

MATHEMATICAL AND STATISTICAL STUDIES OF SOME STOCHASTIC MODELS IN RELIABILITY

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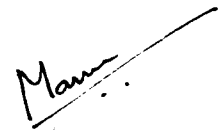
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CERTIFICATE

This is to certify that the work reported in this thesis entitled **MATHEMATICAL AND STATISTICAL STUDIES OF SOME STOCHASTIC MODELS IN RELIABILITY** that is being submitted by Sri. Joseph K.X. for the award of the Doctor of Philosophy, to the University of Calicut, is based on the bonafide research work carried out by him under my supervision and guidance in the Department of Statistics, University of Calicut. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.



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DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

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Chapter 1

Introduction

1.1 Reliability - an introduction

Reliability considerations are important factors in any engineering systems. The problems of failures, repairs and maintenance etc. are to be viewed seriously in the use of any equipments or systems, because the future performance or future life is entirely depending on such reliability characteristics which are not always deterministic due to the complexity, sophistication and automation inherent in the modern technology. They are stochastic in nature . So to solve reliability problems it is highly essential to make use of stochastic models suitable for the situations.

The applied probabilists are concerned with the evaluation of operating characteristic of systems by using appropriate models. The models proposed in the literature for the analysis of such systems vary depending upon the context and background of the analysis. Some of the basic tools like Markov process, Semi Markov process, Renewal theory, Regenerative stochastic process etc. are found to be very useful in different models of such systems.

The thesis deals with mathematical and statistical studies of some stochastic models in reliability. Here we are mainly concerned with multistate reliability models and certain associated inferential problems. The transient probabilities and steady state probabilities of the multistate systems are obtained using Markov model formulations. Also we have determined some performance measures like Mean Time to System Failure (MTSF) and Availability for some models which are of interest from systems design point of view and analysis. The analysis of the models are carried out using standard renewal theoretic arguments. The optimality problems under specified cost structures associated with some of the models are also discussed. Further the problem of estimating the exponential parameters under order restriction has been addressed through Bayesian approach and limit Bayesian approach. Comparison of these estimators are carried out with the usual estimators like Maximum Likelihood Estimator (MLE),

Uniformly Minimum Variance Unbiased Estimator(UMVUE) etc.

This chapter reviews briefly the earlier developments of the theories in the field related to the current work. In section 2 binary coherent systems (BCS) are discussed. Section 3 of this chapter deals with Multistate Coherent systems (MCS). Maintenance policies including repair and replacement problems of these models are given in section 4 and preliminaries of Renewal theory are given in section 5. In Section 6, a brief outline of the procedure of estimating parameters under restriction using Bayesian and limit Bayesian approach is discussed. In the last section an overview of the main contributions of the thesis is provided.

Reliability, as a human attribute has been praised for a very long time. The theory of reliability has grown out of demands of modern technology experienced with complex military systems. In the early 1930's certain areas of reliability, viz. life testing and reliability problems started to receive a great deal of attention from statisticians and engineers. After world war II the development continued through out the world.

In 1951 Epstein and Sobel (1953) began the work in the field of life testing. This work marked the beginning of the wide spread assumption of exponential distribution in life testing research. The mathematically more important paper of Moore and Shanon appeared in 1956. This was concerned with relay network reliability. In 1958 an elegant

summation of the known mathematical results in renewal theory was presented by Smith (1958). In 1959 and 1960's a class of reliability and logistic problems were solved in papers by Black and Proschan (1959) with Proschan using some of the techniques of total positivity. Until 1960's reliability was defined as the probability that an item will perform a required function under stated conditions for a stated period of time. Research on coherent structures began with the paper published in 1961 by Birnbaum, Esary and Sanders (1961). In a sense it puts the work of Moore and Shannon in a more abstract setting. Research on coherent structures and their extensions is still very active today. The publication of *Mathematical Theory of Reliability* by Barlow and Proschan (1965) stimulated the research work on the failure rate function and classes of life distributions defined in terms of this function. Thus it was about 1970's the reliability began to interest a wider class of people. The second book by Barlow and Proschan (1975) emphasized coherent systems and their reliability. Many recent results concerning multivariate life distributions and their properties were included in this book. During the last two decades a substantial progress has been occurred in reliability theory.

Today design and production engineers discuss methods of improving reliability in design and production stages respectively. Testing engineers simulates the realis-

tic environmental conditions. Statisticians produce the optimal experimental methods of life testing. Mathematicians construct mathematical methods of system reliability. Operation research workers are concerned with the best methods of scheduling maintenance, replacement, and repair of the equipment to obtain great reliability matched with adequate availability and costs. Biologists proudly display facts discriminating high reliability of living organisms. A new science was even invented with the aim of discussing how this great reliability is achieved by nature and of trying to imitate such methods in building similar electronic or mechanical devices. All these investigations fall under the name of reliability studies, every year opens new vistas, but all the time probability concepts and statistical methods are used.

1.2 Reliability Models: Binary System.

Consider a system which is composed of n different components and it is called a system of order n . In the traditional way, the system and its components are considered to be in one of two states: functioning and failed. The state of component i , $i = 1, 2, 3, \dots, n$ can then be described by a binary variable x_i , where

$$x_i = \begin{cases} 1, & \text{if component } i \text{ is functioning} \\ 0, & \text{if component } i \text{ is in a failed state.} \end{cases}$$

$\underline{x} = (x_1, x_2, \dots, x_n)$ is called the state vector. Further more we assume that by knowing the states of all the n components, we also know whether the system is functioning or not.

Similarly the state of the system can then be described by a binary function

$$\phi(\underline{x}) = \phi(x_1, x_2, \dots, x_n)$$

where

$$\phi(x) = \begin{cases} 1, & \text{if the system is functioning} \\ 0, & \text{if the system is in a failed state.} \end{cases}$$

$\phi(\underline{x})$ is called *structure function* of the system. Here we mention some simple structures of systems, viz. series structure, parallel structure, k -out of- n structure and coherent structure.

Consider a system with n different components. A system that is functioning if and only if all of its n components are functioning is called a series structure. A system that is functioning if atleast one of its n components is functioning is called a parallel structure. A system that is functioning if and only if at least k of the n components

are functioning is called a k -out-of- n structure. A series structure is therefore an n -out-of- n structure, and a parallel structure is a 1-out-of- n structure.

For a series structure

$$\phi(x) = \prod_{i=1}^n x_i = \min(x_1, x_2, \dots, x_n).$$

For a parallel structure

$$\phi(x) = \prod_{i=1}^n x_i = \max(x_1, x_2, \dots, x_n).$$

For a k -out-of- n structure

$$\phi(x) = \begin{cases} 1, & \text{if } \sum_{i=1}^n x_i \geq k \\ 0, & \text{if } \sum_{i=1}^n x_i < k. \end{cases}$$

When establishing the structure of a system it seems reasonable first to leave out all components that do not play any direct role for the functioning ability of the system. The components we are left with are called *relevant*. The components which are not relevant are called *irrelevant*.

A system of components is said to be *coherent* if all of its components are relevant and the structure function $\phi(x)$ is non decreasing in each argument.

The dual of a structure function ϕ is defined as $\phi^D(x) = 1 - \phi(1 - x)$, where $1 - x = (1 - x_1, 1 - x_2, \dots, 1 - x_n)$. The dual of a series (parallel) system of n components is a parallel (series) system of n components.

A coherent system can also be described through minimal paths or minimal cut representations. A path set is a set of components such that the functioning of these components ensures the functioning of the system. A minimal path set is a path set such that all of its components must be functioning to ensure the functioning of the system. Similarly a cut set is a set of components such that the failure of these components ensures the failure of the system. A minimal cut set is a cut set such that all of its components must fail to ensure the failure of the system. Let p_1, p_2, \dots, p_l denote all possible minimal path sets, and c_1, c_2, \dots, c_k denote all possible minimal cut sets of a coherent system with structure function ϕ . Then ϕ can be represented as a parallel-series or a series-parallel structure. In this case

$$\phi(x) = \begin{cases} 1 - \prod_{i=1}^l \left(1 - \prod_{j \in p_i} x_j \right) \\ \prod_{i=1}^k \left(1 - \prod_{j \in c_i} (1 - x_j) \right) \end{cases}$$

1.2.1 Reliability of coherent systems:

Consider a system consisting of n components. Assume that components are statistically independent. Suppose that the state x_i of the i -th component is random with

$$p(x_i = 1) = p_i = E(x_i), i = 1, 2, 3, \dots, n.$$

Here p_i is the probability that the component i functions, which is called reliability of i . Similarly the reliability of the system is given by

$$P[\phi(x) = 1] = h = E[\phi(x)].$$

We can express system reliability as a function of component reliabilities, as $h = h(p)$ where $p = (p_1, p_2, \dots, p_n)$

$h(p)$ is called *reliability function* of the structure ϕ . If the components are not independent the system reliability is not a function of p alone, in this case $h(p)$ will not be used.

For the series structure $\phi(x) = \prod_{i=1}^n x_i$, the reliability function is $h(p) = \prod_{i=1}^n p_i$, where as for parallel structure $\phi(x) = \prod_{i=1}^n x_i$ has the reliability function $h(p) = \prod_{i=1}^n p_i = 1 - \prod_{i=1}^n (1 - p_i)$ The reliability of a k - out of - n system is

$$h(p) = \sum_{i=k}^n \binom{n}{i} p^i (1-p)^{n-i}.$$

In other words, a coherent system is a system that is not functioning if all of its components are not functioning, is functioning if all of its components are functioning and if it is initially functioning whenever some initially failed components are restored to functioning states.

If the life times of the n components of a system are statistically independent and

T denotes the life time of the system, then the system reliability at time t is given by

$$\begin{aligned} R(t) &= P(T > t) \\ &= \sum_{\{x:\phi(x)=1\}} \prod_{i=1}^n [x_i p_i(t) + (1 - x_i) q_i(t)], \end{aligned}$$

where $p_i(t)$ is the probability that the i -th component is still functioning at time t and $q_i(t) = 1 - p_i(t)$.

In particular if the i -th component has an exponential life distribution with $p_i(t) = \exp(-\lambda_i t)$, for $t > 0$,

$$\text{then } R(t) = \sum_{\{x:\phi(x)=1\}} \prod_{i \in A(x)} \exp(-\lambda_i t) \prod_{j \in B(x)} (1 - \exp(-\lambda_j t)),$$

where $A(x) = \{i : x_i = 1\}$ and $B(x) = \{j : x_j = 0\}$.

The book *Statistical theory of Reliability and life testing* by Barlow and Proschan (1975) emphasized binary coherent structures and their reliability which is the fundamental concept of reliability analysis.

1.3 Reliability Models: Multistate

The theory of binary coherent systems serves as a unifying foundation for a mathematical and statistical theory of reliability. In this theory systems and components are assumed to be in one of two states: functioning and failed. In many real life situations,

however, systems and components are actually capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure. In these situations the dichotomised model is an over simplification of the actual situation. In order to describe more adequately the performance of such 'degradable' systems and components, the theory of multistate coherent systems has been developed. In the last two decades, many different multistate coherent systems have been investigated. Some of the earlier works in this direction can be found in S.M. Ross (1979), Barlow and Wu (1978), El-Newehi (1978) and W.S. Griffith (1980). Basically the ideas underlying the different models proposed are the same. These models assume that the system and its n components can be described separately by a totally ordered set S called the states space. The set of states is $S = \{0, 1, 2, \dots, M\}$. The $M + 1$ states represent successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0. Let $x_i, i = 1, 2, \dots, n$ denote the state or performance level of the i -th component. The vector $\underline{x} = (x_1, x_2, \dots, x_n)$ representing the states of the n components takes its value in S^n , where S^n is the n -th cartesian power of S . It is assumed that the state ϕ of the system is a deterministic function of \underline{x} ; i.e., $\phi = \phi(\underline{x})$, which takes the values in S . The function ϕ is called the structure function of the system. Set theoretic and axiomatic approaches are adopted by various authors to introduce a

variety of classes of multistate systems. They have studied different models and their structural properties. In late seventies Barlow and Wu (1978), El-Newehi *et.al* (1978), Ross (1979) have initiated research in this area.

In 1984 an excellent survey of multistate system theory was presented by E.El-Newehi and Proschan (1984) on degradable systems. In this paper they investigated the deterministic, probabilistic and dynamic aspects of multistate models with suitable examples.

DEFINITION 1.3.1 *A system of n components is said to be a multistate coherent system (MCS) if its structure function satisfies.*

1. ϕ is increasing.
2. For level j and component i , there exists a vector (\cdot, \underline{x}) such that $\phi(j, \underline{x}) = j$ while $\phi(l, \underline{x}) \neq j$ for $l \neq j, i = 1, 2, 3, \dots, n$ and $j = 0, 1, \dots, M$
3. $\phi(j) = j$ for $j = 0, 1, 2, \dots, M$.

This definition is due to El. Newehi *et.al* (1978). Several authors have contributed much to the development of this theory, both deterministic and stochastic. Some of the recent works in this direction are Griffith (1980), Block and Savits (1982), Natvig (1982), Ebrahimi (1984), Ohi and Nishida (1984) and Natvig (1985). In these papers

the relationship between various definitions of MCS have been established, occasionally comparing and contrasting them.

Two recent papers on multistate coherent systems are by Dharmadhikari and Nimbalkar (1991) and Kai Yu, *et.al*(1994). In the paper by Dharmadhikari, *et.al* a measure of importance of an item and that of a given state of an item are discussed with respect to the expected utility function. Kai Yu, *et.al* presented a new MCS model, a generalised multistate coherent system (GMCS) model, which is based on a more general assumption that the state sets of the system and its components are partially ordered.

1.4 Failure and repair models - An Introduction

We know that reliability is the probability of a device giving satisfactory performance for a specified period under specified operating conditions. When a component or system does not function satisfactorily it is said to have failed. The pattern of failure can be obtained from life-testing experiments. These results provide a basis for formulating or constructing mathematically a failure model for analysis.

Many of the currently used failure models are developed under the premise that the operating environment is static. Now new models have been developed when the environment under which the items operate is dynamic. These new failure models are

derived by describing the underlying failure-causing mechanisms, such as degradation and wear, using suitable stochastic processes. Because dynamic environments induce changes in the physics of failure, a stochastic process approach to failure modelling provides flexibility in describing the failure generating mechanisms. This flexibility results in a better description of the failure data and an improved assessment of system reliability. Some of the works in this direction are given by Deshmond (1985), Cinlar and Ozekici (1987), Singpurwalla and Younggreen (1993) and Singpurwalla (1995).

The notion of aging is an important aspect in reliability theory. If X denotes the life time of a fresh unit, the reliability or survival probability of it corresponding to a mission of duration x is given by

$$\bar{F}(x) = 1 - F(x) \text{ where } F \text{ is the life distribution of the unit.}$$

The conditional reliability of a unit of age t is

$$\bar{F}(x | t) = \frac{\bar{F}(t+x)}{\bar{F}(t)}, \text{ if } \bar{F}(t) > 0.$$

The conditional probability of failure during the next interval of duration x of a unit of age t is

$$\begin{aligned} F(x | t) &= \frac{F(t+x) - F(t)}{\bar{F}(t)} \\ &= 1 - \bar{F}(x | t) \end{aligned}$$

The conditional failure rate $r(t)$ at time t is

$$\begin{aligned} r(t) &= \lim_{x \rightarrow 0} \frac{F(t+x) - F(t)}{x \bar{F}(t)} \\ &= \frac{f(t)}{\bar{F}(t)}, \text{ where } f(t) \text{ exists and } \bar{F}(t) > 0. \end{aligned}$$

$r(t)$ is also known as *hazard rate*, *force of mortality* or *intensity*. We can write reliability in terms of hazard. Integrating the above, we have

$$\begin{aligned} \int_0^x r(t) dt &= -\log \bar{F}(x) \\ \text{Therefore } \bar{F}(x) &= \exp \left\{ - \int_0^x r(t) dt \right\} \\ &= \exp \{ -R(x) \}, \end{aligned}$$

where R is called *hazard function*.

Now consider a device which does not age stochastically. In this case failure rate will be a constant and the life distribution will be exponential. If a unit ages adversely

i.e, $\bar{F}(x | t) = \frac{\bar{F}(t+x)}{\bar{F}(t)}$ is decreasing in $t \geq 0$ for each $x > 0$, then F is an *Increasing Failure Rate* (IFR) distribution. Suppose aging is beneficial, i.e, $\bar{F}(x | t)$ is increasing in $t \geq 0$ for each $x \geq 0$, then F is a *Decreasing Failure Rate* (DFR) distribution.

Similar to IFR and DFR distributions we can define a number of classes of life distributions namely *Increasing Failure Rate Average* (IFRA), *Decreasing Failure Rate*

Average (DFRA), New Better than Used (NBU), New Better than Used in Expectation (NBUE), Harmonically New Better than Used in Expectation (HNBUE), Increasing Mean Residual Life (IMRL), Decreasing Mean Residual Life (DMRL) etc., which are of very great significance in the aspects of aging.

Another measure to characterise a failure model is the Mean Time to System Failure. If one is interested in a summarised measure of the random variable X , representing the life time of the system, MTSF is to be computed. It corresponds to the average duration between successive system failures. By using standard manipulations, we get

$$E(X) = \int_0^{\infty} x f(x) dx = \int_0^{\infty} \bar{F}(u) du.$$

Now we discuss reliability improvement by means of repair. In general, whenever the average repair cost in time and money of a piece of system is a fraction of the initial cost of the system, one considers system repair. If such a system can be rapidly returned to service, the effect of failure is minimised. Analysis of a repairable systems has been the concern of many industrial engineers, applied probabilists and statisticians. Consequently several directions of research have been developed on this topic. In such a system the time between failures, repair times, number of failures in an interval, and percentage of operating time in an interval, are figures of merit which are to be

considered along with the system reliability.

In order to describe the beneficial features of repair in a system that tolerates shutdown times, the system function called *availability* is introduced. The availability function $A(t)$ is defined as the probability that the system is operating at time t . By contrast, the reliability function $R(t)$ is the probability that the system has operated over the interval $(0, t)$. Thus reliability is an interval function while availability is a point function, describing the behaviour of the system at a specified epoch. Secondly, the reliability function precludes the failure of the system during the interval under consideration, while availability function does not impose any such restriction on the behaviour of the system. An important difference between $A(t)$ and $R(t)$ is their steady state behaviour. As t becomes large, all reliability functions approach zero, whereas availability functions reach some steady state value. So in the steady state

$$A(\infty) = \lim_{t \rightarrow \infty} A(t).$$

The performance of a repairable system can be better or worse with the passage of time. A repairable system deteriorates with time if the time between two successive repairs tend to get smaller in some sense. Asher and Feingold (1984) defined system deterioration in terms of orderings between interarrival times. Deterioration in systems can also take place due to usage or age and are subject to shocks which cause

failures. Due to the inevitable deterioration and shocks, manufacturing systems might not retain a condition in which acceptable goods can be produced. Restoring a highly deteriorated or failed system is usually time consuming and costly. Frequent inspection and replacement might reduce the chances of high deterioration and failure, but it also incurs maintenance costs. Maintenance policies of stochastically failing systems have been widely investigated in the literature. Valdez-Flores and Feldman (1989) surveyed recent works and Pierskalla Voelker (1976) and Sherif and Smith (1981) reviewed the earlier papers in this area excellently. Many articles in the literature concentrate on modelling the deteriorating process of a stochastically failing system by a Markov Process, because of the tractability of the resulting mathematical problems.

During the last decade, after the introduction of multistate reliability systems, many researchers have studied the maintenance policies of multistate deteriorating systems. Here the operating condition of the system is characterised by the degree of deterioration and can be classified into a finite number of states. In each state, the system is assumed to deteriorate gradually to the next higher state. Without maintenance action the system will eventually fail. It is assumed that the system is monitored continuously such that current state of the system is always known. When failures occur, the system must be replaced with a new identical system. The system

can also be replaced at any point of time. The cost and time required to replace the system depends on the state of the system. In general, the operating cost rate, replacement cost and replacement time increase as the system deteriorates. Therefore it is desirable to replace the system before it fails to avoid operating or replacing the system in a highly deteriorating condition. Some of the studies in this direction are Ohnishi, *et.al* (1986) and Lam and Yeh (1994).

1.5 Renewal Theory - An outline.

All the replacement problems can be treated by the techniques of renewal theory. Here we give a brief account of the relevant aspects of renewal theory.

Renewal theory provides elegant and powerful tools for analysing stochastic process. Renewal theory had its origin in the study of strategies for replacement of technical components. Renewal theory is concerned with the replacement of an item which deteriorates with time.

Consider a sequence of independent and identically distributed random variables $\{X_n\}$ which are nonnegative. At time 0 an item starts functioning. On its failure at time X_1 , a new item starts operation. This is replaced at time $X_1 + X_2$. The process continues in this fashion.

Write $S_n = \sum_{i=1}^n X_i$. Then S_n is the time at which the n -th renewal takes place.

Denote by $N(t)$, the number of renewals at time t measured from the time 0. Clearly $\{S_n \leq t\}$ if and only if $\{N(t) \geq n\}$. Then $\{S_n, n = 0, 1, 2, \dots\}$ with $S_0 = 0$ or equivalently $\{N(t), t \geq 0\}$ is called a *renewal process*.

Let $M(t) = E\{N(t)\}$. It is called the *renewal function*.

Now assuming that X_i have distribution $F(\cdot)$ we get

$$\begin{aligned} P\{N(t) = n\} &= P\{X_1 + X_2 + \dots + X_n \leq t \text{ and } X_1 + X_2 + \dots + X_{n+1} > t\} \\ &= F^{*(n)}(t) - F^{*(n+1)}(t) \end{aligned}$$

When $F^{*(n)}(\cdot)$ is the n fold convolution of $F(\cdot)$ with itself.

Thus

$$\begin{aligned} M(t) &= \sum_{n=0}^{\infty} n \cdot P\{N(t) = n\} \\ &= \sum_{n=0}^{\infty} n [F^{*(n)}(t) - F^{*(n+1)}(t)] \\ &= \sum_{n=1}^{\infty} F^{*(n)}(t), \end{aligned}$$

which is a convergent series. This means that the mean number of renewals is finite for every finite interval of time.

Next we shall get an integral equation satisfied by the renewal function. By con-

ditioning on the time of first renewal, we get

$$\begin{aligned}
 M(t) &= \int_0^{\infty} E\{N(t) \mid X_1 = x\}dF(x) \\
 &= \int_0^t E\{N(t) \mid X_1 = x\}dF(x) + \int_t^{\infty} E\{N(t)/X_1 = x\}dF(x) \\
 &= \int_0^t E\{N(t) \mid X_1 = x\}dF(x) \\
 &= \int_0^t \{1 + M(t-x)\}dF(x) \\
 M(t) &= F(t) + \int_0^t M(t-x)dF(x).
 \end{aligned}$$

Here an integral equation of the form

$$A(t) = a(t) + \int_0^t A(t-x)dF(x), t \geq 0, \quad (1.5.1)$$

where $a(t)$ and $F(t)$ are known functions and $A(t)$ is an unknown function to be determined as a solution to the integral equation is called a *renewal equation*. The solution of an arbitrary renewal equation can be represented in terms of the renewal function.

When $a(t)$ is a bounded function, there exists one and only one solution given by

$$M(t) = \sum_{k=1}^{\infty} F^{*(k)}(t)$$

for (1.5.1).

Now we are interested in the asymptotic behaviour of $\frac{M(t)}{t}$ as $t \rightarrow \infty$. The elementary renewal theorem states that $\frac{M(t)}{t} \rightarrow \frac{1}{\mu}$ as $t \rightarrow \infty$, where $\mu = E(X_1)$. The renewal

theorem states that $M(t+h) - M(t) \rightarrow \frac{h}{\mu}$ as $t \rightarrow \infty$ for any $h > 0$. Using renewal theorem we can determine the limit distribution of the excess life.

Now we define certain variations of the renewal process.

1. A delayed renewal process is one in which the initial component's life time has a distribution different from that of the remaining components. Thus if X has distribution $G(\cdot)$ and X_2, X_3, \dots have distribution $F(\cdot)$ and setting

$$M(t) = \sum_{k=1}^{\infty} F^{*(k)}(t)$$

we see that $M_D(t) = E\{N(t)\}$, the renewal function associated with the delayed renewal process satisfies the renewal equation

$$M_D(t) = G(t) + \int_0^t M_D(t-x) dF(x).$$

2. A stationary renewal process is one in which the initial component's life has the distribution function

$$G(x) = \frac{1}{\mu} \int_0^x \{1 - F(y)\} dy.$$

3. A renewal process with possibly infinite interoccurrence time is called a terminating renewal process.

1.6 Estimation of ordered parameters

Inference concerned with ordered parameters has received considerable attention in statistical literature during the last two decades. Frequently one may come across situations in reliability where one is interested in the estimation of parameters which are restricted. For example, one may be interested in the estimation of mean lives of two components having exponential life distributions, out of which one is produced by a standard company and other is produced by a local company where it is apriori known that the mean life of the component of the standard company is not less than that of the component produced by the local company. For more details on estimation of ordered parameters the reader may refer to Blumenthal and Cohen (1968), Kushary and Cohen (1991), Sackrovtz (1970), Kaur and Singh (1991).

There are mainly two approaches in statistical inference . One is called frequentist approach and other is called Bayesian approach. In the frequentist approach, the inference is completely based on data and no further information about the nature of parameter(s) is used. The minimum variance unbiased estimation, maximum likelihood method of estimation and method of least squares are the common frequentist approach in estimation. On the other hand, in the Bayesian approach the inference is made by incorporating the information from the data with the prior information

regarding the parameter. This approach rests on the belief that in most practical situations the statisticians will possess some subjective apriori information concerning the probable value of the parameter(s). We make use of such information in the form of a probability distribution on the parameter space. This probability distribution is called prior distribution of the parameter(s). The application of Baye's theorem leads to the conditional distribution of the parameter(s) given the data. This distribution is called posterior distribution and all the inference is made from this distribution.

In several practical situations the frequentist method of estimation such as minimum variance unbiased estimation and maximum likelihood estimation may not give satisfactory estimators. For example, in the case of UMVUE when the parameters are ordered, the estimators may be lying outside the parameter space with non zero probability. In the case of MLE also such limitation exists. The Bayesian approach is free of this draw back as the support of posterior distribution never lie outside the parameter space.

1.7 An overview of the main contribution of the thesis.

The thesis is organised into five chapters and each chapter is then divided into many sections. In the introductory chapter the concept of 'Reliability' is introduced and a survey of reliability literature confined to our study is provided. In sections 2 and 3 the preliminaries of binary and multistate reliability models are given respectively. An introduction to failure and repair models are given in section 4. Section 5 discusses an outline of Renewal theory. Section 6 deals with the estimation of ordered parameters and in the last section an overview of the main contributions of the thesis is provided.

In chapter 2, analysis of a repairable multistate system having different failure modes is considered under Markov model formulation. (See Joseph and Manoharan (1997)). Here we study a system which can be repaired at different failure modes and the repair is assumed to be complete. That means the system is having $M + 1$ states with one good and M mutually exclusive failure modes $M - 1, M - 2, \dots, 2, 1, 0$. From the perfect functioning level M , the system can go to any of these failure modes with the corresponding failure rate. After repair is completed, from the failure mode the system can reach the perfect functioning level M with the corresponding repair

rate. The system is characterised by the state of the system at time t . The transient and steady state distributions of the system state are obtained and the results are illustrated through a numerical example.

In chapter 3 we are dealing with the explicit solutions of a Markovian k -out of $-n$ system. Many authors have investigated the reliability analysis of series, parallel and k -out of $-n$ systems. Some of the works in this direction can be found in Ravichandran (1990). A k -out of $-n$ system is a system which functions if at least k components function. The system is described by means of its failed components. Explicit solutions for transient and steady state probabilities are computed for the k -out of $-n$ system using the underlying stochastic process. Performance measures like MTSF and availability are computed for the model under study as these measures are of very much interest from the view point of system design and analysis. Finally a numerical example is provided.

Chapter 4 deals with a multistate deteriorating system with distinct performance levels. Here we study a multistate system that deteriorates over time such that the duration through which the system has a given level of performance is a random variable following an arbitrary distribution where we refer to the performance level as the state of the system. The state of the system is continuously monitored and whenever it

exceeds a particular control limit c , replace the system instantaneously. There are many real life situations in which replacements are made based on a control limit policy. The transient and steady state distributions of the system state are obtained. An associated optimal replacement strategy under a specific cost structure is considered and some numerical illustrations are provided.

In chapter 5 we discuss the estimation of reliability parameters under order restriction. The multistate deteriorating system described in Chapter 4 presents itself as a reliability model where the parameters are restricted. Starting from the perfect functioning level, the performance level of the system goes down as time elapses and when it reaches a particular level c the system is being replaced. The system stays for some time at each performance level called sojourn time. For instance, consider a multistate system having performance levels 0, 1 and 2 with perfect functioning level at 2. Let X_1 and X_2 be the sojourn times at levels 2 and 1 respectively with exponential distribution. It is natural to assume that $E(X_1) \geq E(X_2)$. Here the underlying parameters will be ordered and so their estimation will be of much practical importance. We see that the usual methods of estimation such as minimum variance unbiased (MVU) and maximum likelihood possess certain limitations. In the case of MVU method when $\theta_1 \leq \theta_2$, if we compute $Pr(\hat{\theta}_1 > \hat{\theta}_2)$ we can see that the estimators are lying outside

the parameter space with non zero probability . In the case of MLE also $Pr(\hat{\theta}_1 = \hat{\theta}_2)$ is significant for moderately small sample size. Due to these drawbacks we propose Bayesian approach to estimate the parameters in which $Pr(\hat{\theta}_1 > \hat{\theta}_2) = 0$ for any sample size. Here we choose two families of prior distributions suitable for this context and the Baye's estimators are derived under invariant squared error loss function. We have obtained general expression for Baye's estimators and their risks for any prior distribution. In particular cases of prior distributions that we have chosen these are seen to be computationally tractable. We compare the limit Baye's estimator of the two cases with respect to the risk using invariant squared error loss function and also established numerically the superiority of these estimators over the MLE.

We conclude the thesis pointing out the salient features of our studies and scope for further work. A fairly comprehensive bibliography including all papers and books mentioned in this thesis are given at the end.

Chapter 2

A repairable multistate system

2.1 Introduction

The engineering systems used today can be classified as simple and complex. We assume that the components constituting the system are repairable. In systems, repair is a common operation since the cost of repair is much less than that of replacing the entire system. In this chapter a repairable system having different failure modes is considered. For the investigation of the time dependent behaviour of the repairable systems, stochastic process is a powerful tool. Under reasonable assumptions on the

^oSome part of this chapter is appeared in *Int. Jl. of Information and Management Sciences* 8, No. 1, 1997.

failure and repair distributions a time homogeneous Markov process with finite state space may be used. The transient distribution as well as steady state distribution of the system state can then be easily studied.

Here the analysis is carried out with respect to a multistate system, which is repairable. Consider a system consisting of n units. Here by a system we mean a technical unit, namely an automatic machine, or an electronic product. In the traditional way the system is considered to be in one of two states; functioning and failed. In many real life situations, the system is capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure. In these situations the dichotomised model is an oversimplification of the actual situations, and some kind of multistate consideration is needed. Over the past years, many different multistate coherent system models have been proposed. One may refer to EI- Neweihi and Proschan (1984) for an excellent survey of multistate (degradable) systems. The state of the system could be $M, M - 1, M - 2, \dots, 1, 0$ where M is the perfect functioning level and 0 is the complete failure level. Here we assume that M is the only operating mode of the system and $M - 1, M - 2, \dots, 2, 1, 0$ are the states of the system at different failure modes. This means that the system functions only at the perfect functioning level M , but at any level less than M the system fails. We assume constant failure rates to

each of the failure modes and constant repair rates at each failure mode.

In many practical situations there are systems which possess more than one performance parameter, and each of the parameters can degrade separately or fail separately. A device is said to have $M + 1$ states if it has one good state and M mutually exclusive failure modes $M - 1, M - 2, \dots, 2, 1, 0$. Some examples of such devices include a fluid flow valve, an automatic machine and an explosive. A repairable 3-state device has been developed using Markov model in Procter and Singh (1976). For further references, one can see John. B. deMercado (1971), Procter et.al. (1976), Yamashiro (1980 a, 1980 b, 1981 a, & 1981 b).

In this chapter we are examining a system which can be repaired at different failure modes and the repair is assumed to be complete. Let $a_{M-1}, a_{M-2}, \dots, a_1, a_0$ be the constant failure rates at modes $M - 1, M - 2, \dots, 1, 0$ respectively and $b_{M-1}, b_{M-2}, \dots, b_1, b_0$ be the constant repair rates respectively at modes $M - 1, M - 2, \dots, 1, 0$. From the perfect functioning level M the system can go to any of the failure modes with the corresponding failure rate. Also after repair, from that failure mode the system can reach the perfect functioning level M with the corresponding repair rate.

Here we model the system using Markov process and obtain the probability distribution of the state of the system at time t under the assumption that initially the

system is at state M . In section-2 the transient behaviour of the system is studied and explicit solutions for the Laplace transforms of the transient probabilities are obtained. Further, when the parameters of the problem are known, the roots of the polynomial which are known to be real and distinct are obtained using standard suitable computer routines. In section 3, steady state solutions are obtained and a numerical illustration is given for $M = 2$ in the last section.

2.2 Transient Analysis

Define $\{X(t), t \geq 0\}$ as a right continuous stochastic process representing the state of the system at time t , where t ranges over nonnegative real numbers. The state space is $\{M, M - 1, \dots, 1, 0\}$. Also define $p_j(t) = \text{Prob}\{X(t) = j\}$, for $j = 0, 1, 2, \dots, M$.

Assume that $X(0) = M$ so that $\text{Prob}\{X(0) = M\} = 1$. Repair is started as soon as the failure occurs and the repair is assumed to be complete. Let $a_{M-1}, a_{M-2}, \dots, a_1, a_0$ be the constant failure rates and $b_{M-1}, b_{M-2}, \dots, b_1, b_0$ be the constant repair rates corresponding to the failure modes $M - 1, M - 2, \dots, 1, 0$ respectively. Because of the constant failure and repair rates the stochastic process involved is a time homogeneous Markov process with the states $0, 1, 2, \dots, M$.

Using the standard arguments, we have

$$p_j(t + \Delta t) = p_j(t)(1 - b_j\Delta t) + p_M(t)a_j\Delta t + o(\Delta t), j = 0, 1, \dots, M - 1$$

$$p_M(t + \Delta t) = p_M(t)[1 - (a_{M-1} + a_{M-2} + \dots + a_0)\Delta t] \\ + p_{M-1}(t)b_{M-1}\Delta t + \dots + p_0(t)b_0\Delta t + o(\Delta t)$$

which lead to the following system of differential equations.

$$p'_j(t) = -b_j p_j(t) + a_j p_M(t), j = 0, 1, 2, \dots, M - 1 \\ \text{and } p'_M(t) = -(a_{M-1} + a_{M-2} + \dots + a_0)p_M(t) + \sum_{j=0}^{M-1} b_j p_j(t) \tag{2.2.1}$$

with initial condition $p_M(0) = 1, p_j(0) = 0, j = 0, 1, 2, \dots, M - 1$.

Note that

$$L [p'_j(t)] = s\bar{p}_j(s), \text{ since } p_j(0) = 0.$$

Using (2.2.1), we get

$$s\bar{p}_j(s) = -b_j\bar{p}_j(s) + a_j\bar{p}_M(s).$$

$$(s + b_j)\bar{p}_j(s) = a_j\bar{p}_M(s).$$

Hence

$$\bar{p}_j(s) = \frac{a_j}{s + b_j}\bar{p}_M(s), j = 0, 1, 2, \dots, M - 1.$$

Since $\sum_{j=0}^M p_j(t) = 1$, on taking Laplace Transform, we get $\sum_{j=0}^M \bar{p}_j(s) = \frac{1}{s}$.

Substituting the value of $\bar{p}_j(s)$, we get

$$\sum_{j=0}^{M-1} \frac{a_j}{s+b_j} \bar{p}_M(s) + \bar{p}_M(s) = \frac{1}{s},$$

which yields

$$\bar{p}_M(s) = \frac{1}{s \left[1 + \sum_{k=0}^{M-1} \frac{a_k}{s+b_k} \right]}. \quad (2.2.2)$$

Hence

$$\bar{p}_j(s) = \frac{a_j}{s(s+b_j) \left(1 + \sum_{k=0}^{M-1} \frac{a_k}{s+b_k} \right)}, \quad j = 0, 1, \dots, M-1. \quad (2.2.3)$$

The above system of equations obtained by Laplace Transform technique can be re-written as

$$(s+b_0)\bar{p}_0(s) - a_0\bar{p}_M(s) = 0$$

$$(s+b_1)\bar{p}_1(s) - a_1\bar{p}_M(s) = 0$$

$$(s+b_2)\bar{p}_2(s) - a_2\bar{p}_M(s) = 0$$

$$\vdots \quad \quad \quad \vdots = \vdots$$

$$\vdots \quad \quad \quad \vdots = \vdots$$

$$(s+b_{M-1})\bar{p}_{M-1}(s) - a_{M-1}\bar{p}_M(s) = 0$$

and finally

$$-b_0\bar{p}_0(s) - b_1\bar{p}_1(s) - \dots - b_{M-1}\bar{p}_{M-1}(s) + \left(s + \sum_{j=0}^{M-1} a_j \right) \bar{p}_M(s) = 1.$$

In matrix form,

$$AP = I_{M+1} \tag{2.2.4}$$

where,

$$A = A(s) = \begin{bmatrix} s + b_0 & 0 & 0 & \dots & 0 & -a_0 \\ 0 & s + b_1 & 0 & \dots & 0 & -a_1 \\ 0 & 0 & s + b_2 & \dots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & s + b_{M-1} & -a_{M-1} \\ -b_0 & -b_1 & -b_2 & \dots & -b_{M-1} & s + \sum_{j=0}^{M-1} a_j \end{bmatrix}$$

$$P = P(s) = \begin{bmatrix} \bar{p}_0(s) \\ \bar{p}_1(s) \\ \vdots \\ \bar{p}_M(s) \end{bmatrix} \quad \text{and} \quad I_{M+1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

By using Cramer's rule, the Laplace transform of the function $p_j(t)$ can be explicitly

determined as

$$\bar{p}_j(s) = \frac{|A_j(s)|}{|A(s)|}, \quad j = 0, 1, 2, \dots, M-1,$$

where $A_j(s)$ is obtained from $A(s)$ by replacing the j -th column vector by the unit vector in the right hand side of (2.2.4). Under the assumption that the polynomial $|A(s)|$ has real and distinct roots, a partial fraction expression of $\bar{p}_j(s)$ is feasible. Any standard computer routine can be used for this purpose. Then taking inverse Laplace transform we obtain $p_j(t)$.

2.3 Steady state solutions

Assume that the steady state condition prevails.

Then $\lim_{t \rightarrow \infty} p_j(t) = p_j$ is independent of t , and $p'_j(t) = 0$

Then from (2.2.1),

$$0 = -b_j p_j + a_j p_M.$$

Whence

$$p_j = \frac{a_j}{b_j} p_M, \quad j = 0, 1, \dots, M-1. \quad (2.3.1)$$

Using the normalising condition $\sum_{j=0}^M p_j = 1$, we get

$$\sum_{j=0}^{M-1} \frac{a_j}{b_j} p_M + p_M = 1$$

$$p_M = \frac{1}{1 + \sum_{k=0}^{M-1} \frac{a_k}{b_k}} \quad (2.3.2)$$

Hence

$$p_j = \frac{\frac{a_j}{b_j}}{1 + \sum_{k=0}^{M-1} \frac{a_k}{b_k}}, \quad j = 0, 1, \dots, M - 1 \quad (2.3.3)$$

Equations (2.3.2) and (2.3.3) give the steady state solutions.

REMARK 2.3.1 We observe that the steady state solutions can also be obtained from the Laplace transforms of transient solutions as follows.

$$p_j = \lim_{t \rightarrow \infty} p_j(t) = \lim_{s \rightarrow 0} s \bar{p}_j(s),$$

by final value theorem of Laplace transforms. Thus using equations (2.2.2) and (2.2.3), one can see that, the above limits reduce to the quantities obtained in (2.3.2) and (2.3.3).

2.4 Numerical Illustration

The transient solution of the model described above is determined for a system having 3 states 0,1,and 2 with the parameters $a_0 = 2$, $a_1 = 3$, $b_0 = 1$ and $b_1 = 5$. When $M = 2$,

the system of equations obtained after taking Laplace Transform can be written as

$$(s + b_0)\bar{p}_0(s) - a_0\bar{p}_2(s) = 0$$

$$(s + b_1)\bar{p}_1(s) - a_1\bar{p}_2(s) = 0$$

$$\text{and } -b_0\bar{p}_0(s) - b_1\bar{p}_1(s) + (s + a_0 + a_1)\bar{p}_2(s) = 1$$

In matrix form, this system of equations can be expressed as

$$\begin{pmatrix} s + b_0 & 0 & -a_0 \\ 0 & s + b_1 & -a_1 \\ -b_0 & -b_1 & s + a_0 + a_1 \end{pmatrix} \begin{pmatrix} \bar{p}_0(s) \\ \bar{p}_1(s) \\ \bar{p}_2(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We can solve this system of equations by Crammer's rule; which gives

$$\bar{p}_j(s) = \frac{|A_j(s)|}{|A(s)|}, \quad j = 0, 1, 2.$$

Here

$$\begin{aligned} |A(s)| &= \begin{vmatrix} s + b_0 & 0 & -a_0 \\ 0 & s + b_1 & -a_1 \\ -b_0 & -b_1 & s + a_0 + a_1 \end{vmatrix} \\ &= (s + b_0)(s + b_1)(s + a_0 + a_1) - (s + b_0)a_1b_1 - a_0b_0(s + b_1) \\ |A_0(s)| &= \begin{vmatrix} 0 & 0 & -a_0 \\ 0 & s + b_1 & -a_1 \\ 1 & -b_1 & s + a_0 + a_1 \end{vmatrix} \end{aligned}$$

$$\begin{aligned}
 &= a_0(s + b_1). \\
 |A_1(s)| &= \begin{vmatrix} s + b_0 & 0 & -a_0 \\ 0 & 0 & -a_1 \\ -b_0 & 1 & s + a_0 + a_1 \end{vmatrix} \\
 &= a_1(s + b_0) \\
 |A_2(s)| &= \begin{vmatrix} s + b_0 & 0 & 0 \\ 0 & s + b_1 & 0 \\ -b_0 & -b_1 & 1 \end{vmatrix} \\
 &= (s + b_0)(s + b_1)
 \end{aligned}$$

On substitution, we get

$$\begin{aligned}
 \bar{p}_0(s) &= \frac{a_0(s + b_1)}{s\{s^2 + s(a_0 + a_1 + b_0 + b_1) + a_1b_0 + a_0b_1 + b_0b_1\}} \\
 \bar{p}_1(s) &= \frac{a_1(s + b_0)}{s\{s^2 + s(a_0 + a_1 + b_0 + b_1) + a_1b_0 + a_0b_1 + b_0b_1\}} \\
 \text{and } \bar{p}_2(s) &= \frac{(s + b_0)(s + b_1)}{s\{s^2 + s(a_0 + a_1 + b_0 + b_1) + a_1b_0 + a_0b_1 + b_0b_1\}}
 \end{aligned}$$

For $a_0 = 2, a_1 = 3, b_0 = 1$ and $b_1 = 5$

$$\begin{aligned}
 \bar{p}_0(s) &= \left(\frac{5}{9}\right) \left(\frac{1}{s}\right) - \left(\frac{8}{63}\right) \left(\frac{1}{s+9}\right) - \left(\frac{3}{7}\right) \left(\frac{1}{s+2}\right) \\
 \bar{p}_1(s) &= \left(\frac{3}{18}\right) \left(\frac{1}{s}\right) - \left(\frac{8}{21}\right) \left(\frac{1}{s+9}\right) + \left(\frac{3}{14}\right) \left(\frac{1}{s+2}\right)
 \end{aligned}$$

$$\bar{p}_2(s) = \left(\frac{5}{18}\right) \left(\frac{1}{s}\right) + \left(\frac{32}{63}\right) \left(\frac{1}{s+9}\right) + \left(\frac{3}{14}\right) \left(\frac{1}{s+2}\right)$$

On taking inverse Laplace Transform of $\bar{p}_0(s)$, $\bar{p}_1(s)$ and $\bar{p}_2(s)$, we get the transient probabilities as

$$p_0(t) = \left(\frac{5}{9}\right) - \left(\frac{8}{63}\right) \exp(-9t) - \left(\frac{3}{7}\right) \exp(-2t)$$

$$p_1(t) = \left(\frac{3}{18}\right) - \left(\frac{8}{21}\right) \exp(-9t) + \left(\frac{3}{14}\right) \exp(-2t)$$

$$p_2(t) = \left(\frac{5}{18}\right) + \left(\frac{32}{63}\right) \exp(-9t) + \left(\frac{3}{14}\right) \exp(-2t)$$

Also the steady state distribution of the system is given by

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{10}{18} \\ \frac{3}{18} \\ \frac{5}{18} \end{pmatrix}$$

Chapter 3

Explicit solutions of a Markovian

k - out of n system

3.1 Introduction:

System reliability is depending on the manner in which the subsystems or components are connected to constitute the given system. The simplest combination of components that form a system is a series combination. This is one of the most commonly used structure, where the system functions if all the components function satisfactorily. Another one is a parallel combination of components. Here the system functions

if atleast one of the components function satisfactorily. Also mixed configuration of systems consisting of components connected in series and parallel are available. There are many situations where a system consisting of ' n ' components function satisfactorily when at least k of the n components function. That means a k - out of- n structure is a structure of order ' n ' that function if and only if at least k components function where $1 \leq k \leq n$. For example, an eight cylinder automobile engine in which the successful operation of at least six cylinders are enough for satisfactory performance of the automobile; a shaft lift operated by four cables out of which at least 2 are necessary for safe operation. Such structures are important because among all monotonic structures of order n they display the steepest reliability functions.

In majority of situations, we encounter systems, simple as well as complex, where most of the components constituting the system are repairable. There are engineering systems where repair is a common operation since the cost of repair or replacement is much less than that of replacing the entire system. Generally repair and maintenance make a system available for use for a long period. Availability is the combination of reliability and maintainability. It is the probability that the system will be able to operate with-in the tolerances at a given instant of time.

In this chapter, a k -out of- n system is considered and it is described through the

number of failed components. Here the life times and repair times of the components are exponentially distributed. We examine the situation of a maintained k - out of- n system where each component is separately maintained and undergoes a perfect repair every time it goes down. Repair starts immediately after a component's break down and the repair time is independent of other components. The transient and steady state probabilities of the system are obtained using Markov model formulation. Using this, the operating characteristics like, the system availability, Mean time to system failure (MTSF) etc are also obtained.

The k -out-of- n system has been studied in the past by several authors. See for example, Barlow and Proschan (1965), Linton et.al (1974). The transient analysis of multiple unit systems can be found in Ravichandran (1990). For further studies in this direction one may refer to Derman, et.al (1982), Fu, J.C. (1985), Hwang, F.K. (1986) Papastavridis, S.G. et.al (1992).

For the sake of convenience in description, let us begin with its dual system viz. $(n - k)$ -out-of- n where the life times of the components are assumed to be exponentially distributed. We shall describe the system by means of its failed components. Here the system is incapacitated only if $k + 1$ elements have failed. Let us take for granted, that the system is in state s_k when the number of failed components is k . The states

$s_0, s_1, s_2, \dots, s_k$ are the functioning states of the system and s_{k+1} is the state of failure of the system. The states $s_{k+2}, s_{k+3}, \dots, s_n$ are regarded as impossible.

Initially all the n components are assumed to be in the functioning state. As time elapses, the components may fail and it is repaired and replaced. When the number of failed components exceeds k at any instant, the system will be in down state. Let the time of unfailling operation of a functioning component be distributed according to the exponential law with parameter λ and the time of repair of a failed component be distributed according to the exponential law with parameter μ .

We now summarise the content of this chapter. In Section 2, the transient behaviour of the system is studied and the solutions for the Laplace transforms of the transient probabilities for the system are obtained using matrix method and also Mean Time to System Failure (MTSF) is determined. In Section 3, steady state distributions and system availability are computed. In the last Section, a special case of the model is discussed for $k = 1$ and $n = 2$.

3.2 Transient Analysis

Let $X(t)$ be the number of failed components at time t . Then $\{X(t), t \geq 0\}$ is an integer valued stochastic process taking values in $S = \{0, 1, 2, \dots, k + 1\}$. In the light

of the exponential distribution of the life times and repair times, $X(t)$ is a Markov process and by physical nature it is a birth and death process.

Define $p_j(t) = \text{Prob}\{X(t) = j\}$ for $j = 0, 1, 2, \dots, k + 1$. Assume that

$$p_j(0) = \begin{cases} 1, & \text{for } j = 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Here $p_j(t)$ represents the probability that the system is in state s_j at time t . Using standard arguments, the probability of the system arriving at the state s_j by the moment $(t + \Delta t)$ can be expressed as

$$\left. \begin{aligned} p_j(t + \Delta t) &= p_j(t) [1 - \{(n - j)\lambda + \mu_j\}\Delta t] \\ &\quad + (j + 1)\mu p_{j+1}(t)\Delta t + (n - j + 1)\lambda p_{j-1}(t)\Delta t \\ &\quad + o(\Delta t), \quad j = 0, 1, 2, \dots, k. \\ p_{k+1}(t + \Delta t) &= p_k(t)(n - k)\lambda\Delta t \\ &\quad + p_{k+1}(t) [1 - (k + 1)\mu\Delta t] + o(\Delta t). \end{aligned} \right\} \quad (3.2.1)$$

From the above it is clear that the transient probabilities $p_j(t)$ satisfy the following differential equations.

$$\left. \begin{aligned} p'_j(t) &= -[(n - j)\lambda + \mu_j]p_j(t) + (j + 1)\mu p_{j+1}(t) \\ &\quad + (n - j + 1)\lambda p_{j-1}(t), \quad j = 0, 1, 2, \dots, k. \\ p'_{k+1}(t) &= (n - k)\lambda p_k(t) - (k + 1)\mu p_{k+1}(t). \end{aligned} \right\} \quad (3.2.2)$$

Solving (3.2.2) one may get the explicit solution for $p_j(t)$, for which the techniques of Laplace transform and matrix method are employed.

Taking Laplace transform of (3.2.2) and denoting the Laplace transform of $f(t)$ by $\bar{f}(s)$ we get

$$s\bar{p}_j(s) - p_j(0) = -[(n-j)\lambda + j\mu]\bar{p}_j(s) + (j+1)\mu\bar{p}_{j+1}(s) + (n-j+1)\lambda\bar{p}_{j-1}(s),$$

$$j = 0, 1, 2, \dots, k$$

$$s\bar{p}_{k+1}(s) - p_{k+1}(0) = (n-k)\lambda\bar{p}_k(s) - (k+1)\mu\bar{p}_{k+1}(s)$$

Putting $j = 0, 1, 2, \dots, k$ in the above equations, we get the following system of equations.

$$s\bar{p}_0(s) - p_0(0) = -n\lambda\bar{p}_0(s) + \mu\bar{p}_1(s)$$

$$s\bar{p}_1(s) - p_1(0) = -[(n-1)\lambda + \mu]\bar{p}_1(s) + 2\mu\bar{p}_2(s) + n\lambda\bar{p}_0(s)$$

$$s\bar{p}_2(s) - p_2(0) = -[(n-2)\lambda + 2\mu]\bar{p}_2(s) + 3\mu\bar{p}_3(s) + (n-1)\lambda\bar{p}_1(s)$$

$$s\bar{p}_3(s) - p_3(0) = -[(n-3)\lambda + 3\mu]\bar{p}_3(s) + 4\mu\bar{p}_4(s) + (n-2)\lambda\bar{p}_2(s)$$

$$\vdots = \vdots \quad \vdots$$

$$\vdots = \vdots \quad \vdots$$

$$s\bar{p}_k(s) - p_k(0) = -[(n-k)\lambda + k\mu]\bar{p}_k(s) + (k+1)\mu\bar{p}_{k+1}(s)$$

$$+ \lambda(n-k+1)\bar{p}_{k-1}(s)$$

$$\text{and } s\bar{p}_{k+1}(s) - p_{k+1}(0) = (n-k)\lambda\bar{p}_k(s) - (k+1)\mu\bar{p}_{k+1}(s)$$

vector and I_{k+2} is a column vector. Specifically,

$$A = \begin{bmatrix} a_0 & -\mu & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -n\lambda & a_1 & -2\mu & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -(n-1)\lambda & a_2 & -3\mu & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -(n-k+1)\lambda & a_k & -(k+1)\mu \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -(n-k)\lambda & s+(k+1)\mu \end{bmatrix},$$

where $a_r = s + (n-r)\lambda + r\mu$, $r = 0, 1, 2, \dots, k$.

$$P = \begin{bmatrix} \bar{p}_0(s) \\ \bar{p}_1(s) \\ \vdots \\ \bar{p}_{k+1}(s) \end{bmatrix} \quad \text{and} \quad I_{k+2} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The system of equations (3.2.3) can be solved by Cramer's rule and the Laplace transform of the function $p_j(t)$ can be explicitly determined as

$$\bar{p}_j(s) = \frac{|A_j(s)|}{|A(s)|}, \quad 0 \leq j \leq k+1,$$

where $A_j(s)$ is obtained from $A(s)$ by replacing the j -th column by the unit vector I_{k+2} . Under the assumption that the polynomial $|A(s)|$ has real and distinct roots, a partial fraction expression of $\bar{p}_j(s)$ is feasible. Any standard computer routine can

be used for this purpose. Then taking inverse Laplace transform we obtain explicit solutions for $p_j(t)$.

3.3 Mean Time to System Failure (MTSF):

The performance measure MTSF is very important from system's design point of view and analysis. When one is interested in a summarised measure of the random variable, say, X , representing the life time of the system, namely, its first moment, MTSF is to be computed, because this corresponds to the average duration between successive system failures.

By usual notations,

$$\begin{aligned} \text{MTSF} = E(X) &= \int_0^{\infty} x f(x) dx \\ &= \int_0^{\infty} R(u) du, \end{aligned}$$

where $R(t)$ is the reliability of the system which is equal to $P(X > t)$.

We can compute MTSF using $q_j(t)$ which is a modified version of the $p_j(t)$ functions requiring no transition to the down states during the interval under consideration.

So we define $q_j(t)$ as

$$q_j(t) = \text{Prob}\{X(t) = j, X(u) \neq k + 1, 0 \leq u \leq t \mid X(0) = 0\}, j \in s.$$

That means, $q_j(t)$ is the probability that the system is in state j at time t , given that the failed state $k+1$ has not been visited in $(0, t)$ starting from the initial state 0.

Using standard arguments, the probability of the system arriving at the state s_j , by the moment $(t + \Delta t)$ can be expressed as

$$q_j(t + \Delta t) = q_j(t)[1 - (n - j)\lambda + j\mu]\Delta t + (j + 1)\mu p_{j+1}(t)\Delta t \\ + (n - j + 1)\lambda p_{j-1}(t)\Delta t + o(\Delta t), \quad j = 0, 1, 2, \dots, k - 1$$

$$q_k(t + \Delta t) = q_k(t)[1 - (n - k)\lambda + k\mu]\Delta t + (n - k + 1)\lambda q_{k-1}(t)\Delta t + o(\Delta t).$$

Here it is very clear that the probabilities $q_j(t)$ satisfy the following differential equations.

$$q'_j(t) = -[(n - j)\lambda + j\mu]q_j(t) + (n - j + 1)\lambda q_{j-1}(t) + (j + 1)\mu q_{j+1}(t), \\ j = 0, 1, 2, \dots, k - 1$$

$$q'_k(t) = -[(n - k)\lambda + k\mu]q_k(t) + (n - k + 1)\lambda q_{k-1}(t)$$

Taking the Laplace transform of the above equations we get

$$s\bar{q}_j(s) - q_j(0) = -[(n - j)\lambda + j\mu]\bar{q}_j(s) + (j + 1)\mu\bar{q}_{j+1}(s) + (n - j + 1)\lambda\bar{q}_{j-1}(s), \\ j = 0, 1, 2, \dots, k - 1$$

and $s\bar{q}_k(s) - q_k(0) = (n - k + 1)\lambda\bar{q}_{k-1}(s) - k\mu\bar{q}_k(s).$

The above system of equations can be expressed as

$$BQ = I_{k+1} \tag{3.3.1}$$

When $B = B(s)$ is a $(k + 1) \times (k + 1)$ real symmetric tridiagonal matrix, P and I are column vectors of order $(k + 1)$. Specifically

$$B = B(s) = \begin{bmatrix} a_0 & -\mu & 0 & 0 & \dots & 0 & 0 & 0 \\ -n\lambda & a_1 & -2\mu & 0 & \dots & 0 & 0 & 0 \\ 0 & -(n-1)\lambda & a_2 & -3\mu & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -(n-k+2)\lambda & a_{k-1} & -k\mu \\ 0 & 0 & 0 & \dots & 0 & 0 & -(n-k+1)\lambda & s+k\mu \end{bmatrix}$$

where, $a_r = s + (n - r)\lambda + r\mu$, $r = 0, 1, 2, \dots, k - 1$.

$$Q = \begin{bmatrix} \bar{q}_0(s) \\ \bar{q}_1(s) \\ \vdots \\ \bar{q}_k(s) \end{bmatrix} \text{ and } I_{k+1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The system of equations (3.3.1) can be solved by Crammer's rule and the Laplace

Transform of the function $q_j(t)$ can be explicitly determined as

$$\bar{q}_j(s) = \frac{|B_j(s)|}{|B(s)|}, \quad 0 \leq j \leq k,$$

where $B_j(s)$ is obtained from $B(s)$ by replacing the j -th column by the unit vector I_{k+1} . Under the assumption that the polynomial has real and distinct roots, a partial fraction expression of $\bar{p}_j(s)$ is feasible. Any standard computer routine can be used for this purpose. Taking inverse Laplace Transform we get explicit solutions for $q_j(t)$.

With the knowledge of the Laplace Transform $\bar{q}_j(s)$, the operating characteristic MTSF of the system is computed as

$$\begin{aligned} \text{MTSF} &= \sum_{j \in u} \int_0^{\infty} q_j(u) du \\ &= \sum_{j \in u} \bar{q}_j(0), \text{ where } u \text{ corresponds the up state of the system.} \\ &= \sum_{j=0}^k \bar{q}_j(0) \end{aligned}$$

3.4 Steady State Solutions

In many practical situations one is interested to know the behaviour in the steady state, ie, when the system reaches an equilibrium state, after being in operation for a pretty long time.

For steady state solutions, let

$$p_j = \lim_{t \rightarrow \infty} p_j(t),$$

provided the limit exists. Then from the differential equations satisfied by $p_j(t)$, we have

$$\left. \begin{aligned} [(n-j)\lambda + j\mu]p_j &= (j+1)\mu p_{j+1} + (n-j+1)\lambda p_{j-1}, \quad j = 0, 1, 2, \dots, k \\ \text{and } (k+1)\mu p_{k+1} &= (n-k)\lambda p_k \end{aligned} \right\} \quad (3.4.1)$$

From the first set of (3.4.1), we get

$$\begin{aligned} \text{for } j = 0, \quad p_1 &= \binom{n}{1} \left(\frac{\lambda}{\mu}\right) p_0, \\ \text{for } j = 1, \quad p_2 &= \binom{n}{2} \left(\frac{\lambda}{\mu}\right)^2 p_0, \\ &\vdots = \vdots \\ \text{and for } j = k, \quad p_k &= \binom{n}{k} \left(\frac{\lambda}{\mu}\right)^k p_0, \end{aligned}$$

From the second equation of (3.4.1),

$$\begin{aligned} p_{k+1} &= \left(\frac{n-k}{k+1}\right) \left(\frac{\lambda}{\mu}\right) p_k \\ &= \binom{n}{k+1} \left(\frac{\lambda}{\mu}\right)^{k+1} p_0. \end{aligned}$$

On combining, we get

$$p_j = \binom{n}{j} \left(\frac{\lambda}{\mu}\right)^j p_0, \quad \text{for } j = 1, 2, 3, \dots, k+1.$$

To find p_0 , using the condition $\sum_{j=0}^{k+1} p_j = 1$, we get

$$p_0 = \left[1 + \sum_{j=1}^{k+1} \binom{n}{j} \left(\frac{\lambda}{\mu}\right)^j \right]^{-1}.$$

Therefore

$$p_j = \frac{\binom{n}{j} \left(\frac{\lambda}{\mu}\right)^j}{\left[1 + \sum_{j=1}^{k+1} \binom{n}{j} \left(\frac{\lambda}{\mu}\right)^j\right]}$$

for $j = 1, 2, 3, \dots, k + 1$. These are the steady state solutions.

Availability:

This is also a performance measure defined as the probability that the system is operational at time t . It is denoted by $A(t)$. Recall that the reliability function precludes the failure of the system during the interval under consideration, while availability function does not impose any such restriction on the behaviour of the system.

In our model $A(t)$ is the probability that not more than k components are faulty at time t . Now in the steady state

$$\begin{aligned} A(\infty) &= \lim_{t \rightarrow \infty} A(t) \\ &= \sum_{j=0}^k p_j \\ &= \frac{\sum_{j=0}^k \binom{n}{j} \left(\frac{\lambda}{\mu}\right)^j}{\left[1 + \sum_{j=1}^{k+1} \binom{n}{j} \left(\frac{\lambda}{\mu}\right)^j\right]} \end{aligned}$$

This quantity can be interpreted as the probability that the system is available at any moment of time, which can be considered as a performance measure of the system.

3.5 A Special Case

As a special case, taking $n = 2$ and $k = 1$ the k - out of $-n$ system becomes a parallel system.

Define $p_j(t) = \text{Prob}\{X(t) = j\}$, for $j = 0, 1, 2$. Assume that

$$p_j(0) = \begin{cases} 1 & \text{for } j = 0 \\ 0 & \text{elsewhere} \end{cases}$$

Here $p_j(t)$ represents the probability that the system is in state s_j at time t .

Using standard arguments,

$$p_0(t + \Delta t) = p_0(t)(1 - 2\lambda)\Delta t + \mu p_1(t)\Delta t + o(\Delta t)$$

$$p_1(t + \Delta t) = p_1(t)(1 - \lambda - \mu)\Delta t + 2\mu p_2(t)\Delta t + 2\lambda p_0(t)\Delta t + o(\Delta t)$$

$$\text{and } p_2(t + \Delta t) = (1 - 2)p_2(t)\mu\Delta t + p_1(t)\lambda\Delta t + o(\Delta t)$$

From the above it is clear that the transient probabilities $p_0(t)$, $p_1(t)$ and $p_2(t)$ satisfy the following differential equations.

$$p_0'(t) = -2\lambda p_0(t) + \mu p_1(t)$$

$$p_1'(t) = -(\lambda + \mu)p_1(t) + 2\mu p_2(t) + 2\lambda p_0(t)$$

$$\text{and } p_2'(t) = \lambda p_1(t) - 2\mu p_2(t)$$

On taking the Laplace Transform of the above system of equations,

$$s\bar{p}_0(s) - p_0(0) = -2\lambda\bar{p}_0(s) + \mu\bar{p}_1(s)$$

$$s\bar{p}_1(s) - p_1(0) = -(\lambda + \mu)\bar{p}_1(s) + 2\mu\bar{p}_2(s) + 2\lambda\bar{p}_0(s)$$

$$\text{and } s\bar{p}_2(s) - p_2(0) = \lambda\bar{p}_1(s) - 2\mu\bar{p}_2(s)$$

To solve the system of equations, matrix method is employed. The system of equations can be expressed as

$$AP = I_3.$$

$$\text{i.e., } \begin{bmatrix} s+2\lambda & -\mu & 0 \\ -2\lambda & s+\lambda+\mu & -2\mu \\ 0 & -\lambda & s+2\mu \end{bmatrix} \begin{bmatrix} \bar{p}_0(s) \\ \bar{p}_1(s) \\ \bar{p}_2(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the above system of equations by Cramer's rule, we get

$$\bar{p}_0(s) = \frac{s^2 + (3\mu + \lambda)s + 2\mu^2}{s(s + \lambda + \mu)(s + 2\lambda + 2\mu)}$$

$$\bar{p}_1(s) = \frac{2\lambda(s + 2\mu)}{s(s + \lambda + \mu)(s + 2\lambda + 2\mu)}$$

$$\bar{p}_2(s) = \frac{2\lambda^2}{s(s + \lambda + \mu)(s + 2\lambda + 2\mu)}$$

Deducing the above equations by the method of partial fraction, we get

$$\bar{p}_0(s) = \left(\frac{\mu}{\lambda + \mu}\right)^2 \frac{1}{s} + \frac{2\lambda\mu}{(\lambda + \mu)^2} \frac{1}{s + \lambda + \mu} + \left(\frac{\lambda}{\lambda + \mu}\right)^2 \frac{1}{s + 2\lambda + 2\mu}$$

$$\bar{p}_1(s) = \frac{2\lambda\mu}{(\lambda + \mu)^2} \frac{1}{s} + \frac{2\lambda(\lambda - \mu)}{(\lambda + \mu)^2} \frac{1}{s + \lambda + \mu} - \frac{2\lambda^2}{(\lambda + \mu)^2} \frac{1}{s + 2\lambda + 2\mu}$$

$$\bar{p}_2(s) = \left(\frac{\lambda}{\lambda + \mu}\right)^2 \frac{1}{s} - \frac{2\lambda^2}{(\lambda + \mu)^2} \frac{1}{s + \lambda + \mu} + \left(\frac{\lambda}{\lambda + \mu}\right)^2 \frac{1}{s + 2\lambda + 2\mu}.$$

On taking inverse Laplace transform of $\bar{p}_0(s)$, $\bar{p}_1(s)$ and $\bar{p}_2(s)$ in the above set of equations, we get the transient probabilities as

$$\begin{aligned} p_0(t) &= \left(\frac{\mu}{\lambda + \mu}\right)^2 + \frac{2\lambda\mu}{(\lambda + \mu)^2} \exp\{-(\lambda + \mu)t\} + \left(\frac{\lambda}{\lambda + \mu}\right)^2 \exp\{-2(\lambda + \mu)t\}. \\ p_1(t) &= \frac{2\lambda\mu}{(\lambda + \mu)^2} + \frac{2\lambda(\lambda - \mu)}{(\lambda + \mu)^2} \exp\{-(\lambda + \mu)t\} - \frac{2\lambda^2}{(\lambda + \mu)^2} \exp\{-2(\lambda + \mu)t\} \\ p_2(t) &= \left(\frac{\lambda}{\lambda + \mu}\right)^2 - \frac{2\lambda^2}{(\lambda + \mu)^2} \exp\{-(\lambda + \mu)t\} + \left(\frac{\lambda}{\lambda + \mu}\right)^2 \exp\{-2(\lambda + \mu)t\}. \end{aligned}$$

Thus the steady state probabilities are given by

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{\mu}{\lambda + \mu}\right)^2 \\ \frac{2\lambda\mu}{(\lambda + \mu)^2} \\ \left(\frac{\lambda}{\lambda + \mu}\right)^2 \end{bmatrix}.$$

To compute the performance measure MTSF in terms of $q_j(t)$, we get the following system of equations after taking Laplace transform of the differential equations in $q_j(t)$.

$$\begin{bmatrix} s + 2\mu & -\mu \\ -2\lambda & s + \lambda + \mu \end{bmatrix} \begin{bmatrix} \bar{q}_0(s) \\ \bar{q}_1(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\begin{aligned} \text{whence } \bar{q}_0(s) &= \frac{s + \lambda + \mu}{s^2 + s(\lambda + 3\mu) + 2\mu^2} \\ \bar{q}_1(s) &= \frac{2\lambda}{s^2 + s(\lambda + 3\mu) + 2\mu^2} \end{aligned}$$

$$\begin{aligned}\text{and} \quad \text{MTSF} &= \bar{q}_0(0) + \bar{q}_1(0) \\ &= \frac{\lambda + \mu}{2\mu^2} + \frac{2\lambda}{2\mu^2} \\ &= \frac{3\lambda + \mu}{2\mu^2}\end{aligned}$$

To compute the availability, in this case,

$$\begin{aligned}A(\infty) &= \sum_{j=0}^1 p_j \\ &= \frac{\sum_{j=0}^1 \binom{2}{j} \left(\frac{\lambda}{\mu}\right)^j}{1 + \sum_{j=0}^2 \binom{2}{j} \left(\frac{\lambda}{\mu}\right)^j} \\ &= \frac{\mu(\mu + 2\lambda)}{2\mu^2 + 2\lambda\mu + \lambda^2}\end{aligned}$$

Chapter 4

Optimal replacement policy of a multistate deteriorating system

4.1 Introduction

The main objective of reliability theory is to evaluate the performance of a complex system or its components and to determine the functional relationships between their values. In the traditional way, the system or its components are considered to be in one of two states: functioning and failed, and hence in this case the binary coherent system is a fundamental concept for reliability analysis. The binary theory has been extensively

discussed in Barlow and Proschan (1975). But this approach fails to describe many situations where we can have more than two distinct states. In many real life situations, however, the system or its components are capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure. In these situations, the dichotomised model is an over simplification of the actual situation and some kind of multistate consideration is essential.

In the last decade, many different multistate coherent system models have been suggested by many researchers. They have mainly discussed the deterministic and probabilistic aspects of multistate reliability systems. The pioneering work by Barlow and Wu (1978) on multistate reliability models was a stepping stone for many researchers who worked on multistate reliability systems. Later El. Neweihi, et. al (1978), Ross (1979), Natvig (1982), Griffith (1980), Block and Savits (1982) Fumo and Nishida (1984) have contributed immensely for the vast development of multistate system reliability theory. For an excellent survey on multistate system one may refer to El. Neweihi and Proschan (1984). Recently Kai Yu et. al. (1994) have investigated some interesting properties of generalised multistate monotone coherent systems. But until recently little work had been done on the dynamic aspects of the multistate deteriorating systems. There is a vast literature on preventive replacements for systems subject to stochas-

tic failure. Some comprehensive surveys of this field have been made by Cho and Parlar (1991), Pierskalla and Voelker (1976), Sherif and Smith (1981), Valdez-Flores and Feldman (1989). More recent works of Lam and Yeh (1994 a,b) deal with optimal maintenance and replacement policies for deteriorating multistate systems.

In this chapter we consider multistate systems that deteriorates over time such that the duration through which the system has a given level of performance is a random variable following an arbitrary distribution where we refer to the performance level as the state of the system. The state of the system is continuously monitored and whenever it exceeds a particular control limit, say c , replace the system instantaneously. There are many real life situations in which the replacements are made based on a control limit policy. Such replacement policies for systems subject to stochastic deterioration are typically preferred if the system functioning at lower performance levels is costly or unrewarding. We analyse the multistate system with such a replacement policy and obtain the optimal performance level at which replacement is to be made. In section 2 transient and steady state distributions of the state of the system are obtained. We find that the steady state distribution is independent of the distributional assumptions of X_i 's, but depends only on the first order moments. Section 3 deals with optimal replacement strategy under a specific cost structure, namely, the sum of

operating cost, and replacement cost.

4.2 Analysis of the model

We consider a system which deteriorates at any point of time and the deterioration can be classified into one of a finite number of states $1, 2, \dots, M$ which are the different performance level of the system. Level 1 represents perfect functioning state and level M represents complete down state. The intermediate levels $2, 3, \dots, M - 1$ are ordered to reflect their relative degree of deterioration in ascending order. The system can be described by $S(t)$, the state of the system at time t . Then $S(t) \in \{1, 2, \dots, M\}$. In each level the system remains there for a random duration and gradually deteriorates to the next higher level. The time spent at each performance level is a random variable whose distribution depends on the corresponding level.

It is assumed that the time X_i for the system to stay in level i follows a absolutely continuous distribution F_i with finite mean μ_i and probability density function f_i , ($i = 1, 2, \dots, M$). Also it is assumed that the random variables X_i 's are independent and the system is monitored continuously such that the current state of the system is always known. The system is replaced when the level reaches a control limit c . Here

both the inspection time and replacement time are considered to be negligible. Moreover it is assumed that the inspection does not cause any damage to the system and replacement restores the system to like new.

Clearly $\{S(t), t \geq 0\}$ is a stochastic process with state space $\{1, 2, \dots, c-1\}$. Initially we start with a system at level 1. That is, $S(0) = 1$. Let us look at the time epochs at which the replacement of the system occurs. Let Y_k be the elapsed time between the $(k-1)$ -th and k -th replacement of the system. Note that $Y_k = X_1 + X_2 + \dots + X_{c-1}$. Then, $\{Y_k\}$, $k = 1, 2, \dots$ forms an ordinary renewal process with the underlying distribution

$$F(y) = \text{Prob.}(Y_k \leq y) = \int_0^y f_1 * f_2 \dots * f_{c-1}(u) du, \quad (4.2.1)$$

where $*$ denotes the convolution operator. Let its density function be denoted by $f(\cdot)$.

Probability that k -th replacement is made between t and $t + dt$ is given by

$$\text{Pr} \{t < Y_1 + Y_2 + \dots + Y_k < t + dt\} = f^{*(k)}(t)$$

which can be used for the derivation of transient probabilities.

4.2.1 Transient distribution of the system state

The transient distribution of the system state is defined as

$$p_j(t) = \text{Prob.}\{S(t) = j\}, \quad 1 \leq j \leq c-1$$

We denote the Laplace transform of $g(t)$ as

$$\hat{g}(s) = \int_0^{\infty} \exp(-st)g(t)dt.$$

Then we have the following result.

Theorem 4.2.1 *The Laplace transform of the transient distribution is given by*

$$\hat{p}_j(s) = \begin{cases} \frac{1-f_1(s)}{s(1-f(s))}, & j = 1, \\ \frac{1-f_j(s)}{s(1-f(s))} \prod_{i=1}^{j-1} \hat{f}_i(s), & 2 \leq j \leq c-1. \end{cases} \quad (4.2.2)$$

Proof: Let us first calculate the probability that the system is at level 1 at time t .

This happens in two ways: (i) when there is no replacement in $(0, t)$ (ii) There can be

$k(= 1, 2, \dots)$, replacements in $(0, t-u)$ such that at time $t-u$ the state of the system

will be at 1 and in the remaining duration of u the performance level remains at 1.

Then using the total probability argument we get the expression for $p_1(t)$. In a similar

way conditioning on the time of last replacement we get the expression for $p_j(t)$. Thus,

$$p_1(t) = 1 - \int_0^t f_1(u)du + \sum_{k=1}^{\infty} \int_0^t \left(1 - \int_0^u f_1(v)dv\right) f^{*(k)}(t-u)du;$$

$$p_2(t) = \int_0^t f_1(u)du - \int_0^t f_1 * f_2(u)du$$

$$+ \sum_{k=1}^{\infty} \int_0^t \left(\int_0^u f_1(v)dv - \int_0^u f_1 * f_2(v)dv\right) f^{*(k)}(t-u)du.$$

Similarly we can find $p_3(t)$. In general

$$p_j(t) = \int_0^t f_1 * f_2 * \dots * f_{j-1}(u)du - \int_0^t f_1 * f_2 * \dots * f_j(u)du$$

$$+ \sum_{k=1}^{\infty} \int_0^t \left(\int_0^u f_1 * f_2 * \dots * f_{j-1}(v)dv - \int_0^u f_1 * f_2 * \dots * f_j(v)dv\right) f^{*(k)}(t-u)du, \quad 2 \leq j \leq c-1.$$

On taking the Laplace transform of the above equations, we get $\hat{p}_1(s)$ and $\hat{p}_j(s)$, $2 \leq j \leq c-1$ as

$$\hat{p}_1(s) = \frac{1 - \hat{f}_1(s)}{s} + \sum_{k=1}^{\infty} \frac{1 - \hat{f}_1(s)}{s} \hat{f}(s)^k,$$

$$= \frac{1 - \hat{f}_1(s)}{s} \left[1 + \sum_{k=1}^{\infty} \{\hat{f}(s)\}^k\right],$$

$$= \frac{1 - \hat{f}_1(s)}{s} \left[1 + \frac{\hat{f}(s)}{1 - \hat{f}(s)}\right],$$

$$= \frac{1 - \hat{f}_1(s)}{s(1 - \hat{f}(s))}$$

Similarly,

$$\begin{aligned}\hat{p}_j(s) &= \left(\frac{\prod_{i=1}^{j-1} \hat{f}_i(s) - \prod_{i=1}^j \hat{f}_i(s)}{s} \right) \left[1 + \sum_{k=1}^{\infty} \{ \hat{f}(s) \}^k \right], \\ &= \frac{1 - \hat{f}_j(s)}{s(1 - \hat{f}(s))} \prod_{i=1}^{j-1} \hat{f}_i(s), \quad 2 \leq j \leq c-1.\end{aligned}$$

Hence the proof. □

4.2.2 Steady state distributions

The steady state distribution of the system is given by

$$p_j = \lim_{t \rightarrow \infty} p_j(t) = \lim_{s \rightarrow 0} s p_j(s), \quad (4.2.3)$$

by a Tauberian theorem. (Widder,(1974))

Theorem 4.2.2 *The steady state distribution of the system states is*

$$p_j = \frac{E(X_j)}{\sum_{i=1}^{c-1} E(X_i)}, \quad 1 \leq j \leq c-1. \quad (4.2.4)$$

Proof: We have

$$\begin{aligned}\hat{p}_1(s) &= \frac{1 - \hat{f}_1(s)}{s} + \sum_{k=1}^{\infty} \frac{1 - \hat{f}_1(s)}{s} \hat{f}(s)^k, \\ \text{and } \hat{p}_j(s) &= \left(\frac{\prod_{i=1}^{j-1} \hat{f}_i(s) - \prod_{i=1}^j \hat{f}_i(s)}{s} \right) \left[1 + \sum_{k=1}^{\infty} \{ \hat{f}(s) \}^k \right], \quad 2 \leq j \leq c-1\end{aligned}$$

By Tauberian theorem,

$$p_j = \begin{cases} \lim_{s \rightarrow 0} \left[1 - \hat{f}_1(s) + \sum_{k=1}^{\infty} (1 - \hat{f}_1(s)) \hat{f}(s)^k \right] & \text{for } j = 1 \\ \lim_{s \rightarrow 0} \left[\prod_{i=1}^{j-1} \hat{f}_i(s) - \prod_{i=1}^j \hat{f}_i(s) \right] \left[1 + \sum_{k=1}^{\infty} \left\{ \hat{f}(s) \right\}^k \right] & 2 \leq j \leq c-1 \end{cases}$$

Using (4.2.3) and Theorem 4.2.1, we have

$$p_j = \begin{cases} \lim_{s \rightarrow 0} \frac{1 - \hat{f}_1(s)}{1 - \hat{f}(s)}, & j = 1 \\ \lim_{s \rightarrow 0} \frac{1 - \hat{f}_j(s)}{1 - \hat{f}(s)} \prod_{i=1}^{j-1} \hat{f}_i(s), & 2 \leq j \leq c-1. \end{cases}$$

Clearly $\hat{f}_i(0) = 1, \forall i$. Then by L-Hospital's rule

$$p_j = \begin{cases} \lim_{s \rightarrow 0} \frac{-\hat{f}'_1(s)}{-\hat{f}'(s)}, & j = 1 \\ \lim_{s \rightarrow 0} \frac{-\hat{f}'_j(s)}{-\hat{f}'(s)} \prod_{i=1}^{j-1} \hat{f}_i(s), & 2 \leq j \leq c-1. \end{cases}$$

Note that, $\hat{f}'(0) = -E(X)$.

Then,

$$p_j = \begin{cases} \frac{E(X_1)}{\sum_{i=1}^{c-1} E(X_i)}, & j = 1 \\ \frac{E(X_j)}{\sum_{i=1}^{c-1} E(X_i)}, & 2 \leq j \leq c-1. \end{cases}$$

Thus

$$p_j = \frac{E(X_j)}{\sum_{i=1}^{c-1} E(X_i)}, \quad 1 \leq j \leq c-1.$$

Hence the proof. \square

Remark 4.2.1 The steady state distribution of the system state depends only on the first moments of the underlying distribution.

Remark 4.2.2 As a special case, when X_1, X_2, \dots, X_M are i.i.d random variables with a common distribution function $F(\cdot)$ with mean μ ,

Then

$$p_j = \frac{\mu}{(c-1)\mu} = \frac{1}{c-1}, \quad 1 \leq j \leq c-1$$

which is a uniform distribution on $(1, 2, \dots, c-1)$.

4.3 Cost Analysis

Based on the model described above we develop an optimal replacement policy as follows. We have a multistate system that deteriorates over time such that the duration through which the system has a given level of performance is a random variable following an arbitrary distribution where the system is described by the performance level at time t . The state of the system is continuously monitored and whenever it exceeds a particular control limit the system is replaced instantaneously. The decision on whether the system is to be replaced is based on the performance level at the time of inspection.

The following costs are considered:

- a fixed operating cost incurred per unit time.
- a state dependent cost incurred when the system is replaced in a particular state.
- a fixed cost incurred while the system is being replaced.

Under the above cost structure, it is reasonable to expect that the optimal policy follows a control limit rule which specifies a replacement when the system state exceeds a critical level. The objective is to determine optimal replacement level for a minimum expected total cost rate. Let the total cost (TC) consists of operating cost (OC) and replacement cost (RC). We shall first consider the time dependent cost function.

Let a be the constant operating cost per unit time. The total operating cost in $(0, t)$ is

$$A(t) = a \int_0^t S(u) du.$$

where the above integral can be interpreted as a stochastic integral. Now

$$E\{A(t)\} = a \sum_{n=1}^{c-1} n \int_0^t Pr\{S(u) = n\} du.$$

Let R be the fixed replacement cost and r be the replacement cost per unit degradation in the level. Then the total cost for replacement per cycle (duration between

two successive replacements) is $R + rc$. If $M(t)$ is the renewal function of the renewal process $\{Y_k, k \geq 1\}$, then the expected replacement cost over $(0, t)$ is $(R + rc)M(t)$.

Therefore Total Cost (TC) over $(0, t)$ is

$$TC(c, t) = a \sum_{n=1}^{c-1} n \int_0^t \Pr\{S(u) = u\} du + (R + rc)M(t).$$

4.3.1 Cost Analysis in the Steady State

We choose the objective function as the steady state total expected cost per unit time.

We have to choose the value of 'c' so as to minimise the objective function.

The expected time elapsed between two replacements is

$$E(Y) = \sum_{i=1}^{c-1} E(X_i). \quad (4.3.1)$$

Therefore, expected number of replacements per unit time is

$$\frac{1}{E(Y)} = \frac{1}{\sum_{i=1}^{c-1} E(X_i)}.$$

Also the expected performance level (state) at any instant of time is

$$\begin{aligned} E(S) &= \sum_{n=1}^{c-1} np_n \\ &= \frac{\sum_{n=1}^{c-1} nE(X_n)}{\sum_{i=1}^{c-1} E(X_i)}. \end{aligned} \quad (4.3.2)$$

Therefore the steady state total expected cost per unit time is

$$TC(c) = \frac{R + rc}{E(Y)} + aE(S) \quad (4.3.3)$$

Using (4.3.1), (4.3.2) and (4.3.3), we get

$$TC(c) = \frac{R + rc}{\sum_{i=1}^{c-1} E(X_i)} + a \frac{\sum_{n=1}^{c-1} nE(X_n)}{\sum_{i=1}^{c-1} E(X_i)}.$$

Using this we choose the value of c which minimises $TC(c)$.

Special case (1): Assume that X_i 's are i.i.d random variables with common mean,

$E(X_i) = m, \forall i$. Then,

$$TC(c) = \frac{R + rc}{m(c-1)} + \frac{ac}{2}. \quad (4.3.4)$$

Let c^* be the optimal value of c at which $TC(c)$ is minimum.

Then we have the following theorem.

Theorem 4.3.1 *The optimal value, c^* , satisfies the inequality*

$$c^*(c^* - 3) + 2 \leq \frac{2(R + r)}{am} \leq c^*(c^* - 1). \quad (4.3.5)$$

Proof: If $TC(c)$ is to be minimum at c^* , it should satisfy the following conditions:

$$(i) \quad TC(c^*) \leq TC(c^* + 1) \quad \text{and} \quad (ii) \quad TC(c^*) \leq TC(c^* - 1).$$

From condition (i),

$$\frac{R + rc^*}{m(c^* - 1)} + \frac{ac^*}{2} \leq \frac{R + r(c^* + 1)}{m(c^*)} + \frac{a(c^* + 1)}{2},$$

which reduces to

$$\frac{2(R + r)}{am} \leq c^*(c^* - 1)$$

Similarly the condition (ii) gives

$$c^*(c^* - 3) + 2 \leq \frac{2(R + r)}{am}.$$

On combining these two inequalities we get (4.3.5). □

From the following table we can read c^* for given values of R, r, a, m .

c	2	3	4	5	6	7	8	9	10
$c(c - 3) + 2$	0	2	6	12	20	30	42	56	72
$c(c - 1)$	2	6	12	20	30	42	56	72	90

For example, when $R = 200, r = 10, a = 25$ and $m = 2$, the value of c^* is 4.

Special case (2): Suppose X_i 's are exponentially distributed r.v.s with mean m_n .

For five different sets of values of m_n , the values of c^* and $TC(c^*)$ are tabulated in the

next page for a set of values of R, r , and a .

	Set No.	m_n values	c^*	$TC(c^*)$
$R = 100$	I	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20.	4	70
	II	20,19,18,17,16,15, 14,13,12,11,10,9,8, 7,6,5,4,3,2,1.	2	26
$r = 10$	III	1.2,3.5,4.8,5.6,6.1,6.7,7 8.3,9.67,10.54,11,12.89,14, 17,18.5,21.4,24.5,26,30.1.	4	62.32
$a = 20$	IV	30.7,30.1,26,24.5,21.4,18.5,17, 14,12.89,11,10.54,9.67,8.3,7, 6.7,6.1,5.6,4.8,3.5,1.2.	2	23.99
	V	$\frac{1}{n}, n = 1, 2, \dots, 20$	4	109.09

Chapter 5

Statistical Estimation of Ordered Exponential Parameters

5.1 Introduction

Inference concerned with ordered parameters has received considerable attention in the statistical literature during the last two decades. For references on estimation of ordered parameters one can refer to Robertson *et.al.* (1988), Blumenthal and Cohen (1988), Cohen and Sackrowitz (1970) Sackrowitz (1970) and Kushary and Cohen (1991). Kushary and Cohen, in their paper 'Estimations of ordered Poisson parameters'

have studied the problem of estimating the smallest of a set of ordered Poisson means, when it is known which population corresponds to each mean. They have also discussed the admissibility and minimaxity of the estimators. In reliability there are situations where estimation of means of two exponential life distributions with ordered parameters is desired. For example, we can estimate the mean lives of two mechanical devices having exponential life distributions where one is an improvement of the other and naturally the improved device should not have mean life length less than that of the original device. In another situation the interest may be in the estimation of mean lives of two components having exponential life distributions, out of which one is produced by a standard company and the other is manufactured by a local company where it is apriori known that the mean life of the component of the standard company is not less than that of the component produced by the local company. Such a consideration often comes with regard to certain multistate deteriorating systems. The model discussed in chapter 4 presents itself as a reliability model where the parameters are restricted. The model can be described as follows. Starting from the perfect functioning level, the performance level of the system goes down as time elapses, and when it reaches a particular level, say c , the system is being replaced. The system stays for some time at each performance level called sojourn time. In particular, consider a multistate system

having performance levels 0, 1 and 2 with perfect functioning level at 2. Let X_1 and X_2 be the sojourn times at levels 2 and 1 respectively with exponential distributions. It is natural to assume that $E(X_1) \geq E(X_2)$. Here the underlying parameters will be ordered and so their estimation are of much interest in practice.

In this chapter we are mainly concerned with Bayesian estimation of scale parameters θ_1 and θ_2 of two exponential distributions under the restriction $\theta_1 \leq \theta_2$ using two class of priors and the invariant squared error loss function. We see that the usual methods of estimation such as UMVUE and MLE has certain limitations.

In section 2 of this chapter the statement of the problem is given and the limitations of classical approach are established. In Section 3, we derive general expression for Baye's estimator and Baye's risk. In particular, in the case of two classes of prior distributions which are relevant for reliability situations, we obtained nice expressions for Baye's estimator, Baye's risk, limit Baye's estimator and its frequentist risk. We see that these expressions are computationally tractable. In Section 4, we give a detailed comparison of limit Baye's estimators with the MLE.

Statement of the problem:

Let X and Y be two independent exponential r.v.s having means $\lambda_1 = \frac{1}{\theta_1}$ and $\lambda_2 = \frac{1}{\theta_2}$ such that $\theta_1 \leq \theta_2$. We wish to estimate λ_1 and λ_2 simultaneously with the

order restriction $\theta_1 \leq \theta_2$.

Let $T_1 = \sum_{i=1}^n X_i$ and $T_2 = \sum_{j=1}^n Y_j$ be sufficient statistics of θ_1 and θ_2 , based on n independent observations X_1, X_2, \dots, X_n and Y_1, Y_2, \dots, Y_n respectively.

The joint probability density function (pdf) of (T_1, T_2) at the point (t_1, t_2) (which is the conditional pdf of (T_1, T_2) given (θ_1, θ_2) in Bayesian approach) is

$$f(t_1, t_2; \theta_1, \theta_2) = \frac{\theta_1^n \theta_2^n}{\Gamma(n)^2} \exp\{-(t_1 \theta_1 + t_2 \theta_2)\} t_1^{n-1} t_2^{n-1}, \text{ for } 0 < t_1, t_2 < \infty;$$

$$0 < \theta_1 \leq \theta_2 < \infty \quad (5.1.1)$$

Several researchers have attempted this problem through classical approach. See Kaur and Singh (1991) and the references there in. In the next section we discuss the limitations of estimators obtained through classical approach.

5.2 The limitations of classical approach.

The UMVUE of λ_1 and λ_2 are respectively $\frac{T_1}{n}$ and $\frac{T_2}{n}$. For finite 'n' these estimators lie outside the parameter space with positive probability.

The MLE of λ_1 and λ_2 are respectively

$$\hat{\lambda}_1 = \max\left(\frac{T_1}{n}, \frac{T_1 + T_2}{2n}\right)$$

$$\text{and } \hat{\lambda}_2 = \min \left(\frac{T_2}{n}, \frac{T_1 + T_2}{2n} \right)$$

This is obtained by isotonic regression (see, Kaur & Singh (1991)). Now we compute the probability that the UMVUE lie outside the parameter space, which is same as the probability for $\hat{\lambda}_1 = \hat{\lambda}_2$.

$$\frac{T_1}{n} \leq \frac{T_2}{n} \Leftrightarrow \hat{\lambda}_1 = \hat{\lambda}_2$$

$$\begin{aligned} \text{Hence } P(\hat{\lambda}_1 = \hat{\lambda}_2) &= P(T_1 \leq T_2) \\ &= \sum_{i=0}^{n-1} \binom{n+i-1}{i} \frac{t^n}{(1+t)^{n+i}}, \text{ where } t = \frac{\theta_1}{\theta_2}, 0 < t < 1 \end{aligned}$$

Note that $\max_{0 \leq t < 1} \sum_{i=0}^{n-1} \binom{n+i-1}{i} \frac{t^n}{(1+t)^{n+i}} = \frac{1}{2}$ for all n and which is attained when $t = 1$, i.e., when $\theta_1 = \theta_2$. When $\theta_1 < \theta_2$ (strictly), there is a maximum of 50% chance for $\hat{\lambda}_1 = \hat{\lambda}_2$. We expect that $\hat{\lambda}_1 = \hat{\lambda}_2$ with probability 1 when $\theta_1 = \theta_2$. But it is clear from the above expression that only 50% of the sample points will lead to $\hat{\lambda}_1 = \hat{\lambda}_2$, when $\theta_1 = \theta_2$. i.e., in the case of UMVUE, there is a maximum of 50% chance for the estimator to lie outside the parameter space. In the case of MLE, the estimators coincide with maximum probability $\frac{1}{2}$. When $\theta_1 = \theta_2$, the estimators coincide only in the 50% cases of sample space. In the last section we will see that the performance of MLE, with respect to the scale invariant squared error loss function, is drastically worse when θ_1 and θ_2 are wide apart.

5.3 Baye's and limit Baye's estimation

In the previous section, we have seen that the main drawback of classical estimators is that it lie out side the parameter space with high positive probablity. To overcome this difficulty, the Baye's approach is advocated as in this case the estimators never lie out side the parameter space.

First we derive a general expression of Baye's estimator and Baye's risk.

Let $\pi(\theta_1, \theta_2)$ be the prior distribution of (θ_1, θ_2) with support on $\theta_1 \leq \theta_2$. The posterior distribution of θ_1 and θ_2 is given by

$$\pi(\theta_1, \theta_2 | t_1, t_2) = \frac{f(t_1, t_2; \theta_1, \theta_2)}{h(t_1, t_2)}, \quad (5.3.1)$$

where $f(t_1, t_2; \theta_1, \theta_2)$ is given in (5.1.1) and $h(t_1, t_2)$ is the marginal pdf of (T_1, T_2) given by

$$\begin{aligned} h(t_1, t_2) &= \int_0^\infty \int_{\theta_1}^\infty f(t_1, t_2; \theta_1, \theta_2) \pi(\theta_1, \theta_2) d\theta_1 d\theta_2, \\ &= \int_0^\infty \int_{\theta_1}^\infty \frac{\theta_1^n \theta_2^n}{\Gamma(n)^2} T_1^{n-1} T_2^{n-1} \exp - \{T_1 \theta_1 + T_2 \theta_2\} \pi(\theta_1, \theta_2) d\theta_2 d\theta_1 \\ &= \frac{t_1^{n-1} t_2^{n-1}}{\Gamma(n)^2} \phi_{n,n}(t_1, t_2; \pi), \quad 0 \leq t_1 \leq \infty \text{ and } 0 \leq t_2 \leq \infty, \quad (5.3.2) \end{aligned}$$

where $\phi_{r,s}$ is defined as

$$\phi_{r,s}(T_1, T_2; \pi) = \int_0^\infty \int_{\theta_1}^\infty \theta_1^r \theta_2^s \exp\{-T_1\theta_1 + T_2\theta_2\} \pi(\theta_1, \theta_2) d\theta_1, d\theta_2, \text{ for } r \geq 0 \text{ and } s \geq 0$$
(5.3.3)

We wish to estimate $\lambda_1 = \frac{1}{\theta_1}$ and $\lambda_2 = \frac{1}{\theta_2}$ with respect to the invariant squared error loss function. Since θ_1 and θ_2 are scale parameters it is desirable to take scale invariant squared error loss function. For an estimator (δ_1, δ_2) of (λ_1, λ_2) , the invariant squared error loss function is defined by

$$\begin{aligned} L(\delta_1, \delta_2; \theta_1, \theta_2) &= \left(\frac{\delta_1}{\lambda_1} - 1\right)^2 + \left(\frac{\delta_2}{\lambda_2} - 1\right)^2 \\ &= (\delta_1\theta_1 - 1)^2 + (\delta_2\theta_2 - 1)^2 \end{aligned}$$
(5.3.4)

Denote E_g as the expectation with respect to the pdf g . The frequentist risk of (δ_1, δ_2) as an estimator of λ_1 and λ_2 is given by

$$R(\delta_1, \delta_2; \theta_1, \theta_2) = E_f[L(\delta_1, \delta_2; \theta_1, \theta_2)].$$

For a prior distribution $\pi(\theta_1, \theta_2)$, the Baye's risk is given by

$$r(\delta_1, \delta_2; \pi) = E_\pi[R(\delta_1, \delta_2; \theta_1, \theta_2)].$$

Now we have the following theorem.

THEOREM 5.3.1 For any prior distribution $\pi(\theta_1, \theta_2)$ with support on $\theta_1 \leq \theta_2$, the Baye's estimator of (λ_1, λ_2) with respect to the loss function defined in (5.3.4) is given by

$$\begin{aligned} \delta_1^\pi &= \frac{\phi_{n+1,n}(T_1, T_2; \pi)}{\phi_{n+2,n}(T_1, T_2; \pi)} \\ \delta_2^\pi &= \frac{\phi_{n,n+1}(T_1, T_2; \pi)}{\phi_{n,n+2}(T_1, T_2; \pi)} \end{aligned} \quad (5.3.5)$$

and the Baye's risk of $(\delta_1^\pi, \delta_2^\pi)$ is given by

$$\begin{aligned} r(\delta_1^\pi, \delta_2^\pi; \pi) &= 2 - \int_0^\infty \int_0^\infty \frac{\phi_{n+1,n}^2(t_1, t_2; \pi) t_1^{n-1} t_2^{n-1}}{\phi_{n+2,n}(t_1, t_2; \pi) \Gamma(n)^2} dt_1 dt_2 \\ &\quad - \int_0^\infty \int_0^\infty \frac{\phi_{n,n+1}^2(t_1, t_2; \pi) t_1^{n-1} t_2^{n-1}}{\phi_{n,n+2}(t_1, t_2; \pi) \Gamma(n)^2} dt_1 dt_2 \end{aligned} \quad (5.3.6)$$

where $\phi_{r,s}(T_1, T_2; \pi)$ is defined in (5.3.3).

PROOF. In view of Baye's sufficiency, the Baye's estimator is fully depending on sufficient statistic. Hence to find Baye's estimator it is enough to concentrate on the estimators based on the sufficient statistic (T_1, T_2) . Note that the Baye's estimator minimises the Baye's risk

$$\begin{aligned} r(\delta_1, \delta_2; \pi) &= E_\pi[R(\delta_1, \delta_2; \theta_1, \theta_2)] \\ &= E_h[E_{\pi(\cdot|(t_1, t_2))}\{L(\delta_1, \delta_2; \theta_1, \theta_2)\}] \\ &= E_h[E_{\pi(\cdot|(t_1, t_2))}\{(\delta_1 \theta_1 - 1)^2 + (\delta_2 \theta_2 - 1)^2\}]. \end{aligned}$$

By Fatous lemma, the minimisation of $r(\delta_1, \delta_2; \pi)$ with respect to (δ_1, δ_2) is equivalent to the minimisation of $E_{\pi(\cdot|(t_1, t_2))}\{(\delta_1\theta_1 - 1)^2 + (\delta_2\theta_2 - 1)^2\}$, which is the posterior risk.

Now,

$$\begin{aligned} E_{\pi(\cdot|(t_1, t_2))}\{(\delta_1\theta_1 - 1)^2 + (\delta_2\theta_2 - 1)^2\} &= \delta_1^2 E_{\pi(\cdot|(t_1, t_2))}[\theta_1^2] - 2\delta_1 E_{\pi(\cdot|(t_1, t_2))}[\theta_1] + 1 \\ &\quad + \delta_2^2 E_{\pi(\cdot|(t_1, t_2))}[\theta_2^2] - 2\delta_2 E_{\pi(\cdot|(t_1, t_2))}[\theta_2] + 1 \end{aligned}$$

It can be easily shown that the right hand side of the above expression is minimum when

$$\begin{aligned} \delta_1 = \delta_1^\pi &= \frac{E(\theta_1 | T_1, T_2)}{E(\theta_1^2 | T_1, T_2)} \\ \delta_2 = \delta_2^\pi &= \frac{E(\theta_2 | T_1, T_2)}{E(\theta_2^2 | T_1, T_2)} \end{aligned}$$

Now using the posterior distribution $\pi(\theta_1, \theta_2 | T_1, T_2)$ given in (5.3.1), we get the Bayes estimator $(\delta_1^\pi, \delta_2^\pi)$ in the form (5.3.5).

To find the Baye's risk of this estimator, take the posterior risk

$$\begin{aligned} r_\pi((\delta_1^\pi, \delta_2^\pi) | T_1, T_2) &= E[(\theta_1\delta_1^\pi - 1)^2 | T_1, T_2] + E[(\theta_2\delta_2^\pi - 1)^2 | T_1, T_2] \\ &= (\delta_1^\pi)^2 \frac{\phi_{n+2, n}(T_1, T_2; \pi)}{\phi_{n, n}(T_1, T_2; \pi)} - 2\delta_1^\pi \frac{\phi_{n+1, n}(T_1, T_2; \pi)}{\phi_{n, n}(T_1, T_2; \pi)} + 1 \\ &\quad + (\delta_2^\pi)^2 \frac{\phi_{n, n+2}(T_1, T_2; \pi)}{\phi_{n, n}(T_1, T_2; \pi)} - 2\delta_2^\pi \frac{\phi_{n, n+1}(T_1, T_2; \pi)}{\phi_{n, n}(T_1, T_2; \pi)} + 1 \\ &= \frac{\phi_{n+1, n}^2(T_1, T_2; \pi)}{\phi_{n+2, n}(T_1, T_2; \pi)\phi_{n, n}(T_1, T_2; \pi)} \end{aligned}$$

$$\begin{aligned}
 & -2 \frac{\phi_{n+1,n}(T_1, T_2; \pi)}{\phi_{n+2,n}(T_1, T_2; \pi)\phi_{n,n}(T_1, T_2; \pi)} + 1 \\
 & + \frac{\phi_{n,n+1}^2(T_1, T_2; \pi)}{\phi_{n,n+2}(T_1, T_2; \pi)\phi_{n,n}(T_1, T_2; \pi)} \\
 & -2 \frac{\phi_{n,n+1}(T_1, T_2; \pi)}{\phi_{n,n+2}(T_1, T_2; \pi)\phi_{n,n}(T_1, T_2; \pi)} + 1 \\
 = & 2 - \frac{\phi_{n+1,n}^2(T_1, T_2; \pi)}{\phi_{n+2,n}(T_1, T_2; \pi)\phi_{n,n}(T_1, T_2; \pi)} \\
 & - \frac{\phi_{n,n+1}^2(T_1, T_2; \pi)}{\phi_{n,n+2}(T_1, T_2; \pi)\phi_{n,n}(T_1, T_2; \pi)}
 \end{aligned}$$

Now taking the expectation of posterior risk using the marginal pdf $h(t_1, t_2)$ given in (5.3.2), we get (5.3.6). □

For a given sequence of priors, we can find the corresponding sequence of Baye's estimators using the above theorem. A limit of the sequence of Baye's estimators, if it exists, is called limit Baye's estimator. Such an estimator is not necessarily Baye's with respect to a proper prior. Hence limit Baye's estimator is not exactly Baye's and it is considered as a frequentist estimator. The 'limit Baye's technique' is considered to be a fruitful method to derive reasonable and admissible estimators in most of the cases.

Particular Cases:

There may be some empirical evidence obtained through earlier experiments which would help us to decide on the prior distribution $\pi(\theta_1, \theta_2)$. At this point of time we shall discuss two particular cases of priors given by

$$\pi_1(\theta_1, \theta_2) = \alpha\beta \exp\{-(\alpha - \beta)\theta_1 + \beta\theta_2\}, \quad 0 < \theta_1 \leq \theta_2 < \infty, \quad \alpha, \beta > 0 \quad (5.3.7)$$

$$\pi_2(\theta_1, \theta_2) = \frac{m\theta_1^m}{\theta_2^{1+m}} \beta \exp\{-\beta\theta_1\}, \quad 0 < \theta_1 \leq \theta_2 < \infty, \quad m > 0, \beta > 0 \quad (5.3.8)$$

Case (i): In the case of prior, $\pi_1(\theta_1, \theta_2)$,

$$\begin{aligned} \phi_{r,s}(T_1, T_2; \pi_1) &= \int_0^\infty \int_{\theta_1}^\infty \theta_1^r \theta_2^s \exp\{-(T_1\theta_1 + T_2\theta_2)\} \alpha\beta \exp\{-(\alpha - \beta)\theta_1 + \beta\theta_2\} d\theta_2 d\theta_1 \\ &= \alpha\beta \int_0^\infty \theta_1^r \exp\{-(\alpha - \beta + T_1)\theta_1\} \left\{ \int_{\theta_1}^\infty \theta_2^s \exp\{-(\beta + T_2)\theta_2\} d\theta_2 \right\} d\theta_1. \end{aligned}$$

Now using

$$\int_{\theta_1}^\infty \theta_2^s \exp\{-(\beta + T_2)\theta_2\} d\theta_2 = \frac{s!}{(\beta + T_2)^{s+1}} \exp\{-(\beta + T_2)\theta_1\} \sum_{i=0}^s \frac{((\beta + T_2)\theta_1)^i}{i!},$$

we get

$$\begin{aligned} \phi_{r,s}(T_1, T_2; \pi_1) &= \alpha\beta \frac{s!}{(\beta + T_2)^{s+1}} \sum_{i=0}^s \frac{(\beta + T_2)^i}{i!} \int_0^\infty \theta_1^{r+i} \exp\{-(\alpha + T_1 + T_2)\theta_1\} d\theta_1 \\ &= \alpha\beta \frac{s!}{(\beta + T_2)^{s+1}} \sum_{i=0}^s \frac{(\beta + T_2)^i}{i!} \frac{\Gamma(r + i + 1)}{(\alpha + T_1 + T_2)^{r+i+1}} \end{aligned}$$

$$= \alpha\beta \frac{r!s!}{(\alpha - \beta + T_1)^{r+1}(\beta + T_2)^{s+1}} \sum_{i=0}^s \binom{r+i}{i} q^i p^{r+1}, \quad (5.3.9)$$

where $q = \frac{\beta+T_2}{\alpha+T_1+T_2}$. & $p = \frac{\alpha-\beta+T_1}{\alpha+T_1+T_2}$. Using (5.3.5) and (5.3.9), the Baye's estimator of λ_1 and λ_2 with respect to the prior π_1 and the invariant squared error loss function (5.3.4) is given by

$$\begin{aligned} \delta_1^{\pi_1} &= \frac{\phi_{n+1,n}(T_1, T_2; \pi_1)}{\phi_{n+2,n}(T_1, T_2; \pi_1)} \\ &= \left(\frac{\alpha + T_1 + T_2}{n + 2} \right) \frac{\sum_{i=0}^n \binom{n+i+1}{i} q^i}{\sum_{i=0}^n \binom{n+i+2}{i} q^i} \end{aligned} \quad (5.3.10)$$

$$\begin{aligned} \delta_2^{\pi_1} &= \frac{\phi_{n,n+1}(T_1, T_2; \pi_1)}{\phi_{n,n+2}(T_1, T_2; \pi_1)} \\ &= \left(\frac{\beta + T_2}{n + 2} \right) \frac{\sum_{i=0}^{n+1} \binom{n+i}{i} q^i}{\sum_{i=0}^{n+2} \binom{n+i}{i} q^i} \end{aligned} \quad (5.3.11)$$

Using (5.3.6), the Baye's risk of this estimator is given by

$$\begin{aligned} r(\delta_1^{\pi_1}, \delta_2^{\pi_1}; \pi_1) &= 2 - \frac{\alpha\beta}{\Gamma(n)^2} \int_0^\infty \int_0^\infty \frac{\phi_{n+1,n}^2(t_1, t_2; \pi_1)}{\phi_{n+2,n}(t_1, t_2; \pi_1)} t_1^{n-1} t_2^{n-1} dt_1 dt_2 \\ &\quad - \frac{\alpha\beta}{\Gamma(n)^2} \int_0^\infty \int_0^\infty \frac{\phi_{n,n+1}^2(t_1, t_2; \pi_1)}{\phi_{n,n+2}(t_1, t_2; \pi_1)} t_1^{n-1} t_2^{n-1} dt_1 dt_2 \\ &= 2 - I_1 - I_2, \text{ say} \end{aligned} \quad (5.3.12)$$

For I_1 (see, (5.3.9)),

$$\frac{\phi_{n+1,n}^2(t_1, t_2; \pi_1)}{\phi_{n+2,n}(t_1, t_2; \pi_1)} = \frac{\alpha\beta(n+1)!n!}{(n+2)(\alpha - \beta + t_1)^{n+1}(\beta + t_2)^{n+1}} Q_1 p^{n+1},$$

where,

$$Q_1 = \frac{\left\{ \sum_{i=0}^n \binom{n+i+1}{i} q^i \right\}^2}{\sum_{i=0}^n \binom{n+i+2}{i} q^i}.$$

Therefore,

$$I_1 = \frac{\alpha\beta(n+1)!n!}{(n+2)\Gamma(n)^2} \int_0^\infty \int_0^\infty \frac{Q_1 p^{n+1} t_1^{n-1} t_2^{n-1} dt_1 dt_2}{(\alpha - \beta + t_1)^{n+1} (\beta + t_2)^{n+1}}.$$

Use the transformations

$$q = \frac{\beta + t_2}{\alpha + t_1 + t_2}, \quad p = \frac{\alpha - \beta + t_1}{\alpha + t_1 + t_2} \text{ and } v = \alpha + t_1 + t_2.$$

Then the inverse transformations are $t_1 = pv - \alpha + \beta$ & $t_2 = qv - \beta$, which gives

$v > \max\left(\frac{\beta}{q}, \frac{\alpha - \beta}{p}\right) = a$, (say). Then

$$I_1 = \frac{\alpha\beta(n+1)!n!}{(n+2)(\Gamma(n)^2)} \int_0^1 Q_1 p^{n+1} \int_a^\infty \frac{(pv - \alpha + \beta)^{n-1} (qv - \beta)^{n-1}}{(pv)^{n+1} (qv)^{n+1}} v dv dq.$$

where $a = \frac{\beta}{q}$

On taking $\alpha = \beta$, the integral becomes

$$\begin{aligned} I_1 &= \frac{\beta^2(n+1)!n!}{(n+2)\Gamma(n)^2} \int_0^1 Q_1 p^{n+1} \int_a^\infty \frac{(pv)^{n-1} (qv - \beta)^{n-1}}{(pv)^{n+1} (qv)^{n+1}} v dv dq \\ &= \frac{\beta^2 n!(n+1)!}{(n+2)\Gamma(n)^2} \int_0^1 \frac{Q_1 p^{n-1}}{q^{n+1}} \int_a^\infty \frac{(qv - \beta)^{n-1}}{v^{n+2}} dv dq \\ &= \frac{\beta^2 n^2(n+1)}{n+2} \int_0^1 Q_1 \frac{p^{n-1}}{q^2} \int_a^\infty \frac{(v - \beta/q)^{n-1}}{v^{n+2}} dv dq \end{aligned}$$

In the inside integral, put $u = \frac{v-\beta/q}{v}$ then $1-u = \frac{\beta/q}{v}$ and $dv = \frac{\beta}{q} \frac{du}{(1-u)^2}$.

$$\begin{aligned} I_1 &= \frac{\beta^2 n^2 (n+1)}{n+2} \int_0^1 Q_1 \frac{p^{n-1}}{q^2} \int_0^1 \frac{u^{n-1} (1-u)^3 q^3 \beta}{\beta^3} \frac{du}{q(1-u)^2} dq \\ &= \frac{\beta^2 n^2 (n+1)}{n+2} \int_0^1 Q_1 p^{n-1} \frac{\beta(n, 2)}{\beta^2} dq \\ &= \frac{n^2 (n+1) \Gamma(n) \Gamma(2)}{(n+2) \Gamma(n+2)} \int_0^1 Q_1 p^{n-1} dq \\ &= \frac{n}{n+2} \int_0^1 Q_1 p^{n-1} dq \end{aligned}$$

In a similar way,

$$I_2 = \frac{n}{n+2} \int_0^1 Q_2 p^{n-1} dq,$$

where,

$$Q_2 = \frac{\left\{ \sum_{i=0}^{n+1} \binom{n+i}{i} q^i \right\}^2}{\sum_{i=0}^{n+2} \binom{n+i}{i} q^i}.$$

Substituting the value of I_1 and I_2 in (5.3.12), we get the Bayes risk (when $\alpha = \beta$) as

$$r(\delta_1^{\pi_1}, \delta_2^{\pi_1}; \pi_1) = 2 - \frac{n}{n+2} \left(\int_0^1 Q_1 p^{n-1} dq + \int_0^1 Q_2 p^{n-1} dq \right). \quad (5.3.13)$$

Now consider the limit Bayes estimator obtained by taking $\alpha \rightarrow \beta$ and then $\beta \rightarrow 0$

in (5.3.10) and (5.3.11), under the prior π_1

$$\bar{\delta}_1^{\pi_1} = \left(\frac{T_1 + T_2}{n + 2} \right) \frac{\sum_{i=0}^n \binom{n+i+1}{i} q^i}{\sum_{i=0}^n \binom{n+i+2}{i} q^i} \quad (5.3.14)$$

$$\bar{\delta}_2^{\pi_1} = \left(\frac{T_2}{n + 2} \right) \frac{\sum_{i=0}^{n+1} \binom{n+i}{i} q^i}{\sum_{i=0}^{n+2} \binom{n+i}{i} q^i}, \quad (5.3.15)$$

where $q = \frac{T_2}{T_1+T_2}$ & $p = \frac{T_1}{T_1+T_2}$. As (5.3.13) is independent of β , the limit of the Baye's risk when $\alpha \rightarrow \beta$ and then $\beta \rightarrow 0$, is (5.3.13). Now, the frequentist risk of the limit Baye's estimator $(\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1})$, is

$$R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); \theta_1, \theta_2) = E[(\theta_1 \bar{\delta}_1^{\pi_1} - 1)^2] + E[(\theta_2 \bar{\delta}_2^{\pi_1} - 1)^2].$$

To find this risk we require

$$\begin{aligned} \theta_1^k E[(\bar{\delta}_1^{\pi_1})^k] &= \frac{\theta_1^{n+k} \theta_2^n}{\Gamma(n)^2} \int_0^\infty \int_0^\infty \left\{ \frac{\phi_{n+1,n}}{\phi_{n+2,n}} \right\}^k \exp \{-(t_1 \theta_1 + t_2 \theta_2)\} t_1^{n-1} t_2^{n-1} dt_1 dt_2 \\ &= \frac{(2n+k-1)!}{\Gamma(n)^2 (n+2)^k} \int_0^1 \frac{t^{n+k} Q_1^k q^{n-1} p^{n-1}}{(q+pt)^{2n+k}} dq, \end{aligned} \quad (5.3.16)$$

which is obtained by the simplification as in the derivation of Baye's risk. Here $t = \frac{\theta_1}{\theta_2}$

and

$$Q_1 = \frac{\sum_{i=0}^n \binom{n+i+1}{i} q^i}{\sum_{i=0}^n \binom{n+i+2}{i} q^i}.$$

In a similar way,

$$\theta_2^k E[(\bar{\delta}_2^{\pi_1})^k] = \frac{(2n+k-1)!}{(n+1)^k \Gamma(n)^2} \int_0^1 \frac{t^n Q_2^k q^{n+k-1} p^{n-1}}{(q+pt)^{2n+k}} dq \quad (5.3.17)$$

where,

$$Q_2 = \frac{\sum_{i=0}^{n+1} \binom{n+i}{i} q^i}{\sum_{i=0}^{n+2} \binom{n+i}{i} q^i}.$$

Now, the frequentist risk of $(\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1})$, is

$$\begin{aligned} R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); \theta_1, \theta_2) &= E[(\theta_1 \bar{\delta}_1^{\pi_1} - 1)^2] + E[(\theta_2 \bar{\delta}_2^{\pi_1} - 1)^2] \\ &= \theta_1^2 E[(\bar{\delta}_1^{\pi_1})^2] - 2\theta_1 E[\bar{\delta}_1^{\pi_1}] + 1 + \theta_2^2 E[(\bar{\delta}_2^{\pi_1})^2] - 2\theta_2 E[\bar{\delta}_2^{\pi_1}] + 1 \\ &= \frac{(2n+1)!}{(n+2)^2 (\Gamma(n))^2} \int_0^1 \frac{t^{n+2} Q_1^2 q^{n-1} p^{n-1}}{(q+pt)^{2n+2}} dq \\ &\quad - \frac{2(2n)!}{(\Gamma(n))^2 (n+2)} \int_0^1 \frac{t^{n+1} Q_1 q^{n-1} p^{n-1}}{(q+pt)^{2n+1}} dq \\ &\quad + \frac{(2n+1)!}{(\Gamma(n))^2 (n+1)^2} \int_0^1 \frac{t^n Q_2^2 q^{n+1} p^{n-1}}{(q+pt)^{2n+2}} dq \\ &\quad - \frac{2(2n)!}{(\Gamma(n))^2 (n+1)} \int_0^1 \frac{t^n Q_2 p^{n-1} q^n}{(q+pt)^{2n+1}} dq + 2, \end{aligned} \quad (5.3.18)$$

which is obtained by using (5.3.16) and (5.3.17) with $k = 1$ and $k = 2$.

Case (ii): In the case of prior,

$$\pi_2(\theta_1, \theta_2) = \frac{m\theta_1^m}{\theta_2^{1+m}} \beta \exp\{-\beta\theta_1\}, \quad 0 < \theta_1 \leq \theta_2 < \infty, \quad m > 0, \beta > 0$$

we obtain

$$\begin{aligned} \phi_{r,s}(T_1, T_2; \pi) &= \int_0^\infty \int_{\theta_1}^\infty \theta_1^r \theta_2^s \exp\{-(T_1\theta_1 + T_2\theta_2)\} m \frac{\theta_1^m}{\theta_2^{1+m}} \beta \exp\{-\beta\theta_1\} d\theta_2 d\theta_1 \\ &= m\beta \int_0^\infty \theta_1^{r+m} \exp\{-(\beta + T_1)\theta_1\} \left(\int_{\theta_1}^\infty \theta_2^{s-m-1} \exp\{-T_2\theta_2\} d\theta_2 \right) d\theta_1 \end{aligned} \quad (5.3.19)$$

$$= \frac{m\beta(s-m-1)!}{T_2^{s-m}} \int_0^\infty \theta_1^{r+m} \exp\{-(\beta + T_1 + T_2)\theta_1\} \sum_{i=0}^{s-m-1} \frac{(T_2\theta_1)^i}{i!} d\theta_1,$$

when m is an integer $\leq s - 1$.

$$= \frac{(r+m)!(s-m-1)!m\beta}{T_2^{s-m}(\beta + T_1 + T_2)^{r+m+1}} \sum_{i=0}^{s-m-1} \binom{r+m+i}{i} q^i, \quad (5.3.20)$$

where $q = \frac{T_1}{\beta + T_1 + T_2}$, and m is an integer $\leq s - 1$. Therefore, using (5.3.5) and (5.3.9),

the Bayes estimator of λ_1 and λ_2 with respect to the prior π_2 and the invariant squared

error loss function (5.3.4) is given by

$$\begin{aligned} \delta_1^{\pi_2} &= \frac{\phi_{n+1,n}}{\phi_{n+2,n}} \\ &= \left(\frac{\beta + T_1 + T_2}{n + m + 2} \right) \frac{\sum_{i=0}^{n-m-1} \binom{n+m+i+1}{i} q^i}{\sum_{i=0}^{n-m-1} \binom{n+m+i+2}{i} q^i} \end{aligned} \quad (5.3.21)$$

$$\begin{aligned} \text{and } \delta_2^{\pi_2} &= \frac{\phi_{n,n+1}}{\phi_{n,n+2}} \\ &= \left(\frac{T_2}{n - m + 1} \right) \frac{\sum_{i=0}^{n-m} \binom{n+m+i}{i} q^i}{\sum_{i=0}^{n-m+1} \binom{n+m+i}{i} q^i} \end{aligned} \quad (5.3.22)$$

Baye's risk of the estimator using (5.3.6) is given by

$$r((\delta_1^{\pi_2}, \delta_2^{\pi_2}); \pi_2) = 2 - I_1 - I_2.$$

Here

$$\begin{aligned} I_1 &= \int_0^\infty \int_0^\infty \frac{\phi_{n+1,n}^2 t_1^{n-1} t_2^{n-1}}{\phi_{n+2,n} \Gamma(n)^2} dt_1 dt_2 \\ &= m\beta \frac{(n-m-1)!(n+m+1)!}{(n+m+2)\Gamma(n)^2} \int_0^\infty \int_0^\infty \frac{Q_1 t_1^{n-1} t_2^{m-1}}{(\beta+t_1+t_2)^{n+m+1}} dt_1 dt_2, \end{aligned}$$

$$\text{where } Q_1 = \frac{\left\{ \sum_{i=0}^{n-m-1} \binom{n+m+i+1}{i} q^i \right\}^2}{\sum_{i=0}^{n-m-1} \binom{n+m+i+2}{i} q^i}.$$

Put $q = \frac{t_2}{\beta+t_1+t_2}$, $v = \beta+t_1+t_2$, $p = \frac{\beta+t_1}{\beta+t_1+t_2}$. The inverse transformations are

$t_1 = pv - \beta$. and $t_2 = qv$ Here q varies from 0 to 1 and v varies from $\frac{\beta}{p}$ to ∞ . Therefore

$$\begin{aligned} I_1 &= \frac{m\beta(n-m-1)!(n+m+1)!}{(n+m+2)\Gamma(n)^2} \int_0^1 Q_1 \int_{\beta/p}^\infty \frac{(pv-\beta)^{n-1} (qv)^{m-1}}{v^{n+m+1}} v dv dq \\ &= \frac{m\beta(n-m-1)!(n+m+1)!}{(n+m+2)(\Gamma(n)^2)} \int_0^1 Q_1 q^{m-1} p^{n-1} \int_{\beta/p}^\infty \frac{(v-\beta/p)^{n-1}}{v^{n+1}} dv dq \\ &= \frac{(n-m-1)!(n+m+1)!}{n(n+m+2)\Gamma(n)^2} m \int_0^1 Q_1 q^{m-1} p^n dq, \end{aligned}$$

which is obtained by making the transformation $u = \frac{v-\beta/p}{v}$ in the inside integral, as in

the previous case. Similarly,

$$I_2 = \frac{(n-m)!(n+m)!}{n(n-m+1)\Gamma(n)^2} m \int_0^1 Q_2 q^{m-1} p^n dq,$$

$$\text{where } Q_2 = \frac{\left\{ \sum_{i=0}^{n-m} \binom{n+m+i}{i} q^i \right\}^2}{\sum_{i=0}^{n-m+1} \binom{n+m+i}{i} q^i}.$$

To find the limit of the Baye's risk as $m \rightarrow 0$, note that the right hand side of (5.3.19) is continuous in m . Hence the limit of the Baye's risk as $m \rightarrow 0$ can be obtained from the later expression by putting $m = 0$. Also,

$$\lim_{m \rightarrow 0} \int_0^1 Q_1 m q^{m-1} p^n dq = \lim_{m \rightarrow 0} \int_0^1 Q_1(u) (1 - u^{\frac{1}{m}})^n du,$$

where $u = q^m$. Now,

$$\begin{aligned} \lim_{m \rightarrow 0} Q_1(u) &= \lim_{m \rightarrow 0} \frac{\left\{ \sum_{i=0}^{n-m-1} \binom{n+m+i+1}{i} q^i \right\}^2}{\sum_{i=0}^{n-m-1} \binom{n+m+i+2}{i} q^i} \\ &= \lim_{m \rightarrow 0} \frac{\left\{ \sum_{i=0}^{n-m-1} \binom{n+m+i+1}{i} u^{\frac{i}{m}} \right\}^2}{\sum_{i=0}^{n-m-1} \binom{n+m+i+2}{i} u^{\frac{i}{m}}} \\ &= 1, \end{aligned}$$

$$\text{since } \lim_{m \rightarrow 0} u^{\frac{i}{m}} = \begin{cases} 0, & \text{when } i \geq 1 \\ 1, & \text{when } i = 0 \end{cases}.$$

Hence

$$\lim_{m \rightarrow 0} I_1 = \frac{n+1}{n+2} \quad \& \quad \lim_{m \rightarrow 0} I_2 = \frac{n}{n+1},$$

which gives

$$\lim_{m \rightarrow 0} r((\delta_1^{\pi_2}, \delta_2^{\pi_2}); \pi_2) = \frac{2n+3}{(n+1)(n+3)}$$

The limit Baye's estimators under this prior are given by taking the limit as $\beta \rightarrow 0$ and $m \rightarrow 0$ is

$$\bar{\delta}_1^{\pi_2} = \frac{T_1 + T_2}{n+2} \frac{\sum_{i=0}^{n-1} \binom{n+i+1}{i} q^i}{\sum_{i=0}^{n-1} \binom{n+i+2}{i} q^i} \tag{5.3.23}$$

$$\bar{\delta}_2^{\pi_2} = \frac{T_2}{n+2} \frac{\sum_{i=0}^n \binom{n+i}{i} q^i}{\sum_{i=0}^{n+1} \binom{n+i}{i} q^i} \text{ here } q = \frac{T_2}{T_1+T_2} \tag{5.3.24}$$

The frequentist risk of this estimator is given by

$$\begin{aligned} R(\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}; \theta_1, \theta_2) &= E(\theta_1 \bar{\delta}_1^{\pi_2} - 1)^2 + E(\theta_2 \bar{\delta}_2^{\pi_2} - 1)^2 \\ &= \theta_1^2 E[(\bar{\delta}_1^{\pi_2})^2] - 2\theta_1 E(\bar{\delta}_1^{\pi_2}) + 1 + \theta_2^2 E[(\bar{\delta}_2^{\pi_2})^2] - 2\theta_2 E(\bar{\delta}_2^{\pi_2}) + 1 \end{aligned}$$

To find this, we require

$$\begin{aligned} \theta_1^k E(\bar{\delta}_1^{\pi_2})^k &= \frac{\theta_1^{n+k} \theta_2^n}{\Gamma(n)^2} \int_0^\infty \int_0^\infty \left\{ \frac{\phi_{n+1,n}}{\phi_{n+2,n}} \right\}^k \exp\{-(t_1\theta_1 + t_2\theta_2)\} t_1^{n-1} t_2^{n-1} dt_1 dt_2 \\ &= \frac{\theta_1^{n+k} \theta_2^n}{(n+2)^k \Gamma(n)^2} \int_0^\infty \int_0^\infty (t_1 + t_2)^k Q_1^k \exp\{-(t_1\theta_1 + t_2\theta_2)\} t_1^{n-1} t_2^{n-1} dt_1 dt_2 \end{aligned}$$

Proceeding as in the previous case, we get

$$\theta_1^k E(\bar{\delta}_1^{\pi_2})^k = \frac{\theta_1^{n+k} \theta_2^n (2n+k-1)!}{(n+2)^k \Gamma(n)^2} \int_0^1 \frac{Q_1^k q^{n-1} p^{n-1}}{(q\theta_2 + p\theta_1)^{2n+k}} dq$$

$$= \frac{(2n+k-1)!}{\Gamma(n)^2(n+2)^k} \int_0^1 \frac{t^{n+k} Q_1^k q^{n-1} p^{n-1}}{(q+pt)^{2n+k}} dq$$

where, $t = \frac{\theta_1}{\theta_2}$ and

$$Q_1 = \frac{\sum_{i=0}^{n-1} \binom{n+i+1}{i} q^i}{\sum_{i=0}^{n-1} \binom{n+i+2}{i} q^i}$$

Similarly

$$\begin{aligned} \theta_2^k E(\bar{\delta}_2^{*k}) &= \frac{\theta_1^n \theta_2^{n+k} (2n+k-1)!}{\Gamma(n)^2(n+1)^k} \int_0^1 \frac{Q_2^k q^{n+k-1} p^{n-1}}{(q\theta_2 + p\theta_1)^{2n+k}} dq \\ &= \frac{(2n+k-1)!}{\Gamma(n)^2(n+1)^k} \int_0^1 \frac{t^n Q_2^k q^{n+k-1} p^{n-1}}{(q+pt)^{2n+k}} dq \end{aligned}$$

where,

$$Q_2 = \frac{\sum_{i=0}^n \binom{n+i}{i} q^i}{\sum_{i=0}^{n+1} \binom{n+i}{i} q^i}$$

Putting $k = 1$ and $k = 2$ in the above expression, we get the frequentist risk as

$$\begin{aligned} R(\bar{\delta}_1^{*2}, \bar{\delta}_2^{*2}; \theta_1, \theta_1) &= \frac{(2n+1)!}{\Gamma(n)^2(n+2)^2} \int_0^1 \frac{t^{n+2} Q_1^2 q^{n-1} p^{n-1}}{(q+pt)^{2n+2}} dq \\ &\quad - 2 \frac{(2n)!}{\Gamma(n)^2(n+2)} \int_0^1 \frac{t^{n+1} Q_1 q^{n-1} p^{n-1}}{(q+pt)^{2n+1}} dq + 1 \\ &\quad + \frac{(2n+1)!}{\Gamma(n)^2(n+1)^2} \int_0^1 \frac{t^n Q_2^2 q^{n+1} p^{n-1}}{(q+pt)^{2n+2}} dq \\ &\quad - 2 \frac{(2n)!}{\Gamma(n)^2(n+1)} \int_0^1 \frac{t^n Q_2 q^n p^{n-1}}{(q+pt)^{2n+1}} dq + 1 \end{aligned} \quad (5.3.25)$$

5.4 Numerical Comparison of estimators

In this section we discuss the comparison of the limit Baye's estimators with the MLE. As the expressions for Baye's risk and frequentist risk are much complicated, an analytic comparison is quite difficult. We compute these expressions numerically on a computer, using the numerical integration technique (Weddle's rule) and the performance of these estimators is discussed in detail.

First we derive an expression of the frequentist risk (mean square error) of the MLE. Note that the MLE of (λ_1, λ_2) is

$$\begin{aligned}\hat{\lambda}_1 &= \min\left(\frac{T_1}{n}, \frac{T_1 + T_2}{2n}\right) \\ &= \begin{cases} \frac{T_1}{n} & \text{if } T_1 \leq T_2 \\ \frac{T_1 + T_2}{2n} & \text{if } T_1 > T_2 \end{cases} \\ \hat{\lambda}_2 &= \min\left(\frac{T_2}{n}, \frac{T_1 + T_2}{2n}\right) \\ &= \begin{cases} \frac{T_2}{n} & \text{if } T_1 \leq T_2 \\ \frac{T_1 + T_2}{2n} & \text{if } T_1 > T_2 \end{cases}\end{aligned}$$

The MSE of MLE is

$$R((\hat{\lambda}_1, \hat{\lambda}_2); (\theta_1, \theta_2)) = E((\hat{\lambda}_1\theta_1 - 1)^2) + E((\hat{\lambda}_2\theta_2 - 1)^2)$$

(5.4.1)

To find this risk we require,

$$\begin{aligned} \theta_1^k E(\hat{\lambda}_1)^k &= \int \int_{t_1 < t_2} \left(\frac{T_1}{n}\right)^k \frac{\theta_1^{n+k} \theta_2^n}{\Gamma(n)^2} \exp\{-(t_1 \theta_1 + t_2 \theta_2)\} t_1^{n-1} t_2^{n-1} dt_1 dt_2 \\ &\quad + \int \int_{t_1 > t_2} \left(\frac{t_1 + t_2}{2n}\right)^k \frac{\theta_1^{n+k} \theta_2^n}{\Gamma(n)^2} \exp\{-(t_1 \theta_1 + t_2 \theta_2)\} t_1^{n-1} t_2^{n-1} dt_1 dt_2 \\ &= \frac{\theta_1^{n+k} \theta_2^n}{n^k \Gamma(n)^2} \int_0^\infty \int_{t_1}^\infty t_1^{n+k-1} t_2^{n-1} \exp\{-(t_1 \theta_1 + t_2 \theta_2)\} dt_1 dt_2 \\ &\quad + \frac{\theta_1^{n+k} \theta_2^n}{(2n)^k \Gamma(n)^2} \int_0^\infty \int_0^{t_1} (t_1 + t_2)^k t_1^{n-1} t_2^{n-1} \exp\{-(t_1 \theta_1 + t_2 \theta_2)\} dt_1 dt_2 \end{aligned}$$

Put $q = \frac{t_1}{t_1 + t_2}$, $v = t_1 + t_2$, and $p = \frac{t_1}{t_1 + t_2}$. Then $t_1 = pv$ and $t_2 = qv$. The restriction $t_1 < t_2$ implies $p < q$ which implies $q > 1/2$. Similarly $t_1 > t_2$ implies $q < 1/2$. This transformation gives

$$\begin{aligned} \theta_1^k E(\hat{\lambda}_1)^k &= \frac{\theta_1^{n+k} \theta_2^n}{n^k \Gamma(n)^2} \int_0^\infty \int_{1/2}^1 (pv)^{n+k-1} (qv)^{n-1} \exp\{-v(p\theta_1 + q\theta_2)\} v dv dq \\ &\quad + \frac{\theta_1^{n+k} \theta_2^n}{(2n)^k \Gamma(n)^2} \int_0^\infty \int_0^{1/2} v^k \exp\{-v(p\theta_1 + q\theta_2)\} (pv)^{n-1} (qv)^{n-1} v dv dq \\ &= \frac{(2n+k-1)! \theta_1^{n+k} \theta_2^n}{n^k \Gamma(n)^2} \left\{ \frac{1}{2^k} \int_0^{1/2} \frac{p^{n-1} q^{n-1}}{(p\theta_1 + q\theta_2)^{2n+k}} dq + \int_{1/2}^1 \frac{p^{n+k-1} q^{n-1}}{(p\theta_1 + q\theta_2)^{2n+k}} dq \right\} \\ &= \frac{(2n+k-1)! t^{n+k}}{\Gamma(n)^2 n^k} \left\{ \frac{1}{2^k} \int_0^{1/2} \frac{p^{n-1} q^{n-1}}{(q+pt)^{2n+k}} dq + \int_{1/2}^1 \frac{p^{n+k-1} q^{n-1}}{(q+pt)^{2n+k}} dq \right\} \quad (5.4.2) \end{aligned}$$

where $t = \theta_1/\theta_2$. Similarly,

$$\theta_2^k E(\hat{\lambda}_2)^k = \frac{(2n+k-1)! t^n}{\Gamma(n)^2 n^k} \left\{ \frac{1}{2^k} \int_0^{1/2} \frac{p^{n-1} q^{n-1}}{(q+pt)^{2n+k}} dq + \int_{1/2}^1 \frac{p^{n-1} q^{n+k-1}}{(q+pt)^{2n+k}} dq \right\} \quad (5.4.3)$$

Using (5.4.2) and (5.4.3) with $k = 1$ and $k = 2$ the MSE of the MLE is given by

$$\begin{aligned}
 R((\hat{\lambda}_1, \hat{\lambda}_2); (\theta_1, \theta_2)) &= E((\hat{\lambda}_1\theta_1 - 1)^2) + E((\hat{\lambda}_2\theta_2 - 1)^2) \\
 &= \frac{(2n+1)!t^n(1+t^2)}{4n^2\Gamma(n)^2} \int_0^{1/2} \frac{p^{n-1}q^{n-1}}{(q+pt)^{2n+2}} dq \\
 &\quad - \frac{(2n)!t^n(1+t)}{n\Gamma(n)^2} \int_0^{1/2} \frac{p^{n-1}q^{n-1}}{(q+pt)^{2n+1}} dq \\
 &\quad + \frac{(2n+1)!t^{n+2}}{n^2\Gamma(n)^2} \int_{1/2}^1 \frac{p^{n+1}q^{n-1}}{(q+pt)^{2n+2}} dq \\
 &\quad + \frac{(2n+1)!t^n}{n^2\Gamma(n)^2} \int_{1/2}^1 \frac{p^{n-1}q^{n+1}}{(q+pt)^{2n+2}} dq \\
 &\quad - \frac{2(2n)!t^{n+1}}{n\Gamma(n)^2} \int_{1/2}^1 \frac{p^n q^{n-1}}{(q+pt)^{2n+1}} dq \\
 &\quad - \frac{2(2n)!t^n}{n\Gamma(n)^2} \int_{1/2}^1 \frac{p^{n-1}q^n}{(q+pt)^{2n+1}} dq + 2. \tag{5.4.4}
 \end{aligned}$$

We compute the frequentist risk of $(\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1})$, $(\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2})$ and the MLE numerically by Weddle's rule using the formulae (5.3.18), (5.3.25) and (5.4.4). The result is tabulated in the next page onwards.

n	t	$R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); t)$	$R((\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}); t)$	$R(\hat{\lambda}_1, \hat{\lambda}_1; t)$
1	0.001	1.1125	1.0572	499503.43
	0.003	1.1114	1.0542	55389.67
	0.005	1.1115	1.0533	19900.94
	0.007	1.1115	1.0525	10133.59
	0.009	1.1116	1.0516	6118.27
	0.010	1.1116	1.0511	4951.01
	0.030	1.1112	1.0424	539.91
	0.050	1.1100	1.0339	191.04
	0.070	1.1083	1.0255	95.96
	0.090	1.1062	1.0174	57.25
	0.100	1.1051	1.0133	46.09
	0.300	1.0756	0.9422	5.11
	0.500	1.0441	0.8889	2.3334
	0.700	1.0173	0.8533	1.7179
	0.900	0.9980	0.8356	1.5355
	0.910	0.9973	0.8351	1.5308
	0.930	0.9958	0.8344	1.5224
	0.950	0.9945	0.8339	1.5149
	0.970	0.9933	0.8335	1.5084
	0.990	0.9921	0.8334	1.5027

The limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1}) = 0.5349$

The limit Baye's risk of $(\delta_1^{\pi_2}, \delta_2^{\pi_2}) = 0.8333$

n	t	$R((\delta_1^{\pi_1}, \delta_2^{\pi_1}); t)$	$R((\delta_1^{\pi_2}, \delta_2^{\pi_2}); t)$	$R(\hat{\lambda}_1, \hat{\lambda}_1; t)$
2	0.001	0.7474	0.7037	374467.84
	0.003	0.7499	0.7080	41500.68
	0.005	0.7499	0.7078	14900.71
	0.007	0.7499	0.7073	7582.35
	0.009	0.7498	0.7068	4574.80
	0.010	0.7497	0.7065	3700.73
	0.030	0.7481	0.6985	400.73
	0.050	0.7456	0.6893	140.73
	0.070	0.7429	0.6800	70.11
	0.090	0.7399	0.6712	41.46
	0.100	0.7384	0.6670	33.22
	0.300	0.7087	0.60551	3.2360
	0.500	0.6837	0.5705	1.2778
	0.700	0.6666	0.5555	0.8719
	0.900	0.6594	0.5589	0.7660
	0.910	0.6593	0.5596	0.7637
	0.930	0.6592	0.5611	0.7596
	0.950	0.6593	0.5627	0.7562
	0.970	0.6595	0.5645	0.7534
0.990	0.6598	0.5665	0.7510	

The limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1}) = 0.4016$

The limit Baye's risk of $(\delta_1^{\pi_2}, \delta_2^{\pi_2}) = 0.5833$

n	t	$R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); t)$	$R((\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}); t)$	$R(\hat{\lambda}_1, \hat{\lambda}_1; t)$
3	0.001	0.5401	0.5083	332947.90
	0.003	0.5601	0.5301	36871.03
	0.005	0.5600	0.5300	13233.98
	0.007	0.5600	0.5299	6731.95
	0.009	0.5600	0.5298	4060.32
	0.010	0.5600	0.5298	3283.99
	0.030	0.5593	0.5272	354.35
	0.050	0.5578	0.5226	123.97
	0.070	0.5556	0.5167	61.51
	0.090	0.5531	0.5102	36.2237
	0.100	0.5517	0.5069	28.9569
	0.300	0.5229	0.4489	2.6276
	0.500	0.5015	0.4164	0.9355
	0.700	0.4906	0.4055	0.5941
	0.900	0.4913	0.4144	0.5107
	0.910	0.4917	0.4154	0.5090
	0.930	0.4925	0.4175	0.5062
	0.950	0.4935	0.4197	0.5039
	0.970	0.4946	0.4222	0.5020
	0.990	0.4958	0.4248	0.5006

The limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1}) = 0.3238$

The limit Baye's risk of $(\delta_1^{\pi_2}, \delta_2^{\pi_2}) = 0.4500$

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n	t	$R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); t)$	$R((\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}); t)$	$R(\hat{\lambda}_1, \hat{\lambda}_1; t)$
4	0.001	0.4358	0.4129	311864.81
	0.003	0.4448	0.4226	34556.10
	0.005	0.4445	0.4222	12400.61
	0.007	0.4444	0.4222	6306.74
	0.009	0.4444	0.4222	3803.09
	0.010	0.4445	0.4223	3075.61
	0.030	0.4443	0.4215	331.16
	0.050	0.4435	0.4195	115.60
	0.070	0.4423	0.4164	57.2244
	0.090	0.4405	0.4123	33.6076
	0.100	0.4395	0.4100	26.8290
	0.300	0.4131	0.3598	2.3304
	0.500	0.3933	0.3284	0.7680
	0.700	0.3856	0.3191	0.4564
	0.900	0.3908	0.3303	0.3833
	0.910	0.3915	0.3313	0.3820
	0.930	0.3928	0.3336	0.3797
	0.950	0.3943	0.3361	0.3778
	0.970	0.3959	0.3389	0.3765
0.990	0.3977	0.3418	0.3754	

The limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1}) = 0.2724$

The limit Baye's risk of $(\delta_1^{\pi_2}, \delta_2^{\pi_2}) = 0.3667$

n	t	$R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); t)$	$R((\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}); t)$	$R(\hat{\lambda}_1, \hat{\lambda}_1; t)$
5	0.001	0.3796	0.3631	299483.87
	0.003	0.3672	0.3502	33167.29
	0.005	0.3673	0.3503	11900.59
	0.007	0.3673	0.3503	6051.61
	0.009	0.3673	0.3503	3648.74
	0.010	0.3673	0.3502	2950.59
	0.030	0.3673	0.3502	317.25
	0.050	0.3670	0.3494	110.57
	0.070	0.3663	0.3478	54.6481
	0.090	0.3652	0.3454	32.0389
	0.100	0.3646	0.3440	25.5532
	0.300	0.3415	0.3022	2.1553
	0.500	0.3225	0.2718	0.6692
	0.700	0.3162	0.2629	0.3745
	0.900	0.3240	0.2748	0.3070
	0.910	0.3247	0.2760	0.3058
	0.930	0.3264	0.2784	0.3038
	0.950	0.3282	0.2810	0.3023
0.970	0.3301	0.2837	0.3011	
0.990	0.3322	0.2867	0.3003	

The limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1}) = 0.2356$

The limit Baye's risk of $(\delta_1^{\pi_2}, \delta_2^{\pi_2}) = 0.3095$

n	t	$R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); t)$	$R((\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}); t)$	$R(\hat{\lambda}_1, \hat{\lambda}_1; t)$
6	0.001	0.3460	0.3340	291228.68
	0.003	0.3120	0.2986	32241.31
	0.005	0.3125	0.2991	11567.24
	0.007	0.3125	0.2991	5881.53
	0.009	0.3125	0.2991	3545.85
	0.010	0.3124	0.2990	2867.24
	0.030	0.3125	0.2991	307.97
	0.050	0.3124	0.2988	107.22
	0.070	0.3120	0.2980	52.9307
	0.090	0.3114	0.2967	30.9933
	0.100	0.3110	0.2958	24.7030
	0.300	0.2914	0.2617	2.0402
	0.500	0.2729	0.2325	0.6045
	0.700	0.2672	0.2235	0.3202
	0.900	0.2763	0.2355	0.2562
	0.910	0.2771	0.2366	0.2551
	0.930	0.2789	0.2390	0.2533
	0.950	0.2809	0.2416	0.2520
	0.970	0.2830	0.2444	0.2510
	0.990	0.2852	0.2474	0.2503

The limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1}) = 0.2078$

The limit Baye's risk of $(\delta_1^{\pi_2}, \delta_2^{\pi_2}) = 0.2678$

n	t	$R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); t)$	$R((\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}); t)$	$R(\hat{\lambda}_1, \hat{\lambda}_1; t)$
7	0.001	0.3222	0.3134	285267.62
	0.003	0.2712	0.2604	31579.91
	0.005	0.2716	0.2608	11329.14
	0.007	0.2716	0.2608	5760.04
	0.009	0.2716	0.2608	3472.35
	0.010	0.2716	0.2608	2807.70
	0.030	0.2716	0.2608	301.35
	0.050	0.2716	0.2607	104.83
	0.070	0.2714	0.2603	51.7039
	0.090	0.2711	0.2596	30.3464
	0.100	0.2708	0.2591	24.0957
	0.300	0.2544	0.2316	1.9589
	0.500	0.2364	0.2036	0.5590
	0.700	0.2309	0.1944	0.2817
	0.900	0.2405	0.2060	0.2200
	0.910	0.2414	0.2071	0.2190
	0.930	0.2433	0.2095	0.2173
	0.950	0.2454	0.2121	0.2160
	0.970	0.2476	0.2148	0.2151
	0.990	0.2499	0.2178	0.2145

The limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1}) = 0.1861$

The limit Baye's risk of $(\delta_1^{\pi_2}, \delta_2^{\pi_2}) = 0.2361$

n	t	$R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); t)$	$R((\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}); t)$	$R(\hat{\lambda}_1, \hat{\lambda}_1; t)$
8	0.001	0.3021	0.2956	280762.78
	0.003	0.2399	0.2310	31083.88
	0.005	0.2400	0.2311	11150.55
	0.007	0.2400	0.2311	5668.92
	0.009	0.2400	0.2311	3417.23
	0.010	0.2401	0.2312	2763.05
	0.030	0.2400	0.2311	296.38
	0.050	0.2400	0.2311	103.03
	0.070	0.2399	0.2309	50.7839
	0.090	0.2397	0.2305	29.6863
	0.100	0.2396	0.2302	23.6403
	0.300	0.2261	0.2081	1.8984
	0.500	0.2086	0.1816	0.5254
	0.700	0.2030	0.1720	0.2530
	0.900	0.2128	0.1831	0.1928
	0.910	0.2137	0.1842	0.1918
	0.930	0.2156	0.1865	0.1903
	0.950	0.2177	0.1890	0.1891
	0.970	0.2200	0.1918	0.1882
	0.990	0.2224	0.1947	0.1877

The limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1}) = 0.1686$

The limit Baye's risk of $(\delta_1^{\pi_2}, \delta_2^{\pi_2}) = 0.2111$

n	t	$R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); t)$	$R((\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}); t)$	$R(\hat{\lambda}_1, \hat{\lambda}_1; t)$
9	0.001	0.2828	0.2780	277257.53
	0.003	0.2150	0.2076	20698.08
	0.005	0.2149	0.2075	11011.66
	0.007	0.2149	0.2074	5598.06
	0.009	0.2149	0.2074	3374.35
	0.010	0.2150	0.2075	2728.32
	0.030	0.2149	0.2074	292.51
	0.050	0.2149	0.2074	101.64
	0.070	0.2148	0.2074	50.0682
	0.090	0.2147	0.2072	29.2506
	0.100	0.2146	0.2070	23.2861
	0.300	0.2036	0.1893	1.8517
	0.500	0.1867	0.1642	0.4997
	0.700	0.1809	0.1543	0.2307
	0.900	0.1906	0.1647	0.1717
	0.910	0.1915	0.1658	0.1708
	0.930	0.1935	0.1681	0.1692
	0.950	0.1956	0.1705	0.1681
	0.970	0.1979	0.1732	0.1673
	0.990	0.2003	0.1761	0.1669

The limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1}) = 0.1542$

The limit Baye's risk of $(\delta_1^{\pi_2}, \delta_2^{\pi_2}) = 0.1909$

n	t	$R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); t)$	$R((\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}); t)$	$R(\hat{\lambda}_1, \hat{\lambda}_1; t)$
10	0.001	0.2630	0.2593	274466.28
	0.003	0.1949	0.1886	30389.43
	0.005	0.1944	0.1881	10900.54
	0.007	0.1944	0.1881	5541.36
	0.009	0.1944	0.1881	3340.05
	0.010	0.1945	0.1882	2700.54
	0.030	0.1944	0.1881	289.42
	0.050	0.1944	0.1881	100.52
	0.070	0.1944	0.1881	49.4958
	0.090	0.1944	0.1880	28.9021
	0.100	0.1943	0.1879	23.0027
	0.300	0.1853	0.1737	1.8145
	0.500	0.1691	0.1501	0.4793
	0.700	0.1630	0.1399	0.2131
	0.900	0.1725	0.1497	0.1548
	0.910	0.1734	0.1507	0.1539
	0.930	0.1754	0.1529	0.1524
	0.950	0.1775	0.1554	0.1514
	0.970	0.1797	0.1580	0.1506
	0.990	0.1822	0.1608	0.1502

The limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1}) = 0.1421$

The limit Baye's risk of $(\delta_1^{\pi_2}, \delta_2^{\pi_2}) = 0.1742$

From the above tables, it is clear that the performance of MLE is very poor, especially when the value of t is small (i.e., when the difference of θ_1 and θ_2 is high). Among $(\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1})$ and $(\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2})$, the performance of $(\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2})$ seems to be better.

From the values of limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1})$ and $(\delta_1^{\pi_2}, \delta_2^{\pi_2})$, it is clear that the limit Baye's risk is smaller than their respective frequentist risks. Hence the minimaxity of these two estimators could not be claimed using Theorem 18 of Berger J.O. (1985) p.351. But on a close look at $R((\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1}); t)$, it can be observed that the variation in the values of risks for different values of t is less in comparison with $R((\bar{\delta}_1^{\pi_2}, \bar{\delta}_2^{\pi_2}); t)$. It is well known that an estimator with constant risk is minimax. Also note that the limit Baye's risk of $(\delta_1^{\pi_1}, \delta_2^{\pi_1})$ is smaller than that of $(\delta_1^{\pi_2}, \delta_2^{\pi_2})$. Hence $(\bar{\delta}_1^{\pi_1}, \bar{\delta}_2^{\pi_1})$ may be considered as an almost minimax estimator.

CONCLUDING REMARKS

In this thesis we have studied some stochastic models in Reliability which are physically realizable though complex. The repairable multistate system studied in chapter 2 may be extended with more general distributional assumptions on failure times and repair times. In such situations the underlying stochastic process may be studied through a semi-Markov process. The theory of Markov renewal process then could be applied to find the transient and steady state probabilities. Similarly the k - out of - n system studied in chapter 3 poses an interesting problem with generally distributed failure times and repair times.

Optimal replacement problem considered in chapter 4 may be looked upon under different cost structures depending upon the physical situations. Also the inferential problems associated with the model may be of much interest from the theoretical point of view.

We observed that the limit Baye's estimator with respect to the priors π_1 and π_2 discussed in chapter 5 are superior to the maximum likelihood estimators. Also the limit Baye's estimator with respect to the prior π_1 is seemed to be almost minimax. It is worthwhile to establish these findings analytically.

BIBLIOGRAPHY

- [1] Amarjot Kaur and Singh H. (1991) On the estimation of ordered means of two exponential populations. *Ann. Inst. Stat.Math.*, **43**, 347-356.
- [2] Asher H and Feingold H. (1984) *Repairable Systems Reliability*, Marcel and Dekker.
- [3] Barlow R.E. and Proschan F. (1965) *Mathematical Theory of Reliability*. John Wiley, New York.
- [4] Barlow R.E. and Proschan F. (1975) *Statistical Theory of Reliability and life testing*, Holt, Rinehart and Winston inc, New York.
- [5] Barlow R.E. and Wu.A.S (1978) Coherent systems with multistate components. *Math. Operat. Res*, **4**, 275-281.
- [6] Berger J.O (1988) *Statistical Decision Theory and Bayesian Analysis*, Second Edition, Springer-Verlag, New York.
- [7] Birnbaum Z.W., Esary J.D, and Saunders S.C.(1961) Multi component systems and structures and their reliability, *Technometrics*, **3** , 55-77.

- [8] Black G. and Proschan F. (1959) On optimal redundancy, *Operations Research.*, 7, 581-588.
- [9] Blumenthal S. and Cohen A. (1968) Estimation of two ordered translation parameters. *Ann. Math. Stat.*, 39, 517 - 530.
- [10] Cinlar E and Ozekici S. (1987) Reliability of complex devices in random environments., *Prob. Engineering and Infor. Sci.*, 1, 97-115.
- [11] Derman C., Lieberman G.J. and Ross S.M. (1982) On the consecutive k -out of- r : F system. *IEEE Trans on Rel.*, 34, 127-130.
- [12] Desmond A. (1985) Stochastic models of failure in random environments. *Canad. J. Statist.*, 9, 356-367.
- [13] Dharmadhikari A.D. and Naik Nimbalkar U.V. (1992) The importance of an item in a multistate system, *J. Operat Res. Soc. Japan*, 35, No 1, 31-34.
- [14] El. Neweihi E., Proschan F. and Sethuraman J. (1978) Multistate coherent systems, *J. Appl. Prob.*, 15, 675-688.
- [15] El. Neweihi E. and Proschan F. (1984) Degradable systems: A survey of multistate system theory, *Comm. Statist: Theor. Meth.* 13(4), 405-432.

- [16] Epstein B. and Sobel M.(1953) Life testing *J.Amer.Stat.Assn.*, **48**, 486 - 502.
- [17] Fu J.C. (1985) Reliability of a large consecutive k -out of- n : F system with unequal component reliability., *IEEE Trans on Rel.*, **35**, 316-319.
- [18] Fumio ohi, Nishida T.(1984) On multistate coherent systems, *IEEE Trans. on Rel.*, **18**, 284-288.
- [19] Griffith W.S.(1980) Multistate reliability models, *J.Appl.Prob*, **17**, 735-740 .
- [20] Henry. W. Blocks and Thomas H Savits (1982) A decomposition for multistate monotone systems. *J.Appl.Prob*, **19**,391-402.
- [21] John.B. deMercado (1971) Reliability prediction studies of complex systems having many failed states., *IEEE Trans on Rel.*, **20**, No 4, 223-230.
- [22] Joseph K.X. and Manoharan M.(1997). Analysis of a multistate repairable system. *International Journal of Information and Management Sciences*. **8**, No 1. 25-30.
- [23] Kai Yu, Israel Koren, Yuqing Guo (1994) Generalised multistate monotone coherent systems, *IEEE. Trans. on Rel.* **43**, 242-250.

- [24] Kushary D. and Cohen A. (1991) Estimation of ordered Poisson parameters, *Sankhya*, series A, **53** (3), 334 - 356.
- [25] Lam C.T. Yeh R.H. (1994) Optimal replacement policies for multistate deteriorating system *Naval Res. Log. Quart.*, **41**, 303-315.
- [26] Lam C.T. and Yeh R.H. (1994a) Optimal maintenance policies for deteriorating systems under various maintenance strategies ., *IEEE Trans on Rel.*, **43**, 423-430.
- [27] Linton D.G. (1974) Reliability analysis of the k - out of- n system., *IEEE Trans. on Rel.* , **23**, 97-103.
- [28] Moore E.F and Shannon C.E. (1956) Reliable circuits using less reliable relays, *J. Franklin Inst.*, **262**, 191-200 and 281-297
- [29] Nader Ebrahimi. (1984) Multistate reliability models, *Naval. Res. Logist. Quart.*, **31**, 671-680.
- [30] Natvig B.(1982) Two suggestions of how to define a multistate coherent system *Adv. in Appl. Prob.*, **14**, 434-455.
- [31] Natvig B.(1988) Multistate coherent systems, *Encyclopedia of Statistical Sciences*, John Wiley, New York., **5**, 732-735

- [32] Ohnishi M. Kawai H. and Mine H. (1986) An optimal inspection and replacement policy for a deteriorating system. *J. Appl. Prob.*, **23**, 973-988.
- [33] Papastavridis S.G and Markos V. Koutras (1992) Consecutive k -out of- n systems with maintenances., *Ann. Inst. Statist. Math.*, **44**, 4, 605-612.
- [34] Pierskalla W.P. Voelker J.A (1976) A survey of maintenance models: The control and surveillance of deteriorating systems, *Naval Res. Log. Quart.*, **23**, 353-374.
- [35] Procter C.L and Balbir Singh (1976) A repairable 3 state device, *IEEE Trans on Rel.*, **25**, 3, 210-211.
- [36] Procter C.L. et.al (1976) The analysis of a four state system, *Microelectron Reliab.*, **15**, 53-55.
- [37] Ravichandran N. (1990) *Stochastic methods in Reliability theory.*, Wiley Eastern., New Delhi.
- [38] Robertson T., Wright F.T. and Dykstra R.L. (1988) *Order Restricted Statistical Inference*, John Wiley, New York.
- [39] Ross S.M. (1979) Multivalued state component reliability systems *Ann. of Prob.*, **7**, 379-383.

- [40] Sackrowitz H.B. (1970), Estimation of monotone parameter sequences. The discrete case., *Ann. Math. Statist.*, 41, 609-620.
- [41] Sherif Y.S. and Smith M.L.(1981) Optimal maintenance models for systems subject to failure: A Survey. *Naval Res. Log.Quart.*, 47-74.
- [42] Singapurwalla N.D. and Younggreen M.A. (1993) Multivariate distributions induced by dynamic environments. *Scand. J. Statist.*, 20, 251-261.
- [43] Singapurwalla N.D.(1995) Survival in dynamic environments, *Statistical Science* 10, 1, 86-103.
- [44] Smith W.L.(1958) Renewal theory and its ramifications,*J.Royal Sta.Soc., Series B*, 20, 243-301.
- [45] Valdez-Flores C.and Feldman R.M.(1989) A survey of preventive maintenance models for stochastically deteriorating single unit systems, *Naval Res. Logist. Quart*, 36, 419-446.
- [46] Weiss G.(1956) On the theory of replacement of machinery with a random failure time, *Naval Res. Logist.Quart*, 3, 4, 279-293.
- [47] Widder D.V. (1974) *Advanced Calculus*, 2-nd Edn. Prentice Hall, New Delhi.

- [48] Yamashiro M. (1980a) A repairable multistate device with general repair time, *IEEE Trans on Rel.*, **29**, 276.
- [49] Yamashiro M. (1980b) A multistate system with general repair time distribution, *IEEE Trans. on Rel.* **29**, 453.
- [50] Yamashiro M. (1981a) Analysis of a degraded multistate system with general repair time distributions. *Microelectron Reliab.* **20**, 647- 650
- [51] Yamashiro M. (1981b) . A multistate system with several failure modes and cold stand by units, *Microelectron Reliab.*, **20**, 673-677.
- [52] Yamashiro M. (1981c) Two repairable multistate devices with general repair time distributions, *IEEE Trans on Reliability.*, **30**, 204.