

A STUDY ON THE DOMFORCING NUMBERS OF GRAPHS

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CERTIFICATE

I hereby certify that the thesis entitled "**A Study On The Domforcing Numbers of Graphs**" submitted by **Susanth P** to St. Joseph's College (Autonomous), Devagiri, Kozhikode-673008 for the award of the Degree of **Doctor of Philosophy** is a bona-fide record of the research work carried out by him under my supervision and guidance. The contents of this thesis, in full or parts, have not been submitted and will not be submitted to any other Institute or University for the award of any degree or diploma.



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DECLARATION

I hereby declare that the work presented in the thesis entitled “A Study On The Domforcing Numbers of Graphs” is based on the original work done by me under the guidance of **Dr. Premodkumar K P**, Research Supervisor, Post Graduate Department and Research Centre of Mathematics, St. Joseph’s College (Autonomous), Devagiri, Kozhikode-673008 and has not been included in any other thesis submitted previously for the award of any degree. The contents of the thesis are undergone plagiarism check using iThenticate software at C.H.M.K. Library , University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.



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ABSTRACT

Graph theory has wide-ranging applications across science, engineering, and real-world systems. It studies abstract structures used to model relationships among objects. Significant attention has been given to vertex parameters such as domination, zero forcing number, k -forcing number, and their connected variants in recent years. This thesis introduces a new concept, described as dom-forcing set and links it to connectedness, propagation time and k -forcing sets. Domination and zero forcing are two fundamental and well-studied concepts in graph theory, each with important applications in areas such as network analysis, quantum system control, power grid monitoring, and information dissemination. The notion of a dom-forcing set naturally arises from combining these ideas. It represents a set of vertices that not only dominate the graph locally but also initiate a forcing process that eventually influences the entire graph. This synthesis provides a powerful framework for understanding how control and influence propagate in complex networks. The study develops these concepts by establishing bounds, determining exact values for specific classes of graphs, and presenting various characterizations. A dominating set $D_f \subseteq V(\Gamma)$ in a graph Γ is called a dom-forcing set if the subset D_f forms a zero forcing set. The minimum cardinality of such a set is defined as the dom-forcing number of Γ , denoted by $F_d(\Gamma)$.

The thesis is organized into eleven chapters. Chapters 1 and 2 present the

broad context and necessary preliminaries. Chapters 3, 4, and 5 focus on the dom-forcing number, its behavior under different graph operations, and dom-forcing propagation time. Connected dom-forcing sets are examined in Chapters 6 and 7. Chapters 8 and 9 extend these ideas through generalizations to dom- k -forcing and connected dom- k -forcing sets. The thesis concludes in Chapter 10, followed by directions for future research in Chapter 11. Overall, this work contributes to the growing body of research in graph theory by introducing and systematically analyzing dom-forcing related parameters. The results not only deepen the theoretical understanding of domination and zero forcing processes but also open new directions for further investigation, particularly in the study of dynamic processes on networks. The concepts and techniques developed in this thesis are expected to have potential applications in areas such as network control, information propagation, and complex system analysis.

Key Words: Dom-forcing sets; Connected dom-forcing sets; Dom-forcing propagation time; Connected dom-forcing propagation time; Dom- k -forcing sets; Connected dom- k -forcing sets.

സംഗ്രഹം

ഗ്രാഫ് സിദ്ധാന്തം (Graph Theory) എന്നത് വസ്തുക്കൾ തമ്മിലുള്ള ബന്ധങ്ങളെ പ്രതിനിധീകരിക്കുന്ന അമൂർത്ത ഘടനകളെ പഠിക്കുന്ന ഗണിത ശാസ്ത്രശാഖയാണ്. ശാസ്ത്രം, എഞ്ചിനീയറിംഗ്, വിവിധ ജീവിത സാഹചര്യങ്ങൾ എന്നിവയിൽ വ്യാപകമായ പ്രയോഗങ്ങൾ ഉള്ളതിനാൽ ഈ മേഖലയിലെ ഗവേഷണം വളരെ വേഗത്തിൽ വളർന്നു വരുന്നു. സമീപകാലത്ത്, ഡോമിനേഷൻ (domination), സീറോ ഫോർസിങ്ങ് നമ്പർ (zero forcing number), k -ഫോർസിങ്ങ് നമ്പർ (k -forcing number), അവയുടെ കണക്കുകൾ (connected) രൂപങ്ങൾ എന്നിവയെക്കുറിച്ചുള്ള പഠനങ്ങൾ ഏറെ ശ്രദ്ധ നേടിയിട്ടുണ്ട്.

ഈ തീസിസിൽ “ഡോം-ഫോർസിങ്ങ് സെറ്റ് (dom-forcing set)” എന്ന പുതിയ ആശയം അവതരിപ്പിക്കുകയും അതിനെ കണക്റ്റഡ്നസ്സ്, പ്രൊപ്പഗേഷൻ സമയം, k -ഫോർസിങ്ങ് സെറ്റ് (connectedness, propagation time, k -forcing sets) എന്നിവയുമായി ബന്ധിപ്പിക്കുകയും ചെയ്യുന്നു. ഗ്രാഫ് സിദ്ധാന്തത്തിലെ അടിസ്ഥാനപരവും വ്യാപകമായി പഠിക്കപ്പെട്ടതുമായ രണ്ടാശയങ്ങളാണ് ഡോമിനേഷൻ (domination), സീറോ ഫോർസിങ്ങ് (zero forcing) എന്നിവ. ഇവയ്ക്ക് നെറ്റ് വർക്ക് അനാലിസിസ്, പവർ ഗ്രിഡ് മോണിറ്ററിംഗ്, വിവര സാങ്കേതികവിദ്യ, ക്വാണ്ടം സിസ്റ്റം കൺട്രോൾ തുടങ്ങിയ മേഖലകളിൽ വ്യാപകമായ പ്രയോഗസാധ്യതകളുണ്ട്. ഈ രണ്ടു ആശയങ്ങളുടെ സംയോജനമായാണ് ഡോം-ഫോർസിങ്ങ് സെറ്റ് (dom-forcing set) എന്ന ആശയം രൂപപ്പെടുന്നത്. ഇത് ഗ്രാഫിനെ പ്രാദേശികമായി നിയന്ത്രിക്കുന്നതിനൊപ്പം (local control), മുഴുവൻ ഗ്രാഫിലേക്കും സ്വാധീനം വ്യാപിപ്പിക്കുന്ന ഒരു പ്രക്രിയയും (Propagation) ആരംഭിക്കുന്നു. അതിനാൽ, സങ്കീർണ്ണമായ നെറ്റ് വർക്കുകളിൽ സ്വാധീനം വ്യാപിപ്പിക്കുന്നതെങ്ങനെയെന്ന് പഠിക്കാൻ ഇത് ശക്തമായ ഉപാധിയാണ്. ഈ പഠനത്തിൽ ഈ ആശയങ്ങളെ വികസിപ്പിച്ച് ഇവയുടെ ഗുണങ്ങൾ പഠിക്കുന്നതിനായി ഇവയ്ക്ക് ചില പരിധികൾ (bounds) കണ്ടെത്തുകയും ചില പ്രത്യേക തരം ഗ്രാഫുകളുടെ കൃത്യമായ മൂല്യങ്ങൾ നിർണ്ണയിക്കുന്നതിന് വേണ്ടിയുള്ള സമവാക്യങ്ങൾ തയ്യാറാക്കുകയും വിവിധ സ്വഭാവലക്ഷണങ്ങൾ (characterizations) അവതരിപ്പിക്കുകയും ചെയ്യുന്നു. ഒരു ഗ്രാഫ് യിലെ ശീർഷങ്ങളുടെ ഗണമാണ് $V(G)$. $V(G)$ യുടെ ഉപഗണമായ D_f ഒരു ഡോമിനേറ്റിങ്ങ് സെറ്റും (dominating set) സീറോ ഫോർസിങ്ങ് സെറ്റും (zero forcing set) കൂടിയാണെങ്കിൽ അതിനെ ഡോം-ഫോർസിങ്ങ് സെറ്റ് (dom-forcing set) എന്ന് വിളിക്കുന്നു. ഏറ്റവും കുറവ് അംഗങ്ങളുള്ള ഇത്തരം സെറ്റുകളിലെ അംഗങ്ങളുടെ എണ്ണത്തെ ഡോം-ഫോർസിങ്ങ് നമ്പർ (dom-forcing number) എന്ന് വിളിക്കുന്നു. സാധാരണയായി ഇതിനെ $F_d(G)$ എന്ന് സൂചിപ്പിക്കുകയും ചെയ്യുന്നു.

ഈ ഗവേഷണപ്രബന്ധം പതിനൊന്ന് അധ്യായങ്ങളായി ക്രമീകരിച്ചിരിക്കുന്നു. ഒന്നും രണ്ടും അധ്യായങ്ങൾ പരിചയപ്പെടുത്തലും അടിസ്ഥാന നിർവ്വചനങ്ങളും ഉൾക്കൊള്ളുന്നു. മൂന്നും നാലും അഞ്ചും അധ്യായങ്ങളിൽ ഡോം-ഫോർസിങ് നമ്പർ (dom-forcing number), വിവിധങ്ങളായ ഗ്രാഫ് ഓപ്പറേഷനുകളിൽ ഡോം-ഫോർസിങ് നമ്പറിനു വരുന്ന മാറ്റം, കൂടാതെ ഡോം-ഫോർസിങ് പ്രൊപ്പഗേഷൻ സമയം (dom-forcing propagation time) എന്നിവയെക്കുറിച്ചാണ്. ആറും ഏഴും അധ്യായങ്ങൾ കണക്റ്റഡ് ഡോം-ഫോർസിങ് സെറ്റ് (connected dom-forcing set) നെക്കുറിച്ചാണ്. എട്ടും ഒൻപതും അധ്യായങ്ങൾ ഡോം-k-ഫോർസിങ് സെറ്റ്, കണക്റ്റഡ് ഡോം-k-ഫോർസിങ് സെറ്റ് (dom-k-forcing set, connected dom-k-forcing set) എന്നീ പൊതുവായ രൂപങ്ങളെക്കുറിച്ചാണ്. അധ്യായം പത്തിൽ പഠനത്തിന്റെ കണ്ടെത്തലുകളും നിഗമനങ്ങളും അവതരിപ്പിക്കുന്നു. പതിനൊന്നാം അധ്യായത്തിൽ തുടർ ഗവേഷണത്തിനുള്ള സാധ്യതകളും നിർദ്ദേശങ്ങളും നൽകുകയും ചെയ്യുന്നു. മൊത്തത്തിൽ, dom-forcing സംബന്ധമായ പരാമീറ്ററുകളെ അവതരിപ്പിക്കുകയും ക്രമബദ്ധമായി പഠിക്കുകയും ചെയ്യുകൊണ്ട് ഈ ഗവേഷണം ഗ്രാഫ് സിദ്ധാന്തസംബന്ധമായ പുതിയജ്ഞാനനിർമ്മിതിയിൽ ക്രിയാത്മകമായി ഇടപെടുന്നു. കൂടാതെ domination, zero forcing എന്നീ പ്രക്രിയകളുടെ സിദ്ധാന്തപരമായ അവബോധം വർദ്ധിപ്പിക്കുകയും, നെറ്റ്വർക്ക് നിയന്ത്രണം, വിവരവ്യാപനം, സങ്കീർണ്ണ സിസ്റ്റങ്ങൾ എന്നിവയിൽ കൂടുതൽ പഠനങ്ങൾക്ക് പുതിയ സാധ്യതകൾ തുറക്കുകയും ചെയ്യുന്നു.

സൂചകപദങ്ങൾ: ഡോം-ഫോർസിങ് സെറ്റ്; കണക്റ്റഡ് ഡോം-ഫോർസിങ് സെറ്റ്; ഡോം-ഫോർസിങ് പ്രൊപ്പഗേഷൻ സമയം; കണക്റ്റഡ് ഡോം-ഫോർസിങ് പ്രൊപ്പഗേഷൻ സമയം; ഡോം-k-ഫോർസിങ് സെറ്റ്, കണക്റ്റഡ് ഡോം-k-ഫോർസിങ് സെറ്റ്.

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INTRODUCTION

In mathematics, a graph is an abstract representation of a set of objects. Graph Theory is an important branch of Discrete Mathematics, which is a key tool to model network systems involved in major domains of real life. Graph Theory extends its countless applications to various walks of science like Network Theory, Operational Research, Chemistry, Quantum Physics, Biology, Economics, Artificial Intelligence, Sociology and so on. It marks its origin in 1735 with Leonard Euler's solution [1] to the famous Konigsberg bridge problem. Now it has been evolved as a tool to analyse the structure of networks arising from real world system. This will help to study the impact of the structure on the dynamic processes taking place in it. Due to graph theory's diverse applications in almost all fields of real life, like network analysis, power system and biological system, research has been flourishing in this area since its inception. The study of areas in different vertex parameters such as domination, zero forcing number, k -forcing number, connected k -forcing number and other related problems has significant importance in the contemporary context. In this thesis, new concepts are in-

roduced, such as the dom-forcing set ¹, connected dom-forcing set, dom-forcing propagation time, connected dom-forcing propagation time, dom- k -forcing set, and connected dom- k -forcing set. The study further explores these concepts by deriving bounds, computing exact values, and presenting various characterizations.

A graph $\Gamma = (V, E)$ consists of a finite set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$ joining pairs of vertices. The study of control processes on graphs—where the state of vertices changes according to specific rules—has given rise to important concepts such as domination, propagation, and zero forcing. Among these, zero forcing sets and dominating sets have individually played a significant role in understanding control and influence mechanisms within networks. The interplay between these two notions has led to the concept of dom-forcing sets, a parameter that unifies ideas of control and coverage in graphs.

Zero forcing is a step by step coloring process where at every discrete time step, a black colored vertex with a single white colored neighbor forces that white colored neighbor to become colored black. A zero forcing set of a simple graph Γ is a set of initially colored black vertices which forces the entire graph Γ to become colored black. The zero forcing number is the cardinality of the least (or minimum) zero forcing set.

The zero forcing number of a graph Γ , denoted by $\mathcal{Z}(\Gamma)$, was found by the AIM Minimum Rank Special Graphs Group [2]. They used this graph parameter

¹A *Dom-forcing set* is a portmanteau term coined and used in this thesis to refer to a set that is both dominating and zero forcing. It combines the concepts of domination and zero forcing, meaning that the set satisfies the properties of both. Such a set may also be referred to as a zero forcing dominating set or a dominating zero forcing set.

$\mathcal{Z}(\Gamma)$ to bound the minimum rank for numerous families of graphs. In 2012, Minerva Catral et al. studied the relationship between the maximum nullity, the zero forcing number and the path cover number of a graph Γ [3] and obtained some useful results. They also included the relationship between the zero forcing number and the maximum nullity of edge subdivision graphs in their discussion. In 2015, Amos et al. [4] gave an upper bound of the zero forcing number of a connected graph Γ in terms order n and $\Delta \geq 2$, where Δ is the maximum degree of the graph Γ . In 2016, Michael Gentner et al. verified this conjecture and proved that the result is true only for the graphs C_n , K_n and $K_{\Delta, \Delta}$ [5]. A lower bound for triangle free graphs were also provided by them. More useful studies regarding the zero forcing number were also carried out by Darren D. Row in the paper “A technique for computing the zero forcing number of a graph with a cut-vertex” [6]. In this paper, a technique for computing the zero forcing number of graph with a cut-vertex was established. Darren D. Row also focused his attention in finding graphs having very low or very high zero forcing numbers.

The k -forcing number is a generalization of the zero forcing number. A subset $\mathcal{Z}_k$ of vertices in a graph is called a k -forcing set if its vertices are initially assigned a color while the remaining vertices start as uncolored. The graph undergoes a color propagation process based on the following rule: a colored vertex with at most k uncolored neighbors forces each of those neighbors to become colored. This process continues until all the vertices in the graph are colored. The k -forcing number of a graph, denoted as $\mathcal{Z}_k(\Gamma)$, represents the smallest possible size of a k -forcing set. In [4], Amos et al. proposed the definition of k -forcing set. In [7], the authors present a detailed study on the k -forcing number in the context of splitting graphs. Yair Caro et al. in [8] proved a conjecture post by

Amos et al. in [4].

It is challenging to study $\mathcal{Z}(\Gamma)$ of a graph Γ , without knowing much about the structure of a forcing set. This fact had actually motivated the study of the connected zero forcing number of a graph Γ , denoted by $\mathcal{Z}_c(\Gamma)$ and the connected k -forcing number of Γ , denoted by $\mathcal{Z}_{ck}(\Gamma)$. In 2016, Boris Brimkov and Randy Davila [9] imposed a condition on the zero forcing set that the initially colored set of black vertices should form a connected induced subgraph. This restriction on the zero forcing set had led to the introduction of the concept of connected zero forcing set and thus the connected zero forcing number $\mathcal{Z}_c(\Gamma)$.

A connected k -forcing set of a graph Γ , is a k -forcing set of Γ that induces a sub graph of Γ which is connected. The minimum size of connected k -forcing set is called connected k -forcing number. Connected k -forcing number $\mathcal{Z}_{ck}(\Gamma)$ of several families of graphs including trees, flower snarks, and hyper cubes were studied and analysed. Further studies were carried out and in 2016, Randy Davila, Michael A. Henning, Colton Magnant, and Ryan Pepper [10] provided a sharp lower and upper bounds on the connected forcing number of a graph. In [11], the authors present a detailed study on the connected k -forcing number in the context of splitting graphs.

A set $D \subseteq V$ of vertices in a graph $\Gamma = (V, E)$ is called a dominating set if every vertex $v \in V(\Gamma)$ is either an element of D or is adjacent to an element of D . The rigorous study of the dominating sets in Graph theory began around 1960. According to Hedetniemi and Laskar (1990) [12], the domination problems were studied from the 1950's onwards, but the rate of research on domination significantly increased in the mid 1970's. In 1958, Berge[13] defined the concept

of the domination number of a graph, called as “coefficient of external stability”. In 1962, Ore [14] used the name “dominating set” and “domination number” for the same concept. In 1977, Cockayne and Hedetniemi [15] made an interesting and extensive survey of the results known at the time about dominating sets in graphs [16, 17]. They have used the notation $\gamma(\Gamma)$ for the domination number of a graph, which has become very popular since then [18]. The survey paper of Cockayne and Hedetniemi [15] has generated a lot of interests in the study of dominating sets in graphs. Recent books on dominating set have stimulated a sufficient inspiration leading to the expansive growth of this field to study. It has also put some order into this huge collection of research papers, and organized the study of dominating sets in graphs into meaningful subareas, placing the study of dominating sets in even broader mathematical and algorithmic contexts [19, 16, 20].

This thesis, combine domination, a graph-theoretic parameter, and other graph-theoretic parameters such as zero forcing, connected zero forcing, k -forcing, connected k -forcing, and propagation time. While extensive research has been conducted on each parameter separately, their combined study is a novel approach. The hybrid concept of dom-forcing sets in graphs is introduced here. For a graph Γ , a set $D_f \subseteq V$ of vertices is called a *dom-forcing set* if D_f must form a dominating set as well as a zero forcing set. The minimum cardinality of such a set is called the dom-forcing number of the graph Γ and is denoted by $F_d(\Gamma)$. It also introduce dom-forcing propagation time, connected dom-forcing set, connected dom-forcing propagation time, dom- k -forcing set, and connected dom- k -forcing set.

The study of dom-forcing sets in graphs is motivated by the need to unify and generalize two fundamental concepts in graph theory: domination and zero forcing. Both concepts serve as effective tools for modeling control and influence in networks, but they capture different aspects of these processes. Domination focuses on local or immediate control, requiring every vertex to be either selected or adjacent to a selected vertex. In contrast, zero forcing models a dynamic spread of influence, where control propagates through the network according to prescribed rules. By combining these two perspectives, dom-forcing sets provide a richer framework that simultaneously captures both the static placement of control and its dynamic propagation across the network.

This combination is particularly motivated by real-world applications where influence must be initiated and sustained throughout a system. For example, in power systems, starting sensors (domination) have to blanket the grid as well as facilitate cascading monitoring (forcing). In social networks, it may be the case that only a small group of influential actors need to directly reach their neighbors as well as start a more general chain effect of influence. Likewise, in biological systems or distributed computing, a connected and efficient spread of signals or information is crucial. Dom-forcing sets provide a mathematical model for such dual demands for instant coverage and sustained propagation.

In addition to the introductory Chapter, the thesis is divided into ten other Chapters and Chapters into Sections.

Chapter One is the introduction.

Chapter Two provides a brief description of the basic definitions of graph theory that will be very useful in the upcoming chapters.

In **Chapter Three**, the problem of determining the dom-forcing number of graphs is addressed. The chapter is divided into six sections, with the first section serving as an introduction. In Section 2, certain bounds for the dom-forcing number are discussed. Section 3 is devoted to determining the exact values of the dom-forcing number for several well-known graphs. In Section 4, the dom-forcing number is obtained for graphs satisfying $F_d(\Gamma) = \mathcal{Z}(\Gamma)$. Section 5 deals with graphs for which $F_d(\Gamma) = \gamma(\Gamma)$. Finally, Section 6 presents several inequalities involving the dom-forcing number of splitting graphs of certain classes of graphs.

In **Chapter Four**, the effect of various graph operations on the dom-forcing number, an important parameter related to domination in graphs, is examined. The operations considered include vertex deletion, edge deletion, vertex addition, as well as the sum and product of graphs, each of which alters the structure and properties of a graph. The chapter is divided into three sections. In Section 2, upper and lower bounds for the dom-forcing number in the context of edge operations are established. It is analyzed how the deletion or contraction of an edge in a graph Γ influences its dom-forcing number. Further, vertex operations are investigated by considering the impact of vertex deletion and the vertex sum of graphs on this parameter. In Section 3, the dom-forcing number is determined for various graph products, including corona products and rooted products. Relevant examples and theoretical results are presented, illustrating how these graph operations lead to variations in the dom-forcing number.

Chapter Five introduces a new parameter in graph theory called dom-forcing propagation time. Section 2 establishes the upper and the lower bounds for minimum and maximum dom-forcing propagation time in terms of dom-

forcing number. Section 3 explores the maximum and minimum dom-forcing propagation time for several renowned graphs. Also characterize graph with extreme dom-forcing propagation time.

In **Chapter Six**, a new graph theoretic parameter, namely the connected dom-forcing set, is introduced. Following the introductory section, Section 2 is devoted to establishing upper and lower bounds for this parameter in terms of the connected zero forcing number and the connected domination number. In Section 3, the connected dom-forcing number is studied for various classes of product graphs, including the corona product, generalized corona product, rooted product, and Cartesian product of graphs. Section 4 focuses on the investigation of the connected dom-forcing number of the splitting graph of a given graph. In particular, the connected dom-forcing number is determined for specific graphs such as $S(P_n)$, $S(C_n)$, $S(K_{1,n})$, and $S(L_n)$.

It is stated in **Chapter Seven** that a new parameter in graph theory, namely the connected dom-forcing propagation time, is introduced. In Section 2, upper and lower bounds for the minimum connected dom-forcing propagation time are established in terms of the dom-forcing number. Section 3 explores the minimum connected dom-forcing propagation time for several well-known graphs, while Section 4 provides a characterization of graphs with extreme connected dom-forcing propagation time.

It is described in **Chapter Eight** that it serves as a generalization of **Chapter Three**. Throughout the chapter, the problem of determining the dom- k -forcing number of graphs is addressed. The chapter is organized into six sections. In Section 2, bounds for the dom- k -forcing number are established and graphs

with extreme dom- k -forcing number are characterized. Section 3 determines the dom- k -forcing number for graphs satisfying $F_{dk}(\Gamma) = \mathcal{Z}k(\Gamma)$. In Section 4, exact values of the dom- k -forcing number are obtained for several snake graphs. Section 5 examines graphs for which $Fdk(\Gamma) = \gamma(\Gamma)$. Finally, Section 6 establishes several inequalities involving the dom- k -forcing number of splitting graphs of certain classes of graphs.

It is explained in **Chapter Nine** that it is a generalization of **Chapter Six**. In this chapter, the connected dom- k -forcing number of graphs is studied. Section 2 establishes upper and lower bounds for this parameter in terms of the connected k -forcing number and the connected domination number. Section 3 investigates graphs for which $F_{dk}(\Gamma) = F_{cdk}(\Gamma)$, while Section 4 focuses on graphs where $F_{dk}(\Gamma) < F_{cdk}(\Gamma)$. In Section 5, the connected dom- k -forcing number of the splitting graph of a given graph is examined, with particular attention to graphs such as $S(P_n)$, $S(C_n)$, and $S(L_n)$.

Chapter Ten concludes the thesis and further recommendations discussed in **Chapter Eleven**.

Chapter 2

PRELIMINARIES

This chapter provides a brief account of the preliminary definitions from graph theory which are very useful for the upcoming chapters. For the notations and terminologies not defined directly in the thesis, it may refer to [21] and [22].

2.1 Basic Definitions

Definition 2.1.1. [21] A graph Γ is an ordered triple $\Gamma = (V(\Gamma), E(\Gamma), \psi(\Gamma))$ consisting of a non-empty set $V(\Gamma)$ of vertices, a set $E(\Gamma)$ of edges and an incidence function $\psi(\Gamma)$ which associates with each element of $E(\Gamma)$, an unordered pair of vertices (not necessarily distinct) of Γ .

Definition 2.1.2. [21] Two vertices are said to be adjacent, if there is an edge between them. Otherwise, they are non-adjacent. If a vertex u is adjacent to a vertex v in a graph Γ , denote it as $u \sim v$. If two edges are incident with a common vertex, then they are adjacent. Otherwise, non-adjacent.

Definition 2.1.3. [21] An edge with identical end points is called a loop. Edges joining the same pair of vertices are called multiple edges.

Definition 2.1.4. [21] A graph Γ is finite if both the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$ are finite. Otherwise, the graph Γ is said to be infinite.

Definition 2.1.5. [21] The number of vertices in a graph Γ is known as the order of the graph and it is denoted by $|\Gamma|$.

Definition 2.1.6. [21] A graph which has no loops and multiple edges is called a simple graph. A graph is trivial if it has only one point.

Definition 2.1.7. [21] Two graphs Γ and H are said to be isomorphic if there exists bijections $\theta : V(\Gamma) \rightarrow V(H)$ and $\phi : E(\Gamma) \rightarrow E(H)$ such that $\psi_\Gamma(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$. If Γ and H are isomorphic, write it as $\Gamma \cong H$.

Definition 2.1.8. [21] A graph Γ is complete if every pair of distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n .

Definition 2.1.9. [21] An empty graph (null graph) is a graph which has no edges.

Definition 2.1.10. [21] A graph Γ is called bipartite if the vertex set $V(\Gamma)$ can be partitioned into two subsets X and Y such that each edge of Γ has one end in X and the other end in Y . (X, Y) is called a partition of Γ .

Definition 2.1.11. [21] A graph Γ is said to be complete bipartite if Γ is simple, bipartite with bipartition (X, Y) and each vertex of X is joined to every vertex of Y . If $|X| = m, |Y| = n$, then Γ is denoted by $K_{m,n}$.

Definition 2.1.12. [21] The complement of a simple graph Γ , denoted by $\bar{\Gamma}$, is a simple graph with vertex set $V(\Gamma)$ and such that two vertices are adjacent in $\bar{\Gamma}$ if and only if they are non-adjacent in Γ .

Definition 2.1.13. [21] A graph H is a subgraph of Γ , if $V(H) \subseteq V(\Gamma)$, $E(H) \subseteq E(\Gamma)$ and ψ_H is the restriction of ψ_Γ to $E(H)$.

If H is a subgraph of Γ , then it is denoted by $H \subseteq \Gamma$. Γ is the super graph of H .

If H is a subgraph of Γ and $H \neq \Gamma$, then H is a proper subgraph of Γ ($H \subset \Gamma$).

H is a spanning subgraph of Γ if H is a subgraph and $V(H) = V(\Gamma)$.

Let Γ be a graph. By deleting all loops and for every pair of adjacent vertices all, except one link joining them, obtain a simple spanning subgraph of Γ called the underlying simple graph of Γ .

Definition 2.1.14. [21] Let V^1 be a non-empty subset of the vertex set V of Γ . The subgraph of Γ whose vertex set is V^1 and whose edge set is the set of those edges of Γ that have both ends in V^1 is called the subgraph of Γ induced by V^1 and is denoted by $\Gamma[V^1]$. The graph $\Gamma[V^1]$ is the induced subgraph of Γ .

Definition 2.1.15. [21] If $u \in V(\Gamma)$, the open neighborhood of u ; denoted by $N_\Gamma(u)$ is the set of vertices adjacent to u in Γ .

Definition 2.1.16. [21] Let Γ be a graph and v be it's vertex. Then, the degree of v is the number of edges of Γ incident with v , counting each loop as two edges. The degree of v is denoted as $deg(v)$. For a simple graph, $deg(v)$ is the cardinality of $N_\Gamma(v)$. The vertex with zero degree is called an isolated vertex. A vertex with degree one is called an end vertex or pendant vertex.

$\delta(\Gamma)$ and $\Delta(\Gamma)$ are denoted as, the minimum degree and the maximum degree of vertices in Γ respectively.

Definition 2.1.17. [21] A graph Γ is k -regular if $\deg(v) = k$ for all $v \in V$. A regular graph is a graph which is k -regular for some $k \geq 0$.

Definition 2.1.18. [21] A walk in a graph is a finite, non-null or non-empty sequence $w = v_0e_1v_1e_2v_2, \dots, e_kv_kv_k$ whose terms are alternatively vertices and edges so that for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i . Then, w is a walk from v_0 to v_k or w is a (v_0, v_k) walk. v_0 is called the origin and v_k is called the terminus of w . $v_1, v_2, \dots, v_{k-1}$ are called the internal vertices. The integer k is called the length of w .

Definition 2.1.19. [21] A walk in which every edge is distinct is called a trail.

Definition 2.1.20. [21] A walk in which every vertex is distinct (hence edges are distinct) is called a path.

Definition 2.1.21. [21] A cycle is a closed trail whose origin and internal vertices are different. A cycle of length k is called a k -cycle and it is denoted by C_k .

Definition 2.1.22. [21] A graph is called acyclic if it has no cycles.

Definition 2.1.23. [21] Two vertices u and v of Γ are said to be connected if there is a (u, v) path between them.

Definition 2.1.24. [21] A tree is a acyclic graph which is connected.

Definition 2.1.25. [21] The distance between two vertices u and v , denoted by $d_\Gamma(u, v)$, is the length of a shortest path connecting them. Clearly $d_\Gamma(u, u) = 0$ and $d_\Gamma(u, v) = \infty$, if there is no path connecting u and v .

Definition 2.1.26. [21] Diameter of a graph Γ is the maximum distance between the vertices of Γ . That is, $\text{diam}(\Gamma) = \text{Max}\{d(x, y): x \text{ and } y \text{ are vertices of } \Gamma\}$.

Definition 2.1.27. [21] In a connected graph Γ , the transmission degree of a vertex v is defined as $Tr(v) = \sum_{u \in V(\Gamma)} d_{\Gamma}(u, v)$.

Definition 2.1.28. [21] Clique of a graph is a set of mutually adjacent vertices. The maximum size of a clique of a graph Γ , called the clique number of Γ , is denoted by $\omega(\Gamma)$.

2.2 Graphs Operations

Definition 2.2.1. [23] Let $\Gamma_1 = (V_1; E_1)$ and $\Gamma_2 = (V_2; E_2)$ be two simple graphs. Then, the graph $\Gamma = (V; E)$; where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, is called the union of graphs Γ_1 and Γ_2 and is denoted by $\Gamma_1 \cup \Gamma_2$. If $V_1 \cap V_2 = \emptyset$, then $\Gamma_1 \cup \Gamma_2$ is usually denoted by $\Gamma_1 + \Gamma_2$, called the sum of the graphs Γ_1 and Γ_2 .

Definition 2.2.2. [23] Let $\Gamma_1 = (V_1; E_1)$ and $\Gamma_2 = (V_2; E_2)$ be two simple graphs. Then, the graph $\Gamma = (V; E)$; where $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$, is called the intersection of graphs Γ_1 and Γ_2 and is denoted by $\Gamma_1 \cap \Gamma_2$.

Definition 2.2.3. [24] If K is a subgraph of Γ with $V(K) = V(\Gamma)$ and $E(K) = E(\Gamma) \setminus \{e\}$ (where $e \in E(\Gamma)$), then, K is obtained from Γ by removing an edge e and write $K = \Gamma - e$.

Definition 2.2.4. [24] Consider an edge e that connects two vertices x and y in the graph Γ . Edge contraction Γ/e has two steps. First, the edge e is removed from the graph so x and y are no longer connected. Second, x and y are merged into one vertex. This simplifies the graph by reducing the number of vertices and edges.

Definition 2.2.5. [24] Given a graph $\Gamma(V, E)$, and edge subdivision involves selecting an edge $e = uv$ in E and introduce a new vertex w currently not in V , then replace the edge uv in Γ with two new edges uw and wv . The resulting graph is called a subdivision of Γ .

Definition 2.2.6. [24] When a vertex u is removed from a graph Γ , along with all edges incident to u , the resulting graph is denoted as $\Gamma - u$.

Definition 2.2.7. [23] Let $\Gamma = (V(\Gamma), E(\Gamma))$ and $K = (V(K), E(K))$ be two graphs. The join of two graphs is defined as their union, augmented by all possible edges hk , where $h \in V(\Gamma)$ and $k \in V(K)$. This operation is represented by $\Gamma \vee K$.

Definition 2.2.8. [23] The corona of Γ with K , represented as $\Gamma \circ K$, is a graph constructed by taking one copy of Γ and $|V(\Gamma)|$ copies of K . The resulting graph is an order $|V(\Gamma)||V(K)| + |V(\Gamma)|$. Each vertex in the i^{th} copy of K is connected to the i -th vertex of Γ .

Figure 2.1 illustrates this construction for $C_4 \circ P_3$.

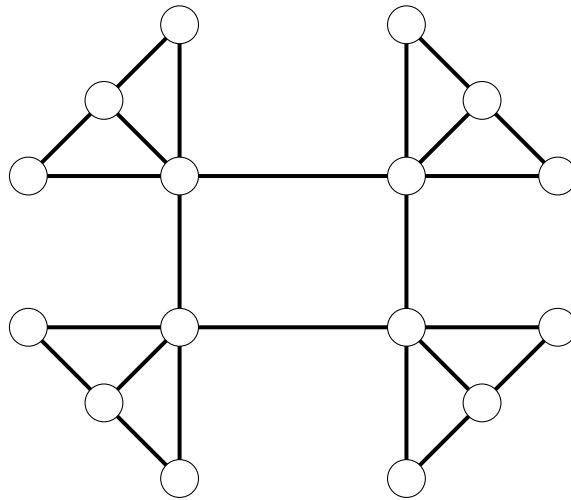


Figure 2.1: $C_4 \circ P_3$.

Definition 2.2.9. [25] Let Γ be a graph with $|\Gamma| = n$, where $V(\Gamma)$ denote the vertex set of Γ , and let $H_1, \dots, H_n$ be the graphs. Define the graph $\Gamma \prec H_1, \dots, H_n \succ$ as the graph obtained by joining all the vertices of the graph H_i to the i -th vertex of Γ , where $i = 1, \dots, n$. Note that $|H_i|$ can be zero, in which case no extra vertices will be joined to the vertex i . The graph $\Gamma \prec H_1, \dots, H_n \succ$ is called a generalized corona of Γ with $H_1, \dots, H_n$.

Definition 2.2.10. [25] Let Γ and K be two graphs of order n_1 and n_2 respectively. For any integer $j \geq 2$, define the graph $\Gamma \circ^j K$ recursively from $\Gamma \circ K$ as $\Gamma \circ^j K = (\Gamma \circ^{j-1} K) \circ K$. It is also noted that $|\Gamma \circ^{j-1} K| = n_1(n_2 + 1)^{j-1}$ and $|\Gamma \circ^j K| = |\Gamma \circ^{j-1} K| + n_1 n_2 (n_2 + 1)^{j-1}$.

See Figure 2.2 for an illustration of $P_3 \circ^2 K_2$.

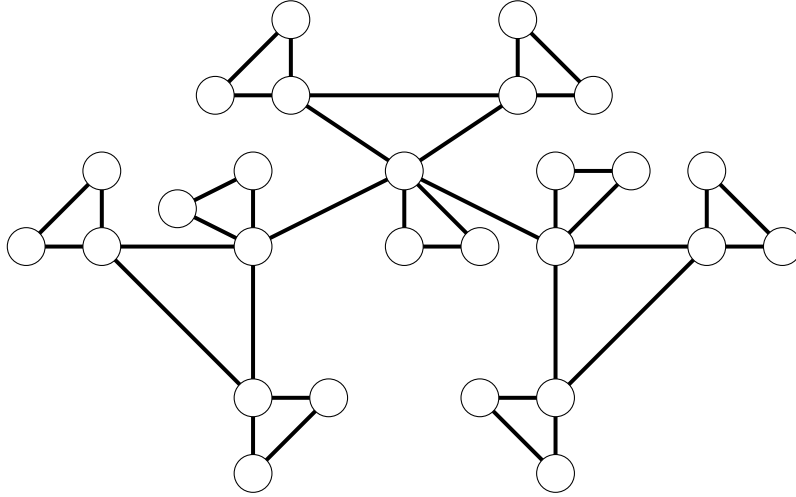


Figure 2.2: $P_3 \circ^2 K_2$.

Definition 2.2.11. [26] Let Γ be a connected graph with vertices $v_1, v_2, \dots, v_n$ and let K be a sequence of n rooted graphs $H_1, H_2, \dots, H_n$. The rooted product of Γ and K is defined as the graph obtained by identifying the root of $H_i, 1 \leq$

$i \leq n$, with the i th vertex of Γ for all i . This graph is denoted by $\Gamma(K)$ and is known as the rooted product of Γ by K .

Definition 2.2.12. [23] The cartesian product of two graphs Γ and H , denoted by $\Gamma \square H$, is the graph with vertex set $V(\Gamma) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if

- (1) $u = u'$ and $vv' \in E(H)$, or
- (2) $v = v'$ and $uu' \in E(\Gamma)$.

2.3 Zero Forcing and Domination

Definition 2.3.1. [2] Zero forcing is a step-by-step coloring process in which, at each discrete time step, a black-colored vertex u with exactly one white-colored neighbor v forces that neighbor to become black. When this rule is applied, it is said that u forces v , denoted by $u \rightarrow v$. It is further defined that a zero forcing set of a simple graph Γ is a set of vertices initially colored black that eventually forces the entire graph Γ to become black. The zero forcing number of Γ is defined as the minimum cardinality of a zero forcing set and is denoted by $\mathcal{Z}(\Gamma)$.

Definition 2.3.2. [27] A connected zero forcing set (or $\mathcal{CZFS}$ for brevity) of a graph Γ , is a zero forcing set of Γ that induces a sub graph of Γ which is connected. The connected zero forcing number of Γ , $\mathcal{Z}_c(\Gamma)$, is the minimum size of connected zero forcing sets. Any connected zero forcing set of order $\mathcal{Z}_c(\Gamma)$ is called a minimum connected zero forcing set.

Definition 2.3.3. [4] A subset $\mathcal{Z}_k$ of vertices in a graph Γ is called a k -forcing set if its vertices are initially assigned a color while the remaining vertices start

as uncolored. The graph undergoes a color propagation process based on the following rule: a colored vertex with at most k uncolored neighbors forces each of those neighbors to become colored. This process continues until all the vertices in the graph are colored. The k -forcing number of a graph, denoted as $\mathcal{Z}_k(\Gamma)$, represents the smallest possible size of a k -forcing set.

Definition 2.3.4. [11] A connected k -forcing set of a graph Γ , is a k -forcing set of Γ that induces a sub graph of Γ which is connected. The connected k -forcing number of Γ , $\mathcal{Z}_{ck}(\Gamma)$, is the minimum size of connected k -forcing sets. Any connected k - forcing set of order $\mathcal{Z}_{ck}(\Gamma)$ is called a minimum connected k -forcing set.

Definition 2.3.5. [13] Let Γ be a graph. A dominating set is a subset S of the graph's vertex set such that every vertex not in S is adjacent to at least one vertex in S . The domination number of a graph Γ , denoted by $\gamma(\Gamma)$, is the minimum number of vertices in a dominating set.

Definition 2.3.6. [28] If a sub graph formed by the vertices of a dominating set in a graph Γ is connected then the set is called a connected dominating set. A minimum connected dominating set of a graph Γ is a connected dominating set with the smallest possible cardinality among all connected dominating sets of Γ . The connected domination number of Γ is the number of vertices in the minimum connected dominating set, and is denoted by $\gamma_c(\Gamma)$.

This chapter discuss at-length all the pre-requisites for the study. The next chapter will focus on the concept of dom-forcing sets in graphs.

Chapter 3

DOM-FORCING SETS IN GRAPHS

This chapter¹ addresses the problem of determining the dom-forcing number of graphs. In Section 2, it establishes bounds for the dom-forcing number of graphs. Section 3 presents the exact values of the dom-forcing number for several well-known graphs. In Section 4, it determines the dom-forcing number for graphs satisfying $F_d(\Gamma) = \mathcal{Z}(\Gamma)$. Section 5 examines graphs for which $F_d(\Gamma) = \gamma(\Gamma)$. In Section 6, it provides several inequalities involving the dom-forcing number of splitting graphs of certain graphs.

3.1 Introduction

This section combines two graph theoretic parameters namely, zero forcing and domination and introduce a new hybrid concept dom-forcing sets of graphs.

Definition 3.1.1. *Let $\Gamma = (V, E)$ be a graph. A set $D_f \subseteq V$ of vertices is called*

¹This chapter has been published in *Asian Research Journal of Mathematics*, Volume 22, Number 3, March 2026, Pages 88-104.

a dom-forcing set if it satisfies the following two conditions.

- i) D_f must form a dominating set.
- ii) D_f must form a zero forcing set.

The minimum cardinality of such a set is called the dom-forcing number of the graph Γ and is denoted by $F_d(\Gamma)$. For instance, contemplate the graph C_5 illustrated in Figure 3.1.

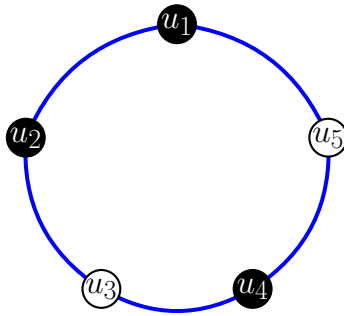


Figure 3.1: In this graph C_5 , $D_f = \{u_1, u_2, u_4\}$ is a dominating as well as a zero forcing set and no subset of the vertex set with cardinality less than three has this property, so $F_d(C_5) = 3$.

3.2 Bounds for Dom-forcing Sets

This section discussed some bounds for the dom-forcing set. The definition makes it evident that the combination of a zero forcing set and a dominating set constitutes a dom-forcing set. Therefore, the following relationship holds:

Proposition 3.2.1. *For any connected graph Γ*

$$i) \mathcal{Z}(\Gamma) \leq F_d(\Gamma) \leq \mathcal{Z}(\Gamma) + \gamma(\Gamma)$$

$$ii) \gamma(\Gamma) \leq F_d(\Gamma) \leq \mathcal{Z}(\Gamma) + \gamma(\Gamma)$$

The following figure illustrates a graph where $\mathcal{Z}(\Gamma) = \gamma(\Gamma) = F_d(\Gamma) = 2$.

Example 3.2.2. *The dom-forcing set and the dom-forcing number of the graph in the Figure 3.2 is $D_f = \{v_1, v_2\}$ and $F_d(\Gamma) = 2$.*

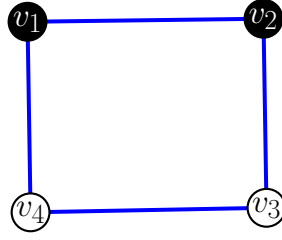


Figure 3.2: The graph $\Gamma(V, E)$.

Proposition 3.2.3. *Let $\Gamma = (V, E)$ be a graph and S be its minimum zero forcing set. Then $F_d(\Gamma) \leq \mathcal{Z}(\Gamma) + \gamma(\Gamma - N[S])$.*

Proof. Let $\Gamma = (V, E)$ be a graph and S be its minimum zero forcing set. There exists a dominating set with cardinality $\gamma(\Gamma - N[S])$, say D , of $\Gamma - N[S]$. Then $S \cup D$ is a dom-forcing set of Γ . Therefore, $F_d(\Gamma) \leq \mathcal{Z}(\Gamma) + \gamma(\Gamma - N[S])$. $\square$

It can be seen that this bound is sharp for Paths, Cycles etc. This is illustrated in the Figure 3.3.

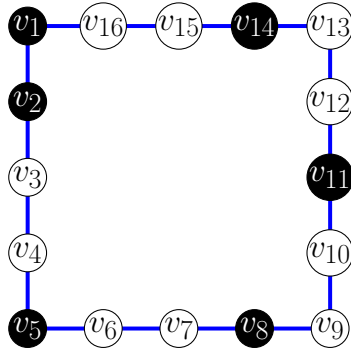


Figure 3.3: In the cycle C_{16} , $S = \{v_1, v_2\}$ will be a minimum zero forcing set and $\Gamma - N[S]$ is a path with vertex set $\{v_4, v_5, \dots, v_{15}\}$. $T = \{v_5, v_8, v_{11}, v_{14}\}$ will be a minimum dominating set of $\Gamma - N[S]$, therefore $\gamma(\Gamma - N[S]) = 4$. Hence $S \cup T$ is a dom-forcing set of C_{16} , which is minimum. Hence $F_d(C_{16}) = 6$.

The next section discusses the exact values of the dom-forcing number of some graphs like, path, cycle, ladder graph, coconut tree graph, and diamond snake graph.

3.3 Exact Values of The Dom-forcing Number of Graphs

This section investigates precise values of the dom-forcing number for several renowned graphs. Start with a path P_n .

Theorem 3.3.1. [29, 30] For a path P_n , $\gamma(P_n) = \lceil n/3 \rceil$ and $\mathcal{Z}(P_n) = 1$.

Example 3.3.2. Consider the graphs given in Figure 3.4. For P_3 , $D_f = \{v_1, v_3\}$ and $F_d = 2$. For P_4 , $D_f = \{u_1, u_4\}$ and $F_d = 2$. The result is generalized as follows.

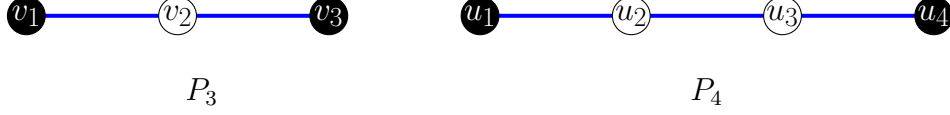


Figure 3.4: Dom-forcing set for P_3 and P_4 .

Theorem 3.3.3. For a path P_n , $F_d(P_n) = \lfloor n/3 \rfloor + 1$.

Proof. Consider a path P_n , with $V(P_n) = \{v_1, v_2, \dots, v_n\}$. For $i = 2, 3, \dots, n-1$ each vertex v_i is adjacent to v_{i-1} and v_{i+1} . It can be easily observed that $F_d(P_1) = F_d(P_2) = 1$. For $n \geq 3$, consider the following cases and the following subset of $V(P_n)$.

Case 1: Assume $n \equiv 0 \pmod{3}$, the set $S = \{v_{3k+2}\}$, where $0 \leq k < \frac{n}{3}$ is a minimum dominating set of P_n which is unique upto isomorphism. The set S cannot zero force the entire graph. Hence $F_d(P_n) \geq \lceil n/3 \rceil + 1$. But the set $D_f = \{v_{3k+1}, v_n\}$ where $0 \leq k < \lfloor n/3 \rfloor$ dominates and zero forces the entire graph P_n . Therefore, $F_d(P_n) = \lfloor n/3 \rfloor + 1$

Case 2: Assume $n \equiv 1 \pmod{3}$, $D_f = \{v_{3k+1}, v_n\}$ where $0 \leq k < \lfloor n/3 \rfloor$.

Case 3: Assume $n \equiv 2 \pmod{3}$, $D_f = \{v_{3k+1}, v_{n-1}\}$ where $0 \leq k < \lfloor n/3 \rfloor$.

In both cases D_f contains one of the end vertices, and therefore it forces P_n . In addition, it dominates P_n . Therefore, the set D_f is a dom-forcing set for P_n . Any pendant vertex or two adjacent vertices forces P_n , and no set with cardinality less than $|D_f|$ have this property, since $|D_f| = \gamma(P_n)$.

Therefore, in all cases, $F_d(P_n) = \lfloor n/3 \rfloor + 1$. □

The following observations are noted in [10].

Observation 3.3.4. [10] For any connected graph $\Gamma = (V; E)$, $\mathcal{Z}(\Gamma) = 1$ if and only if $\Gamma = P_n$.

Using the above observation, a graph with dom-forcing number one can be characterize as follows.

Theorem 3.3.5. For any graph Γ , $F_d(\Gamma) = 1$ if and only if $\Gamma = P_1$ or P_2 .

Proof. For $\Gamma = P_1$ or P_2 , by the Theorem 3.3.3, $F_d(\Gamma) = 1$. Conversely, suppose that $F_d(\Gamma) = 1$. Then $\mathcal{Z}(\Gamma) = \gamma(\Gamma) = 1$. But by Observation 3.3.4, $\mathcal{Z}(\Gamma) = 1$ implies that Γ is a path, also from the Theorem 3.3.3, $\Gamma = P_1$ or P_2 . $\square$

Now let us examine the dom-forcing number of cycle graph C_n .

Theorem 3.3.6. [29, 30] For a cycle C_n , of order $n \geq 3$, $\mathcal{Z}(C_n) = 2$ and $\gamma(C_n) = \lceil n/3 \rceil$.

Theorem 3.3.7. For cycle C_n , $n \geq 3$

$$F_d(C_n) = \begin{cases} \lfloor n/3 \rfloor + 1 & \text{if } n \equiv 0, 1 \pmod{3} \\ \lfloor n/3 \rfloor + 2 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Proof. Let C_n be a cycle with vertices $v_1, v_2, \dots, v_n$. Consider the following cases and the following subset of $V(C_n)$.

Case 1: Assume $n \equiv 0 \pmod{3}$, the set $S = \{v_{3k+2}\}$, where $0 \leq k < \frac{n}{3}$ is a minimum dominating set of C_n which is unique upto isomorphism. The set S cannot zero force the entire graph. Hence $F_d(C_n) \geq \lfloor n/3 \rfloor + 1$. But the set $D_f = \{v_{3k+1}, v_n\}$, where $0 \leq k < \lfloor n/3 \rfloor$ dominates and zero forces the entire

3.3. EXACT VALUES OF THE DOM-FORCING NUMBER OF GRAPHS 27

graph C_n . Therefore, $F_d(C_n) = \lfloor n/3 \rfloor + 1$.

Case 2: Assume $n \equiv 1 \pmod{3}$, $D_f = \{v_{3k+1}, v_n\}$, where $0 \leq k < \lfloor n/3 \rfloor$. In this case, D_f contains two adjacent vertices, and therefore it forces C_n . In addition, for $i = 2, 3, \dots, n-1$, each vertex v_i is adjacent with v_{i-1} and v_{i+1} , D_f dominates C_n . Therefore, D_f is a dom-forcing set. Any pair of adjacent vertices can force the cycle graph C_n . Moreover, no set with cardinality smaller than $|D_f|$ possesses this forcing property, since $|D_f| = \gamma(C_n)$. Hence, $F_d(C_n) = \lfloor n/3 \rfloor + 1$.

Case 3: Assume $n \equiv 2 \pmod{3}$, the set $S = \{v_{3k+1}, v_{n-1}\}$, where $0 \leq k < \frac{n}{3}$ is a minimum dominating set of C_n which is unique upto isomorphism. The set S cannot zero force the entire graph. Hence $F_d(C_n) \geq \lceil n/3 \rceil + 1$. But the set $D_f = \{v_{3k+1}, v_{n-1}, v_n\}$, where $0 \leq k < \lfloor n/3 \rfloor$ dominates and zero forces the entire graph P_n . Therefore, $F_d(C_n) = \lceil n/3 \rceil + 1 = \lfloor n/3 \rfloor + 2$

From these cases,

$$F_d(C_n) = \begin{cases} \lfloor n/3 \rfloor + 1 & \text{if } n \equiv 0, 1 \pmod{3} \\ \lfloor n/3 \rfloor + 2 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

□

The ladder graph L_n is the graph obtained by taking the Cartesian product of P_n with P_2 .

Theorem 3.3.8. [30] For a ladder graph L_n , $n \geq 2$, $\mathcal{Z}(L_n) = 2$.

Theorem 3.3.9. [31] For a ladder graph L_n , $n \geq 2$, $\gamma(L_n) = \lfloor \frac{n}{2} \rfloor + 1$.

Theorem 3.3.10. For $n \geq 2$, let L_n denote the ladder graph, then $F_d(L_n) = \lfloor \frac{n}{2} \rfloor + 1$.

Proof. Let $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ be the vertices of the ladder graph L_n . For the ladder graphs L_1, L_2, L_3 and L_4 , the subset of vertices $\{u_1\}$, $\{u_1, u_2\}$, $\{u_1, u_2, v_3\}$ and $\{u_1, u_2, v_4\}$ respectively constitute a dom-forcing set, and for $n > 4$, consider the following cases and the following subsets of vertex sets.

Case 1: Assume that $\lceil \frac{n}{2} \rceil$ is odd, consider

$$D_f = \{u_1, u_{4k-2}, v_{4k}, u_n\} \text{ for } 1 \leq k < \lceil n/4 \rceil$$

Case 2: Assume $\lceil \frac{n}{2} \rceil$ is even, consider

$$D_f = \{u_1, u_2, v_{4k}, u_{4k+2}, v_n\} \text{ for } 1 \leq k < \lceil n/4 \rceil.$$

Now D_f forms a dom-forcing set. It can be easily verified that no set with less than $|D_f|$ vertices cannot form a dom-forcing set, hence $F_d(L_n) = \lceil \frac{n}{2} \rceil + 1$. $\square$

The above theorem illustrated in the case of L_5 .

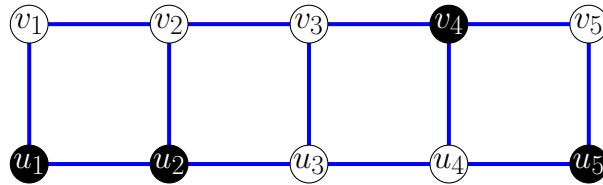


Figure 3.5: The ladder graph L_5 , dom-forcing set $D_f = \{u_1, u_2, v_4, u_5\}$ and $F_d(L_5) = 4$.

A coconut tree graph $CT(m, n)$ is the graph obtained from the path P_m by appending ' n ' new pendant edges at an end vertex of P_m .

Theorem 3.3.11. [32] For any coconut tree graph $CT(m, n)$, the domination number is $1 + \lceil \frac{m-2}{3} \rceil$, where $m \geq 1, n \geq 1$.

Theorem 3.3.12. *For any coconut tree graph $CT(m, n)$, the zero forcing number is n , where $m \geq 1, n \geq 1$.*

Proof. Let $v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n$ be the vertices of $CT(m, n)$, then $\{u_1, u_2, \dots, u_n\}$ forms a zero forcing set. Which is minimum since $CT(m, n)$ contains a star graph of $n + 2$ vertices. $\square$

Theorem 3.3.13. *For any coconut tree $CT(m, n)$, the dom-forcing number is $n + \lceil \frac{m-1}{3} \rceil$, where $m \geq 1, n \geq 1$.*

Proof. Let $v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n$ be the vertices of $CT(m, n)$. Then $\{u_1, u_2, \dots, u_n\}$ forms a minimum zero forcing set.

Case 1: Assume $n = 0, 1 \pmod 3$, $D_f = \{u_1, u_2, \dots, u_n, v_{3k}\}$, where $0 < k \leq \lfloor m/3 \rfloor$.

Case 2: Assume $n = 2 \pmod 3$, $D_f = \{u_1, u_2, \dots, u_n, v_{3k}, v_n\}$, where $0 < k \leq \lfloor m/3 \rfloor$.

In both cases D_f is a dom-forcing set. No set with cardinality less than $|D_f|$ have this property. Therefore, $F_d(CT(m, n)) = n + \lceil \frac{m-1}{3} \rceil$. $\square$

There are graphs having $\mathcal{Z}(\Gamma) = \gamma(\Gamma)$, but its dom-forcing number is larger than $\mathcal{Z}(\Gamma)$. For example, a diamond snake graph D_n is a connected graph obtained from a path P of length n with each edge $e = (u, v)$ in P replaced by a cycle of length 4 with u and v as non-adjacent vertices of the cycle. For any diamond snake graph, $\mathcal{Z}(D_n) = \gamma(D_n) = n + 1$ [25, 32].

Theorem 3.3.14. *For diamond snake graph*

$$F_d(D_n) = \begin{cases} \frac{3n}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $u_1, \dots, u_n, v_1, \dots, v_{n+1}, w_1, \dots, w_n$ be the vertices of diamond snake graph D_n (see Figure 3.6). Consider the following cases and the following subset of the vertex set of D_n .

Case 1: Assume n is even, $D_f = \{u_1, \dots, u_n, v_{2k}\}$, where $1 \leq k \leq \frac{n}{2}$.

Case 2: Assume n is odd, $D_f = \{u_1, \dots, u_n, v_{2k}\}$, where $1 \leq k \leq \frac{n+1}{2}$.

Now D_f forms a dom-forcing set. It can be easily verified that no set with less than $|D_f|$ vertices forms a dom-forcing set. Therefore,

$$F_d(D_n) = \begin{cases} \frac{3n}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

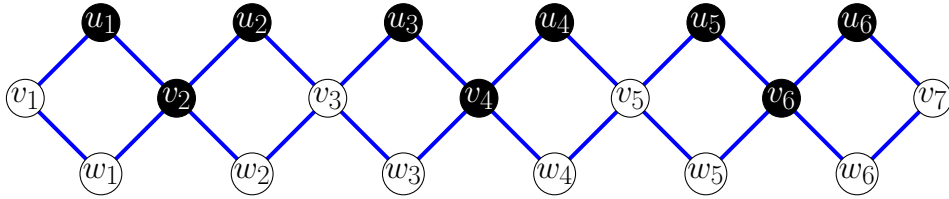


Figure 3.6: The Diamond Snake Graph D_6 . Dom-forcing set $D_f = \{u_1, u_2, \dots, u_6, v_2, v_4, v_6\}$ and $F_d(D_6) = 9$.

□

3.4 Exact Values of Dom-forcing Number of Graphs

Where $\mathcal{Z}(\Gamma) = F_d(\Gamma)$

There are graphs having the same dom-forcing number and the zero-forcing number. For example, consider the triangular snake graph, denoted by, TS_n , obtained from a path P_{n+1} by replacing each edge of the path by a triangle C_3 [33].

Theorem 3.4.1. [33] For a triangular snake graph TS_n , $\mathcal{Z}(TS_n) = n + 1$.

Theorem 3.4.2. For a triangular snake graph TS_n , $F_d(TS_n) = \mathcal{Z}(TS_n) = n + 1$.

Proof. Let $v_1, v_2, \dots, v_{n+1}, u_1, u_2, \dots, u_n$ be the vertices of the triangular snake graph TS_n (see Figure 3.7). $D_f = \{v_1, u_1, u_2, \dots, u_n\}$ form a dom-forcing set, which is minimum since $\mathcal{Z}(TS_n) = n + 1$. Therefore $F_d(TS_n) = \mathcal{Z}(TS_n) = n + 1$.

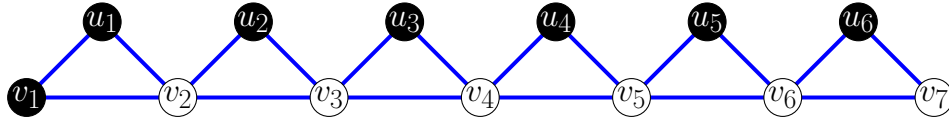


Figure 3.7: The triangular snake graph TS_6 , dom-forcing set $D_f = \{v_1, u_1, u_2, \dots, u_6\}$ and $F_d(TS_6) = 7$.

□

A hypercube of dimension k , represented by, Q_k , has vertex set $\{0, 1\}^k$, with vertices adjacent when they differ in exactly one coordinate. Or, the constructive definition:

Let Q_0 be a single vertex. For $k \geq 1$, Q_k is formed by taking two copies of Q_{k-1} and adding a matching joining the corresponding vertices in the two copies. This is equivalent to taking the Cartesian product of Q_{k-1} and K_2 to form Q_k [34]. Zero forcing number of hypercube graph Q_k is 2^{k-1} [2].

Theorem 3.4.3. For a hypercube graph Q_k , $F_d(Q_k) = \mathcal{Z}(Q_k) = 2^{k-1}$.

Proof. The set of all vertices having first coordinate zero (all vertices of Q_{k-1}) forms a zero forcing set with cardinality 2^{k-1} . Clearly, it dominates Q_k (see Figure 3.8). Hence $F_d(Q_k) = \mathcal{Z}(Q_k) = 2^{k-1}$.

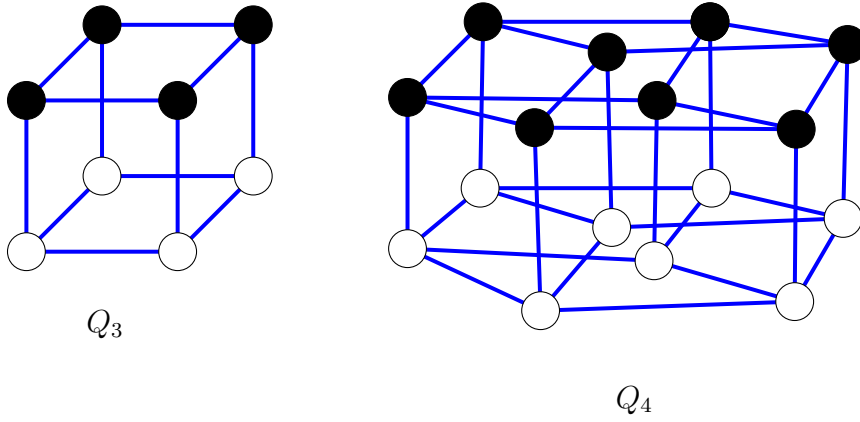


Figure 3.8: Dom-forcing sets (as well as zero forcing sets) in hypercube graphs Q_3 and Q_4 .

□

Theorem 3.4.4. *Let Γ be a graph with n vertices and $\Delta(\Gamma) = n - 1$. Then $\mathcal{Z}(\Gamma) \leq F_d(\Gamma) \leq \mathcal{Z}(\Gamma) + 1$.*

Proof. Let v be the vertex of Γ having degree $n - 1$. If the minimum zero forcing set S contains v , then $\mathcal{Z}(\Gamma) = F_d(\Gamma)$. Otherwise, $S \cup \{v\}$ forms a dom-forcing set. □

By using Theorem 3.4.4, the following results can be verified.

Theorem 3.4.5. *For a complete graph K_n , $\mathcal{Z}(K_n) = F_d(K_n) = n - 1$.*

The wheel graph, W_n , is a graph obtained by connecting a single vertex to all vertices of a cycle graph C_{n-1} .

Theorem 3.4.6. *For wheel graph W_n , $\mathcal{Z}(W_n) = F_d(W_n) = 3$.*

3.4. EXACT VALUES OF DOM-FORCING NUMBER OF GRAPHS WHERE $\mathcal{Z}(\Gamma) = F_d(\Gamma)$ 33

The dove tail graph, DT_n , is the graph $P_n + K_1$, $n \geq 2$. The dove tail graph has $n + 1$ vertices and $2n - 1$ edges [35].

Theorem 3.4.7. *Let DT_n denote the dove tail graph. Then, $\mathcal{Z}(DT_n) = F_d(DT_n) = 2$.*

In the case of complete bipartite graph every zero forcing set dominates the entire graph and, hence, the following result.

Theorem 3.4.8. *Let $K_{m,n}$ denote the complete bipartite where $n, m \geq 2$. Then $\mathcal{Z}(K_{m,n}) = F_d(K_{m,n}) = m + n - 2$.*

From Theorems 3.3.1, 3.3.3, 3.3.6 and 3.3.7 here are the following observations.

Observation 3.4.9. *For a path P_n and a cycle C_n ,*

- $\mathcal{Z}(P_n) = F_d(P_n)$ if and only if $n = 1$ or 2 .
- $\mathcal{Z}(C_n) = F_d(C_n)$ if and only if $n = 3$ or 4 .

Theorem 3.4.10. [30] *For the star graph $K_{1,n}$, $\gamma(K_{1,n}) = 1$ and $\mathcal{Z}(K_{1,n}) = n - 1$, for $n \geq 2$.*

Theorem 3.4.11. *For the star graph $K_{1,n}$, $F_d(K_{1,n}) = n$, for $n \geq 2$.*

Proof. It is noted that the star graph $K_{1,n}$ have n vertices of degree 1 and one vertex of degree n . Any minimum zero forcing set does not contain the vertex of degree n , and $\mathcal{Z}(K_{1,n}) = n - 1$. Therefore by Theorem 3.4.4, $F_d(K_{1,n}) = n$.

□

The pineapple graph, denoted by K_m^n , is the graph formed by coalescing any vertex of the complete graph K_m with the star graph $K_{1,n}$, ($m \geq 3, n \geq 2$). The number of vertices in K_m^n is $m + n$ and the number of edges in K_m^n is $\frac{m^2 - m + 2n}{2}$ [36].

Theorem 3.4.12. *Let Γ be the pineapple graph K_m^n with $n \geq 2, m \geq 3$. Then $F_d(\Gamma) = m + n - 2$.*

Proof. The pineapple graph comprises $m + n$ vertices. In [25], the zero forcing number of the pineapple graph is $m + n - 3$, when $n \geq 2, m \geq 3$. Pineapple graph has a vertex v with degree $m + n - 1$, and the vertex v does not belong to any minimum zero forcing set. Therefore by Theorem 3.4.4, $F_d(\Gamma) = m + n - 3 + 1 = m + n - 2$. $\square$

Definition 3.4.13. [37] Let $\Gamma = (V, E)$ be a graph and B a zero forcing set of Γ . Define $B^{(0)} = B$, and for $t \geq 0$, $B^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \bigcup_{s=0}^t B^{(s)}$ such that w is the only neighbor of b not in $\bigcup_{s=0}^t B^{(s)}$. The propagation time of B in Γ , denoted $pt(\Gamma, B)$, is the smallest integer t_0 such that $V = \bigcup_{s=0}^{t_0} B^{(s)}$. Two minimum zero forcing sets of the same graph may have different propagation times. The minimum propagation time of Γ is

$$pt(\Gamma) = \min\{pt(\Gamma, B) \mid B \text{ is a minimum zero forcing set of } \Gamma\}.$$

Example 3.4.14. Let Γ be the graph in Figure 3.9. Let $B_1 = \{v_1, v_2\}$ and $B_2 = \{v_5, v_8\}$. Then $B_1^{(1)} = \{v_3, v_6\}$, $B_1^{(2)} = \{v_4\}$, $B_1^{(3)} = \{v_5\}$, $B_1^{(4)} = \{v_7\}$, and $B_1^{(5)} = \{v_8\}$, so $pt(\Gamma, B_1) = 5$. However $B_2^{(1)} = \{v_7\}$, $B_2^{(2)} = \{v_6\}$, $B_2^{(3)} = \{v_1, v_4\}$, and $B_2^{(4)} = \{v_2, v_3\}$, so $pt(\Gamma, B_2) = 4$.

3.4. EXACT VALUES OF DOM-FORCING NUMBER OF GRAPHS WHERE $\mathcal{Z}(\Gamma) = F_d(\Gamma)$ 35

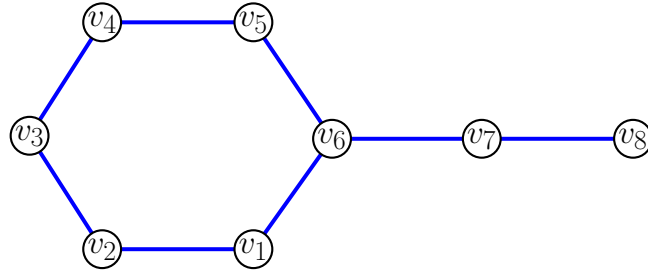


Figure 3.9: Graph for Example 3.4.14.

Theorem 3.4.15. *Let $\Gamma = (V, E)$ be a graph with minimum propagation time one. Then, $F_d(\Gamma) = \mathcal{Z}(\Gamma)$.*

Proof. From the definition of minimum propagation time, there exist a minimum zero forcing set B such that $V = B \cup B^{(1)}$. Every vertex of $B^{(1)}$ is adjacent to some vertex in B , so B dominates Γ . Hence B is a dom-forcing set which is minimum. $\square$

The converse of the above theorem is false. For example, let Γ be the wheel graph with six vertices. Then $F_d(\Gamma) = \mathcal{Z}(\Gamma)$, but minimum propagation time 2.

Theorem 3.4.16. *Let P denote the Petersen graph shown in Figure 3.10. Then, $F_d(P) = 5 = \mathcal{Z}(P)$.*

Proof. The minimum propagation time of the Petersen graph is one and minimum zero forcing number is five [2]. Hence by theorem 3.4.15, here is the result.

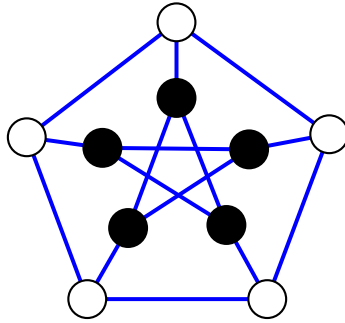


Figure 3.10: Dom-forcing set and zero forcing set for the Petersen graph.

□

3.5 Graphs Where $\gamma(\Gamma) = F_d(\Gamma)$

There are some graphs with $\gamma(\Gamma) = F_d(\Gamma)$. For example, a helm graph is a graph that is created by attaching a pendant edge to each vertex of an n -wheel graph's cycle, and is denoted by H_m , where $m \geq 4$ [38].

Theorem 3.5.1. [38] Let H_m denote the helm graph. Then the domination number of graph H_m is m .

Theorem 3.5.2. The dom-forcing number of helm graph H_m is m . ie $F_d(H_m) = m = \gamma(H_m)$.

Proof. Let v_0 denote the vertex in helm graph such that its degree is equal to m , and let $v_1, v_2, \dots, v_m$ represent the vertices in the helm graph, each with a degree of 3. Also $u_1, u_2, \dots, u_m$ represent the pendant vertices of the helm graph H_m (See Figure 3.11). It can be easily verified that $D_f = \{v_1, u_2, u_3, \dots, u_m\}$

is a dom-forcing set of H_m , which is minimum since $\gamma(H_m) = m$. Therefore, $F_d(H_m) = m = \gamma(H_m)$.

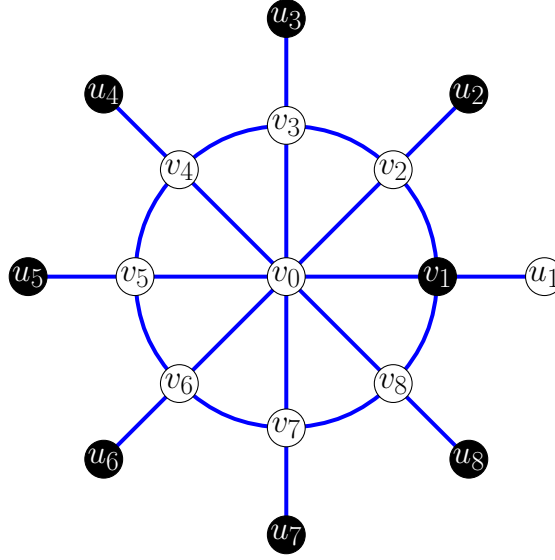


Figure 3.11: The Helm graph H_8 , dom-forcing set $D_f = \{v_1, u_2, u_3, \dots, u_8\}$ and $F_d(H_8) = 8$.

□

From Theorem 3.3.1, 3.3.3, 3.3.6 and 3.3.7, the following results can be derived.

Theorem 3.5.3. *For any path P_n , $F_d(P_n) = \gamma(P_n)$ if and only if n is not a multiple of three.*

Theorem 3.5.4. *For any cycle C_n , $F_d(C_n) = \gamma(C_n)$ if and only if $n = 3k + 1$, where k is any positive integer.*

3.6 Dom-forcing Number of Splitting Graph of a Graph Γ

The splitting graph of a graph Γ is the graph $S(\Gamma)$ obtained by taking a vertex v' corresponding to each vertex $v \in \Gamma$ and join v' to all vertices of Γ adjacent to v [39].

Theorem 3.6.1. *Let Γ be a connected graph of order $n > 1$ and $S(\Gamma)$ be its splitting graph. Then, $F_d[S(\Gamma)] \leq 2F_d(\Gamma)$.*

Proof. Consider any minimum dom-forcing set D_f of Γ . Let $D_f = \{v_1, v_2, \dots, v_k\}$ for $1 \leq k < n$ be a minimum zero forcing set of Γ . Now consider the set $D'_f = \{v_1, v_2, \dots, v_k\} \cup \{u_1, u_2, \dots, u_k\}$, where $u_1, u_2, \dots, u_k$ are vertices corresponding to $v_1, v_2, \dots, v_k$ which are added to obtain $S(\Gamma)$. As in [30], D'_f forces $S(\Gamma)$. From the definition of splitting graph $N(u_1, u_2, \dots, u_k) = N(D_f)$, where $N(D_f)$ denotes the vertices which are adjacent to $\{v_1, v_2, \dots, v_k\}$, and $N[v_1, v_2, \dots, v_k] = V(S(\Gamma)) - \{u_1, u_2, \dots, u_k\}$. Hence D'_f dominates $S(\Gamma)$, it is a dom-forcing set. Therefore, $F_d[S(\Gamma)] \leq 2F_d(\Gamma)$. $\square$

Definition 3.6.2. [24] Given a graph $\Gamma = (V, E)$, a path cover is a set of disjoint induced paths in Γ such that every vertex $v \in V$ belongs to exactly one path. The path cover number $P(\Gamma)$ is the minimum number of paths in a path cover.

Theorem 3.6.3. [24] Let Γ be a graph. If $X \subset V(\Gamma)$ is a zero forcing set for Γ , then X induces a path cover for Γ and $P(\Gamma) \leq \mathcal{Z}(\Gamma)$.

Theorem 3.6.4. [40, 30] Let $S(P_n)$ denote the splitting graph of the path P_n .

$$\text{Then, } \mathcal{Z}[S(P_n)] = 2\mathcal{Z}(P_n) = 2 \text{ and } \gamma[S(P_n)] = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4} \end{cases}.$$

Theorem 3.6.5. *Let $S(P_n)$ denote the splitting graph of the path P_n . Then for $2 \leq n \leq 4$, $F_d[S(P_n)] = n$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the graph P_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(P_n)$. Depending upon the number of vertices of P_n , consider the following subsets of the vertex set of $S(P_n)$.

For $S(P_2)$, $D_f = \{u_1, v_1\}$, for $S(P_3)$, $D_f = \{u_1, v_1, v_2\}$, D_f forms a dom-forcing set which is minimum. For $S(P_4)$, $D_f = \{u_1, v_1, v_4, u_4\}$ forms a dom-forcing set, therefore $F_d[S(P_4)] \leq 4$. The domination number of $S(P_4)$ is 2. Hence $2 \leq F_d[S(P_4)]$. Let $A = \{v_2, v_3\}$ be a minimum dominating set of $S(P_4)$. Then $S(P_4) - A$ has four components (two components contains only single vertex and two of them is a path of length one), hence the set A cannot induces a path cover for $S(P_4)$. Hence $F_d[S(P_4)]$ cannot be 2. Claim the following

Claim 3.6.6. *Any dominating set of cardinality 3 must contain A .*

Proof of the Claim: Assume that there exists a dominating set D of $S(P_4)$ which does not contain A and the cardinality of D is 3. In $S(P_4)$, $\deg(u_1) = \deg(u_4) = 1$, and v_2, v_3 be the vertices having degree 4. All other vertices are of degree 2. Then consider the following cases.

Case 1: D contains a vertex of degree 4.

Then D must contain any one of the pendant vertices and the remaining non-dominating vertices are of degree two and which are non-adjacent. Hence, it is

concluded that a dominating set of cardinality 3 cannot be formed.

Case 2: D cannot contain a vertex of degree 4

Then D must contain the pendant vertices u_1, u_4 , and all other vertices are of degree two. Hence, a dominating set of cardinality 3 cannot be formed.

In both cases there is a contradiction, hence our assumption is wrong, and the claim is proved.

By the above claim $F_d[S(P_4)]$ cannot be 3. Therefore, $F_d[S(P_4)] = 4$. That is $F_d[(S(P_n))] = n$, for $2 \leq n \leq 4$. $\square$

Let me now consider the case where the number of vertices in the splitting graph of the path is ≥ 5 .

Theorem 3.6.7. *For $n \geq 5$, let $S(P_n)$ be the splitting graph of the path P_n .*

Then

$$\begin{aligned} \frac{n+2}{2} \leq F_d[(S(P_n))] &\leq \frac{n+4}{2} \quad \text{if } n \equiv 0 \pmod{4} \\ F_d[(S(P_n))] &= \frac{n+3}{2} \quad \text{if } n \equiv 1, 3 \pmod{4} \\ F_d[(S(P_n))] &= \frac{n+2}{2} \quad \text{if } n \equiv 2 \pmod{4}. \end{aligned}$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the graph P_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(P_n)$. At least one vertex from each pair u_1, v_2 and u_n, v_{n-1} belong to any dominating set, since $N(u_1) = \{v_2\}$ and $N(u_n) = \{v_{n-1}\}$. Also at least one vertex from v_{i-1}, v_{i+1} belongs to the dominating set, since $N(u_i) = \{v_{i-1}, v_{i+1}\}$ and $N(v_i) = \{v_{i-1}, v_{i+1}, u_{i-1}, u_{i+1}\}$. Thus any dominating set contains at least $\frac{n}{2}$ number of vertices. The set $\{v_1, u_1\}$ forces $S(P_n)$. Now depending upon the number of vertices of P_n , consider the following subsets of the vertex set of

$S(P_n)$.

For $n \geq 5$, consider the following cases.

Case 1: Assume $n \equiv 0 \pmod{4}$, $D_f = \{u_1, v_1, v_{4k}, v_{4k+1}, v_n, u_n\}$ where $0 < k < \lceil \frac{n}{4} \rceil$ forms a dom-forcing set, therefore $F_d[S(P_n)] \leq \frac{n+4}{2}$. The domination number of $S(P_n)$ is $\frac{n}{2}$. Hence $\frac{n}{2} \leq F_d[S(P_n)]$. Let A be a minimum dominating set of $S(P_n)$. Then $S(P_n) - A$ has $3 + \frac{n}{4}$ components (two components contain only a single vertex, two of them is the path of length one and $\frac{n}{4} - 1$ components having six vertices which include four pendant vertices), the set A cannot induce a path cover for $S(P_n)$. Hence $F_d[S(P_n)]$ cannot be $\frac{n}{2}$. Therefore, $\frac{n+2}{2} \leq F_d[S(P_n)] \leq \frac{n+4}{2}$.

Case 2: Assume $n \equiv 1, 2 \pmod{4}$, $D_f = \{u_1, v_1, v_{4k}, v_{4k+1}\}$ where $0 < k < \lceil \frac{n}{4} \rceil$, forms a dom-forcing set, therefore

$$F_d[S(P_n)] \leq \begin{cases} \frac{n+3}{2} & \text{if } n \equiv 1 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

If $n \equiv 2 \pmod{4}$, then the domination number of $S(P_n)$ is $\frac{n+2}{2}$. Hence $F_d[S(P_n)] = \frac{n+2}{2} = |D_f|$. If $n \equiv 1 \pmod{4}$, then the domination number of $S(P_n)$ is $\frac{n+1}{2}$. Hence $F_d[S(P_n)] \geq \frac{n+1}{2}$. Let A be a minimum dominating set of $S(P_n)$. Then $S(P_n) - A$ has $4 + \frac{n-1}{4}$ components (three components contain only a single vertex, two of them are paths of length one and $\frac{n-1}{4} - 1$ components having six vertices which include four pendant vertices), the set A cannot induce a path cover for $S(P_n)$. Hence $F_d[S(P_n)]$ cannot be $\frac{n+1}{2}$. Therefore, $F_d[S(P_n)] = \frac{n+3}{2}$.

Case 3: Assume $n \equiv 3 \pmod{4}$, $D_f = \{u_1, v_1, v_{4k}, v_{4k+1}, v_{n-1}\}$ where $0 < k < \lceil \frac{n}{4} \rceil$

form a dom-forcing set, therefore $F_d[S(P_n)] \leq \frac{n+3}{2}$. The domination number of $S(P_n)$ is $\frac{n+1}{2}$. Hence $F_d[S(P_n)] \geq \frac{n+1}{2}$. Let A be a minimum dominating set of $S(P_n)$. Then $S(P_n) - A$ has at least $3 + \frac{n-3}{4}$ components, the set A cannot induce a path cover for $S(P_n)$. Hence $F_d[S(P_n)]$ cannot be $\frac{n+1}{2}$. Therefore, $F_d[S(P_n)] = \frac{n+3}{2}$.

From all the cases, for $n \geq 5$,

$$\begin{aligned} \frac{n+2}{2} \leq F_d[S(P_n)] &\leq \frac{n+4}{2} \quad \text{if } n \equiv 0 \pmod{4} \\ F_d[S(P_n)] &= \frac{n+3}{2} \quad \text{if } n \equiv 1, 3 \pmod{4} \\ F_d[S(P_n)] &= \frac{n+2}{2} \quad \text{if } n \equiv 2 \pmod{4}. \end{aligned}$$

□

Theorem 3.6.8. [40, 30] Let $S(C_n)$ denote the splitting graph of the cycle C_n .

$$\text{Then } \mathcal{Z}(S(C_n)) = 2\mathcal{Z}(C_n) = 4 \text{ and } \gamma(S(C_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

$$\text{Theorem 3.6.9. For } n \geq 4, F_d[S(C_n)] \leq \begin{cases} \frac{n+4}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+5}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+6}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proof. For $n \geq 4$, let $v_1, v_2, \dots, v_n$ be the vertices of the graph C_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(C_n)$. It can be seen that $\{v_1, v_2, u_2, u_3\}$ forms a zero forcing set. Now depending upon the number of vertices of C_n , consider the following cases and following subsets of the vertex set of $S(C_n)$.

Case 1: Assume $n \equiv 0, 2, 3 \pmod{4}$, $D_f = \{u_2, u_3, v_{4k+1}, v_{4k+2}\}$, where $0 \leq k < \lceil \frac{n}{4} \rceil$

Case 2: Assume $n \equiv 1 \pmod{4}$, $D_f = \{u_2, u_3, v_{4k+1}, v_{4k+2}, v_n\}$, where $0 \leq k < \frac{n-1}{4}$.

In both cases D_f forms a dom-forcing set. Hence

$$F_d[S(C_n)] \leq \begin{cases} \frac{n+4}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+5}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+6}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

□

Theorem 3.6.10. [30] Let $S(K_{1,n})$ denote the splitting graph of a Star Graph $K_{1,n}$. Then, for $n \geq 2$, $\gamma[S(K_{1,n})] = 2$ and $\mathcal{Z}[S(K_{1,n})] = 2n - 2$.

Theorem 3.6.11. Let $S(K_{1,n})$ denote the splitting graph of a Star Graph $K_{1,n}$. Then, for $n \geq 2$, $F_d[S(K_{1,n})] = 2n - 1$.

Proof. For $n \geq 2$, let $u_1, v_1, v_2, \dots, v_n$ be the vertices of the star graph $K_{1,n}$ with $\deg(u_1) = n$ and $u'_1, v'_1, v'_2, \dots, v'_n$ be the vertices corresponding to $u_1, v_1, v_2, \dots, v_n$ which are added to obtain $S(K_{1,n})$. From the above theorem, $\mathcal{Z}[S(K_{1,n})] = 2n - 2$ and $B = \{v_2, \dots, v_n, v'_2, \dots, v'_n\}$ forms a zero forcing set which is minimum and any minimum zero forcing set does not contain u_1 . B is not a dom-forcing set. So adding u_1 to B , the considered dom-forcing set which is minimum. Hence $F_d[S(K_{1,n})] = 2n - 1$. □

Theorem 3.6.12. [30] Let $S(L_n)$ denote the splitting graph of ladder graph L_n . Then for $n \geq 2$, $\mathcal{Z}[S(L_n)] = 4$.

Theorem 3.6.13. Let $S(L_n)$ denote the splitting graph of ladder graph L_n . Then, $\gamma[S(L_n)] = 2\lceil \frac{n}{3} \rceil$.

Proof. Let $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ be the vertices of the ladder graph L_n and let $\{u'_1, u'_2, \dots, u'_n, v'_1, v'_2, \dots, v'_n\}$ be the vertices corresponding to $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ which are added to obtain $S(L_n)$. At least one vertex from $\{u'_1, u_2, v_1\}$, $\{v'_1, v_2, u_1\}$, $\{u'_n, u_{n-1}, v_n\}$, and $\{v'_n, v_{n-1}, u_n\}$ belonging to any dominating set, since $N(u'_1) = \{u_2, v_1\}$, $N(v'_1) = \{v_2, u_1\}$, $N(u'_n) = \{u_{n-1}, v_n\}$, and $N(v'_n) = \{v_{n-1}, u_n\}$. For $0 < k < \lceil \frac{n}{3} \rceil$ at least one vertex from $\{u_{3k}, u_{3k+2}\}$ and $\{v_{3k}, v_{3k+2}\}$ belong to any dominating set, since

$$N(u_{3k+1}) = \{u_{3k}, u_{3k+2}, v_{3k+1}, u'_{3k}, u'_{3k+2}, v'_{3k+1}\},$$

$$N(v_{3k+1}) = \{v_{3k}, v_{3k+2}, u_{3k+1}, v'_{3k}, v'_{3k+2}, u'_{3k+1}\},$$

$$N(u'_{3k+1}) = \{u_{3k}, u_{3k+2}, v_{3k+1}\},$$

$$N(v'_{3k+1}) = \{v_{3k}, v_{3k+2}, u_{3k+1}\}.$$

Thus, any dominating set contains at least $2\lceil \frac{n}{3} \rceil$ number of vertices. For $S(L_1)$, the set $\{u_1, v_2\}$ be a minimum dominating set. For $n \geq 2$, depending upon the number of vertices of L_n , consider the following subset.

For $n \equiv 0, 2 \pmod{3}$, $S = \{u_{3k+2}, v_{3k+2}\}$, where $0 \leq k < \lceil \frac{n}{3} \rceil$ and for $n \equiv 1 \pmod{3}$, $S = \{u_{3k+2}, v_{3k+2}, u_n, v_n\}$, where $0 \leq k < \lceil \frac{n-1}{3} \rceil$. Now the set S forms a dominating set. So $\gamma[S(L_n)] = 2\lceil \frac{n}{3} \rceil$. $\square$

Theorem 3.6.14. For $n \geq 2$,

$$2\lceil \frac{n}{3} \rceil \leq F_d[S(L_n)] \leq 2 + 2\lceil \frac{n}{3} \rceil.$$

Proof. Let $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ be the vertices of the ladder graph L_n , and let $\{u'_1, u'_2, \dots, u'_n, v'_1, v'_2, \dots, v'_n\}$ be the vertices corresponding to $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ which are added to obtain $S(L_n)$. For $S(L_1)$,

$\{u_1, v_2\}$ be a minimum dom-forcing set. For $n \geq 2$, depending upon the number of vertices of L_n , consider the following cases and following subsets of the vertex set of $S(L_n)$.

Case 1: For $n \equiv 0, 2 \pmod{3}$, $D_f = \{u'_1, v'_1, u_{3k+2}, v_{3k+2}\}$, where $0 \leq k < \lceil \frac{n}{3} \rceil$ and

Case 2: For $n \equiv 1 \pmod{3}$, $D_f = \{u'_1, v'_1, u_{3k+2}, v_{3k+2}, u_n, v_n\}$, where $0 \leq k < \lceil \frac{n-1}{3} \rceil$.

Now D_f forms a dom-forcing set. Therefore, $F_d[S(L_n)] \leq 2 + 2\lceil \frac{n}{3} \rceil$. Domination number of $S(L_n)$ is $2\lceil \frac{n}{3} \rceil$. Hence

$$2\lceil \frac{n}{3} \rceil \leq F_d[S(L_n)] \leq 2 + 2\lceil \frac{n}{3} \rceil.$$

□

This Chapter elaborated the concept of dom-forcing with illustrations. The next Chapter continues it by explaining how different graph operations affect the dom-forcing number.

Chapter 4

DOM-FORCING SETS AND GRAPH OPERATIONS

This chapter looks at how different graph operations affect the dom-forcing number, a key parameter in graph theory that relates to domination sets in graphs. The operations considered are vertex deletion, edge deletion and vertex addition and the sum and product of graphs, each of which can change the structure and properties of a graph. It contains three sections, Section 1 introduces it, Section 2 provides upper and lower bounds for the dom-forcing number in the context of edge operations, and Section 3, enplanes the dom-forcing number for different types of graph products including corona products and rooted products.

The study investigate how the dom-forcing number of a graph Γ changes under edge operations such as edge deletion and edge contraction In addition, it examines vertex-based operations, specifically vertex deletion and the vertex sum of graphs, and analyze their impact on the dom-forcing number. The thesis pro-

vides theoretical results and shows how these graph operations lead to different dom-forcing numbers.

4.1 Introduction

The concept of dom-forcing number is introduced in Chapter 3. In this chapter, the effect of different graph operations on the dom-forcing number is studied. By analyzing how these operations influence the dom-forcing number, the study aim to gain deeper insight into the behavior and structural properties of dom-forcing sets across various classes of graphs.

4.2 Dom-forcing Number and Graph Operations

In this section, the thesis examine how various graph operations influence the dom-forcing number. Specifically, it analyzes the effects of vertex deletion, where a single vertex is removed, and edge deletion, where an edge between two vertices is removed. It also studies vertex addition by introducing new vertices and observing how these modifications change the structure of the graph.

Furthermore, the analysis explores graph sum and graph product operations to understand how combining two graphs impacts the dom-forcing number. Overall, this section aims to provide a clear understanding of the relationship between these graph operations and the behavior of the dom-forcing number. The study considers edge operations and vertex operations in the analysis of graph opera-

tions.

4.2.1 Edge Operations

Here, the thesis analyses the effect of edge deletion in dom-forcing number. Removing an edge can change the dom-forcing number by at most one. Consider the following examples.

Decrease

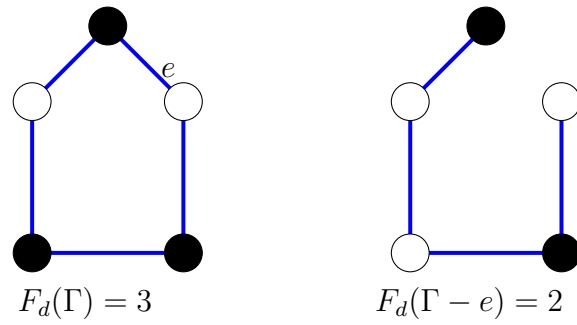


Figure 4.1: Removing an edge lowers the dom-forcing number.

No Change

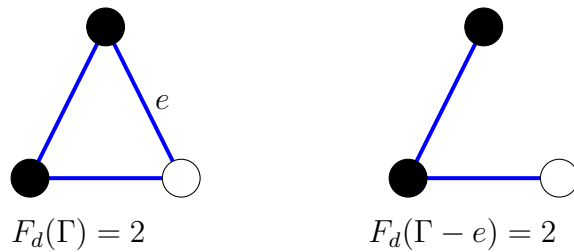


Figure 4.2: Removing an edge does not alter the dom-forcing number.

Increase

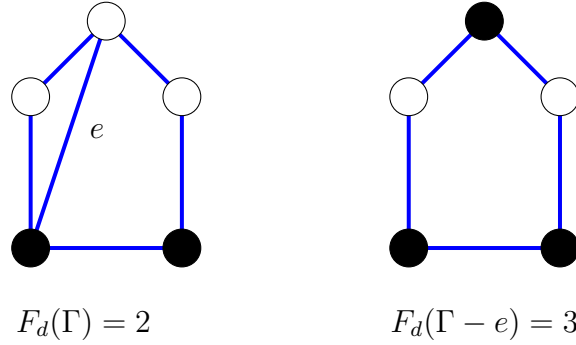


Figure 4.3: Removing an edge raises the dom-forcing number.

Theorem 4.2.1. [24] Let Γ be a connected graph. If $e = uv$ represents an edge of Γ , then

$$\mathcal{Z}(\Gamma) - 1 \leq \mathcal{Z}(\Gamma - e) \leq \mathcal{Z}(\Gamma) + 1$$

Theorem 4.2.2. [41] Let $e = uv$ be an edge in a graph Γ . If there is a minimum zero forcing process for Γ in which neither u forces v nor v forces u , then $\mathcal{Z}(\Gamma) - 1 \leq \mathcal{Z}(\Gamma - e) \leq \mathcal{Z}(\Gamma)$. Otherwise, $\mathcal{Z}(\Gamma) \leq \mathcal{Z}(\Gamma - e) \leq \mathcal{Z}(\Gamma) + 1$.

Theorem 4.2.3. [42] Let Γ be a connected graph. If $e = uv$ represents an edge of Γ , then

$$\gamma(\Gamma) \leq \gamma(\Gamma - e) \leq \gamma(\Gamma) + 1.$$

Theorem 4.2.4. Let Γ be a connected graph. If $e = uv$ represents an edge of Γ , then

$$F_d(\Gamma) - 1 \leq F_d(\Gamma - e) \leq F_d(\Gamma) + 1.$$

Proof. Let D_f be the minimum dom-forcing set of Γ . Consider the following cases.

Case 1: If both u and v are in D_f , then deletion of e does not affect the

domination property of $\Gamma - e$, forcing number may be differ by one (Theorem 4.2.2). So $F_d(\Gamma) - 1 \leq F_d(\Gamma - e) \leq F_d(\Gamma)$.

Case 2: If either u or v is not in D_f and D_f is a dom-forcing set for $\Gamma - e$ then $F_d(\Gamma) - 1 \leq F_d(\Gamma - e) \leq F_d(\Gamma)$. If either u or v is not in D_f and D_f is not a dom-forcing set for $\Gamma - e$, then $D_f \cup \{v\}$ (or $D_f \cup \{u\}$) forms a dom-forcing set of $\Gamma - e$.

Case 3: If both u and v are not in D_f , then the deletion of e does not affect the domination of $\Gamma - e$, forcing number may be differ by one (Theorem 4.2.2). Therefore, $F_d(\Gamma) - 1 \leq F_d(\Gamma - e) \leq F_d(\Gamma) + 1$. Therefore, in all cases

$$F_d(\Gamma) - 1 \leq F_d(\Gamma - e) \leq F_d(\Gamma) + 1.$$

□

The dom-forcing number can increase or decrease due to edge contraction depending on the original graph and the edge being contracted. This shows how graph properties are connected and complex in relation to edge operations.

Theorem 4.2.5. [24] Consider an edge e that connects two vertices, u and v , in a graph Γ . Then,

$$\mathcal{Z}(\Gamma) - 1 \leq \mathcal{Z}(\Gamma/e) \leq \mathcal{Z}(\Gamma) + 1.$$

Theorem 4.2.6. Let $e = uv$ be an edge in a graph Γ . Then

$$F_d(\Gamma) - 1 \leq F_d(\Gamma/e) \leq F_d(\Gamma) + 1.$$

Proof. Let D_f be the minimum dom-forcing set of Γ . Consider the following cases.

Case 1: If both u and v are in D_f , then $D_f - \{u\}$ (or $D_f - \{v\}$) dominates Γ/e .

So by theorem 4.2.5, $F_d(\Gamma) - 1 \leq F_d(\Gamma/e)$.

Case 2: If either u or v is not in D_f , and D_f is a dom-forcing set for Γ/e , then $F_d(\Gamma) - 1 \leq F_d(\Gamma/e)$. If either u or v is not in D_f and D_f is not a dom-forcing set for Γ/e , then $F_d(\Gamma/e) \leq F_d(\Gamma) + 1$.

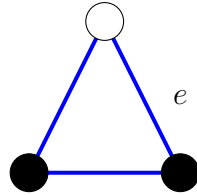
Case 3: If both u and v not in D_f , then D_f dominates Γ/e . So by Theorem 4.2.5. $F_d(\Gamma) - 1 \leq F_d(\Gamma/e) \leq F_d(\Gamma) + 1$. Therefore, in all cases

$$F_d(\Gamma) - 1 \leq F_d(\Gamma/e) \leq F_d(\Gamma) + 1.$$

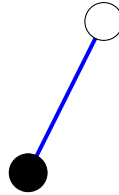
□

The following examples illustrate each of the three possible cases.

Decrease



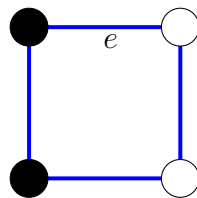
$$F_d(G) = 2$$



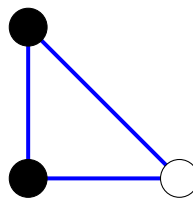
$$F_d(G/e) = 1$$

Figure 4.4: Contracting an edge reduces the dom-forcing number.

No change



$$F_d(\Gamma) = 2$$



$$F_d(\Gamma/e) = 2$$

Figure 4.5: Contracting an edge does not alter the dom-forcing number.

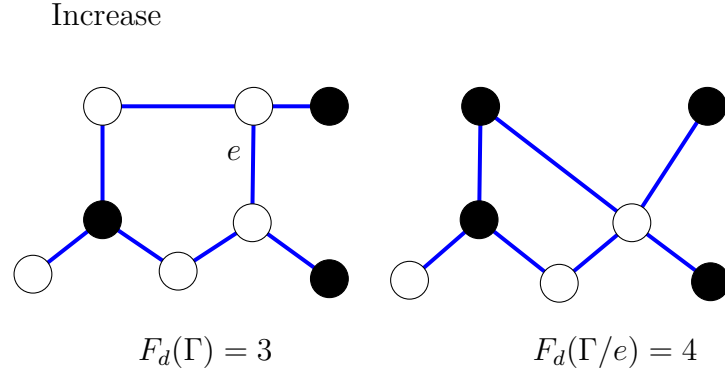


Figure 4.6: Contracting an edge raises the dom-forcing number.

Subdividing an edge may increase the dom-forcing number.

Theorem 4.2.7. [24] If K is obtained from Γ by subdividing an edge $e = uv$, then

$$\mathcal{Z}(\Gamma) \leq \mathcal{Z}(K) \leq \mathcal{Z}(\Gamma) + 1.$$

Theorem 4.2.8. If K is obtained from Γ by subdividing an edge $e = uv$, then

$$F_d(\Gamma) \leq F_d(K) \leq F_d(\Gamma) + 1.$$

Proof. Let D_f be the minimum dom-forcing set for Γ . If D_f is a dom-forcing set for K , then $F_d(\Gamma) = F_d(K)$. Otherwise $D_f \cup \{v'\}$, where v' is the new vertex added, forms a dom-forcing set for K . Hence $F_d(K) \leq F_d(\Gamma) + 1$. $\square$

The following are examples of the two possible cases in the Theorem above.

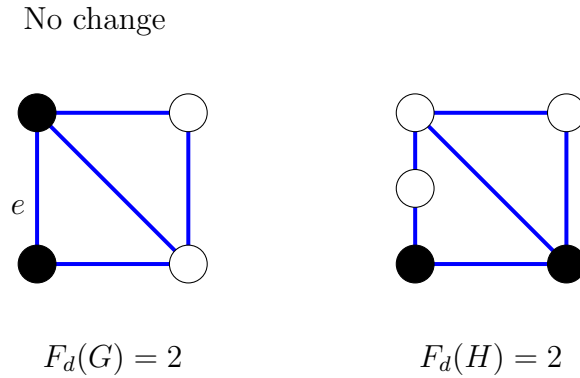


Figure 4.7: An example of a graph where edge subdivision leaves the dom-forcing number unchanged.

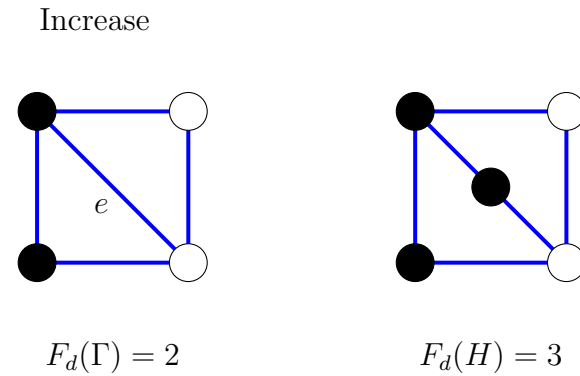


Figure 4.8: An example of a graph where subdividing an edge increases the dom-forcing number.

4.2.2 Vertex Operations

This section discusses the effect of vertex operations on dom-forcing number.

It is observed that the removal of a vertex from a graph can either increase or decrease its dom-forcing number. It is noted that there exist graphs for which $F_d(\Gamma)$ and $F_d(\Gamma - u)$ differ by an arbitrarily large amount. As an illustration, the

wheel graph W_n of order n is considered. It has already been shown in Chapter 3 that $F_d(W_n) = 3$. When the vertex of degree $n - 1$ is removed, the resulting graph becomes isomorphic to C_{n-1} , and hence $F_d(W_n - u) = F_d(C_{n-1})$. The following examples are then considered.

Decrease

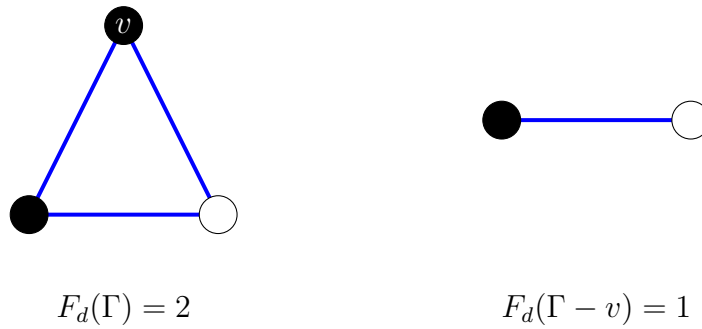


Figure 4.9: Vertex deletion decreases the dom-forcing number.

No change

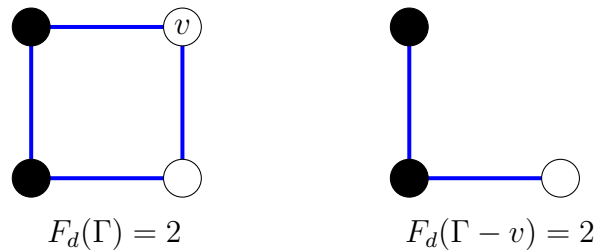


Figure 4.10: Vertex deletion does not change the dom-forcing number.

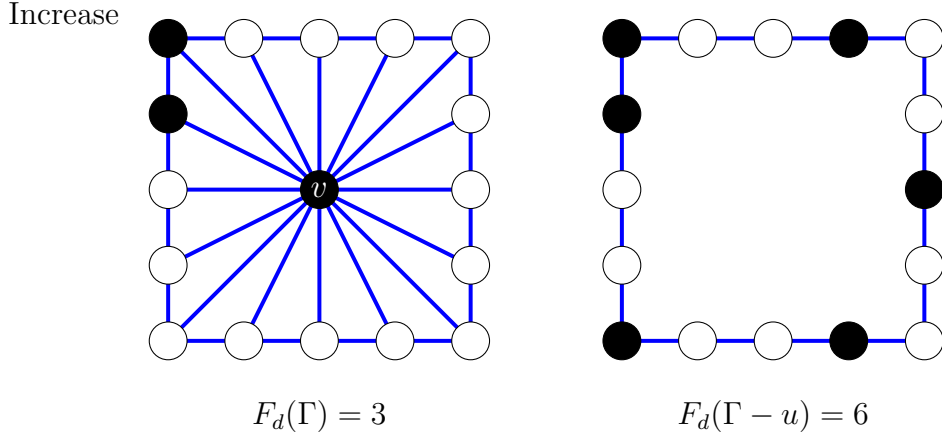


Figure 4.11: $F_d(\Gamma)$ and $F_d(\Gamma - u)$ can be arbitrary far apart.

Theorem 4.2.9. [24] Let u be a vertex of the graph Γ . Then

$$\mathcal{Z}(\Gamma - u) - 1 \leq \mathcal{Z}(\Gamma) \leq \mathcal{Z}(\Gamma - u) + 1.$$

Theorem 4.2.10. Consider a graph Γ with a vertex labeled as v . If there is a minimum dom-forcing set of Γ not including v , then

$$F_d(\Gamma) - 1 \leq F_d(\Gamma - v) \leq F_d(\Gamma) + 1.$$

Proof. Let D_f be a minimum dom-forcing set of Γ and $v \notin D_f$. Then the deletion of v does not affect the domination, therefore D_f dominates $\Gamma - v$. By Theorem 4.2.9, the zero forcing number of $\Gamma - v$ differ by maximum of one. If D_f forces $\Gamma - v$, then $F_d(\Gamma) - 1 \leq F_d(\Gamma - v) \leq F_d(\Gamma)$. If D_f does not forces $\Gamma - v$, adding one more vertex to D_f , by [24], got a new set that forces $\Gamma - v$, then $F_d(\Gamma) \leq F_d(\Gamma - v) \leq F_d(\Gamma) + 1$. Hence $F_d(\Gamma) - 1 \leq F_d(\Gamma - v) \leq F_d(\Gamma) + 1$. $\square$

Corollary 4.2.11. *Let Γ be a graph with a vertex labeled v . If $F_d(\Gamma - v) > F_d(\Gamma) + 1$, then every minimum dom-forcing set of Γ contains the vertex v .*

Let Γ and K be two simple undirected graphs with $V(\Gamma) \cap V(K) = \emptyset$ and $E(\Gamma) \cap E(K) = \emptyset$. If $v_1 \in V(\Gamma)$ and $v_2 \in V(K)$, the process of identifying the vertices v_1 and v_2 forms the vertex-sum of the graphs Γ and K at a new vertex u , denoted as $\Gamma +_u K$. The resulting simple graph has the vertex set $V(\Gamma) \cup (V(K) \setminus \{v_1, v_2\}) \cup \{u\}$ and the edge set $E(\Gamma) \cup E(K)$. The open neighborhood of u , denoted $N(u)$, is given by $N(u) = N(v_1) \cup N(v_2)$ [41].

Theorem 4.2.12. [41] Let Γ and K be two graphs each with a vertex labeled u . Then the following results hold.

1. If u is in some minimal zero forcing set of K but not in any minimal zero forcing set of Γ , then

$$\mathcal{Z}(\Gamma +_u K) = \mathcal{Z}(\Gamma) + \mathcal{Z}(K).$$

2. If u is in some minimal zero forcing set of K and some minimal zero forcing set of Γ , then

$$\mathcal{Z}(\Gamma +_u K) = \mathcal{Z}(\Gamma) + \mathcal{Z}(K) - 1.$$

3. If u is neither in any minimal zero forcing set of K nor in any minimal zero forcing set of Γ , then

$$\mathcal{Z}(\Gamma +_u K) = \mathcal{Z}(\Gamma) + \mathcal{Z}(K) + 1.$$

Theorem 4.2.13. *Let Γ and K be two graphs each with a vertex labeled v . Then,*

$$F_d(\Gamma) + F_d(K) - 1 \leq F_d(\Gamma +_v K) \leq F_d(\Gamma) + F_d(K) + 1.$$

Proof. Let D_{f_Γ} be a minimum dom-forcing set for Γ and D_{f_K} be a minimum dom-forcing set for K . Consider the following cases.

Case 1: Let $v \in D_{f_\Gamma}$ and $v \in D_{f_K}$. If $D_{f_\Gamma} \cup D_{f_K}$ be a dom-forcing set for $\Gamma +_v K$, then $F_d(\Gamma +_v K) = F_d(\Gamma) + F_d(K) - 1$. Otherwise, the union of D_{f_Γ} and the reversal of the forcing sequence of D_{f_K} form a dom-forcing set for $\Gamma +_v K$, hence $F_d(\Gamma +_v K) = F_d(\Gamma) + F_d(K)$.

Case 2: If either $v \in D_{f_\Gamma}$ or $v \in D_{f_K}$, then $D_{f_\Gamma} \cup D_{f_K}$ is a dom-forcing set for $\Gamma +_v K$. Therefore $F_d(\Gamma +_v K) \leq F_d(\Gamma) + F_d(K)$. Consider the graph $\Gamma +_v K$. If a vertex is removed from $N(v) \cap (D_{f_\Gamma} \cup D_{f_K})$ and the resulting set remains a dom-forcing set for $\Gamma +_v K$, then $F_d(\Gamma +_v K) = F_d(\Gamma) + F_d(K) - 1$, otherwise, $F_d(\Gamma +_v K) = F_d(\Gamma) + F_d(K)$.

Case 3: If $v \notin D_{f_\Gamma}$ and $v \notin D_{f_K}$. If $D_{f_\Gamma} \cup D_{f_K}$ is a dom-forcing set for $\Gamma +_v K$, then $F_d(\Gamma +_v K) \leq F_d(\Gamma) + F_d(K)$. Otherwise, $D_{f_\Gamma} \cup D_{f_K} \cup \{u\}$, where u be a vertex that is forced by v in a zero forcing process started with the vertices of D_{f_Γ} , form a dom-forcing set. Thus, $F_d(\Gamma +_v K) \leq F_d(\Gamma) + F_d(K) + 1$.

Hence from the above cases,

$$F_d(\Gamma) + F_d(K) - 1 \leq F_d(\Gamma +_v K) \leq F_d(\Gamma) + F_d(K) + 1.$$

□

The next section discusses the dom-forcing number of joint and product of graphs.

4.3 Join and Product of Graphs

This section explores the dom-forcing number for the join of two graphs, as well as for various types of product graphs, including corona products and rooted products.

The study proceed to analyze the dom-forcing number associated with join of two graphs $\Gamma \vee K$.

Theorem 4.3.1. [41] Let Γ and K be two graphs which are connected . Then,

$$\mathcal{Z}(\Gamma \vee K) = \min\{|K| + \mathcal{Z}(\Gamma); |\Gamma| + \mathcal{Z}(K)\}.$$

Theorem 4.3.2. Let Γ and K be two graphs which are connected. Then,

$$F_d(\Gamma \vee K) = \mathcal{Z}(\Gamma \vee K) = \min\{|K| + \mathcal{Z}(\Gamma); |\Gamma| + \mathcal{Z}(K)\}.$$

Proof. Every zero forcing set dominates $\Gamma \vee K$, since by the definition of join, one vertex from both Γ and K dominates $\Gamma \vee K$. Therefore,

$$F_d(\Gamma \vee K) = \mathcal{Z}(\Gamma \vee K) = \min\{|K| + \mathcal{Z}(\Gamma); |\Gamma| + \mathcal{Z}(K)\}.$$

□

Consider the dom-forcing number of generalized corona product of graphs.

Theorem 4.3.3. [41, 25] For the graph Γ with $|\Gamma| = n$ and $H_1, \dots, H_n$ be graphs with order at least two,

$$\mathcal{Z}(\Gamma \prec H_1, \dots, H_n \succ) = \mathcal{Z}(\Gamma) + \sum_{i=1}^n \mathcal{Z}(H_i).$$

Also for any graph K with order at least two,

$$\mathcal{Z}(\Gamma \circ K) = \mathcal{Z}(\Gamma) + n\mathcal{Z}(K).$$

Theorem 4.3.4. For the graph Γ with $|V(\Gamma)| = n$ and graphs $H_1, \dots, H_n$, each of order at least two, the dom-forcing number $F_d(\Gamma \prec H_1, \dots, H_n \succ)$ is given by:

$$F_d(\Gamma \prec H_1, \dots, H_n \succ) = \min \left\{ \mathcal{Z}(\Gamma) + \sum_{v_i \in \mathcal{Z}_\Gamma} \mathcal{Z}(H_i) + \sum_{v_i \notin \mathcal{Z}_\Gamma} a_i \right\}$$

where $a_i = \inf\{\mathcal{Z}(H_i) + 1, F_d(H_i)\}$, and minimum is taken over all possible minimum zero forcing sets of Γ .

Proof. Let $\mathcal{Z}_\Gamma = \{v_1, \dots, v_t\}$ be a minimum zero forcing set for Γ , and $\mathcal{Z}_{H_1}, \dots, \mathcal{Z}_{H_n}$, be the minimum zero forcing sets for $H_1, \dots, H_n$ and $D_{f_{H_1}}, \dots, D_{f_{H_n}}$, be the minimum dom-forcing sets for $H_1, \dots, H_n$ respectively. Let

$$A = (\mathcal{Z}_\Gamma \cup \mathcal{Z}_{H_1} \cup \dots \cup \mathcal{Z}_{H_t}) \cup (\cup_{i=t+1}^n A_i),$$

where A_i is either $\mathcal{Z}_{H_i} \cup \{v_i\}$, or $D_{f_{H_i}}$ forms a dom-forcing set for $\Gamma \prec H_1, \dots, H_n \succ$. Hence $F_d(\Gamma \prec H_1, \dots, H_n \succ) \leq |A|$, that is,

$$F_d(\Gamma \prec H_1, \dots, H_n \succ) \leq \min \left\{ \mathcal{Z}(\Gamma) + \sum_{v_i \in \mathcal{Z}_\Gamma} \mathcal{Z}(H_i) + \sum_{v_i \notin \mathcal{Z}_\Gamma} a_i \right\}$$

where $a_i = \inf\{\mathcal{Z}(H_i) + 1, F_d(H_i)\}$.

Note that the degree of each vertex v_i of Γ is increased by $|H_i|$ and the degree of each vertex of H_i is increased by 1 in $\Gamma \prec H_1, \dots, H_n \succ$. Let $v_i \in \mathcal{Z}_\Gamma$ and consider the corresponding graph H_i . It is observed that at least $\mathcal{Z}(H_i) + 1$ vertices are required as initially colored black to start the zero forcing process in each of these H_i 's, $1 \leq i \leq t$. Also these vertices dominates these H_i 's. Then v_t forces v_{t+1} and to continue the process, at least $\mathcal{Z}(H_{t+1})$ more vertices are required in H_{t+1} as initially colored black to zero force. Also $\min\{\mathcal{Z}(H_{t+1}) + 1, F_d(H_{t+1})\}$ vertices are required to dom-force all the vertices of H_{t+1} in the graph $\Gamma \prec H_1, \dots, H_n \succ$. Continue the process until all the vertices are turned black. Hence,

$$F_d(\Gamma \prec H_1, \dots, H_n \succ) \geq \min \left\{ \mathcal{Z}(\Gamma) + \sum_{v_i \in \mathcal{Z}_\Gamma} \mathcal{Z}(H_i) + \sum_{v_i \notin \mathcal{Z}_\Gamma} a_i \right\}$$

where $a_i = \inf\{\mathcal{Z}(H_i) + 1, F_d(H_i)\}$.

From the above,

$$F_d(\Gamma \prec H_1, \dots, H_n \succ) = \min \left\{ \mathcal{Z}(\Gamma) + \sum_{v_i \in \mathcal{Z}_\Gamma} \mathcal{Z}(K_i) + \sum_{v_i \notin \mathcal{Z}_\Gamma} a_i \right\}$$

where $a_i = \inf\{\mathcal{Z}(H_i) + 1, F_d(H_i)\}$, and minimum is taken over all possible minimum zero forcing sets of Γ . $\square$

For the graph Γ with $|V(\Gamma)| = n$ and for any graph K , from the Definition 2.2.8 it can be seen that $\Gamma \circ K = \Gamma \prec K, \dots, K \succ$. Therefore from the above theorem, the following result can be derived.

Corollary 4.3.5. *For the graph Γ with $|V(\Gamma)| = n$ and for any graph K with order at least two,*

$$F_d(\Gamma \circ K) = \min\{n(\mathcal{Z}(K) + 1), \mathcal{Z}(\Gamma) + nF_d(K)\}.$$

From Corollary 4.3.5, the following result can be derived.

Corollary 4.3.6. *For the graph Γ with $|\Gamma| = n$ and for any graph K with order at least two, if $F_d(K) = \mathcal{Z}(K)$, then*

$$F_d(\Gamma \circ K) = \mathcal{Z}(\Gamma \circ K) = \mathcal{Z}(\Gamma) + n\mathcal{Z}(K).$$

Otherwise

$$F_d(\Gamma \circ K) = n(\mathcal{Z}(K) + 1).$$

Proof. If $F_d(K) = \mathcal{Z}(K)$, then

$$\min\{n(\mathcal{Z}(K) + 1), \mathcal{Z}(\Gamma) + nF_d(K)\} = \mathcal{Z}(\Gamma) + nF_d(K) = \mathcal{Z}(\Gamma) + n\mathcal{Z}(K).$$

Moreover if $F_d(K) > \mathcal{Z}(K)$, then

$$\min\{n(\mathcal{Z}(K) + 1), \mathcal{Z}(\Gamma) + nF_d(K)\} = n(\mathcal{Z}(K) + 1).$$

Hence the result. $\square$

By the repeated application of Corollary 4.3.6, the following results obtained.

Theorem 4.3.7. *Let Γ and K be two graphs of order n_1 and n_2 , ($n_2 \geq 2$), respectively. If $F_d(K) = \mathcal{Z}(K)$, then*

$$F_d(\Gamma \circ^j K) = \mathcal{Z}(\Gamma \circ^j K) = \mathcal{Z}(\Gamma \circ^{j-1} K) + n_1(n_2 + 1)^{j-1} \mathcal{Z}(K).$$

Otherwise

$$F_d(\Gamma \circ^j K) = n_1(n_2 + 1)^{j-1}(\mathcal{Z}(K) + 1).$$

Theorem 4.3.8. *For the graph Γ with $|V(\Gamma)| = n$, $F_d(\Gamma \circ K_1) = n$.*

Proof. Let $\{v_i\}$ be the vertex set of i^{th} copy of K_1 . Then $A = \{v_1 \cdots v_n\}$ be a dom-forcing set for $\Gamma \circ K_1$. Therefore, $F_d(\Gamma \circ K_1) \leq n$.

Every vertex v_i in $\Gamma \circ K_1$ is a pendant vertex and no two v_i is adjacent to a common vertex. Hence at least n vertices needed to dominate $\Gamma \circ K_1$. Therefore, $F_d(\Gamma \circ K_1) \geq n$.

From the above, $F_d(\Gamma \circ K_1) = n$. $\square$

By the repeated application of Theorem 4.3.8, the following results obtained.

Corollary 4.3.9. *For the graph Γ with $|V(\Gamma)| = n$, $F_d(\Gamma \circ^j K_1) = 2^{j-1}n$.*

It is noted that, $K_1 \circ C_n$ be the wheel graph of order $n + 1$, denoted by $W_{1,n}$ [25]. Hence, the following result.

Remark 4.3.10. Let $W_{1,n}$, $n \geq 3$, be a wheel graph. Then, $F_d(W_{1,n}) = 3$.

The fan graph F_{n_1, n_2} is defined as the join graph $K_{n_1} + P_{n_2}$. The case $n_1 = 1$ corresponds to the usual fan graph F_{1, n_2} . Note that $F_{1, n_2} = K_1 \circ P_{n_2}$, where K_1 is the singleton graph. Hence

Remark 4.3.11. Let $F_{1,n}$, $n \geq 2$, be a fan graph. Then, $F_d(F_{1,n}) = 2$.

A k -Comb graph is obtained by appending a leaf to each vertex of a path on k vertices. That is a k -Comb graph is $P_k \circ K_1$. Hence,

Remark 4.3.12. Let Γ , be a k -Comb graph. Then $F_d(\Gamma) = k$.

The thesis investigates the dom-forcing number of rooted product of graphs.

Theorem 4.3.13. Let Γ be a graph with order n and K be a graph rooted with any vertex and its dom-forcing number is $F_d(K)$. Then the dom-forcing number of rooted product of Γ by K is given by:

$$F_d[\Gamma(K)] \leq nF_d(K).$$

Proof. Let $D_{f_{K_i}}$ be the minimum dom-forcing set for i^{th} copy of K . Then $A = \bigcup_{i=1}^n D_{f_{K_i}}$ forms a dom-forcing set for rooted product of Γ by K . Hence

$$F_d[\Gamma(K)] \leq nF_d(K).$$

□

The inequality is strict for some graphs. For example, consider the rooted product of P_2 by P_3 , where the pendant vertex of P_3 is the root. The graph thus obtained is isomorphic to P_6 . It can be derived that $F_d(P_3) = 2$ and

$F_d[P_2(P_3)] = 3$, hence the inequality is strict. Also consider the example of the rooted product of P_2 by C_4 rooted with any vertex of C_4 . It can be noted that $F_d(C_4) = 2$ and $F_d[P_2(C_4)] = 4$, hence the upper bound is sharp. It is an open problem to characterize the graph Γ with $F_d[\Gamma(K)] = nF_d(K)$.

Chapter 4 discussed the effect of graph operations in dom-forcing set. The study identified that the dom-forcing numbers bounded under these graph operations. The next Chapter further explains the minimum number of steps required to complete the coloring process starting with minimum dom-forcing set.

Chapter 5

DOM-FORCING PROPAGATION TIMES IN GRAPHS

In this chapter introduces a new parameter in graph theory called dom-forcing propagation time. Section 1 introduces the concept, Section 2 establishes upper and lower bounds for minimum and maximum dom-forcing propagation time in terms of dom-forcing number and Section 3 explores the maximum and minimum dom-forcing propagation time for several renowned graphs. The graphs with extreme dom-forcing propagation time is characterized .

5.1 Introduction

The propagation time of a dom-forcing set of graph Γ is the minimum number of steps that it takes to force all the vertices black, starting with the vertices in the dom-forcing set and performing independent forces simultaneously. The minimum and maximum dom-forcing propagation times of a graph are taken

over all minimum dom-forcing sets of the graph. It discusses the minimum dom-forcing propagation time of a graph Γ . Additionally, it delves into the precise determination of minimum dom-forcing propagation times for certain well-known graphs.

5.2 Dom-forcing Propagation Time of Graphs

This section discusses the minimum and the maximum dom-forcing propagation time for a graph Γ . Also, the study finds the exact value of minimum dom-forcing propagation time for certain well-known graphs. It define the propagation time in Chapter 3 (Definition 3.4.13). The thesis extend the concept of propagation time into dom-forcing sets. Every dom-forcing set is also a zero forcing set. Hence it can find the propagation time for dom-forcing set of a graph Γ . The study defines the minimum and maximum dom-forcing propagation time of Γ as follows.

Definition 5.2.1. *Let $\Gamma = (V, E)$ be a graph and D_f be a dom-forcing set of Γ . Define $D_f^{(0)} = D_f$, and for $t \geq 0$, $D_f^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \bigcup_{s=0}^t D_f^{(s)}$ such that w is the only neighbor of b not in $\bigcup_{s=0}^t D_f^{(s)}$. The propagation time of D_f in Γ , denoted $pt(\Gamma, D_f)$, is the smallest integer t_0 such that $V = \bigcup_{s=0}^{t_0} D_f^{(s)}$. Two minimum dom-forcing sets of the same graph may have different propagation times. The minimum dom-forcing propagation time of Γ is defined as follows*

$$dpt(\Gamma) = \min\{pt(\Gamma, D_f) | D_f \text{ is a minimum dom-forcing set of } \Gamma\}.$$

Definition 5.2.2. *A subset D_f of vertices of Γ is an efficient dom-forcing set*

for Γ if D_f is a minimum dom-forcing set of Γ and $pt(\Gamma, D_f) = dpt(\Gamma)$.

Definition 5.2.3. The maximum dom-forcing propagation time of Γ is

$$DPT(\Gamma) = \max\{pt(\Gamma, D_f) | D_f \text{ is a minimum dom-forcing set of } \Gamma\}.$$

Consider the following example to show that two minimum dom-forcing sets of the same graph may have different propagation times.

Example 5.2.4. Let Γ be the graph in Figure 5.1. Let $B_1 = \{v_2, v_3, v_6, v_8\}$ and $B_2 = \{v_3, v_4, v_7, v_9\}$ be two minimum dom-forcing sets for Γ . Then $B_1^{(1)} = \{v_1, v_4\}$, $B_1^{(2)} = \{v_5, v_7\}$, $B_1^{(3)} = \{v_9\}$, so $pt(\Gamma, B_1) = 3$. However $B_2^{(1)} = \{v_2, v_5, v_8\}$, $B_2^{(2)} = \{v_1, v_6\}$, so $pt(\Gamma, B_2) = 2$. Hence $dpt(\Gamma) = 2$, and $DPT(\Gamma) = 3$.

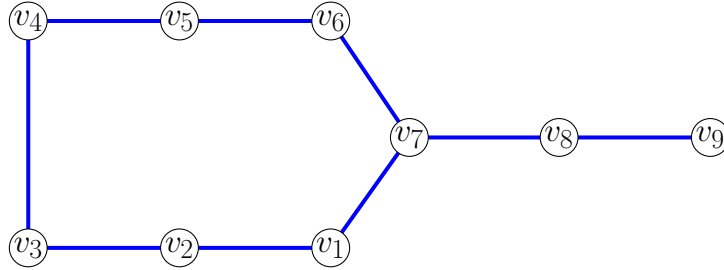


Figure 5.1: Graph for Example 5.2.4.

From the definition, it is clear that $dpt(\Gamma) \leq DPT(\Gamma)$. The bounds for dom-forcing propagation time is given below.

Proposition 5.2.5. Let Γ be a graph with dom-forcing number $F_d(\Gamma)$. If n is the number of vertices in Γ , then

$$\left\lceil \frac{n - F_d(\Gamma)}{F_d(\Gamma)} \right\rceil \leq dpt(\Gamma) \leq DPT(\Gamma) \leq n - F_d(\Gamma).$$

Proof. Let Γ be a graph with dom-forcing number $F_d(\Gamma)$. If n is the number of vertices of Γ , then a total of $n - F_d(\Gamma)$ vertices are white in the first stage. In each stage at least one vertex and at most $F_d(\Gamma)$ number of vertices colored black. Hence

$$\left\lceil \frac{n - F_d(\Gamma)}{F_d(\Gamma)} \right\rceil \leq dpt(\Gamma) \leq DPT(\Gamma) \leq n - F_d(\Gamma).$$

□

The above bounds are sharp. For example, consider a path P_5 with the vertex set $\{v_1, v_2, v_3, v_4, v_5\}$, each v_i is adjacent with v_{i-1} and v_{i+1} , $1 < i < 5$. $D_f = \{v_1, v_4\}$ is a minimum dom-forcing set and $dpt(P_5) = DPT(P_5) = 3 = 5 - F_d(P_5)$. Also consider a k regular hypercube, Q_k , having vertex set $\{0, 1\}^k$, with vertices adjacent when they differ in exactly one coordinate. For a hypercube graph Q_k , $F_d(Q_k) = 2^{k-1}$. Here $dpt(Q_k) = DPT(Q_k) = 1 = \left\lceil \frac{n - F_d(Q_k)}{F_d(Q_k)} \right\rceil$. Again, consider the path graph P_6 with the dom-forcing set $D_f = \{v_1, v_4, v_5\}$. In this case, $dpt(P_6) = 1 = \frac{n - F_d(P_6)}{F_d(P_6)}$. Now consider another dom-forcing set of P_6 , namely $D_f = \{v_1, v_3, v_5\}$. In this case, $DPT(P_6) = 3 = 6 - F_d(P_6)$.

It is noted that every dom-forcing set is also a zero forcing set, hence, the following observations.

Observation 5.2.6. *For any graph Γ , $dpt(\Gamma) \leq pt(\Gamma)$.*

Observation 5.2.7. *For any graph Γ , $dpt(\Gamma) = 0$ if and only if Γ is a totally disconnected graph.*

Theorem 5.2.8. *Let Γ be a graph with $pt(\Gamma) = 1$. Then $dpt(\Gamma) = 1$.*

Proof. Given that Γ is a graph with $pt(\Gamma) = 1$. Hence, there exists a minimum zero forcing set that forces all the vertices to turn black in one step. Therefore, this zero forcing set also dominates the graph, establishing $dpt(\Gamma) = 1$. $\square$

Converse of the above proposition is not true. Consider the graph P_6 , $dpt(P_6) = 1$ but $pt(\Gamma) = 5$. From the Proposition 5.2.5, the following proposition derived.

Proposition 5.2.9. *For any graph Γ with $dpt(\Gamma) = 1$, then $F_d(\Gamma) \geq \frac{|V(\Gamma)|}{2}$.*

The converse of this proposition is false. Let Γ be the graph obtained from K_4 by appending a leaf to one vertex. Then $F_d(\Gamma) = 3 > |V(\Gamma)|/2$ and $dpt(\Gamma) = 2$.

In Chapter 3, proved that for any graph Γ , $F_d(\Gamma) = 1$ if and only if $\Gamma = P_1$ or P_2 . Using this result, the following theorem established.

Theorem 5.2.10. *For any connected graph Γ , $dpt(\Gamma) = |V(\Gamma)| - 1$ if and only if Γ is either P_1 or P_2 .*

Proof. Let Γ be a connected graph with $dpt(\Gamma) = |V(\Gamma)| - 1$. Then by Proposition 5.2.5, it is observed that $F_d(\Gamma) \leq 1$. Since Γ is connected and $F_d(\Gamma) = 1$. Therefore, $\Gamma = P_1$ or P_2 . Converse part is trivial. $\square$

Proposition 5.2.11. *Let Γ be a disconnected graph. Then the following are equivalent.*

1. $dpt(\Gamma) = |V(\Gamma)| - 2$.
2. $DPT(\Gamma) = |V(\Gamma)| - 2$.
3. $\Gamma = P_2 \cup P_1$ or $\Gamma = P_1 \cup P_1$.

Proof. If $\Gamma = P_2 \cup P_1$, then $dpt(\Gamma) = DPT(\Gamma) = 1 = |V(\Gamma)| - 2$. Also if $\Gamma = P_1 \cup P_1$, then $dpt(\Gamma) = DPT(\Gamma) = 0 = |V(\Gamma)| - 2$. Now assume $DPT(\Gamma) =$

$|V(\Gamma)| - 2$. Then by Proposition 5.2.5, $F_d(\Gamma) = 2$, Γ has exactly two components. At least one component of Γ is an isolated vertex otherwise, more than one force occurs at time step one, and so $\Gamma = P_2 \cup P_1$ or $\Gamma = P_1 \cup P_1$. $\square$

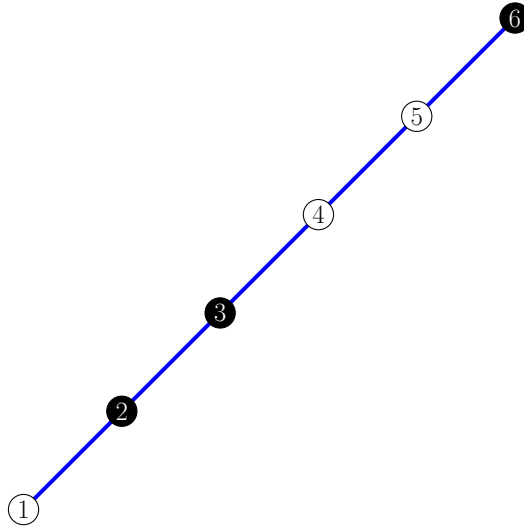
Remark 5.2.12. *In the case of a connected graph the following results observed.*

1. *If $dpt(\Gamma) = |V(\Gamma)| - 2$, then $DPT(\Gamma) = |V(\Gamma)| - 2$, but converse is not true. For example, consider the graph P_4 , $DPT(P_4) = 2 = |V(\Gamma)| - 2$ but $dpt(P_4) = 1$.*
2. *$dpt(\Gamma) = |V(\Gamma)| - 2$ if and only if $F_d(\Gamma) = 2$ and exactly one force is performed at each time for every minimum dom-forcing set. For example, P_3 and P_5 are graphs with $dpt(\Gamma) = |V(\Gamma)| - 2$.*
3. *$DPT(\Gamma) = |V(\Gamma)| - 2$ if and only if $F_d(\Gamma) = 2$ and there exists a minimum dom-forcing set such that exactly one force performed at each time.*

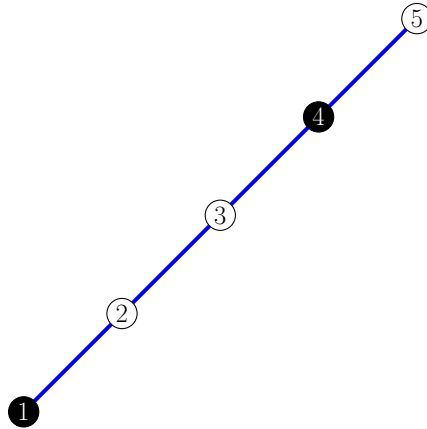
Minimum dom-forcing propagation time of a graph Γ and its sub graph H are not comparable. Consider the following examples.

Let Γ be a graph. If H is a sub graph of Γ , then $dpt(H) \leq dpt(\Gamma)$ is not true in general.

Consider the path $\Gamma = P_6$ given below.

Figure 5.2: The Graph Γ .

Let $D_f = \{2, 3, 6\}$. Then D_f is a minimum dom-forcing set for P_6 and $dpt(\Gamma) = 1$. Now consider a sub graph H of Γ given below.

Figure 5.3: The Graph H .

Here $D'_f = \{1, 4\}$ be a minimum dom-forcing set for H and $dpt(H) = 3$. Therefore, the assertion follows.

Next example shows that if Γ is a graph and H is a sub graph of Γ , then

$dpt(H) \geq dpt(\Gamma)$ is not true in general. Consider the wheel graph $\Gamma = W_8$ given below

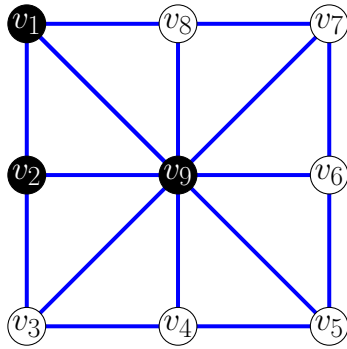


Figure 5.4: The wheel graph W_8 .

Let $S_1 = \{v_1, v_2, v_9\}$. Then S_1 is the minimum dom-forcing set. Hence $dpt(\Gamma) = 3$. Now consider a sub graph $H = C_8$ of Γ given below.

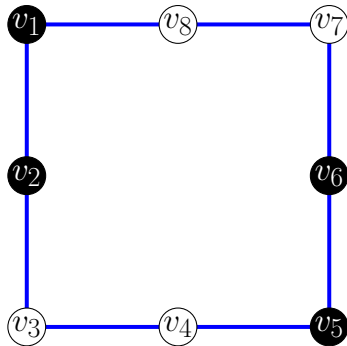


Figure 5.5: The Cycle C_8 .

Let $S_2 = \{v_1, v_2, v_5, v_6\}$. Then S_2 is the minimum dom-forcing set for H . Hence $dpt(H) = 1$. Therefore, the assertion follows.

5.3 Exact Values of Dom-forcing Propagation Time

This section computes the exact values of dom-forcing propagation time of certain graphs such as the path P_n , cycle C_n , complete graph K_n , complete bipartite $K_{m,n}$, Petersen graph P , wheel graph W_n , triangular snake graph TS_n , helm graph H_m and dove tail graph DT_n . The discussion begins with the path graph P_n . The following theorem establishes its dom-forcing propagation time.

Theorem 5.3.1. *Let Γ be the path P_n , of order $n \geq 2$. Then $DPT(P_n) = n - F_d(P_n)$.*

Proof. Consider a path P_n , with $V(P_n) = \{v_1, v_2, \dots, v_n\}$. For $i = 2, 3, \dots, n-1$ each vertex v_i is adjacent with v_{i-1} and v_{i+1} . It can be easily observed that for $n = 2$ and 3 , $DPT(P_2) = DPT(P_3) = 1 = n - F_d(P_n)$. For $n \geq 4$, consider the following cases and the following subsets of $V(P_n)$.

Case 1: Assume $n \equiv 0 \pmod{3}$, $D_f = \{v_{3k+1}, v_{n-3}, v_{n-1}\}$, where $0 \leq k < \lfloor n/3 - 1 \rfloor$.

Case 2: Assume $n \equiv 1 \pmod{3}$, $D_f = \{v_{3k+1}, v_{n-1}\}$, where $0 \leq k < \lfloor n/3 \rfloor$.

Case 3: Assume $n \equiv 2 \pmod{3}$, $D_f = \{v_{3k+1}, v_{n-1}\}$, where $0 \leq k < \lfloor n/3 \rfloor$.

In all the cases above, D_f is a minimum dom-forcing set, and need any pendant black vertex or two adjacent black vertices to force the remaining white

vertices in the path P_n . Since it is not considering the adjacent black vertices to force the path P_n hence only one force occurs at any time, therefore $pt(P_n, D_f) = n - F_d(P_n)$, and which is maximum by Proposition 3.4.14. Hence $DPT(P_n) = n - F_d(P_n)$. $\square$

Now let me consider the minimum dom-forcing propagation time for the path P_n .

Theorem 5.3.2. *For the path graph P_n ,*

1. $dpt(P_n) = n - F_d(P_n)$, for $n \equiv 2 \pmod{3}$.
2. $dpt(P_n) = \left\lceil \frac{n - F_d(P_n)}{2} \right\rceil$, for $n \equiv 1 \pmod{3}$.
3. $dpt(P_n) = \left\lceil \frac{n - F_d(P_n)}{3} \right\rceil$, for $n \equiv 0 \pmod{3}$.

Proof. Consider a path P_n , with $V(P_n) = \{v_1, v_2, \dots, v_n\}$. For $i = 2, 3, \dots, n-1$, each vertex v_i is adjacent with v_{i-1} and v_{i+1} . It is noted that any pendant black vertex or any two adjacent black vertices forces the entire vertex set of the graph P_n . Consider the following cases.

Case 1: Assume $n \equiv 2 \pmod{3}$. In this case, every minimum dom-forcing set contains exactly one pendant black vertex and no two adjacent black vertices. Hence only one force occurs at each time. Therefore, $dpt(P_n) = n - F_d(P_n)$.

Case 2: Assume $n \equiv 1 \pmod{3}$. In this case, there exist a minimum dom-forcing set containing two pendant vertices. Hence two simultaneous forces occur at each time step.

Claim: There does not exist a minimum dom-forcing set with more than two simultaneous forces occurring in at least one time step.

Proof. Assume that there exists a minimum dom-forcing set D_f , containing one pendant vertex and one set of two adjacent vertices. Also, assume that three simultaneous forces happen in at least one time step using this dom-forcing set. Now D_f must contain the vertices v_1 or v_n . Also need to choose the adjacent black vertices as v_i, v_{i+1} , for $4 \leq i \leq n-3$. Now $|N[v_i, v_{i+1}, v_n]| = 6$. To dominate the remaining $n - 6$ vertices, need

$$\left\lceil \frac{n-6}{3} \right\rceil = \frac{n-4}{3}$$

vertices. Now $|D_f| = \frac{n-4}{3} + 3 = \frac{n+5}{3}$, contradiction to our assumption that $F_d(P_n) = \lfloor n/3 \rfloor + 1$. $\square$

$$\text{Hence } dpt(P_n) = \left\lceil \frac{n-F_d(P_n)}{2} \right\rceil.$$

Case 3: Assume $n \equiv 0 \pmod{3}$. In this case, there exists a minimum dom-forcing set containing one set of two adjacent vertices and one pendant vertex, three forces occur in at least one time step. Moreover, there does not exist a minimum dom-forcing set with more than three forces occurs at a time (it can be proved in similar manner, as in the proof of the above claim). Hence

$$dpt(P_n) \geq \left\lceil \frac{n-F_d(P_n)}{3} \right\rceil. \quad (5.1)$$

For the path P_3 , $D_f = \{v_2, v_3\}$ and $dpt(P_3) = 1$, and for $n \geq 6$, consider $D_f = \{v_{3k+2}, v_{3\lceil n/9 \rceil}, v_{3j}\}$, $0 \leq k < \lceil n/9 \rceil$ and $\lceil n/9 \rceil < j \leq \frac{n}{3}$. Then D_f is a minimum dom-forcing set with propagation time $\left\lceil \frac{n-F_d(P_n)}{3} \right\rceil$. Hence

$$dpt(P_n) \leq \left\lceil \frac{n-F_d(P_n)}{3} \right\rceil. \quad (5.2)$$

From (5.1) and (5.2), $dpt(P_n) = \left\lceil \frac{n-F_d(P_n)}{3} \right\rceil$. $\square$

Theorem 5.3.3. *For the cycle C_n , $n \geq 3$, $DPT(C_n) = \left\lceil \frac{n-F_d(C_n)}{2} \right\rceil$.*

Proof. Let C_n be the cycle with vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Consider the following cases and the following subsets of $V(C_n)$.

Case 1: Assume $n \equiv 0, 1 \pmod{3}$, $D_f = \{v_{3k+1}, v_n\}$, where $0 \leq k < \lfloor n/3 \rfloor$.

Case 2: Assume $n \equiv 2 \pmod{3}$, $D_f = \{v_{3k+1}, v_{n-2}, v_n\}$, where $0 \leq k < \lfloor n/3 \rfloor$.

In both cases, D_f is a minimum dom-forcing set and $pt(C_n, D_f) = \left\lceil \frac{n-F_d(C_n)}{2} \right\rceil$. Any two adjacent vertices force the entire vertex set of C_n , also at most two forces occur at each time step. Hence $DPT(C_n) = \left\lceil \frac{n-F_d(C_n)}{2} \right\rceil$. $\square$

Now let me consider the minimum dom-forcing propagation time for the cycle C_n .

Theorem 5.3.4. *For the cycle C_n of order $n \geq 3$,*

1. $dpt(C_n) = \left\lceil \frac{n-F_d(C_n)}{2} \right\rceil$ for $n \equiv 0, 1 \pmod{3}$.
2. $dpt(C_n) = \left\lceil \frac{n-F_d(C_n)}{4} \right\rceil$ for $n \equiv 2 \pmod{3}$.

Proof. Let C_n be a cycle with vertices $v_1, v_2, \dots, v_n$. Any two adjacent vertices forces C_n . Consider the following cases.

Case 1: Assume $n \equiv 0, 1 \pmod{3}$, every minimum dom-forcing set contains exactly two adjacent vertices, two forces occur at each time step. Hence $dpt(C_n) =$

$$\left\lceil \frac{n - F_d(C_n)}{2} \right\rceil.$$

Case 2: Assume $n \equiv 2 \pmod{3}$. It can be noted that $dpt(C_5) = 1$, and for $n > 5$, there exists a minimum dom-forcing set with two set of two adjacent vertices. Maximum of four forces can occur in at least one time step, hence

$$dpt(C_n) \geq \left\lceil \frac{n - F_d(C_n)}{4} \right\rceil. \quad (5.3)$$

Now consider $D_f = \{v_{3k+1}, v_{3j+2}\}$ where $0 \leq k \leq \lfloor n/6 \rfloor$ and $\lfloor n/6 \rfloor \leq j \leq \lfloor n/3 \rfloor$. Here D_f is a minimum dom-forcing set with $pt(C_n, D_f) = \left\lceil \frac{n - F_d(C_n)}{4} \right\rceil$. Hence

$$dpt(C_n) \leq \left\lceil \frac{n - F_d(C_n)}{4} \right\rceil \quad (5.4)$$

Therefore from (5.3) and (5.4), $dpt(C_n) = \left\lceil \frac{n - F_d(C_n)}{4} \right\rceil$ for $n \equiv 2 \pmod{3}$. $\square$

From the definition of dom-forcing set and by the Theorem 5.2.8, the following theorems can be verified.

Theorem 5.3.5. *Let K_n be the complete graph with n vertices. Then $dpt(K_n) = DPT(K_n) = 1$.*

Theorem 5.3.6. *Let $K_{m,n}$ be a complete bipartite graph, where $n, m \geq 2$. Then $dpt(K_{m,n}) = DPT(K_{m,n}) = 1$.*

Theorem 5.3.7. *For the Petersen graph P , $dpt(P) = 1$.*

Theorem 5.3.8. *For the star graph $K_{1,n}$, $dpt(K_{1,n}) = DPT(K_{1,n}) = 1$.*

Theorem 5.3.9. *Let W_n be the wheel graph with n vertices. Then $dpt(W_n) = DPT(W_n) = \left\lceil \frac{n-3}{2} \right\rceil$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the wheel graph W_n with degree $d(v_1) = n - 1$ and all the other vertices have degree 3. Then $D_f = \{v_1, v_2, v_3\}$ be a minimum dom-forcing set and which is unique (up to isomorphism). Hence two simultaneous forces occur at each time step, $dpt(W_n) = DPT(W_n) = \left\lceil \frac{n - F_d(W_n)}{2} \right\rceil = \left\lceil \frac{n-3}{2} \right\rceil$. $\square$

Next the study investigates the dom-forcing propagation time of the triangular snake graph TS_n , where n denotes the number of triangles in TS_n .

Theorem 5.3.10. *For the triangular snake graph TS_n , with triangles $n > 2$, $dpt(TS_n) = 2$ and $DPT(TS_n) = n$.*

Proof. Let $v_1, v_2, \dots, v_{n+1}, u_1, u_2, \dots, u_n$ be the vertices of the triangular snake graph TS_n (see Figure 5.6). If $D_f = \{v_1, u_1, u_2, \dots, u_n\}$, then D_f is a minimum dom-forcing set with $pt(TS_n, D_f) = n = |TS_n| - F_d(TS_n)$, which is maximum by Proposition 5.2.5. Hence $DPT(TS_n) = n$.

There exists no minimum dom-forcing set in TS_n with propagation time one. Now consider the following cases.

Case 1: Assume n is odd, $D'_f = \{v_1, u_1, v_3, u_3, \dots, v_n, u_n\}$.

Case 2: Assume n is even, $D'_f = \{v_1, u_1, v_3, u_3, \dots, v_{n-1}, u_{n-1}, u_n\}$.

In both the cases, D'_f act as a minimum dom-forcing set with $pt(TS_n, D'_f) = 2$. Hence $dpt(TS_n) = 2$.

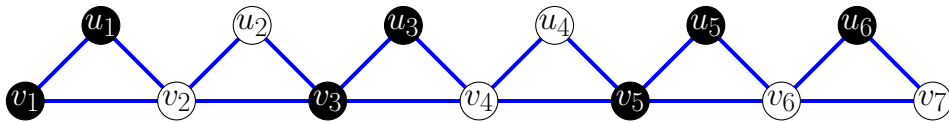


Figure 5.6: The triangular snake graph TS_6 , dom-forcing set $D'_f = \{v_1, u_1, v_3, u_3, v_5, u_5, u_6\}$ and $pt(TS_6, D'_f) = 2$.

□

Remark 5.3.11. *It can be easily verified that $dpt(TS_1) = dpt(TS_2) = 1$.*

A double triangular snake graph $D(T_n)$ consists of two triangular snake graph that have a common path. Specifically, double triangular snake graph is obtained by attaching two new vertices u_i and w_i to the consecutive vertices v_i and v_{i+1} of a path, for $1 \leq i \leq n$ [43]. Thus, $D(T_n)$ contains $2n$ triangles.

Theorem 5.3.12. [33] Let Γ be a double triangular snake graph $D(T_n)$, and $n \geq 2$. Then $\mathcal{Z}(\Gamma) = n + 1$.

Lemma 5.3.13. [33] For a double triangular snake graph, the vertices u_i or w_i must contain in every zero forcing set.

Lemma 5.3.14. *For a double triangular snake graph, either u_i and w_i or one of the vertices v_i or v_{i+1} must contained in every minimum dominating set.*

Proof. Assume that v_i and v_{i+1} does not belong to the domination set. Then u_i and w_i are in the dominating set, because both u_i and w_i are adjacent with v_i and v_{i+1} for $1 \leq i \leq n$. □

Theorem 5.3.15. *For the double triangular snake graph $D(T_n)$, $F_d[D(T_n)] = n + \lceil \frac{n}{2} \rceil$.*

Proof. Let $v_1, \dots, v_{n+1}, u_1, \dots, u_n, w_1, \dots, w_n$ be the vertices of the double triangular snake graph $D(T_n)$. Consider the following cases.

Case 1: Assume n is odd, $D_f = \{u_1, \dots, u_n, v_2, v_4 \dots, v_{n-1}, v_n\}$.

Case 2: Assume n is even, $D_f = \{u_1, \dots, u_n, v_2, v_4 \dots, v_n\}$.

In both the cases, D_f must be a dom-forcing set. Both u_i and w_i are adjacent with v_i and v_{i+1} for $1 \leq i \leq n$. By Lemma 5.3.13, the vertices u_i or w_i must contained in every zero forcing set. With out the loss of generality, let these vertices be $u_1, \dots, u_n$. Also by Lemma 5.3.14, either u_i and w_i or one of the vertices v_i or v_{i+1} must contained in every minimum dominating set. Hence D_f is a minimum dom-forcing set. Since the researcher is choosing the minimum dom-forcing set, to find the dom-forcing number, he chose the vertices for the dom-forcing set as described in case 1 and 2. Hence $F_d[D(T_n)] = n + \lceil \frac{n}{2} \rceil$. $\square$

Theorem 5.3.16. *Let $D(T_n)$ be the double triangular snake graph with $n > 1$. Then $DPT[D(T_n)] = 1 + \lceil \frac{n}{2} \rceil$ and*

$$dpt[D(T_n)] = \begin{cases} 1 + \lceil \frac{n-1}{4} \rceil & \text{if } n \text{ is odd} \\ 1 + \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $v_1, \dots, v_{n+1}, u_1, \dots, u_n, w_1, \dots, w_n$ be the vertices of the double triangular snake graph $D(T_n)$. Consider the following cases.

Case 1: Assume n is even, $D_f = \{u_1, \dots, u_n, v_2, v_4, \dots, v_n\}$ is a minimum dom-forcing set, which is unique up to isomorphism. In first time step, color all the vertices v_i , $1 \leq i \leq n+1$ into black and with the remaining n vertices, two simultaneous forces occur at each time step. Hence $DPT[D(T_n)] = dtp[D(T_n)] = 1 + \lceil \frac{n}{2} \rceil$.

Case 2: Assume n is odd, consider $D_f = \{u_1, \dots, u_n, v_2, v_4, \dots, v_{n-1}, v_n\}$. Then D_f is a minimum dom-forcing set and in first time step, color all the vertices v_i , $1 \leq i \leq n+1$ into black. And the remaining n vertices, two simultaneous forces occur at each time step, which is maximum. Hence $DPT[D(T_n)] = 1 + \lceil \frac{n}{2} \rceil$.

Case 3: For the minimum dom-forcing propagation time, it can be easily ob-

served that for $n = 3$, $D'_f = \{u_1, u_2, u_3, v_2, v_3\}$ be a minimum dom-forcing set with propagation time 2 and for $n > 3$, consider

$$D'_f = \{u_1, \dots, u_n, v_{2i}, v_{2j-1}, w_{2\lceil \frac{n}{4} \rceil - 1}\},$$

$1 \leq i < \lceil \frac{n}{4} \rceil$ and $\lceil \frac{n}{4} \rceil < j \leq \lceil \frac{n}{2} \rceil$. Here D'_f is a minimum dom-forcing set and in the first time step, color all the vertices v_i , $1 \leq i \leq n+1$ into black. For the remaining $n-1$ white vertices, four simultaneous forces occur at each time step, which is minimum. Hence $dpt[D(T_n)] = 1 + \lceil \frac{n-1}{4} \rceil$. $\square$

The above theorem is illustrated in the case of $D(T_5)$.

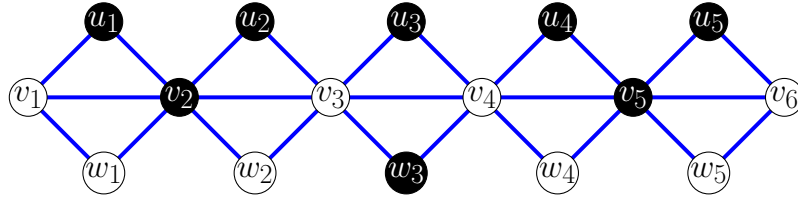


Figure 5.7: The double triangular snake graph $D(T_5)$, dom-forcing set $D'_f = \{u_1, \dots, u_5, v_2, v_5, w_3\}$ and $dpt[D(T_5)] = 2$.

Alternate triangular snake graph $A(T_n)$ is a graph obtained from the path P_n by replacing each alternate edge of the path P_n by a cycle C_3 [43].

Theorem 5.3.17. *For the alternate triangular snake graph $A(T_n)$, $F_d[A(T_n)] = 1 + \text{number of triangles in } A(T_n)$.*

Proof. Let $A(T_n)$ denote the alternate triangular snake graph. From the definition, $A(T_n)$ is obtained from the path P_n by replacing each alternate edge of P_n by a cycle C_3 . One of the vertices of each C_3 is contained in every dominating set of $A(T_n)$. But exactly one vertex in each C_3 cannot zero force the entire

graph. Hence, every dom-forcing set of $A(T_n)$ must contain at least one vertex from each cycle of $A(T_n)$, together with one additional vertex. Therefore, every dom-forcing set of $A(T_n)$ contains at least $(\text{number of cycles in } A(T_n)) + 1$ vertices. Now consider the following cases.

Case 1: Assume that n is even. Consider the case in which the first edge is replaced by a cycle C_3 . Since alternate edges are replaced by a cycle C_3 , there will be only $\frac{n}{2}$ cycles and the rest of the edges will remain as the paths connecting these cycles. Let $v_1, \dots, v_n$ be the vertices in the path P_n and $w_1, \dots, w_{\frac{n}{2}}$ be the vertices added to obtain $A(T_n)$. Now, $D_f = \{v_1, w_1, \dots, w_{\frac{n}{2}}\}$ be a dom-forcing set of $A(T_n)$.

Now consider the case when second edge is replaced by the cycle C_3 . It is clear that there are $\frac{n}{2} - 1$ triangles in each $A(T_n)$. Let $v_1, \dots, v_n$ be the vertices of the path P_n and $w_1, \dots, w_{\frac{n}{2}-1}$ be the vertices added to obtain $A(T_n)$. Now $D_f = \{v_1, w_1, \dots, w_{\frac{n}{2}-2}, v_{n-1}\}$ be a dom-forcing set of $A(T_n)$.

Case 2: Assume that n is odd. Consider the case when the first edge is replaced by the cycle C_3 . Since alternate edges are replaced by a cycle C_3 , there will be only $\frac{n-1}{2}$ cycles and rest of the edges will remain as the paths connecting these cycles. Let $v_1, \dots, v_n$ be the vertices in the path P_n and $w_1, \dots, w_{\frac{n-1}{2}}$ be the vertices added to obtain $A(T_n)$. Now $D_f = \{w_1, \dots, w_{\frac{n-1}{2}}, v_n\}$ be a dom-forcing set of $A(T_n)$.

Consider the case when second edge is replaced by the cycle C_3 . Since n is odd, the graph contains same number of triangles. Let $v_1, \dots, v_n$ be the vertices

in the path P_n and $w_1, \dots, w_{\frac{n-1}{2}}$ be the vertices added to obtain $A(T_n)$. Now $D_f = \{v_1, w_1, \dots, w_{\frac{n-1}{2}}\}$ be a dom-forcing set of $A(T_n)$.

In all the cases, there exists a dom-forcing set with cardinality equal to the number of triangles in $A(T_n) + 1$. Therefore, $F_d[A(T_n)] = 1 + \text{number of triangles in } A(T_n)$. $\square$

The dom-forcing set described in the above theorem has propagation time $n - 1$, which is maximum by Proposition 5.2.5. Hence the following result.

Theorem 5.3.18. *Let $A(T_n)$ denote the alternate triangular snake graph. Then $DPT[A(T_n)] = n - 1$.*

Consider the minimum propagation time of the alternate triangular snake graph.

Theorem 5.3.19. *Let n be an odd number and $A(T_n)$ denote the alternate triangular snake graph. Then $dpt[A(T_n)] = \frac{n+1}{2}$.*

Proof. Let $A(T_n)$ be the alternate triangular snake graph and n be an odd number. The alternate triangular snake graph obtained by replacing the first edge or the second edge of the path P_n by the cycle C_3 are isomorphic. Hence consider the case when the first edge is replaced by the cycle C_3 . Let $v_1, \dots, v_n$ be the vertices in the path P_n and $w_1, \dots, w_{\frac{n-1}{2}}$ be the vertices added to obtain $A(T_n)$. It can be noted that every minimum dom-forcing set contains one vertex from each of the cycle C_3 and an additional vertex. Hence in the color changing process maximum two forces occur at each time step. Also note that, each w_i is adjacent with v_{2i-1} and v_{2i} . Now for $n = 3$, $D_f = \{w_1, v_2\}$ with propagation

time 2 and for $n = 5$, $D_f = \{w_1, w_2, v_4\}$ with $dpt[A(T_5)] = 3$. For $n > 5$, the set $D_f = \{w_1, \dots, w_{\frac{n-3}{2}}, v_{n-1}, v_{\frac{n+1}{2}}\}$ be a minimum dom-forcing set of $A(T_n)$ with propagation time $\frac{n+1}{2}$. Which is minimum since only one forcing occurs at first time step. Therefore $dpt[A(T_n)] = \frac{n+1}{2}$. $\square$

Theorem 5.3.20. *Let n be an even number and $A(T_n)$ denote the alternate triangular snake graph with first edge is replaced by the cycle C_3 . Then*

$$dpt[A(T_n)] = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 0 \pmod{4} \\ \frac{n}{2} & \text{if } n \not\equiv 0 \pmod{4}. \end{cases}$$

Proof. Let n be an even number and $A(T_n)$ denote the alternate triangular snake graph with the first edge is replaced by the cycle C_3 . Then there will be only $\frac{n}{2}$ cycles. Let $v_1, \dots, v_n$ be the vertices in the path P_n and $w_1, \dots, w_{\frac{n}{2}}$ be the vertices added to obtain $A(T_n)$. It can be noted that every minimum dom-forcing set contains one vertex from each of the cycle C_3 and an additional vertex. Hence maximum of two forces occur at each time step. Now consider the following cases.

Case 1: $n \equiv 0 \pmod{4}$. $D_f = \{w_1, \dots, w_{\frac{n}{2}}, v_{\frac{n+2}{2}}\}$ be a minimum dom-forcing set with propagation time $\frac{n}{2} + 1$. Which is minimum since only one force occurs at first time step and there are even number of cycles. In this case $dpt[A(T_n)] = \frac{n}{2} + 1$.

Case 2: $n \not\equiv 0 \pmod{4}$. $D_f = \{w_1, \dots, w_{\frac{n}{2}}, v_{\frac{n}{2}}\}$ be a minimum dom-forcing set with propagation time $\frac{n}{2}$. Which is minimum since only one force occurs at the first time step and there are odd number of cycles. In this case $dpt[A(T_n)] = \frac{n}{2}$.

Therefore,

$$dpt[A(T_n)] = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 0 \pmod{4} \\ \frac{n}{2} & \text{if } n \not\equiv 0 \pmod{4}. \end{cases}$$

□

Next consider the alternate triangular snake graph with second edge is replaced by the cycle C_3 .

Theorem 5.3.21. *Let n be an even number greater than 4 and $A(T_n)$ denote the alternate triangular snake graph with second edge is replaced by the cycle C_3 . Then*

$$dpt[A(T_n)] = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n}{2} + 1 & \text{if } n \not\equiv 0 \pmod{4}. \end{cases}$$

Proof. Let n be an even number greater than 4 and $A(T_n)$ denote the alternate triangular snake graph with second edge is replaced by the cycle C_3 . Then there will be only $\frac{n}{2} - 1$ cycles. Let $v_1, \dots, v_n$ be the vertices in the path P_n and $w_1, \dots, w_{\frac{n}{2}-1}$ be the vertices added to obtain $A(T_n)$. It can be noted that every minimum dom-forcing set contains one vertex from each of the cycle C_3 and an additional vertex. Hence maximum of two forces occur at each time step. Now consider the following cases.

Case 1: $n \equiv 0 \pmod{4}$. $D_f = \{v_2, w_2, \dots, w_{\frac{n}{2}-2}, v_{n-1}, v_{\frac{n}{2}}\}$ be a minimum dom-forcing set with propagation time $\frac{n}{2}$. Which is minimum since only one force occurs at first time step and there are odd number of cycles. In this case $dpt[A(T_n)] = \frac{n}{2}$.

Case 2: $n \not\equiv 0 \pmod{4}$. For $n = 6$, $D_f = \{v_2, w_1, v_5\}$ and for $n > 6$, $D_f = \{v_2, w_2, \dots, w_{\frac{n}{2}-2}, v_{n-1}, v_{\frac{n}{2}}\}$. Then D_f be a minimum dom-forcing set with propagation time $\frac{n}{2} + 1$. Which is minimum since only one force occurs at first time step and there are even number of cycles. In this case $dpt[A(T_n)] = \frac{n}{2} + 1$.

Therefore,

$$dpt[A(T_n)] = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n}{2} + 1 & \text{if } n \not\equiv 0 \pmod{4}. \end{cases}$$

□

Remark 5.3.22. Consider the case of $A(T_4)$ with second edge is replaced by C_3 . Then $D_f = \{v_1, v_3\}$ be the minimum dom-forcing set, which is unique up to isomorphism. Hence, $dpt[A(T_4)] = 3$.

The Figure 5.8 depicts an alternate triangular snake graph $A(T_7)$ with dom-forcing number 4 which gives the maximum propagation time 6 and minimum propagation time 4.



Figure 5.8: The alternate triangular snake graph $A(T_7)$ with first edge is replaced by the cycle C_3 .

The Figure 5.9 depicts an alternate triangular snake graph $A(T_8)$ with first edge is replaced by the cycle C_3 . The dom-forcing number of $A(T_8)$ is 5, which gives the maximum propagation time 7 and minimum propagation time 5.

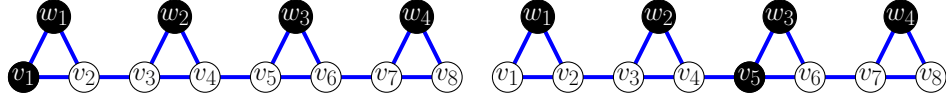


Figure 5.9: The alternate triangular snake graph $A(T_8)$ with first edge is replaced by the cycle C_3 .

The Figure 5.10 depicts an alternate triangular snake graph $A(T_8)$ with second edge is replaced by the cycle C_3 . The dom-forcing number of $A(T_8)$ is 4, which gives the maximum propagation time 7 and minimum propagation time 4.

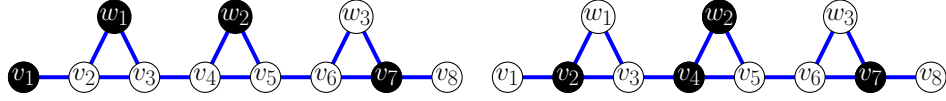


Figure 5.10: The alternate triangular snake graph $A(T_8)$ with second edge is replaced by the cycle C_3 .

Consider the helm graph H_m . the dom-forcing number of helm graph H_m has been proved and is included in Chapter 3.

Theorem 5.3.23. *For the helm graph, H_m , $dpt(H_m) = 3$.*

Proof. Let v_0 denote the vertex in helm graph such that its degree is equal to m , and let $v_1, v_2, \dots, v_m$ represent the vertices in the helm graph, each with a degree of 3. Also $u_1, u_2, \dots, u_m$ represent the pendant vertices of the helm graph H_m . To complete the colour changing process in two steps, it is needed that, all pendant vertices as black in the initial step, but it not possible because the pendant vertices does not dominates the entire graph. Therefore, $dpt(H_m) \geq 3$. Here $D_f = \{v_1, v_2, v_3, \dots, v_m\}$ be a minimum dom-forcing set with propagation time 3. Hence $dpt(H_m) = 3$. $\square$

Now the study investigates the dom-forcing propagation time of the dove tail graph, DT_n . It can be easily observed that the dom-forcing number of the dove tail graph is 2.

Theorem 5.3.24. *Let DT_n denote the dove tail graph. Then $dpt(DT_n) = DPT(DT_n) = n - 1$.*

Let DT_n be a dove tail graph and $\{u, v_1, \dots, v_n\}$ be the vertex set of DT_n with degree $\deg(u) = n$. Also let $e_1 = uv_1$ and $e_2 = uv_2$. Then the following result.

Proposition 5.3.25. *Let Γ be a connected graph. Then $dpt(\Gamma) = |V(\Gamma)| - 2$ if and only if Γ is one of the following (up to isomorphism):*

1. P_5
2. DT_n , where $n \geq 2$
3. $DT_n - e_1$ or $DT_n - e_2$, where $n \geq 2$.
4. $DT_n - \{e_1, e_2\}$, where $n \geq 4$.
5. Γ be the graph obtained from 2 to 4 (which is not a path) by appending a leaf to vertex u .

Proof. Let Γ be the one of above graphs. Then verify that $dpt(\Gamma) = |V(\Gamma)| - 2$. Conversely, assume that $dpt(\Gamma) = |V(\Gamma)| - 2$, then by Proposition 5.2.5, $dpt(\Gamma) \leq |V(\Gamma)| - F_d(\Gamma)$, ie $F_d(\Gamma) \leq 2$. Here $F_d(\Gamma)$ cannot be one, since $F_d(\Gamma) = 1$ only for P_1 and P_2 , in both cases $dpt(\Gamma) = 1$. Hence $F_d(\Gamma) = 2$ and exactly one force is performed at each time step for every minimum dom-forcing set. Without loss of generality, assume that $\{u, v\}$ be a minimum dom forcing set for Γ . If Γ is a path then $\Gamma = P_3$ or P_5 , otherwise there are two cases.

Case 1: There exists a minimum dom-forcing set having forcing sequence with $n - 1$ vertices starting with v . Therefore the graph is of the form 2 to 4 (Proposition 5.3.25).

Case 2: Every minimum dom-forcing set having forcing sequence with $n - 2$ vertices starting with v , and forcing sequence with 2 vertices starting with u . Therefore the graph is of the form 5 (Proposition 5.3.25). Hence the result. $\square$

The Chapter 5, introduced and examine the dom-forcing propagation time. The study detected the exact value of minimum and maximum dom-forcing propagation time of different snake graphs. It has also characterized graphs with extreme dom-forcing propagation time. The next chapter takes it further to explain how connectedness will change the dom-forcing set.

Chapter 6

CONNECTED DOM-FORCING SETS IN GRAPHS

This chapter¹ elaborates the concept of the connected dom-forcing number. Section 1 lays out the preliminary idea, Section 2 establishes upper and lower bounds for this parameter in terms of the connected zero forcing number and the connected domination number. Section 3 deals with the connected dom-forcing number for various types of product graphs, including the corona product, generalized corona product, rooted product, and Cartesian product of graphs. And Section 4 examines the connected dom-forcing number of the splitting graph of a given graph. the chapter considers the connected dom-forcing number for specific graphs such as $S(P_n)$, $S(C_n)$, $S(K_{1,n})$ and $S(L_n)$.

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6.1 Introduction

The concept of dom-forcing number is introduced in Chapter 3. This section integrate the concepts of connected domination and connected zero forcing to introduce a new graph-theoretic parameter: the connected dom-forcing set. In this chapter considers only connected graphs. First define the hybrid concept of connected dom-forcing set.

Definition 6.1.1. *Let $\Gamma = (V, E)$ be a graph. A set $CD_f \subseteq V$ of vertices is called a connected dom-forcing set if it satisfies the following two conditions.*

- i) CD_f must form a connected dominating set.*
- ii) CD_f must form a connected zero forcing set.*

The minimum cardinality of such a set is called the connected dom-forcing number of the graph Γ and is denoted by $F_{cd}(\Gamma)$. For instance, contemplate the graph C_5 illustrated in Figure 6.1.

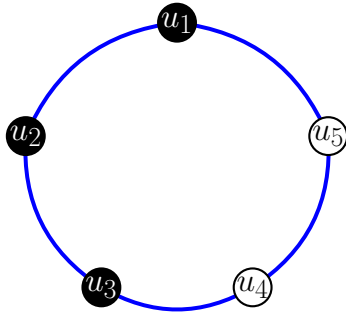


Figure 6.1: In this graph C_5 , $CD_f = \{u_1, u_2, u_3\}$ is a connected dominating as well as a zero forcing set and no subset of the vertex set with cardinality less than 3 has this property, so $F_{cd}(C_5) = 3$.

The focus of this chapter is on determining the connected dom-forcing number, denoted by $F_{cd}(\Gamma)$, for various well-known graphs and their specific products, including the corona product, rooted product, and Cartesian product of graphs. This investigation aims to explore the structural properties of $F_{cd}(\Gamma)$ and provide new insights into its applications in graph theory.

6.2 Exact Values of Connected Dom-forcing Number

This section compares the dom-forcing set and the connected dom-forcing set in the context of specific types of graphs.

From the definition, it is evident that the union of a connected zero forcing set and a connected dominating set forms a connected dom-forcing set. Consequently, the following relationship holds.

Proposition 6.2.1. *For any connected graph Γ*

$$i) \mathcal{Z}_c(\Gamma) \leq F_{cd}(\Gamma) \leq \mathcal{Z}_c(\Gamma) + \gamma_c(\Gamma)$$

$$ii) \gamma_c(\Gamma) \leq F_{cd}(\Gamma) \leq \mathcal{Z}_c(\Gamma) + \gamma_c(\Gamma)$$

Also from the definition, every connected dom-forcing set is a dom-forcing set and if a dom-forcing set is connected in components, then it is a connected dom-forcing set. Hence, the following result.

Proposition 6.2.2. *For any connected graph Γ ,*

$$F_d(\Gamma) \leq F_{cd}(\Gamma).$$

A connected acyclic spanning sub graph of a graph is called a spanning tree of a graph. A vertex in a spanning tree is called a leaf if it has degree one. A maximum leaf spanning tree is a spanning tree that has the largest possible number of leaves among all spanning trees of Γ . The maximum leaf number of Γ is the number of leaves in the maximum leaf spanning tree [44].

If d is the connected domination number of an n -vertex graph Γ , where $n > 2$, and l is its max leaf number, then the three quantities d, l , and n obey the simple equation $n = d + l$ [45]. Therefore by Proposition 6.2.1, $n - l \leq F_{cd}(\Gamma)$.

There are graphs having connected domination number is equal to the connected dom-forcing number. From the relation connecting connected domination number and max leaf, the following results obtained.

Proposition 6.2.3. *Let P_n , $n \geq 4$ be a path of order n . Then*

$$F_{cd}(P_n) = \gamma_c(P_n) = n - 2.$$

Proposition 6.2.4. *Let C_n , $n \geq 4$ be a cycle of order n . Then*

$$F_{cd}(C_n) = \gamma_c(C_n) = n - 2.$$

Proposition 6.2.5. *Let Γ be any connected graph with order greater than two. Then $F_{cd}(\Gamma) \geq 2$.*

Proof. If possible assume that $F_{cd}(\Gamma) = 1$. Then by Proposition 3.3.5, Γ must be P_1 or P_2 , both have order less than or equal to two. Hence the result. $\square$

From the definition of connected dom-forcing number, dom-forcing number and the results in Chapter 3, the following results can be verified.

Proposition 6.2.6. .

- For a complete graph K_n , $F_d(K_n) = F_{cd}(K_n) = n - 1$.
- For a complete bipartite graph $K_{m,n}$, $F_d(K_{m,n}) = F_{cd}(K_{m,n}) = m + n - 2$, where $m, n \geq 2$.
- For the star graph $K_{1,n}$, $F_d(K_{1,n}) = F_{cd}(K_{1,n}) = n$, for $n \geq 2$.
- For the Petersen graph P , $F_d(P) = F_{cd}(P) = 5$
- For a wheel graph W_n , $F_d(W_n) = F_{cd}(W_n) = 3$.
- For a hyper cube graph Q_k , $F_d(Q_k) = F_{cd}(Q_k) = 2^{k-1}$.
- For a Coconut tree graph $CT(m, n)$, $F_{cd}[CT(m, n)] = m + n - 2$.
- For a Helm graph H_m , $F_{cd}(H_m) = m + 1$.

The connected dom-forcing number of a graph and its sub graphs are not comparable. Consider the following propositions.

Proposition 6.2.7. Let Γ be a graph. If H is a sub graph of Γ , then $F_{cd}(H) \leq F_{cd}(\Gamma)$ is not true in general.

Proof. Consider the wheel graph $\Gamma = W_{16}$ given below

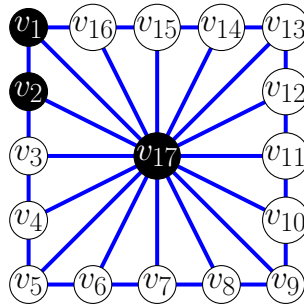


Figure 6.2: The wheel graph W_{16} .

Let $S_1 = \{v_1, v_2, v_{17}\}$. Then S_1 is the minimum connected dom-forcing set. Hence $F_{cd}(\Gamma) = 3$. Now consider a subgraph $H = C_{16}$ of Γ given below.

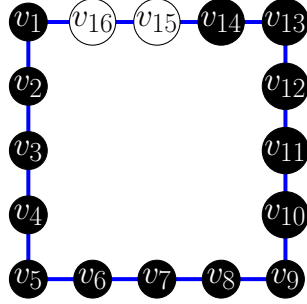


Figure 6.3: The Cycle C_{16} .

Let $S_2 = \{v_1, v_2, \dots, v_{14}\}$. Then S_2 is the minimum connected dom-forcing set. Hence $F_{cd}(H) = 14$. Therefore, the assertion follows. $\square$

Proposition 6.2.8. *Let Γ be a graph. If H is a sub graph of Γ , then $F_{cd}(H) \geq F_{cd}(\Gamma)$ is not true in general.*

Proof. Consider the graph Γ given below

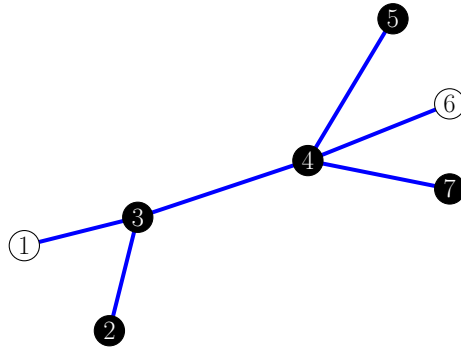


Figure 6.4: The Graph Γ .

Let $S_1 = \{2, 3, 4, 5, 7\}$. Then S_1 is the minimum connected dom-forcing set. Hence $F_{cd}(\Gamma) = 5$. Now consider a subgraph H of Γ given below.

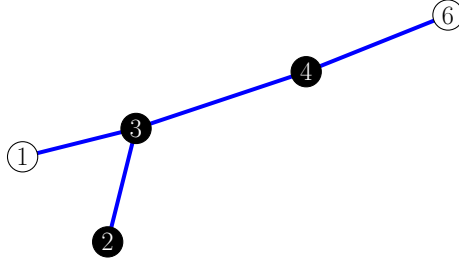


Figure 6.5: The Graph H.

Let $S_2 = \{2, 3, 4\}$. Then S_2 is the minimum connected dom-forcing set. Hence $F_{cd}(H) = 3$. Therefore, the result follows. $\square$

For a tree T , $F_{cd}(T) \geq F_{cd}(T_2)$. where T_2 is any sub tree of T .

Proposition 6.2.9. *Let T be a tree with $n \geq 3$ vertices and m leaves. Among these m leaves, n_i leaves are adjacent to a vertex v_i of T , for $i = 1, 2, \dots, r$. Then the connected dom-forcing number of T is $n - r$.*

Proof. Given T is a graph with n vertices and m leaves, hence every connected domination set contains all $n - m$ non-leaf vertices. Also, the vertex v_1 is adjacent to n_1 leaves. To force all these n_1 leaves, must color $n_1 - 1$ leaves black initially. Therefore every connected dom-forcing set must contain $n - m + (n_1 - 1) + (n_2 - 1) + \dots + (n_r - 1)$ vertices. But $n_1 + \dots + n_r = m$, hence

$$F_{cd}(T) = n - r.$$

$\square$

The above theorem is illustrated in Figure 6.6.

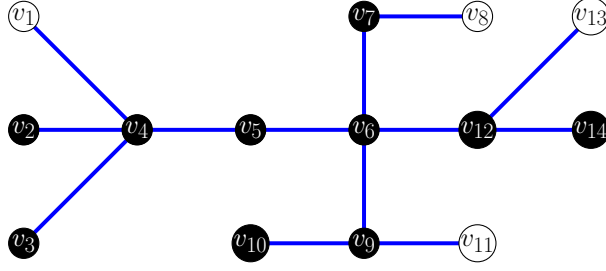


Figure 6.6: A tree T with 14 vertices and 8 leaves. Among these leaves, v_1, v_2, v_3 are adjacent to v_4 ; v_8 is adjacent to v_7 ; v_{10}, v_{11} are adjacent to v_9 and v_{13}, v_{14} are adjacent to v_{12} . Hence $F_{cd}(T) = 14 - 4 = 10$.

Now consider the connected dom-forcing number of $\Gamma \vee H$, the join of two graphs.

Proposition 6.2.10. *Let Γ and H be two connected graphs. Then*

$$F_{cd}(\Gamma \vee H) = F_d(\Gamma \vee H) = \mathcal{Z}(\Gamma \vee H) = \min\{|H| + \mathcal{Z}(\Gamma); |\Gamma| + \mathcal{Z}(H)\}.$$

Proof. Every zero forcing set dominates $\Gamma \vee H$, since by the definition of join of graphs, one vertex from both Γ and H dominates $\Gamma \vee H$. Also it is connected. Therefore,

$$F_{cd}(\Gamma \vee H) = F_d(\Gamma \vee H) = \mathcal{Z}(\Gamma \vee H) = \min\{|H| + \mathcal{Z}(\Gamma); |\Gamma| + \mathcal{Z}(H)\}.$$

□

6.3 Connected Dom-forcing Number of Product of Graphs

This section discusses the connected dom-forcing number of different types of product of graphs. Starting with the generalized corona product.

Theorem 6.3.1. [27] Let Γ be a graph with $|V(\Gamma)| = n$, and let $H_1, \dots, H_n$ be the graphs with order at least two,

$$\mathcal{Z}_c(\Gamma \prec H_1, \dots, H_n \succ) = |V(\Gamma)| + \sum_{i=1}^n \mathcal{Z}(H_i).$$

Theorem 6.3.2. Let Γ be a graph with $|V(\Gamma)| = n$, and let $H_1, \dots, H_n$ be the graphs with order at least two,

$$F_{cd}(\Gamma \prec H_1, \dots, H_n \succ) = \mathcal{Z}_c(\Gamma \prec H_1, \dots, H_n \succ) = |V(\Gamma)| + \sum_{i=1}^n \mathcal{Z}(H_i).$$

Proof. Let $\mathcal{Z}_i$ be a minimum zero forcing set of H_i for all $1 \leq i \leq n$. It can be noted that, $V(\Gamma) \cup \mathcal{Z}_1 \cup \dots \cup \mathcal{Z}_n$ is a minimum connected zero forcing set of the graph $\Gamma \prec H_1, \dots, H_n \succ$. From the definition of generalized corona product, this set dominates $\Gamma \prec H_1, \dots, H_n \succ$. Hence

$$F_{cd}(\Gamma \prec H_1, \dots, H_n \succ) = \mathcal{Z}_c(\Gamma \prec H_1, \dots, H_n \succ) = |V(\Gamma)| + \sum_{i=1}^n \mathcal{Z}(H_i).$$

□

For the graph Γ with $|V(\Gamma)| = n$ and for any graph H , from the definition, it is evident that $\Gamma \circ H = \Gamma \prec H, \dots, H \succ$. Therefore from the above theorem, the following results obtained.

Corollary 6.3.3. *For the graph Γ with $|V(\Gamma)| = n$ and any graph H with order at least two,*

$$F_{cd}(\Gamma \circ H) = n(1 + \mathcal{Z}(H)).$$

Let Γ and H be two graphs of order n_1 and n_2 respectively. For any integer $k \geq 2$, define the graph $\Gamma \circ^k H$ recursively from $\Gamma \circ H$ as $\Gamma \circ^k H = (\Gamma \circ^{k-1} H) \circ H$. It is also noted that $|\Gamma \circ^{k-1} H| = n_1(n_2 + 1)^{k-1}$ and $|\Gamma \circ^k H| = |\Gamma \circ^{k-1} H| + n_1 n_2 (n_2 + 1)^{k-1}$.

By the repeated application of Corollary 6.3.3, the following result derived.

Theorem 6.3.4. *Let Γ and H be two graphs of order n_1 and n_2 , ($n_2 \geq 2$), respectively. Then*

$$F_{cd}(\Gamma \circ^k H) = n_1(n_2 + 1)^{k-1}(\mathcal{Z}(H) + 1).$$

Theorem 6.3.5. *If Γ is a graph with $|V(\Gamma)| = n$, then $F_{cd}(\Gamma \circ K_1) = n$.*

Proof. Let $\{v_1, v_2, \dots, v_n\}$ be the vertex set of Γ . Then $A = \{v_1, \dots, v_n\}$ be a connected dom forcing set for $\Gamma \circ K_1$. Therefore $F_{cd}(\Gamma \circ K_1) \leq n$.

Note that every connected dominating set contains all vertices of Γ . Therefore, $F_{cd}(\Gamma \circ K_1) \geq n$.

From the above, $F_{cd}(\Gamma \circ K_1) = n$. □

By the repeated application of Theorem 6.3.5, the following results obtained.

Corollary 6.3.6. *For the graph Γ with $|\Gamma| = n$, $F_{cd}(\Gamma \circ^k K_1) = 2^{k-1}n$.*

Consider the connected dom-forcing number of rooted product of two graphs.

Theorem 6.3.7. *Let Γ be a graph of order n , and let H be a graph of order at least two, rooted at some vertex. Suppose the rooted vertex belongs to a minimum connected dom-forcing set of H . Then the connected dom forcing number of the rooted product of Γ by H ,*

$$F_{cd}[\Gamma(H)] = n(F_{cd}(H)).$$

Proof. Let $CD_{f_{H_i}}$ be the minimum connected dom-forcing set for the i th copy of H . Then $A = \bigcup_{i=1}^n CD_{f_{H_i}}$ forms a connected dom-forcing set for the rooted product of Γ by H . Hence

$$F_{cd}[\Gamma(H)] \leq n(F_{cd}(H)). \quad (6.1)$$

Given that the rooted vertex is an element of the minimum connected dom-forcing set, every connected dom-forcing set contains $CD_{f_{H_i}}$ for all i . Hence

$$F_{cd}[\Gamma(H)] \geq n(F_{cd}(H)). \quad (6.2)$$

From (6.1) and (6.2),

$$F_{cd}[\Gamma(H)] = n(F_{cd}(H)).$$

□

Now consider the ladder graph L_n .

Theorem 6.3.8. [26] Connected domination number of the ladder graph L_n is

$$\gamma_c(L_n) = \gamma_c(\Gamma_{2,n}) = \begin{cases} 2 & \text{if } n = 2, 3 \\ n & \text{if } n \geq 4. \end{cases}$$

Theorem 6.3.9. *The connected dom-forcing number of the ladder graph L_n is*

$$F_{cd}(L_n) = F_{cd}(\Gamma_{2,n}) = n.$$

Proof. The graph $\Gamma_{2,2}$ is cycle C_4 , and it is noted that $F_{cd}(C_4) = 2$. In the case of $\Gamma_{2,3}$, no connected domination set with cardinality 2 forces $\Gamma_{2,3}$. However the set $A = \{(1,1), (1,2), (1,3)\}$ is a dom-forcing set. Hence $F_{cd}(\Gamma_{2,3}) = 3$. For $n \geq 4$, the set $A = \{(1,1), (1,2), \dots, (1,n)\}$ forms a connected dom-forcing set, which is the minimum by Theorem 6.3.8. Therefore

$$F_{cd}(L_n) = F_{cd}(\Gamma_{2,n}) = n.$$

□

Theorem 6.3.10. [26] Connected domination number of $3 \times p$ grid graph $\Gamma_{3,p}$ is $\gamma_c(\Gamma_{3,p}) = p$.

Theorem 6.3.11. *The connected dom-forcing number of $3 \times p$ grid graph $\Gamma_{3,p}$ is*

$$F_{cd}(\Gamma_{3,p}) = p + 1.$$

Proof. It is clear that the set $A = \{(2,1), (2,2), \dots, (2,p)\}$ is a minimum connected domination set, and A is unique. The set A cannot force $\Gamma_{3,p}$. But $A \cup \{(1,1)\}$ forms a connected dom-forcing set, which is minimum. Hence

$$F_{cd}(\Gamma_{3,p}) = p + 1.$$

□

Theorem 6.3.12. [46] Connected domination number of $p \times q$ ($p \geq 4, q \geq 4$) grid graph $\Gamma_{p,q}$ is

$$\gamma_c(\Gamma_{p,q}) = \frac{pq - a'_{p,q}}{3} + \bar{r}'_{p,q} + c'_{p,q}.$$

in which

$$a'_{p,q} = p(\bmod 3) \cdot q(\bmod 3)$$

$$\bar{r}'_{p,q} = \begin{cases} 3 & p(\bmod 3).q(\bmod 3) = 4 \\ 2 & p(\bmod 3).q(\bmod 3) = 2 \\ 1 & p(\bmod 3).q(\bmod 3) = 1 \\ 0 & p(\bmod 3).q(\bmod 3) = 0 \end{cases}$$

$$c'_{p,q} = \begin{cases} \min\{\frac{p}{3}, \frac{q}{3}\} & p(\bmod 3) = 0 \text{ and } q(\bmod 3) = 0 \\ \frac{p}{3} & p(\bmod 3) = 0 \text{ and } q(\bmod 3) \neq 0 \\ \frac{q}{3} & p(\bmod 3) \neq 0 \text{ and } q(\bmod 3) = 0 \\ \lfloor \frac{p}{3} \rfloor + \lfloor \frac{q}{3} \rfloor - 1 & p(\bmod 3) \neq 0 \text{ and } q(\bmod 3) \neq 0 \end{cases}$$

Theorem 6.3.13. For $p \times q$ ($p \geq 4, q \geq 4$) grid graph $\Gamma_{p,q}$,

$$\gamma_c(\Gamma_{p,q}) \leq F_{cd}(\Gamma_{p,q}) \leq \gamma_c(\Gamma_{p,q}) + 1.$$

Proof. Consider the following cases.

Case 1: For any q and $p \bmod 3 = 0$, consider the vertex set $S = A \cup B \cup C$ in which $A = \{(1, 2), (2, 2), \dots, (p, 2)\}$, $B = \{(2, 3), (2, 4), \dots, (2, q)\}$, and $C = \{(x, 3), \dots, (x, q) | x = 5, 8, \dots, p-1\}$. Theorem 6.3.12 says that S is a minimum connected dominating set with cardinality $\frac{p(q+1)}{3}$.

Case 2: For any p and $q \bmod 3 = 0$, consider the vertex set $S = A \cup B \cup C'$ in which $C' = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q-1\}$. Then S is a minimum connected dominating set with cardinality $\frac{(p+1)q}{3}$ (By Theorem 6.3.12). In both cases, the set $A \cup \{(1, 1)\}$ is sufficient to force the entire graph. Hence

$$\gamma_c(\Gamma_{p,q}) \leq F_{cd}(\Gamma_{p,q}) \leq \gamma_c(\Gamma_{p,q}) + 1.$$

Case 3: For $p \pmod 3 = 1$ and $q \pmod 3 = 1$, consider the vertex set

$$S = \begin{cases} A \cup B_1 \cup C_1 \cup E_1 & \text{for } p = 4 \\ A \cup B \cup C_1 \cup D_1 \cup E_1 & \text{for } p > 4, \end{cases}$$

where $B_1 = \{(2, 3), (2, 4), \dots, (2, q-2)\}$, $C_1 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q-2\}$, $D_1 = \{(x, q-1), (x, q) | x = 5, 8, \dots, p-5\}$ and $E_1 = \{(p-1, q-1), (p-1, q), (p-2, q)\}$. Then S is a minimum connected dominating set with cardinality $\frac{pq+p+q-3}{3}$ (By Theorem 6.3.12). The set S forces the entire graph, as explained below.

Color the vertices in S black. The only white neighbor of $(p-1, q)$ is (p, q) . Therefore, $(p-1, q) \rightarrow (p, q) \rightarrow (p, q-1)$. Since the vertices in C_1 and D_1 are already black, the top three rows can be forced using C_1 and D_1 . Hence, the entire graph can be forced to become black. Therefore,

$$\gamma_c(\Gamma_{p,q}) = F_{cd}(\Gamma_{p,q}).$$

Case 4: For $p \pmod 3 = 1$ and $q \pmod 3 = 2$, consider the vertex set $S = A \cup B \cup C_2 \cup D_2 \cup E_2$, in which $C_2 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q-3\}$, $D_2 = \{(x, q-2), (x, q-1), (x, q) | x = 5, 8, \dots, p-2\}$ and $E_2 = \{(p, q-2), (p, q-1)\}$. Then S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (by Theorem 6.3.12). This set forces the entire graph, as explained below.

Color the vertices in S black. The only white neighbor of $(p, q-2)$ is $(p-1, q-2)$; therefore, $(p, q-2) \rightarrow (p-1, q-2) \rightarrow (p-1, q-1) \rightarrow (p-1, q) \rightarrow (p, q)$.

Since the vertices in C_2 and D_2 are black, the top four rows are forced by C_2 and D_2 . Hence, the entire graph is forced to be black. Therefore,

$$\gamma_c(\Gamma_{p,q}) = F_{cd}(\Gamma_{p,q}).$$

Case 5: For $p \pmod{3} = 2$ and $q \pmod{3} = 1$, consider the vertex set $S = A \cup B \cup C_3 \cup D_3 \cup E_3$, in which $C_3 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q - 2\}$, $D_3 = \{(x, q - 2), (x, q - 1), (x, q) | x = 5, 8, \dots, p - 3\}$ and $E_3 = \{(p, q - 1), (p, q)\}$. Then S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (By Theorem 6.3.12). This set forces the entire graph, as follows.

Color the vertices in S as black. The only white neighbor of $(p, q - 1)$ is $(p - 1, q - 1)$, and the only white neighbor of (p, q) is $(p - 1, q)$. Therefore $(p, q - 1) \rightarrow (p - 1, q - 1) \rightarrow (p - 2, q - 1); (p, q) \rightarrow (p - 1, q) \rightarrow (p - 2, q)$. Since the vertices in B_3, C_3 and D_3 are black, the top three rows are forced by B_3, C_3 and D_3 . Hence, the entire graph is forced to be black. Therefore,

$$\gamma_c(\Gamma_{p,q}) = F_{cd}(\Gamma_{p,q}).$$

Case 6: For $p \pmod{3} = 2$ and $q \pmod{3} = 2$, consider the vertex set $S = A \cup B \cup C_4 \cup D_4 \cup E_4$, in which $C_4 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q - 3\}$, $D_4 = \{(x, q - 2), (x, q - 1), (x, q) | x = 5, 8, \dots, p - 3\}$ and $E_4 = \{(p, q - 2), (p, q - 1), (p, q)\}$. Then S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (by Theorem 6.3.12). This set forces the entire graph as follows.

Color the vertices in S as black. The only white neighbor of $(p, q-2)$ is $(p-1, q-2)$, the only white neighbor of $(p, q-1)$ is $(p-1, q-1)$ and the only white neighbor of (p, q) is $(p-1, q)$. Therefore $(p, q-2) \rightarrow (p-1, q-2) \rightarrow (p-2, q-2)$; $(p, q-1) \rightarrow (p-1, q-1) \rightarrow (p-2, q-1)$; $(p, q) \rightarrow (p-1, q) \rightarrow (p-2, q)$. Again vertices in B_4 , C_4 and D_4 are black, the top four rows are forced by B_4 , C_4 and D_4 . Hence, the entire graph is forced to be black. Therefore,

$$\gamma_c(\Gamma_{p,q}) = F_{cd}(\Gamma_{p,q}).$$

In all cases,

$$\gamma_c(\Gamma_{p,q}) \leq F_{cd}(\Gamma_{p,q}) \leq \gamma_c(\Gamma_{p,q}) + 1.$$

□

Corollary 6.3.14. *For p and $q \equiv 1 \pmod{3}$, the connected dom-forcing number of a $p \times q$ grid graph $\Gamma_{p,q}$ is*

$$F_{cd}(\Gamma_{p,q}) = \frac{pq + p + q - 3}{3}.$$

Corollary 6.3.15. *If p, q is not a multiple of 3 and p or $q \equiv 2 \pmod{3}$, then connected dom-forcing number of a $p \times q$ grid graph $\Gamma_{p,q}$ is*

$$F_{cd}(\Gamma_{p,q}) = \frac{pq + p + q - 2}{3}.$$

Theorem 6.3.16. *The connected dom-forcing number of Cartesian product of cycle C_n and path P_2 is n . That is,*

$$F_{cd}(C_n \square P_2) = n.$$

Proof. Let $v_1, \dots, v_n$ be the vertices of one copy of C_n in $C_n \square P_2$ and $v'_1, \dots, v'_n$ be the vertices of other copy of C_n in $C_n \square P_2$, each v_i is adjacent to v'_i . It can

be noted that every connected dominating set contains at least n vertices, and let these vertices be $\{v_1, \dots, v_n\}$. Also these vertices force the entire graph as black. Hence

$$F_{cd}(C_n \square P_2) = n.$$

□

6.4 Connected Dom-forcing Number of Splitting Graph of a Graph

The splitting graph of a graph Γ is the graph $S(\Gamma)$ obtained by taking a vertex v' corresponding to each vertex $v \in \Gamma$ and join v' to all the vertices of Γ adjacent to v [39].

Theorem 6.4.1. [47] Let Γ be a connected graph of order n . Then

$$\gamma_c(S(\Gamma)) = \begin{cases} 2 & \text{if } \gamma_c(\Gamma) = 1 \\ \gamma_c(\Gamma) & \text{if } \gamma_c(\Gamma) \geq 2. \end{cases}$$

Theorem 6.4.2. [30] Let $S(P_n)$ denote the splitting graph of the path P_n . Then $\mathcal{Z}[S(P_n)] = 2\mathcal{Z}(P_n) = 2$.

Theorem 6.4.3. Let $S(P_n)$ denote the splitting graph of the path P_n . Then

$$n - 1 \leq F_{cd}[S(P_n)] \leq n.$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the path graph P_n in $S(P_n)$, and let $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(P_n)$. Then $A = \{v_2, \dots, v_{n-1}\}$ be a minimum connected dominating

set by Theorem 6.4.1 and is unique. It can be noted that this set is not a zero forcing set. Hence $F_{cd}[S(P_n)] \geq n - 1$. The set $B = \{v_1, u_1\}$ zero forces $S(P_n)$ by Theorem 6.4.2. Therefore, $A \cup B$ forms a connected dom-forcing set of $S(P_n)$. Hence $F_{cd}[S(P_n)] \leq n$. From these, $n - 1 \leq F_{cd}[S(P_n)] \leq n$. $\square$

The Chapter 3 derived the dom-forcing number of $S(P_n)$, the splitting graph of the path P_n , is $F_d[S(P_n)] = n$, for $2 \leq n \leq 4$. From the above two Theorems, the following result can be verified.

Corollary 6.4.4. *Let $S(P_n)$ denote the splitting graph of the path P_n . Then for $2 \leq n \leq 4$, $F_{cd}[S(P_n)] = n$.*

Theorem 6.4.5. *Let $S(P_n)$ denote the splitting graph of the path P_n . Then for $n \geq 5$, $F_{cd}[S(P_n)] = n - 1$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the path graph P_n in $S(P_n)$, and let $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(P_n)$. For $S(P_5)$, $CD_f = \{v_2, u_3, v_4, v_5\}$ and for $S(P_6)$, $CD_f = \{v_2, v_3, u_4, v_5, v_6\}$ form a connected dom-forcing set and is minimum. For $n \geq 7$, consider the subsets $CD_f = \{v_2, v_3, u_3, u_4, v_5, \dots, v_{n-1}\}$ of the vertex set of $S(P_n)$. The set CD_f forms a connected dom-forcing set of the graph $S(P_n)$ and is minimum. Hence for $n \geq 7$, $F_{cd}[S(P_n)] = n - 1$. $\square$

Theorem 6.4.6. [30] *Let $S(C_n)$ denote the splitting graph of the cycle C_n . Then $\mathcal{Z}(S(C_n)) = 2\mathcal{Z}(C_n) = 4$.*

Theorem 6.4.7. *Let $S(C_n)$ denote the splitting graph of the cycle C_n . Then*

$$n - 1 \leq F_{cd}[S(C_n)] \leq n.$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle graph C_n in $S(C_n)$, and let $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(C_n)$. Then $A = \{v_1, \dots, v_{n-2}\}$ is a minimum connected dominating set by Theorem 6.4.1. This set is unique up to isomorphism. It can be noted that this set is not a zero forcing set. Hence

$$F_{cd}[S(C_n)] \geq n - 1. \quad (6.3)$$

The set $B = \{v_1, v_2, u_2, u_3\}$ zero forces $S(C_n)$ by Theorem 6.4.6. Therefore $A \cup B$ be a connected dom-forcing set of $S(C_n)$. Hence

$$F_{cd}[S(C_n)] \leq n. \quad (6.4)$$

From (6.3) and (6.4), $n - 1 \leq F_{cd}[S(C_n)] \leq n$. □

Corollary 6.4.8. *Let $S(C_n)$ denote the splitting graph of the cycle C_n . Then for $n \geq 7$, $F_{cd}[S(C_n)] = n - 1$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle graph C_n in $S(C_n)$, and let $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(C_n)$. For $n \geq 7$, consider the subset $CD_f = \{v_1, v_2, u_2, u_3, v_4, \dots, v_{n-2}\}$ of the vertex set of $S(C_n)$. The set CD_f forms a connected dom-forcing set of the graph $S(C_n)$ and is minimum. Hence for $n \geq 7$, $F_{cd}[S(C_n)] = n - 1$. □

Chapter 3 proved $F_d[S(K_{1,n})] = 2n - 1$. Now investigates its connected dom-forcing number.

Theorem 6.4.9. *Let $S(K_{1,n})$ denote the splitting graph of the star graph $K_{1,n}$. Then for $n \geq 2$, $F_{cd}[S(K_{1,n})] = 2n - 1$.*

Proof. For $n \geq 2$, let $u_1, v_1, v_2, \dots, v_n$ be the vertices of the star graph $K_{1,n}$ with $\deg(u_1) = n$ and $u'_1, v'_1, v'_2, \dots, v'_n$ be the vertices corresponding to $u_1, v_1, v_2, \dots, v_n$ which are added to obtain $S(K_{1,n})$. From the above theorem $B = \{u_1, v_2, \dots, v_n, v'_2, \dots, v'_n\}$ forms a dom-forcing set which is connected. Hence $F_{cd}[S(K_{1,n})] = 2n - 1$. $\square$

Theorem 6.4.10. [30] Let $S(L_n)$ denote the splitting graph of the ladder graph L_n . Then for $n \geq 2$, $\mathcal{Z}[S(L_n)] = 4$.

Theorem 6.4.11. For $n \geq 2$,

$$n + 1 \leq F_{cd}[S(L_n)] \leq n + 2.$$

Proof. Let $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ be the vertices of the ladder graph L_n and let $\{u'_1, u'_2, \dots, u'_n, v'_1, v'_2, \dots, v'_n\}$ be the vertices corresponding to $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ which are added to obtain $S(L_n)$. Then $A = \{v_1, \dots, v_n\}$ is a minimum connected dominating set by Theorem 6.4.1. It can be noted that no minimum connected dominating set zero forces the entire graph. Hence

$$F_{cd}[S(L_n)] \geq n + 1. \quad (6.5)$$

The set $B = \{v_1, v_2, u'_1, v'_1\}$ zero forces $S(L_n)$ by the above Theorem. Therefore, the set $A \cup B$ is a connected dom-forcing set of $S(L_n)$. Hence

$$F_{cd}[S(L_n)] \leq n + 2. \quad (6.6)$$

From (6.5) and (6.6),

$$n + 1 \leq F_d[S(L_n)] \leq n + 2.$$

$\square$

Chapter 6 examined the way in which connectedness changed the dom-forcing set in various type of graphs. The next chapter will examine how connectedness changed the dom-forcing propagation time.

CONNECTED DOM-FORCING PROPAGATION TIMES IN GRAPHS

This chapter¹ computes the connected dom forcing propagation time of several classes of graphs, such as paths, cycles, the star graph, etc. Section 1 introduces the concept, section 2 explains the connected dom-forcing propagation. Section 3 elaborates the notion of *one – dfpt* graphs and identify several classes of such graphs, and section 4 investigates the graphs having connected dom-forcing propagation time more than one. Also, characterize graphs having $\text{dpt}_c(G) = |V(G)| - 1$ and $\text{dpt}_c(G) = |V(G)| - 2$.

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7.1 Introduction

The propagation time of a zero forcing set is defined as the number of steps required to color all vertices of a graph black, starting from the initial zero forcing set. At each step, all possible independent forcing actions are applied simultaneously; that is, every vertex that is able to force a neighboring vertex to change color does so in parallel until no further forcing can occur. The propagation time of a zero forcing set is investigated in [37] and the connected propagation time is discussed in [27]. This chapter extends connected propagation time into connected dom-forcing sets and introduce the connected dom-forcing propagation time of a graph. Additionally, it delves into the precise determination of minimum connected dom-forcing propagation time for certain well-known graphs.

7.2 Connected Dom-forcing Propagation Time of Graphs

This section introduces the concept of connected dom-forcing propagation time of graphs. Also investigated some bounds for connected dom-forcing propagation time of graphs. Connected propagation time is already defined in [27].

Definition 7.2.1. [27] Let $\Gamma = (V, E)$ be a graph and A is a connected zero forcing set of Γ . Define $A^{(0)} = A$, and for $t \geq 0$, $A^{(t+1)}$ be the set of vertices w for which there exists a vertex $a \in \bigcup_{s=0}^t A^{(s)}$ such that w is the only neighbor of the vertex a not in $\bigcup_{s=0}^t A^{(s)}$. The connected propagation time of A in Γ , denoted $pt_c(\Gamma, A)$, is the smallest integer t_0 such that $V = \bigcup_{s=0}^{t_0} A^{(s)}$. Two minimum

connected zero forcing sets of the same graph may have different propagation times. The minimum connected propagation time of Γ is

$$pt_c(\Gamma) = \min\{pt_c(\Gamma, A) | A \text{ is a minimum connected zero forcing set of } \Gamma\}.$$

This concept is extended to connected dom-forcing sets. Every connected dom-forcing set is a connected dominating set as well as a connected zero-forcing set. Hence, to find out $pt_c(\Gamma, A)$, where A is a connected dom-forcing set. The connected dom-forcing propagation time of a graph Γ is defined as follows.

Definition 7.2.2. Let $\Gamma = (V, E)$ be a graph and C is a connected dom-forcing set of Γ . Define $C^{(0)} = C$, and for $t \geq 0$, $C^{(t+1)}$ is the set of vertices w for which there exists a vertex $a \in \bigcup_{s=0}^t C^{(s)}$ such that w is the only neighbor of the vertex a not in $\bigcup_{s=0}^t C^{(s)}$. The connected dom-forcing propagation time of C in Γ , denoted $pt_c(\Gamma, C)$, is the smallest integer t_0 such that $V = \bigcup_{s=0}^{t_0} C^{(s)}$. Two minimum connected dom-forcing sets of the same graph may have different propagation times. The minimum connected dom-forcing propagation time of Γ is

$$dpt_c(\Gamma) = \min\{pt_c(\Gamma, C) | C \text{ is a minimum connected dom-forcing set of } \Gamma\}.$$

Every connected dom-forcing set is also a dom-forcing set, a connected zero forcing set, and a zero forcing set. Hence, the following observations.

Observation 7.2.3. For any graph Γ ,

1. $dpt_c(\Gamma) \leq dpt(\Gamma) \leq pt(\Gamma)$.
2. $dpt_c(\Gamma) \leq pt_c(\Gamma) \leq pt(\Gamma)$.

Similar to the dom-forcing propagation time, the connected dom-forcing propagation time of a graph Γ and its sub graph H are not comparable. Consider the following examples.

Let Γ be a graph. If H is a sub graph of Γ , then $dpt_c(H) \leq dpt_c(\Gamma)$ is not true in general.

Consider the graph $\Gamma = K_4$ depicted in Figure 7.1.

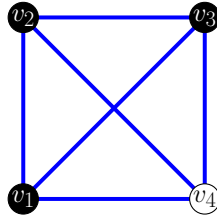


Figure 7.1: The Graph Γ .

Let $CD_f = \{v_1, v_2, v_3\}$ be the connected dom-forcing set for K_4 and $dpt_c(\Gamma) =$

1. Now consider a sub graph $H = K_4 - e$ of Γ given below.

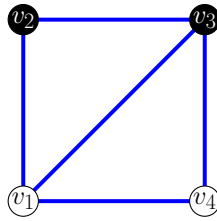


Figure 7.2: The Graph H .

Here $CD'_f = \{v_2, v_3\}$ is a minimum connected dom-forcing set for H and $dpt_c(H) = 2$. Therefore, the assertion follows. Converse of the above assertion is need not be true in general. See the following example.

Let Γ be a graph. If H is a sub graph of Γ , then $dpt(H) \geq dpt(\Gamma)$ is not true

in general.

Consider the wheel graph $\Gamma = W_8$ depicted in Figure 3.

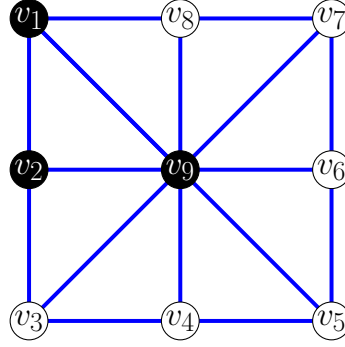


Figure 7.3: The wheel graph W_8 .

Consider the set $S_1 = \{v_1, v_2, v_9\}$. Then S_1 is the minimum connected dom-forcing set of W_8 . Hence $dpt_c(\Gamma) = 3$. Now consider a sub graph $H = C_8$ of Γ given below.

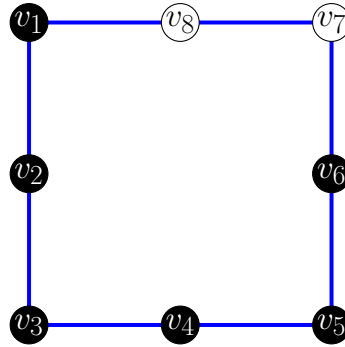


Figure 7.4: The Cycle C_8 .

Let $S_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then S_2 is the minimum connected dom-forcing set for H . Hence $dpt_c(H) = 1$. Therefore, the assertion follows.

The bounds for the connected dom-forcing propagation time are given below.

Proposition 7.2.4. *Let Γ be a connected graph with connected dom-forcing number $F_{cd}(\Gamma)$. If $|V(\Gamma)|$ is the number of vertices of Γ , then*

$$\left\lceil \frac{|V(\Gamma)| - F_{cd}(\Gamma)}{F_{cd}(\Gamma)} \right\rceil \leq dpt_c(\Gamma) \leq |V(\Gamma)| - F_{cd}(\Gamma).$$

Proof. Let Γ be a graph with connected dom-forcing number $F_{cd}(\Gamma)$. Also let $|V(\Gamma)|$ be the number of vertices of Γ , then a total of $|V(\Gamma)| - F_{cd}(\Gamma)$ vertices are white in the first stage. In each stage, at least one vertex and at most $F_{cd}(\Gamma)$ number of vertices colored black. Hence

$$\left\lceil \frac{|V(\Gamma)| - F_{cd}(\Gamma)}{F_{cd}(\Gamma)} \right\rceil \leq dpt_c(\Gamma) \leq |V(\Gamma)| - F_{cd}(\Gamma).$$

□

These bounds are sharp, for example consider the dove tail graph DT_n , (The dove tail graph DT_n , is the graph $P_n + K_1, n \geq 2$. The dove tail graph has $n + 1$ vertices and $2n - 1$ edges [35]). $F_{cd}(DT_n) = 2$ and $dpt_c(DT_n) = n - 1 = |V(DT_n)| - F_{cd}(DT_n)$. Also for the path P_n , $F_{cd}(P_n) = n - 2$ and $dpt_c(P_n) = 1 = \left\lceil \frac{n - F_{cd}(P_n)}{F_{cd}(P_n)} \right\rceil$.

Theorem 7.2.5. *Let Γ be a graph with $pt_c(\Gamma) = 1$. Then $dpt_c(\Gamma) = 1$.*

Proof. Given that Γ is a graph with $pt_c(\Gamma) = 1$. Hence, there exists a minimum connected zero forcing set that forces all the vertices to turn black in one step. This zero forcing set also dominates the graph, establishing $dpt_c(\Gamma) = 1$. □

The converse of the above Theorem is not true. Consider the path graph P_6 . Then $dpt_c(P_6) = 1$, but $pt_c(P_6) = 5$.

Proposition 7.2.6. *Let Γ be any graph with $dpt_c(\Gamma) = 1$. Then $F_{cd}(\Gamma) \geq \frac{|V(\Gamma)|}{2}$.*

The converse of the above Proposition is not true. For example, consider the graph Γ be the graph obtained from K_4 by appending a leaf to one vertex. Then $F_{cd}(\Gamma) = 3 > |V(\Gamma)|/2$ and $dpt_c(\Gamma) = 2$.

From results in Chapter 6 and the definition of connected dom-forcing propagation time, it can be noted that the following graphs have connected dom-forcing propagation time 1.

1. Path P_n .
2. Cycle C_n .
3. Tree.
4. Complete graph K_n .
5. Complete bipartite graph $K_{m,n}$.
6. Petersen graph P .
7. Star graph $K_{1,n}$.
8. Comb graph $P_n \circ K_1$.
9. Hyper cube graph Q_k .
10. Coconut tree graph $CT(m, n)$.
11. Helm graph H_m .

It is an open problem to characterize the graph Γ for which $dpt_c(\Gamma) = 1$. For brevity, call these graphs as *one-dfpt* graphs.

7.3 Graphs Where The Connected Dom-forcing Propagation Time Greater Than One

This section investigates the graphs having connected dom-forcing propagation time greater than one.

There are graphs having connected dom-forcing propagation time greater than one. For example, consider the wheel graph W_n with n vertices.

Theorem 7.3.1. *Let W_n denote the wheel graph with n vertices. Then $dpt_c(W_n) = \lceil \frac{n-3}{2} \rceil$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the wheel graph W_n with degree $d(v_1) = n - 1$ and all other vertices have degree 3. Then $D_f = \{v_1, v_2, v_3\}$, is a minimum connected dom-forcing set and which is unique (up to isomorphism). Therefore, two forces occur simultaneously at every time step, hence $dpt_c(W_n) = \left\lceil \frac{n - F_{cd}(W_n)}{2} \right\rceil = \left\lceil \frac{n-3}{2} \right\rceil$. $\square$

The dove tail graph DT_n , is the graph $P_n + K_1$, where $n \geq 2$. The dove tail graph has $n + 1$ vertices and $2n - 1$ edges [35]. It can be easily observed that the connected dom-forcing number of the dove tail graph is 2.

Theorem 7.3.2. *Let DT_n denote the dove tail graph. Then $dpt_c(DT_n) = n - 1$.*

An antiprism graph [48] is a graph having $2n$ vertices on two cycles labeled with $\{v_1, v_2, \dots, v_n\}$ in the inner cycle, $\{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$ in the outer cycle. Joining these two cycles with edges of the form $(v_i v_j : 1 \leq i \leq n, j = n+i), (v_1 v_{2n})$ and $(v_i v_{j-1} : 2 \leq i \leq n, j = n+i)$ such that v_i and $v_{i+1(mod n)}$ are adjacent.

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It is denoted as A_n . Now consider the antiprism graph A_n and its connected propagation time.

Theorem 7.3.3. [49] Let A_n denote an antiprism graph with $2n$ vertices. Then $\gamma_c(A_n) = n - 1$.

First discuss the connected dom-forcing number of the antiprism graph.

Theorem 7.3.4. Let A_n , where $n > 3$, be an antiprism graph. Then $F_{cd}(A_n) = n$.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the inner cycle and let $\{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$ be the corresponding vertices in A_n . Consider the set $A = \{v_1, v_2, \dots, v_{n-1}\}$, which dominates A_n . The set A is a minimum connected dominating set by the above Theorem. It can be noted that the set A cannot force the entire graph A_n . But the set $A \cup \{v_{n+2}\}$ forces the entire graph. Hence $F_{cd}(A_n) = n$. $\square$

In the case of A_3 , the connected dom-forcing set $CD_f = \{v_1, v_2, v_3, v_4\}$ is the minimum connected dom-forcing set and hence $F_{cd}(A_3) = 4$.

Theorem 7.3.5. Let A_n , where $n > 4$, be an antiprism graph. Then $dpt_c(A_n) = \lceil \frac{n}{2} \rceil$.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the inner cycle and let $\{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$ be the corresponding vertices in A_n . Consider the set $C = \{v_1, v_2, \dots, v_{n-1}, v_{n+3}\}$, which is a minimum connected dom-forcing set for A_n , by the above Theorem. It can be noted that two simultaneous forces occur at each time step. Hence $dpt_c(A_n) = \lceil \frac{n}{2} \rceil$. $\square$

In the case of A_8 , the above Theorem is illustrated in Figure 7.5.

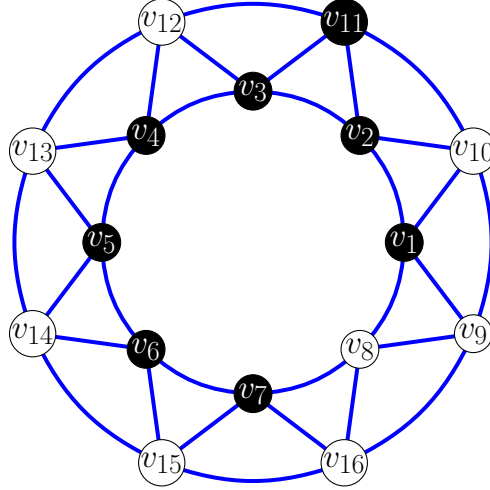


Figure 7.5: The antiprism graph A_8 , the connected dom-forcing set $A = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{11}\}$, $F_{cd}(A_8) = 8$ and $dpt_c(A_n) = 4$.

In the case of A_3 , $\{v_1, v_2, v_3, v_4\}$ is a minimum connected dom-forcing set with $dpt_c(A_3) = 1$. For A_4 , $\{v_1, v_2, v_3, v_6\}$ is a minimum connected dom-forcing set with $dpt_c(A_4) = 3$.

The gear graph is a wheel graph with a vertex added between each pair of adjacent vertices of the outer cycle. The gear graph Γ_n has $2n + 1$ vertices and $3n$ edges [49].

Theorem 7.3.6. [49] For the gear graph Γ_n , $\gamma_c(\Gamma_n) = 1 + \lceil n/2 \rceil$.

Theorem 7.3.7. For the gear graph Γ_n ,

$$F_{cd}(\Gamma_n) = 2 + \lceil n/2 \rceil.$$

Proof. Let $v, v_1, v_2, \dots, v_n$ be the vertices of the wheel graph with $\deg(v) = n$ and $u_1, u_2, \dots, u_n$ be the vertices added to get the gear graph G_n . Now consider

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the set $A = \{v, v_1, v_3, \dots, v_n\}$, if n is odd and $A = \{v, v_1, v_3, \dots, v_{n-1}\}$, if n is even. In both cases, the set A is the minimum connected dominating set by the above Theorem and which is unique up to isomorphism. Since every vertex in the set A has two white neighbors, it cannot zero forces the graph G_n . But the set $A \cup \{u_1\}$ zero forces G_n . Hence

$$F_{cd}(\Gamma_n) = 2 + \lceil n/2 \rceil.$$

□

Theorem 7.3.8. *Let Γ_n denote the gear graph. Then*

$$dpt_c(\Gamma_n) = \left\lceil \frac{|V(\Gamma)| - F_{cd}(\Gamma)}{2} \right\rceil = \begin{cases} \left\lceil \frac{3n-2}{4} \right\rceil & \text{for } n \text{ is even} \\ \left\lceil \frac{3n-3}{4} \right\rceil & \text{for } n \text{ is odd.} \end{cases}$$

Proof. Let $v, v_1, v_2, \dots, v_n$ be the vertices of the wheel graph with $\deg(v) = n$ and $u_1, u_2, \dots, u_n$ be the vertices added to get the gear graph Γ_n . Now consider the set $A = \{v, v_1, v_3, \dots, v_n, u_1\}$, where n is odd and let $A = \{v, v_1, v_3, \dots, v_{n-1}, u_1\}$, where n is even. In both the cases, the set A is the minimum connected dominating set by the above Theorem. It can be noted that two simultaneous forces occur at each time step. Hence

$$dpt_c(\Gamma_n) = \left\lceil \frac{|V(\Gamma)| - F_{cd}(\Gamma)}{2} \right\rceil = \begin{cases} \left\lceil \frac{3n-2}{4} \right\rceil & \text{for } n \text{ is even} \\ \left\lceil \frac{3n-3}{4} \right\rceil & \text{for } n \text{ is odd.} \end{cases}$$

□

The above Theorem is illustrated in Figure 9.6.

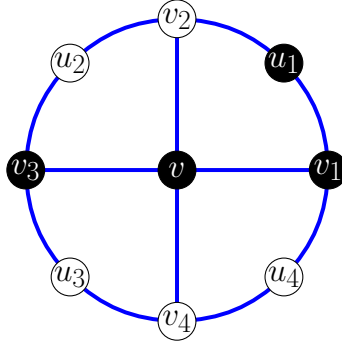


Figure 7.6: For the gear graph Γ_4 , $A = \{v, v_1, u_1, v_3\}$ is a minimum connected dom-forcing set. Here $F_{cd}(\Gamma_4) = 4$ and $dpt_c(\Gamma_4) = 3$.

Next, consider the connected dom-forcing propagation time of the generalized corona and the corona product of graphs. In Chapter 6, find the connected dom-forcing number of the generalized corona and the corona product of graphs. From these, the following results obtained.

Theorem 7.3.9. *Let Γ be a graph with $|V(\Gamma)| = n$ and $H_1, \dots, H_n$ be connected graphs with order at least two. Then*

$$dpt_c(\Gamma \prec H_1, \dots, H_n \succ) = \max\{pt(H_1), pt(H_2), \dots, pt(H_n)\}.$$

Theorem 7.3.10. *For a graph Γ with $|V(\Gamma)| = n$ and for any graph H of order at least two,*

$$dpt_c(\Gamma \circ H) = pt(H).$$

7.4 Extreme Values of Connected Dom-forcing Propagation Time

This section characterizes the graphs having connected propagation time $|V(\Gamma)| - 1$ and $|V(\Gamma)| - 2$. It can be noted that for any graph Γ , $F_d(\Gamma) = 1$ if and only if $\Gamma = P_1$ or P_2 (Proposition 3.3.5). Using this result, the following results obtained.

Proposition 7.4.1. *Let Γ be any connected graph. Then $dpt_c(\Gamma) = |V(\Gamma)| - 1$, if and only if $\Gamma = P_1$ or P_2 .*

Proof. Consider any connected graph Γ with $dpt_c(\Gamma) = |V(\Gamma)| - 1$. Then by the Proposition 7.2.4, $F_{cd}(\Gamma) \leq 1$. Since Γ is connected and $F_{cd}(\Gamma) = 1 = F_d(\Gamma)$, either $\Gamma = P_1$ or P_2 . Converse part is trivial. $\square$

Let DT_n be a dove tail graph, and $\{u, v_1, \dots, v_n\}$ be the vertex set of DT_n with degree $\deg(u) = n$. Also let $e_1 = uv_1$ and $e_2 = uv_2$. Then, the following result obtained.

Proposition 7.4.2. *Let Γ be a connected graph. Then $dpt_c(\Gamma) = |V(\Gamma)| - 2$ if and only if Γ is one of the following (up to isomorphism).*

- i) DT_n , where $n \geq 2$.
- ii) $DT_n - e_2$, where $n \geq 2$.
- iii) Γ be the graph obtained from either i or ii (which is not a path) by appending a leaf to the vertex u .

Proof. Let Γ be any one of the graphs described above. Then, it can be verified that $dpt_c(\Gamma) = |V(\Gamma)| - 2$. Conversely, assume that $dpt_c(\Gamma) = |V(\Gamma)| - 2$, then

$F_{cd}(\Gamma) = 2$ and exactly one force is performed at each time. Without loss of generality, assume that $\{u, v\}$ be a minimum connected dom-forcing set for Γ . Since $\{u, v\}$ is connected, there exists an edge connecting u and v . Consider the following cases.

Case 1: Every minimum connected dom-forcing set having forcing sequence with $n - 1$ vertices starting at v . Therefore, the graph is of the form either i or ii.

Case 2: Every minimum connected dom-forcing set having forcing sequence with $n - 2$ vertices starting with v , and forcing sequence with 2 vertices starting with u . Therefore, the graph is of the form iii. Hence the result follows. $\square$

The next chapter generalize the concept of dom-forcing sets and deals with dom- k -forcing sets in graphs.

DOM- k -FORCING SETS IN GRAPHS

This chapter¹ addresses the problem of determining the dom- k -forcing number of graphs. After introducing the concept in Section 1, Section 2 explains some bounds for the dom- k -forcing number. Also characterize graphs with extreme dom- k -forcing number. Section 3 determines the dom- k -forcing number of some graphs, where $F_{dk}(\Gamma) = \mathcal{Z}_k(\Gamma)$. Section 4 finds precise values of the dom- k -forcing number for several snake graphs. Section 5 computes the dom-forcing number of some graphs, where $F_{dk}(\Gamma) = \gamma(\Gamma)$. And the last Section 6 provides some inequalities containing the dom- k -forcing number of spitting graphs of some graphs.

8.1 Introduction

The dom-forcing set of graphs are discussed in Chapter 3. This chapter generalizes this concept. The Chapter combines k - forcing and domination, two graph

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theoretic parameters and introduce a new hybrid concept in graph theory called the dom- k -forcing set, which is defined as follows.

Definition 8.1.1. *A subset of vertices D_{kf} in a graph Γ is referred to as a dom- k -forcing set if it satisfies the properties of both a k -forcing set and a dominating set. The dom- k -forcing number of a graph Γ , denoted as $F_{dk}(\Gamma)$, represents the smallest possible size of a dom- k -forcing set.*

When $k = 1$, this is equivalent to the dom-forcing number, denoted by $F_d(\Gamma)$. When $k = 2$, this is known as the dom-2-forcing number and is denoted by $F_{d2}(\Gamma)$.

For instance, consider the graph Γ illustrated in Figure 8.1. The set $D_{fk} = \{v_2, v_5\}$ is a valid 2-forcing set as well as a dominating set, whereas $F_k = \{v_1, v_6\}$ does not satisfy the condition. The dom-2-forcing number of Γ is 2.

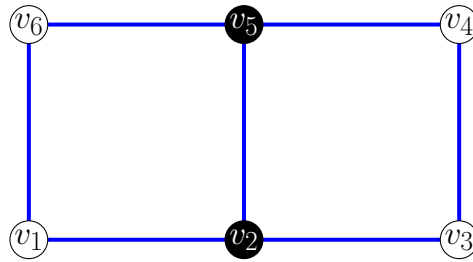


Figure 8.1: The graph $\Gamma = (V, E)$ with dom-2- forcing set $\{v_2, v_5\}$.

This chapter introduces the study of dom- k -forcing sets in graphs. It studies the dom- k -forcing sets in certain special graphs, deriving formulas for dom- k -forcing number based on these studies. Additionally, explore the relationships between this parameter and other well-known graph theoretic parameters.

8.2 Exact Values of $F_{dk}(\Gamma)$

In this section, precise values of the dom- k -forcing number for several renowned graphs are presented. Also investigating some bounds for $F_{dk}(\Gamma)$.

By definition, the union of a k -forcing set and a dominating set forms a dom- k -forcing set. Consequently, the following relationship holds.

Proposition 8.2.1. *For a graph Γ*

$$i) \mathcal{Z}_k(\Gamma) \leq F_{dk}(\Gamma) \leq \mathcal{Z}_k(\Gamma) + \gamma(\Gamma)$$

$$ii) \gamma(\Gamma) \leq F_{dk}(\Gamma) \leq \mathcal{Z}_k(\Gamma) + \gamma(\Gamma)$$

It is noted that for any graph Γ with minimum degree δ and for any positive integer k , $\mathcal{Z}_k(\Gamma) \geq \delta - k + 1$ see [4]. Combining this inequality and above Proposition, the following results obtained.

Proposition 8.2.2. *Let Γ be a graph with minimum degree δ and let k be any positive integer. Then, $F_{dk}(\Gamma) \geq \delta - k + 1$.*

From the definition it is also evident that, for a graph Γ , every dom- k -forcing set is also a dom- $k + 1$ -forcing set. Hence, the following result.

Proposition 8.2.3. *Let Γ be a graph. Then $F_{d(k+1)}(\Gamma) \leq F_{dk}(\Gamma)$, for all $k \in \mathbb{N}$.*

Theorem 8.2.4. [7] *If Γ is a connected graph with $\Delta(\Gamma) \leq 2$, then $\mathcal{Z}_2(\Gamma) = 1$.*

Theorem 8.2.5. *Let Γ be a connected graph with $\Delta(\Gamma) \leq 2$. Then $F_{d2}(\Gamma) = \gamma(\Gamma)$.*

Proof. Let Γ be a graph with $\Delta(\Gamma) \leq 2$. Then any single vertex in $V(\Gamma)$ 2-forces the graph. Hence every dominating set is also a dom-2-forcing set. Therefore, $F_{d2}(\Gamma) = \gamma(\Gamma)$. □

The above theorem can be generalized as follows.

Theorem 8.2.6. *Let Γ be a connected graph with maximum degree Δ . Then $F_{d\Delta}(\Gamma) = \gamma(\Gamma)$.*

Proof. Let Γ be a graph with maximum degree Δ . Then any single vertex in $V(\Gamma)$ Δ -forces the graph. Hence every dominating set is also a dom- Δ -forcing set. Therefore, $F_{d\Delta}(\Gamma) = \gamma(\Gamma)$. $\square$

By the above theorem it can be observed that if $k > \Delta$, then $F_{dk}(\Gamma) = \gamma(\Gamma)$. Henceforth, k is assumed to be lie strictly between 1 and Δ .

There are graphs with $\mathcal{Z}_2(\Gamma) = 1$ and $F_{d2}(\Gamma) > \gamma(\Gamma)$. Consider the graph given below. The set $\mathcal{Z}_k = \{v_1\}$ is a 2- forcing set, but it cannot dominates Γ . The set $S = \{v_2, v_5\}$ is a dominating set for Γ , but it is not a 2-forcing set. The set $\mathcal{Z}_k \cup S$ form a 2-forcing as well as a dominating set for Γ . Hence Γ is a graph with $\mathcal{Z}_2(\Gamma) = 1$, $\gamma(\Gamma) = 2$ and $F_{d2}(\Gamma) = 3$.

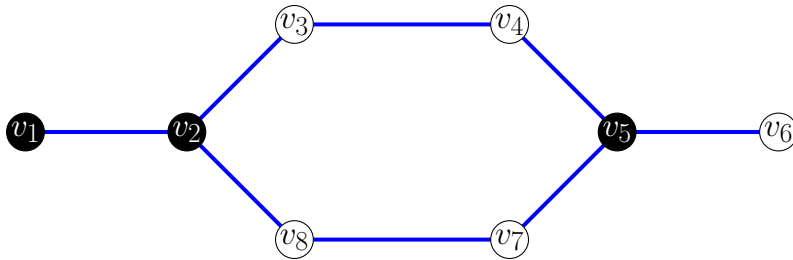


Figure 8.2: A graph Γ having $\mathcal{Z}_2(\Gamma) = 1$, $\gamma(\Gamma) = 2$ and $F_{d2}(\Gamma) = 3$.

From the above example, it is evident that the upper bound given in Proposition 6.2.1 is sharp. The following result can be observed.

Proposition 8.2.7. *If Γ is a connected graph with minimum degree $\delta(\Gamma) \geq k+1$, then $F_{dk}(\Gamma) > 1$.*

The case of $F_{d2}(\Gamma) = 1$ is investigated below.

Proposition 8.2.8. *For a graph Γ , $F_{d2}(\Gamma) = 1$ if and only if Γ is one of the following.*

- i) A path P_1, P_2 or P_3 .
- ii) The cycle C_3 .

Proof. Let Γ be one of the graphs listed above. It can be seen that $F_{d2}(\Gamma) = 1$. Conversely, assume $F_{d2}(\Gamma) = 1$. Let $\{v\}$ be the corresponding dom-2-forcing set. Then every other vertex must adjacent to v . Also the set $\{v\}$ must 2-forces Γ , the degree of v is atmost 2. Hence the graph Γ must be one of the following.

- i) A path P_1, P_2 or P_3 .
- ii) The cycle C_3 . □

The dom- k -forcing number is exactly equal to the order of the graph when the graph is totally disconnected. This result will be proved in the following proposition.

Proposition 8.2.9. *Let Γ be a graph of order $n > 1$. Then $F_{dk}(\Gamma) = n$ if and only if $\Gamma = \overline{K_n}$.*

Proof. Let $\Gamma = \overline{K_n}$. Then clearly $F_{dk}(\Gamma) = n$ for all positive integer k . If $\Gamma \neq \overline{K_n}$, then there exists a component of Γ having more than one vertex. Hence $F_{dk}(\Gamma) < n$. Therefore, $F_{dk}(\Gamma) = n$ if and only if $\Gamma = \overline{K_n}$. □

The friendship graph F_n is formed by connecting n copies of the complete graph K_3 to a single common vertex [23]. It is evident that $\gamma(F_n) = 1$ and $\mathcal{Z}_2(F_n) = n - 1$ [7].

Proposition 8.2.10. *Let F_p denote the friendship graph with $p \geq 2$ triangles. Then $F_{d2}(F_p) = p$.*

Proof. Let $v, v_1, v'_1, \dots, v_p, v'_p$ be the vertices of F_p with common vertex v . Consider the subset $A = \{v_1, \dots, v_{p-1}\}$ of $V(F_p)$. Then A is a minimum 2-forcing set, which is unique up to isomorphism. But A does not dominate F_p . Hence $F_{d2}(F_p) \geq p$. It is evident that the set $A \cup \{v\}$ dominates as well as 2-forces F_p , therefore $F_{d2}(F_p) = p$. $\square$

8.3 Exact Values of Dom- k -forcing Number of Graphs Where $\mathcal{Z}_k(\Gamma) = F_{dk}(\Gamma)$

There are graphs having the same dom- k -forcing number and the k -forcing number.

Let $\Gamma = K_n$ be a complete graph with n vertices. Then $\mathcal{Z}_k(K_n) = \max\{n - k, 1\}$ for any positive integer k [4].

Theorem 8.3.1. *For a complete graph K_n , $F_{dk}(K_n) = \mathcal{Z}_k(K_n) = \max\{n - k, 1\}$.*

Proof. For any complete graph K_n , every k -forcing set dominates K_n . Hence $F_{dk}(K_n) = \mathcal{Z}_k(K_n) = \max\{n - k, 1\}$. $\square$

Let $K_{m,n}$ be a complete bipartite where $n, m \geq 3$. Then, $\mathcal{Z}_2(K_{m,n}) = m + n - 4$ [7]. Here every 2-forcing set of $K_{m,n}$ dominates $K_{m,n}$, hence the result follows.

Theorem 8.3.2. *Let $K_{m,n}$ be a complete bipartite, where $n, m \geq 3$. Then $\mathcal{Z}_2(K_{m,n}) = F_{d2}(K_{m,n}) = m + n - 4$.*

The wheel graph, W_n , is a graph obtained by connecting a single vertex to all the vertices of a cycle graph C_{n-1} . Also, $\mathcal{Z}_2(W_n) = 2$ for all $n \geq 4$ and $\mathcal{Z}_k(W_n) = 1$ for $k \geq 3$ [7].

Theorem 8.3.3. *For a wheel graph W_n , $F_{dk}(W_n) = 2$ for $2 \leq k < n - 1$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the wheel graph W_n with degree $d(v_1) = n - 1$ and all other vertices have degree 3. Then $D_{fk} = \{v_1, v_2\}$, $2 \leq k < n - 1$ forms a dominating as well as a k -forcing set for W_n , which is minimum. Hence $F_{dk}(W_n) = 2$ for $2 \leq k < n - 1$. $\square$

Now consider the dove tail graph DT_n .

Theorem 8.3.4. *Let DT_n be a dove tail graph. Then, $F_{dk}(DT_n) = \mathcal{Z}(DT_n) = 2$, for $1 \leq k \leq n - 1$.*

Proof. Let DT_n be a dove tail graph and $\{u, v_1, \dots, v_n\}$ be the vertex set of DT_n with degree $\deg(u) = n$. The set $D = \{u\}$ is a minimum dominating set of DT_n , which is unique. For $1 \leq k \leq n - 1$, D cannot k -force DT_n . But the set $\{u, v_1\}$ k -forces the entire graph DT_n . Hence $F_{dk}(DT_n) = 2$, for $1 \leq k \leq n - 1$. $\square$

8.4 Dom- k -forcing Number of Snake Graphs

This section investigates the dom- k -forcing number of some snake graphs like diamond snake graph, triangular snake graph, double triangular snake graph and alternate triangular snake graph. Starts with the diamond snake graph D_n .

Theorem 8.4.1. *For the diamond snake graph D_n , the dom- k -forcing number $F_{dk}(D_n) = \gamma(D_n) = n + 1$, for $2 \leq k \leq 4$.*

Proof. Let $u_1, \dots, u_n, v_1, \dots, v_{n+1}, w_1, \dots, w_n$ be the vertices of the diamond snake graph D_n (see Figure 3.6). Now consider the subset $D_{fk} = \{v_1, \dots, v_{n+1}\}$ of the vertex set of D_n . The vertex set D_{fk} dominates the graph D_n as well as it k -forces the graph D_n . Hence

$$F_{dk}(D_n) \leq n + 1 \quad (8.1)$$

From proposition 8.2.1

$$F_{dk}(D_n) \geq n + 1 \quad (8.2)$$

Hence from (8.1) and (8.2), $F_{dk}(D_n) = \gamma(D_n)$. $\square$

The dom- k -forcing number of the diamond snake graph D_6 is illustrated below.

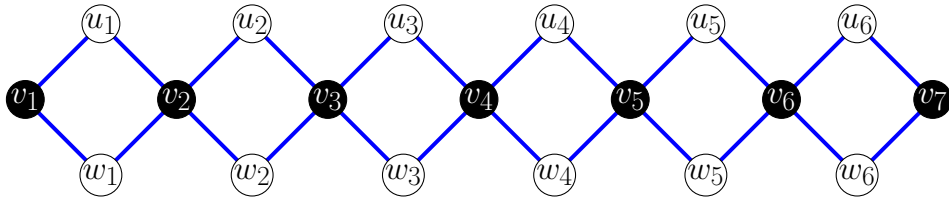


Figure 8.3: The Diamond Snake Graph D_6 , and its dom- k -forcing set $D_{fk} = \{v_1, v_2, \dots, v_7\}$ with $F_{dk}(D_6) = 7$, for $2 \leq k \leq 4$.

The triangular snake graph, TS_n , is obtained from a path P_{n+1} by replacing each edge of the path by a triangle C_3 [33].

Theorem 8.4.2. *For a triangular snake graph TS_n , $F_{d2}(TS_n) = \lceil \frac{n+1}{2} \rceil$.*

Proof. Let $v_1, v_2, \dots, v_{n+1}, u_1, u_2, \dots, u_n$ be the vertices of the triangular snake graph TS_n (see Figure 8.4). Each u_i is adjacent with v_i and v_{i+1} , for $1 \leq i \leq n$. Now consider the following cases.

Case 1: Assume n is odd, the set $D_{f2} = \{v_2, v_4, \dots, v_{n-1}, v_{n+1}\}$ forms a dominating set, which is minimum. Also this set 2-forces TS_n . Therefore, $F_d(TS_n) = \frac{n+1}{2}$.

Case 2: Assume n is even, the set $D = \{v_2, v_4, \dots, v_n\}$ forms a minimum dominating set, which is unique. This set does not 2-force TS_n . But $D_{f2} = \{v_2, v_4, \dots, v_n, v_{n+1}\}$ forms a dom-2-forcing set of TS_n . Hence $F_d(TS_n) = \frac{n+2}{2}$.

In both cases, $F_{d2}(TS_n) = \lceil \frac{n+1}{2} \rceil$.

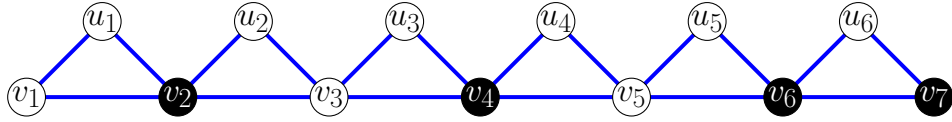


Figure 8.4: The triangular snake graph TS_6 , dom-2-forcing set $D_f = \{v_2, v_4, v_6, v_7\}$ and $F_{d2}(TS_6) = 4$.

□

Now consider the double triangular snake graph $D(T_n)$.

Theorem 8.4.3. *For the double triangular snake graph $D(T_n)$, the dom-2-forcing number $F_{d2}[D(T_n)] = n + 1$.*

Proof. Let $v_1, \dots, v_{n+1}, u_1, \dots, u_n, w_1, \dots, w_n$ be the vertices of the double triangular snake graph $D(T_n)$. It is evident that either u_i and w_i or one of the

vertices v_i or v_{i+1} must be contained in every minimum dominating set. The set $D_{f2} = \{v_1, \dots, v_{n+1}\}$ forms a dom-2-forcing set for $D(T_n)$. This set is minimum, since removing any vertex v_i from D_{f2} prevents the remaining vertices in D_{f2} from forcing either u_i or w_i . To complete the 2-forcing process, one additional vertex must be added to $D_{f2} \setminus v_i$. If both v_i and v_{i+1} are removed from D_{f2} , the resulting set fails to dominate $D(T_n)$. Hence, $F_{d2}[D(T_n)] = n + 1$.

□

The above theorem is illustrated in the case of $D(T_5)$.

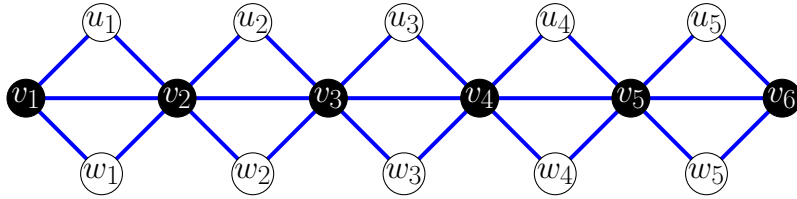


Figure 8.5: The double triangular snake graph $D(T_5)$, dom-2-forcing set $D_{f2} = \{v_1, \dots, v_6\}$ and $F_{d2}[D(T_5)] = 6$.

Investigate the dom- k -forcing number of alternate triangular snake graph $A(T_n)$.

Theorem 8.4.4. *Let $A(T_n)$ denote the alternate triangular snake graph which is obtained from the path P_n with $n > 6$ vertices. Then $F_{dk}[A(T_n)] = \gamma[A(T_n)] =$ number of triangles in $A(T_n)$, for $2 \leq k \leq 3$.*

Proof. Let $A(T_n)$ denote the alternate triangular snake graph with $n > 6$. From the definition, $A(T_n)$ is obtained from the path P_n by replacing each alternate

edge of the path P_n by a cycle C_3 . One vertex of each C_3 is contained in every dominating set of the alternate triangular snake graph $A(T_n)$. Now consider the following cases.

Case 1: Assume that n is even. Consider the case in which the first edge is replaced by a cycle C_3 . Since alternate edges are replaced by a cycle C_3 , there are only $\frac{n}{2}$ cycles and the rest of the edges will remain as the paths connecting these cycles. Let $v_1, \dots, v_n$ be the vertices in the path P_n and $w_1, \dots, w_{\frac{n}{2}}$ be the vertices added to obtain $A(T_n)$. Now it can be easily verified that $D_{fk} = \{w_1, \dots, w_{\frac{n}{2}}\}$ acts as a dom- k -forcing set for $A(T_n)$, $2 \leq k \leq 3$.

Now consider the case when second edge is replaced by the cycle C_3 . It is clear that there are $\frac{n}{2} - 1$ triangles in each $A(T_n)$. Let $v_1, \dots, v_n$ be the vertices of the path P_n and $w_1, \dots, w_{\frac{n}{2}-1}$ be the vertices added to obtain $A(T_n)$. Now $D_{fk} = \{v_2, w_2, \dots, w_{\frac{n}{2}-2}, v_{n-1}\}$ is a dom- k -forcing set of $A(T_n)$, for $2 \leq k \leq 3$.

Case 2: Assume that n is odd. Consider the case when first edge is replaced by the cycle C_3 . Since alternate edges are replaced by a cycle C_3 , there will be only $\frac{n-1}{2}$ cycles and the rest of the edges will remain as the paths connecting these cycles. Let $v_1, \dots, v_n$ be the vertices in the path P_n and $w_1, \dots, w_{\frac{n-1}{2}}$ be the vertices added to obtain $A(T_n)$. Now it can be easily observed that $D_{fk} = \{w_1, \dots, w_{\frac{n-3}{2}}, v_{n-1}\}$ acts as a dom- k -forcing set for $A(T_n)$, $2 \leq k \leq 3$.

Consider the case when the second edge is replaced by the cycle C_3 . Since n is odd, the graph contains the same number of triangles. Let $v_1, \dots, v_n$ be the vertices in the path P_n and $w_1, \dots, w_{\frac{n-1}{2}}$ be the vertices added to obtain $A(T_n)$.

Now $D_{fk} = \{v_2, w_2, \dots, w_{\frac{n-1}{2}}\}$ is a dom- k -forcing set of $A(T_n)$, for $2 \leq k \leq 3$.

In all the cases, there exists a dom-2-forcing set with the cardinality equal to the number of triangles in $A(T_n)$. Hence, $F_{d2}[A(T_n)] = \text{number of triangles in } A(T_n)$. $\square$

Now examine the value of $F_{d2}[A(T_n)]$, for $n \leq 6$.

For $n = 2$, the graph $A(T_2)$ is the cycle C_3 , and hence $F_{d2}[A(T_2)] = 1$.

When $n = 3$, the set $\{v_1, v_2\}$ forms a minimum dom-2-forcing set, giving $F_{d2}[A(T_3)] = 2$.

For $n = 4$, let $A(T_4)$ denote the alternate triangular snake graph in which either the first or the second edge is replaced by the cycle C_3 . In this case, the set $\{v_2, v_3\}$ is a minimum dom-2-forcing set, and thus $F_{d2}[A(T_4)] = 2$.

In the case $n = 5$, two non-isomorphic forms of $A(T_5)$ arise. If the first edge is replaced by the cycle C_3 , then $\{w_1, v_4\}$ is a minimum dom-2-forcing set. Alternatively, if the second edge is replaced by C_3 , then $\{v_2, w_2\}$ is a minimum dom-2-forcing set. In both configurations, $F_{d2}[A(T_5)] = 2$.

Finally, when $n = 6$, consider the alternate triangular snake graph $A(T_6)$. If the first edge is replaced by the cycle C_3 , then $\{w_1, w_2, w_3\}$ forms a minimum dom-2-forcing set. On the other hand, if the second edge is replaced by C_3 , the set $\{v_1, v_2, v_5\}$ serves as a minimum dom-2-forcing set (see Figure 6). In both cases, $F_{d2}[A(T_6)] = 3$.

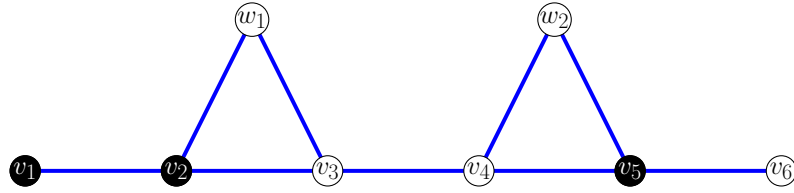


Figure 8.6: The alternate triangular snake graph $A(T_6)$ with the second edge is replaced by the cycle C_3 . The set $\{v_1, v_2, v_5\}$ is a minimum dom-2-forcing set.

8.5 Graphs Where $F_{dk}(\Gamma) = \gamma(\Gamma)$

There are some graphs with $\gamma(\Gamma) = F_{dk}(\Gamma)$. In general, the following result obtained.

Theorem 8.5.1. *Let Γ be a graph with $F_{dp}(\Gamma) = \gamma(\Gamma)$ for some positive integer p . Then $F_{dk}(\Gamma) = \gamma(\Gamma)$ for all $k \geq p$.*

From the Theorem 8.2.5, the following result established.

Theorem 8.5.2. .

1. For the path P_n , $F_{d2}(P_n) = \gamma(P_n)$.
2. For the cycle C_n , $F_{d2}(C_n) = \gamma(C_n)$.

Now consider the ladder graph L_n .

Theorem 8.5.3. *Let L_n denote the Ladder graph which is obtained from the path P_n with $n \geq 2$ vertices. Then $F_{d2}(L_n) = \gamma(L_n)$.*

Proof. Let $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ be the vertex set of the ladder graph L_n . The vertex u_1 is adjacent with u_2 and v_1 , and similarly, v_1 is adjacent with

v_2 and u_1 . Consequently, any dominating set of L_n must include either u_1 or v_1 , or both u_2 and v_2 . Moreover, each of these choices is sufficient to initiate a 2-forcing process that eventually colors the entire graph. Therefore, the dom-2-forcing number of the ladder graph equals its domination number, that is, $F_{d2}(L_n) = \gamma(L_n)$. $\square$

The sunlet graph with $2n$ vertices [50], denoted as S_n , is formed by attaching n pendant edges to the cycle C_n . The dom- k -forcing number of the sunlet graph S_n is investigated.

Theorem 8.5.4. *For the sunlet graph S_n of order $2n$, $F_{dk}(S_n) = n$, where k is any positive integer.*

Proof. Let $v_1, v_2, \dots, v_n$ represent the vertices in the sunlet graph, each with degree 3. Also let $u_1, u_2, \dots, u_n$ represent the pendant vertices of the sunlet graph S_n (See Figure 8.7). It can be easily verified that $D_{fk} = \{u_1, u_2, u_3, \dots, u_n\}$ is a dom- k -forcing set of S_n . Which is minimum, since S_n has n pendant vertices. Therefore, $F_{dk}(S_n) = n = \gamma(S_n)$.

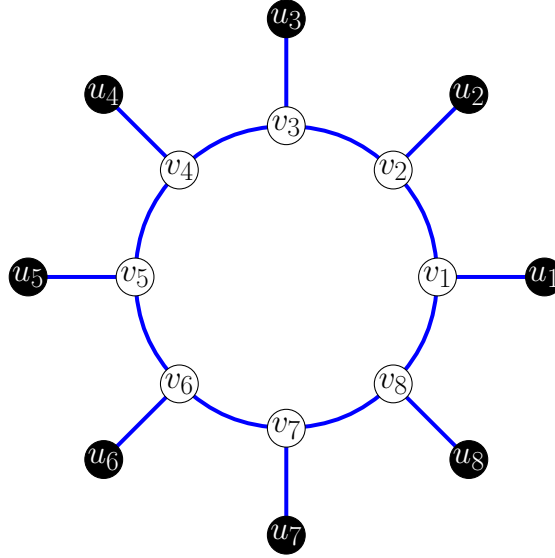


Figure 8.7: The sunlet graph S_8 , dom- k -forcing set $D_{fk} = \{u_1, u_2, u_3, \dots, u_8\}$ and $F_{dk}(S_8) = 8$.

□

The line graph of a graph Γ [51], denoted as $L(\Gamma)$, is a graph where each vertex represents an edge of Γ . Two vertices in $L(\Gamma)$ are adjacent if their corresponding edges in Γ share a common vertex. The dom-2-forcing number of line graph of sunlet graph is investigated.

Theorem 8.5.5. *The dom-2-forcing number of the line graph of the sunlet graph is*

$$F_{d2}[L(S_n)] = \begin{cases} \frac{n+2}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ represent the vertices in the line graph of the sunlet graph, each with a degree 4. Also $u_1, u_2, \dots, u_n$ represent the vertices with degree 2 (See Figure 8.8). Each u_i is adjacent to v_{i-1} and v_i , hence at least one

of the vertices from v_{i-1}, v_i belongs to every dominating set of $L(S_n)$. Since the degrees of the vertices v_i and v_{i-1} are 4, it is not possible to initiate the 2-forcing process from these vertices. Now consider the following subset of the vertex set of $L(S_n)$ and the following cases.

Case 1: If n is odd, $D_{f2} = \{u_1, v_2, v_4, \dots, v_{n-1}\}$.

Case 2: If n is even, $D_{f2} = \{u_1, v_2, v_4, \dots, v_n\}$.

In both the cases D_{f2} is a dom-2-forcing set of $L(S_n)$, which is minimum. Therefore,

$$F_{d2}[L(S_n)] = \begin{cases} \frac{n+2}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

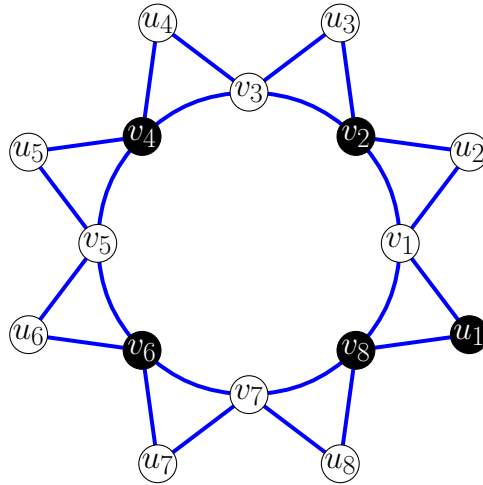


Figure 8.8: The line graph of the sunlet graph $L(S_8)$, dom-2-forcing set $D_{f2} = \{u_1, v_2, v_4, v_6, v_8\}$ and $F_{d2}(L(S_8)) = 5$.

□

Now consider the dom- k -forcing number of the grid graph $\Gamma_{m,n}$.

Theorem 8.5.6. For $m \times n$ ($m \geq 2, n \geq 2$) grid graph $\Gamma_{m,n}$,

$$\gamma(\Gamma_{m,n}) \leq F_{d2}(\Gamma_{m,n}) \leq \gamma(\Gamma_{m,n}) + 1.$$

Proof. Let $\Gamma_{m,n}$ be the grid graph. Also let S be the minimum dominating set for $\Gamma_{m,n}$, that is $|S| = \gamma(\Gamma_{m,n})$. If the set S is a 2-forcing set of $\Gamma_{m,n}$, then $F_{d2}(\Gamma_{m,n}) = \gamma(\Gamma_{m,n})$. Otherwise, $S \cup \{(1, 1)\}$ dom-2-forces $\Gamma_{m,n}$. Hence

$$\gamma(\Gamma_{m,n}) \leq F_{d2}(\Gamma_{m,n}) \leq \gamma(\Gamma_{m,n}) + 1.$$

□

Theorem 8.5.7. Let $\Gamma_{m,n}$ denote a grid graph of order mn . Then $F_{dk}(\Gamma_{m,n}) = \gamma(\Gamma_{m,n})$, for any integer $k \geq 3$.

Proof. Let $\Gamma_{m,n}$ denote a grid graph of order mn . In $\Gamma_{m,n}$, $(1, 1)$ is adjacent with $(1, 2)$ and $(2, 1)$, any one of these vertices must belong to every dominating set of $\Gamma_{m,n}$. Hence every dominating set k -forces $\Gamma_{m,n}$ for any integer $k \geq 3$. Therefore, $F_{dk}(\Gamma_{m,n}) = \gamma(\Gamma_{m,n})$, for any integer $k \geq 3$. □

Now consider the prism graph $C_n \square K_2$.

Theorem 8.5.8. Let Γ be the prism graph $C_n \square K_2$, where $v_1, v_2, \dots, v_n$ are the vertices of the cycle C_n . Let $v'_1, v'_2, \dots, v'_n$ be the corresponding vertices in $C_n \square K_2$. Now, consider a graph H obtained by subdivides the edges $v_i v'_i$ (for $i = 1, 2, \dots, n$) exactly once ($[\gamma]$). Then, $F_{d2}(H) = \gamma(H) = n$.

Proof. Let $w_1, w_2, \dots, w_n$ be the vertices which subdivides the edges $v_i v'_i$ (for $i = 1, 2, \dots, n$), respectively, exactly once. Here w_i is adjacent to v_i and v'_i , one of the vertices must belong to every minimum dominating set of H . Otherwise,

the dominating set would fail to be minimum. Hence $\gamma(H) \geq n$. Also $D = \{w_1, w_2, \dots, w_n\}$ dominates H , so $\gamma(H) = n$. Here the set D dominates and 2-forces the graph H . Therefore, $F_{d2}(H) = \gamma(H) = n$. $\square$

8.6 Dom- k -forcing Number of the Splitting Graph of a Graph

The splitting graph of a graph Γ is the graph $S(\Gamma)$ obtained by taking a vertex v' corresponding to each vertex $v \in \Gamma$ and join v' to all the vertices of Γ adjacent to v [39]. First, consider a bound for the dom- k -forcing number of the splitting graph of a graph Γ .

Theorem 8.6.1. *Let Γ be a connected graph of order $n > 1$ and $S(\Gamma)$ be its splitting graph. Then $F_{dk}[S(\Gamma)] \leq 2F_{dk}(\Gamma)$, for all positive integer k .*

Proof. Let D_{fk} be a minimum dom- k -forcing set of Γ . Suppose $D_{fk} = \{v_1, v_2, \dots, v_j\}$, where $1 \leq j < n$. Now consider the set $D'_{fk} = \{v_1, v_2, \dots, v_j\} \cup \{u_1, u_2, \dots, u_j\}$, where $u_1, u_2, \dots, u_j$ are the vertices corresponding to $v_1, v_2, \dots, v_j$ which are added to obtain $S(\Gamma)$. As in [7] D'_{fk} , k -forces $S(\Gamma)$. From the definition of splitting graph, $N(u_1, u_2, \dots, u_j) = N(D_{fk})$, where $N(D_{fk})$ denotes the vertices which are adjacent to $\{v_1, v_2, \dots, v_j\}$, and $N[v_1, v_2, \dots, v_j] = V(S(\Gamma)) - \{u_1, u_2, \dots, u_j\}$. Hence D'_{fk} dominates $S(\Gamma)$, it is a dom- k -forcing set. Therefore, $F_{dk}[S(\Gamma)] \leq 2F_{dk}(\Gamma)$, for all positive integer k . $\square$

Consider the exact values of the dom- k -forcing number of the splitting graph of a graph Γ . Start with the splitting graph of a path P_n .

Theorem 8.6.2. *Let $S(P_n)$ denote the splitting graph of the path P_n . Then*

$$\gamma[S(P_n)] \leq F_{d2}[S(P_n)] \leq \gamma[S(P_n)] + 1.$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the graph P_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(P_n)$. Also let S be the minimum dominating set for $S(P_n)$, that is $|S| = \gamma(S(P_n))$. If the set S is a 2-forcing set of $S(P_n)$, then $F_{d2}(S(P_n)) = \gamma(S(P_n))$. Otherwise, $S \cup \{u_2\}$ dom-2-forces $S(P_n)$ [7]. Hence

$$\gamma[S(P_n)] \leq F_{d2}[S(P_n)] \leq \gamma[S(P_n)] + 1.$$

□

Proposition 8.6.3. *For $n \equiv 2(\text{mod } 4)$, $F_{d2}[S(P_n)] = \gamma[S(P_n)] = \frac{n+2}{2}$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the graph P_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(P_n)$. For $S(P_2)$, $\{v_1, v_2\}$ form a dom-2-forcing set, which is minimum since $\gamma[S(P_2)] = 2$. For $n \geq 3$ and $n \equiv 2(\text{mod } 4)$, consider the subset of the vertex set of $S(P_n)$, as $D = \{v_{2+4j}, v_{3+4j}, v_{n-1}, v_n\}$ where $0 \leq j < \lfloor \frac{n}{4} \rfloor$. Clearly D dominates $S(P_n)$, which is minimum since $\gamma[S(P_n)] = \frac{n+2}{2}$. Since v_n is in D , it 2-forces $S(P_n)$. Hence $F_{d2}[S(P_n)] = \gamma[S(P_n)] = \frac{n+2}{2}$. □

For any integer $3 \leq k \leq 4$, every dominating set of $S(P_n)$ k -forces the entire graph. Hence, the following Proposition derived.

Proposition 8.6.4. *For any integer $k \geq 3$, $F_{dk}[S(P_n)] = \gamma[S(P_n)]$.*

Next, consider the splitting graph of the cycle C_n . In [7] it can be seen that, for a fixed j , $\{v_j, u_j\}$ or $\{v_{j-1}, v_j, v_{j+1}\}$ 2- forces $S(C_n)$. Where $v_1, v_2, \dots, v_n$ are the vertices of the graph C_n and $u_1, u_2, \dots, u_n$ are the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(C_n)$. Using this, the following Theorems obtained.

Theorem 8.6.5. *Let $S(C_n)$ denote the splitting graph of the cycle C_n . Then*

$$\gamma[S(C_n)] \leq F_{d2}[S(C_n)] \leq \gamma[S(C_n)] + 1.$$

Proposition 8.6.6. *If n is not a multiple of 4, then $F_{d2}[S(C_n)] = \gamma[S(C_n)]$.*

Proof. For $n \geq 3$, let $v_1, v_2, \dots, v_n$ be the vertices of the graph C_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(C_n)$. It is noted that either $\{v_1, u_1\}$ or $\{v_1, v_n, v_{n-1}\}$ forms a 2-forcing set.

For $S(C_3)$, $D_{f2} = \{u_1, v_1\}$, forms a dom-2-forcing set which is minimum. For $n \geq 5$, depending upon the number of vertices of C_n , consider the following cases and following subsets of the vertex set of $S(C_n)$.

Case 1: Assume $n \equiv 1(mod 4)$, $D_{f2} = \{v_1, v_{4m}, v_{4m+1}\}$, where $0 < m \leq \lfloor \frac{n}{4} \rfloor$.

Case 2: Assume $n \equiv 2(mod 4)$, $D_{f2} = \{v_1, v_{4m}, v_{4m+1}, v_n\}$, where $0 < m \leq \lfloor \frac{n}{4} \rfloor$.

Case 3: Assume $n \equiv 3(mod 4)$, $D_{f2} = \{u_1, v_1, v_{4m}, v_{4m+1}\}$, where $0 < m \leq \lfloor \frac{n}{4} \rfloor$.

In all the cases D_{f2} form a dom-2-forcing set. Hence $F_{d2}[S(C_n)] = \gamma[S(C_n)]$, where n is not a multiple of 4. $\square$

It can be noted that every dominating set of $S(C_n)$ k -forces, for all $k \geq 3$. Hence, the result.

Corollary 8.6.7. *For any integer $k \geq 3$, $F_{dk}[S(C_n)] = \gamma[S(C_n)]$.*

Now, consider the splitting graph of ladder graph and its dom- k -forcing number. The result for 2-forcing number and domination number is already established.

Theorem 8.6.8. [7] Let $S(L_n)$ denote the splitting graph of ladder graph L_n . Then for $n \geq 2$, $\mathcal{Z}_2[S(L_n)] = 2$.

Using Proposition 8.2.1, a bound for the dom-2-forcing number of the splitting graph of a ladder graph $S(L_n)$ can be easily obtained.

Theorem 8.6.9. For $n \geq 2$,

$$2 \left\lceil \frac{n}{3} \right\rceil \leq F_{d2}[S(L_n)] \leq 2 + 2 \left\lceil \frac{n}{3} \right\rceil.$$

For a wheel graph, $F_{d2}(W_n) = 2$ (By the Theorem 8.3.3). Also in [7], $\mathcal{Z}_2(S(W_n)) = 3$ for all $n \geq 5$. Combining the Proposition 8.2.1 and Theorem 8.6.1, the following results obtained.

Theorem 8.6.10. For a wheel graph W_n , $3 \leq F_{d2}(S(\Gamma)) \leq 4$.

This chapter investigates the dom- k -forcing number of graphs and establishes several bounds. It determines the exact values of the dom- k -forcing number and characterizes graphs for which the dom- k -forcing number is equal to the k -forcing number. The next chapter discusses the concept of the connected dom- k -forcing number.

Chapter 9

CONNECTED DOM- k -FORCING SETS IN GRAPHS

This chapter¹ introduces the study of connected dom- k -forcing sets in graphs. Connected dom- k -forcing sets are characterized for certain special graphs, and formulas for the connected dom- k -forcing number are derived based on these characterizations. Moreover, the relationships between this parameter and other well-known graph-theoretic parameters are explored.

9.1 Introduction

The connected dom-forcing set of graphs is discussed in Chapter 6. In this chapter, this concept is generalized. The concepts of connected domination and connected k -forcing are combined to introduce a new graph theoretic parameter,

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called the connected dom- k -forcing set.

Definition 9.1.1. For any $k \in \mathbb{N}, 1 \leq k \leq \Delta$, a connected dom- k -forcing set of a graph Γ , is a dom- k -forcing set of Γ that induces a sub graph of Γ which is connected. The connected dom- k -forcing number of Γ , $F_{cdk}(\Gamma)$, is the minimum size of a connected dom- k - forcing set.

For instance, consider the graph Γ illustrated in Figure 1. The set $D_{fk} = \{v_2, v_5\}$ is a valid 2-forcing set as well as a dominating set, the sub graph induced by D_{fk} is connected. Here the connected dom-2-forcing number of Γ is 2.

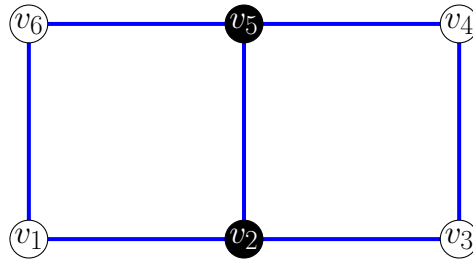


Figure 9.1: $\Gamma(V, E)$.

When $k = 1$, the connected dom- k -forcing number is equivalent to the connected dom-forcing number, denoted by $F_{cd}(\Gamma)$. When $k = 2$, this is known as the connected dom-2-forcing number and is denoted by $F_{cd2}(\Gamma)$.

9.2 Bounds for Connected Dom- k -forcing Number

This section derives some bounds for connected dom- k -forcing number. Also examine the graph having $F_{cdk}(\Gamma) = \gamma_c(\Gamma)$. From the definition, it is clear that

the union of a connected k - forcing set and a connected dominating set forms a dom- k -forcing set. Consequently, the following relationship holds.

Proposition 9.2.1. *For a graph Γ*

$$i) \mathcal{Z}_{ck}(\Gamma) \leq F_{cdk}(\Gamma) \leq \mathcal{Z}_{ck}(\Gamma) + \gamma_c(\Gamma)$$

$$ii) \gamma_c(\Gamma) \leq F_{cdk}(\Gamma) \leq \mathcal{Z}_{ck}(\Gamma) + \gamma_c(\Gamma)$$

From the definition it is also evident that, for a graph Γ , every connected dom- k -forcing set is also a connected dom- $k+1$ -forcing set. Hence, the following results obtained.

Proposition 9.2.2. *Let Γ be a graph. Then $F_{cd(k+1)}(\Gamma) \leq F_{cdk}(\Gamma)$, for all $k \in \mathbb{N}$.*

From the definition every connected dom- k -forcing set is also a dom- k -forcing set. Hence, the following results obtained.

Proposition 9.2.3. *For any connected graph Γ ,*

$$F_{dk}(\Gamma) \leq F_{cdk}(\Gamma).$$

Theorem 9.2.4. [7] *If Γ is a connected graph with $\Delta(\Gamma) \leq 2$, then $\mathcal{Z}_2(\Gamma) = 1$.*

Theorem 9.2.5. *Let Γ be a connected graph with $\Delta(\Gamma) \leq 2$. Then $F_{cd2}(\Gamma) = \gamma_c(\Gamma)$.*

Proof. Let Γ be a graph with $\Delta(\Gamma) \leq 2$. Then any single vertex in $V(\Gamma)$ is a 2-forcing set of the graph. Hence every connected dominating set is also a connected dom-2-forcing set. Therefore, $F_{cd2}(\Gamma) = \gamma_c(\Gamma)$. $\square$

There are graphs with $\mathcal{Z}_{c2}(\Gamma) = 1$ and $F_{d2}(\Gamma) > \gamma_c(\Gamma)$. Consider the graph given below. The set $\mathcal{Z}_2 = \{v_1\}$ is a connected 2- forcing set, but it cannot

dominates Γ . The set $S = \{v_2\}$ is a dominating set for Γ , but it is not a 2-forcing set. The set $\mathcal{Z}_2 \cup S$ forms a 2-forcing as well as a dominating set for Γ . Hence Γ is a graph with $\mathcal{Z}_{c2}(\Gamma) = 1$, $\gamma_c(\Gamma) = 1$ and $F_{cd2}(\Gamma) = 2$.

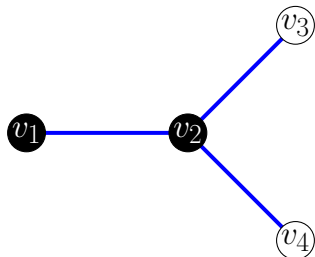


Figure 9.2: A graph Γ having $\mathcal{Z}_{c2}(\Gamma) = 1$, $\gamma_c(\Gamma) = 1$ and $F_{cd2}(\Gamma) = 2$.

From the above example, it is evident that the upper bound given in Proposition 9.2.1 is sharp. The following result can easily be observed.

Proposition 9.2.6. *If Γ is a connected graph with minimum degree $\delta \geq k + 1$, then $F_{cdk}(\Gamma) > 1$.*

In general, the following results are obtained.

Theorem 9.2.7. *Let Γ be a graph with $F_{cdp}(\Gamma) = \gamma_c(\Gamma)$ for some positive integer p . Then $F_{cdk}(\Gamma) = \gamma_c(\Gamma)$ for all $k \geq p$.*

9.3 Graphs Where $F_{dk}(\Gamma) = F_{cdk}(\Gamma)$

There exist graphs having the same dom- k -forcing number and connected dom- k -forcing number. Consider the friendship graph F_n . In Chapter 8, it has been proved that $F_{d2}(F_p) = p$. The connected dom-2-forcing set of this graph is now investigated.

Proposition 9.3.1. *Let F_p denote the friendship graph with $p \geq 2$ triangles. Then $F_{d2}(F_p) = F_{cd2}(F_p) = p$.*

Proof. Let $v, v_1, v'_1, \dots, v_p, v'_p$ be the vertices of F_p with common vertex v . Consider the subset $A = \{v, v_1, \dots, v_{p-1}\}$ of $V(F_p)$. Then A is a connected dom-2-forcing set, which is minimum and $F_{d2}(F_p) = p$. Hence $F_{d2}(F_p) = F_{cd2}(F_p) = p$. $\square$

Let $\Gamma = K_n$ be a complete graph with n vertices. It is evident that $F_{dk}(K_n) = \max\{n - k, 1\}$, for any positive integer k .

Theorem 9.3.2. *For the complete graph K_n , $F_{dk}(K_n) = F_{cdk}(K_n) = \max\{n - k, 1\}$, where k is any positive integer.*

Proof. For any complete graph K_n , every dom- k -forcing set is connected. Hence, $F_{dk}(K_n) = F_{cdk}(K_n) = \max\{n - k, 1\}$, for any positive integer k . $\square$

Let $K_{m,n}$ be a complete bipartite, where $n, m \geq 3$. Then, $F_{d2}(K_{m,n}) = m + n - 4$. Here every dom-2-forcing set of $K_{m,n}$ is connected. Hence, the following results established.

Theorem 9.3.3. *Let $K_{m,n}$ be a complete bipartite, where $n, m \geq 3$. Then $F_{d2}(K_{m,n}) = F_{cd2}(K_{m,n}) = m + n - 4$.*

Let W_n be the wheel graph. Then, $F_{dk}(W_n) = 2$, for $2 \leq k < n - 1$.

Theorem 9.3.4. *For the wheel graph W_n , $F_{dk}(W_n) = F_{cdk}(W_n) = 2$, where $2 \leq k < n - 1$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the wheel graph W_n . Assume that v_1 has degree $n - 1$ and all other vertices have degree 3. Then for $2 \leq k < n - 1$, every minimum dom- k -forcing set contains v_1 and the set connected. Therefore, $F_{dk}(W_n) = F_{cdk}(W_n) = 2$, for $2 \leq k < n - 1$. $\square$

Let H_m be the helm graph. Then, $F_{d2}(H_m) = m$. Now, consider its connected dom-2-forcing number.

Theorem 9.3.5. *Let H_m be the helm graph. Then $F_{d2}(H_m) = F_{cd2}(H_m) = m$.*

Let Γ be the prism graph $C_n \square K_2$, where $v_1, v_2, \dots, v_n$ are the vertices of the cycle C_n and $v'_1, v'_2, \dots, v'_n$ be the corresponding vertices in $C_n \square K_2$. Now, consider a graph H obtained by subdividing each edge $v_i v'_i$ (for $i = 1, 2, \dots, n$) exactly once [7].

Theorem 9.3.6. *Under the above assumptions, let H be the graph obtained from the prism graph by subdividing each edge $v_i v'_i$ (for $i = 1, 2, \dots, n$) exactly once. Then, $F_{d2}(H) = F_{cd2}(H) = n$.*

Proof. Let $w_1, w_2, \dots, w_n$ be the vertices obtained by subdividing each edges $v_i v'_i$ (for $i = 1, 2, \dots, n$) exactly once. Each vertex w_i is adjacent to both v_i and v'_i . Consequently, to dominate w_i , any dominating set of H must contain at least one of the vertices v_i, w_i , or v'_i . Hence $\gamma(H) \geq n$. Also the set $D = \{w_1, w_2, \dots, w_n\}$ dominates H , $\gamma(H) = n$. Here the set D is a 2-forcing set of the graph H and the subgraph induced by D is connected. Therefore, $F_{d2}(H) = F_{cd2}(H) = n$. $\square$

Theorem 9.3.7. *Let Γ be the graph $C_n \square P_3$, $n > 3$. Then $F_{cd2}(\Gamma) = \gamma_c(\Gamma) = n$.*

Proof. Let $u_1, \dots, u_n, v_1, \dots, v_n, w_1, \dots, w_n$ be the vertices of the three copies of C_n in $C_n \square P_3$, with $\deg(u_i) = \deg(w_i) = 3$ and $\deg(v_i) = 4$. The set $A = \{v_1, \dots, v_n\}$ is a connected dominating set for Γ . Here, $|A|$ is minimum, since removing any vertex v_i from A affects the domination property. Hence $\gamma_c(\Gamma) = n$. Moreover, the set A is a 2-forcing set of the graph Γ , since it 2-forces the entire graph. Therefore, $F_{cd2}(\Gamma) = \gamma_c(\Gamma) = n$. $\square$

9.4 Graphs Where $F_{dk}(\Gamma) < F_{cdk}(\Gamma)$

There are graphs with $F_{dk}(\Gamma) < F_{cdk}(\Gamma)$. From the Theorem 9.2.5, the following results obtained.

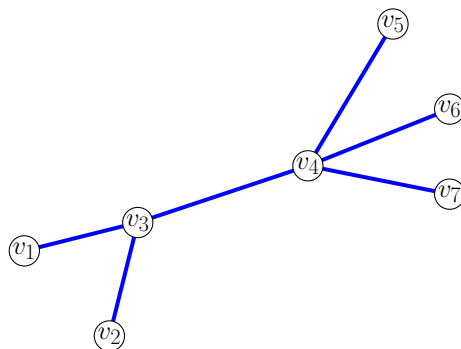
Theorem 9.4.1.

1. For a path P_n of order n , $F_{cd2}(P_n) = \gamma_c(P_n) = n - 2$.
2. For a cycle C_n of order n , $F_{cd2}(C_n) = \gamma_c(C_n) = n - 2$.

Now, consider the connected dom- k -forcing number of trees.

Definition 9.4.2. Let T be a tree. Consider the vertices which are adjacent to the leaf (pendant) vertex and call these vertices as parent leaf vertices.

For instance, consider the tree T illustrated in figure 9.3. Here the parent leaf vertices of T are v_3 and v_4 .

Figure 9.3: The tree T .

Theorem 9.4.3. *Let T be a tree. The number of leaves adjacent to each parent leaf vertex is at most k if and only if $F_{cdk}(T) = \gamma_c(T) = n - m$, where n is the number of vertices in T and m is the number of leaves in T .*

Proof. First, assume that the total number of leaves adjacent to each parent leaf vertex of the tree T is at most k . It is evident that all the vertices of T , except the pendant vertices, form a unique minimum connected dominating set, which k -forces the graph T . Hence $F_{cdk}(T) = \gamma_c(T) = n - m$, where n is the number of vertices in T and m is the number of leaves in T .

Conversely, assume that $F_{cdk}(T) = \gamma_c(T) = n - m$, where n is the number of vertices in T and m is the number of leaves in T . Let H be the subset of the vertex set of T that does not contain the pendant vertices of T . Hence $|H| = n - m$. If possible, assume that there exists a parent leaf vertex, say v , which is adjacent to at least $k + 1$ pendant vertices. All vertices, except the pendant vertices, belong to the unique minimum connected dominating set of T . Moreover, from v , it is not possible to k -force all the pendant vertices adjacent to v . Therefore, the set H cannot be a connected dom- k -forcing set of T , a contradiction. Hence, the

total number of leaves adjacent to each parent leaf vertex of the tree T is at most k . $\square$

Now consider the ladder graph L_n . It is known that for a ladder graph L_n of order n , $F_{cd}(L_n) = n = \gamma_c(L_n)$, for $n \geq 4$ (see Chapter 6). Therefore, the following Proposition.

Proposition 9.4.4. *Let L_n be the Ladder graph of order $n \geq 4$. Then $F_{cdk}(L_n) = n$, for all $k \in \mathbb{N}$.*

Let us recall the following result from [26] to prove the succeeding Theorem.

Theorem 9.4.5. *The connected dom-2-forcing number of $3 \times p$ grid graph $\Gamma_{3,p}$ is*

$$F_{cd2}(\Gamma_{3,p}) = p.$$

Proof. It is evident that the set $A = \{(2, 1), (2, 2), \dots, (2, p)\}$ is a minimum connected domination set, and is unique. The set A is a 2-forcing set of the graph $\Gamma_{3,p}$. Hence

$$F_{cd2}(\Gamma_{3,p}) = p.$$

$\square$

Theorem 9.4.6. *Let $\Gamma_{p,q}$ be a $p \times q$ ($p \geq 4, q \geq 4$) grid graph. Then*

$$F_{cd2}(\Gamma_{p,q}) = \gamma_c(\Gamma_{p,q}).$$

Proof. Consider the following cases.

Case 1: For any q and $p \pmod{3} = 0$, consider the vertex set $S = A \cup B \cup C$

in which $A = \{(1, 2), (2, 2), \dots, (p, 2)\}$, $B = \{(2, 3), (2, 4), \dots, (2, q)\}$, and $C = \{(x, 3), \dots, (x, q) | x = 5, 8, \dots, p-1\}$. Theorem 6.3.12 says that the set S is a minimum connected dominating set with cardinality $\frac{p(q+1)}{3}$.

Case 2: For any p and $q \pmod{3} = 0$, consider the vertex set $S = A \cup B \cup C'$ in which $C' = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q-1\}$. Then the set S is a minimum connected dominating set with cardinality $\frac{(p+1)q}{3}$ (By Theorem 6.3.12).

Case 3: For $p \pmod{3} = 1$ and $q \pmod{3} = 1$, consider the vertex set $S = A \cup B_1 \cup C_1 \cup E_1$ for $p = 4$ and $S = A \cup B \cup C_1 \cup D_1 \cup E_1$ for $p > 4$, in which $B_1 = \{(2, 3), (2, 4), \dots, (2, q-2)\}$, $C_1 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q-2\}$, $D_1 = \{(x, q-1), (x, q) | x = 5, 8, \dots, p-5\}$ and $E_1 = \{(p-1, q-1), (p-1, q), (p-2, q)\}$. Then the set S is a minimum connected dominating set with cardinality $\frac{pq+p+q-3}{3}$ (By Theorem 6.3.12).

Case 4: For $p \pmod{3} = 1$ and $q \pmod{3} = 2$, consider the vertex set $S = A \cup B \cup C_2 \cup D_2 \cup E_2$, in which $C_2 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q-3\}$, $D_2 = \{(x, q-2), (x, q-1), (x, q) | x = 5, 8, \dots, p-2\}$ and $E_2 = \{(p, q-2), (p, q-1)\}$. Then the set S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (By Theorem 6.3.12).

Case 5: For $p \pmod{3} = 2$ and $q \pmod{3} = 1$, consider the vertex set $S = A \cup B \cup C_3 \cup D_3 \cup E_3$, in which $C_3 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q-2\}$, $D_3 = \{(x, q-2), (x, q-1), (x, q) | x = 5, 8, \dots, p-3\}$ and $E_3 = \{(p, q-1), (p, q)\}$. Then the set S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$.

(By Theorem 6.3.12).

Case 6: For $p \pmod{3} = 2$ and $q \pmod{3} = 2$, consider the vertex set $S = A \cup B \cup C_4 \cup D_4 \cup E_4$, in which $C_4 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q - 3\}$, $D_4 = \{(x, q - 2), (x, q - 1), (x, q) | x = 5, 8, \dots, p - 3\}$ and $E_4 = \{(p, q - 2), (p, q - 1), (p, q)\}$. Then the set S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (By Theorem 6.3.12).

In all the cases the set S is a 2-forcing set the graph $\Gamma_{p,q}$. Hence

$$F_{cd2}(\Gamma_{p,q}) = \gamma_c(\Gamma_{p,q}).$$

□

Now consider the prism graph $C_n \square K_2$.

Theorem 9.4.7. [49] Let Γ be a prism graph $C_n \square K_2$, $n > 3$. Then $\gamma_c(\Gamma) = n$.

Theorem 9.4.8. Let Γ be a prism graph $C_n \square K_2$, $n > 3$. Then $F_{cdk}(\Gamma) = \gamma_c(\Gamma) = n$, for all positive integer $1 \leq k \leq 3$.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle C_n and let $v'_1, v'_2, \dots, v'_n$ be the corresponding vertices in $C_n \square K_2$. Consider the set $A = \{v_1, v_2, \dots, v_n\}$, which dominates Γ . The set A is a minimum connected dominating set by the above Theorem. It is noted that the set A forces the entire graph Γ . Hence $F_{cdk}(\Gamma) = \gamma_c(\Gamma) = n$, for all positive integer $1 \leq k \leq 3$. □

Now consider the antiprism graph A_n and its connected 2-forcing set.

Theorem 9.4.9. [49] Let A_n be an antiprism graph with $2n$ vertices. Then $\gamma_c(A_n) = n - 1$.

Theorem 9.4.10. *Let A_n , $n > 3$, be an antiprism graph. Then $F_{cd2}(A_n) = \gamma_c(A_n) = n - 1$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the inner cycle and let $\{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$ be the corresponding vertices in A_n . Consider the set $A = \{v_1, v_2, \dots, v_{n-1}\}$, which dominates A_n . The set A is a minimum connected dominating set by the above Theorem. It can be observed that the set A is a 2-forcing set of A_n , as it 2-forces the entire graph with the forcing process initiated at the vertex v_2 . Hence $F_{cd2}(A_n) = \gamma_c(A_n) = n - 1$. $\square$

In the case of A_8 , the above Theorem is illustrated in Figure 9.4.

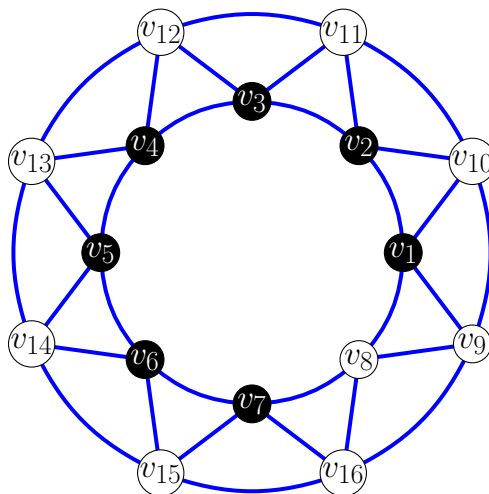


Figure 9.4: The antiprism graph A_8 , the connected dom-2-forcing set $A = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $F_{cd2}(A_8) = 7$.

The n -crossed prism graph [49], R_n , of positive even n vertices is the graph obtained by taking two disjoint cycle graphs C_n labeled with $\{v_1, v_2, \dots, v_n\}$ in the inner cycle, $\{v'_1, v'_2, \dots, v'_n\}$ in the outer cycle. Adding edges (v_k, v'_{2k+1}) and (v_{k+1}, v'_{2k}) , for $k = 1, 3, \dots, (n-1)$. The graph R_n is a 3-regular graph.

Theorem 9.4.11. [49] Let R_n be the n -crossed prism graph with $2n$ vertices. Then $\gamma_c(R_n) = n$.

Theorem 9.4.12. Let R_n be the n -crossed prism graph. Then $F_{cdk}(R_n) = \gamma_c(R_n) = n$, for all positive integer k .

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the inner cycle and let $\{v'_1, v'_2, \dots, v'_n\}$ be the corresponding vertices of the outer circle. Consider the set $A = \{v_1, v_2, \dots, v_n\}$, which dominates R_n . The set A is a minimum connected dominating set by the above Theorem. It is noted that the set A forces the entire graph R_n . Hence $F_{cdk}(R_n) = \gamma_c(R_n) = n$, for all positive integer k . $\square$

In the case of 8- crossed prism graph, R_8 , the above Theorem is illustrated in Figure 9.5.

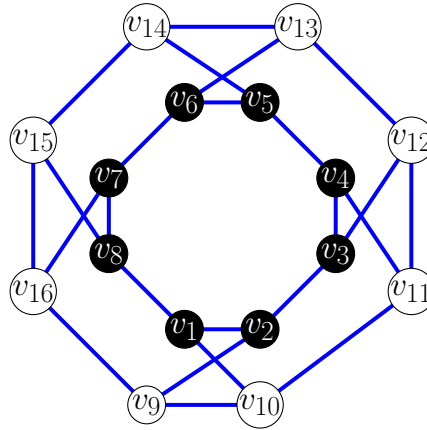


Figure 9.5: The 8-crossed prism graph R_8 , the connected dom-2-forcing set $A = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $F_{cd2}(R_8) = 8$.

Now consider the gear graph G_n .

Theorem 9.4.13. [49] For the gear graph G_n , $\gamma_c(G_n) = 1 + \lceil n/2 \rceil$.

Theorem 9.4.14. *For the gear graph G_n ,*

$$F_{cdk}(G_n) = \begin{cases} 2 + \lceil n/2 \rceil & \text{for } k = 1 \\ 1 + \lceil n/2 \rceil & \text{for } k > 1. \end{cases}$$

Proof. Let $v, v_1, v_2, \dots, v_n$ be the vertices of the wheel graph with $\deg(v) = n$ and $u_1, u_2, \dots, u_n$ be the vertices added to get the gear graph G_n . Now consider $A = \{v, v_1, v_3, \dots, v_n\}$ if n is odd and $A = \{v, v_1, v_3, \dots, v_{n-1}\}$ if n is even. In both cases, the set A is the minimum connected dominating set by the above Theorem and which is unique up to isomorphism. It is noted that the set A is a 2-forcing set of the graph G_n . Since every vertex in the set A has two white neighbors, it cannot zero forces the graph G_n . But $A \cup \{u_1\}$ zero forces G_n . Hence

$$F_{cdk}(G_n) = \begin{cases} 2 + \lceil n/2 \rceil & \text{for } k = 1 \\ 1 + \lceil n/2 \rceil & \text{for } k > 1. \end{cases}$$

□

The above theorem is illustrated in Figure 7.6.

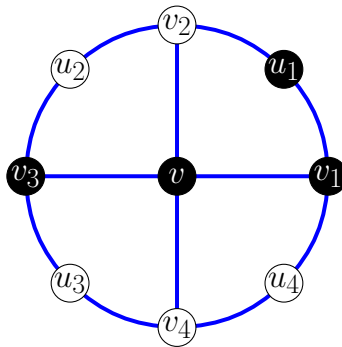


Figure 9.6: The gear graph G_4 , $A = \{v, v_1, u_1, v_3\}$ is a minimum connected dominating set and $B = \{v, v_1, v_3\}$ is a minimum connected dom-2-forcing set. Hence $F_{cd}(G_4) = 4$ and $F_{cd2}(G_4) = 3$.

9.5 Connected Dom- k -forcing Number of Splitting Graph of a Graph

The splitting graph of a graph Γ is the graph $S(\Gamma)$ obtained by taking a vertex v' corresponding to each vertex $v \in \Gamma$ and join v' to all the vertices of Γ adjacent to v [39].

Theorem 9.5.1. *Let $S(P_n)$, $n \geq 4$, denote the splitting graph of the path P_n . Then $F_{cd2}[S(P_n)] = \gamma_c[S(P_n)] = n - 2$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the path graph P_n in $S(P_n)$, and let $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S(P_n)$. Then the set $A = \{v_2, \dots, v_{n-1}\}$ is a minimum connected dominating set by the Theorem 6.4.1. It is noted that this set 2-forces $S(P_n)$. Hence $F_{cd}[S(P_n)] = n - 2$. $\square$

Remark 9.5.2. *The graph $S(P_2)$, which is isomorphic to P_4 , hence $F_{cd2}[S(P_2)] = \gamma_c[S(P_2)] = 2$. For the graph $S(P_3)$, $A = \{v_1, v_2\}$ is a minimum connected dominating set which 2-forces the entire graph. Hence $F_{cd2}[S(P_3)] = \gamma_c[S(P_3)] = 2$. Therefore in all cases, $F_{cd2}[S(P_n)] = \gamma_c[S(P_n)]$ for all n .*

Theorem 9.5.3. *Let $S(C_n)$, $n \geq 5$, denote the splitting graph of the cycle C_n . Then*

$$F_{cd2}[S(C_n)] = \gamma_c[S(C_n)] = n - 2.$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle graph C_n in $S(C_n)$, and let $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added

to obtain $S(C_n)$. Then $A = \{v_1, \dots, v_{n-2}\}$ is a minimum connected dominating set by the Theorem 6.4.1. It is noted that this set 2-forces $S(C_n)$. Hence $F_{cd}[S(C_n)] = n - 2$. $\square$

In the case of $S(C_3)$, $A = \{v_1, u_1\}$ is a minimum connected dominating set which 2-forces the entire graph. Hence $F_{cd2}[S(C_3)] = \gamma_c[S(C_3)] = 2$. For $S(C_4)$, $A = \{v_1, v_2\}$ is a minimum connected dominating set and is unique. It can be seen that this set is not a 2-forcing set. But $A' = \{v_1, v_2, u_1\}$ is a minimum connected dom-2-forcing set. Hence $F_{cd2}[S(C_3)] = 3$.

Theorem 9.5.4. For $n \geq 2$, $F_{cd3}[S(L_n)] = \gamma_c[S(L_n)] = n$.

Proof. Let $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ be the vertices of the ladder graph L_n and let $\{u'_1, u'_2, \dots, u'_n, v'_1, v'_2, \dots, v'_n\}$ be the vertices corresponding to $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ which are added to obtain $S(L_n)$. Then $A = \{v_1, \dots, v_n\}$ is a minimum connected dominating set by the Theorem 6.4.1. It can be noted that the set A is a 3-forcing set of the graph $S(L_n)$. Hence $F_{cd3}[S(L_n)] = n$. $\square$

This chapter established several bounds for the connected dom- k -forcing number and determined its exact values for certain classes of graphs. Furthermore, it was shown that there exist graphs whose dom- k -forcing number and connected dom- k -forcing number are equal.

Chapter 10

CONCLUSION

In this thesis, several new concepts are introduced, such as the dom-forcing set, connected dom-forcing set, dom-forcing propagation time, connected dom-forcing propagation time, dom- k -forcing set, and connected dom- k -forcing set. The study further explores these concepts by deriving bounds, computing exact values, and presenting various characterizations.

10.1 Summary of the Thesis

The first Chapter is the introductory Chapter. This Chapter introduces the core concepts and sets the stage for the research, providing background on the problem and its significance in graph theory.

In the second Chapter, the preliminaries, provides the essential definitions and foundational concepts in graph theory that will be used throughout the thesis.

The third Chapter focuses on calculating the dom-forcing number of graphs. It includes bounds, precise values for well-known graphs, and explores cases where

$F_d(\Gamma) = \mathcal{Z}(\Gamma)$ and $F_d(\Gamma) = \gamma(\Gamma)$. It also provides inequalities related to the dom-forcing number of splitting graphs.

Chapter 4 investigates how various graph operations (vertex/edge deletion, vertex addition, and graph sums/products) affect the dom-forcing number. It includes upper and lower bounds for edge operations and explores the impact of graph products, such as the corona and the rooted products.

Chapter 5 introduces a new parameter, the dom-forcing propagation time. This Chapter investigates bounds on the minimum and maximum propagation times in terms of the dom-forcing number and determines the propagation time for several well-known classes of graphs.

The concept of a connected dom-forcing set is introduced in Chapter 6. The Chapter explores bounds for this parameter, its relationship with the connected zero forcing number, and investigates the connected dom-forcing number for different types of graph products and splitting graphs.

Similar to Chapter 5, Chapter 7 focuses on connected dom-forcing propagation time. Also compute bounds and investigates the propagation time for various well-known graphs, as well as characterizing graphs with extreme propagation times.

Chapter 8 generalizes the dom-forcing number by introducing the dom- k -forcing number. It establishes bounds for this new parameter, investigates extreme cases, and applies the concept to specific graph classes such as snake graphs. The Chapter also explores inequalities involving the dom- k -forcing number of splitting graphs.

Chapter 9 extends the notion of the connected dom-forcing number by introducing the connected dom- k -forcing number. The Chapter investigates bounds

for this parameter, analyzes its relationship with the dom- k -forcing number, and studies how it behaves across various classes of graphs, including splitting graphs.

This study reveals that, for certain classes of graphs, the zero forcing number and the dom-forcing number coincide. It is also observed that the removal of a vertex from a graph may lead to a significant increase in its dom-forcing number, which can, in some cases, become arbitrarily large. Furthermore, there appears to be a notable tendency in graphs whereby, for sufficiently large values of k , the dom- k -forcing set coincides with the dominating set.

In addition, the comparative table presented in this study serves as an effective tool for analyzing and contrasting the dom-forcing number, connected dom-forcing number, dom- k -forcing number, and connected dom- k -forcing number for a selection of graphs. This comparative analysis provides deeper insights into the relationships among these parameters and enhances the overall understanding of their structural behavior.

Table 10.1: Dom-forcing number caparison table

Graph Γ	$F_d(\Gamma)$	$F_{cd}(\Gamma)$	$F_{dk}(\Gamma)$	$F_{cdk}(\Gamma)$
Complete Graph K_n	$n - 1$	$n - 1$	$\max\{n - k, 1\}$	$\max\{n - k, 1\}$
Complete bipartite Graph $K_{m,n}$	$m + n - 2$	$m + n - 2$	$m + n - 4$, for $k = 2$	$m + n - 4$, for $k = 2$
Helm Graph H_m	m	$m + 1$	m	m , for $k \geq 2$
Ladder Graph L_n	$\lceil \frac{n}{2} \rceil + 1$	n	$\gamma(L_n)$, for $k \geq 2$	n
Wheel graph W_n	3	3	2, for $2 \leq k \leq n - 1$	2, for $2 \leq k \leq n - 1$
Dove tail graph DT_n	2	2	2, for $1 \leq k \leq n - 1$	2, for $1 \leq k \leq n - 1$
Path P_n	$\lfloor \frac{n}{3} \rfloor + 1$	$n - 2$	$\gamma(P_n)$, for $k \geq 2$	$n - 2$
Cycle C_n	$\lfloor n/3 \rfloor + 1$ if $n \equiv 0, 1 \pmod 3$ $\lfloor n/3 \rfloor + 2$ if $n \equiv 2 \pmod 3$	$n - 2$	$\gamma(C_n)$, for $k \geq 2$	$n - 2$

RECOMMENDATIONS

This study has opened multiple meaningful directions for future research to deepen the understanding of dom-forcing sets and their connected parameters. A few emerging challenges with its potential for development are furnished below to provide a lead for future researchers.

One line of investigation is the characterization of graphs Γ for which $F_d(\Gamma) = \mathcal{Z}(\Gamma)$ and $F_d(\Gamma) = \gamma(\Gamma)$. Such characterizations would unravel the inter connect- edness of dom-forcing, zero forcing, and domination, which indeed will reveal the structural properties that unify these concepts. In addition, in-depth analysis is required to determine the exact value of the dom- forcing number for splitting graphs of standard graph classes such as paths, cycles, and ladder graphs.

Although the dom-forcing number has been established for a set of few well-known graphs, a detailed, analytical and problem-solving understanding across broader classes of graphs is yet to be accomplished. Hence, extending these results to more general families such as bipartite graphs and random graphs is

an area yet to be explored. Such extensions could provide new insights and potentially lead to general results applicable to wider classes. Moreover, the characterization of graphs Γ satisfying $F_d[\Gamma(H)] = nF_d(H)$, as well as those for which the dom-forcing propagation time $dpt(\Gamma) = 1$ or 2 , presents robust opportunities for theoretical development, which is essential for progress in the application of it in both simulation as well as problem solving in real life contexts.

The development of efficient algorithms to compute the dom-forcing number for arbitrary graphs is an important and challenging problem from a computational perspective. Given the combinatorial complexity inherent in both domination and zero forcing, designing polynomial-time algorithms or effective approximation techniques would be highly valuable. Additionally, constructing algorithms to determine the maximum and minimum dom-forcing propagation time for connected graphs would significantly enhance the practical applicability of these concepts in multiple emerging domains.

Another promising direction is the exploration of relationships between $F_d(\Gamma)$ and other graph parameters such as chromatic number, independence number, and spectral properties. Investigating these connections may contribute to a more unified theory of graph invariants. Further, it is worthwhile to characterize graphs in which $F_{cd}(H) = F_{cd}(\Gamma)$, where H is a subgraph of Γ , and to study ladder graphs satisfying conditions such as $F_{cd}[S(Ln)] = n + 1$ and $F_{cd}[S(Ln)] = n + 2$.

Last but not the least, additional attention should be given to connected variants and generalized forms of dom-forcing. In particular, characterizing graphs for which the connected dom-forcing propagation time $dpt_c(\Gamma) = 1$ or 2 , as well as graphs Γ satisfying $F_{dk}(\Gamma) = \mathcal{Z}_k(\Gamma)$ and $F_{dk}(\Gamma) = \gamma(\Gamma)$, would further extend

the theory in meaningful ways. Along with this, determining the connected dom-
 k -forcing number of the splitting graph of an arbitrary graph remains an open
and challenging problem, offering significant scope for future investigation. In
short, developing further the new concept of dom-forcing sets would be a game
changer in a world which is evolving into technology driven one where artificial
intelligence will be a force to reckon with. A true integration and application
of the concept would certainly enhance the potentials unleashed through the
algorithm-based transformations of human lives of our times in the digital world
and beyond.

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APPENDIX

List of Publications

1. Susanth P, Charles Dominic, and K. P. Premodkumar, *Dom-forcing sets in graphs*, Asian Research Journal of Mathematics 22(3)(2026), pp. 88-104.
2. Susanth P, Charles Dominic, and K. P. Premodkumar, *Connected dom-forcing sets in graphs*, Advances and Applications in Discrete Mathematics 42(7)(2025), pp. 713-736.
3. Susanth P, Charles Dominic, and K. P. Premodkumar, *Connected dom-forcing propagation times in graphs*, Advances and Applications in Discrete Mathematics 43(1)(2026), pp. 77-95.
4. Susanth P, Charles Dominic, and K. P. Premodkumar, *Connected dom-K-forcing sets in graphs*, Journal of Advances in Mathematics and Computer Science 41(2)(2026), pp. 15-27.
5. Susanth P, Charles Dominic, and K. P. Premodkumar, *Dom-k-forcing sets in graphs*, Advances and Applications in Discrete Mathematics 43(2)(2026), pp. 241-266.
6. Susanth P, Charles Dominic, and K. P. Premodkumar, *Dom-forcing sets and graphs operations: A study* (Communicated).
7. Susanth P, Charles Dominic, and K. P. Premodkumar, *Dom-forcing propagation time in graphs* (Communicated).