Ph.D. THESIS **MATHEMATICS**

FUZZY GRAPHIC AND REPRESENTABLE MATROIDS: DUALITY, CONNECTIVITY AND APPLICATIONS

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CERTIFICATE

I hereby certify that the thesis entitled "**Fuzzy Graphic and Representable Matroids: Duality, Connectivity and Applications**" is a bonafide work carried out by **Smt. Shabna O. K,** under my guidance for the award of Degree of Ph.D. in Mathematics of the M.E.S Mampad College(Autonomous) and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

I hereby declare that the work presented in the thesis, entitled "**Fuzzy Graphic and Representable Matroids: Duality, Connectivity and Applications**" is based on the original work done by me under the guidance of **Dr. Sameena Kalathodi**, Assistant Professor, PG and Research Department of Mathematics, M.E.S Mampad College(Autonomous) and it has not been included in any other thesis submitted previously for the award of any degree. The contents of the thesis are undergone plagiarism check using iThenticate software at C.H.M.K Library, University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.

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ABSTRACT

Fuzzy matroids mark a significant advancement in mathematical theory, providing an effective framework for modeling complex systems where partial truths and uncertainty prevail. Classical matroid theory often struggles to represent contexts involving vagueness, highlighting the need for more flexible structures. This research focuses on fuzzy graphic and representable matroids, offering a thorough analysis of their properties and applicability in various domains.

A comprehensive conceptual basis is developed for fuzzy graphic matroids, emphasizing their distinct characteristics compared to classical graphic matroids. Key findings reveal how bases, circuits, and connectivity vary with different threshold applications, demonstrating the adaptable nature of fuzzy structures. Furthermore, the study of fuzzy representable matroids uncovers important insights into their representability, this challenges traditional crisp matroid theory and underscores the unique complexities inherent in fuzzy settings.

The thesis also explores the construction of fuzzy duals within fuzzy graphic matroids, offering valuable insights into their connectivity and establishing a link between fuzzy duals and geometric duals. These findings advance the theoretical understanding of dualization processes in fuzzy contexts.

The results emphasize the need for advanced approximation techniques in fuzzy matroid theory, particularly for applications in network theory, decision-making, and optimization under uncertainty.

സംഗ്രഹം

ഫസ്സി മട്രോയ്ഡുകൾ ഗണിതശാസ്ത്ര സിദ്ധാന്തത്തിലെ ഒരു പ്രധാന വികാസത്തെ പ്രതിനിധീകരിക്കുന്നു, ഭാഗിക സത്യങ്ങളും അനിശ്ചിതത്വവും സാധാരണമായ സങ്കീർണ്ണ സംവിധാനങ്ങളെ മാതൃകയാക്കുന്നതിനുള്ള ശക്തമായ ചട്ടക്കൂട് വാഗ്ദാനം ചെയ്യുന്നു. അവ്യക്തത നിലനിൽക്കുന്ന സന്ദർഭങ്ങളെ വേണ്ടത്ര പ്രതിനിധീകരിക്കുന്നതിൽ ക്ലാസിക്കൽ മട്രോയിഡ് സിദ്ധാന്തത്തിന് പലപ്പോഴും പ്രശ്നമുണ്ട്, ഇത് കൂടുതൽ വഴക്കമുള്ള ഘടനകളുടെ ആവശ്യകതയെ എടുത്തുകാണിക്കുന്നു. ഈ ഗവേഷണം ഫസ്സി ഗ്രാഫിക്, ഫസ്സി റപ്രസൻഡബ്ൾ മട്രോയ്ഡുകളിൽ ശ്രദ്ധ കേന്ദ്രീകരിക്കുന്നു, വിവിധ ഡൊമെയ്നുകളിലുടനീളമുള്ള അവയുടെ ഗുണങ്ങളെയും പ്രയോഗത്തെയും കുറിച്ച് വിപുലമായ വിശകലനം നൽകുന്നു.

ക്ലാസിക്കൽ ഗ്രാഫിക് മട്രോയ്ഡുകളുമായി താരതമ്യപ്പെടുത്തുമ്പോൾ അവയുടെ വ്യതിരിക്തമായ സ്വഭാവസവിശേഷതകൾ എടുത്തുകാണിച്ചുകൊണ്ട് ഫസ്സി ഗ്രാഫിക് മാട്രോയ്ഡുകൾക്കായി സമഗ്രമായ ഒരു ആശയപരമായ അടിസ്ഥാനം സ്ഥാപിച്ചിട്ടുണ്ട്. ബെയ്സിസ്, സർക്യൂട്ടുകൾ, കണക്റ്റിവിറ്റി എന്നിവയുടെ ആശയങ്ങൾ വ്യത്യസ്ത ത്രഷോൾഡ് വിലകൾക്കനുസരിച്ച് എങ്ങനെ വ്യത്യാസപ്പെടാം എന്ന് പ്രധാന കണ്ടെത്തലുകൾ വെളിപ്പെടുത്തുന്നു. കൂടാതെ, ഫസ്സി റപ്രസൻഡബ്ൾ മട്രോയ്ഡുകളുടെ പഠനം അവയുടെ പ്രാതിനിധ്യത്തിലേക്കുള്ള പ്രധാന കണ്ടെത്തലുകൾ വെളിപ്പെടുത്തുന്നു, എല്ലാ ഫസ്സി ഗ്രാഫിക് മാട്രോയ്ഡുകളും എല്ലാ ഫീൽഡുകളിലും ഫസ്സി റപ്രസൻഡബ്ൾ അല്ല എന്ന് ഇത് വ്യക്തമാക്കുന്നു. ഇത് ക്രിസ്പ് മാട്രോയിഡ് സിദ്ധാന്തത്തിൽ നിലവിലുള്ള ആശയങ്ങളെ വെല്ലുവിളിക്കുകയും ഫസ്സി ക്രമീകരണങ്ങളിൽ സവിശേഷമായ സങ്കീർണ്ണതകൾക്ക് ഊന്നൽ നൽകുകയും ചെയ്യുന്നു.

ഫസ്സി ഗ്രാഫിക് മട്രോയിഡുകൾക്കുള്ളിൽ ഫസ്സി ഡ്യുവലുകളുടെ നിർമ്മാണവും അവയുടെ കണക്റ്റിവിറ്റി സവിശേഷതകളിലേക്ക് വിലയേറിയ ധാരണകൾ അവതരിപ്പിക്കുന്നതും ഫസ്സി ഡ്യുവലുകളും ജ്യോമെട്രിക് ഡ്യുവലുകളും തമ്മിലുള്ള ബന്ധം സ്ഥാപിക്കുന്നതും ഇവിടെ പരിശോധിക്കുന്നു. ഈ കണ്ടെത്തലുകൾ ഫസ്സി സെറ്റിംഗുകളിൽ ഡ്യുവൽ കണ്ടെത്തുന്ന പ്രക്രിയകളെക്കുറിച്ചുള്ള തിയററ്റിക്കൽ ധാരണ മെച്ച പ്പെടുത്തുന്നു.

ഫസ്സി നെറ്റ്വർക്ക് സിദ്ധാന്തം, തീരുമാനമെടുക്കൽ, ഒപ്റ്റിമൈസേഷൻ എന്നിവയിലെ ആപ്ലിക്കേഷനുകൾക്കായി, ഫസ്സി മട്രോയിഡ് സിദ്ധാന്തത്തിനുള്ളിലെ വിപുലമായ സാങ്കേതിക വിദ്യകളുടെ ആവശ്യകത റിസൽറ്റുകൾ ചൂണ്ടിക്കാണിക്കുന്നു.

Contents

List of symbols

t	Threshold value
M	Matroid
E	A finite set
2^E	Collection of all subsets of E
$\mathcal{I}$	A collection of subsets of E
$\mid I \mid$	Cardinality of the set I
$I_2 - I_1$	Relative complement of set I_1 w.r.t I_2
B	A basis for a matroid
$\mathcal{B}$	Collection of bases of M
C	Circuit of a matroid
$\mathcal{C}$	Collection of circuits of M
$G^{'}$	A finite graph
$V^{'}$	Vertex set of $G^{'}$
$E^{'}$	Edge set of $G^{'}$
$M(G^{'})$	Cycle matroid on $G^{'}$
$M(U)$	Vector Matroid

e_i	ith edge of a graph
$r(M)$	Rank of M
cl	Closure operator
M^*	Dual matroid of M
$M^*(G)$	Cocycle matroid
C^*	Cocircuit
K_5	Complete graph with 5 vertices
$K_{3,3}$	Complete bipartite graph with 6 vertices
μ	Fuzzy subset
$\mathcal{F}(X)$	Family of all fuzzy sets on the set X
$\mu \vee \nu$	μ join ν
$\mu \wedge \nu$	μ wedge ν
μ^c	Complement of μ
σ	Fuzzy relation
$S \times T$	S cross T
G	A fuzzy graph
$\overline{G}$	Partial fuzzy subgraph of G
μ^∞	Strength of connectedness
$\mu^{'}$	Underlying set of μ
τ	Fuzzy subset on a set S
P	A path
$\mathcal{M}$	A Fuzzy matroid
$\mathcal{I}$	Family of independent fuzzy sets of $\mathcal{M}$
$\mathcal{F}(X)$	Family of all fuzzy subsets on X
$\mu \setminus \setminus a$	A fuzzy subset of μ with $\mu(a) = 0$

ρ	Fuzzy rank function
$\mathfrak{B}$	Family of fuzzy bases for $\mathcal{M}$
$\mathcal{M}^*$	Dual fuzzy matroid of $\mathcal{M}$
$\mathcal{C}(\mu)$	Circuit interval
γ	A mapping from E_1 to E_2
Γ	A mapping from $\mathcal{F}(E_1)$ to $\mathcal{F}(E_2)$
γ^{-1}	Inverse of γ
$\mathcal{M}_1 \cong \mathcal{M}_2$	$\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$
$\mathcal{M}_F(G)$	Fuzzy cycle matroid of G
χ_B	Characteristic function of the set B
W	A vector space
$\widehat{W}$	A fuzzy vector space
$\mathcal{M}_F[X]$	Fuzzy vector matroid induced by X
$< B >$	Subspace spanned by B
$\mathcal{M}_F{}^*(G)$	Dual cycle matroid of G
Ψ	Fuzzy multi graph
f	Fuzzy planarity value
G^*	Geometrical fuzzy dual of G
$\sim$	An equivalence relation on $\mathcal{F}(E)$
$\widehat{a}$	Pythagorean fuzzy number
$\mathcal{P}$	Pythagorean fuzzy set

List of Figures

List of Tables

Chapter 1

Introduction

1.1 History and Development

Fuzzy matroids, an extension of classical matroid theory, integrate the concepts of fuzzy set theory with the combinatorial structures of matroids. This interdisciplinary approach aims to handle uncertainty and vagueness inherent in various real-world problems. The study of fuzzy matroids bridges the gap between crisp mathematical structures and the need for flexible modeling in areas such as decision-making, optimization, and network theory.

Matroid theory, introduced by Whitney in 1935 in [64], is a branch of combinatorial mathematics that generalizes the concept of linear independence in vector spaces. The term "matroid" was coined to describe essential dependency properties common to graphs and matrices. Around the same time, Birkhoff proved that a matroid can be viewed as a geometric lattice, while Maclane established its geometric representation using points, planes, and three-dimensional spaces. Since then, it has been recognized that matroids naturally appear in combinatorial optimization, providing a framework to address a wide range of combinatorial challenges. Sometimes, the term “combinatorial geometry” is used interchangeably with simple matroids. Matroids have applications in graph theory, optimization, and various algorithmic problems. They are characterized by properties such as the independence axioms, the rank function, and the closure operator.

Fuzzy set theory, introduced by Zadeh in 1965 in [27], extends classical set theory to accommodate the concept of partial membership. This theory is essential in modeling uncertainty and imprecision, providing a foundation for fuzzy logic and various applications in control systems, decision-making, and artificial intelligence. Compared to other mathematical theories, fuzzy set theory is arguably the most flexible in practice. This is primarily because a fuzzy set represents relativity, variability, and inexactness in the definition of its elements.

The integration of fuzzy set theory into matroid theory led to the development of fuzzy matroids. Several researchers have contributed to defining and exploring this combined structure.

The study of matroids in combination with fuzzy logic has opened new avenues for managing uncertainty in system analysis, operations research, and economics. The theory of fuzzy matroids was first introduced by Goetschel and Voxman in their seminal work, “Fuzzy Matroids” (1988)[44], where they laid the foundation by using the notion of fuzzy independent sets to define fuzzy matroids. Over the years, Goetschel and Voxman have made several contributions to the field, refining and expanding on the theory of fuzzy matroids.

In their work [42], Goetschel and Voxman (1989) introduced the concept of bases for fuzzy matroids and developed algorithms for finding optimal solutions to problems under uncertainty. This was further expanded in their investigation of fuzzy circuits (1989) in [43], where they proposed a framework for circuit analysis in fuzzy systems. Their subsequent works in "Fuzzy Matroid Structures"[46] (1991) and "Fuzzy Matroid Sums and a Greedy Algorithm"[45] (1992) advanced the application of fuzzy logic to greedy algorithms, demonstrating that fuzzy matroids can extend classical matroid theory into domains where uncertainty plays a major role.

One of their significant contributions was the introduction of fuzzy rank functions (1991) in [48] and the study of spanning properties in fuzzy matroids (1992) in [49], which laid the foundation for future investigations into fuzzy optimization systems. Their focus on the structures and axioms governing fuzzy matroids made a substantial impact on the mathematical understanding of these systems.

Parallel to the work of Goetschel and Voxman,, other scholars also made major contributions. For example, Y.Hsueh (1993) in [73] explored the general fuzzification of matroids, presenting alternative approaches for handling fuzzy independence and dependence. Ladislav Novak offered important insights on the foundational work by Goetschel and Voxman, with papers [28], [29] and [30] in 1997 and 2001 critically analyzing and extending their theories. Novak's contributions addressed limitations in the definition of fuzzy independent sets and clarified axioms related to the theory.

Further developments emerged through the introduction of new structures and procedures in fuzzy matroid theory. Sheng-Gang Li and colleagues (2007) in

[52] proposed closure axioms for a class of fuzzy matroids, which played a significant role in extending the classical closure operator to the fuzzy context. This group also introduced co-towers of matroids, offering new ways to understand the structure and function of fuzzy matroids within a comprehensive mathematical framework. Fu-Gui Shi (2009) in [10] presented a new perspective on the fuzzification of matroids, providing a noval approach to extending classical matroid theory into the fuzzy domain.

Another important development was the study of M-fuzzifying bases by Xiu Xin and Fu-Gui Shi (2009) in [68], which contributed to a deeper understanding of the axiomatic structures necessary for fuzzy matroids. Their work aligned with that of Yao-Long Li, Guo-Jun Zhang, and Ling-Xia Lu (2010) in [70], who examined bases of closed regular fuzzy matroids, establishing foundational axioms to explain how fuzzy matroids operate under specific closure conditions.

In recent years, fuzzy matroids have found various applications, ranging from route optimization to wireless network analysis, where uncertainty and optimization are key challenges. Muhammad Asif and Muhammad Akram (2020) explored the concept of Pythagorean fuzzy matroids and demonstrated their applications in decision-making systems, extending fuzzy matroid theory into practical domains in [36]. Their work, along with that of Musavarah Sarwar and Muhammad Akram (2017) in [37], further developed the role of fuzzy matroids in m-polar fuzzy systems, highlighting the adaptability and flexibility of fuzzy matroid structures.

Talal Al-Hawary's contributions are also notable in the literature, particularly his work on fuzzy closure matroids (2016) in [59] and fuzzy greedoids (2011) in [60]. Al-Hawary's research on closure axioms and the interaction between matroid theory and greedy algorithms within the fuzzy framework has opened new avenues for both theoretical and applied research. His studies of fuzzy flats, fuzzy strong maps, and fuzzy hesitant maps in [59] demonstrate the increasing complexity and effectiveness of fuzzy matroids in mathematical theory and real-world applications.

Additionally, the work of Yonghong Li, Sidong Xian, and Dong Qiu (2014) in [72] has focused on developing algorithms related to closed fuzzy matroids, providing practical solutions for complex systems involving uncertainty and optimization. Their improved algorithm for fuzzy circuits in [71] (2011) has become a valuable tool for researchers and engineers working with closed fuzzy systems.

In brief, fuzzy matroids have grown considerably since their introduction by Goetschel and Voxman in 1988. Over the past few years, the field has expanded to include a variety of axioms, algorithms, and applications, with scholars like Li, Shi, and Novak building on the initial framework to broaden the applicability of fuzzy matroids in both theoretical and practical settings. Whether in the fuzzification of classical matroid theory, the application of fuzzy logic in systems analysis, or the development of new greedy algorithms, the literature on fuzzy matroids continues to be a rich and evolving field with wide-ranging implications.

In this study, we explore the concept of fuzzy matroids, focusing on their formation from fuzzy graphs and fuzzy vector spaces. We begin by analyzing the characteristics of matroids derived from the threshold graph of a fuzzy graph. Subsequently, we construct two distinct types of fuzzy matroids: fuzzy graphic matroids, derived from fuzzy graphs, and fuzzy representable matroids, constructed from fuzzy vector spaces. We then examine the properties of these fuzzy matroids and explore the relationships between them. Additionally, we construct the fuzzy dual of fuzzy graphic matroids and determine the conditions under which the fuzzy dual of a fuzzy graphic matroid remains fuzzy graphic. Furthermore, we investigate the connectivity of fuzzy graphic matroids in detail, highlighting several fundamental properties. Finally, we identify and discuss key practical applications of both fuzzy graphic and representable matroids, demonstrating their relevance and utility in various contexts.

1.2 Motivation

In a context where data uncertainty and complexity are increasingly prominent, traditional mathematical structures like crisp matroids often fall short in representing real-world problems that involve vagueness and partial truth. The limitations of classical matroid theory, particularly in areas such as optimization, decision-making under uncertainty, and network analysis, necessitate the development of more adaptive and flexible frameworks. Fuzzy set theory, with its ability to handle uncertainty through degrees of membership, offers a natural extension to matroid theory. The introduction of fuzzy matroids, especially fuzzy graphic and representable matroids, presents a promising approach to overcoming these limitations. Incorporating fuzziness into core matroid concepts such as bases, circuits, and duals broadens the scope of matroid theory, enabling it to address problems involving imprecise or incomplete data.

This research is motivated by the need to bridge the gap between crisp mathematical structures and the uncertain environments in which many real-world problems arise. The study of fuzzy matroids is not only theoretically compelling but also practically significant, with potential applications in network theory, optimization, wireless communication, and financial portfolio management. The ability of fuzzy matroids to model uncertainty provides a fundamental tool for addressing complex, dynamic systems that traditional approaches often fail to capture effectively.

1.3 An Overview of the Thesis

This thesis consists of ten chapters that introduce and comprehensively analyze two new types of fuzzy matroids: fuzzy graphic matroids and fuzzy representable matroids.

Chapter 1 is an introductory chapter that includes a literature review, a brief explanation of the motivations for the study, and an outline of the subsequent chapters.

In **Chapter 2**, the basic definitions, notations and results from matroid theory, fuzzy graph theory and fuzzy matroid theory, that will be used in the following chapters are outlined.

In **chapter 3**, we study the matroids that can be induced from the threshold graphs of a fuzzy graph. The major structural and notational differences between classical matroids and the fuzzy induced matroids derived from a fuzzy graph for different threshold values t are identified.

Chapter 4 introduces a new approach to constructing a particular category of fuzzy matroids originating from fuzzy graphs, termed fuzzy graphic matroids. The isomorphic and structural characteristics of fuzzy graphic matroids are studied, and several notable results and illustrative examples are highlighted in this chapter.

In **Chapter 5**, we present another novel class of fuzzy matroids derived from fuzzy vector spaces, introducing the notion of fuzzy representable matroids. We study the necessary conditions for the existence of analogs of some well-established results in classical matroid theory and illustrate these with extensive examples.

Chapter 6 introduces the construction of the fuzzy dual of a fuzzy graphic matroid. We begin with fundamental examples to illustrate how fuzzy duals are constructed and to identify the unique aspects of duality within the context of fuzzy graphic matroids. These examples help develop a clear understanding of the structural characteristics of the fuzzy dual of a fuzzy graphic matroid. A significant focus of this chapter is determining the conditions under which the fuzzy dual of a fuzzy graphic matroid retains the property of being fuzzy graphic. Specific examples are provided to demonstrate cases where the fuzzy dual diverges from being fuzzy graphic, thereby highlighting the complexities and structural deviations that can arise during the dualization process in fuzzy matroid theory. Additionally, we construct the geometric dual of a fuzzy graph and examine the fuzzy matroid corresponding to this geometric dual fuzzy graph.

Our research of fuzzy graphic matroids reveals a deep connection between connectivity in fuzzy graph theory and fuzzy matroid theory, inspiring us to extend this concept to fuzzy graphic matroids. In **Chapter 7**, by introducing an equivalence relation derived from fuzzy graphic matroids, we construct a framework to understand connectivity more effectively. We discuss the necessary and sufficient conditions for the connectedness of a fuzzy graphic matroid, linking structural characteristics with overall connectivity.

Chapter 8 presents key applications of fuzzy graphic and representable matroids in solving complex problems such as optimal route mapping, portfolio optimization, and Color Mixing. This highlights the significant role that these mathematical structures play in addressing real-world challenges characterized by uncertainty. The flexibility of fuzzy matroids in modeling uncertainty proves invaluable for tackling these kinds of practical contexts.

Chapter 9 concludes the key findings of the study, while **Chapter 10** offers recommendations for future research in this area.

Chapter 2

Preliminaries

The chapter presents the fundamental definitions, notations, and results in both crisp and fuzzy matroid theory, which serve as the foundational concepts for the subsequent chapters.

This chapter is organized into three sections. The first section presents basic definitions and key results from matroid theory that will be referenced in the subsequent chapters. The second section outlines fundamental definitions and terminologies from fuzzy graph theory. The third section introduces essential definitions and results from fuzzy matroid theory, which form the basis for discussions in the chapters that follow.

2.1 Matroid Theory

In combinatorics, a branch of mathematics, a matroid, also known as an independence structure, provides a generalized framework for the concept of "independence", extending beyond linear independence in vector spaces. One of the most fundamental and widely used formulations of a matroid is based on the notion of independent sets.

Definition 2.1.1. [15] A *Matroid* M is an ordered pair $(E,\ \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:

(1) $\emptyset \in \mathcal{I}$

(2) If $I \in \mathcal{I}$ and $I^{'} \subseteq I$, then $I^{'} \in \mathcal{I}$

(3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$

The members of $\mathcal{I}$ are the *independent sets* of M, the subsets of E which are not members of $\mathcal{I}$ are the *dependent sets* of M and E is the *ground set* of M.

Definition 2.1.2. [15] A maximal independent set in M is called a *basis* or a *base* of M, and we denote the collection of bases of M by $\mathcal{B}$.
A minimal dependent set in an arbitrary matroid M is called a *circuit* of M, and we denote a set of circuits of M by $\mathcal{C}$ or $\mathcal{C}(M)$.

Definition 2.1.3. [15] The matroid derived from a graph $G^{'}$ is called the *cycle matroid* or *polygon matroid* of $G^{'}$, denoted by $M(G^{'})$. Clearly a set X of edges is independent in $M(G^{'})$ if and only if X does not contain the edge set of a cycle. A matroid that is isomorphic to the cycle matroid of a graph is called *graphic matroid*.

Definition 2.1.4. [15] The matroid derived from a subset U of a vector space V is called the *vector matroid* induced by U, symbolized by $M[U]$. Understandably a set W of vectors is independent in $M[U]$ if and only if W is linearly independent in V. Thus, a *representable matroid* is a matroid which is isomorphic to a vector matroid.

Example 2.1.1. *[15] Consider the graph $G^{'}$ with 4 vertices and 7 edges as shown in Figure 2.1. Let E be the edge set of $G^{'}$, that is, $\{e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7\}$, and let $\mathcal{I}$ be the collection of subsets of E that do not contain all of the edges of any simple closed path or cycle of $G^{'}$. The cycles of $G^{'}$ have edge sets $\{e_7\}$, $\{e_5,\ e_6\}$, $\{e_1,\ e_2,\ e_4\}$, $\{e_2,\ e_3,\ e_5\}$, $\{e_2,\ e_3,\ e_6\}$, $\{e_1,\ e_3,\ e_4,\ e_5\}$ and $\{e_1,\ e_3,\ e_4,\ e_6\}$.*

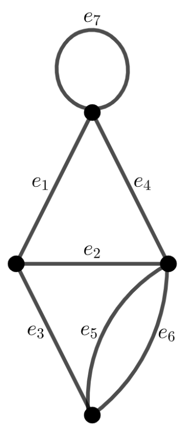


Figure 2.1: The graph $G^{'}$

Example 2.1.2. *For the cyclic matroid given in Example 2.1.1, the collection of bases $\mathcal{B}$ is given by,*

$$\mathcal{B} = \{\{e_1,\ e_2,\ e_3\},\ \{e_1,\ e_2,\ e_5\},\ \{e_1,\ e_2,\ e_6\},\ \{e_1,\ e_3,\ e_4\},\ \{e_1,\ e_3,\ e_5\},$$

$$\{e_1,\ e_3,\ e_6\},\ \{e_1,\ e_4,\ e_5\},\ \{e_1,\ e_4,\ e_6\},\ \{e_2,\ e_3,\ e_4\},\ \{e_2,\ e_4,\ e_5\},$$

$\{e_2,\ e_4,\ e_6\},\ \{e_3,\ e_4,\ e_5\},\ \{e_3,\ e_4,\ e_6\}\}.$

Lemma 2.1.1. *[15] If B_1 and B_2 are two bases of a matroid M, then $|B_1| = |B_2|$.*

Definition 2.1.5. [15] The *rank* $r(M)$ of M is the size of a basis of M.

Lemma 2.1.2. *[15] Let $M = M(G^{'})$ where $G^{'}$ be a connected graph, then*

$$r(M) =\mid V(G^{'}) \mid -1.$$

Definition 2.1.6. [15] Let M be a matroid having ground set E and rank function r. Let cl be the function from 2^E into 2^E defined, for all $X \subseteq E$, by

$$cl(X) = \{x \in E : r(X \cup x) = r(X)\}.$$

Lemma 2.1.3. *[15] The closure operator cl of a matroid on the set E has the following properties:*

(1) If $X \subseteq E$, then $X \subseteq cl(X)$.

(2) If $X \subseteq Y \subseteq E$, then $cl(X) \subseteq cl(Y)$.

(3) If $X \subseteq E$, then $cl(cl(X)) = cl(X)$.

(4) If $X \subseteq E$, $x \in E$, and $y \in cl(X \cup x) - cl(X)$, then $x \in cl(X \cup y)$.

Definition 2.1.7. [15] Let M be a matroid and $\mathcal{B}^*(M)$ be $\{E(M) - B : B \in \mathcal{B}(M)\}$. Then $\mathcal{B}^*(M)$ is the set of bases of a matroid on $E(M)$, is called the *dual* of M and is denoted by M^*.

Lemma 2.1.4. *[15] Let M be a matroid, then $(M^*)^* = M$.*

Proposition 2.1.1. *[15] For all subsets X of the ground set E of a matroid M,*

$$r^*(X) = \mid X \mid - r(M) + r(E - X),$$

where r^* is the rank function of M^*.

Definition 2.1.8. [15] If G' is a graph, the dual of the cycle matroid of G' is denoted by $M^*(G')$ and is called the *bond matroid* of G' or the *cocycle matroid* of G'. An arbitrary matroid that is isomorphic to the bond matroid of some graph is called *cographic.*

Proposition 2.1.2. *[15] Neither $M^*(K_5)$ nor $M^*(K_{3,3})$ is graphic.*

Theorem 2.1.1. *If G' is planar, then $M^*(G')$ is graphic.*

Proposition 2.1.3. *[15] The matroid M is connected if and only if, for every pair of distinct elements of $E(M)$, there is a circuit containing both.*

Proposition 2.1.4. *[15] Let G' be a loopless graph without isolated vertices. If G' has at least three vertices, then $M(G')$ is a connected matroid if and only if G' is a 2-connected graph.*

Proposition 2.1.5. *[15] If x and y are distinct elements of a circuit C of a matroid M, then M has a cocircuit C^* such that $C \cap C^* = \{x, y\}$.*

2.2 Fuzzy Graph Theory

A fuzzy set is mathematically defined by assigning to each element in the domain of interest a value that represents its degree of membership in the set. This value, known as the membership grade, indicates the extent to which an element aligns with or belongs to the concept represented by the fuzzy set. Unlike classical sets, where membership is binary, fuzzy sets allow elements to belong to a set to varying degrees. These membership grades are real numbers in the closed interval from 0 to 1, where 1 signifies full membership and 0 denotes complete non-membership.

Definition 2.2.1. [56] Let X be a set. A *fuzzy subset* μ on X is a function $\mu : X \longrightarrow [0,1]$

We denote the family of fuzzy sets on X by $\mathcal{F}(X)$. If $\mu,\ \nu \in \mathcal{F}(X)$, then

$$supp(\mu) = \{x \in X \mid \mu(x) > 0\},$$

$$m(\mu) = inf\{\ \mu(x) \mid x \in supp(\mu)\},$$

$$C_r(\mu) = \{x \in X \mid \mu(x) \geq r\}, where\ 0 \leq r \leq 1,$$

$$\mu \vee \nu = max\{\ \mu,\ \nu\},$$

$$\mu \wedge \nu = min\{\ \mu,\ \nu\},$$

$$\mu^c(x) = 1 - \mu(x),\ \forall x \in X$$

The cardinality of a fuzzy set μ is defined as $\mid \mu \mid = \sum_{x \in X} \mu(x)$.
If $\mu,\ \nu \in \mathcal{F}(X)$, then we write $\mu < \nu$ if

i. $\mu(x) \leq \nu(x)$ for each x in X ,

ii. $\mu(x) < \nu(x)$ for some x in X

Definition 2.2.2. [34] Let S and T be two sets and let μ and ν be fuzzy subsets of S and T respectively. A fuzzy relation σ from the fuzzy set μ in to the fuzzy set ν is a fuzzy subset of $S \times T$ such that $\sigma(x,\ y) \leq \mu(x) \wedge \nu(y),\ \forall x \in S$ and $y \in T$.

Definition 2.2.3. [34] A fuzzy relation σ is called reflexive if $\sigma(x,\ x) = \mu(x),\ \forall x \in S$.

Definition 2.2.4. [34] A fuzzy relation σ is called symmetric if $\sigma(x,\ y) = \sigma(y,\ x),\ \forall x,\ y \in S$.

A graph is traditionally defined as a symmetric binary relation on a nonempty set S. In parallel, a fuzzy graph is a binary fuzzy relation on a fuzzy set. The concept of a fuzzy graph was first introduced by Kaufmann, drawing on Zadeh's foundational work on fuzzy relation.Building on this, Rosenfeld extended the theory by applying fuzzy relations to fuzzy sets, thereby formalizing the structure of fuzzy graphs and establishing fuzzy counterparts to many classical graph theoretical concepts. Further advancements were made by R. T. Yeh and S. Y. Bang, who introduced various notions of connectedness for fuzzy graphs and fuzzy digraphs. Since then, numerous researchers have contributed to the field by exploring fuzzy analogues of key graph-theoretical constructs, including fuzzy trees, graph operations, cycles and co-cycles, fuzzy line graphs, and the metric dimensions of fuzzy graphs.

Definition 2.2.5. [56] Let S be a non empty set. A *fuzzy graph* is a pair of functions $G = (\sigma,\ \mu)$ where σ is a fuzzy subset of S and μ is a symmetric fuzzy relation on σ. That is, $\sigma : S \longrightarrow [0,\ 1]$ and $\mu : S \times S \longrightarrow [0,\ 1]$ such that $\mu(u,\ v) \leq \sigma(u) \wedge \sigma(v)$ for all $u,\ v \in S$.

We represent the underlying graph of $G : (\sigma,\ \mu)$ by $G' : (\sigma',\ \mu')$ where σ' is the non empty set S of vertices and $\mu' \subseteq S \times S$. It's important to note that a crisp graph G' is a special case of a fuzzy graph, where each vertex and edge of G' has a membership degree of 1. We do not consider loops, and we assume that μ is reflexive. Additionally, the underlying set σ' is assumed to be finite, and the σ can be chosen in any way that satisfies the definition of fuzzy graphs in all examples.

Definition 2.2.6. [56] The fuzzy graph $H : (\tau,\ \upsilon)$ is called a *partial fuzzy subgraph* of $G : (\sigma,\ \mu)$ if $\tau \subseteq \sigma$ and $\upsilon \subseteq \mu$. In particular, we call $H : (\tau,\ \upsilon)$ a fuzzy subgraph of $G : (\sigma,\ \mu)$ if $\tau(u) = \sigma(u),\ \forall u \in \tau'$ and $\upsilon(u,\ v) = \mu(u,\ v),\ \forall (u,\ v) \in \upsilon'$.

Definition 2.2.7. [56] A partial fuzzy subgraph $(\tau,\ \upsilon)$ spans the fuzzy graph $(\sigma,\ \mu)$ if $\sigma = \tau$. In this case $(\tau,\ \upsilon)$ is called a *partial fuzzy spanning subgraph* of $(\sigma,\ \mu)$.

Definition 2.2.8. [56] A fuzzy graph $(\tau,\ \upsilon)$ spans the fuzzy graph $(\sigma,\ \mu)$ if $\sigma = \tau$ and

$$\upsilon(u,\ v) = \begin{cases} \mu(u,\ v), & \text{if } (u,\ v) \in \sigma' \\ 0, & \text{otherwise} \end{cases}$$

In this case we call $(\tau,\ \upsilon)$, a *fuzzy spanning subgraph* of $G = (\sigma,\ \mu)$.

Definition 2.2.9. [56] A *path* P in a fuzzy graph $G = (\sigma,\ \mu)$ is a sequence of distinct vertices $x_0,\ x_1,\ \ldots x_n$ (except possibly x_0 and x_n) such that $\mu(x_{i-1},\ x_i) > 0,\ i = 1, \ldots, n$.

Here n is called the *length* of the path.

The strength of a path P of length n is defined as $\wedge_{i=1}^{n} \mu(x_{i-1},\ x_i)$.

Definition 2.2.10. [56] A fuzzy graph $G = (\sigma,\ \mu)$ is *connected* if any two vertices are joined by a path. Maximal connected partial fuzzy subgraphs are called *components*.

Definition 2.2.11. [56] A *maximum spanning tree* of a connected fuzzy graph $G = (\sigma,\ \mu)$ is a fuzzy spanning subgraph $T : (\sigma,\ \nu)$, such that T' is a tree, and for which $\sum_{u \neq v} \nu(u,\ v)$ is maximum.

Definition 2.2.12. [56] An *arc* $(u,\ v)$ is a fuzzy bridge of $G = (\sigma,\ \mu)$ if the deletion of $(u,\ v)$ reduces the strength of connectedness between some pair of nodes. That is, $(u,\ v)$ is a fuzzy bridge if and only if there are nodes $x,\ y$ such that $(u,\ v)$ is an arc of every strongest xy path.

Definition 2.2.13. [56] A node is a *fuzzy cut node* of $G : (\sigma,\ \mu)$ if removal of it reduces the strength of connectedness between some other pair of nodes. That is, w is a fuzzy cut node if and only if there exist nodes $u,\ v$ distinct from w such that w is on every strongest uv path.

Definition 2.2.14. [56] A connected fuzzy graph $G : (\sigma,\ \mu)$ with no fuzzy cutnodes is called a *block*.

We call P a *cycle* if $x_0 = x_n$ and $n \geq 3$.

Definition 2.2.15. [56] A fuzzy graph $G = (\sigma,\ \mu)$ is called a *cycle* if the pair $(supp(\sigma),\ supp(\mu))$ is a cycle, and G is called a *fuzzy cycle* if $(supp(\sigma),\ supp(\mu))$ is a cycle and $\nexists$ unique $xy \in supp(\mu)$ such that $\mu(xy) = \wedge\{\mu(uv)|uv \in supp(\mu)\}$.

Definition 2.2.16. [56] A fuzzy graph G is called a *complete fuzzy graph* if $\mu(u,\ v) = \sigma(u) \wedge \sigma(v),\ \forall u,\ v \in \sigma'$.

2.3 Fuzzy Matroid Theory

Fuzzy matroids extend classical matroid theory by incorporating principles from fuzzy set theory, thereby enabling the modeling of systems characterized by imprecise boundaries and partial memberships. This enriched framework significantly contributes to fields such as theoretical computer science and combinatorial optimization by offering robust tools to handle uncertainty in complex, real-world scenarios.

Geotschel and Voxman defined the concept of fuzzy matroids as follows.

Definition 2.3.1. [44] Suppose that E is a finite set and that $\mathcal{I} \subseteq \mathcal{F}(E)$ is a nonempty family of fuzzy sets satisfying:

(i) (Hereditary property) If $\mu(x) \in \mathcal{I}$, $\nu \in \mathcal{F}(E)$, and $\nu < \mu$, then $\nu \in \mathcal{I}$

(ii) (Exchange property) If $\mu,\ \nu \in \mathcal{I}$ and $|supp(\mu)| < |supp(\nu)|$, then there exists $\omega \in \mathcal{I}$ such that

  (a) $\mu < \omega < \mu \vee \nu$

  (b) $m(\omega) \geq min\{m(\mu),\ m(\nu)\}$.

Then the pair $\mathcal{M} = (E,\ \mathcal{I})$ is a *fuzzy matroid* on E, and $\mathcal{I}$ is the family of *independent fuzzy sets* of $\mathcal{M}$.

Theorem 2.3.1. *[44] Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy matroid, and for each r, $0 < r \leqslant 1$, let*

$$\mathcal{I}_r = \{C_r(\mu) \mid \mu \in \mathcal{I}\}.$$

Then for each r, $0 < r \leqslant 1$, $M_r = (E,\ \mathcal{I}_r)$ is a (crisp) matroid on E.

Suppose that $\mathcal{M} = (E,\ \mathcal{I})$ is a fuzzy matroid, and for each r, $0 < r \leq 1$, let $M_r = (E,\ \mathcal{I}_r)$ be the (crisp) matroids on E defined in Theorem 2.3.1. Since E is a finite set, there is at most a finite number of matroids that can be defined on E. Thus there is a finite sequence $r_0 < r_1 < \cdots < r_n$ such that

(i) $r_0 = 0;\ r_n \leq 1$

(ii) $\mathcal{I}_s \neq \emptyset$ if $0 < s \leq r_n$; $\mathcal{I}_s = \emptyset$ if $s > r_n$

(iii) If $r_i < s,\ t < r_{i+1}$, then $\mathcal{I}_s = \mathcal{I}_t,\ 0 \leq i \leq n-1$

(iv) If $r_i < s < r_{i+1} < t < r_{i+2}$, then $\mathcal{I}_s \supsetneq \mathcal{I}_t,\ 0 \leq i \leq n-2$.

The sequence $r_0,\ r_1,\ \cdots\ r_n$ is called the *fundamental sequence*[44] for $\mathcal{M}$.

Definition 2.3.2. [42] A *fuzzy basis* for a fuzzy matroid $\mathcal{M} = (E,\ \mathcal{I})$ is a maximal member μ in $\mathcal{I}$ (where μ is said to be maximal in $\mathcal{I}$ if whenever $\nu \in \mathcal{I}$ and $\mu \leq \nu$ then $\mu = \nu$).

Theorem 2.3.2. *[42] Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy matroid. Then $\mathcal{M}$ is closed if and only if for each $\mu \in \mathcal{I}$, there is a fuzzy basis $\nu \in \mathcal{I}$ such that $\mu \leq \nu$.*

Definition 2.3.3. [42] Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy matroid with fundamental sequence $r_0 < r_1 < \cdots < r_n$. $\mathcal{M}$ is said to be *regular* if whenever $r_i < r_j$ and **B** is a basis of $(E,\ \mathcal{I}_{ri})$, there is a basis **A** of $(E,\ \mathcal{I}_{rj})$ such that $A \subseteq B$.

Theorem 2.3.3. *[42] Let $\mathcal{M} = (E,\ \mathcal{I})$ be a closed fuzzy matroid with fundamental sequence $r_0 < r_1 < \cdots < r_n$. Then $\mathcal{M}$ is regular if and only if fuzzy bases for $\mathcal{M}$ have the same cardinality.*

Definition 2.3.4. [43] Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy matroid. Then $\mu \in \mathcal{F}(E)$ is a *fuzzy circuit* of $\mathcal{M}$ if $\mu \notin \mathcal{I}$ and $\mu \setminus \setminus a \in \mathcal{I}$ for each $a \in supp(\mu)$, where $\mu \setminus \setminus a$ is defined by

$$(\mu \setminus \setminus a)(x) = \begin{cases} \mu(x), & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$$

Theorem 2.3.4. *[43] If $\mathcal{M} = (E,\ \mathcal{I})$ is a fuzzy matroid, then μ is a fuzzy circuit in $\mathcal{M}$ if and only if $\mu = \nu \vee \omega$ where ν is an elementary fuzzy circuit, $\omega \in \mathcal{I}$, and $supp(\omega) \subsetneq supp(\nu)$.*

Theorem 2.3.5. *[43] Suppose that $\mathcal{M} = (E,\ \mathcal{I})$ is a fuzzy matroid and that $\mu \in \mathcal{F}(E)$. If $\mu \notin \mathcal{I}$, then there is a fuzzy circuit ν such that $\nu \leq \mu$.*

Theorem 2.3.6. *[43] If μ_1 and μ_2 are fuzzy circuits in a fuzzy matroid $\mathcal{M}$ and if $\mu_1 \leq \mu_2$, then $supp(\mu_1) = supp(\mu_2)$.*

Definition 2.3.5. [48] Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy matroid. The function $\rho : \mathcal{F}(E) \longrightarrow [0,\ \infty)$ defined by

$$\rho(\mu) = sup\{\mid \nu \mid : \nu \leq \mu \text{ and } \nu \in \mathcal{I}\}$$

is the *fuzzy rank function* for $\mathcal{M}$.

Theorem 2.3.7. *[43] If ρ is the fuzzy rank function for a (closed) fuzzy matroid $\mathcal{M} = (E,\ \mathcal{I})$, then:*

(i) $\rho(\mu) = \mid \mu \mid$ if and only if $\mu \in \mathcal{I}$.

(ii) If $\mu \in \mathcal{F}(E)$, then there exists $\nu \leq \mu$ such that $\rho(\mu) = \rho(\nu) = \mid \nu \mid$.

Definition 2.3.6. [46] Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy matroid and let $\mathfrak{B}$ be the family of fuzzy bases for $\mathcal{M}$. Let $\mathbf{1} : E \longrightarrow [0,1]$ be the fuzzy set defined by

$\mathbf{1}(e) = \mathbf{1}$ for each $e \in E$, and if β is a fuzzy set on E, let $\beta^c = 1 - \beta$. Let $\mathfrak{B}^* = \{\beta^c \mid \beta \in \mathfrak{B}\}$. If $\mathcal{M}$ is closed and regular, then $\mathfrak{B}^*$ forms the family of fuzzy bases for a closed, regular fuzzy matroid $\mathcal{M}^*$; $\mathcal{M}^*$ is called the *dual* of $\mathcal{M}$.

Theorem 2.3.8. *[46] If $\mathcal{M}$ is closed and regular fuzzy matroid , then $\mathcal{M}^{**} = \mathcal{M}$.*

Theorem 2.3.9. *[46] Let $\mathcal{M} = (E,\ \mathcal{I})$ be a closed, regular fuzzy matroid with rank function ρ. Let $\mathcal{M}^*$ be the dual of $\mathcal{M}$ and let ρ^* be the rank function for $\mathcal{M}^*$. Then for each $\mu \in \mathcal{F}(E)$,*

$$\rho^*(\mu) = \mid \mu \mid + \rho(\mu^c) - \rho(1).$$

Theorem 2.3.10. *[43] Suppose that $\mathcal{M} = (E,\ \mathcal{I})$ is a closed fuzzy matroid and that μ_1 and μ_2 are fuzzy circuits in $\mathcal{M}$ with circuit intervals $\mathcal{C}(\mu_1)$ and $\mathcal{C}(\mu_2)$. Suppose further that $a \in supp(\mu_1) \cap supp(\mu_2)$, $b \in supp(\mu_1) \setminus supp(\mu_2)$, and $\mathcal{C}(\mu_1) \cap \mathcal{C}(\mu_2) \neq \emptyset$. Then there is a fuzzy circuit ω such that*

$$\omega < (\mu_1 \vee \mu_2) \setminus \backslash a \text{ and } b \in supp(\omega).$$

Definition 2.3.7. [43] Suppose that $\mathcal{M}$ is a closed fuzzy matroid and that μ is a fuzzy circuit in $\mathcal{M}$. Then $\mathcal{C}(\mu) = (\tau(\mu),\ m(\mu)]$ is the circuit interval of μ.

Theorem 2.3.11. *[43] Suppose that $\mathcal{M}$ is a closed fuzzy matroid and that μ is a fuzzy circuit in $\mathcal{M}$. Let $\mathcal{C}(\mu) = (\tau(\mu),\ m(\mu)]$ be the circuit interval of μ.*

(i) If $\alpha \in \mathcal{C}(\mu)$, then $C_\alpha(\mu)$ is a crisp circuit in $M_\alpha = (E,\ \mathcal{I}_\alpha)$.

(ii) If $\alpha \notin \mathcal{C}(\mu)$, then $C_\alpha(\mu) \in \mathcal{I}_\alpha$.

Chapter 3

A Comparative Study of Classic and Fuzzy-Induced Graphic Matroids

3.1 Introduction

A graphic matroid is constructed from the edge set of a graph, where the independent sets consists of acyclic subsets of edge set, and the dependent sets correspond to edge sets that form cycles within the graph. This structure encapsulates key combinatorial properties of graphs and has found extensive applications in various domains, including optimization problems, the analysis of electrical circuits, and the design and evaluation of networks.

However, in many real-world scenarios, graph structures are not crisply defined. Fuzzy graphs offer a more nuanced representation by associating membership values with vertices and edges, indicating the degree to which each element is present. Within this framework, the corresponding matroid structures become more intricate, and classical results from traditional graphic matroid theory may no longer hold when extended to graphic matroids induced from fuzzy graphs. This shift necessitates the development of new theoretical tools and adaptations to address the added complexity introduced by fuzziness.

This chapter investigates the fundamental differences between classical graphic matroids and those induced from fuzzy graphs. A detailed comparative analysis is provided to illustrate how key results in classical matroid theory may transform or fail to hold when extended to the context of fuzzy graph structures. The chapter concludes with illustrative examples that emphasize the impact of fuzziness on core matroid properties, thereby underlining the need for revised definitions and frameworks in fuzzy settings.

3.2 A Study on Divergence of Classical Results in Fuzzy Induced Graphic Matroids

We begin this section by recalling the definition of the threshold graph associated with a fuzzy graph G.

Definition 3.2.1. [56] Let $G = (\sigma,\ \mu)$ be a fuzzy graph. Let $0 \leq t \leq 1$. Let $\sigma^t = \{x \in \sigma^* \mid \sigma(x) \geq t\}$ and $\mu^t = \{uv \in \mu^* \mid \mu(uv) \geq t\}$. Then, $H = (\sigma^t,\ \mu^t)$ is called the threshold graph of the fuzzy graph G, corresponding to t.

Thus, unlike a crisp graph, which yields a single unique cycle matroid, a fuzzy graph can induce multiple cycle matroids corresponding to different threshold values.

Example 3.2.1. *Consider the fuzzy graph G given in Figure 3.1, where the membership value of each edge indicating the certainty or strength of the connection:*

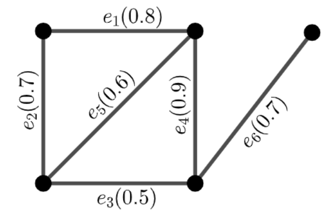


Figure 3.1: The fuzzy graph G

We can construct different cyclic matroids from this fuzzy graph by changing the threshold value t. For each t in the threshold ranges $0 \leq t \leq 0.5,\ 0.5 < t \leq 0.6,\ 0.6 < t \leq 0.7,\ 0.7 < t \leq 0.8$ *and* $0.8 < t \leq 0.9$*, we obtain the cyclic matroids corresponding to the graphs* $(a),\ (b),\ (c),\ (d),$ *and* (e)*, respectively, given in figure 3.2.*

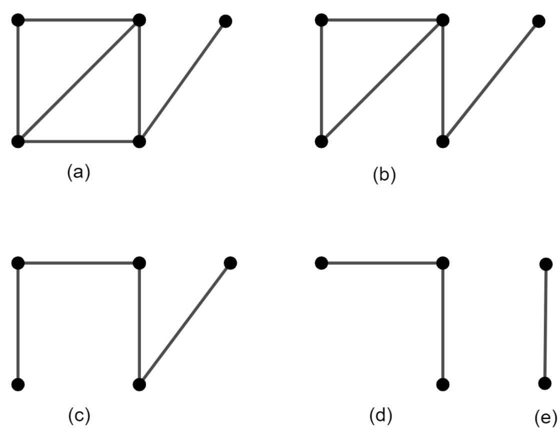


Figure 3.2: The graphs corresponding to different values of t

3.2.1 Basis and Circuit Structures

For a given graph G', the collection of bases of the associated cyclic matroid $M(G')$ is well defined and can be determined using methods such as greedy algorithms.

However, in the case of a fuzzy-induced graphic matroid, the selection of a membership threshold t significantly influences which fuzzy edges are included. Different threshold values yield different sets of independent edges, resulting in varied collections of bases. Consequently, unlike in the crisp case, a unique collection of bases is not guaranteed, highlighting a key deviation from the classical property.

Example 3.2.2. *Consider a social network with 5 individuals given in Figure 3.3 and fuzzy edges denoting the strength of professional collaboration:*

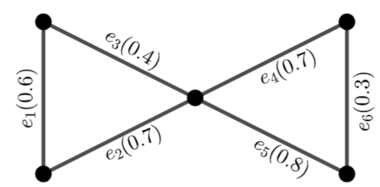


Figure 3.3: A social network with 5 individuals

Ignore the fuzzy weights for a moment and construct a crisp version of the graph:

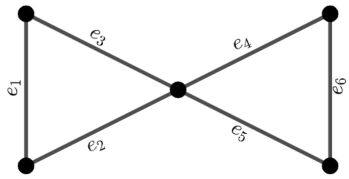


Figure 3.4: Crisp version of the social network

The possible basis are:

$B_1 = \{e_1,\ e_2,\ e_4,\ e_5\},\ B_2 = \{e_1,\ e_2,\ e_4,\ e_6\},\ B_3 = \{e_1,\ e_2,\ e_5,\ e_6\},$

$B_4 = \{e_1,\ e_3,\ e_4,\ e_5\},\ B_5 = \{e_1,\ e_3,\ e_4,\ e_6\},\ B_6 = \{e_1,\ e_3,\ e_5,\ e_6\},$

$B_7 = \{e_2,\ e_3,\ e_4,\ e_5\},\ B_8 = \{e_2,\ e_3,\ e_4,\ e_6\},\ B_9 = \{e_2,\ e_3,\ e_5,\ e_6\}$

Thus the collection $\mathcal{B}$ can be uniquely determined as

$$\mathcal{B} = \{B_1,\ B_2,\ B_3,\ B_4,\ B_5,\ B_6,\ B_7,\ B_8,\ B_9\}.$$

Now, depending on the values of t we set, there may obtain different collections of bases:

$$\textit{For}\ 0 \le t \le 0.3,\ \mathcal{B}_1 = \mathcal{B}.$$

$$\textit{For}\ 0.3 < t \le 0.4,\ \mathcal{B}_2 = \{\{e_1,\ e_2,\ e_4,\ e_5\},\ \{e_1,\ e_3,\ e_4,\ e_5\},\ \{e_2,\ e_3,\ e_4,\ e_5\},\}.$$

$$\textit{For}\ 0.4 < t \le 0.6,\ \mathcal{B}_3 = \{\{e_1,\ e_2,\ e_4,\ e_5\}\}.$$

$$\textit{For}\ 0.6 < t \le 0.7,\ \mathcal{B}_4 = \{\{e_2,\ e_4,\ e_5\}\}$$

$$\textit{For}\ 0.7 < t \le 0.8,\ \mathcal{B}_5 = \{\{e_5\}\}.$$

$$\textit{For}\ 0.8 < t \le 1,\ \mathcal{B}_6 = \{\emptyset\}.$$

The collection of circuits in a graphic matroid is uniquely defined as the set of minimal edge subsets that form cycles. However, in a fuzzy-induced graphic matroid, applying a threshold to convert fuzzy edges into a crisp form can alter the cycle structure. As a result, the notion of a circuit may differ from that in the original fuzzy graph, depending on the threshold value used. This introduces variability and complexity not present in classical graphic matroids.

Example 3.2.3. *In the fuzzy induced cyclic matroid from the fuzzy graph given in Figure 3.3, if* $0 < t \leq 0.3$*, then the collection* $\mathcal{C}$ *of circuits is given by,*
$\mathcal{C} = \{\{e_1,\ e_2,\ e_3\},\ \{e_4,\ e_5,\ e_6\}\}$.
If $0.3 < t \leq 0.4$*, then* $\mathcal{C} = \{\{e_1,\ e_2,\ e_3\}\}$.
For $0.4 < t \leq 1$*, the matroid does not contain any circuits.*

3.2.2 Connectivity

We now state a fundamental result concerning the connectedness of a graphic matroid:

Proposition 3.2.1. *[15] Let* G' *be a loopless graph without isolated vertices. If* G' *has at least three vertices, then* $M(G')$ *is a connected matroid if and only if* G' *is a 2-connected graph.*

Thus, in a graphic matroid $M(G')$, connectivity directly reflects the connectivity of the underlying graph G'. However, in the case of fuzzy graphs, when fuzzy edges are filtered using a threshold value, certain edges may be excluded from consideration. This exclusion can lead to a loss of connectivity in the resulting threshold graph, and consequently, a loss of connectivity in the induced matroid structure as well.

Example 3.2.4. *Consider the graph given in Figure 3.5, clearly it is 2- connected, and by Proposition 3.2.1, the cyclic matroid* $M(G')$ *is connected.*

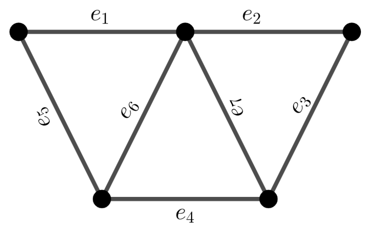


Figure 3.5: A 2-connected graph $G^{'}$

Let us give fuzzy membership values to the edges of $G^{'}$*, and the resultant fuzzy graph is given in Figure 3.6:*

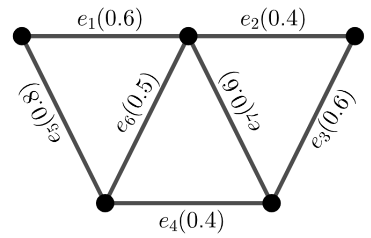


Figure 3.6: Fuzzy representation of $G^{'}$ in Figure 3.5

Choose a threshold value for t from the interval $(0.4,\ 0.5]$*, and apply it to the fuzzy graph shown in Figure 3.6, this results in the thresholded graph depicted in Figure 3.7. Let* $M_2(G^{'})$ *be the cyclic matroid induced from the graph in Figure 3.7. According to 3.2.1, since the threshold graph* $G^{'}$ *is not 2-connected, the matroid* $M_2(G^{'})$ *is consequently not connected.*

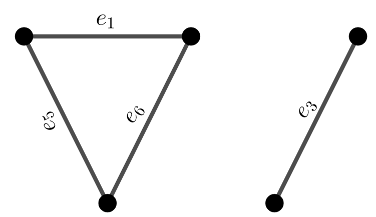


Figure 3.7: Thresholded graph of G' in Figure 3.6 where $t \in (0.4,\ 0.5]$

3.2.3 Rank Function

The rank r for a graphic matroid is defined as follows:

Definition 3.2.2. [15] Let $M = M(G')$, where G' is a graph. If G' is connected, then the rank of M, denoted as $r(M)$ is given by

$$r(M) = \mid V(G') \mid -1 \tag{3.1}$$

If G' has $\omega(G)$ connected components, then

$$r(M) = \mid V(G') \mid -\omega(G') \tag{3.2}$$

When deriving a cyclic matroid from a fuzzy graph, the rank of the matroid depends on the chosen threshold value t.

Example 3.2.5. *Consider the fuzzy graph G in Figure 3.8, we apply different threshold values t to its fuzzy edges and evaluate the rank of the induced cyclic matroid corresponding to each threshold.*

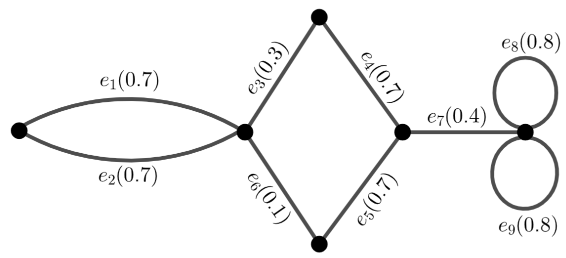


Figure 3.8: Fuzzy graph G to evaluate the rank of the induced matroid

If $0 < t \leq 0.1$*, then* $M = M(G')$ *where* G' *is the graph in Figure 3.9(a). As* G' *is connected, by (3.1),* $r(M) = \mid V(G') \mid -1 = 6 - 1 = 5$*. If* $0.1 < t \leq 0.3$*, then* M *is induced from the graph in Figure 3.9(b). Thus,* $r(M) = 5$*. For* $0.3 < t \leq 0.4$*, equivalently, we can see from Figure 3.9(c) that* $\mid V(G') \mid = 6$ *and* $\omega(G) = 2$*, so, by (3.2),* $r(M) = 6 - 2 = 4$*. If* $0.4 < t \leq 0.7$*,* M *is induced by the graph in Figure 3.9(d), then* $r(M) = 6 - 3 = 3$*. Finally, for* $0.7 < t \leq 1$*, then* $r(M) = 0$ *as* $\emptyset$ *is a basis for* M*.*

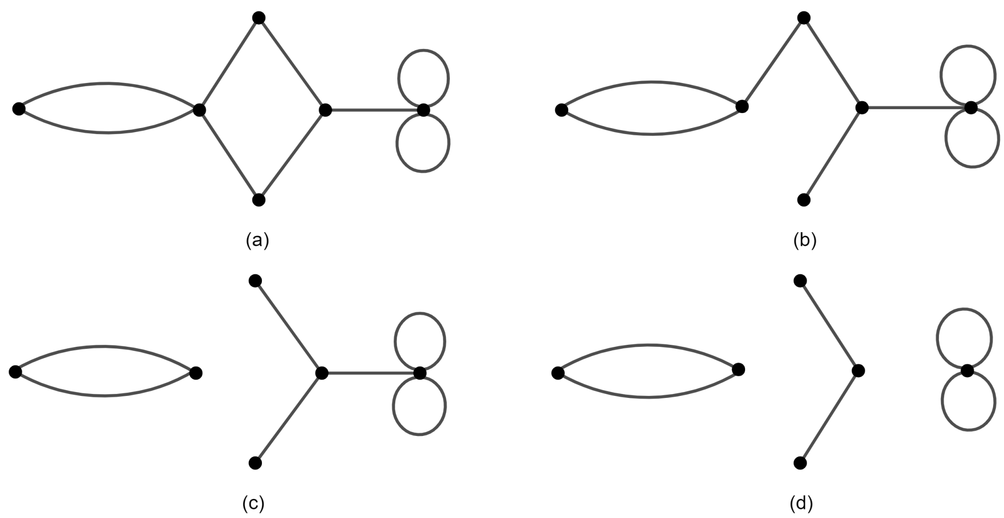


Figure 3.9: Graphs at different threshold levels of t

3.2.4 Dual Matroid

From Definitions 6.2.1, it is clear that, for any graphic matroid $M(G^{'})$, the dual matroid $M^*(G^{'})$ is well-defined and unique up to isomorphism.

However, when considering a fuzzy graph G and applying a threshold value t to crisp the graph, the structure of the dual matroid $M^*(G)$ of $M(G)$ may vary depending on the chosen threshold t.

Example 3.2.6. *Let G be the fuzzy graph depicted in Figure 3.10. By varying the threshold value t, we can generate different thresholded versions of G and observe how the structure of the corresponding dual matroid changes with each choice of t.*

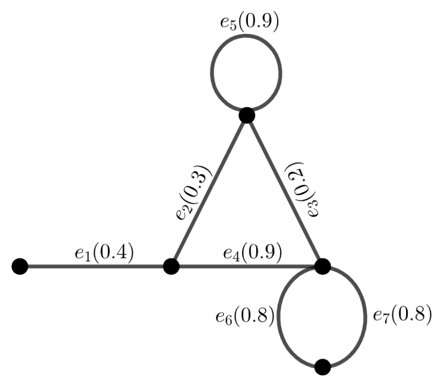


Figure 3.10: The fuzzy graph to be thresholded for finding dual matroid

If we assign the threshold value t from any of the intervals, $[0,\ 0.2]$, $(0.2,\ 0.4]$, $(0.4,\ 0.5]$, $(0.5,\ 0.7]$, $(0.7,\ 0.8]$, *and* $(0.8,\ 0.9]$*, the corresponding dual matroids* M_1^*, M_2^*, M_3^*, M_4^*, M_5^*, *and* M_6^* *are the graphic matroids induced from the graphs labeled* (a), (b), (c), (d), (e), *and* (e) *respectively, shown in Figure 3.11. For* $0.9 < t \leq 1$*, the dual matroid* $M_7^* = \emptyset$.

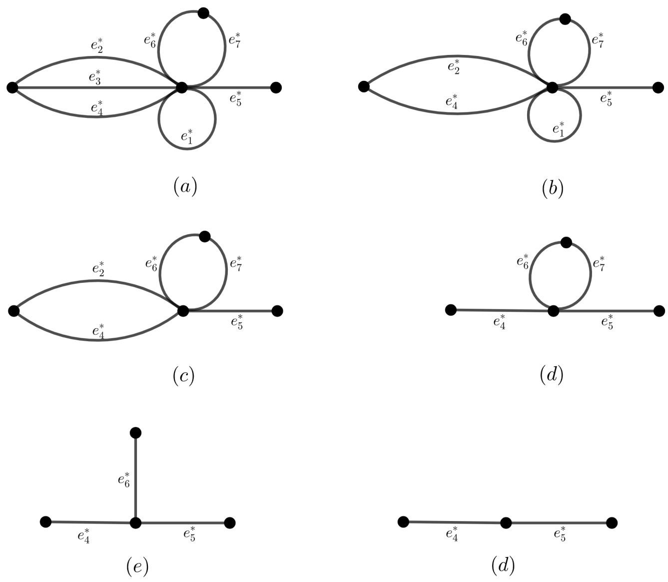


Figure 3.11: The dual graphs

3.3 Algorithm to Construct a Fuzzy Induced Graphic Matroid

To construct a graphic matroid from a fuzzy graph using thresholding, we follow a step-by-step process: first, we decide which edges to preserve based on their membership values relative to a chosen threshold, and then we construct the corresponding cyclic matroid from the resulting crisp graph.

Below is an algorithm that summarizes the process of constructing a graphic matroid from a fuzzy graph by applying threshold values to determine which edges to include.

Algorithm:

Step 1: Input the vertex set V and edge set E of the fuzzy graph G.

Step 2: Input the fuzzy membership values $\mu(e)$ of each edge $e \in E$.

Step 3: Input the threshold value $t \in [0,\ 1]$.

Step 4: Construct an edge set E_t such that

$$E_t = \{e \in E \mid\ \mu(e) \geq t\}.$$

Step 5: Construct the crisp graph $G_t^{'} = (V,\ E_t)$.

Step 6: Construct a family $\mathcal{I}_t$ of subsets of E_t such that

$$\mathcal{I}_t = \{I \subseteq E_t \mid\ I \ \text{ do not form a cycle in } G\}.$$

Step 7: Construct a family $\mathcal{C}_t$ of subsets of E_t such that

$$\mathcal{C}_t = \{C \subseteq E_t \mid \text{C is a Cycle and} C \setminus e \text{ is acyclic for any } e \in C\ in\ G\}.$$

Step 8: Identify the graphic matroid $M(G_t^{'})$ as

- E_t is the ground set
- $\mathcal{I}_t$ is the collection of independent sets
- $\mathcal{C}_t$ is the collection of circuits.

Step 9: Repeat the process for different values of t.

Proof

The algorithm for constructing a fuzzy-induced graphic matroid employs a thresholding process that converts a fuzzy graph into a crisp graph by selecting edges based on a chosen threshold t. We aim to prove that for any threshold t, the resulting structure is a well-defined matroid, satisfying all the fundamental axioms of matroid theory.

The ground set E_t of the graphic matroid is constructed by applying the threshold t to the fuzzy graph G, where each edge e has a membership value $\mu(e) \in [0,\ 1]$. The algorithm selects the ground set as $E_t = \{e \in E \mid\ \mu(e) \geq t\}$. Since E is finite and $t \in [0,\ 1]$, the subset $E_t \subseteq E$ is well-defined and finite for every choice of t.

By construction, for any $I \in \mathcal{I}_t$, the subset I is an acyclic set of edges in E. Since any subset $I^{'} \subseteq I$ cannot contain a cycle (being a subset of an acyclic set), $I^{'}$ is also acyclic and hence independent. This shows that the hereditary property holds for the family $\mathcal{I}_t$.

Now, let $I_1,\ I_2 \in \mathcal{I}_t$ such that $\mid I_1 \mid < \mid I_2 \mid$. Then, there must exist an edge $e \in I_2 \setminus I_1$, $I_1 \cup \{e\}$ is acyclic. Thus the exchange property holds for any independent sets I_1 and I_2. Also, by the construction of $\mathcal{C}$, the algorithm ensures that for any set of edges which forms a cycle, the deletion of any edge from this set results in an acyclic graph, satisfying the minimality condition of circuits.

3.4 Conclusion

This chapter highlights several key differences between classical graphic matroid theory and its fuzzy-induced counterpart. One of the most significant distinctions lies in the concept of bases. In crisp graphic matroids, the collection of bases is uniquely defined and fixed. However, in fuzzy-induced graphic matroids, the collection of bases varies depending on the threshold chosen for the fuzzy graph, illustrating the inherent flexibility and adaptability of these fuzzy structures.

We also examined how the identification of circuits and the notion of connectivity differ between crisp and fuzzy-induced graphic matroids. While circuits and connectivity are clearly defined and straightforward in the classical case, the fuzzy nature of edges, with their partial memberships, introduces variability. This can lead to changes in the set of circuits or even loss of connectivity when thresholds are applied. Furthermore, the concept of dual matroids diverges in fuzzy contexts, since thresholding a fuzzy graph may yield a dual matroid that no longer corresponds to any crisp graphic matroid.

Finally, we presented an algorithm for constructing a fuzzy-induced graphic matroid by applying threshold values. This approach demonstrates that although fuzzy-induced matroids provide greater flexibility, it is crucial to understand the effects of thresholding in order to manage the resulting variability effectively.

Chapter 4

Fuzzy Graphic Matroids

4.1 Introduction

Fuzzy matroid theory offers a flexible framework for addressing uncertainty within various mathematical structures, extending beyond the capabilities of traditional crisp matroid theory. In this chapter, we focus on fuzzy matroids, with particular emphasis on the concept of isomorphism between two fuzzy matroids. We provide a precise definition of isomorphism in this context, establishing a foundation for a comprehensive study of their structural properties.

Building on this foundation, we extend our study to fuzzy matroids induced from fuzzy graphs. This chapter introduces a novel approach to constructing fuzzy matroids directly from fuzzy graphs. This innovative concept not only broadens the applicability of fuzzy matroids but also forges a meaningful link between fuzzy graph theory and fuzzy matroid theory.

Throughout the chapter, we analyze the fundamental properties of isomorphic fuzzy matroids, revealing their distinctive characteristics and interrelationships. Moreover, the study of fuzzy matroids derived from fuzzy graphs paves the way for a deeper understanding of the interplay between fuzzy graphic structures and fuzzy matroid representations. Exploring the properties unique to these fuzzy matroids enriches our comprehension of this evolving field and provides valuable insights into the intricate connections between fuzzy graph theory and fuzzy matroid theory.

4.2 Isomorphism of Fuzzy Matroids

In this section, we define the concept of isomorphism between fuzzy matroids and explore several of their key properties.

Definition 4.2.1. Let $\mathcal{M}_1 = (E_1,\ \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2,\ \mathcal{I}_2)$ be two fuzzy matroids. $\mathcal{M}_1$ and $\mathcal{M}_2$ are fuzzy isomorphic if there exists a mapping $\gamma : E_1 \longrightarrow E_2$ such that γ satisfies the following conditions:

(i) γ is a one-to-one correspondence,

(ii) Set a fuzzy set mapping Γ from $\mathcal{F}(E_1)$ to $\mathcal{F}(E_2)$ corresponding to γ such that $\forall\ \mu \in \mathcal{F}(E_1),\ \forall x \in E_2$,

$$\Gamma(\mu)(x) = \mu\left(\gamma^{-1}(x)\right).$$

Now, for $\mu \in \mathcal{F}(E_1),\ \mu \in \mathcal{I}_1$ if and only if $\Gamma(\mu) \in \mathcal{I}_2$,

denoted by $\mathcal{M}_1 \cong \mathcal{M}_2$.

Example 4.2.1. *Consider the following weighted graph* $G^{'}$ *with edge set* $E = \{e_1,\ e_2,\ e_3,\ e_4,\ e_5\}$*, and weight function* $w : E \longrightarrow [0, 1]$ *such that* $w(e_1) = 0.3,\ w(e_2) = 0.8,\ w(e_3) = 0.7,\ w(e_4) = 0.5,\ w(e_5) = 0.2$*. Let* $r = 0.2$*, then* $E_r = E$*. Let*

$\mathcal{I}_1 = \{\{e_1\},\ \{e_2\},\ \{e_3\},\ \{e_4\},\ \{e_1,\ e_2\},\ \{e_1,\ e_3\},\ \{e_1,\ e_4\},\ \{e_2,\ e_3\},\ \{e_3,\ e_4\}\}$.

Then the pair $\mathcal{M}_1 = (E,\ \mathcal{I}_1)$ *is a fuzzy matroid.*

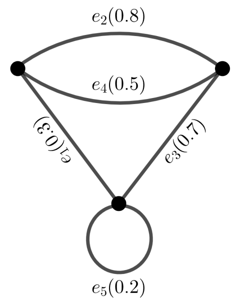


Figure 4.1: The weighted graph $G^{'}$

Let A *be the matrix with weighted column vectors*

$$\begin{array}{ccccc} v_1(0.5) & v_2(0.3) & v_3(0.8) & v_4(0.2) & v_5(0.7) \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Let $V = \{v_1,\ v_2,\ v_3,\ v_4,\ v_5\}$ *and*

$\mathcal{I}_2 = \{\{v_1\},\ \{v_2\},\ \{v_3\},\ \{v_5\},\ \{v_1,\ v_2\},\ \{v_1,\ v_5\},\ \{v_2,\ v_3\},\ \{v_2,\ e_5\},\ \{v_3,\ v_5\}\}$.

Then the pair $\mathcal{M}_2 = (V,\ \mathcal{I}_2)$ *is another fuzzy matroid. Also,* $\mathcal{M}_1 \cong \mathcal{M}_2$.

Theorem 4.2.1. *Let* $\mathcal{M}_1 = (E_1,\ \mathcal{I}_1)$ *and* $\mathcal{M}_2 = (E_2,\ \mathcal{I}_2)$ *be two fuzzy matroids such that* $\mathcal{M}_1 \cong \mathcal{M}_2$*. Then*

(i) $\forall\ \mu_1,\ \mu_2 \in \mathcal{F}(E_1),\ \mu_1 \leq \mu_2$ *if and only if* $\Gamma(\mu_1) \leq \Gamma(\mu_2)$.

(ii) for each $r,\ 0 < r \leq 1,\ (E_1,\ (\mathcal{I}_1)_r) \cong (E_2,\ (\mathcal{I}_2)_r)$.

Proof. Since $\mathcal{M}_1 \cong \mathcal{M}_2$, there is an isomorphic map $\gamma : E_1 \longrightarrow E_2$. Also, corresponding to γ we have a map $\Gamma : \mathcal{F}(E_1) \longrightarrow \mathcal{F}(E_1)$ such that
$\forall\ \mu \in \mathcal{F}(E_1),\ \forall x \in E_2,$

$$\Gamma(\mu)(x) = \mu\left(\gamma^{-1}(x)\right).$$

Then, for $\mu \in \mathcal{F}(E_1),\ \mu \in \mathcal{I}_1$ if and only if $\Gamma(\mu) \in \mathcal{I}_2$.
Let $\mu_1,\ \mu_2 \in \mathcal{F}(E_1)$. Then,

(i)

$$\begin{aligned}\mu_1 \leq \mu_2 \Leftrightarrow \mu_1\left(\gamma^{-1}(x)\right) &\leq \mu_2\left(\gamma^{-1}(x)\right),\ \forall x \in E_2\\ &\Leftrightarrow \Gamma(\mu_1)(x) \leq \Gamma(\mu_2)(x),\ \forall x \in E_2\\ &\Leftrightarrow \Gamma(\mu_1) \leq \Gamma(\mu_2)\end{aligned}$$

(ii) Let $A \subseteq E_1$, and $A \in (\mathcal{I}_1)_r$, for some $r,\ 0 < r \leq 1$.
$A \in (\mathcal{I}_1)_r$ if and only if $A = C_r(\mu)$ for some $\mu \in \mathcal{I}_1$. For this $\mu \in \mathcal{I}_1$, by the fuzzy isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$, we have $\Gamma(\mu) \in \mathcal{I}_2$ and $\gamma(A) = C_r(\Gamma(\mu))$. This is if and only if $\gamma(A) \in (\mathcal{I}_2)_r$.
That is, if $A \in (\mathcal{I}_1)_r$, then $\gamma(A) \in (\mathcal{I}_2)_r$.

□

4.3 Fuzzy Graphic Matroids

In this section, the concept of a fuzzy graphic matroid is introduced, and several of its fundamental properties are discussed.

A *fuzzy graph* [56] $G = (\sigma,\ \mu)$ consists of a nonempty set V together with a pair of functions $\sigma : V \longrightarrow [0,\ 1]$ and $\mu : E \longrightarrow [0,\ 1]$ such that for all $x,\ y \in V$,

$$\mu(xy) \leq \sigma(x) \bigwedge \sigma(y).$$

A fuzzy graph $G = (\sigma,\ \mu)$ is *connected* if the corresponding graph G' is connected. if G' is a tree, then the fuzzy graph $(\sigma,\ \mu)$ is also called a *fuzzy tree*[56].
Let G' be a simple graph and let $\sigma(x) = 1,\ \forall x \in V$, then we denote the fuzzy graph by $G = (V,\ \mu)$.
Now, a fuzzy matroid can be derived from a fuzzy graph using the following procedure.

Theorem 4.3.1. *Let $G = (\sigma,\ \mu)$ be a fuzzy graph with corresponding graph $G' = (V,\ E)$. For each $r,\ 0 < r \leq 1,$ let*

$$E_r = \{e \in E \mid \mu(e) \geq r\}$$

$$\mathfrak{F}_r = \{F \mid F\ is\ a\ forest\ in\ the\ (crisp)graph\ (V,\ E_r)\}$$

$$\mathfrak{E}_r = \{\mathcal{E}(F) \mid F\ \in \mathfrak{F}_r\}, where\ \mathcal{E}(F)\ is\ the\ edge\ set\ of\ F.$$

If

$$\mathfrak{I} = \{\mu \in \mathcal{F}(E) \mid C_r(\mu) \in \mathfrak{E}_r\ for\ each\ r,\ 0 < r \leq 1\ \}$$

then, $(E,\ \mathfrak{I})$ is a fuzzy matroid.

Proof. We prove the properties (i) and (ii) of definition 2.3.1.

(i) Suppose $\mu \in \mathfrak{I},\ \nu \in \mathcal{F}(E)$, and $\nu \leq \mu$. Then $C_r(\mu) \in \mathfrak{E}_r$ for each $0 < r \leq 1$. Also, for each $r,\ C_r(\nu) \subseteq C_r(\mu)$ and $(E,\ \mathfrak{E}_r)$ is a crisp matroid by Theorem 2.3.1.

$$\Rightarrow C_r(\nu) \in \mathfrak{E}_r\ for\ each\ r.$$

$$\Rightarrow \nu \in \mathfrak{I}$$

Thus, $(E,\ \mathfrak{I})$ satisfies the hereditary property.

(ii) Suppose $\mu,\ \nu \in \mathfrak{I}$ and $|supp(\mu)| < |supp(\nu)|$. Let $\delta = min\{m(\mu),\ m(\nu)\}$.
Then we have, $supp(\mu) \in I_\delta$ and $supp(\nu) \in I_\delta$.
I_δ is a crisp matroid, therefore $\exists$ a set $C \in I_\delta$ such that
$supp(\mu) \subseteq C \subseteq supp(\mu) \cup supp(\nu)$.
Let

$$\omega(x) = \begin{cases} \mu(x), & \text{if } x \in supp(\mu) \\ \delta, & \text{if } x \in C \setminus supp(\mu) \\ 0, & \text{otherwise} \end{cases}$$

Then clearly $\omega \in \mathfrak{I}$, $\mu < \omega \leq \mu \vee \nu$ and $m(\omega) \geq min\{m(\mu),\ m(\nu)\}$.

Thus, $(E,\ \mathfrak{I})$ is a fuzzy matroid. □

The fuzzy matroid $(E,\ \mathfrak{I})$ constructed in Theorem 4.3.1 is referred to as the *fuzzy cycle matroid* of G, and is denoted by $\mathcal{M}_F(G)$.

Distinct fuzzy graphs can induce the same fuzzy cycle matroid, even if the corresponding fuzzy graphs are not isomorphic. This indicates that the fuzzy cycle matroid does not necessarily preserve all structural details of the original fuzzy graph. The following example illustrates this observation.

Example 4.3.1. *Consider the following fuzzy graphs* $G_1 = (V_1,\ \mu_1)$ *and* $G_2 = (V_2,\ \mu_2)$*, where* $\mu_1(e_i) = \mu_2(e_i') = 0.8,\ i = 1, \ldots\ldots, 6$. *Then, easily we can get* $\mathcal{M}_F(G_1) = \mathcal{M}_F(G_2)$. *But, clearly,* G_1 *is not isomorphic to* G_2.

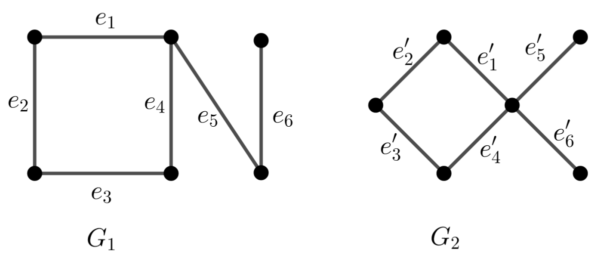


Figure 4.2: Non-isomorphic fuzzy graphs with same fuzzy cycle matroid

Definition 4.3.1. Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy matroid. If there exists a fuzzy graph $G = (\sigma,\ \mu)$ such that $\mathcal{M} \cong \mathcal{M}_F(G)$, then $\mathcal{M}$ is called a *fuzzy graphic matroid.*

Theorem 4.3.2. *Let $\mathcal{M}$ be a fuzzy graphic matroid. Then $\mathcal{M} \cong \mathcal{M}_F(G)$ for some connected fuzzy graph G.*

Proof. As $\mathcal{M}$ is a fuzzy graphic matroid, $\mathcal{M} \cong \mathcal{M}_F(G)$ for some fuzzy graph $G = (\sigma,\ \mu)$. If G is connected, the result is proved. If G is not a connected fuzzy graph, then the corresponding graph G' is also not connected.

Now, we construct a connected graph from G' as follows.

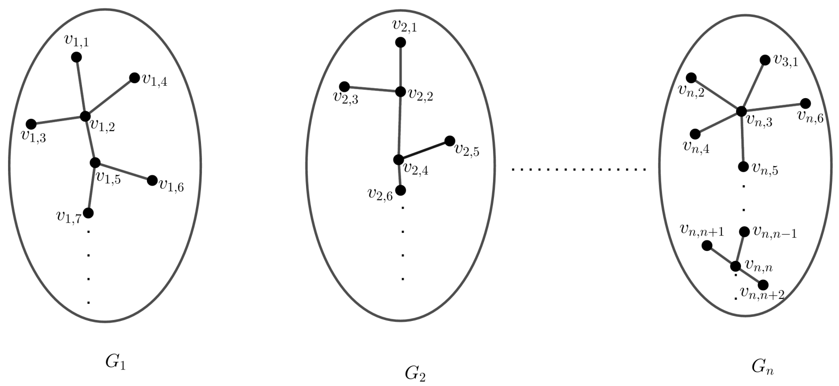


Figure 4.3: Underlying graph G' of the fuzzy graph G

We suppose that $G_1,\ G_2,\ ...,\ G_n$ are the connected components of $G^{'}$. From each connected component G_i, we choose a vertex $v_{i,i}$. Form a new graph $H^{'}$ by identifying $v_{1,1},\ v_{2,2},\ ...,$ and $v_{n,n}$ as a single vertex v.

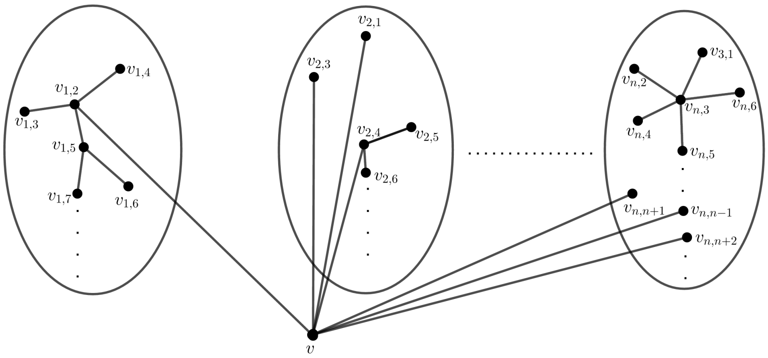


Figure 4.4: Underlying graph $H^{'}$ of the fuzzy graph graph H

Obviously, $E(H^{'}) = E(G^{'})$, and $H^{'}$ is connected. Thus, H is a new connected fuzzy graph with $H^{'}$ as the underlying graph. Moreover, $\forall X \subseteq E(G^{'})$, If X does not contain any cycles of $G^{'}$ then X does not contain any cycles of $H^{'}$ and vice versa. Thus, we have the two fuzzy cycle matroids $\mathcal{M}_F(G)$ and $\mathcal{M}_F(H)$ are isomorphic. Then by tansitivity, we have $\mathcal{M} \cong \mathcal{M}_F(H)$. $\square$

4.4 Structural Properties

4.4.1 Fuzzy Basis

In the context of fuzzy graphic matroids, a fuzzy basis serves as a generalization of the classical notion of a basis in crisp matroid theory. In crisp matroids, a basis is defined as a maximal independent set, meaning a set of elements that is independent, and to which the addition of any further element would result in dependence. In fuzzy graphic matroids, however, the notion of independence is extended to incorporate degrees of membership, and as such, a fuzzy basis reflects not only the maximality of independence but also the influence of fuzziness on the structure.

For fuzzy graphic matroids, the concept of a fuzzy basis extends the classical idea by incorporating varying degrees of independence, as determined by the membership values associated with elements in the fuzzy graph.

Definition 4.4.1. A *fuzzy base* of a fuzzy graphic matroid $\mathcal{M}$, induced by a fuzzy graph $G = (\sigma,\ \mu)$, is a combination of spanning forests constructed from a sequence of membership levels of the fuzzy graph. Let $\{b_1,\ b_2,\ \cdots\ b_n\} = \{\mu(e) \mid e \in supp(\mu)\}$ with $0 < b_1 \leq b_2 \leq \cdots \leq b_n \leq 1$. The fuzzy base is defined as a family of spanning forests $\{B_1,\ B_2,\ \cdots,\ B_n\}$, where B_1 is a spanning forest of $\mu[b_1]$, and each subsequent forest B_i is a maximal spanning forest of $\mu[b_i]$ such that $B_{i+1} \subseteq B_i$ and contains no cycles. The fuzzy base ν is given by:
$\nu = \vee_{i=1}^{n} \chi_{B_i} \wedge [b_i]$, where χ_{Bi} is the characteristic function of the forest B_i.

In the context of fuzzy graph theory, a fuzzy basis may exhibit properties associated with the connectivity of the underlying fuzzy graph. For instance, the edges comprising a fuzzy basis might form a connected fuzzy subgraph and, in certain cases, resemble a fuzzy analog of a spanning tree. While in classical matroid theory all bases have equal cardinality, this uniformity does not necessarily extend to fuzzy graphic matroids. Due to the graded nature of membership values and thresholding effects, fuzzy bases can differ in size, highlighting a significant departure from traditional matroid structures.

Example 4.4.1. *Consider the following fuzzy graph and let* $\mathcal{M} = (E,\ \mathcal{I})$ *be the fuzzy cyclic matroid induced from it.*

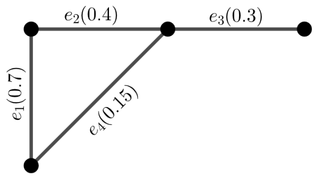


Figure 4.5: fuzzy matroids having bases of different cardinalities

We now construct two fuzzy bases by following the above definition. Obviously, $\{\mu(e) \mid e \in supp\ \mu\} = \{0.15,\ 0.3,\ 0.4,\ 0.7\} = \{b_1,\ b_2,\ b_3,\ b_4\}$. *The fuzzy graphs* $\mu[a_i],\ i = 1,\ 2,\ 3,\ 4$ *are presented in the following figure.*

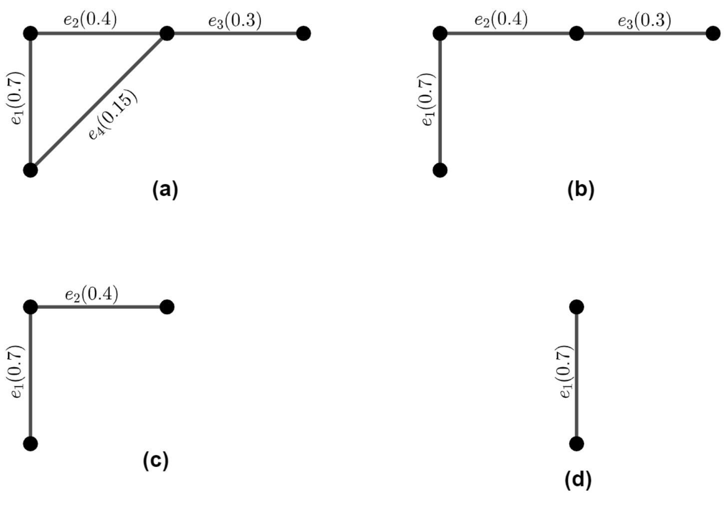


Figure 4.6: fuzzy graphs $\mu[a_i],\ i = 1,\ 2,\ 3,\ 4$

First we choose $B_1 = \{e_1,\ e_2,\ e_3\}$, *and then we choose* $B_2 = \{e_1,\ e_2,\ e_3\}$, $B_3 = \{e_1,\ e_2\}$ *and* $B_4 = \{e_1\}$.

Let $\nu_1 = (\chi_{A_1} \wedge [0.15]) \vee (\chi_{A_2} \wedge [0.3]) \vee (\chi_{A_3} \wedge [0.4]) \vee (\chi_{A_4} \wedge [0.7])$, *that is,*

$$\nu_1(e_i) = \begin{cases} 0.7, & if i = 1 \\ 0.4, & if i = 2 \\ 0.3, & if i = 3 \\ 0, & otherwise \end{cases}$$

then ν_1 *is a fuzzy base.*

Similarly, the following ν_2 *is another fuzzy base:*

$\nu_2 = (\chi_{A_1} \wedge [0.15]) \vee (\chi_{A_2} \wedge [0.3]) \vee (\chi_{A_3} \wedge [0.4]) \vee (\chi_{A_4} \wedge [0.7])$

where $B_1 = \{e_2,\ e_3,\ e_4\}, B_2 = \{e_2,\ e_3\}, B_3 = \{e_2\}$ *and* $B_4 = \emptyset$.

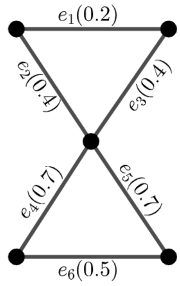
That is ,

$$\nu_2(e_i) = \begin{cases} 0.4, & if i = 2 \\ 0.3, & if i = 3 \\ 0.15, & if i = 4 \\ 0, & otherwise \end{cases}$$

Thus, ν_1 *and* ν_2 *are fuzzy bases of* $\mathcal{M} = (E,\ \mathcal{I})$*. While we have* $\mid \nu_1 \mid= 1.4$ *and* $\mid \nu_2 \mid= 0.85$

In classical graphic matroids, a tree, being an acyclic subgraph, naturally forms an independent set. However, this result does not directly extend to the fuzzy setting.

Example 4.4.2. *Let* $\mathcal{M} = (E,\ \mathcal{I})$ *be the fuzzy cyclic matroid induced from following fuzzy graph* G*.*



G

Figure 4.7: Fuzzy tree but not independent in $\mathcal{M}$

Here, $\mu = \{e_1(0.2),\ e_2(0.4),\ e_3(0.4),\ e_4(0.7),\ e_5(0.),\ e_6(0.5)\}$ *forms a fuzzy tree, but* $\mu \notin \mathcal{I}$.

The following theorem leads to the closure property of fuzzy graphic matroids.

Theorem 4.4.1. *[42] Let* $\mathcal{M} = (E,\ \mathcal{I})$ *be a fuzzy matroid. Then* $\mathcal{M}$ *is closed if and only if for each* $\mu \in \mathcal{I}$*, there is a fuzzy basis* $\nu \in \mathcal{I}$ *such that* $\mu \leq \nu$*.*

Theorem 4.4.2. *Every fuzzy graphic matroid is closed*

Proof. Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy graphic matroid induced from the fuzzy graph G. Let $\mu \in \mathcal{I}$. If μ is maximal, the theorem is thereby proven. Suppose μ is not maximal. We have from the definition of fuzzy base for a fuzzy graphic matroid, there exist atleast one fuzzy base ν for $\mathcal{M}$. So, we can find a fuzzy edge $e \in supp(\nu) \setminus supp(\mu)$ and that can be added to μ without creating a fuzzy cycle to get another fuzzy independent set ω. If ω is maximal, we have done. Repeat this process, extending μ by adding fuzzy edges from ν, until we get a maximal fuzzy independent set ξ, which is a fuzzy base with $\mu \leq \xi$. □

4.4.2 Fuzzy Circuit

In fuzzy graphic matroids, a fuzzy circuit redefines the concept of a circuit from crisp matroid theory to the fuzzy setting.

Definition 4.4.2. Let $\mathcal{M}$ be a fuzzy graphic matroid. $\mu \in \mathcal{F}(E)$ is a *fuzzy circuit* of $\mathcal{M}$ if μ forms a cycle and $\mu \setminus \backslash e$ forms a tree in G for each $e \in supp(\mu)$, where $\mu \setminus \backslash e$ is given by

$$\mu \setminus \backslash e(x) = \begin{cases} \mu(x), & \text{if } x \neq e, \\ 0, & \text{if } x = e \end{cases}$$

In a fuzzy graphic matroid $\mathcal{M}$, a fuzzy circuit μ might not necessarily correspond to a fuzzy cycle in the underlying fuzzy graph G.

Example 4.4.3. *Consider the fuzzy cyclic matroid $\mathcal{M}$ induced from the following fuzzy graph G. $\mu = \{e_2(0.3),\ e_3(0.4),\ e_4(0.5),\ e_5(0.2)\}$ is a fuzzy circuit in $\mathcal{M}$, but μ is not a fuzzy cycle in G.*

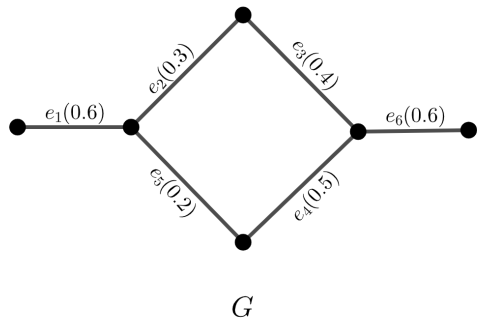


Figure 4.8: Fuzzy circuit in $\mathcal{M}$ but not a fuzzy cycle in G

Proposition 4.4.1. *Suppose that $\mathcal{M} = (E,\ \mathcal{I})$ is a fuzzy graphic matroid and β is a fuzzy base of $\mathcal{M}$ with $e \in E \setminus supp(\beta)$. Then, there exists a unique fuzzy circuit ν such that $supp\ \nu \subseteq supp(\beta) \cup \{e\}$ and $e \in supp(\nu)$.*

Proof. Since β is a fuzzy base, we have, $supp(\beta) \cup \{e\} \notin \mathcal{I}$, which implies that it contains some fuzzy circuit ν, where $e \in supp(\nu)$. If there exists some fuzzy circuit ω distinct from ν such that $supp(\omega) \subseteq supp(\beta) \cup \{e\}$, we have $(\omega \vee \nu) \setminus \backslash e$ contains a fuzzy circuit, which contradicts $(\omega \vee \nu) \setminus \backslash e \leq \beta$. We can conclude that ν is the unique fuzzy circuit such that $supp(\nu) \leq supp(\beta) \cup \{e\}$. □

Proposition 4.4.2. *Suppose that $\mathcal{M} = (E, \mathcal{I})$ is a fuzzy graphic matroid and β is a fuzzy base of $\mathcal{M}$. Then for any $e_1 \in E \setminus supp(\beta)$, the fuzzy set ν such that $supp(\nu) = supp(\beta \setminus \backslash e_2) \cup \{e_1\}$ is a fuzzy base of $\mathcal{M}$ if and only if e_2 is an element in the fuzzy circuit ω, where $supp(\omega) = supp(\beta) \cup \{e_1\}$.*

Proof. Let ψ be the fuzzy set such that $supp(\psi) = supp(\beta \setminus \backslash e_2) \cup \{e_1\}$ and assume the negation of the statement that ψ is a fuzzy base of $\mathcal{M}$ for some $e_2 \in supp(\beta) \setminus supp(\nu)$. Then $supp(\nu) \subseteq supp(\beta \setminus \backslash e_2) \cup \{e_1\}$, a contradiction.

Now, assume $e_2 \in \nu$ and suppose that $supp(\beta \setminus \backslash e_2) \cup e_1$ contains a cycle in the underlying fuzzy graph. Then, there exists a fuzzy circuit ω other than ν such that $supp(\omega) \subseteq supp(\beta) \cup \{e_1\}$, which contradicts Proposition 4.4.1. □

Lemma 4.4.1. *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two fuzzy graphic matroids induced from the same fuzzy graph G, such that any fuzzy circuit of $\mathcal{M}_1$ contains a fuzzy circuit of $\mathcal{M}_2$ and vice versa, then $\mathcal{M}_1 = \mathcal{M}_2$.*

Proof. If μ is a fuzzy circuit of $\mathcal{M}_1$, there exists a fuzzy circuit ν of $\mathcal{M}_2$ such that $\nu \leq \mu$ and for ν there exist a fuzzy circuit ω of $\mathcal{M}_1$ such that $\omega \leq \nu$. Thus, we have

$$\omega \leq \nu \leq \mu.$$

Then by Theorem 2.3.6 , we must have $supp(\omega) = supp(\mu)$, which implies that $supp(\mu) = supp(\nu)$. Similarly we show that any fuzzy circuit of $\mathcal{M}_2$ is a fuzzy circuit of $\mathcal{M}_1$, thus, both fuzzy graphic matroids have the same collection of fuzzy circuits and they are equal. □

4.5 Conclusion

In conclusion, this chapter has introduced a solid theoretical basis for the study of fuzzy matroids by first defining an isomorphism between two fuzzy matroids and proving some fundamental properties. The concept of isomorphism plays a key role in understanding the structural similarities between different fuzzy matroids, allowing for a detailed study of their equivalence under various transformations.

The introduction of fuzzy graphic matroids, a new class of fuzzy matroids induced from fuzzy graphs, represents an important development in the field. By associating concepts from fuzzy graphs with fuzzy matroid theory, this section opens up new avenues for applications, particularly in areas where uncertainty or fuzziness of data plays a significant role, such as network theory and optimization problems. The results established regarding fuzzy graphic matroids affirm their place within the broader framework of fuzzy matroids, highlighting their unique characteristics and how they differ from traditional, crisp cases.

The final section of the chapter extends the study to the structural properties of fuzzy matroids, focusing on fuzzy bases and fuzzy circuits. By examining these properties, the chapter provides insight into the internal structure and behavior of fuzzy matroids, which is essential for their practical applications. The discussion of counterexamples from the crisp case helps to highlight the detailed differences between fuzzy and crisp matroids, further expanding the theoretical framework.

Chapter 5

Fuzzy Representable Matroids

5.1 Introduction

Representable matroids are a class of matroids that can be defined using vectors over a field, illustrating the dependency structure among them. A matroid is representable if its independent sets correspond to linearly independent sets of vectors. This establishes a connection between the abstract concepts of matroid theory and the well-established principles of linear algebra. Representable matroids can be defined over various fields, such as the real numbers or finite fields, while preserving the core notions of dependence and independence. This chapter analyzes the interdependent relationship between fuzzy matroids and fuzzy vector spaces, introducing a novel perspective on the construction of fuzzy matroids.

The main focus of this study lies in the derivation of a fuzzy matroid from a fuzzy vector space, illuminating the relationship between these mathematical structures. In particular, it introduces the concept of fuzzy representable matroids and examines their fundamental properties. The concept of representability holds principal importance in matroid theory, as it reveals the structural characteristics and constraints of these mathematical components.

A remarkable result presented in this chapter concerns the representation of fuzzy graphic matroids over various fields. The study reveals a compelling finding that, in general, a fuzzy graphic matroid cannot be represented over any field. This significant result prompts a deeper exploration of the limitations and challenges associated with the representability of fuzzy graphic matroids.

Moreover, the chapter provides important insights by explaining the conditions under which a fuzzy graphic matroid can be considered representable over a given field. Analyzing these conditions not only aids the theoretical understanding of fuzzy matroids but also has practical implications for applications where representing uncertainty plays a fundamental role.

5.2 Fuzzy Representable Matroids

This section covers the construction of a fuzzy matroid induced from a fuzzy vector space, followed by a study of its connection to fuzzy graphic matroids.

Definition 5.2.1. [40] Let W be a vector space over the field of real numbers and let $\nu : W \longrightarrow [0,1]$. Then the pair $\widehat{W} = (W,\ \nu)$ is called a *fuzzy vector space* if it satisfies the property that for all $a,\ b \in \mathbb{R}$ and $w_1,\ w_2 \in W$, we have $\nu(aw_1 + bw_2) \geq \nu(w_1) \wedge \nu(w_2)$.

Example 5.2.1. *[40] Let $W = \mathbb{R}^2$. Define the fuzzy subset $\nu : W \longrightarrow [0,1]$ by*

$\nu[(0,0)] = 1,\ \nu[(0, \mathbb{R} \setminus \{0\})] = \frac{1}{2}$ *and* $\nu[\mathbb{R}^2 \setminus (0, \mathbb{R})] = \frac{1}{4}$. *Then,* $\widehat{W} = (W,\ \nu)$ *is a fuzzy vector space.*

Definition 5.2.2. [40] If $\widehat{W} = (W,\ \nu)$ is a fuzzy vector space. Then, a finite set of vectors $\{w_k\}_{k=1}^n$ is fuzzy linearly independent in $\widehat{W}$ if and only if $\{w_k\}_{k=1}^n$ is linearly independent in W and for all $\{a_i\}_{k=1}^n \subset \mathbb{R}$,

$$\nu\left(\sum_{k=1}^{n} a_k w_k\right) = \wedge_{k=1}^{n} \nu(a_k w_k)$$

Proposition 5.2.1. *[40] Consider a fuzzy vector space* $\widehat{W} = (W,\ \nu)$*. Then any set of vectors* $\{w_k\}_{k=1}^N \subset W \setminus \{0\}$ *which has distinct* ν*-values is linearly and fuzzy linearly independent.*

Below, we obtain a fuzzy matroid from a fuzzy vector space.

Theorem 5.2.1. *Let* $\widehat{W} = (W,\ \nu)$ *be a fuzzy vector space, where* W *is a vector space over* $\mathbb{R}$ *and* X *be a subset of* W*. For each* r*,* $0 < r \leq 1$*, let*

$$X_r = \{x \in X \mid \nu(x) \geq r\}$$

$$\mathcal{I}_r = \{I = \{x_i\}_{i=1}^N \subset X \setminus \{0\} \mid x_i \textit{ has distinct } \nu \textit{ values }\}$$

If

$$\mathfrak{I}_X = \{\nu \in \mathcal{F}(X) \mid C_r(\nu) \in \mathcal{I}_r \ for\ each\ r,\ where\ 0 < r \leq 1\ \}$$

then, $(X,\ \mathfrak{I}_X)$ *is a fuzzy matroid.*

Proof. We prove the properties (i) and (ii) of definition 2.3.1

(i) Suppose $\mu \in \mathfrak{I}_X,\ \nu \in \mathcal{F}(E)$, and $\nu \leq \mu$. Then $C_r(\mu) \in \mathcal{I}_r$ for each $0 < r \leq 1$.

Also, for each r, $C_r(\nu) \subseteq C_r(\mu)$ and $(X,\ \mathcal{I}_r)$ is a crisp matroid by Theorem 2.3.1.

$$\Rightarrow C_r(\nu) \in \mathcal{I}_r \ for\ each\ r.$$

$$\Rightarrow \nu \in \mathfrak{I}_X$$

Thus, $(X,\ \mathfrak{I}_X)$ satisfies the first property.

(ii) Suppose that $\alpha,\ \beta \in \mathfrak{I}_X$ and $|supp(\alpha)| < |supp(\beta)|$. Let $\delta = min\{m(\alpha),\ m(\beta)\}$.
Then we have, $supp(\alpha) \in I_\delta$ and $supp(\beta) \in I_\delta$.
Thus, $\exists$ a set $Y \in I_\delta$ such that $supp(\alpha) \subseteq Y \subseteq supp(\alpha) \cup supp(\beta)$, as $(X,\ \mathfrak{I}_\delta)$ is a crisp matroid,
Let

$$\xi(w) = \begin{cases} \alpha(w), & \text{if } w \in supp(\alpha) \\ \delta, & \text{if } w \in Y \setminus supp(\alpha) \\ 0, & \text{otherwise} \end{cases}$$

Then verily $\xi \in \mathfrak{I}_X$, $\alpha < \xi \leq\ \alpha \vee \beta$ and $m(\xi) \geq min\{\ m(\alpha),\ m(\beta)\}$.

Thus, $(X,\ \mathfrak{I}_X)$ is a fuzzy matroid. □

In the above theorem $(X,\ \mathfrak{I}_X)$ is called the *fuzzy vector matroid* induced by X, denoted by $\mathcal{M}_F\,[X]$.

Example 5.2.2. *Consider* $\widehat{W} = (\mathbb{R}^3,\ \nu)$, *where* $\nu\,[(0,0,0)] = 1,\ \nu\,[(0,0,\mathbb{R}\setminus\{0\})] = 1/2,\ \nu\,[(0,\mathbb{R}\setminus\{0\},0)] = 1/4$ *and* $\nu\,[\mathbb{R}\setminus(0,\mathbb{R},\mathbb{R})] = 1/3$.
Now, consider the following matrix

$$\begin{array}{ccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 \end{array}$$
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 4 & 2 & 2 \\ 0 & 0 & 0 & 4 & 8 & 4 & 0 \end{bmatrix}$$

Let $X \subseteq \mathbb{R}^3$ be the set $\{w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6,\ w_7\}$ of column labels of the matrix. Then, clearly $\nu(w_1) = 1,\ \nu(w_2) = 1/3,\ \nu(w_3) = 1/4,\ \nu(w_4) = 1/2,\ \nu(w_5) = 1/4,\ \nu(w_6) = 1/4,\ \nu(w_7) = 1/4.$
Let $\mathcal{I}_X$ be the family of fuzzy subsets of X under ν such that each w_i has distinct ν values. Then $\mathcal{I}_X$ consists of all fuzzy subsets of $X \setminus \{w_1\}$ with atmost three elements except for $\{w_2,\ w_3,\ w_7\}$, $\{w_4,\ w_6,\ w_7\}$, $\{w_4,\ w_5,\ w_7\}$ and any fuzzy subset containing $\{w_5,\ w_6\}$.
Thus, $(X,\ \mathcal{I}_X)$ is a fuzzy vector matroid.

Definition 5.2.3. Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy matroid. If we can find a subset X of some fuzzy vector space $\widehat{W} = (W,\ \mu)$ such that $\mathcal{M} \cong \mathcal{M}_F[X]$, then $\mathcal{M}$ is called a *fuzzy representable matroid*, we also say that $\mathcal{M}$ is fuzzy representable.

Theorem 5.2.2. *Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy representable matroid, and for each $r,\ r \in (0,\ 1]$, let $\mathcal{I}_r = \{C_r(\mu) : \mu\ \in \mathcal{I}\}$. Then for each $r,\ r \in (0,\ 1]$, $\mathcal{M}_r = (E,\ \mathcal{I}_r)$ is a representable matroid.*

Proof. By Theorem 2.3.1, it follows that, for each $r,\ r \in (0,\ 1]$, $\mathcal{M}_r$ is a matroid. As $\mathcal{M}$ is a fuzzy represenrtable matroid, there must be a fuzzy vector subspace $\widehat{W} = (W,\ \nu)$ such that $\mathcal{M} \cong \mathcal{M}_F[X]$, where $X \subseteq W$. Thus, by definition 4.2.1, there exists a bijective mapping $\gamma : E \longrightarrow X$ and corresponding to γ, we can set a map $\Gamma : \mathcal{F}(E) \longrightarrow \mathcal{F}(X)$ such that, for $\mu \in \mathcal{F}(E)$, $\mu \in \mathcal{I}$ if and only if $\Gamma(\mu) \in \mathcal{I}_X$. For each subspace $r \in (0,\ 1]$, $C_r(\nu)$ is a subspace of W and let $\mathfrak{J}_r$ be the collection of linearly independent vectors of $C_r(\nu)$. Let us denote the (crisp) vector matroid $M[C_r(\nu)] = (C_r(\nu),\ \mathfrak{J}_r)$. Next, define a mapping $\Upsilon : E \longrightarrow C_r(\nu)$ by

$$\Upsilon(e) = \gamma(e),\ \forall e \in E.$$

We have to show that $\mathcal{M}_r \cong M[C_r(\nu)]$. For that, it is enough to prove that $A \in \mathcal{I}_r$ if and only if $\Upsilon(A) \in \mathfrak{J}_r$.

Let $A \in \mathcal{I}_r \Rightarrow$ the membership degree of A in $\mathcal{I}$ is atleast r, and since $\mathcal{M} \cong \mathcal{M}_F[X]$, the membership degree of $\Upsilon(A)$ in $\mathcal{I}_X$ is also atleast r. Thus,

$\vee\{s \in (0, 1] : \gamma(A) \subseteq C_s(\mu)$ and $\Upsilon(A)$ is linearly independent$\} \geq r$. That is $\exists\ b \geq a$ such that$\Upsilon(A) \subseteq C_s(\mu) \subseteq C_r(\mu)$ and $\Upsilon(A)$ is linearly independent. This results in $\Upsilon(A) \in \mathfrak{J}_r$.

Conversely, let $\Upsilon(A) \in \mathfrak{J}_r\ \Rightarrow \Upsilon(A) \subseteq C_r(\nu)$ and $\Upsilon(A)$ does not contain any linearly dependent vectors.

$\Rightarrow$ the membership degree of $\Upsilon(A)$ in $\mathcal{I}_X \geq r$, this implies the membership degree of A in $\mathcal{I} \geq r$. It defines $A \in \mathcal{I}_r$.

Thus, $\mathcal{M}_r = (E,\ \mathcal{I}_r)$ is representable. □

Remark: In crisp matroid theory, it is known that a graphic matroid is representable over any field. However, the following example shows that not all fuzzy matroids are representable over every field. This statement is justified by the example below.

Example 5.2.3. *Let* $X = \{x_1,\ x_2,\ x_3\}$, $\mathcal{I}_1 = \{\Phi\}$, $\mathcal{I}_{0.6} = \{\Phi,\ \{x_1\},\ \{x_2\},$ $\{x_1,\ x_2\}\}$, $\mathcal{I}_{0.25} = \{\Phi,\ \{x_1\},\ \{x_2\},\ \{x_3\},\ \{x_1,\ x_2\},\ \{x_1,\ x_3\},\ \{x_2,\ x_3\}\}$.

Then clearly, $(X,\ \mathcal{I}_1)$*,* $(X,\ \mathcal{I}_{0.25})$ *and* $(X,\ \mathcal{I}_{0.6})$ *are crisp matroids, and* $\mathcal{I}_1 \subset \mathcal{I}_{0.6} \subset \mathcal{I}_{0.25}$.

Let

$$\mathcal{I}_r = \begin{cases} \mathcal{I}_{0.25}, & r \in (0,\ 0.25] \\ \mathcal{I}_{0.6}, & r \in (0.25,\ 0.6] \\ \mathcal{I}_1, & r \in (0.6,\ 1] \end{cases}$$

and let

$$\mathcal{I} = \{\mu \in \mathcal{F}(X) |\ C_r(\mu) \in \mathcal{I}_r,\ r \in (0,\ 1]\ \}.$$

Then, $(X,\ \mathcal{I})$ *is a fuzzy matroid.*

Let G *be the fuzzy graph given below.*

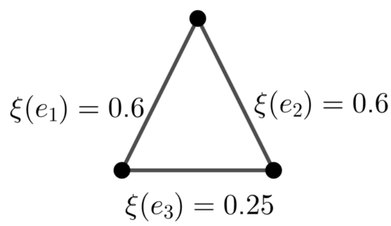


Figure 5.1: Fuzzy graph G with $(X,\ \mathcal{I}) \cong \mathcal{M}_F(G)$

It is asy to see that $(X,\ \mathcal{I}) \cong \mathcal{M}_F(G)$. *Thus,* $(X,\ \mathcal{I})$ *is a fuzzy graphic matroid.*

Suppose that there exist a fuzzy vector space $\widehat{W} = (W,\ \nu)$ *and* $Y \subseteq W$ *such that* $\mathcal{M}_F(G) \cong \mathcal{M}_F[Y]$. *By definition of fuzzy graphic and fuzzy representable matroids, we obtain that* Y *consists of three linearly dependent elements, in which any two are linearly independent. Let* $Y = \{y_1,\ y_2,\ y_3\}$ *by isomorphism between two fuzzy matroids, there should exist a vector such that its value in* ν *is exactly* 0.25*. Let us assume that,* $\nu(u_1) = 0.25$*. Since* y_1 *is the linear combination of*

y_2 and y_3, it follows that $0.25 = \nu(y_1) \geq \nu(y_2) \wedge \nu(y_3) = 0.6$, a contradiction. Thus, $(X,\ \mathcal{I})$ is not a fuzzy representable matroid.

Next, we consider the conditions that allow a fuzzy graphic matroid to be fuzzy representable over any field.

Theorem 5.2.3. *[40] Given a vector space E with basis $B = \{v_\alpha\}_{\alpha \in A}$, constant $\mu_0 \in (0,\ 1]$ and any set of constants $\{\mu_\alpha\}_{\alpha \in A} \subset (0,\ 1]$ such that $\mu_0 \geq \alpha$ for all $\alpha \in A$. Let us construct a function $\mu : E \longrightarrow [0,\ 1]$ in the following way. Any $z \neq 0,\ z \in E$ can be uniquely written as $z = \sum_{i=1}^{N} a_i \nu_{\alpha_i}$, with $a_i \neq 0$. Define, $\mu(z) = \wedge_{i=1}^{N} \mu_{\alpha_i}$ and $\mu(0) = \mu_0$.*
Clearly μ is defined for all $z \in E$ and is well-defined. Thus, $\widehat{E} = (E,\ \mu)$ is a fuzzy vector space with fuzzy basis B.

Theorem 5.2.4. *Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy matroid. If $\mathcal{M} \cong \mathcal{M}_F(G)$ where $G = (\sigma,\ \mu)$ is a fuzzy graph with supp(μ) is a tree, then $\mathcal{M}$ is a fuzzy representable matroid.*

Proof. Let $supp(\sigma) = \{v_1,\ v_2,\ ...,\ v_n\}$, where $n > 0$, and let $m =\mid supp(\mu) \mid$. Let F be any field and suppose 0 and 1 are respectively the additive and multiplicative identity elements of F. Let $W_n(F)$ be an n dimensional vector space over F.
Now consider the collection of vectors,

$$\mathcal{E} = \{e_i \in W_n(F) |\ i\text{th component of } e_i \text{ is 1 and all other components are } 0\}.$$

Now, we have to obtain a fuzzy vector space $\widehat{W}$ such that the fuzzy matroid induced by $\widehat{W}$ is isomorphic to $\mathcal{M}$.

For that, set a map ψ from $supp(\mu)$ to $W_n(F)$ such that $\psi(e) = e_j - e_i$, where e is the edge connecting the vertices v_i and v_j.
Now, consider the collection

$$B = \{\psi(e) \mid e \in supp(\mu)\}$$

and let $W =< B >$, that is, W is the subspace spanned by B.
Then, we have, for any subset Y of $supp(\mu)$, Y does not contain any cycles in G if and only if the vector family of $\psi(Y)$ is independent on W.

Obviously, the vectors in B are linearly independent, since $supp(\mu)$ is a tree. By Theorem 5.2.3, we can get a fuzzy vector space $\widehat{W}$ with a fuzzy basis ν such that

$$supp(\nu) = B.$$

Thus by definitions of fuzzy graphic and fuzzy representable matroids

$$\mathcal{M}_F(G) \cong \mathcal{M}_F[W].$$

Therefore, $\mathcal{M}$ is fuzzy representable. □

This result highlights a fundamental distinction between crisp and fuzzy matroid theories. In crisp matroid theory, graphic matroids are always representable over any field, reflecting a well-defined and consistent linear structure. In contrast, fuzzy graphic matroids introduce variability and uncertainty through membership values, which complicates their representability. Unlike their crisp counterparts, fuzzy graphic matroids may not be representable over any field due to the fuzzy nature of edges and the dependence on thresholding. Moreover, while crisp representable matroids strictly adhere to linear independence in vector spaces, fuzzy representable matroids generalize this concept by incorporating degrees of membership, allowing for more flexible and nuanced structural relationships. These differences underscore the richer, yet more complex, framework of fuzzy matroid theory in modeling uncertainty and partial information.

5.3 Conclusion

In summary, this study has contributed to the field of fuzzy matroid theory by introducing a new category derived from fuzzy vector spaces. The concept of fuzzy representable matroids has been developed as a central focus, presenting intriguing properties compared to crisp matroid theory. While it is well-established in crisp matroid theory that a graphic matroid is representable over any field, this study has demonstrated a clear divergence from this expectation within the context of fuzzy matroids.

Through a striking example, it is highlighted that not all fuzzy graphic matroids are representable over every field. This divergence from crisp matroid theory emphasizes the unique challenges and complexities inherent in the fuzzy setting, offering significant insights into the limitations of representability within this specific category of matroids.

By analyzing the representability of fuzzy graphic matroids, this study offers a nuanced perspective that extends beyond the boundaries of classical matroid theory. This divergence from the crisp model encourages a re-evaluation of theoretical assumptions, fostering a deeper understanding of the intricate relationship between fuzzy matroids and representability.

In brief, this chapter extends the theoretical foundations of fuzzy matroid theory by exploring its association with fuzzy vector spaces. The concept of fuzzy representable matroids serves as a key foundation for further investigations into the complex relationships and limitations within this evolving mathematical domain. The identification of the non-representability of certain fuzzy graphic matroids over various fields opens new avenues for future research and invites a reassessment of existing models.

Chapter 6

Duality in Fuzzy Graphic Matroids

6.1 Introduction

The existence of a theory of duality is a major strength of matroid theory. This theory extends the concept of planar duality in plane graphs. This chapter explores the duals of fuzzy graphic matroids, presenting a symmetrical approach to fuzzy independence within the framework of fuzzy graphs. Analogous to classical matroid theory, fuzzy dual matroids emphasize the complementary aspects of fuzzy independence.

We begin by presenting examples and results that highlight the structure and properties of the dual of a fuzzy graphic matroid. Next, we investigate whether the dual of a fuzzy graphic matroid retains its fuzzy graphic nature, providing counterexamples that demonstrate cases where it does not. Finally, we identify and examine specific conditions under which the dual of a fuzzy graphic matroid can also be classified as fuzzy graphic.

6.2 Dual of Fuzzy Graphic Matroids

Definition 6.2.1. [46] Let $\mathcal{M}$ be a fuzzy matroid and let $\mathfrak{B}$ be the family of fuzzy bases for $\mathcal{M}$. Let $\mathbf{1} : E \longrightarrow [0,1]$ be the fuzzy set defined by $\mathbf{1}(e) = \mathbf{1}$ for each $e \in E$, and if β is a fuzzy set on E, let $\beta^c = 1 - \beta$. Let $\mathfrak{B}^* = \{\beta^c \mid \beta \in \mathfrak{B}\}$. If $\mathcal{M}$ is closed and regular, then $\mathfrak{B}^*$ forms the family of fuzzy bases for a closed fuzzy matroid $\mathcal{M}^*$, is called the dual of $\mathcal{M}$.

If G is a fuzzy graph, the dual of fuzzy cycle matroid of G is denoted as $\mathcal{M}_F{}^*(G)$. This particular fuzzy matroid is named as the *fuzzy cocycle matroid* of G. Any fuzzy matroid showing isomorphism with the fuzzy cocycle matroid of a certain fuzzy graph is referred as a *fuzzy cographic matroid.* In this chapter, our study focuses on fuzzy cographic matroids, specifically aiming to determine the conditions under which this class of fuzzy matroids can be classified as fuzzy graphic.

Let G be a fuzzy graph with fuzzy edge weights. A *fuzzy edge cut* is a set of edges A whose removal results in a fuzzy separation of the graph into two or more components, and is denoted as $G \setminus A$.

With the definition 6.2.1 in mind, we illustrate the concept of the dual of a fuzzy graphic matroid using the following example.

Example 6.2.1. *The fuzzy graphic matroids in the second column are the fuzzy* *duals of those in the first column.*

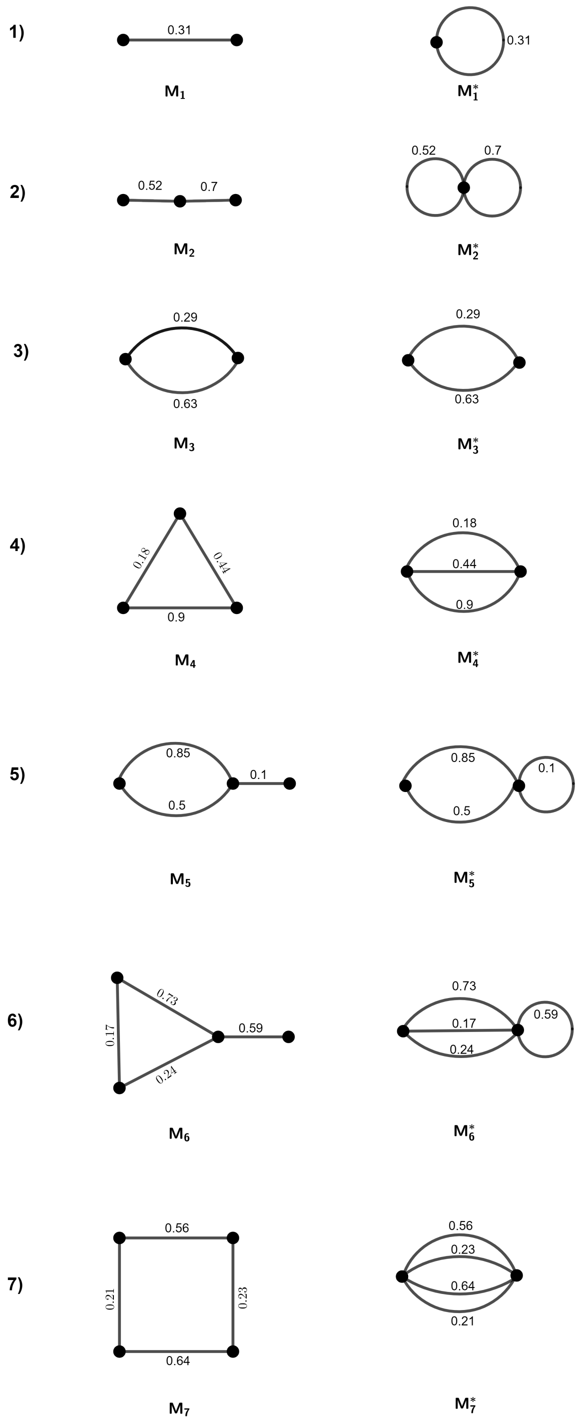


Table 6.1: Examples of fuzzy matroids with dual structures

Theorem 6.2.1. *If $\mathcal{M}$ is a regular fuzzy graphic matroid, then $\mathcal{M}^*$ is also regular.*

Proof. Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy graphic matroid with fundamental sequence $0 = r_0 < r_1 < \cdots < r_n \leq 1$. Then $\mathcal{M} \cong \mathcal{M}_F(G)$ for some fuzzy graph G. Let $r_i < r_j$ and B^* be a basis for the crisp matroid $M_{r_i^*} = (E,\ \mathcal{I}_{ri}^*)$. That is B^* contains the maximal edge set of the fuzzy graph G with no cycles.

$\Rightarrow$ $B = E \setminus B^*$ is a maximal edge set of G with no cycles in the matroid

$M_{r_i} = (E,\ \mathcal{I}_{ri})$.

$\Rightarrow$ B forms a basis for M_{r_i}

Since $\mathcal{M}$ is closed and regular, there is a basis A of the matroid $M_{r_j} = (E,\ \mathcal{I}_{rj})$ such that $A \subseteq B$.

$\Rightarrow$ $A^* = E \setminus A$ forms a basis for the matroid $M_{r_j^*} = (E,\ \mathcal{I}_{rj}^*)$, and it is straightforward that $A^* \subseteq B^*$. □

In Example 6.2.1, the fuzzy graphic matroid retains a characteristic wherein its dual is also a fuzzy graphic matroid. We now consider the problem of determining the precise conditions under which the dual of a fuzzy graphic matroid remains fuzzy graphic. The following example demonstrates that this is not always the case. Before proceeding, it is necessary to review the following definition and results.

Definition 6.2.2. [54] Let $\Psi = (V,\ \sigma,\ E)$ be a fuzzy multigraph, then a value $\mathrm{I}_{(x,y)}$ is assigned for an edge $(x,y)_{\mu_p}$, p is an integer, where

$$\mathrm{I}_{(x,y)} = \frac{(x,\ y)_{\mu_p}}{min\{\sigma(x),\ \sigma(y)\}}.$$

In fuzzy multigraph, when two edges intersect at a point, a value is assigned to that point in the following way. Let E contains two edges $((a,\ b),(a,\ b)_{\mu^k})$ and $((c,\ d),(c,\ d)_{\mu^l})$ which are intersected at a point P, where k and l are fixed integers. The intersecting value at the point P is:

$$\mathrm{I}_P = \frac{\mathrm{I}_{(a,b)} + \mathrm{I}_{(c,d)}}{2}.$$

Definition 6.2.3. [54] LetΨ be a fuzzy multigraph and for a certain geometrical representation $P_1,\ P_2,\ \cdots,\ P_z$ be the points of intersections between the edges. Then Ψ is said to be fuzzy planar graph with fuzzy planarity value f, where

$$f = \frac{1}{1 + \{\mathrm{I}_{P_1}, \mathrm{I}_{P_2}, \cdots \mathrm{I}_{P_z}\}}.$$

It is obvious that f is bounded and $0 < f \leq 1$.

Theorem 6.2.2. *[54] Let Ψ be a fuzzy planar graph with fuzzy planarity value greater than 0.5. The number of points of intersection between effective edges in Ψ is at most one.*

Theorem 6.2.3. *[54] Let Ψ be a fuzzy planar graph with fuzzy planarity value f. If $f \geq 0.67$, then no two effective edges of Ψ intersect.*

Definition 6.2.4. [54] A Fuzzy planar graph with fuzzy planarity value is more than or equal to 0.67 is called a *strong fuzzy planar graph.*

Example 6.2.2. *Consider the fuzzy graph and one of its geometrical representation given in Figure 6.1 and 6.2 respectively. Let $\mathcal{M}$ be the fuzzy graphic matroid induced from the following fuzzy graph and let $\mathcal{M}^*$ be the fuzzy dual matroid of $\mathcal{M}$. Assume that $\mathcal{M}^*$ is fuzzy graphic matroid, and hence there exist some fuzzy* *graph G such that* $\mathcal{M}^* \cong \mathcal{M}_F(G)$. *By Theorem 4.3.2, the fuzzy graph G must be connected.*

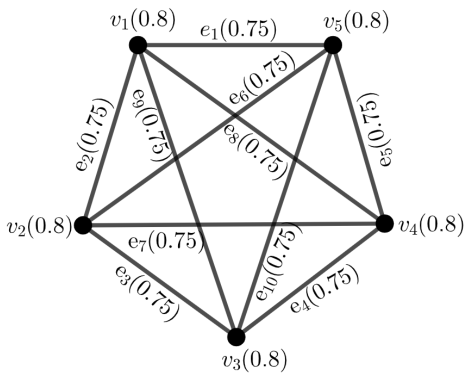


Figure 6.1: Non-fuzzy planar graph G

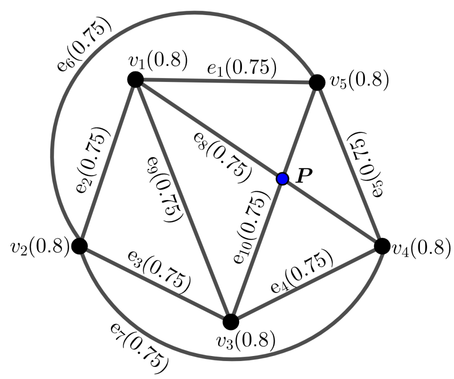


Figure 6.2: A geometrical representation of G

Here, for each e_i, $i = 1,\ 2,\ \cdots,\ 10$,

$$\mathrm{I}_{ei} = \frac{0.75}{0.8} = 0.9375.$$

Also, P *is the intersecting point of the fuzzy edges* $\mathrm{e}_8(0.75)$ *and* $\mathrm{e}_{10}(0.75)$. *Therefore,*

$$\mathrm{I}_P = \frac{0.9375 + 0.9375}{2} = 0.9375.$$

Thus, the fuzzy planarity value f for the fuzzy graph is,

$$f = \frac{1}{1 + 0.9375} = 0.516.$$

From these, we can conclude that all the fuzzy edges are effective, the intersection between two effective edges has occurred and it is not a strong fuzzy planar graph. Then, the geometrical representation of this fuzzy planar graph is similar to the crisp K_5 and by Proposition 2.1.2 such a fuzzy graph G does not exist.

Example 6.2.3. *Consider the fuzzy graph and one of its geometrical representation given in Figure 6.3. As demonstrated in the preceding example, we can verify the fuzzy dual matroid of $\mathcal{M}$ also not a fuzzy graphic matroid, where $\mathcal{M}$ is the fuzzy graphic matroid induced by the given fuzzy graph.*

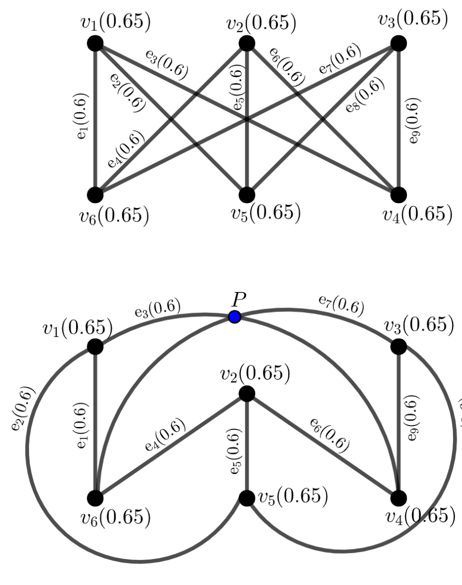


Figure 6.3: A fuzzy graph and its geometrical representation

Here, for each $e_i, \ i = 1, \ 2, \ \cdots, \ 9,$

$$\mathrm{I}_{ei} = 0.923,$$

$$\mathrm{I}_P = 0.923,$$

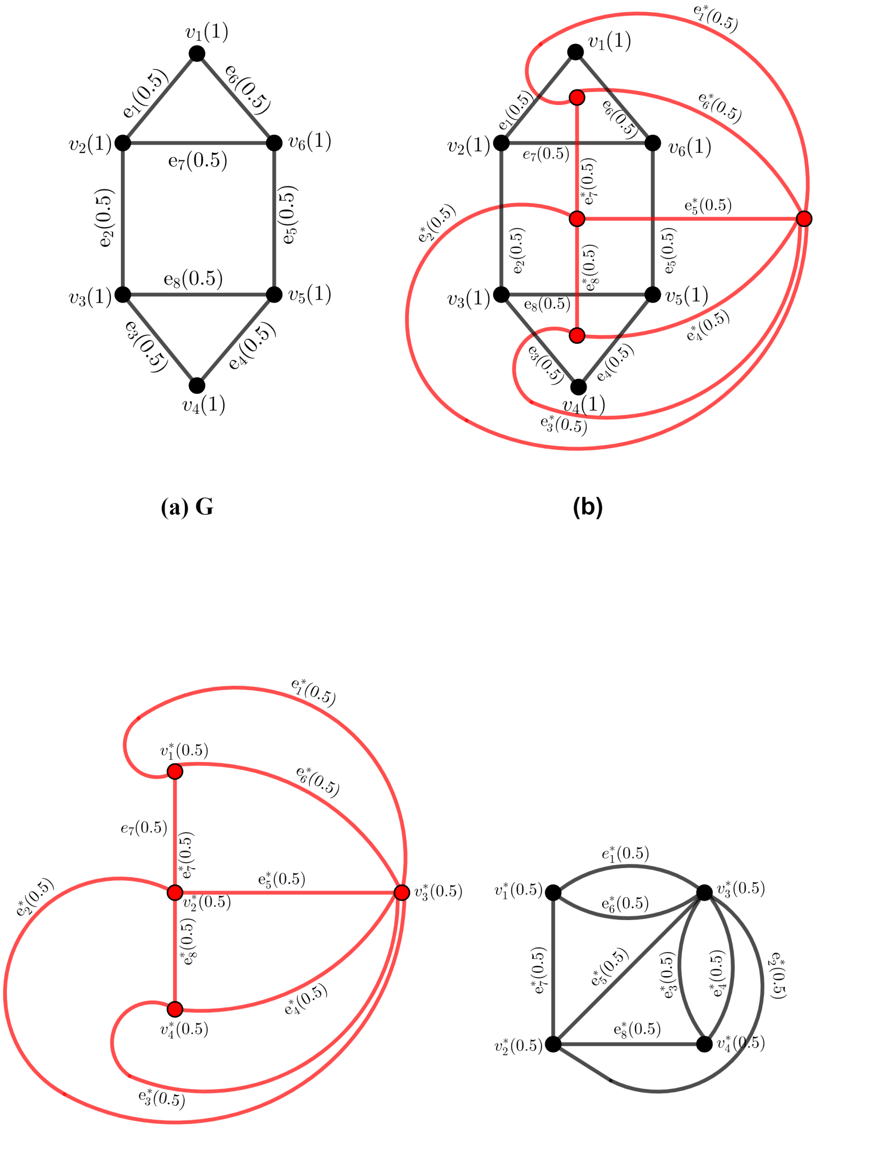
$$f = 0.52.$$

Now we explore fuzzy graphs whose fuzzy dual matroids are also fuzzy graphic. To obtain the desired result, for a given fuzzy graph G, we construct another fuzzy graph whose fuzzy cycle matroid is isomorphic to fuzzy dual matroid of G. We demonstrate the method of this construction using the following definition and example.

Definition 6.2.5. [5] Let $\Psi = (V, \sigma, E)$ be a fuzzy planar graph with planarity value 1 and $E = \{((x, y), (x, y)_{\mu_j}), j = 1, 2, \cdots P_{xy} \mid (x, y) \in V \times V\}$. Again, let $F_1, F_2, \cdots F_k$ be the fuzzy faces of Ψ. The fuzzy dual graph of Ψ is a fuzzy planar graph $\Psi' = (V', \sigma', E')$, where $V' = \{x_i, i = 1, 2, \cdots, k\}$, x_i is a vertex in Ψ' corresponding to the face F_i in Ψ; E' is the set of edges of Ψ' and $(x_i, x_j) \in E'$ if there is a common boundary for the faces F_i and F_j . The function σ' is given by: $\sigma' : V' \longrightarrow [0, 1]$ such that $\sigma'(x_i)$ is the maximum among all the membership values of the edges of the boundary of F_i. Between two faces F_i and F_j of Ψ, there may exist more than one common edge. Thus between two vertices x_i and x_j in fuzzy dual graph Ψ', there may be more than one edges. If $(a, b) \in E$ is a common edge between the fuzzy faces F_i and F_j , then the membership value of the fuzzy edge $(x_i, x_j) \in V'$ is equal to the membership value of the edge (a, b).

Example 6.2.4. *Consider the Figure 6.4 given below, part (a) shows the strong fuzzy plannar graph G and (c) shows its geometrical fuzzy dual G^*. In (b), G*

and G^* have been overlaid to illustrate the construction of G^* as per the above definition 6.2.5. The fuzzy graph in (d) is another geometrical representation of G^*.



(a) G **(b)**

(c) G^* (d) Geometrical embedding of G^*

Figure 6.4: Geometrical dual construction of fuzzy graph G

Lemma 6.2.1. *Let G^* be the geometric fuzzy dual of the strong fuzzy plannar graph G, then*

$$\mathcal{M}^*(G) \cong \mathcal{M}(G^*).$$

Proof. From definition 6.2.5, constructing the geometric fuzzy dual G^* from the original strong fuzzy planar graph G determines a bijection ψ from $E(G)$ to $E(G^*)$. We aim to prove that the fuzzy graphic matroid $\mathcal{M}(G)$ is isomorphic to $\mathcal{M}^*(G^*)$ under the mapping ψ.

Let β be a fuzzy basis in $\mathcal{M}(G)$. That is β is a maximal fuzzy independent set in $\mathcal{M}(G)$. From the construction of G^*, $\psi(\beta)$ constitutes a minimal fuzzy circuit in $\mathcal{M}(G^*)$. Then by definition 6.2.1, $1 - \psi(\beta)$ represents the complement of this fuzzy circuit and therefore is a maximal fuzzy independent set in the dual matroid $\mathcal{M}^*(G^*)$. That is, $1 - \psi(\beta)$ is a fuzzy basis for $\mathcal{M}^*(G^*)$.
Moreover, the cardinality is preserved:

$$\mid 1 - \psi(\beta) \mid = \mid 1 - (1 - \beta) \mid = \mid \beta \mid .$$

So the transformation preserves the structure of the basis in terms of size and fuzziness. Hence,

$$\mathcal{M}(G) \cong \mathcal{M}^*(G^*).$$

Now by theorem 2.3.8,

$$\mathcal{M}^*(G) = \mathcal{M}(G^*).$$

□

This lemma readily leads to the following result.

Theorem 6.2.4. *If G is a strong fuzzy planar graph, then $\mathcal{M}^*(G)$ is a fuzzy graphic matroid.*

Proof. From the Lemma 6.2.1, we have:

$$\mathcal{M}^*(G) = \mathcal{M}(G^*).$$

Since G^* is constructed as the geometric fuzzy dual of a planar graph G, G^* is also a fuzzy graph. Also, it retains the planarity and fuzziness in a structurally dual manner. Therefore, G^* is also fuzzy graphic. Hence, $\mathcal{M}^*(G)$ is a fuzzy graphic matroid. □

Example 6.2.5. *Let us consider the following fuzzy graph having a point of intersection P between two fuzzy edges $e_8(0.15)$ and $e_{10}(0.25)$.*

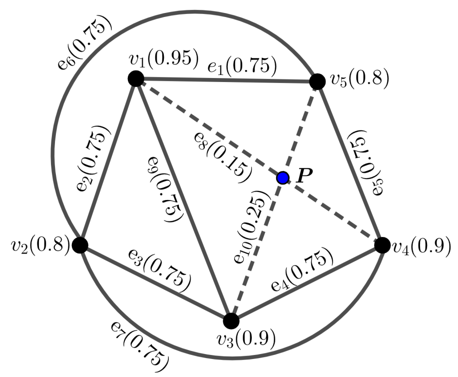
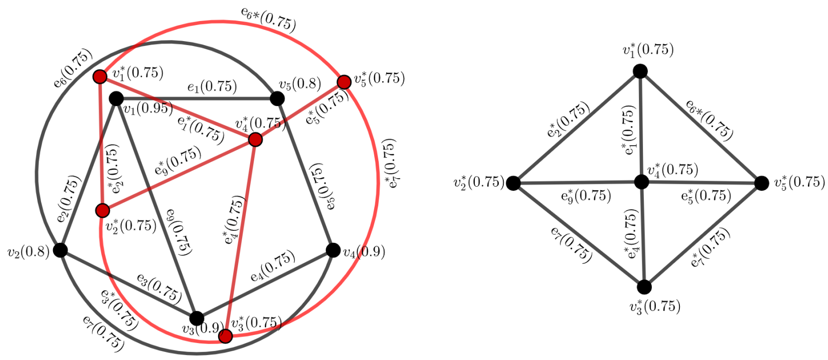


Figure 6.5: A strong fuzzy planar graph G with non-considerable edges

Here, $\mathrm{I}_{e_8} = 0.166$ *and* $\mathrm{I}_{e_{10}} = 0.312$*. Also, the fuzzy planarity of G is $f = 0.6765$.*

This implies that the intersection occurs between two non-considerable edges, and G is a strong fuzzy planar graph. Therefore, $\mathcal{M}^*(G) \cong \mathcal{M}(G^*)$ where G^* with its construction is presented in the following graph.



(b) (c) G^*

Figure 6.6: Construction of geometrical dual of G in Figure 6.5

In crisp matroid theory, the dual of a graphic matroid is always cographic, and this duality is well-structured and predictable. However, in the fuzzy setting, the dual of a fuzzy graphic matroid is not necessarily fuzzy graphic. The presence of fuzzy membership values introduces uncertainties in cycle and cocycle structures, breaking the symmetry observed in the crisp case. As a result, while duality still exists in fuzzy matroids, the preservation of the graphic nature under duality is not guaranteed, revealing a fundamental distinction between crisp and fuzzy graphic matroids.

6.3 Conclusion

In this chapter, the fuzzy dual of fuzzy graphic matroids is constructed. With the help of several examples, it is shown how the fuzzy dual is formed within the setting of fuzzy graphic matroids. These examples provide a clear understanding of the characteristics of the fuzzy dual of a fuzzy graphic matroid.

An important result in this chapter is that the fuzzy dual of a fuzzy graphic matroid does not necessarily retain the property of being fuzzy graphic. This is demonstrated through specific examples where the fuzzy dual of a fuzzy graphic matroid diverges from being fuzzy graphic, thereby highlighting the complexities involved in the dualization process within fuzzy matroid theory.

Further, the geometric dual of a fuzzy graph is constructed, and the fuzzy matroid corresponding to this geometric dual fuzzy graph is considered. This approach provides a new perspective and a method to systematically study the dualization process within fuzzy graphic matroids.

A key result is the proof that for strong fuzzy planar graphs, the fuzzy dual of a fuzzy graphic matroid is isomorphic to the fuzzy matroid obtained from the geometric dual graph. This finding not only establishes a clear relationship between the fuzzy dual and the geometric dual but also confirms that the dualization process preserves the graphic nature under certain conditions.

Chapter 7

Connectivity in Fuzzy Graphic Matroids

7.1 Introduction

The study of connectivity within the context of fuzzy matroids represents a significant endeavor, given the fundamental role that connectivity plays in the practical applications of these mathematical structures. As a core concept, connectivity enhances the theoretical understanding of fuzzy matroids and contributes valuable insights for real-world applications. Building upon the previous chapters on fuzzy graphic matroids, this chapter extends the exploration by examining connectivity from the perspective of this specific class of fuzzy matroids.

To define connectivity in fuzzy graphic matroids, we introduce an equivalence relation that provides a structured framework for analyzing connectivity. By using equivalence classes, we can explore the internal relationships within fuzzy graphic matroids. We also discussed the necessary and sufficient conditions for a fuzzy graphic matroid to be connected. Furthermore, fuzzy analogs of certain results from classical matroid theory were reviewed, with counterexamples presented where these results fail in the fuzzy case, and conditions examined under which they remain valid. Several illustrative examples were provided to clarify these concepts.

7.2 Connectivity in Fuzzy Graphic Matroids

In this section, we discuss the connectedness of fuzzy graphic matroids. We begin by reviewing some definitions and results related to the connectedness of fuzzy graphs.

Definition 7.2.1. [56] Let $G = (V,\ \sigma)$ be a fuzzy graph, let $x,\ y$ be two distinct vertices and let $\overline{G}$ be the *partial fuzzy subgraph* of G obtained by deleting the edge xy. That is, $\overline{G} = (V,\ \sigma^{'})$, where $\mu^{'}(xy) = 0$ and $\mu^{'} = \mu$ for all other pairs. We call xy a fuzzy bridge in G if $\mu^{'\infty}(u,\ v) < \mu^{\infty}(u,\ v)$ for some $u,\ v$ in $\sigma^{'}$.

Definition 7.2.2. [56] Let w be any vertex and w is a *fuzzy cutvertex* if deleting the vertex w reduces the strength of connectedness between some other pair of vertices. Hence, w is a fuzzy cutvertex if and only if there exists $u,\ v$ distinct from w such that w is on every strongest path from u to v.

Definition 7.2.3. [56] A partial fuzzy subgraph $\overline{G}$ of G is called *nonseparable* or a *block* if it has no fuzzy cutvertices. Sometimes we refer to a block in a fuzzy graph as a *fuzzy block*.

Definition 7.2.4. [56] A maximum spanning tree of a connected fuzzy graph $(\sigma,\ \mu)$ is a *fuzzy spanning subgraph* $T = (\sigma,\ \nu)$ of G, which is a tree, such that $\mu^{\infty}(u,\ v)$ is the strength of the unique strongest $u-v$ path in T for all $u,\ v \in G$.

Theorem 7.2.1. *[66] Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy matroid, $e_1,\ e_2,\ e_3 \in \mathcal{F}(E)$ and $\mu,\ \nu \in \mathfrak{C}(\mathcal{M})$. If $e_1 \vee e_2 \leq \mu, e_2 \vee e_3 \leq \nu$ and $\mathcal{C}(\mu) \cap \mathcal{C}(\nu) \neq \emptyset$, then there is a fuzzy circuit ω such that $e_1 \vee e_3 \leq \omega$.*

We now define connectedness in fuzzy graphic matroids and present some useful properties.

Let $\mathcal{M}$ be a fuzzy graphic matroid. Define a relation $\sim$ in such a way that, for any two fuzzy edges $e_a,\ e_b \in \mu$, $e_a \sim e_b$ if and only if either there is a fuzzy circuit ω in $\mathfrak{C}(\mathcal{M})$ containing both e_a and e_b or $e_a = e_b$.

Proposition 7.2.1. *The relation $\sim$ is an equivalence relation on the family of fuzzy edge sets of E.*

Proof. Clearly $\sim$ is reflexive and symmetric. To show that $\sim$ is transitive, suppose that $e_a,\ e_b,\ e_c \in \mathcal{F}(E)$ and $\omega_1,\ \omega_2 \in \mathfrak{C}(\mathcal{M})$ such that $e_a \vee e_b \leq \omega_1$ and $e_b \vee e_c \leq \omega_2$. Clearly, $supp(\omega_1) \cap supp(\omega_2) \neq \emptyset$. By the construction of fuzzy graphic matroids, corresponding to the fuzzy circuits ω_1 and ω_2, there exist circuits C_1 and C_2 respectively in the crisp matroid $M = (E,\ \mathcal{I}_0)$ such that $C_1 \cap C_2 \neq \emptyset$. Assume that, $C_1 \cup C_2$ is the minimal circuit in the collection of all such pairs of circuits.

Claim: There is a fuzzy circuit ω such that $e_a \vee e_c \leq \omega$.

Assume that there is no such a fuzzy circuit. Then obviously $\omega_1 \neq \omega_2$, that implies $C_1 \neq C_2$. Choose an element $\alpha \in \omega_1 \cap \omega_2$. In the following Figure, given a Venn diagram useful in keeping track of the rest of the proof.

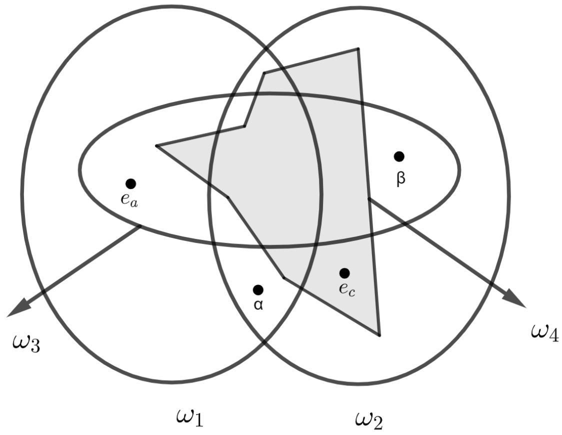


Figure 7.1: Venn diagram to understand the proof

By Theorem 2.3.10, $\mathcal{M}$ has a fuzzy circuit, let it be ω_3, such that $e_a \in \omega_3 \leq (\omega_1 \wedge \omega_2) \setminus \setminus \alpha$. It is clear that, by assumtion, $e_c \notin \omega_3$. As $\omega_3 \nleq \omega_1$, there exists an element β of $\omega_2 \setminus \omega_1$ that is in ω_3. Applying Theorem 2.3.10 to ω_2 and ω_3, we find that $\mathcal{M}$ has a fuzzy circuit ω_4 such that $e_c \in \omega_4 \leq (\omega_2 \vee \omega_3) \setminus \setminus \beta$. Since $\omega_4 \not< \omega_2$, the fuzzy set $\omega_4 \wedge (\omega_3 - \omega_2) \neq \emptyset$. Therefore $\omega_4 \wedge \omega_1 \neq \emptyset$. But $\omega_1 \vee \omega_4 \leq (\omega_1 \vee \omega_2) \setminus \setminus \beta$ and so $| C_1 \bigcup C_3 | < | C_1 \bigcup C_2 |$, where C_3 is the circuit in the matroid M corresponding to the fuzzy circuit ω_4. Therefore the pair $(C_1,\ C_3)$ contradicts the choice of $(C_1,\ C_2)$ because $e_a \in \omega_1,\ e_c \in \omega_4$ and $\omega_1 \wedge \omega_4 \neq \emptyset$. □

If there is only one equivalence class determined by the equivalence relation $\sim$, then $\mathcal{M}$ is called a *connected fuzzy graphic matroid*. Suppose that $\nu \in \mathcal{F}(E)$, if for every pair of fuzzy points $e_a,\ e_b \leq \nu$, there exists $\omega \in \mathfrak{C}(\mathcal{M})$ such that $e_a,\ e_b \leq \omega \leq \nu$, then ν is said to be connected in $\mathcal{M}$. The following result can be obtained as a consequence of the proof of Proposition 7.2.1.

Proposition 7.2.2. *Let $\mathcal{M} = (E, \ \mathcal{I})$ be a fuzzy graphic matroid, then $\mathcal{M}$ is connected if and only if for every pair of distinct fuzzy edges e_1 and e_2 of $\mathcal{M}$, there is a fuzzy circuit $\omega \in \mathfrak{C}$ containing both e_1 and e_2.*

Proof. Suppose $\mathcal{M}$ is connected. Let e_1 and e_2 be two distinct fuzzy edges. Suppose, for contradiction, that there is no fuzzy circuit $\omega \in \mathfrak{C}$.

Then, in the fuzzy graph G from which $\mathcal{M} \cong \mathcal{M}_F(G)$ is derived, e_1 and e_2 are not simultaneously contained in any fuzzy cycle. This implies that the edge set E can be partitioned into two disjoint non-empty fuzzy subsets E_1 and E_2, such that all fuzzy circuits are contained entirely within E_1 or E_2, but not across. But this contradicts the connectedness of $\mathcal{M}$, since $\mathcal{M}$ would then be the direct sum of two fuzzy matroids on E_1 and E_2, respectively, violating the definition of a connected fuzzy matroid.

Therefore, for every pair $e_1, \ e_2 \in E$, there must exist a fuzzy circuit $\omega \in \mathfrak{C}$ such that $e_1, \ e_2 \in \omega$.

Now, suppose for every pair $e_1, \ e_2 \in E$, there must exist a fuzzy circuit $\omega \in \mathfrak{C}$ such that $e_1, \ e_2 \in \omega$.

Suppose, to the contrary, that $\mathcal{M}$ is not connected. Then $\mathcal{M}$ can be expressed as a direct sum of two non-empty fuzzy matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are fuzzy matroids on disjoint ground sets E_1 and E_2 such that $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$, and all fuzzy circuits are either in E_1 or in E_2, but not spanning both.

Let $e_1 \in E_1$ and $e_2 \in E_2$. By the assumption, there must be a fuzzy circuit $\omega \in \mathfrak{C}$ such that $e_1, \ e_2 \in \omega$. But this is a contradiction, since ω must be either entirely within E_1 or within E_2, as per the decomposition.

Therefore, no such decomposition exists, and $\mathcal{M}$ must be connected. $\square$

The following result characterizes a connected fuzzy graphic matroid in terms of connected crisp cycle matroids.

Proposition 7.2.3. *Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy graphic matroid, then $\mathcal{M}$ is connected if and only if the underlying cyclic matroid $M = (E,\ \mathcal{I}_0)$ is connected.*

Proof. Suppose that $\mathcal{M}$ is connected. Let $e_a,\ e_b \in E$ with $e_a \neq e_b$. Since $\mathcal{M}$ is connected, there exists a fuzzy circuit $\mu \in \mathfrak{C}(\mathcal{M})$ satisfying $e_a \vee e_b \leq \mu$. By Theorem 2.3.11, $\mu_{[m(\mu)]}$ is a circuit in the cyclic matroid $M_{m(\mu)} = (E,\ \mathcal{I}_{m(\mu)})$. By definition of fuzzy graphic matroids, $\mu_{m(\mu)}$ is a circuit in the underlying cyclic matroid $M = (E,\ \mathcal{I}_0)$. Therefore M is connected.

Conversely suppose $M = (E,\ \mathcal{I}_0)$ is connected cyclic matroid. Let $e_a,\ e_b \in \mathcal{F}(E)$. Since M is connected, there exists a circuit C in M containing the edges corresponding to the fuzzy edges e_a and e_b. Let $\omega = \chi_C$, then $\omega \in \mathfrak{C}(\mathcal{M})$ and $e_a,\ e_b \leq \omega$, so $\mathcal{M}$ is connected. $\square$

Theorem 7.2.2. *Let $\mathcal{M} = (E,\ \mathcal{I})$ be a fuzzy cyclic matroid induced by the fuzzy graph G. Then $\mathcal{M}$ is connected if and only if G is 2-connected.*

Proof. Let $M = (E,\ \mathcal{I})$ be the crisp matroid induced by G. Now, consider the $\mathcal{M}$ induced 0 level crisp matroid $M_0 = (E, I_0)$, where, $\mathcal{I}_0 = \{C_0(\mu) \mid \mu \in \mathcal{I}\}$ and $C_0(\mu) = \{e \in E \mid \mu(e) \geq 0\}$. From the construction of M_0, it is clear that the elements of $\mathcal{I}_0$ are exact the elements of $\mathcal{I}$.
That is, $M_0 = M$.
From Proposition 7.2.3 and Proposition 2.1.4, we get $\mathcal{M}$ is connected if and only if M_0 is connected if and only if the underlying graph G is 2-connected. $\square$

Example 7.2.1. *Let G be the graph shown in the following figure. Then the corresponding cycle matroid is $M = (E,\ \mathcal{I})$, where $E = \{e_1,\ e_2,\ e_3,\ e_4\}$ and $\mathcal{I} = 2^{\{e_1,\, e_3,\, e_4\}} \cup 2^{\{e_2,\, e_3,\, e_4\}} - \{\{e_1,\ e_3,\ e_4\},\ \{e_2, e_3,\ e_4\}\}$. Let*

$$\mathcal{I}_a = \begin{cases} 2^E, & \text{if } 0 < a \ \leq \dfrac{1}{4} \\ 2^{\{e_1,e_3,e_4\}} \cup 2^{\{e_2,e_3,e_4\}}, & \text{if } \dfrac{1}{4} < a \ \leq \dfrac{3}{4} \\ \mathcal{I}, & \dfrac{3}{4} < a \ \leq 1 \end{cases}$$

and $\mathcal{I} = \{\mu \in [0,1]^E \ \mid \mu_{[a]} \in \mathcal{I}_a, 0 < a \ \leq 1\}$, then $\mathcal{M} = (E,\ \mathcal{I})$ is a fuzzy cyclic matroid.

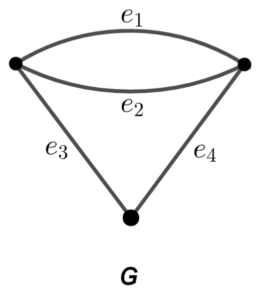
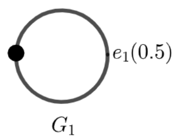


Figure 7.2: 2-connected graph G that induces $\mathcal{M}$

Note that G in the figure is 2-connected, so the corresponding cycle matroid M is connected. By Theorem 7.2.2, $\mathcal{M}$ is connected.

Example 7.2.2. *The fuzzy cycle matroids induced by the fuzzy graphs in Table 7.1 are examples of connected fuzzy cycle matroids with at most four elements. Moreover, the underlying crisp graph of any connected fuzzy cycle matroid with up to four edges is always isomorphic to the crisp graph of one of the fuzzy graphs listed in the table.*

<table>
<tr><th>Number of fuzzy edges</th><th>Fuzzy graph</th></tr>
<tr><td>0</td><td></td></tr>
<tr><td rowspan="2">1</td><td>
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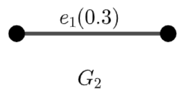
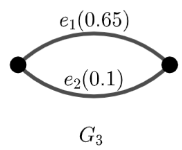
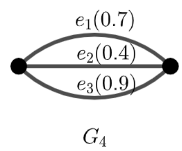
<tr><td>
</td></tr>
<tr><td>2</td><td>
</td></tr>
<tr><td rowspan="2">3</td><td>
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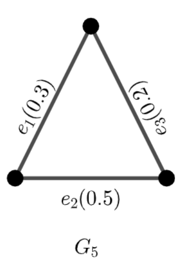
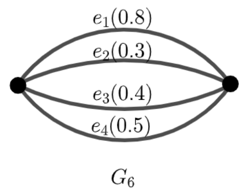
<tr><td>
</td></tr>
</table>

<table>
<tr><th>Number of fuzzy edges</th><th>Fuzzy graph</th></tr>
<tr><td rowspan="3">4</td><td>
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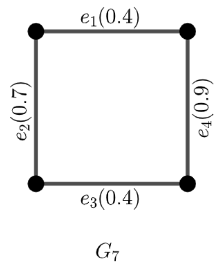
<tr><td>
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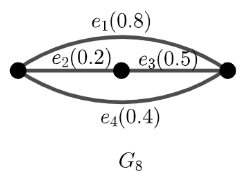
<tr><td>
</td></tr>
</table>

Table 7.1: Some connected fuzzy cyclic matroids on at most 4 fuzzy edges

Theorem 7.2.3. *Let $\mathcal{M}$ be a regular fuzzy cyclic matroid induced by a strong fuzzy planar graph G. Then $\mathcal{M}$ is connected if and only if $\mathcal{M}^*$ is connected.*

Proof. Since G is strong fuzzy planar graph and $\mathcal{M}$ is regular, by Theorem 6.2.4, $\mathcal{M}^*$ is also closed and regular. Hence, from Theorem 2.3.8 we have $\mathcal{M}^{**} = \mathcal{M}$, it is enough to prove the property for the dual matroid $\mathcal{M}^*$.

Let e_a and e_b be two fuzzy edges, as $\mathcal{M}$ is connected, there exist a fuzzy circuit ω such that $e_{1_a} \vee e_{2_b} \leq \omega$. This implies that the edges $e_1,\ e_2 \in supp(\omega)$ and $supp(\omega)$ is a circuit in the underlying crisp matroid M. By Proposition 2.1.5, there exist a cocircuit C^*, such that $C^* \cap supp(\omega) = \{e_1,\ e_2\}$. Clearly, the fuzzy edge set ξ with $supp(\xi) = C^*$ is a fuzzy circuit in $\mathcal{M}^*$ and $e_{1_a} \vee e_{2_b} \leq\ \xi$, the proof is thus concluded. $\square$

Example 7.2.3. *Consider the fuzzy graph G and its dual graph G^* given below and let $\mathcal{M}$ and $\mathcal{M}^*$ be the fuzzy cyclic matroids induced by G and G^* respectively. Clearly $\mathcal{M}^*$ is the fuzzy dual matroid of $\mathcal{M}$. Since G and G^* are connected fuzzy graphs, $\mathcal{M}$ and $\mathcal{M}^*$ are connected fuzzy cyclic matroids.*

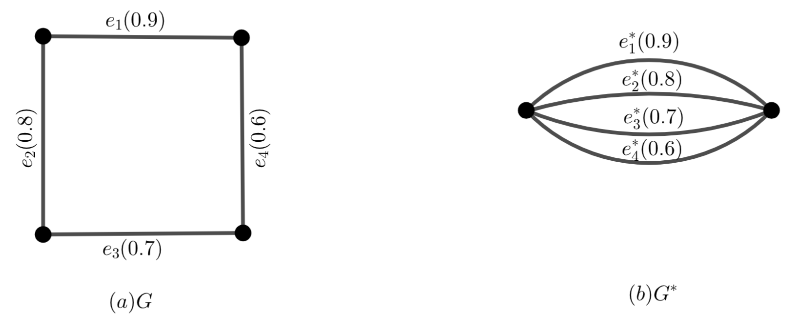


Figure 7.3: A connected fuzzy cyclic matroid with its connected fuzzy dual matroid.

The following result related to connectivity is well known in the context of crisp matroid theory; however, Example 7.2.4 demonstrates that it does not hold in the case of fuzzy cycle matroids.

Proposition 7.2.4. *[9] Let $M = (E,\ \mathcal{I})$ be a matroid and if X and Y are connected subsets of E and $X \cap Y \neq \emptyset$ then $X \cup Y$ is a connected set.*

Example 7.2.4. *Consider the cycle matroids $M_1 = (E,\ \mathcal{I}_1)$, $M_2 = (E,\ \mathcal{I}_2)$ and $M_3 = (E,\ \mathcal{I}_3)$ corresponding to the graphs $G_1,\ G_2$ and G_3 respectively, where $E = \{e_1,\ e_2,\ e_3,\ e_4,\ e_5\},\ I_1 = 2^E,\ I_2 = 2^{\{e_1,\, e_2,\, e_3,\, e_5\}} \bigcup 2^{\{e_1,\, e_2,\, e_4,\, e_5\}} \bigcup 2^{\{e_1,\, e_3,\, e_4,\, e_5\}}$, and $I_3 = I_2 - \{\{e_1,\ e_2,\ e_5\},\ \{e_1,\ e_3,\ e_4,\ e_5\}\}$.*

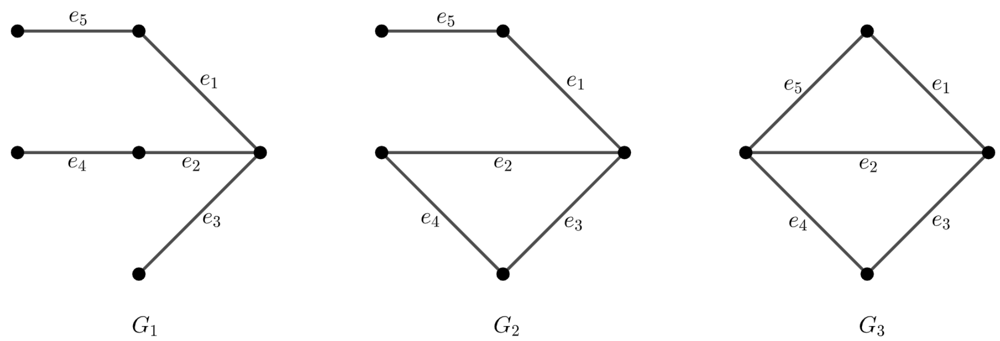


Figure 7.4: The graphs $G_1,\ G_2$ and G_3

.

Let

$$I_r = \begin{cases} I_1, & 0 < r \leq 0.3 \\ I_2, & 0.3 < r \leq 0.7 \\ I_3, & 0.7 < r \leq 1 \end{cases}$$

and $\mathfrak{I} = \{\mu \in [0,1]^E \mid C_r(\mu) \in I_r\},\ 0 < r \leq 1$. Then $\mathcal{M} = (E,\ \mathfrak{I})$ is a fuzzy cyclic matroid.

Let $\mu = \chi_{\{e_1, e_2, e_5\}}$ *and* $\nu = \chi_{\{e_2, e_3, e_4\}} \wedge [0.4]$. *Here* $\mu \vee \nu$ *is not connected even though* μ *and* ν *are connected and* $\mu \wedge \nu \neq [0]$.

Theorem 7.2.4. *Let* $\mathcal{M} = (E,\ \mathcal{I})$ *be a fuzzy graphic matroid induced from a fuzzy graph* G *with fundamental sequence* $r_0 < r_1 < \cdots r_n$. *Consider two connected fuzzy sets* μ *and* ν *of* $\mathcal{M}$. *If there exists a fuzzy edge* e *of* G *such that* $m(\mu) = \mu(e)$ *and* $m(\nu) = \nu(e)$ *and for some* $j \in \{0, 1, \cdots, n-1\}$, $\mu(e),\ \nu(e) \in (r_j,\ r_{j+1})$, *then* $\mu \vee \nu$ *is connected in* G.

Proof. Let $e_{a_1},\ e_{a_2} \in \mu \vee \nu$. If $e_{a_1},\ e_{a_2} \leq \mu$ and $e_{a_1},\ e_{a_2} \leq \nu$, since both μ and ν are connected, the result is straightforward.

Suppose $e_{a_1} \leq \mu,\ e_{a_1} \nleq \nu$ and $e_{a_2} \leq \nu,\ e_{a_2} \nleq \mu$. Assume $\mu(e) \leq \nu(e)$. Since μ is connected, there exist a fuzzy circuit ω_1 which contains the fuzzy edges e_{a_1} and e with μ-fuzzy value such that $\omega_1 \leq \mu$. Similarly, we can find another fuzzy circuit ω_2 containing e_{a_2} and e with ν-fuzzy value with $\omega_2 \leq \nu$. Clearly, *supp* $\omega_1 \cap$ *supp* $\omega_2 \neq \emptyset$. Thus by Theorem 2.3.10, there exists a fuzzy circuit ω such that ω contains both e_{a_1} and e_{a_2}, and $\omega \leq \mu \vee \nu$.

The result is true for the case $\nu(e) \leq \mu(e)$ also, by applying the similar argument. □

7.3 Conclusion

In conclusion, our study of fuzzy graphic matroids has revealed a strong connection between the concept of connectivity in graph theory and matroid theory. Recognizing the central role of connectivity in both domains, we have extended this foundational idea into the broader framework of fuzzy matroids, offering deeper insight into their structure and potential applications.

By defining an equivalence relation on fuzzy graphic matroids, we developed a clear framework for understanding connectivity within these structures. One major outcome is that for a fuzzy graphic matroid to be connected, its underlying cycle matroid must also be connected. Additionally, we examined fuzzy analogs of several well-established results from classical matroid theory, supporting them with illustrative examples. The connectivity of the dual of a fuzzy graphic matroid was also analyzed through examples. This detailed study of connectivity in fuzzy graphic matroids not only strengthens the theoretical foundation but also provides valuable insights for researchers working in fuzzy combinatorics, network theory, and related fields.

Chapter 8

Applications of Fuzzy Graphic and Representable Matroids

8.1 Introduction

This chapter focuses on practical applications of fuzzy graphic and representable matroids, particularly in generating different shades of secondary colors through various combinations of primary colors. The goal is to demonstrate how these matroid structures offer a systematic approach to color mixing by selecting optimal combinations of primary colors.

Graphic matroids are used to model different paths or combinations of primary colors, ensuring that each mix is efficient and free from redundancy. Representable matroids, on the other hand, help define linear combinations of primary color shades within a vector space, guaranteeing the independence of each secondary shade generated. Together, these matroid concepts provide an effective framework for understanding and optimizing the process of color mixing in a structured way.

This chapter will explore these applications step-by-step, emphasizing their simplicity and their practical contributions to solving real-world problems in color mixing.

8.2 Applications of Fuzzy Graphic and Representable Matroids

Fuzzy matroids have demonstrated wide-ranging applications in network theory, combinatorial optimization, and decision support systems, particularly in structures characterized by uncertainty. In our daily lives, we frequently encounter various forms of vagueness and uncertainty that cannot be completely eliminated. This makes fuzzy structures especially valuable for effectively managing such imprecision. Graphical models, particularly fuzzy graphic and representable matroids, offer a direct way to visualize and handle multi-faceted uncertainty in data and information. By organizing complex relationships in a structured manner, they assist in solving real-world problems involving vagueness or incomplete information.

Among the various applications of fuzzy graphic and representable matroids, three stand out prominently: optimal route mapping, portfolio optimization, and color mixing. In optimal route mapping, fuzzy matroids help identify the most efficient pathways within networks where connections are uncertain or subject to fluctuating constraints. In portfolio optimization, fuzzy matroids manage assets with uncertain and varying returns, offering a framework to balance risks and rewards amid unstable market conditions. For color mixing, fuzzy matroids model the blending of primary colors by treating color intensities as fuzzy variables, enabling the generation of a richer and more diverse range of secondary hues. These applications demonstrate the flexibility and power of fuzzy matroids in addressing complex, real-world optimization challenges.

8.2.1 Optimal Route Mapping Problem

Fuzzy graphic matroids can be effectively applied to optimization problems such as determining the most efficient route for traveling between multiple locations, often aiming to minimize cost, distance, or time. For example, consider the classic problem of minimizing the total time and cost for a salesman who needs to visit a list of cities, ensuring that each city is visited exactly once to distribute products efficiently.

The algorithm to determine an optimal route, minimizing both time and cost, is outlined as follows:

Algorithm 1: To find optimal route

Step 1: Input the n number of cities $C_1,\ C_2,\ \cdots\cdots C_n$ which represent the nodes of the graph.

Step 2: Input the edges $e_i,\ 1 \leq i \leq \dfrac{n(n-1)}{2}$ between the cities.

Step 3: Calculate the membership grade function for each edge.

Step 4: Determine a collection $\mathfrak{I}$ of fuzzy edge sets so that any member of $\mathfrak{I}$ do not form a circuit.

Step 5: Determine $\mathfrak{B} \subseteq \mathfrak{I}$ such that $\mathfrak{B} = \{\beta_k = \{e_i\}_{i=1}^{n-1} \mid \beta_k \text{ is maximal}\}$.

Step 6: Compute the sum of membership grade function of edges in each $\beta_k \in \mathfrak{B}$

Step 7: Select β_k having minimum membership grade value.

Before demonstrating the algorithm with a specific example, we first review some relevant definitions that form the foundation for understanding the process.

Definition 8.2.1. [50] Let X be a fix set. A Pythagorean fuzzy set is an object having the form

$$\mathcal{P} = \{\langle x, (\mu_P(x),\ \nu_P(x))\rangle \mid x \in X\},$$

where the function $\mu_{\mathcal{P}} : X \longrightarrow [0,\ 1]$ defines the degree of membership and the function $\nu_{\mathcal{P}} : X \longrightarrow [0,\ 1]$ defines the degree of non-membership of the element $x \in X$ to $\mathcal{P}$, respectively, and for every $x \in X$, it holds that

$$(\mu_P(x))^2 + (\nu_P(x))^2 \leq 1.$$

For convenience, we call $\widehat{a} = (\mu,\ \nu)$ a Pythagorean Fuzzy Number(PFN), where

$$\mu \in [0,\ 1],\ \nu \in [0,\ 1],\ \mu^2 + \nu^2 \leq 1.$$

Definition 8.2.2. [53] Let $\widehat{a} = (\mu,\ \nu)$ be a Pythagorean fuzzy number, a score function S of a a PFN can be represented as follows:

$$S(\widehat{a}) = \frac{1}{2}(1 + \mu^2 - \nu^2),\ S(\widehat{a}) \in [0,\ 1]. \tag{8.1}$$

Proof

- **Input validity**

  The algorithm takes as input a finite collection of destinations $\{C_1,\ C_2,\ \cdots,\ C_n\}$, where each destination corresponds to a vertex in the fuzzy graph. The routes between these destinations are correspond to fuzzy edges, each represented by $e_i(\mu_i,\ \nu_i)$, where μ_i and ν_i are the fuzzy attributes associated with the edge e_i. Thus, the inputs to the algorithm are finite and well-defined.

- **Calculation of membership grade value**

  The membership grade value $r(e_i)$ for each edge is calculated using Equation (8.1). This value estimates the strength of the connection between two destinations by considering both their positive and negative fuzzy information.

- **Construction of independent set**

  This step always finds a maximal independent set using well-known graph-theoretic methods. In this context, the maximal independent set consists of fuzzy edges that do not form any cycles, visit each vertex exactly once, and contain no repeated edges. Therefore, identifying such maximal independent sets corresponds to finding spanning paths.

- **Calculation of total membership grade value**

  For each maximal independent spanning path, the algorithm calculates the total value by summing the membership grade values of the fuzzy edges included in the path.

- **Selection of minimal path**

  At the end, the algorithm selects the route with the minimum total value, ensuring the chosen path is optimal for visiting all destinations efficiently.

Time Complexity Analysis

To evaluate the efficiency of the proposed algorithm, we analyze its time complexity step by step. Each stage of the algorithm is examined based on the input size, and both the practical and theoretical scenarios are considered to provide a complete picture of its computational performance.

- **Step 1:** The algorithm reads the list of n cities, which are the nodes of the graph. This operation involves a single pass over the input list of cities and therefore takes linear time with respect to the number of nodes. Hence, the time complexity for this step is $O(n)$.

- **Step 2:** The algorithm takes as input the edges between the cities. Since the graph is complete, the total number of edges is $m = \dfrac{n(n-1)}{2}$, which is $O(n^2)$ in terms of input size. Thus, this step requires $O(n^2)$ time to process all edge data.

- **Step 3:** The membership grade function is computed for each edge. This involves evaluating a predefined fuzzy function for each of the m edges, which is again $O(n^2)$ operations. Therefore, the time complexity for this step is also $O(n^2)$.

- **Step 4:** In this step the identification of all fuzzy edge sets that do not form a circuit—that is, subsets of edges that are acyclic. Sorting the edges and checking cycles using the optimized approach takes $O(n^2 logn)$.

- **Step 5:** Generating all possible spanning trees in a complete graph is an exponential task, the number of spanning trees in a complete graph with n nodes is n^{n-2}, according to Cayley's formula. However, in practical implementations where only a single maximal independent set is generated, this step can be completed in $O(n^2 logn)$ time.

- **Step 6:** This step involves computing the total membership grade for each fuzzy spanning tree $\beta_k \in \mathfrak{B}$. For each such tree, the algorithm sums the membership values of $n - 1$ edges. If all possible spanning trees are considered, this step becomes $O(n^{n-1})$, as the sum operation is performed for an exponential number of trees. In contrast, if only one tree is used, the computation requires just $O(n)$ time.

- **Step 7:** The algorithm selects the tree with the minimum total membership grade value from the collection $\mathfrak{B}$. If all spanning trees are stored, this requires a linear scan over $| \mathfrak{B} |$ trees and hence takes $O(n^{n-2})$ time in the worst case. If only one tree is generated and evaluated, this step takes constant time, that is, $O(1)$.

Since the algorithm is implemented using a greedy approach that constructs only a single fuzzy spanning tree based on sorted membership grades, then the dominant steps are edge sorting and cycle checking, both of which contribute a time complexity of $O(n^2 logn)$. Therefore, the overall time complexity in the efficient case is $O(n^2 logn)$.

Here, we illustrate the application of a graphic fuzzy matroid to the traveling salesman problem. Consider a set of 5 cities $\{A,\ B,\ C,\ D,\ E\}$ where each city is directly connected to every other city. The fuzzy graphical representation of the route connections between the five cities is shown in Figure 8.1.

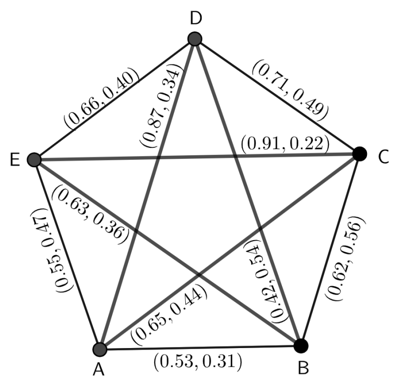


Figure 8.1: The fuzzy graphical representation of the route connection between the five cities.

The membership components of the fuzzy edges represent the estimated time and cost to travel between cities, along with the degree of uncertainty or the possibility of failing to meet those estimates due to certain constraints or disruptions. The objective is to determine a path that allows the salesman to visit all five cities exactly once, while ensuring that the overall time and cost are minimized. To achieve this, we construct the following table, which outlines the fuzzy information associated with each edge in Figure 8.1, along with the computed membership grade function for each edge.

Sl No	Routes	$e_i(\mu_i,\ \nu_i)$	$r(e_i) = \frac{1}{2}\{1 + [\mu_i^2 - \nu_i^2]\}$
1	$A \longleftrightarrow B$	(0.53, 0.31)	0.5924
2	$A \longleftrightarrow C$	(0.65, 0.44)	0.61445
3	$A \longleftrightarrow D$	(0.87, 0.34)	0.82065
4	$A \longleftrightarrow E$	(0.55, 0.47)	0.5408
5	$B \longleftrightarrow C$	(0.62, 0.56)	0.5354
6	$B \longleftrightarrow D$	(0.42, 0.54)	0.4424
7	$B \longleftrightarrow E$	(0.63, 0.36)	0.63365
8	$C \longleftrightarrow D$	(0.71, 0.49)	0.632
9	$C \longleftrightarrow E$	(0.91, 0.22)	0.88985
10	$D \longleftrightarrow E$	(0.66, 0.40)	0.6378

Table 8.1: Fuzzy membership grade values of edges of the fuzzy given in Figure 8.1

From Figure 8.1, we observe that the salesman must traverse at least four edges in such a way that no cycle is formed with the previously selected edges, and each city is visited exactly once. These 4-element fuzzy edge sets together constitute the collection of all maximal independent subsets of the graphic fuzzy matroid induced from the fuzzy graph in Figure 8.1. Let this collection be denoted by $\mathfrak{B}$. Next, we eliminate all elements of $\mathfrak{B}$ that do not correspond to spanning paths, yielding a refined collection $\mathfrak{B}'$, which contains only those sets that form valid spanning paths. Once all the elements of $\mathfrak{B}'$ are determined, we compute the total membership grade function for each $\beta_i \in \mathfrak{B}'$ by summing the membership values of the edges in β_i. The optimal route is then selected by identifying the β_i with the minimum total membership grade value.

The following table presents the membership value of each $\beta \in \mathfrak{B}'$, as generated by the C program provided in Appendix I.

Sl No	Spanning paths	Membership grade value
1	$A \to B \to C \to D \to E$	2.3976
2	$A \to B \to C \to E \to D$	2.65545
3	$A \to B \to D \to C \to E$	2.55665
4	$A \to B \to D \to E \to C$	2.56245
5	$A \to B \to E \to D \to C$	2.7479
6	$A \to B \to E \to D \to C$	2.49585
7	$A \to C \to B \to D \to E$	2.23005
8	$A \to C \to B \to E \to D$	2.4213
9	$A \to C \to D \to B \to E$	2.3225
10	$A \to C \to D \to E \to B$	2.5179
11	$A \to C \to E \to B \to D$	2.58035
12	$A \to C \to E \to D \to B$	2.5845
13	$A \to D \to B \to C \to E$	2.6883
14	$A \to D \to B \to E \to C$	2.78655
15	$A \to D \to C \to B \to E$	2.6217
16	$A \to D \to C \to E \to B$	2.97615
17	$A \to D \to E \to B \to C$	2.6275
18	$A \to D \to E \to C \to B$	2.8837
19	$A \to E \to B \to C \to D$	2.34185

Table 8.2: Fuzzy membership grade values of spanning paths (a)

Sl No	Spanning paths	Membership grade value
20	$A \to E \to B \to D \to C$	2.24885
21	$A \to E \to C \to B \to D$	2.40845
22	$A \to E \to C \to D \to B$	2.50505
23	$A \to E \to D \to B \to C$	2.1564
24	$A \to E \to D \to C \to B$	2.346
25	$B \to A \to C \to D \to E$	2.47665
26	$B \to A \to C \to E \to D$	2.7345
27	$B \to A \to D \to C \to E$	2.9349
28	$B \to A \to D \to E \to C$	2.9407
29	$B \to A \to E \to C \to D$	2.65505
30	$B \to A \to E \to D \to C$	2.403
31	$B \to C \to A \to D \to E$	2.6083
32	$A \to C \to D \to B \to E$	2.3225
33	$B \to C \to A \to E \to D$	2.32845
34	$B \to C \to D \to A \to E$	2.52885
35	$B \to C \to E \to A \to D$	2.7867
36	$B \to D \to A \to C \to E$	2.76735
37	$B \to D \to A \to E \to C$	2.6937
38	$B \to D \to C \to A \to E$	2.22965
39	$B \to D \to E \to A \to C$	2.23545
40	$B \to E \to A \to C \to D$	2.4209
41	$B \to E \to A \to D \to C$	2.6271
42	$B \to E \to C \to A \to D$	2.9586

Table 8.3: Fuzzy membership grade values of spanning paths (b)

Sl No	Spanning paths	Membership grade value
43	$B \to E \to D \to A \to C$	2.70655
44	$C \to A \to B \to D \to E$	2.28705
45	$C \to A \to B \to E \to D$	2.4783
46	$C \to A \to D \to B \to E$	2.51115
47	$C \to A \to E \to B \to D$	2.2313
48	$C \to B \to A \to D \to E$	2.58625
49	$C \to B \to A \to E \to D$	2.3064
50	$C \to B \to D \to A \to E$	2.33925
51	$C \to B \to E \to A \to D$	2.5305
52	$C \to D \to A \to B \to E$	2.6787
53	$C \to D \to B \to A \to E$	2.2076
54	$C \to E \to A \to B \to D$	2.46545
55	$C \to E \to B \to A \to D$	2.93655
56	$D \to A \to B \to C \to E$	2.8383
57	$D \to A \to C \to B \to E$	2. 60415
58	$D \to B \to A \to C \to E$	2.5391
59	$D \to B \to C \to A \to E$	2.13305
60	$D \to C \to A \to B \to E$	2.4725

Table 8.4: Fuzzy membership grade values of spanning paths (c)

From the table, we observe that the minimum total membership value is 2.13305, and the corresponding optimal route with minimum time and cost is illustrated in Figure 8.2. Therefore, under the given constraints, the most convenient path for the salesman to visit all the cities exactly once is either: $D \to B \to C \to A \to E$ or $E \to A \to C \to B \to D$.

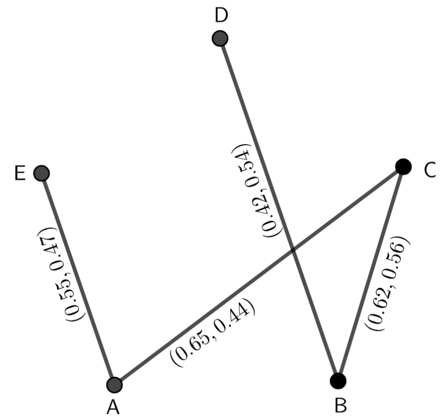


Figure 8.2: The minimal route

The C programming code to generate all the spanning paths of a given fuzzy graph G, which do not form a cycle with membership values, and also displays the minimal path is provided in Appendix I.

8.2.2 Portfolio Optimization Problem

A portfolio optimization problem involves determining the optimal combination of financial assets, such as bonds, stocks, or other investments, with the goal of maximizing returns while minimizing risk, all within a set of predefined constraints. The objective is to construct a balanced portfolio that aligns with an investor's financial goals and risk tolerance. In this section, we explore how fuzzy matroids can be effectively applied to portfolio optimization, offering a structured approach to handle uncertainties and imprecise information commonly found in financial decision-making.

Algorithm 2: To solve portfolio optimization problem

Step 1: Input the n number of assets $A_1,\ A_2,\ \cdots\cdots, A_n$.

Step 2: Input the return and risk values of each asset $A_i,\ i = 1,\ 2,\ \cdots,\ n$.

Step 3: Set the maximum risk value that can be allowed as $risk_{max}$.

Step 4: Calculate the total return and total risk values for each subsets of the collection $\mathcal{A} = \{A_1,\ A_2,\ \cdots\cdots A_n\}$.

Step 5: For each subset $S \subseteq \mathcal{A}$, check the validity condition $risk(S) \leq risk_{max}$.

Step 6: If the constraint is satisfied, then mark the subset as valid and set the collection of all valid subsets as the collection of independent subsets.

Step 7: Among all independent sets, select the subset with highest return and lower risk as the optimal solution.

Proof

- **Input validity**

  The algorithm takes a finite set of assets $\{A_1,\ A_2,\ \cdots,\ A_n\}$, and systematically generates all 2^n possible subsets, ensuring that no valid combination is overlooked. The algorithm is guaranteed to terminate, as it completes its execution after processing all possible subsets.

- **Calculation of total risk**

  For each subset, the algorithm calculates total risk by summing the risk values of all the assets included in that subset. Specifically, for a subset $S = \{A_{i1},\ A_{i2},\ \cdots A_{im}\}$,

Total risk$(S) = \sum_{j=1}^{m} r_{ij}$, where r_{ij} is the risk value of A_{ij}.

- **Validation of risk constraint**

  The algorithm filters the generated subsets based on a risk constraint. A subset S is considered valid (or independent) if its total risk is less than or equal to the given risk threshold, denoted as $risk_{max}$. This step ensures that only those subsets whose total risk does not exceed $risk_{max}$ are retained for further evaluation, while all subsets violating the constraint are discarded.

- **Finding the optimal subset**

  Among the valid (independent) subsets, the algorithm identifies the one that maximizes the total return. This ensures that, out of all subsets satisfying the risk constraint, the algorithm selects the portfolio with the highest possible return, thereby guaranteeing an optimal investment strategy within the given risk limit.

Time Complexity Analysis

Let n be the total number of assets available for portfolio selection. The algorithm involves checking all possible subsets of these assets and selecting the one that maximizes return while keeping risk below a specified threshold.

- **Step 1:** The algorithm reads the list of n assets, which takes linear time with respect to the input size. Hence, the time complexity of this step is $O(n)$.

- **Step 2:** In this step the return and risk values for each of the n assets are input. Assuming that both return and risk are given as numerical values per asset, this step requires $O(n)$ time.

- **Step 3:** This step involves setting a maximum allowable risk value, denoted as $risk_{max}$. This is a constant-time operation and therefore takes $O(1)$ time.

- **Step 4:** In this step the algorithm becomes computationally intensive. It requires evaluating the total return and total risk for each subset of the collection $\mathcal{A} = \{A_1,\ A_2,\ \cdots\cdots A_n\}$. The number of all possible subsets of a set with n elements is 2^n. For each subset, computing the total return and total risk involves summing over at most n elements. Therefore, the time complexity for this step is $O(n.2^n)$.

- **Step 5:** The algorithm checks the validity of each subset by comparing the total risk with the predefined threshold $risk_{max}$. Since this is a single comparison per subset, the total time required is $O(2^n)$.

- **Step 6:** Here the subsets that satisfy the risk constraint are marked as valid and stored as independent subsets. Marking and storing each valid subset is at most $O(1)$ per subset, so this step also has a time complexity of $O(2^n)$.

- **Step 7:** The algorithm selects the optimal subset among all valid (independent) subsets by comparing return and risk values. Assuming that there are at most 2^n valid subsets, scanning through them to find the one with maximum return and minimal risk takes $O(2^n)$ time.

The dominant part of the algorithm is the enumeration of all possible subsets and the evaluation of their risk and return values, which occurs in Step 4. This gives an overall exponential time complexity of $O(n.2^n)$.

Example 8.2.1. *Consider the following collection $\mathcal{A}$ of 4 assets, each with fuzzy values for return and risk:*
$\mathcal{A} = \{(A_1,\ 0.8,\ 0.7),\ (A_2,\ 0.3,\ 0.2),\ (A_3,\ 0.3,\ 0.3),\ (A_4,\ 0.5,\ 0.4)\}$.
Here, each asset is represented as (Asset, Return, Risk). We aim to find a subset of $\mathcal{A}$ that maximizes the total return while ensuring that the total risk does not exceed a specified threshold, taken here as $risk_{max} = 0.6$.
To do this, we calculate the total return and total risk for each possible subset of assets. The valid subsets, i.e., those whose total risk is less than or equal to 0.6, form the collection of independent subsets:

$$\mathcal{I} = \{\emptyset,\ A_2,\ A_3,\ A_4,\ \{A_2,\ A_3\},\ \{A_2,\ A_4\}\}.$$

Among these, the subset $\{A_2,\ A_4\}$ yields the maximum total return while keeping the total risk within the threshold.
Thus,$\{A_2,\ A_4\}$ is the optimal portfolio under the given constraint.

The C program code used to generate the optimal solution for the portfolio optimization problem, maximizing return while minimizing risk—is provided in Appendix II.

8.2.3 Color Mixing Problem

While fuzzy matroids are mainly applied in optimization, decision-making, and machine learning, they also hold potential for applications in color mixing, depending on the nature of the problem being addressed.

Color mixing often involves blending various colors to achieve a specific hue, saturation, or brightness. Fuzzy matroids can be applied in such scenarios to effectively model the inherent uncertainty or imprecision present in the color blending process.

The following color wheel represents primary, secondary, and tertiary colors, where P stands for primary, S for secondary, and T for tertiary colors. We know that primary colors cannot be created by mixing any other colors. By mixing two primary colors together, a secondary color is formed. Furthermore, mixing one primary color with one secondary color produces a tertiary color.

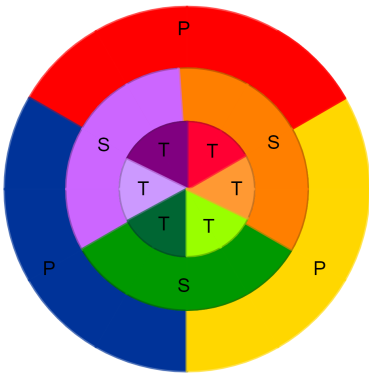


Figure 8.3: Color wheel

To create a secondary color, one can combine any two of the primary pigments, Red, Yellow, or Blue. The exact shade of the resulting secondary color depends on the proportions used in the mixture. For instance, mixing more Yellow than Blue produces a yellowish Green, while more Blue than Yellow results in a bluish Green. Similar variations occur when mixing primary and secondary colors to create tertiary colors, where the precise balance determines the final hue.

If we represent the pigmentation and the proportions of each color used as fuzzy sets, then color mixing can be modeled using fuzzy graphs as follows.

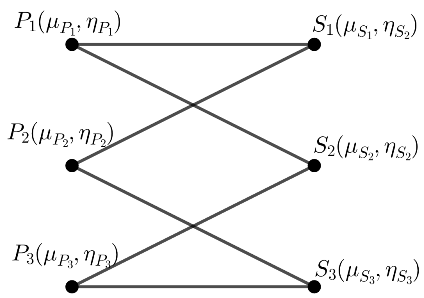


Figure 8.4: Fuzzy graphic representation of creation of secondary colors

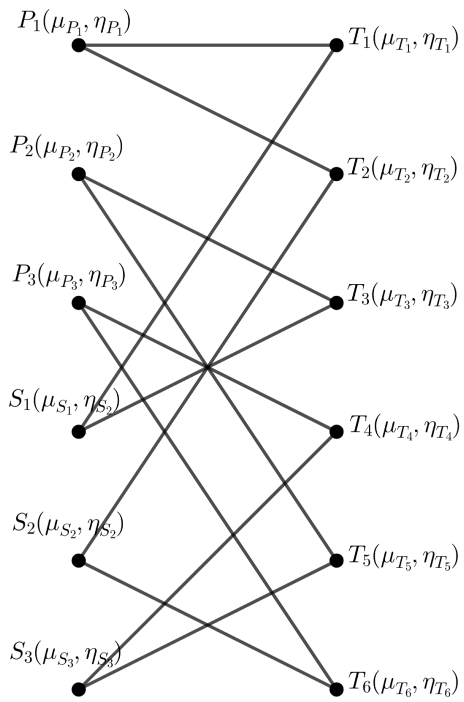


Figure 8.5: Fuzzy graphic representation of creation of tertiary colors

Here, $P_i,\ S_i,\ T_i$ represent primary, secondary and tertiary colors, respectively. The fuzzy values μ and η denote the pigmentation intensity of each color and the proportion of each color used in the mixing process.

In the fuzzy matroid structure for constructing secondary colors, the ground set consists of the primary colors, red, blue, and yellow, or various shades of these primary colors. When constructing tertiary colors, the ground set is extended to include both primary and secondary colors. In both cases, the colors do not have fixed intensities; instead, they are associated with fuzzy membership values. These membership values represent the degree to which each color or shade contributes to the final mixed color, capturing the inherent uncertainty and variability in color intensity.

An independent set in a fuzzy matroid represents a selection of colors that can be combined without violating certain constraints. These constraints might include limits on the maximum allowed intensity of the color mixture or restrictions on the overall proportions required to achieve a target shade. For example, to maintain a balanced blend, we might avoid having one color dominate the mix with an excessively high membership value. The independent sets thus ensure that the combination of colors remains within acceptable fuzzy thresholds, enabling controlled and harmonious color mixing.

The concept of a dual fuzzy matroid can be applied effectively to color decomposition. Given a mixed color, the dual fuzzy matroid helps identify the underlying primary color components along with their corresponding membership degrees. This approach is particularly useful for analyzing and understanding how a specific shade was created, by revealing the contributions and intensities of each primary color involved in the mixture.

8.3 Conclusion

In conclusion, this chapter has explored significant applications of fuzzy graphic and representable matroids in various optimization problems. We began with the optimal route mapping problem, demonstrating how the fuzzy framework effectively manages uncertainty in network connections. By leveraging fuzzy matroids, it becomes possible to identify efficient paths despite unknowns or fluctuations in data. The algorithms, supported by the C program and illustrative example provided, clearly showcased the practical application of these concepts to real-world scenarios. This approach not only enhances routing efficiency but also facilitates improved decision-making in the analysis and management of complex networks.

Next, we considered portfolio optimization, where fuzzy matroids play a crucial role in balancing risks and returns amid uncertain financial markets. Through the inclusion of algorithms, a C program, and a detailed example, we demonstrated how fuzzy structures can aid in crafting smarter investment strategies. These methods enable investors to make informed decisions, even when confronted with unpredictable market conditions. Finally, we explored the application of fuzzy matroids in color mixing, highlighting their versatility across diverse fields. Overall, the examples presented in this chapter showcase the effectiveness of fuzzy matroids in managing complex optimization problems, establishing them as valuable tools for both researchers and practitioners.

Chapter 9

Conclusion

The study of fuzzy matroids presented in this thesis introduces several critical advancements in both theoretical foundations and practical applications. Throughout the chapters, we have explored how fuzzy-induced structures extend and enrich the classical concepts of matroid theory, particularly focusing on their graphic and representable forms. This research delves into fundamental aspects such as bases, circuits, duality, connectivity, and optimization, providing novel insights into addressing uncertainty and fuzziness within the mathematical framework of matroids.

A major observation arises from comparing crisp graphic matroid theory with its fuzzy counterpart. While the collection of bases in crisp graphic matroids is uniquely determined, the bases of a graphic matroid induced from the threshold graph of a fuzzy graph vary depending on the chosen threshold values. This flexibility allows fuzzy frameworks to adapt to diverse conditions, but it also necessitates a thorough understanding of the effects of thresholding, particularly when analyzing circuits and connectivity. Developing algorithms to manage these structures underscores the critical need to carefully navigate such variability to ensure the effective application of fuzzy matroids.

The introduction of fuzzy graphic matroids as a new category, alongside the formalization of isomorphisms between fuzzy matroids, marks a significant advancement in the field. By bridging fuzzy matroid theory with fuzzy graph theory, this research paves the way for novel applications in areas where data uncertainty plays a critical role, such as optimization and network analysis. This work not only builds upon existing theoretical frameworks but also highlights important departures from crisp matroid cases, particularly concerning representability.

Regarding representability, a key distinction between fuzzy and crisp matroid theory emerges. Unlike crisp graphic matroids, not all fuzzy graphic matroids are representable over any field, prompting a critical reevaluation of established results in this domain. This divergence underscores both the opportunities and challenges in developing fuzzy representable matroids, requiring a thorough investigation into the strengths and constraints of fuzzy frameworks within a broader mathematical setting.

The study of fuzzy duals adds another dimension to this research, revealing that the dual of a fuzzy graphic matroid does not always preserve its fuzzy graphic nature. This distinction highlights the complexities involved in the dualization process within fuzzy matroid theory. However, for strong fuzzy planar graphs, the fuzzy dual of a graphic matroid remains isomorphic to the matroid derived from the geometric dual graph, establishing a significant connection between fuzzy duality and geometric duality under specific conditions.

Connectivity, a core concept in both graph and matroid theory, has been successfully extended into the fuzzy domain through this research. A key outcome is the establishment of conditions for connectivity in fuzzy graphic matroids, specifically that the underlying cycle matroid must be connected. By exploring fuzzy counterparts of classical connectivity results, this work strengthens the theoretical framework of fuzzy connectivity and underscores its significance in addressing complex network and combinatorial problems.

Finally, this thesis demonstrates practical applications of fuzzy graphic and representable matroids in solving real-world optimization problems. From optimal route mapping to portfolio optimization and color mixing, the versatility of fuzzy matroids is showcased across diverse fields. The algorithms and examples provided offer concrete approaches for handling uncertainty within various frameworks, reinforcing the significance of fuzzy matroids as powerful tools in both theoretical research and applied mathematics.

In conclusion, this thesis makes a significant contribution to the field of fuzzy matroid theory by expanding its scope through the introduction of new structures, comprehensive theoretical analysis, and practical applications. It lays a solid foundation for future research and establishes fuzzy matroids as a vital framework for addressing complexity and uncertainty in optimization and network theory.

Chapter 10

Recommendations

As this thesis presents significant developments in the study of fuzzy matroids, it also opens up numerous avenues for future research. The findings highlight key areas where deeper theoretical exploration and expanded practical applications can further advance the field. Below are some recommended directions for future research in fuzzy matroids:

- **Generalizing Representability in Fuzzy Matroids:**
  This thesis highlights the limitations of representability in fuzzy graphic matroids compared to their crisp counterparts. Future research could aim to develop a generalized theory of representability tailored specifically for fuzzy matroids, addressing the gaps and challenges identified in this study.

- **Extending the Study of Fuzzy Dual**
  The findings on fuzzy dual matroids in this thesis underscore the complexities involved in the dualization process, particularly the deviations from crisp graphic matroid behavior. Future research could focus on de-

veloping a comprehensive framework to characterize the conditions under which fuzzy duals preserve key properties such as planarity, connectivity, and representability. Additionally, extending the investigation of duals to other classes of fuzzy matroids beyond fuzzy graphic matroids may yield significant theoretical advancements and practical insights.

- **Application of Fuzzy Matroid in Multi-Criteria**

  Further research could explore the application of fuzzy matroids in multi-criteria optimization problems across diverse fields such as artificial intelligence, logistics, and supply chain management. Investigating how fuzzy matroid frameworks can effectively balance competing objectives and handle uncertainty in these complex environments may lead to more robust and adaptable optimization techniques.

- **Fuzzy Matroid in Topology and Geometry**

  The study of fuzzy graphic matroids and their duals reveals a profound connection between fuzzy matroid theory and geometric or topological structures. Future research could formally investigate this relationship, exploring the potential of fuzzy matroids to model topological or geometric properties in spaces characterized by uncertainty or imprecision. Such work could open new pathways for applying fuzzy matroids in fields like computational topology, spatial analysis, and uncertain data modeling.

- **Fuzzy Matroid in Machine Learning and Data Science**

  Given the extensive application of fuzzy systems in machine learning, there is significant potential to integrate fuzzy matroids into algorithm design for tasks such as feature selection, clustering, and classification. By leveraging the inherent ability of fuzzy matroids to handle uncertainty and vagueness, these algorithms could enhance decision-making processes in environments with imprecise or noisy data, leading to more robust and adaptive machine learning models.

Bibliography

[1] A. K. Katsaras and D. B. Liu, Fuzzy Vector Spaces and Fuzzy Topological Vector Spaces, Journal of Mathematical Analysis and Applications,Vol. 58, 1977, pages 135-146.

[2] A. Muneera and R. V. N. Srinivas Rao, Study on fuzzy graphs and applications, Bulletin of Mathematics and statistics Research Vol. 4, 2016, pages 19-22.

[3] A.Nagoor Gani and J.Malarvizhi, Isomorphism on Fuzzy Graphs , International Journal of Mathematical and Computational Sciences vol 2 No. 11, 2008, Pages 825-831.

[4] András Recski, Applications of matroids in electric network theory, Optimization Methods and Software, 2020, https://doi.org/10.1080/10556788.2020.1740220.

[5] A. Pal, S. Samanta and M.Pal, Concept of Fuzzy Planar Graphs, Proceedings of Science and Information Conference, , UK, October 7-9, 2013, pp. 557-563.

[6] Chun-E. Huang, Graphic and representable fuzzifying matroids, Proyecciones Journal of Mathematics Vol. 29, No. 1, 2010, pages 17-30.

[7] Chun-E. Huang and Fu-Gui Shi, Bases of [0, 1]- Matroids, Hacettepe Journal of Mathematics and Statistics 39(2), 2010, Pages 233-240.

[8] Daniel C. Slilaty, On cographic matroids and signed-graphic matroids, Discrete Mathematics 301, 2005, Pages 207 – 217.

[9] D. J. A Welsh, Matroid Theory, Academic Press, London, New York, 1976.

[10] Fu-Gui Shi, A new approach to the fuzzification of matroids, Fuzzy Sets and Systems 160, 2009, Pages 696-705.

[11] Fu-Gui Shi, (L, M)-fuzzy matroids, Fuzzy Sets and Systems 160, 2009, Pages 2387-2400.

[12] Fu-Gui Shi and Chun-E Huang, Fuzzy bases and the fuzzy dimension of fuzzy vector spaces, Mathematical Communications, Vol 15, No. 2, 2010, Pages 303-310.

[13] Fu-Gui Shi and Lan Wang, Characterizations and applications of M-fuzzifying matroids, Journal of Intelligent & Fuzzy Systems 25, 2013, Pages 919-930.

[14] Guanrong Chen and Trung Tat Pham, Introduction to Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems, CRC Press, Boca Raton London New York Washington, D.C, 2001.

[15] James G. Oxley, Matroid Theory, Oxford University Press, New York, Tokyo, 1992.

[16] James R. Evans,Graphic matroids and the multicommodity transportation problem, Mathematical Programming 13, 1977, Pages 323-328.

[17] J. Jon Arockiaraj and V. Chandrasekaran, Double Layered Complete Fuzzy Graph (DLCFG), Global Journal of Pure and Applied Mathematics Vol. 13, No. 9, 2017, Pages 6633-6646.

[18] John Mighton, A new characterization of graphic matroids, Journal of Combinatorial Theory, Series B 98, 2008, Pages 1253–1258.

[19] Kiran R. Bhutani and Azriel Rosenfeld, Strong arcs in fuzzy graphs, Information Sciences 152, 2003, Pages 319-322.

[20] K.R. Sandeep Narayan1 and M.S. Sunitha, Connectivity in a Fuzzy Graph and its Complement, Gen. Math. Notes, Vol. 9, No. 1, 2012, Pages 38-43.

[21] K.S. Abdukhalikov, The dual of a fuzzy subspace, Fuzzy Sets and Systems 82, 1996, Pages 375-381.

[22] K. Sameena and M. S Sunitha, Fuzzy Graphs in Fuzzy Neural Networks, Proyecciones Journal of Mathematics, Vol. 28, No. 3, 2009, 239 - 252.

[23] K. Sameena and M. S Sunitha, On g-distance in fuzzy trees, International Journal of Mathematics and Systems, Vol. 19, No. 4, 2011, 787 - 791.

[24] K. Shriram, R. Sujatha and S. Narasimman, Fuzzy combinatorial dual graph, International Journal of Pure and Applied Mathematics, Vol. 109, No. 7, 2016, Pages 35-41.

[25] Krasnoproshin. V and Obraztsov. V, Management optimization problems on fuzzy graphs, Investigaciones Europeas de Dirección y Economía de la Empresa vol 1 No 3, 1995, Pages 63-69.

[26] K. Truemper, Matroid Decomposition, Leibniz, U.S.A, 1998.

[27] L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.

[28] Ladislav A. Novak, On fuzzy independence set systems, Fuzzy Sets and Systems 91, 1997, Pages 365-374.

[29] Ladislav Novak, A comment on Bases of fuzzy matroids,Fuzzy Sets and Systems 87 , 1997, Pages 251-252.

[30] Ladislav Novak, On Goetschel and Voxman fuzzy matroids, Fuzzy Sets and Systems 117, 2001, Pages 407-412.

[31] L. Wang andF. G. Shi, Characterization of *L*-fuzzifying matroids by *L*-fuzzifying closure operators, Iranian Journal of Fuzzy Systems 7(1), 2010, Pages 47-58.

[32] L. X. Lu and W. W. Zheng, Categorical relations among matroids, Iranian Journal of Fuzzy Systems 7(1), 2010, Pages 81-89.

[33] M. Durgadevi, G. Bhavana Priya and S.Naga durga, Graph theory In Linear programming, Journal of Emerging Technologies and Innovative Research Volume 9, Issue 3, 2022, Pages 468-481.

[34] Mordeson. J. N and Nair. P. S, Fuzzy graphs and Fuzzy Hyper Graphs, Physica-Verlag, 2000.

[35] M. S. Sunitha and A. Vijayakumar, A characterization of fuzzy trees, Information Sciences 113, 1999, Pages 293-300.

[36] Muhammad Asif, Muhammad Akram and Ghous Ali, Pythagorean Fuzzy Matroids with Application, Symmetry 2020,https://doi.org/10.3390/sym12030423.

[37] Musavarah Sarwar and Muhammad Akram, New Applications of m-Polar Fuzzy Matroids, Symmetry 2017, https://doi.org/10.3390/sym9120319.

[38] Nuha abdul-jabbar, Jehan H. Naoom and Eman H. Ouda,Fuzzy Dual Graph, Journal of Al-Nahrain University vol 12(4), 2009, Pages 168-171.

[39] P. D. Seymour, Recognizing Graphic Matroids , Combinatorica, Vol. 1, 1981, Pages 75-78.

[40] P. Lubczonok, Fuzzy Vector Spaces, Fuzzy Sets and Systems 38, 1990, Pages 329-343.

[41] Qingyin Li and William Zhu, Covering Cycle Matroid, Hindawi Publishing Corporation 2013, http://dx.doi.org/10.1155/2013/539401.

[42] Roy Goetschel and William Voxman, Bases of Fuzzy Matroids, Fuzzy Sets and Systems 31, 1989, Pages 253-261.

[43] Roy Goetschel and William Voxman, Fuzzy Circuits, Fuzzy Sets and Systems 32, 1989, Pages 35-43.

[44] Roy Goetschel and William Voxman, Fuzzy Matroids, Fuzzy Sets and Systems 27, 1988, Pages 291-302.

[45] Roy Goetschel and William Voxman, Fuzzy Matroids and a Greedy Algorithm, Fuzzy Sets and Systems 37, 1990, Pages 201-213.

[46] Roy Goetschel and William Voxman, Fuzzy Matroid Structures, Fuzzy Sets and Systems 41, 1991, Pages 343-357.

[47] Roy Goetschel and William Voxman, Fuzzy matroid sums and a greedy algorithm, Fuzzy Sets and Systems 52, 1992, Pages 189-200.

[48] Roy Goetschel and William Voxman, Fuzzy Rank Functions, Fuzzy Sets and Systems 42, 1991, Pages 245-258.

[49] Roy Goetschel and William Voxman, Spanning properties for fuzzy matroids, Fuzzy Sets and Systems 51, 1992, Pages 313-321.

[50] R.R. Yager, Pythagorean membership grades in multicriteria decision making, IEEE Transactions on Fuzzy Systems 22(2014), 958–965.

[51] Shapla Shirin and Kamrunnahar,Application of Fuzzy Optimization Problem in Fuzzy Environment, Dhaka Univ. J. Sci. 62(2), 2014, Pages 119-125.

[52] Sheng-Gang Li, Xiu Xin and Yao-Long Li, Closure axioms for a class of fuzzy matroids and co-towers of matroids, Fuzzy Sets and Systems 158, 2007, Pages 1246-1257.

[53] Sheng-Jun Wu and Gui-Wu Wei, Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making, International Journal of Knowledge-based Intelligent Engineering Systems 21(2017), 189–201.

[54] Sovan Samanta and Madhumangal Pal, Fuzzy Planar Graphs, IEEE Transactions on Fuzzy Systems, 2013, DOI: 10.1109/TFUZZ.2014.2387875.

[55] S. R. Kingan, Almost-Graphic Matroids, Advances in Applied Mathematics 28, 2002, Pages 438-477.

[56] Sunil Mathew, John N. Mordeson and Davender S. Malik, Fuzzy Graph Theory, Springer International Publishing AG 2018.

[57] Takao Asano, A Note on Nongraphic Matroids, Journal of Combinatorial Theory, Series B 37, 1984, Pages 290-293.

[58] Talal Al-Hawary, Complete Fuzzy Graphs, International J.Math. Combin. Vol.4, 2011, Pages 26-34.

[59] Talal Al-Hawary, Fuzzy closure matroids, Matematika, Vol 32, 2016, Pages 69-74.

[60] Talal Al-Hawary, Fuzzy Greedoids, International Journal of Pure and Applied Mathematics, Vol 70 No. 3, 2011, Pages 285-295.

[61] Talal Ali AL-Hawary, On Fuzzy Matroids, International J.Math. Combin, Vol 1, 2012, Pages 13-21.

[62] T. Lakshmi, Excellent and Perfect Fuzzy Graphs, Taga Journal Vol. 14, 2018, Pages 2069-2073.

[63] Wei Yao and Fu-Gui Shi, Bases axioms and circuits axioms for fuzzifying matroids, Fuzzy Sets and Systems 161, 2010, Pages 3155-3165.

[64] Whitney. H, On the abstract properties of linear dependence, Amer. J. Math. 57, 1935, Pages 509-533.

[65] X. Li and H. Yi,Structural properties of fuzzy graphs, Iranian Journal of Fuzzy Systems Vol. 14, No. 1, 2017, Pages 131-144.

[66] Xiao-Nan Li, San-yang Liu and Sheng-gang Li,Connectedness of refined Goetschel–Voxman fuzzy matroids, Fuzzy Sets and Systems 161, 2010, Pages 2709-2723.

[67] Xiu Xin and Fu-Gui Shi, Categories of bi-fuzzy pre-matroids, Computers and Mathematics with Applications 59, 2010, Pages 1548-1558.

[68] Xiu Xin and Fu-Gui Shi, M-Fuzzifying Bases, Proyecciones Journal of Mathematics 28, 2009, Pages 271-283.

[69] X. Xin, F. G. Shi and S. G. Li, M-Fuzzifying derived operators and difference derived operators, Iranian Journal of Fuzzy Systems 7(2), 2010, Pages 71-81.

[70] Yao-Long Li,Guo-Jun Zhang and Ling-Xia Lu, Axioms for bases of closed regular fuzzy matroids, Fuzzy Sets and Systems 161, 2010, Pages 1711-1725.

[71] Yonghong Li, Sidong Xian and Dong Qiu, An Improved algorithm of Fuzzy Circuit for Closed Fuzzy Matroids, Eighth International Conference on Fuzzy Systems and Knowledge Discovery (FSKD), 2011.

[72] Yonghong Li, Yingjie Shi and Dong Qiu, The Research of the Closed Fuzzy Matroids, Journal of Mathematics and Informatics Vol 2, 2014, Pages 17-23.

[73] Yuang-Cheh Hsueh, On fuzzification of matroids, Fuzzy Sets and Systems 53, 1993, Pages 319-327.

APPENDIX I

C Program for Generating the Minimal Path

The following C program generates all the spanning paths of a given fuzzy graph G, which do not form a cycle with membership values, and also displays the minimal path.

```
#include <stdio.h>
#include <limits.h>
#include <math.h>
#define MAX 100
// Function to input the fuzzy graph (mu: membership values, nu: non-membership values)
// Function prototypes
void inputGraph(int n, float mu[MAX][MAX], float nu[MAX][MAX]);
// Function to calculate the final membership grades for edges
```

```
void calculateMembershipGrades(int n, float mu[MAX][MAX], float nu[MAX][MAX],
float membership[MAX][MAX]);
// Function to find all spanning paths using the calculated membership grades
void findAllSpanningPaths(int n, float membership[MAX][MAX]);
// Main function
int main() {
int n;
float membership[MAX][MAX]; // Membership grades
float mu[MAX][MAX], nu[MAX][MAX]; // Edge parameters (mu and nu)
printf("Enter the number of cities (nodes):");
scanf("%d", &n);
inputGraph(n, mu, nu);
calculateMembershipGrades(n, mu, nu, membership);
findAllSpanningPaths(n, membership);
return 0;
}
// Function to input the graph and node parameters
void inputGraph(int n, float mu[MAX][MAX], float nu[MAX][MAX]) {
printf("Enter the mu and nu values for each edge between cities:\n");
for (int i = 0; i < n; i++){
for (int j = i + 1; j < n;j++){
printf("Edge between city %d and city %d:\n", i + 1, j + 1);
printf("Mu value for edge (%d, %d): ", i + 1, j + 1);
scanf("%f", & mu[i][j]);
mu[j][i] = mu[i][j]; // Symmetric for undirected graph
```

```
printf("Nu value for edge (%d, %d): ", i + 1, j + 1);
scanf("%f", & nu[i][j]);
nu[j][i] = nu[i][j]; // Symmetric for undirected graph

                    }

            }

    }
// Function to calculate membership grades using the formula
void calculateMembershipGrades(int n, float mu[MAX][MAX], float nu[MAX][MAX],
float membership[MAX][MAX]) {
printf("Calculating membership grades... \n");
for (int i = 0; i < n; i++) {
for (int j = i + 1; j < n; j++) {
// Calculating the membership grade using the given formula
membership[i][j] = 0.5 * (1 + (pow(mu[i][j], 2) - pow(nu[i][j], 2)));
membership[j][i] = membership[i][j]; // Symmetric for undirected graph

                    }

            }
```

}

void findAllSpanningPathsUtil(int n, float membership[MAX][MAX], int path[ ],

int visited[ ], int pathIndex, float minTotal, int finalPath[ ]) {

if ($pathIndex == n$) {

float $totalMembership = 0$;

// Calculate total membership for the current path for ($int\ i = 0;\ i < n - 1;\ i+$

$+$){

$totalMembership + = membership[path[i]][path[i + 1]]$;

}

// Print the spanning path printf("Spanning Path: ");

for ($int\ i = 0;\ i < n;\ i + +$) {

printf("%d ", $path[i] + 1$);

if ($i\ ! = n - 1$){

printf("$- >$ ");

}

}

printf("with total membership grade: $\%.2f \setminus n$", $totalMembership$);

// Update the minimum total and store the path if it's better

if ($totalMembership < minTotal$){

$*minTotal = totalMembership$;

$for(int\ i = 0;\ i < n;\ i + +)${

$finalPath[i] = path[i];\ //Update\ the\ final\ minimal\ path$

}

}

return;

}

$for(int\ i = 0;\ i < n;\ i++)\{$

if (!visited[i]) { $//If\ the\ city\ is\ not\ visited$

$visited[i] = 1;\ //Mark\ city\ as\ visited$

$path[pathIndex] = i;\ //Add\ to\ the\ path$

findAllSpanningPathsUtil(n, membership, path, visited, $pathIndex + 1$, minTotal, finalPath);

$visited[i] = 0;$

}

}

}

void findAllSpanningPaths(int n, float membership[MAX][MAX]) {

int path[MAX];

int $visited[MAX] = \{0\}$; // Proper array initialization

float $minTotal = INT\_MAX;$

int finalPath[MAX];

for $(int\ start = 0;\ start < n;\ start + +)\{$

$visited[start] = 1;$

$path[0] = start;$

findAllSpanningPathsUtil(n, membership, path, visited, 1, & minTotal, final-Path);

$visited[start] = 0;$

}

// Output the best path found printf("$\backslash nMinimum\ total\ membership\ grade\ path:$ ");

for $(int\ i = 0;\ i < n;\ i + +)\{$

$printf(\text{“}\%d\text{“},\ finalPath[i] + 1);$

if $(i\ ! = n - 1)\{$

printf("$- >$ ");

}

}

printf(" with total membership grade: $\%.2f \backslash n$", minTotal);

}

APPENDIX II

C Program for Finding the Optimal Solution for a Portfolio Optimization Problem

The following C program generates the optimal solution for a portfolio optimization problem in terms of maximum return while minimizing the risk value.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
// Structure to represent an asset
typedef struct {
char name[10];
float return_val;
float risk_val;
} Asset;
// Pass total_return and total_risk as pointers to store computed values
```

```
// Function to calculate total return and risk for a given subset
void calculateSubset(Asset assets[ ], int subset[ ], int n, float
*total_return, float * total_risk) {
*total_return = 0.0;
*total_risk = 0.0;
for (int i = 0; i < n; i++){
if (subset[i] == 1){
*total_return += assets[i].return_val;
*total_risk += assets[i].risk_val;

                    }

            }

    }
// Function to print the assets in the subset
void printSubset(Asset assets[ ], int subset[ ], int n) {
printf("{ ");
for (int i = 0; i < n; i++){
if (subset[i] == 1){
printf("%s ", assets[i].name);

            }
```

}

printf("}"\ n);

}

// Function to find all subsets and determine independent and optimal sets based on the risk constraint

void findOptimalSubsets(Asset assets[ ], int n, float risk_max) {

int total_subsets = $pow(2, n)$;

float optimal_return = -1.0;

float optimal_risk = 100.0;

printf(" \ nIndependent Subsets (Valid based on Risk <= %.2f) : \n", risk_max);

for ($int\ i = 0;\ i <$ total_subsets; $i++$){

int subset[n];

float total_return, total_risk;

// Generate subset using bits of i for ($int\ j = 0;\ j < n;\ j++$){

$subset[j]$ = (i & (1 << j))? 1 : 0;

}

// Calculate total return and total risk for the current subset

calculateSubset(assets, subset, n, &total_return, &total_risk);

// Check if the subset is independent (valid based on the risk constraint)

if (total_risk <= risk_max){

printSubset(assets, subset, n);

printf("− > Total Return : %.2f, Total Risk : %.2f\n", total_return, total_risk);

```
// Check if this is the optimal subset
if (total_return > optimal_return || (total_return == optimal_return&&total_risk <
optimal_risk)){
optimal_return = total_return;
optimal_risk = total_risk;

                    }

            }

    }
// Print the optimal subset(s)
printf(" \ nOptimal Subset(s) (Maximizing Return with Risk <= %.2f) :
\n", risk_max);
for (int i = 0; i < total_subsets; i++){
int subset[n];
float total_return, total_risk;
for (int j = 0; j < n; j++) {
subset[j] = (i & (1 << j)) ?1 : 0;

    }
// Calculate total return and total risk for the current subset
calculateSubset(assets, subset, n, &total_return, &total_risk);
// Check if the subset is optimal
```

```
if (total_return == optimal_return && total_risk <= risk_max){
printSubset(assets, subset, n);
printf("-> Optimal Return : %.2f, Optimal Risk : %.2f\n", total_return, total_risk);

                    }

            }

    }
int main() {
int n;
float risk_max;
// Input number of assets
printf("Enter the number of assets: ");
scanf("%d", &n);
// Dynamically allocate memory for assets
Asset *assets = (Asset*)malloc(n * sizeof(Asset));
for (int i = 0; i < n; i++){
printf("\nEnter name of asset %d : ", i + 1);
scanf("%s", assets[i].name);
printf("Enter return value of asset %d : ", i + 1);
scanf("%f", &assets[i].return_val);
printf("Enter risk value of asset %d : ", i + 1);
scanf("%f", &assets[i].risk_val);
```

```
    }
// Input the maximum risk threshold
printf("\nEnter the maximum risk allowed: ");
scanf("%\f", &risk_max);
// Find the optimal subsets based only on the risk constraint
findOptimalSubsets(assets, n, risk_max);
free(assets);
return 0;

    }
```

APPENDIX III

List of publications

1. O. K. Shabna and K. Sameena, *Matroids from fuzzy graphs*, Malaya Journal of Matematik, Volume s, Number 1, 2019, Pages 500-504.

2. Shabna O. K. and Sameena K., *Graphic Fuzzy Matroids*, South East Asian J. of Mathematics and Mathematical Sciences, Volume 17, Number 1, 2021, Pages 223 - 232.

3. Shabna O. K. and Sameena K., *Fuzzy Matroids from Fuzzy Vector Spaces*, South East Asian J. of Mathematics and Mathematical Sciences, Volume 17, Number 3, 2021, Pages 381 - 390.

4. Shabna O. K. and Sameena K., *Connectivity Analysis and Applications of Graphic Fuzzy Matroids*, South East Asian J. of Mathematics and Mathematical Sciences, Volume 19, Number 3, 2023, Pages 333 - 346.

5. Shabna O. K. and Sameena K., *Fuzzy Graphic Matroids: Some Applications*, (communicated).

6. Shabna O. K. and Sameena K., *A Constructive Approach to Characterize Dual Fuzzy Matroids as Fuzzy Graphic*, (communicated).

7. Shabna O. K. and Sameena K., *A Comparative Study of Classic and Fuzzy-Induced Graphic Matroids*, (communicated).