ distribution using negative binomial compounding.

Remark 5.1. Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random vectors. Then $(p \oplus U_N, p \oplus V_N)$ follows BNBD $(c_1, c_2, 1, v)$ distribution if and only if $(X_i, Y_i), i \geq 1$ have BGD $(c_1, c_2, 1)$ where N is independent of $(X_i, Y_i), i \geq 1$ and follows negative binomial distribution.

Proof of the Remark 5.1 is omitted as it is obvious.

Now we introduce BDL $(c_1, c_2, \alpha_1, \alpha_2, \theta, v)$ using negative binomial sum of a set of independent random vectors in which the components are independently distributed as discrete Mittag-Leffler.

Theorem 5.3. *Suppose that* $\{(X_i, Y_i), i \geq 1\}$ *is a sequence of independently and identically distributed random vectors and* N, *independent of* $(X_i, Y_i), i \geq 1$ *has the negative binomial distribution in (3.5). Then* $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ *follows* $BDL(c_1, c_2, \alpha_1, \alpha_2, 1-p, v)$ *distribution if and only if the components* $X_i's$ *and* $Y_i's$ *are independently distributed according discrete Mittag-Leffler with parameters* (α_1, c_1) *and* (α_2, c_2) *respectively.*

Proof. Assume that the components of $(X_i, Y_i), i \geq 1$ are independently distributed according to discrete Mittag-Leffler with parameters (α_1, c_1) and (α_2, c_2) respectively. Therefore the joint p.g.f. of $(X_i, Y_i), i \geq 1$ is

$$Q(s_1, s_2) = \frac{1}{(1 + c_1(1 - s_1)^{\alpha_1})(1 + c_2(1 - s_2)^{\alpha_2})}.$$

From (5.6), the p.g.f. of $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ is

$$\begin{aligned} P(s_1, s_2) &= \left(\frac{p}{(1+pc_1(1-s_1)^{\alpha_1})(1+pc_2(1-s_2)^{\alpha_2})-1+p}\right)^v \\ &= \left(\frac{p}{pc_1(1-s_1)^{\alpha_1}+pc_2(1-s_2)^{\alpha_2}+p^2c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}+p}\right)^v \\ &= \left(\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}+pc_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}\right)^v. \end{aligned}$$

Comparing with (5.1), we get $\theta = 1 - p$.

Conversely, suppose that $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ follows BDL$(c_1, c_2, \alpha_1, \alpha_2, 1-p, v)$. From (5.6), we get

$$\left(\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}+pc_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}\right)^v = \left(\frac{pQ(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1, 1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2)}{1-(1-p)Q(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1, 1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2)}\right)^v.$$

Solving, we obtain that $X_i's$ and $Y_i's$ are independently distributed according to discrete Mittag-Leffler with parameters (α_1, c_1) and (α_2, c_2) respectively. □

Jayakumar and Mundassery (2007) obtained BNBD (c_1, c_2, θ, v) distribution as the negative binomial sum of independently and identically distributed random vectors.

Remark 5.2. Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors. Let N be independent of $(X_i, Y_i), i \geq 1$ and follow negative binomial distribution stated in (3.5). Then $(p \oplus U_N,\ p \oplus V_N)$ follows BNBD $(c_1, c_2, 1-p, v)$ distribution if and only if $(X_i, Y_i), i \geq 1$ are BGD $(c_1, c_2, 0)$ random vectors.

Proof of the Remark 5.2 follows easily.

Now, we obtain a characterization of the negative binomial distribution.

Theorem 5.4. *Consider a sequence* $\{(X_i, Y_i), i \geq 1\}$ *of independent and identical random vectors following BDML* $(c_1, c_2, \alpha_1, \alpha_2, 1)$ *distribution. Then for* $0 < p < 1$, $(p^{\frac{1}{\alpha_1}} \oplus U_N, p^{\frac{1}{\alpha_2}} \oplus V_N)$ *has BDL* $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ *distribution if and only if* N *follows the negative binomial distribution in (3.5).*

Proof of the theorem is analogous to that of Theorem 3.6.

A characterization of negative binomial distribution along the lines of Theorem 5.4 can be obtained using BNBD $(c_1, c_2, 1, v)$ and BGD $(c_1, c_2, 1)$.

5.4 Autoregressive Processes with BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ Marginals

In the following theorem we obtain a necessary and sufficient condition for a first order autoregressive process with marginals have BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ distribution to be stationary.

Theorem 5.5. *Let* $(X_n, Y_n), \geq 1$ *constitute a first order autoregressive process with structure*

$$(X_n, Y_n) = (\rho^{\frac{1}{\alpha_1}} \oplus X_{n-1} + \epsilon_n, \rho^{\frac{1}{\alpha_2}} \oplus Y_{n-1} + \psi_n), \quad 0 \leq \rho < 1 \tag{5.7}$$

where $(\epsilon_n, \psi_n), n \geq 1$ *is a sequence of independently and identically distributed random vectors . Then the process in (5.7) is stationary with BDL* $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$

marginals if and only if innovation random vectors, $(\epsilon_n, \psi_n), n \geq 1$ *have p.g.f.*

$$P_{\epsilon,\psi}(s_1, s_2) = \left(\rho + \frac{1-\rho}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2}}\right)^v \tag{5.8}$$

provided (X_0, Y_0) *has the BDL* $(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ *distribution.*

Proof. The p.g.f. of (5.7), is

$$P_{X_n,Y_n}(s_1, s_2) = P_{X_{n-1},Y_{n-1}}(1 - \rho^{\frac{1}{\alpha_1}} + \rho^{\frac{1}{\alpha_1}} s_1, 1 - \rho^{\frac{1}{\alpha_2}} + \rho^{\frac{1}{\alpha_2}} s_2) P_{\epsilon_n,\psi_n}(s_1, s_2). \tag{5.9}$$

Suppose that the process is stationary with BDL$(c_1, c_2, \alpha_1, \alpha_2, 1, v)$ marginals. Then from (5.9) we have,

$$\left(\frac{1}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2}}\right)^v = \left(\frac{1}{1 + \rho c_1(1-s_1)^{\alpha_1} + \rho c_2(1-s_2)^{\alpha_2}}\right)^v P_{\epsilon,\psi}(s_1, s_2).$$

Hence we get,

$$P_{\epsilon,\psi}(s_1, s_2) = \left(\rho + \frac{1-\rho}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2}}\right)^v.$$

Proof of the converse is obtained by induction method. When $n = 1$, from (5.9), we get

$$P_{X_1,Y_1}(s_1, s_2) = P_{X_0,Y_0}(1 - \rho^{\frac{1}{\alpha_1}} + \rho^{\frac{1}{\alpha_1}} s_1, 1 - \rho^{\frac{1}{\alpha_2}} + \rho^{\frac{1}{\alpha_2}} s_2) P_{\epsilon_n,\psi_n}(s_1, s_2).$$

Suppose that $(\epsilon_n, \psi_n), n \geq 1$ have the p.g.f. given in (5.8). Therefore,

$$\begin{aligned} P_{X_1,Y_1}(s_1, s_2) &= \left(\frac{1}{1 + \rho c_1(1-s_1)^{\alpha_1} + \rho c_2(1-s_2)^{\alpha_2}}\right)^v \\ &\qquad \left(\rho + \frac{1-\rho}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2}}\right)^v \\ &= \left(\frac{1}{1 + c_1(1-s_1)^{\alpha_1} + c_2(1-s_2)^{\alpha_2}}\right)^v. \end{aligned}$$

By mathematical induction, it follows that the process $\{(X_n, Y_n), n \geq 1\}$ is stationary with BDL$(\mu_1, \mu_2, \alpha_1, \alpha_2, 1, v)$ marginals. $\square$

Now we develop a first order autoregressive process with BNBD $(c_1, c_2, 1, v)$ marginals.

Remark 5.3. Suppose that an autoregressive process $\{(X_n, Y_n), n \geq 1\}$ has the structure

$$(X_n, Y_n) = (\rho \oplus X_{n-1} + \epsilon_n, \rho \oplus Y_{n-1} + \psi_n), 0 \leq \rho < 1$$

where $(\epsilon_n, \psi_n), n \geq 1$ is a sequence of independently and identically distributed random vectors with p.g.f.

$$P_{\epsilon_n,\psi_n}(s_1, s_2) = \left(\rho + \frac{1-\rho}{1 + c_1(1-s_1) + c_2(1-s_2)}\right)^v. \tag{5.10}$$

Let (X_0, Y_0) have BNBD $(c_1, c_2, 1, v)$ distribution. Then the process $(X_n, Y_n), n \geq 1$ is stationary with BNBD $(c_1, c_2, 1, v)$ marginals if and only if $(\epsilon_n, \psi_n), n \geq 1$ have the p.g.f. in (5.10).

We omit the proof as it is obvious.

Jayakumar and Gadag (1999) developed a first order autoregressive random coefficient model with structure

$$X_n = U_n X_{n-1} + \epsilon_n, n = 1, 2, 3, \ldots.$$

where $\{\epsilon_n, n \geq 1\}$ and $\{U_n, \geq 1\}$ are two independent sequences of independently and identically distributed random vectors and U_n has distribution function $F(u) = u^{\alpha v}$, $0 < \alpha \leq 1,\ v > 0,\ 0 < u < 1$.

In the following theorem we develop a first order autoregressive process with this structure and having BDL $(c_1, c_2, \alpha, \alpha, \theta, v+1)$ marginals.

Theorem 5.6. *Consider a first order autoregressive process* $\{(X_n, Y_n), n \geq 1\}$ *with structure*

$$(X_n, Y_n) = (T_n X_{n-1} + \epsilon_n, T_n Y_{n-1} + \psi_n). \quad for \;\; n = 1, 2, 3, ... \tag{5.11}$$

Let $\{T_n, n \geq 1\}$ *be a sequence of independent random variables distributed according to* $F(t) = t^{\alpha v}$, $0 < \alpha \leq 1, v > 0$, $0 < t < 1$ *and* $(\epsilon_n, \psi_n), n \geq 1$ *denote the sequence of innovations which are independent of* $T_n, n \geq 1$. *Then* $(X_n, Y_n), n \geq 1$ *define a first order stationary autoregressive process with marginals* $BDL(c_1, c_2, \alpha, \alpha, 1, v + 1)$ *distribution if and only if* $(\epsilon_n, \psi_n), n \geq 1$ *are distributed as* $BDML(c_1, c_2, \alpha, \alpha, 1)$.

Proof is analogous to that of Theorem 3.7.

SOME BIVARIATE DISTRIBUTIONS GENERATED THROUGH COMPOUNDING

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Chapter 6

Bivariate Tailed Distributions

6.1 Introduction

In real life we encounter a number of situations where the multivariate forms of tailed distributions are applied for their modeling. For example, consider the study of reliability of a two component system. Let the random variables X and Y represent lifetimes of the components. Suppose that the system will fail because of the instantaneous failure of the components. In this situation, we can apply the bivariate tailed distributions to study probability behavior of the random vector (X, Y). Tail of the non negative random vector (X, Y) includes the positive part of the sample space excluding the point $(X = 0, Y = 0)$. Therefore in bivariate tailed distributions, we express the probability distribution of (X, Y) as mixture of a probability 'σ' at

$(X = 0, Y = 0)$ and the remaining part with probability '$1-\sigma$'. In Section 2, we introduce the tailed form of BML$(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ distribution and develop autoregressive process with bivariate tailed Mittag-Leffler marginals. A discrete analogue of bivariate tailed Mittag-Leffler distribution is studied in Section 3 and the corresponding autoregressive model is developed.

6.2 Bivariate Tailed Mittag-Leffler Distribution

We define now the tailed form of BML$(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ distribution given in (2.1).

Definition 6.1. A non negative random vector (X, Y) is said to follow bivariate tailed Mittag-Leffler distribution if its Laplace transform is

$$\phi(\lambda_1, \lambda_2) = \frac{1 + \sigma(\mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2} + (1-\theta)\mu_1\lambda_1^{\alpha_1}\mu_2\lambda_2^{\alpha_2})}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2} + (1-\theta)\mu_1\lambda_1^{\alpha_1}\mu_2\lambda_2^{\alpha_2}}, \tag{6.1}$$

$$0 \leq \sigma < 1 \; ; \; \mu_1, \mu_2 > 0 \; ; \; 0 < \alpha_1, \alpha_2 \leq 1 \; ; \; 0 \leq \theta \leq 1.$$

It is denoted by BTML$(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$.

A non negative random vector (X, Y) following BTML$(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ distribution has a probability 'σ' at $(X = 0, Y = 0)$ and for $(X > 0, Y > 0)$ it admits $(1-\sigma)$ times the bivariate Mittag-Leffler distribution with Laplace transform in (2.1). That is, the following expression leads to (6.1)

$$\phi(\lambda_1, \lambda_2) = \sigma + (1-\sigma)\frac{1}{(1 + \mu_1\lambda_1^{\alpha_1})(1 + \mu_2\lambda_2^{\alpha_2}) - \theta\mu_1\lambda_1^{\alpha_1}\mu_2\lambda_2^{\alpha_2}}.$$

When $\sigma = 0$, we get (2.1).

The Laplace transform of BTML$(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution is

$$\phi(\lambda_1, \lambda_2) = \frac{1 + \sigma\left(\mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}\right)}{1 + \left(\mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}\right)}. \tag{6.2}$$

In the following theorem we develop autoregressive models with marginals have BTML$(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution.

Theorem 6.1. *Consider a bivariate first order autoregressive process*

$$\begin{aligned} (X_0, Y_0) &\stackrel{d}{=} BTML(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1) \text{ and for } n = 1, 2, 3, \ldots \\ (X_n, Y_n) &= \begin{cases} (\epsilon_n, \psi_n) & \text{with probability } \rho \\ (a^{\frac{1}{\alpha_1}} X_{n-1} + \epsilon_n, a^{\frac{1}{\alpha_2}} Y_{n-1} + \psi_n) & \text{with probability } 1-\rho \end{cases} \\ & \qquad 0 \le a < 1\ ,\ 0 \le \rho \le 1. \end{aligned} \tag{6.3}$$

where $\{(\epsilon_n, \psi_n), n \ge 1\}$ is a sequence of independently and identically distributed random vectors. Suppose that $\{(U_n, V_n), n \ge 1\}$ and $\{(R_n, S_n), n \ge 1\}$ are two independent sequences distributed as $BTML(a, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ and $BTML(\sigma', \mu_1', \mu_2', \alpha_1, \alpha_2, 1)$ respectively such that $\sigma' = \frac{\sigma}{b}, \mu_1' = \mu_1 b$ and $\mu_2' = \mu_2 b$ where $b = a(\rho + (1-\rho)\sigma)$. Then the process is stationary with marginals follow $BTML$ $(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution if and only if

$$(\epsilon_n, \psi_n) \stackrel{d}{=} (U_n, V_n) + (R_n, S_n). \tag{6.4}$$

Proof. Suppose that $(X_n, Y_n), n \ge 1$ have BTML $(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution. The Laplace transform of the model in (6.3) is

$$\phi_{X_n, Y_n}(\lambda_1, \lambda_2) = \rho\phi_{\epsilon_n, \psi_n}(\lambda_1, \lambda_2) + (1-\rho)\phi_{X_{n-1}, Y_{n-1}}(a^{\frac{1}{\alpha_1}}\lambda_1, a^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon_n, \psi_n}(\lambda_1, \lambda_2). \tag{6.5}$$

If the process is stationary,

$$\phi_{X,Y}(\lambda_1,\lambda_2)=\rho\phi_{\epsilon,\psi}(\lambda_1,\lambda_2)+(1-\rho)\phi_{X,Y}(a^{\frac{1}{\alpha_1}}\lambda_1,a^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon,\psi}(\lambda_1,\lambda_2).$$

Substituting $\phi_{X,Y}(\lambda_1,\lambda_2)$, we get

$$\begin{aligned}\frac{1+\sigma\left(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}\right)}{1+\left(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}\right)} &= \rho\phi_{\epsilon,\psi}(\lambda_1,\lambda_2)+(1-\rho)\frac{1+\sigma\left(a\mu_1\lambda_1^{\alpha_1}+a\mu_2\lambda_2^{\alpha_2}\right)}{1+\left(a\mu_1\lambda_1^{\alpha_1}+a\mu_2\lambda_2^{\alpha_2}\right)}\phi_{\epsilon,\psi}(\lambda_1,\lambda_2)\\ &= \phi_{\epsilon,\psi}(\lambda_1,\lambda_2)\left(\frac{1+(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2})(a(\rho+(1-\rho)\sigma))}{1+a\left(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}\right)}\right).\end{aligned}$$

Simplifying, we get

$$\phi_{\epsilon,\psi}(\lambda_1,\lambda_2)=\left(\frac{1+a\left(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}\right)}{1+\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}}\right)\left(\frac{1+\sigma\left(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}\right)}{1+b(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2})}\right). \tag{6.6}$$

where $b=a(\rho+(1-\rho)\sigma)$.

Now, finding the Laplace transform of (6.4)

$$\phi_{U_n,V_n}(\lambda_1,\lambda_2)=\frac{1+a\left(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}\right)}{1+\left(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}\right)}.$$

$$\phi_{R_n,S_n}(\lambda_1,\lambda_2)=\frac{1+\sigma'\left(\mu_1'\lambda_1^{\alpha_1}+\mu_2'\lambda_2^{\alpha_2}\right)}{1+\left(\mu_1'\lambda_1^{\alpha_1}+\mu_2'\lambda_2^{\alpha_2}\right)}.$$

Substituting $\sigma'=\frac{\sigma}{b}$, $\mu_1'=\mu_1 b$ and $\mu_2'=\mu_2 b$ in $\phi_{R_n,S_n}(\lambda_1,\lambda_2)$, we get

$$\phi_{R_n,S_n}(\lambda_1,\lambda_2)=\frac{1+\sigma\left(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}\right)}{1+b\left(\mu_1\lambda_1^{\alpha_1}+\mu_2\lambda_2^{\alpha_2}\right)}.$$

Thus we can see that Laplace transform of $(U_n,V_n)+(R_n,S_n)$ coincides with (6.6).

Conversely we can show that the process is stationary with marginals BTML $(\sigma,\mu_1,\mu_2,\alpha_1,\alpha_2,1)$ when (ϵ_n,ψ_n) satisfies (6.4). Suppose that (X_0,Y_0) is distributed as BTML $(\sigma,\mu_1,\mu_2,\alpha_1,\alpha_2,1)$.

Put $n=1$ in (6.5)

$$\phi_{X_1,Y_1}(\lambda_1,\lambda_2)=\rho\phi_{\epsilon_1,\psi_1}(\lambda_1,\lambda_2)+(1-\rho)\phi_{X_0,Y_0}(a^{\frac{1}{\alpha_1}}\lambda_1,a^{\frac{1}{\alpha_2}}\lambda_2)\phi_{\epsilon_1,\psi_1}(\lambda_1,\lambda_2).$$

Substituting the Laplace transform of (X_0, Y_0) and (ϵ_1, ψ_1) and simplifying, we get

$$\phi_{X_1,Y_1}(\lambda_1, \lambda_2) = \frac{1 + \sigma\left(\mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}\right)}{1 + \mu_1\lambda_1^{\alpha_1} + \mu_2\lambda_2^{\alpha_2}}.$$

Hence by mathematical induction, we get the process is stationary with marginals follow BTML $(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution. □

As a remark of Theorem 6.1, we can obtain bivariate first order autoregressive process with marginals have tailed form of Moran's bivariate exponential distribution given in (1.17).

We obtain the tailed form of Moran's bivariate exponential distribution by putting $\alpha_1 = \alpha_2 = 1$ in (6.1). Therefore, Laplace transform of Moran's bivariate tailed exponential distribution is

$$\phi(\lambda_1, \lambda_2) = \frac{1 + \sigma(\mu_1\lambda_1 + \mu_2\lambda_2 + (1-\theta)\mu_1\lambda_1\mu_2\lambda_2)}{1 + \mu_1\lambda_1 + \mu_2\lambda_2 + (1-\theta)\mu_1\lambda_1\mu_2\lambda_2},$$

$$0 \le \sigma < 1;\ \mu_1, \mu_2 > 0;\ 0 \le \theta \le 1.$$

We denote the distribution with the above Laplace transform as MBTE $(\sigma, \mu_1, \mu_2, \theta)$. When $\theta = 1$, we have

$$\phi(\lambda_1, \lambda_2) = \frac{1 + \sigma\left(\mu_1\lambda_1 + \mu_2\lambda_2\right)}{1 + \left(\mu_1\lambda_1 + \mu_2\lambda_2\right)}.$$

In the following remark, we develop a bivariate first order autoregressive process with MBTE $(\sigma, \mu_1, \mu_2, 1)$ marginals.

Remark 6.1. Let $\{(\epsilon_n, \psi_n), n \ge 1\}$ be a sequence of independently and identically distributed random vectors such that

$$(\epsilon_n, \psi_n) = (U_n, V_n) + (R_n, S_n) \tag{6.7}$$

where $\{(U_n, V_n), n \geq 1\}$ and $\{(R_n, S_n), n \geq 1\}$ are two independent sequences of independently and identically distributed random vectors. Consider a first autoregressive process with structure:

$$
\begin{aligned}
(X_0, Y_0) &\stackrel{d}{=} MBTE(\sigma, \mu_1, \mu_2, 1) \quad \text{and} \quad \text{for } n; 1, 2, 3, ... \\
(X_n, Y_n) &= \begin{cases} (\epsilon_n, \psi_n) & \text{with probability } \rho \\ (aX_{n-1} + \epsilon_n, aY_{n-1} + \psi_n) & \text{with probability } 1 - \rho \end{cases} \qquad (6.8) \\
& \qquad\qquad\qquad 0 \leq a < 1 \ , 0 \leq \rho \leq 1.
\end{aligned}
$$

Then the process in (6.8) is stationary with marginals follow MBTE $(\sigma, \mu_1, \mu_2, 1)$ distribution if and only if $(\epsilon_n, \psi_n), n \geq 1$ satisfies (6.7) such that (U_n, V_n) and (R_n, S_n) are distributed as MBTE $(a, \mu_1, \mu_2, 1)$ and MBTE $(\sigma', \mu_1', \mu_2', 1)$ respectively where $\sigma' = \frac{\sigma}{b}$, $\mu_1' = \mu_1 b$ and $\mu_2' = \mu_2 b$ and $b = a(\rho + (1 - \rho)\sigma)$.

Proof of the Remark 6.1 is omitted as it is obvious from the proof of Theorem 6.1.

A generalization of BTML $(\sigma, \mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ distribution could be obtained by considering the tailed form of bivariate semi Mittag-Leffler distribution defined in (2.38). For this, we introduce a scale parameter 'δ'(> 0) in BSML (α_1, α_2, p) and therefore (2.38) becomes

$$\phi(\lambda_1, \lambda_2) = \frac{1}{1 + \delta\xi(\lambda_1, \lambda_2)} \tag{6.9}$$

where $\xi(\lambda_1, \lambda_2)$ is as stated in Definition 2.3. Denoting this bivariate semi Mittag-Leffler distribution as BSML $(\delta, \alpha_1, \alpha_2, p)$.

Now, we define the tailed form of the distribution with Laplace transform in (6.9).

Definition 6.2. A non negative random vector (X, Y) is said to follow bivariate tailed semi Mittag-Leffler distribution if it has Laplace transform

$$\phi(\lambda_1, \lambda_2) = \frac{1 + \delta\sigma\xi(\lambda_1, \lambda_2)}{1 + \delta\xi(\lambda_1, \lambda_2)}, \quad 0 \leq \sigma \leq 1. \tag{6.10}$$

and we denote the respective distribution as BTSML $(\delta, \sigma, \alpha_1, \alpha_2, p)$. Note that (6.10) is obtained by assigning a probability 'σ' at $(X = 0, Y = 0)$ and for $(X > 0, Y > 0)$, the probability distribution is $(1 - \sigma)$ times the bivariate semi Mittag-Leffler distribution with Laplace transform in (6.9). That is, (6.10) is obtained from

$$\phi(\lambda_1, \lambda_2 = \sigma + (1 - \sigma)\frac{1}{1 + \delta\xi(\lambda_1, \lambda_2)}.$$

In the following theorem we obtain a first order stationary autoregressive process with BTSML $(\delta, \sigma, \alpha_1, \alpha_2, p)$ marginals.

Theorem 6.2. *The bivariate first order tailed semi Mittag-Leffler autoregressive process can be defined as*

$$(X_n, Y_n) = \begin{cases} (\epsilon_n, \psi_n) & \text{with probability } \rho \\ (a^{\frac{1}{\alpha_1}} X_{n-1} + \epsilon_n, a^{\frac{1}{\alpha_2}} Y_{n-1} + \psi_n) & \text{with probability } 1 - \rho, \end{cases}$$

$$0 \leq a < 1 \ , \ 0 \leq \rho \leq 1,$$

where $\{(\epsilon_n, \psi_n), n \geq 1\}$ *is a sequence of independently and identically distributed random vectors. Then the process is stationary with BTSML* $(\delta, \sigma, \alpha_1, \alpha_2, \rho)$ *marginals if and only if* $(\epsilon_n, \psi_n), n \geq 1$ *satisfies*

$$(\epsilon_n, \psi_n) = (U_n, V_n) + (R_n, S_n)$$

where $\{(U_n, V_n), n \geq 1\}$ and $\{(R_n, S_n), n \geq 1\}$ are two independent sequences of independent and identical random vectors following BTSML $(\delta, a, \alpha_1, \alpha_2, \rho)$ and BTSML $(\delta', \sigma', \alpha_1, \alpha_2, \rho)$ distributions respectively, $\delta' = \delta b$, $\sigma' = \dfrac{\sigma}{b}$ and $b = a(\rho + (1-\rho)\sigma)$, provided (X_0, Y_0) is distributed as BTSML $(\delta, \sigma, \alpha_1, \alpha_2, \rho)$.

Proof of the Theorem 6.3 can obtained by following the arguments used in Theorem 6.1.

The following theorem gives a characterization of BTSML $(\delta, \sigma, \alpha_1, \alpha_2, p)$.

Theorem 6.3. *Consider a first order autoregressive process*

$$(X_n, Y_n) = (\sigma^{\frac{1}{\alpha_1}} X_{n-1} + \epsilon_n, \sigma^{\frac{1}{\alpha_2}} Y_{n-1} + \psi_n), \tag{6.11}$$

for n=1,2,3,..., $0 \leq \sigma < 1$.

When the process is stationary, the innovations $\{(\epsilon_n, \psi_n, n \geq 1\})$ is a sequence of independently and identically distributed random vectors according to BTSML $(\delta, \sigma, \alpha_1, \alpha_2, p)$ if and only if $\{(X_n, Y_n), n \geq 1\}$ are distributed as BSML$(\delta, \alpha_1, \alpha_2, \sigma)$.

Proof. The Laplace transform of (6.11) is

$$\phi_{X_n, Y_n}(\lambda_1, \lambda_2) = \phi_{X_{n-1}, Y_{n-1}}(\sigma^{\frac{1}{\alpha_1}} \lambda_1, \sigma^{\frac{1}{\alpha_2}} \lambda_2) \phi_{\epsilon_n, \psi_n}(\lambda_1, \lambda_2).$$

When the process is stationary

$$\phi_{X,Y}(\lambda_1, \lambda_2) = \phi_{X,Y}(\sigma^{\frac{1}{\alpha_1}} \lambda_1, \sigma^{\frac{1}{\alpha_2}} \lambda_2) \phi_{\epsilon,\psi}(\lambda_1, \lambda_2). \tag{6.12}$$

Suppose that $(\epsilon_n, \psi_n), n \geq 1$ follow BTSML $(\sigma, \delta, \alpha_1, \alpha_2, p)$ distribution.

Therefore,

$$\phi_{\epsilon,\psi}(\lambda_1, \lambda_2) = \frac{1 + \delta\sigma\xi(\lambda_1, \lambda_2)}{1 + \delta\xi(\lambda_1, \lambda_2)}.$$

Substituting $\phi_{\epsilon,\psi}(\lambda_1, \lambda_2)$ in (6.12) and using the fact that any bivariate Laplace transform, $\phi(\lambda_1, \lambda_2)$ can be expressed as $\frac{1}{1 + \delta\xi(\lambda_1, \lambda_2)}$, we get

$$\frac{1}{1 + \delta\xi(\lambda_1, \lambda_2)} = \phi_{X,Y}(\sigma^{\frac{1}{\alpha_1}}\lambda_1, \sigma^{\frac{1}{\alpha_2}}\lambda_2)\frac{1 + \delta\sigma\xi(\lambda_1, \lambda_2)}{1 + \delta\xi(\lambda_1, \lambda_2)}.$$

Therefore,

$$\phi_{X,Y}(\lambda_1, \lambda_2) = \frac{1}{1 + \delta\xi(\lambda_1, \lambda_2)}.$$

Hence $\{(X_n, Y_n), n \geq 1\}$ are distributed as BSML$(\delta, \alpha_1, \alpha_2, \sigma)$.

To prove the converse, substituting the Laplace transform of $(X_n, Y_n), n \geq 1$ in (6.12) and solving, we get

$$\phi_{\epsilon,\psi}(\lambda_1, \lambda_2) = \frac{1 + \delta\sigma\xi(\lambda_1, \lambda_2)}{1 + \delta\xi(\lambda_1, \lambda_2)}.$$

Thus $(\epsilon_n, \psi_n), n \geq 1$ are distributed according to BTSML $(\delta, \sigma, \alpha_1, \alpha_2, p)$. □

6.3 Bivariate Tailed Discrete Mittag-Leffler Distribution

We consider a non negative integer random vector (X, Y) such that it assumes a probability 'σ' when $(X = 0, Y = 0)$ and for other pairs of values it admits '$1 - \sigma$' times the bivariate discrete Mittag-Leffler distribution stated in (4.2). In the following definition we obtain the tailed form of the bivariate discrete Mittag-Leffler distribution that introduced in (4.2).

Definition 6.3. A non negative integer valued random vector (X, Y) is said to follow bivariate tailed discrete Mittag-Leffler distribution if its p.g.f. is

$$P(s_1, s_2) = \frac{1+\sigma(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}+(1-\theta)c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2})}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}+(1-\theta)c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}, \tag{6.13}$$

$$c_1, c_2 > 0, \;\; 0 < \alpha_1, \alpha_1 \leq 1, \;\; 0 \leq \theta \leq 1, \;\; |s_1| \leq 1, \; |s_2| \leq 1.$$

It is denoted by BTDML $(\sigma, c_1, c_2, \alpha_1, \alpha_2, \theta)$. As stated earlier, we can obtain (6.13) from

$$P(s_1, s_2) = \sigma+(1-\sigma)\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}+(1-\theta)c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}.$$

When $\theta = 1$, (6.13) becomes

$$P(s_1, s_2) = \frac{1+\sigma(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2})}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}. \tag{6.14}$$

Now, in the following theorem we generate (6.14) as the distribution of random sum of independently and identically distributed random vectors according to bivariate Sibuya distribution with p.g.f. in (4.33). We assume that the number of summands follow the 'zero modified' geometric distribution in (1.21).

Theorem 6.4. *Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random vectors. Let N be independent of $(X_i, Y_i), i \geq 1$, and have the 'zero modified' geometric distribution in (1.21). Let $U_N = \sum_{i=1}^{N} X_i$ and $V_N = \sum_{i=1}^{N} Y_i$. Then*

(U_N, V_N) is distributed as BTDML $(\sigma, c, c, \alpha_1, \alpha_2, 1)$ where $c = \dfrac{1-p}{p}$ if and only if $(X_i, Y_i), i \geq 1$ follow bivariate Sibuya distribution with p.g.f. in (4.33).

Proof. Suppose that $(X_i, Y_i), i \geq 1$ are independently and identically distributed random vectors and have p.g.f. $Q(s_1, s_2)$. Then the p.g.f. of (U_N, V_N) is

$$\begin{aligned}
P(s_1, s_2) &= E(s_1^{U_N} s_2^{V_N}) \\
&= \sum_{n=0}^{\infty} E(s_1^{X_1+X_2+\ldots+X_N} s_2^{Y_1+Y_2+\ldots+Y_N} / N = n) P(N = n) \\
&= \sum_{n=0}^{\infty} E(s_1^{X_1+X_2+\ldots+X_n} s_2^{Y_1+Y_2+\ldots+Y_n}) P(N = n) \\
&= \sum_{n=0}^{\infty} \left(E s_1^{X_i} s_2^{Y_i}\right)^n P(N = n) \\
&= \sum_{n=0}^{\infty} (Q(s_1, s_2))^n P(N = n) \\
&= \sigma + (1-\sigma) \frac{p}{1-(1-p)Q(s_1, s_2)}. \qquad (6.15)
\end{aligned}$$

Putting $Q(s_1, s_2) = 1 - (1-s_1)^{\alpha_1} - (1-s_2)^{\alpha_2}$ in (6.15), we get

$$P(s_1, s_2) = \frac{1 + \sigma(c(1-s_1)^{\alpha_1} + c(1-s_2)^{\alpha_2})}{1 + c(1-s_1)^{\alpha_1} + c(1-s_2)^{\alpha_2}}$$

where $c = \dfrac{1-p}{p}$. Hence we get (U_N, V_N) is distributed as BTDML $(\sigma, c, c, \alpha_1, \alpha_2, 1)$.

In order to prove the converse, substituting $P(s_1, s_2)$ in (6.15), we have

$$\frac{1 + \sigma(c(1-s_1^{\alpha_1}) + c(1-s_2^{\alpha_2}))}{1 + c(1-s_1^{\alpha_1}) + c(1-s_2^{\alpha_2})} = \sigma + (1-\sigma)\frac{p}{1-(1-p)Q(s_1, s_2)}.$$

Solving, we get

$$Q(s_1, s_2) = 1 - (1-s_1)^{\alpha_1} - (1-s_2)^{\alpha_2}.$$

□

In the following theorem we develop a first order stationary autoregressive process in which the marginals follow BTDML $(\sigma, c_1, c_2, \alpha_1, \alpha_2, 1)$ distribution.

Theorem 6.5. *Let a bivariate autoregressive process have the structure*

$$(X_n, Y_n) = \begin{cases} (\epsilon_n, \psi_n) & \textit{with probability } \rho \\ (a^{\frac{1}{\alpha_1}} \oplus X_{n-1} + \epsilon_n, a^{\frac{1}{\alpha_2}} \oplus Y_{n-1} + \psi_n) & \textit{with probability } 1-\rho \end{cases} \tag{6.16}$$

$$0 \le a < 1 \ , \ 0 \le \rho \le 1.$$

where $\{(\epsilon_n, \psi_n), n \ge 1\}$ *is a sequence of independently and identically distributed random vectors and satisfies*

$$(\epsilon_n, \psi_n) \stackrel{d}{=} (U_n, V_n) + (R_n, S_n). \tag{6.17}$$

where $\{(U_n, V_n), n \ge 1\}$ *and* $\{(R_n, S_n), n \ge 1\}$ *are two independent sequences. Then the process is stationary with marginals follow BTDML* $(\sigma, c_1, c_2, \alpha_1, \alpha_2, 1)$ *distribution if and only if* $\{(U_n, V_n), n \ge 1\}$ *and* $\{(R_n, S_n), n \ge 1\}$ *are distributed according to BTDML* $(a, c_1, c_2, \alpha_1, \alpha_2, 1)$ *and BTDML* $(\sigma', c_1', c_2', \alpha_1, \alpha_2, 1)$ *respectively where* $\sigma' = \frac{\sigma}{b}, c_1' = c_1 b, c_2' = c_2 b$, $b = a(\rho + (1-\rho)\sigma)$ *and provided*

$$(X_0, Y_0) \stackrel{d}{=} BTDML(\sigma, c_1, c_2, \alpha_1, \alpha_2, 1).$$

Proof. The p.g.f. of (6.16) is

$P_{X_n, Y_n}(s_1, s_2)$

$$= \rho P_{\epsilon_n, \psi_n}(s_1, s_2) + (1-\rho) P_{X_{n-1}, Y_{n-1}}(1 - a^{\frac{1}{\alpha_1}} + a^{\frac{1}{\alpha_1}} s_1, 1 - a^{\frac{1}{\alpha_2}} + a^{\frac{1}{\alpha_2}} s_2) P_{\epsilon_n, \psi_n}(s_1, s_2). \tag{6.18}$$

Assume that the process $(X_n, Y_n, \ge 1)$ is stationary with BTDML $(\sigma, c_1, c_2, \alpha_1, \alpha_2, 1)$ marginals. Then (6.18) becomes

$$\frac{1+\sigma(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2})}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}$$

$$= P_{\epsilon,\psi}(s_1,s_2)\left(\rho+1-\rho\frac{1+\sigma a(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2})}{1+a(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2})}\right).$$

Solving we get,

$$P_{\epsilon,\psi}(s_1,s_2)=\left(\frac{1+a(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2})}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}\right)\left(\frac{1+\sigma(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2})}{1+b(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2})}\right) \tag{6.19}$$

where $b=a(\rho+(1-\rho)\sigma)$. Now finding the p.g.f. of (6.17)

$$P_{U_n,V_n}(s_1,s_2)=\frac{1+a\left(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}\right)}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}.$$

$$P_{R_n,S_n}(s_1,s_2)=\frac{1+\sigma'\left(c_1'(1-s_1)^{\alpha_1}+c_2'(1-s_2)^{\alpha_2}\right)}{1+c_1'(1-s_1)^{\alpha_1}+c_2'(1-s_2)^{\alpha_2}}.$$

Replacing σ', c_1', c_2' as given, we get

$$P_{R_n,S_n}(s_1,s_2)=\frac{1+\sigma\left(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}\right)}{1+b\left(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}\right)}.$$

Since $\{(U_n,V_n),n\geq 1\}$ and $\{(R_n,S_n),n\geq 1\}$ are two independent sequences of random vectors, we get that p.g.f. of (6.17) coincides with (6.19).

In order to prove the converse, assume that $(\epsilon_n,\psi_n),n\geq 1$ satisfies (6.17). Put $n=1$ in (6.18), we get

$$P_{X_1,Y_1}(s_1,s_2)=\rho\phi_{\epsilon_1,\psi_1}(s_1,s_2)+(1-\rho)P_{X_0,Y_0}(1-a^{\frac{1}{\alpha_1}}+a^{\frac{1}{\alpha_1}}s_1,1-a^{\frac{1}{\alpha_2}}+a^{\frac{1}{\alpha_2}}s_2)P_{\epsilon_1,\psi_1}(s_1,s_2).$$

Substituting the p.g.f. of (X_0,Y_0) and (ϵ_1,ψ_1) and simplifying, we get

$$P_{X_1,Y_1}(s_1s_2)=\frac{1+\sigma\left(c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}\right)}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}.$$

By induction, we get the process is stationary with marginals have BTDML $(\sigma,c_1,c_2,\alpha_1,\alpha_2,1)$ distribution. □

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Chapter 7

Summary and Conclusion

In the present work, we have introduced bivariate Mittag-Leffler (BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$) distribution, its generalizations, discrete analogues and studied their properties. BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ distribution gives a generalization to the well known Moran's bivariate exponential distribution. Distributional properties of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ like, distribution function, density function, product moments, etc are obtained. The distribution is characterized using a random summation in which the number of summands have geometric or bivariate geometric distribution. Using these compoundings, we have obtained BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ as the distribution of sum of random vectors in which the components are independently distributed as Mittag-Leffler. Estimates of the parameters are obtained using log moments of the distribution. First order stationary autoregressive processes with BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, 1)$ marginals are developed. The bivariate Mittag-Leffler forms of different important bivariate exponential distributions like Marshall-Olkin's bivariate exponential, Hawkes' bivariate exponential and Paulson's bivariate exponential are also introduced. As a generalization to the BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ distribution, bivariate quasi factorial gamma distribution is introduced. This distribution is a generalization to the Moran's bivariate gamma distribution. Distributional properties of bivariate quasi factorial gamma distribution are studied. We have obtained the bivariate quasi factorial gamma as the negative binomial compound of independently and identically distributed random vectors. Time series models with bivariate quasi factorial gamma marginals are developed. As a generalization to bivariate quasi factorial gamma, bivariate semi quasi factorial gamma distribution is introduced and studied.

Bivariate discrete Mittag-Leffler distribution and bivariate discrete Linnik distributions are introduced as the discrete analogues of BML $(\mu_1, \mu_2, \alpha_1, \alpha_2, \theta)$ and bivariate quasi factorial gamma distributions. As a special case of the bivariate discrete Mittag-Leffler distribution, a bivariate geometric distribution is studied. Characterizations of bivariate discrete Mittag-Leffler distribution are obtained using the geometric compounding. Autoregressive processes with bivariate discrete Mittag-Leffler distribution marginals are developed. Using bivariate geometric compounding, discrete analogues of the bivariate Mittag-Leffler distributions that generalize Marshall-Olkin's bivariate exponential distribution and Hawkes' bivariate exponential distribution are introduced. Bivariate discrete Linnik distribution is introduced as a generalization to the bivariate discrete Mittag-Leffler distribution and its properties are obtained. As a special case of bivariate discrete Linnik distribution, bivariate negative binomial distribution is studied. Characterization of bivariate discrete Linnik distribution is obtained using the negative binomial compounding. First order stationary autoregressive models with bivariate discrete Linnik marginals are developed. Bivariate tailed Mittag-Leffler distribution and bivariate tailed discrete Mittag-Leffler distribution are introduced and the corresponding autoregressive models are developed.

Random summation arises in many contexts. It is mainly applied in modeling practical problems that deal with certain phenomena in which the respective mathematical models are sums of random number of independent random variables. Gnedenko and Korolev (1996) gave a number of situations where we usually come across random summation, especially geometric summation and describe the modeling of such situations with respective physical terminology.

Geometric summation arises naturally in many applied problems. Kozubowski and Panorska (1999a) established the applications of geometric summation in financial portfolio modeling. Kozubowski and Rachev (1994) used geometric random sums as an adequate device to model the foreign currency exchange rate data. Distribution of geometric sums appear in queuing theory and reliability in connection to 'regenerating processes with rare events' (see Gertsbakh (1984) and Jacobs (1986)). Gnedenko and Korolev (1996) expressed the waiting time of a telephone customer calling at an arbitrary time to talk to the operator as a compound of geometric distribution. The applications of random summation in insurance are discussed in Rolski et al. (1999). The probability distributions, we have introduced and studied in this work may be appropriate in modeling bivariate data sets which are closed under geometric summation.

Even though the constant hazard rate and memoryless property causes much applications of exponential distribution in reliability studies and renewal theory, it is inadequate to model heavy tailed data. The Mittag-Leffler distribution being heavy tailed as compared to exponential distribution is a most suitable fit in such situations. Jayakumar (2003) used Mittag-Leffler distribution to model the rate of flow of water in Kallada river, Kerala, India. The semi Mittag-Leffler distribution and semi quasi factorial gamma distributions are useful in modeling data sets that exhibit periodic movements. Recently much focus is given on developing different time series models with non Guassian marginals (see Balakrishna and Jayakumar (1997) and Block et al. (1988)). The first order stationary autoregressive processes having bivariate Mittag-Leffler and bivariate discrete Mittag-Leffler marginals are developed in Mundassery and Jayakumar (2006, 2007b)

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