

D 52411



(Pages : 3)

Name.....

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION  
DECEMBER 2013

(CCSS)

Mathematics

MAT 3E 02—OPERATIONS RESEARCH

Time : Three Hours

Maximum : 80 Marks

**Part A**

*Answer all questions.  
Each question carries 4 marks.*

1. Define a convex function. Show that the sum of two convex functions is a convex function.
2. Prove that a basic feasible solution of an LP problem is a vertex of the convex set of feasible solutions.
3. What are simplex multipliers in an LP Problem.
4. Prove that the value of the objective function  $f(x)$  for any feasible solution of the primal is not less than the value of the objective function  $\theta(y)$  for any feasible solution of the dual.
5. Describe the Caterer problem in Operations Research.
6. What is integer linear programming? Give an example of an integer linear programming problem.
7. When will you say that a subproblem is fathomed while using branch and bound method to solve an integer linear programming problem.
8. Write Kuhn-Tucker conditions for the LP :

$$\text{Maximize } 3x_1 + 2x_2$$

$$\text{subject to } 2x_1 - x_2 \leq 4$$

$$x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0.$$

(8 × 4 = 32 marks)

**Part B**

*Answer either A or B of each question.  
Each question carries 16 marks.*

9. A (a) Let  $f(X)$  be defined in a convex domain  $K \subseteq E_n$  and be differentiable. Prove that  $f(X)$  is a convex function iff  $f(X_2) - f(X_1) \geq (X_2 - X_1)' \nabla f(X_1)$  for all  $X_1, X_2$  in  $K$ .

Turn over

- (b) Solve the following problem by simplex method:

$$\begin{aligned} &\text{Minimize } x_1 - 2x_2 - 3x_3 \\ &\text{subject to } -2x_1 + 4x_3 \leq 12 \\ &\quad -4x_1 + 8x_2 + 3x_3 \leq 10 \\ &\quad 3x_1 + 2x_2 - x_3 \leq 7 \\ &\quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

- B (a) Explain the simplex procedure to solve a linear programming problem.

- (b) Solve the following problem by big M method :

$$\text{Minimize } f(X) = 4x_1 + 5x_2$$

$$\begin{aligned} &\text{subject to } 2x_1 + x_2 \leq 6, \\ &\quad x_1 + 2x_2 \leq 5 \\ &\quad x_1 + x_2 \geq 1, \\ &\quad x_1 + 4x_2 \geq 2, \\ &\quad x_1, x_2 \geq 0. \end{aligned}$$

10. A (a) Write the dual of the LP problem :

$$\begin{aligned} &\text{Maximize } 2x_1 + 5x_2 + 3x_3 \\ &\text{subject to } 4x_1 + x_3 \leq 420, \\ &\quad 2x_2 + 3x_3 \leq 460 \\ &\quad 2x_1 + x_2 + x_3 \leq 500, \\ &\quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

Without carrying out the simplex computation on either the primal or the dual problem, estimate a range for the optimal value of the objective function.

- (b) Solve by dual simplex method :

$$\begin{aligned} &\text{Minimize } f = 3x_1 + 5x_2 + 2x_3 \\ &\text{subject to } -x_1 + 2x_2 + 2x_3 \geq 3, \\ &\quad x_1 + 2x_2 + x_3 \geq 2, \\ &\quad -2x_1 - x_2 + 2x_3 \geq -4, \\ &\quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

- B (a) Describe a method to obtain a basic feasible solution to a transportation problem.
- (b) Food bags have to be lifted by three different types of aircraft  $A_1, A_2, A_3$ , from an airport and dropped in flood affected villages  $V_1, V_2, V_3, V_4, V_5$ . The quantity of food (in suitable units) that can be carried in one trip by aircraft  $A_i$  to village  $V_j$  is given in the following table. The total number of trips that  $A_i$  can make in a day is given in the last column. The number of trips possible each day to village  $V_j$  is given in the last row. Find the number of trips each aircraft should make on each village so that the total quantity of food transported in a day is maximum

	$V_1$	$V_2$	$V_3$	$V_4$	$V_5$	
$A_1$	10	8	6	9	12	50
$A_2$	5	3	8	4	10	90
$A_3$	7	9	6	10	4	60
	100	80	70	40	20	

11. A (a) Describe the cutting plane method to solve an integer linear programming problem.

- (b) Solve by branch and bound method :

$$\text{Maximize } x_1 + 2x_2$$

$$\text{subject to } 5x_1 + 7x_2 \leq 21$$

$$-x_1 + 3x_2 \leq 8,$$

$$x_1, x_2$$

non-negative integers.

- B (a) Find the minimum of  $f(X) = (x_1 + 1)^2 + (x_2 - 2)^2$

$$\text{subject to } g_1(X) = x_1 - 2 \leq 0$$

$$g_2(X) = x_2 - 1 \leq 0$$

$$x_1, x_2 \geq 0.$$

- (b) Solve by the method of quadratic programming :

$$\text{Maximize } 8x_1 + 10x_2 - x_1^2 - x_2^2$$

$$\text{subject to } 3x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0.$$

(3 × 16 = 48 marks)

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Name.....

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THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2013

(CCSS)

Mathematics

MAT 3E 01—ADVANCED TOPOLOGY

Time : Three Hours

Maximum : 80 Marks

**Part A**

Answer all questions

Each question carries 4 marks

- I. 1. Prove that a compact subset in a Hausdorff space is closed.
2. Prove that the unit circle in  $\mathbb{R}^2$  with usual topology is compact.
3. Prove that  $T_4$  spaces are completely regular.
4. Define large box in topological spaces. Prove that intersection of a family of large boxes is a large box.
5. Prove that Tietze extension theorem holds for maps into Euclidean spaces with the usual topology.
6. Prove that a product of topological spaces is completely regular if each coordinate is so.
7. Show that in a Hausdorff space no filter can converge to more than one point in it.
8. Prove that a subset  $A$  of a space  $X$  is open if and only if no net in the complement  $X - A$  converges to a point in  $A$ .

**Part B**

Answer either A or B of each question

Each question carries 12 marks

- II. A. (a) Prove that every regular, Lindelöf space is normal.  
(b) Prove that every compact Hausdorff space is  $T_4$ .
- B. (a) If a space has the property that for any two mutually disjoint closed subsets  $A, B$  of it, there exists a continuous function  $f : X \rightarrow [0, 1]$  taking the value 0 at all points on  $A$  and value 1 at all points on  $B$ , then prove that  $X$  is normal.  
(b) Prove that  $T_4$  spaces are Tychonoff spaces.

Turn over

- III. A. (a) Prove that topological product of regular spaces is regular.  
(b) Let  $X$  be a completely regular space. Suppose  $F$  is a compact subset in  $X$ ,  $C$  is a closed subset of  $X$  and  $F \cap C = \emptyset$ . Then prove that there exists a continuous function from  $X$  into the unit interval which takes the value 0 at all points of  $F$  and the value 1 at all points of  $C$ .
- B. (a) Define locally connected space. Prove that every quotient space of a locally connected space is locally connected.  
(b) Prove that product of topological spaces is path-connected if and only if each coordinate space is path-connected.
- IV. A. (a) Prove that a topological space is Hausdorff if and only if no filter can converge to more than one point in it.  
(b) Prove that every complete metric space with no isolated points is countable.
- B. (a) Let  $S$  be a family of subsets of a set  $X$ . Then prove that there exists a filter on  $X$  having  $S$  as a subbase if and only if  $S$  has the finite intersection property.  
(b) Prove that a topological space is compact if and only if every family of closed subsets of it which has the finite intersection property has a non-empty intersection.
- V. A. Prove that a metric space  $X$  is complete if and only if it contains a dense subset  $D$  such that every Cauchy sequence in  $D$  has a limit point in  $X$ .
- B. Prove that every locally compact Hausdorff space is completely regular.

D 52409



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Name.....

Reg. No.....

**THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2013**

(CCSS)

Mathematics

**MAT 3C 12—PARTIAL DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS**

Time : Three Hours

Maximum : 80 Marks

**Part A**

*Answer all questions.*

*Each question carries 4 marks.*

1. Show that  $2z = (ax + y)^2 + b$  is a complete integral of  $px + qy - z^2 = 0$ .
2. Find the general integral of  $(z^2 - 2yz - y^2) p + x(y + z) q = x(y - z)$ .
3. Show that the equations  $p^2 + q^2 - 1 = 0$  and  $(p^2 + q^2)x - pz = 0$  are compatible and find the one-parameter family of common solutions.
4. Solve the Cauchy problem for  $2z_x + yz_y = z$ , when the initial data curve C is given by  $x_0 = s, y_0 = s^2, z_0 = s, 1 \leq s \leq 2$ .
5. Determine the Monge cone with vertex at  $(0, 0, 0)$  for  $p^2 + q^2 = 1$ .
6. Reduce the equation  $x^2 u_{xx} - y^2 u_{yy} = 0$  into its canonical form.
7. Suppose that  $u(x, y)$  is harmonic in a bounded domain D and continuous in  $\bar{D} = D \cup B$ , where B is the boundary of D. Show that  $u$  attains its maximum on B.
8. Solve the Neumann problem for the upper half plane.
9. Solve the Dirichlet exterior problem for a circle.
10. Show that if  $y_m(x)$  and  $y_n(x)$  are characteristic function of the Fredholm equation  $y(x) = \lambda \int_a^b k(x, \xi) y(\xi) d\xi$  corresponding to distinct characteristic numbers, then  $y_m(x)$  and  $y_n(x)$  are orthogonal over the interval  $(a, b)$ .
11. Transform the problem  $\frac{d^2 y}{dx^2} + y = x, y(0) = 0, y'(1) = 0$  to a Fredholm integral equation.
12. Determine the resolvent associated with  $k(x, \xi) = \cos(x + \xi)$  in  $(0, 2\pi)$ , in the form of a power series in  $\lambda$ .

(12 × 4 = 48 marks)

**Turn over**

## Part B

Answer either Part A or Part B of each question.  
Each question carries 8 marks.

13. A. (a) Show that if there is a functional relation between two functions  $u(x, y)$  and  $v(x, y)$  not involving  $x$  and  $y$  explicitly then  $\frac{\partial(u, v)}{\partial(x, y)} = 0$ .

- (b) Show that a necessary and sufficient condition that the Pfaffian differential equation :

$$\vec{X} \cdot d\vec{r} = P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz = 0$$

be integrable is that  $(\vec{X} \cdot \text{curl } \vec{X}) = 0$ .

- B. (a) Explain Charpit's method to find the complete integral of a first order partial differential equation.

- (b) Find the complete integral of  $p^2 + q^2 = x + y$ .

14. A. (a) Find the general integral of the differential equation :

$$(x - y) y^2 p + (y - x) x^2 q = (x^2 + y^2) z$$

and the particular solution through  $xz = a^2, y = 0$ .

- (b) Find the characteristic strips of the equation  $x p + y q - p q = 0$ , where the initial data curve is :

$$x_0(s) = 2s, y_0(s) = 0, z_0(s) = s.$$

- B. (a) Derive D'Alembert's solution which describes the vibrations of an infinite string.

- (b) Write short notes on :

(i) Domain of dependence.

(ii) Range of influence.

15. A. (a) State the Dirichlet problem.

- (b) Show that the solution for the Dirichlet problem for a circle of radius  $a$  is given by the Poisson integral formula.

- B. (a) State the heat conduction problem in an infinite rod.

- (b) Solve the heat conduction problem in a finite rod of length  $l$ .

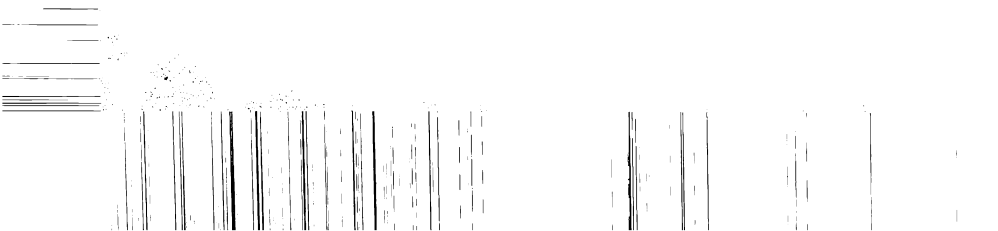
A. Show that any solution of the integral equation  $y(x) = \lambda \int_0^1 (1 - 3x\xi) y(\xi) d\xi + F(x)$  can be expressed as the sum of  $F(x)$  and some linear combination of the characteristic function.

B. (a) Show that the characteristic values of  $\lambda$  for the equation  $y(x) = \lambda \int_0^{2\pi} \sin(x + \xi) y(\xi) d\xi$  are  $\lambda_1 = 1/\pi$  and  $\lambda_2 = -1/\pi$ , with corresponding characteristic functions of the form  $y_1(x) = \sin x + \cos x$  and  $y_2(x) = \sin x - \cos x$ .

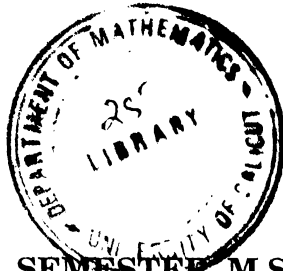
(b) Obtain the most general solution of the equation  $y(x) = \lambda \int_0^{2\pi} \sin(x + \xi) y(\xi) d\xi + x$  under the assumption that  $\lambda \neq \pm \frac{1}{\pi}$ .

(4 × 8 = 32 marks)

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(Pages : 3)

Name.....

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THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2013

(CCSS)

Mathematics

MAT 3C 11—FUNCTIONAL ANALYSIS

Time : Three Hours

Maximum : 80 Marks

Part A

Answer all questions.

Each question carries 4 marks.

1. Show that if  $X$  is a separable metric space and  $Y \subset X$ , then  $Y$  is separable in the induced metric.
2. State and prove Holders inequality for measurable functions.
3. Let  $X = \mathbb{K}^3$ . For  $x = (x(1), x(2), x(3)) \in X$ , let  $\|x\| = \left[ (|x(1)|^2 + |x(2)|^2)^{3/2} + |x(3)|^3 \right]^{1/3}$ . Show that  $\| \cdot \|$  is a norm on  $\mathbb{K}^3$ .
4. Let  $X$  be a normed space and  $E$  be a convex subset of  $X$ . Show that the interior  $E^\circ$  of  $E$  and the closure  $\bar{E}$  of  $E$  are also convex.
5. Let  $X$  and  $Y$  be normed spaces and  $\| \cdot \|$  be the operator norm on  $BL(X, Y)$ . Show that if  $X \neq \{0\}$ , then for any  $F \in BL(X, Y)$ ;  
$$\|F\| = \sup \{ \|F(x)\| : x \in X, \|x\| = 1 \}$$
$$= \sup \{ \|F(x)\| : x \in X, \|x\| < 1 \}.$$
6. Let  $X = C_{00}$  with the norm  $\| \cdot \|_1$ . Show that the linear functional on  $X$  defined by  $f(x) = \sum_{j=1}^{\infty} \frac{x(j)}{j}$ ,  $x \in X$  is continuous and  $\|f\| = 1$ .
7. Let  $\{a_1, a_2, \dots, a_m\}$  be a linearly independent set in a normed space  $X$ . Show that there exist  $f_1, f_2, \dots, f_m$  in  $X'$  such that  $f_j(a_i) = \delta_{ij}$ ,  $1 \leq i, j \leq m$ .
8. Show that a Banach space cannot have a denumerable basis.

Turn over

9. Let  $D$  denote the open unit disc in  $\mathbb{C}$  and let  $A = \{x \in C(\overline{D}) : x \text{ analytic on } D\}$ . For  $x \in A$ , let  $\|x\| = \max\{|x(e^{it})| : t \in [-\pi, \pi]\}$ . Show that  $A$  is a Banach space in the norm  $\|\cdot\|$ .
10. Let  $X$  be a normed space and  $E$  be a subset of  $X$ . Show that  $E$  is bounded in  $X$  iff  $f(E)$  is bounded in  $K$  for every  $f \in X'$ .
11. Let  $X$  and  $Y$  be Banach spaces and  $F_n \in BL(X, Y)$ ;  $n = 1, 2, \dots$  show that there is some  $F \in BL(X, Y)$  such that  $F_n(x) \rightarrow F(x)$  for every  $x \in X$  iff  $(F_n(x))$  converges for every  $x$  in some set  $E$  whose span is dense in  $X$  and the set  $\{\|f_n\| : n = 1, 2, \dots\}$  is bounded.
12. Let  $X$  be a normed space and  $f : X \rightarrow K$  be linear. Show that  $f$  is closed iff  $f$  is continuous.

(12 × 4 = 48 marks)

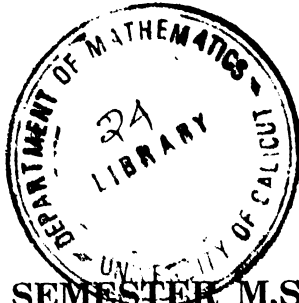
**Part B***Answer either (a) or (b) of each questions.**Each question carries 8 marks.*

13. (a) Show that a metric space  $X$  is compact iff every sequence in  $X$  has a convergent subsequence.
- (b) Show that for  $1 \leq p \leq \infty$ , the metric space  $L^p(E)$  is complete.
14. (a) (i) Let  $X$  and  $Y$  be normed spaces and  $F : X \rightarrow Y$  be a linear map. Show that  $F$  is continuous at zero iff  $F$  is continuous on  $X$ .
- (ii) Show that every bijective linear map on a finite dimensional normed space is a homeomorphism.
- (b) (i) Let  $X$  be a normed space over  $K$ . Show that the subset  $\{x \in X : \|x\| \leq 1\}$  of  $X$  is compact iff  $X$  is finite dimensional.
- (ii) Show that the closed unit ball in  $l^2$  is convex, closed and bounded, but not compact.

15. (a) (i) Let  $Y$  be a subspace of a normed space  $X$  and  $g \in Y'$ . Show that there is some  $f \in X'$  such that  $f|_Y = g$  and  $\|f\| = \|g\|$ .
- (ii) Let  $X = \mathbb{K}^2$  with the norm  $\|\cdot\|_\infty$ . Let  $Y = \{(x(1), x(2)) \in X : x(1) = x(2)\}$ . Define  $g \in Y'$  by  $g(x(1), x(2)) = x(1)$ . Show that the Hahn-Banach extensions of  $g$  to  $X$  are given by  $f(x(1), x(2)) = tx(1) + (1-t)x(2)$ , where  $t \in [0, 1]$  is fixed.
- (b) (i) Let  $X$  be a normed space and  $Y$  be a closed subspace of  $X$ . Show that  $X$  is a Banach space iff  $Y$  and  $X/Y$  are Banach spaces in the induced norm and the quotient norm, respectively.
- (ii) Show that  $C_{00}$  is not closed in  $l^\infty$ .
16. (a) (i) Let  $X$  and  $Y$  be Banach spaces and  $F : X \rightarrow Y$  be a closed linear map. Show that  $F$  is continuous.
- (ii) Give an example of a closed linear map which is not continuous.
- (b) (i) State and prove open mapping theorem.
- (ii) Give an example to show that the open mapping theorem may not hold if the normed spaces  $X$  and  $Y$  are not Banach spaces.

(4 × 8 = 32 marks)

D 52407



(Pages : 4)

Name.....

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2013

(CCSS)

Mathematics

MAT 3C 10—COMPLEX ANALYSIS

Time : Three Hours

Maximum : 80 Marks

Part A

Answer all questions.

Each question carries 4 marks.

1. Show that the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$  has radius of convergence 1. Discuss the convergence at  $z = 1, -1$  and  $i$ .
2. Discuss the mapping properties of  $f(z) = z^2$ .
3. Let S be the Riemann sphere. For the points  $1+i, 1, \frac{i}{2}$  of  $\mathbb{C}$ , find the corresponding points of S.  
What is the set in S corresponding to the circle  $\{z : |z| = 3\}$  in  $\mathbb{C}$ .
4. Let  $r$  be a closed rectifiable curve in  $\mathbb{C}$ . Show that for all integers  $n \geq 2$ ,  $\int_r (z-a)^{-n} dz = 0$ .
5. If  $f$  and  $g$  are analytic functions in a region  $G$  such that  $I_g$  is also analytic on  $G$ , then show that either  $f$  is a constant or  $g \equiv 0$ .
6. Let  $f$  be an analytic function on a region  $G$ . If there exists  $a$  in  $G$  such that  $|f(a)| \leq |f(z)|$  for all  $z$  in  $G$ , show that  $f(a) = 0$  or  $f$  is constant on  $G$ .
7. Let  $G$  be a star-shaped region. Prove that every closed rectifiable curve in  $G$  is homotopic to zero.
8. Let  $r$  be a closed rectifiable curve in an open set  $G$ . Show that if  $r$  is homotopic to zero in  $G$ , then  $r$  is homologous to zero in  $G$ .

Turn over

9. Let  $f$  be analytic on a region  $G$  except for poles. Show that the poles of  $f$  cannot have a limit point in  $G$ .
10. If  $f$  has a pole of order  $m$  at  $z = a$  and if  $g(z) = (z - a)^m f(z)$ , then show that :

$$\text{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

11. Suppose  $f$  is analytic in  $\bar{B}(0; 1)$  and satisfy  $|f(z)| < 1$  for  $|z| = 1$ . Find the number of solutions (counting multiplicity) of the equation  $f(z) = z^5$ .
12. Let  $f$  be analytic in the disk  $B(0, R)$ . For  $0 < r < R$ , let :

$$A(r) = \max\{ \operatorname{Re} f(z) : |z| \leq r \}.$$

Show that unless  $f$  is a constant,  $A(r)$  is a strictly increasing function of  $r$ .

(12 × 4 = 48 marks)

### Part B

Answer either part A or in part B in each question.

Each question carries 8 marks.

13. A (a) Show that a power series  $\sum_{n=0}^{\infty} a_n (z - a)^n$  converges absolutely for each  $z$  in  $B(a, R)$ ,

$$\text{where } \frac{1}{R} = \limsup |a_n|^{1/n}.$$

(2 marks)

- (b) If  $u$  is a harmonic function in  $\mathbb{C}$ , prove the  $u$  has a harmonic conjugate in  $\mathbb{C}$ .

(3 marks)

- (c) If  $f$  is a branch of the logarithm in a region  $G$ , show that  $f$  is analytic in  $G$  and the derivative is  $1/z$ .

(3 marks)

- B (a) State and prove the Chain rule for analytic functions.

(3 marks)

- (b) Prove that a power series is infinitely many times differentiable in the disk of convergence.

(3 marks)

- (c) Show that every linear transformation preserves cross-ratios.

(2 marks)

14. A (a) State and prove Liouville's theorem. (3 marks)

(b) If  $r$  is piecewise smooth on  $[a, b]$  and if  $f : [a, b] \rightarrow \mathbb{C}$  is continuous, prove that :

$$\int_a^b f dr = \int_a^b f(t) r'(t) dt.$$

(3 marks)

(c) If  $f$  and  $g$  are analytic in a region  $G$  and if they coincide on a set of points with a limit point in  $G$ , prove that  $f \equiv g$ .

(2 marks)

B (a) Suppose  $f$  is analytic in  $B(a; R)$ . Show that  $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$  for  $|z-a| < R$ , where

$$a_n = \frac{1}{n!} f^{(n)}(a).$$

(3 marks)

(b) Let  $f$  be a function analytic in a region  $G$ . If there exists a point  $a$  in  $G$ , such that  $f^{(n)}(a) = 0$  for all integers  $n \geq 0$ , prove that  $f \equiv 0$ .

(3 marks)

(c) Show that if  $f$  is analytic in a region  $G$  and if  $|f(z)|$  attains a maximum in  $G$ , then  $f$  is constant.

(2 marks)

15. A (a) State and prove the theorem on the Laurent series development in the annulus.

(5 marks)

(b) If  $f$  has an isolated singularity at  $z = a$  and if  $\lim_{z \rightarrow a} (z-a) f(z) = 0$ , prove that  $z = a$  is a removable singularity of  $f$ .

(3 marks)

B (a) State and prove Goursat's theorem.

(5 marks)

(b) If  $G$  is a simply connected region and if  $f$  is analytic in  $G$ , prove that  $f$  has a primitive in  $G$ .

(3 marks)

16. A (a) State and prove the Argument principle.

(4 marks)

(b) Show that :  $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ .

(4 marks)

Turn over

- B (a) State and prove the Residue theorem. (3 marks)**
- (b) State Ronche's theorem and deduce the fundamental theorem of algebra from it. (2 marks)**
- (c) State and prove Maximum modulus theorem (Third version). (3 marks)**
- [4 × 8 = 32 marks]**

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(Pages : 4)

Name.....

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**THIRD SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2013**

(CCSS)

Mathematics

**MAT 3E 02—OPERATIONS RESEARCH**

Time : Three Hours

Maximum : 80 Marks

**Part A**

*Answer all questions.*

*Each question carries 4 marks.*

1. Obtain a necessary and sufficient condition for a differentiable function in a convex domain to be convex.
2. Write a short note on degeneracy in linear programming problems.
3. Existence of dual provides some practically useful suggestions in L.P. problems. Give an example.
4. What is the transportation problem ? Write the standard form.
5. Define the following terminologies :
  - (a) Convex function.
  - (b) Basic feasible solution.
  - (c) Convexlinear combination.
  - (d) Optimal solution.
6. Prove that  $f(x) = x^2$ ,  $x \in \mathbb{R}$  is a convex function.
7. There are  $n$  objects,  $j = 1, 2, \dots, n$ , whose weights are  $w_j$  and values  $v_j$ . They have to be chosen to be packed in a knapsack so that the total value of the objects chosen is maximum subject to their total weight not exceeding  $W$ . Formulate this problem as an integer linear programming problem.
8. Obtain Kuhn-Tucker conditions for the problem.

$$\text{Minimize } f = x_1^2 + x_2^2$$

$$\text{subject to } g = (x-1)^3 - x^3 \geq 0.$$

(8 × 4 = 32 marks)

**Part B**

*Answer either Part A or Part B of each question.*

*Each question carries 16 marks.*

9. A. (i) Describe a systematic procedure for finding a feasible solution to a system of linear inequalities.

Turn over

(ii) Use the simplex method to solve :

$$\text{Minimize } 3x_1 + x_2 + x_3 + x_4$$

subject to

$$-2x_1 + 2x_2 + x_3 = 4$$

$$3x_1 + x_2 + x_4 = 6$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

(8 + 8 = 16 marks)

B. (i) What are artificial variables and why we need them ? How do they differ from slack or surplus variables ?

(ii) Solve the following problem using simplex method :

$$\text{Maximize } 3x_1 + 2x_2 + 3x_3$$

subject to

$$x_1 + x_2 + x_3 \leq 9$$

$$2x_1 + 3x_2 + 5x_3 \leq 30$$

$$2x_1 - x_2 - x_3 \leq 8$$

$$x_1, x_2, x_3 \geq 0.$$

(8 + 8 = 16 marks)

10. A. (i) What is dualing in a linear programming problem ? Prove that the dual of the dual is primal.

(ii) There are three reservoirs with daily supplies of 15, 20 and 25 million liters of fresh water, respectively. On each day water must be supplied to four cities A, B, C, D whose demands are 8, 10, 12 and 15 respectively. The cost of pumping per million liters is given below :

	A	B	C	D
1	2	3	4	5
2	3	2	5	2
3	4	1	2	3

Find the cheapest pumping schedule, if excess water can be disposed of at no cost.

(8 + 8 = 16 marks)

B. (i) Solve the integer programming problem optimally

$$\begin{aligned} &\text{Maximize } 8x_1 + 6x_2 + 10x_3 \\ &\text{subject to } 8x_1 + 4x_2 + 2x_3 \leq 155 \\ &\quad 3x_1 + 6x_2 + 12x_3 \leq 135 \\ &\quad x_1, x_2, x_3 \geq 0 \text{ and integers} \end{aligned}$$

(ii) Use Kuhn-Tucker conditions to find the minima and maxima of  $(x_1 - 4)^2 + (x_2 - 3)^2$  subject to  $36(x_1 - 2)^2 + (x_2 - 3)^2 \leq 9$ .

(8 + 8 = 16 marks)

11. A. (i) Solve by branch and bound method :

$$\begin{aligned} &\text{Minimize } 9x_1 + 10x_2 \\ &\text{subject to } 0 \leq x_1 \leq 10 \\ &\quad 0 \leq x_2 \leq 8 \\ &\quad 3x_1 + 5x_2 \geq 45 \\ &\quad x_2 \text{ integer} \end{aligned}$$

(ii) Solve by cutting plane method :

$$\begin{aligned} &\text{Minimize } -2x_1 - 3x_2 \\ &\text{subject to } 2x_1 + 2x_2 \leq 7 \\ &\quad 0 \leq x_1 \leq 2 \\ &\quad 0 \leq x_2 \leq 2 \\ &\quad x_1, x_2 \text{ integers} \end{aligned}$$

(8 + 8 = 16 marks)

B. (i) Solve by Kuhn-Tucker conditions and verify geometrically :

$$\begin{aligned} &\text{Minimize } x_1 \\ &\text{subject to } (x_1 - 4)^2 + x_2^2 \leq 16 \\ &\quad (x_1 - 3)^2 + (x_2 - 2)^2 = 13 \end{aligned}$$



Turn over

(ii) Solve by the method of quadratic programming :

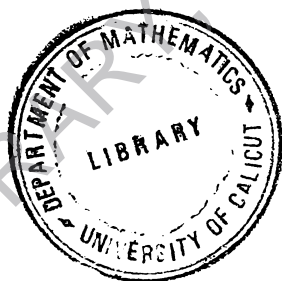
$$\text{Minimize } -6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$$

$$\text{subject to } x_1 + x_2 \leq 2$$

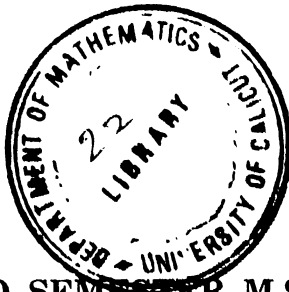
$$x_1, x_2 \geq 0.$$

(8 + 8 = 16 marks)

[3 × 16 = 48 marks]



D 32233



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Name.....

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**THIRD SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2013**

Mathematics

MAT 3E 01—ADVANCED TOPOLOGY

Time : Three Hours

Maximum : 80 Marks

**Part A**

*Answer all questions.  
Each question carries 4 marks.*

1. Prove that compact subsets of a Hausdorff space are closed.
2. Let  $X$  be a regular space,  $C$  a closed subset of  $X$  and  $F$  a compact subset of  $X$  such that  $C \cap F = \emptyset$ . Prove that there exists disjoint open sets  $U, V$  such that  $C \subset U$  and  $F \subset V$ .
3. Define locally connected space. Give an example of a space which is connected but not locally connected.
4. Prove that product of topological spaces is path connected iff each co-ordinate space is path connected.
5. Let  $X$  be any set of  $\mathcal{F}$  the set of all finite subsets of  $X$ . For  $F, G \in \mathcal{F}$  define  $F \geq G$  to mean  $F \supset G$ . Prove that  $\geq$  directs  $\mathcal{F}$ .
6. Let  $X$  be a space and suppose  $A \subset X, x \in X$ . Prove that  $x \in \bar{A}$  iff there exists a filter  $\mathcal{F}$  on  $X$  such that  $A \in \mathcal{F}$  and  $\mathcal{F}$  converges to  $x$ .
7. Prove that a closed subset of a complete metric space is complete with respect to the induced metric.
8. Prove that, in a locally compact Hausdorff space, a subset of first category can have no interior points.

(8 × 4 = 32 marks)

**Part B**

*Answer either Part A or Part B in each question.  
Each question carries 12 marks.*

9. A (a) Show that a continuous bijection from a compact space into a Hausdorff space is an embedding.  
(b) Prove that a subset of  $\mathbb{R}$  with the usual topology is compact iff it is closed and bounded.  
B State and prove Tietze extension theorem.
10. A (a) Prove that a topological space  $X$  is locally connected iff components of open subsets of  $X$  are open in  $X$ .

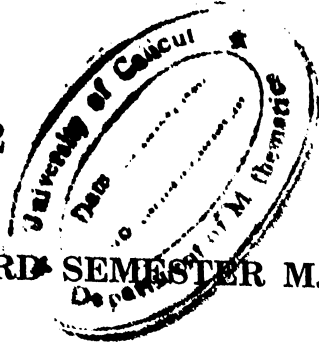
Turn over

- (b) Prove that every path connected space is connected.
- B (a) Prove that a topological product of spaces is Tychonoff iff each co-ordinate space is Tychonoff.
- (b) Let  $f: X \rightarrow Y$  be continuous and open and  $x_0 \in X$ . If  $X$  is first countable at  $x_0$ , prove that  $Y$  is first countable at  $f(x_0)$ .
- 11 A (a) Let  $S: D \rightarrow X$  be a net in a topological space and let  $x \in X$ . Prove that  $x$  is a cluster point of  $S$  iff there exists a subnet of  $S$  which converges to  $x$  in  $X$ .
- (b) Let  $X$  be the topological product of an indexed family of spaces  $\{X_i : i \in I\}$ . Let  $\mathcal{F}$  be a filter on  $X$  and let  $x \in X$ . Prove that  $\mathcal{F}$  converges to  $x \in X$  iff for each  $i \in I$ , the filter  $\pi_{i\#}(\mathcal{F})$  converges to  $\pi_i(x)$  in  $X_i$ .
- B (a) Prove that a topological space is Hausdorff iff no filter can converge to more than one point in it.
- (b) Let  $S: D \rightarrow X$  be a net and  $\mathcal{F}$  the filter associated with it. Let  $x \in X$ . Prove that  $S$  converges to  $x$  as a net iff  $\mathcal{F}$  converges to  $x$  as a filter.
12. A (a) Prove that a metric space is compact iff it is complete and totally bounded.
- (b) Prove that a  $G_\delta$ -set in a complete metric space is metrically topologically complete.
- B (a) Let  $(X, d)$  be a complete metric space. Prove that a subset of first category in  $X$  cannot have any interior points.
- (b) Prove that every metric space can be isometrically embedded as a dense subspace of a complete metric space.

(4 × 12 = 48 marks)



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Name.....

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THIRD SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2013

(CCSS)

Mathematics



MAT 3C 12—PARTIAL DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

Time : Three Hours

Maximum : 80 Marks

**Part A**

*Answer all questions.  
Each question carries 4 marks.*

1. Find the general integral of  $y^2 p - xy q = x(z - 2y)$ .
2. Show that the following Pfaffian differential equation is integrable and find its integral :

$$ydx + xdy + 2z dz = 0.$$

3. Show that the two equations  $xp - yq - x = 0$  and  $x^2 p + q - xz = 0$  are compatible.
4. Solve the Cauchy problem for  $2z_x + yz_y = z$ , when the initial data curve is C :

$$x_0 = s, y_0 = s^2, z_0 = s, 1 \leq s \leq 2.$$

5. Find the characteristic strips of the equation  $xp + yq - pq = 0$ .
6. Reduce the equation  $x^2 u_{xx} - y^2 u_{yy} = 0$  into canonical form.
7. Suppose that  $u(x, y)$  is harmonic in a bounded domain D and continuous in  $\bar{D} = D \cup B$ . Show that  $u$  attains its maximum on the boundary B of D.
8. Solve the Neumann problem for the upper half plane.
9. Solve the Dirichlet Exterior problem for a circle.

10. Show that if  $y(x)$  satisfies the differential equation  $\frac{d^2 y}{dx^2} + xy = 1$  and the conditions

$y(0) = 0 = y'(0)$ , then  $y$  also satisfies

The Volterra equation  $y(x) = \int_0^x \xi (\xi - x) y(\xi) d\xi + \frac{1}{2} x^2$ .

Turn over

11. Show that the Kernel  $K(x, \xi) = 1 + 3x\xi$  has a double characteristic number associated with  $(-1, 1)$ , with two independent characteristic functions.
12. Determine the iterated Kernel  $K_2(x, \xi)$  associated with  $K(x, \xi) = |x - \xi|$  in  $(0, 1)$ .  
(12 × 4 = 48 marks)

### Part B

Answer either Part A or Part B of each question.

Each question carries 8 marks.

13. A (a) Find the partial differential equation by eliminating the arbitrary function  $F$  from the equation  $F(x + y, x - \sqrt{z}) = 0$ .  
(2 marks)
- (b) Show that the Pfaffian differential equation  $\bar{X} \cdot d\bar{r} = p(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz = 0$  is integrable iff  $(\bar{X} \cdot \text{curl } \bar{X}) = 0$ .  
(6 marks)
- B (a) Explain Charpit's method to find a complete integral of a first order partial differential equation  $f(x, y, z, p, q) = 0$ .  
(5 marks)
- (b) Find the complete integral of  $p + q = pq$ .  
(3 marks)
14. A (a) Find the integral surface of the equation  $(p^2 + q^2)x = pz$  corresponding to the initial data curve  $C: x_0 = 0, y_0 = s^2, z_0 = 2s$ .  
(6 marks)
- (b) Write a short note on characteristic strip of non-linear first order partial differential equation  $f(x, y, z, p, q) = 0$ .  
(2 marks)
- B (a) Derive d'Alembert's solution which describes the vibrations of an infinite string.  
(4 marks)
- (b) Using d'Alembert's solution obtain the solution which describes the vibrations of a semi-infinite string.  
(4 marks)

15. (A) (a) State Dirichlet problem and show that the solution to the Dirichlet problem is stable.

(3 marks)

(b) Solve the Dirichlet problem for a rectangle.

(B) State the heat conduction problem for an infinite rod and solve it.

(2 + 6 = 8 marks)

16 (A) (a) Transform the problem :

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\lambda x^2 - 1) y = 0; y(0) = 0, y(1) = 0$$

$$\text{to the relation } y(x) = \lambda \int_0^1 G(x, \xi) \xi y(\xi) d\xi$$

$$\text{where } G(x, \xi) = \begin{cases} \frac{x}{2\xi} (1 - \xi^2) & \text{when } x < \xi \\ \frac{\xi}{2x} (1 - x^2) & \text{when } x > \xi. \end{cases}$$

(5 marks)

(b) Obtain an approximate solution of the integral equation  $y(x) = \int_0^1 \sin(x\xi) y(\xi) d\xi + x^2$ ,

by replacing  $\sin(x\xi)$  by the first two terms of its power series  $\sin(x\xi) = x\xi - \frac{(x\xi)^3}{3!} + \dots$

(3 marks)

(B) (a) Obtain the most general solution of the equation  $y(x) = \lambda \int_0^{2\pi} \sin(x+\xi) y(\xi) d\xi + x$ ,

where  $\lambda \neq \pm 1/\pi$ .

(4 marks)

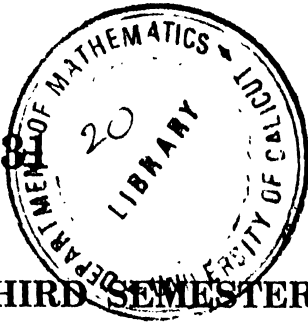
(b) Solve the integral equation by iterative method :

$$y(x) = \lambda \int_0^1 (x + \xi) y(\xi) d\xi + 1.$$

(4 marks)

[4 × 8 = 32 marks]

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Name.....

Reg. No.....

**THIRD SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2013**

(CCSS)

Mathematics

**MAT 3C 11—FUNCTIONAL ANALYSIS**

Time : Three Hours

Maximum : 80 Marks

**Part A**

*Answer all questions.*

*Each question carries 4 marks.*

1. Show that every Cauchy sequence  $(x_n)$  in a metric space is bounded. Is the converse true ? Justify your answer.
2. Show that the metric space  $L^\infty([a, b])$  is not separable.
3. Let  $\| \cdot \|_1, \dots, \| \cdot \|_m$  be norms on a linear space  $X$ . Let  $r_1, \dots, r_m$  be positive real numbers for  $x \in X$ , let  $\|x\| = \max \{r_1 \|x\|_1, \dots, r_m \|x\|_m\}$ . Show that  $\| \cdot \|$  is a norm on  $X$ , and  $\|x_n - x\| \rightarrow 0$  iff  $\|x_n - x\|_j \rightarrow 0$  for each  $j = 1, 2, \dots, m$ .
4. Let  $Y$  be a subspace of a normed space  $X$ . Show that  $Y^\circ \neq \phi$  iff  $Y = X$ .
5. Show that every linear map on a finite dimensional normed linear space is continuous.
6. Show that the linear functional  $f$  on  $l^\infty$  given by  $f(x) = \sum_{j=1}^{\infty} \frac{x(j)}{j^2}$  is continuous and  $\|f\| = \frac{\pi^2}{6}$ .
7. Let  $X = k^2$  with the norm  $\| \cdot \|_1$ . Consider  $Y = \{(x(1), x(2)) \in X : x(2) = 0\}$  and define  $g \in Y'$  by  $g(x(1), x(2)) = x(1)$ . Determine all the Hahn Banach extensions of  $g$  to  $X$ .
8. Prove that the dual space  $X'$  of every normed space  $X$  is a Banach space.
9. Give an example of a normed linear space which is not a Banach space. Determine the Banach space in which the given normed space can be embedded as a dense subspace.
10. Let  $X$  and  $Y$  be normed spaces and  $F : X \rightarrow Y$  be linear. Show that  $F$  is continuous iff  $g \circ F$  is continuous for every  $g \in Y'$ .

Turn over

11. Let  $X$  be a normed space and  $f : X \rightarrow K$  be linear. Show that  $f$  is closed iff  $f$  is continuous.
12. Let  $X$  and  $Y$  be Banach spaces and  $F : X \rightarrow Y$  be a linear map which is closed and surjective. Show that  $F$  is continuous and open.

(12 × 4 = 48 marks)

**Part B**

*Answer either part (a) or part (b) of each question.  
Each question carries 8 marks.*

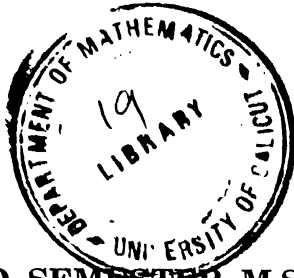
13. (a) (i) Show that the intersection of a countable number of dense open subsets of a metric space  $X$  is dense in  $X$ .
- (ii) Show that every complete and totally bounded metric space  $X$  is compact.
- (b) (i) Show that for  $1 \leq p \leq \alpha$ , the metric space  $L^p(E)$  is complete.
- (ii) Define  $m^{\text{th}}$  Fejer kernel  $k_m(t)$  and evaluate  $\int_{-\pi}^{\pi} k_m(t) dt$ .
14. (a) (i) Show that every finite dimensional subspace of a normed linear space  $X$  is complete.
- (ii) Show that the closed unit ball in  $l^2$  is convex, closed and bounded, but not compact.
- (b) (i) Let  $X$  and  $Y$  be normed spaces. For  $F \in BL(X, Y)$ , define  $\|F\| = \sup \{\|F(x)\| : x \in X, \|x\| \leq 1\}$ .
- Show that  $\|F\| = \inf \{\alpha \geq 0 : \|F(x)\| \leq \alpha \|x\| \text{ for all } x \in X\}$ .
- (ii) Show that a linear map from a normed linear space into a normed linear space need not be continuous always.
15. (a) (i) Let  $X$  be a normed space over  $K$  and  $a \neq 0 \in X$ . Show that there is some  $f \in X'$  such that  $f(a) = \|a\|$  and  $\|f\| = 1$ .
- (ii) Let  $X$  be a normed space. Show that for every subspace  $Y$  of  $X$  and every  $g \in Y'$ , there is a unique Hahn-Banach extension of  $g$  to  $X$  iff  $X'$  is strictly convex.
- (b) (i) Show that the linear space  $C_{\infty}$  cannot be a Banach space in any norm.
- (ii) Let  $Z$  be a dense subspace of a normed space  $X$ . Show that if  $Y$  is a Banach space, then every  $F \in BL(Z, Y)$  has unique norm preserving extension to  $X$ .

16. (a) (i) State and prove uniform boundedness principle.
- (ii) Let  $X$  be a normed space and  $(x_n)$  be a sequence in  $X$  such that  $(f(x_n))$  converges in  $\mathbb{K}$  for every  $f \in X'$ . Show that the sequence  $(x_n)$  is bounded.
- (b) (i) Let  $X$  be a normed space and  $P : X \rightarrow X$  be a projection. Show that  $P$  is a closed map iff the subspaces  $R(P)$  and  $Z(P)$  are closed in  $X$ .
- (ii) Let  $Y$  be a finite dimensional subspace of a Banach space  $X$ . Show that there is a continuous projection  $P$  defined on  $X$  such that  $R(P) = Y$ .

(4 × 8 = 32 marks)



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Name.....

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THIRD SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2013

(CCSS)

Mathematics

MAT 3C 10—COMPLEX ANALYSIS

Time : Three Hours

Maximum : 80 Marks

Part A

Answer all the questions.  
Each question carries 4 marks.

1. Let  $S$  be the Riemann sphere. Let  $Z$  and  $Z'$  be the points of  $S$  corresponding to the points  $z$  and  $z'$  respectively of  $\mathbb{C}$ . If  $W$  is the point on  $S$  corresponding to  $z+z'$ , find the coordinates of  $W$  in terms of  $Z$  and  $Z+Z'$ .
2. Let  $f$  be a function analytic on a region  $G$ . If the derivative of  $f$  vanishes identically on  $G$ , prove that  $f$  is constant on  $G$ .
3. Show that a Möbius transformation has  $\infty$  as its only fixed point if and only if it is a translation.
4. Let  $r(t) = 1 + e^{it}$  for  $0 \leq t \leq 2\pi$ .

Find  $\int_r (z^2 - 1)^{-1} dz$ .

5. Show that  $\int_0^{2\pi} \frac{e^{is}}{e^{is} - z} dz = 2\pi$ , if  $|z| < 1$ .
6. Let  $f$  be an entire function. Suppose there exists a constant  $M$ , an  $R > 0$  and an integer  $n \geq 1$  such that  $|f(z)| \leq M|z|^n$  for  $|z| > R$ . Show that  $f$  is a polynomial of degree  $\leq n$ .
7. Let  $G$  be an open, star shaped set in  $\mathbb{C}$ . If  $\gamma$  is any closed rectifiable curve  $G$ , prove that  $\gamma \sim 0$ .
8. Let  $f$  be a one-one, analytic function on a region  $G$ . Show that  $f'(z) \neq 0$  for any  $z$  in  $G$ .
9. If  $f$  has an isolated singularity at  $z = a$ , then show that this is a removable singularity of  $f$  if and only if

$$\lim_{z \rightarrow a} (z - a) f(z) = 0.$$

Turn over

10. If  $f$  has a pole of order  $m$  at  $z = a$  and if  $g(z) = (z-a)^m f(z)$ , show that  $\text{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a)$ .
11. Suppose  $f$  is analytic in  $B(0; 1)$  and satisfies  $|f(z)| < 1$  for  $|z| = 1$ . Find the number of solutions (counting multiplicity) of the equation  $f(z) = z^n$ , where  $n$  is an integer,  $n \geq 1$ .
12. Let  $f$  be a non constant analytic function in the disc  $B(0; R)$ . For  $0 \leq r < R$ , define  $A(r) := \max\{\text{Re } f(z) \mid |z| = r\}$ . Prove that  $A(r)$  is a strictly increasing function of  $r$ .  
(12  $\times$  4 = 48 marks)

### Part B

Answer either Part A or Part B in each question.

Each question carries 8 marks.

13. (A) (a) State and prove the chain rule for analytic functions. (3 marks)
- (b) If  $f$  is a branch of  $\log z$  in a region  $G$ , show that the totality of all branches of  $\log z$  in  $G$  are the functions  

$$f(z) + 2\pi k i, k \in \mathbb{Z}.$$
(2 marks)
- (c) If  $z_1, z_2, z_3, z_4$  are four distinct points in  $\mathbb{C}_\infty$ , prove that  $(z_1, z_2, z_3, z_4)$  is a real number if and only if all the four points be on a circle. (3 marks)
- (B) (a) Find the radius of convergence of the power series  $\sum_{n=0}^{\infty} \left( \frac{4n^2}{2n+1} - \frac{6n^2}{3n+4} \right) z^n$ . (2 marks)
- (b) Let  $G$  be a region and  $u, v; G \rightarrow \mathbb{R}$  be functions having continuous partial derivatives in  $G$ . Let  $f$  be defined on  $G$  by  $f = u + iv$ . Show that  $f$  is analytic on  $G$  if and only if  $u$  and  $v$  satisfy the Cauchy-Riemann equations in  $G$ . (4 marks)
- (c) If  $z_2, z_3, z_4$  are distinct points in  $\mathbb{C}$  and if  $T$  is any Möbius transformation, show that for any  $z_1$  in  $\mathbb{C}$   $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ . (2 marks)
14. (A) (a) Suppose  $\gamma$  is a rectifiable curve in  $\mathbb{C}$  and  $f$  is a continuous function on  $\{r\}$ . Prove that  

$$\left| \int_r f(z) dz \right| \leq \int_r |f(z)| |dz|.$$
(2 marks)
- (b) Show that a bounded entire function is a constant. (2 marks)
- (c) State and prove the first version of Cauchy's Integral Formula. (4 marks)

(B) (a) Let  $f$  be analytic in the disc  $B(a; R)$ . If  $\gamma$  is any closed rectifiable curve in  $B(a; R)$ , prove that  $\int_{\gamma} f = 0$ . (2 marks)

(b) Let  $f$  be a closed rectifiable curve in  $\mathbb{C}$ . Show that  $n(r, z)$  is a constant  $m$  each component of  $\mathbb{C} - \{r\}$ . (3 marks)

(c) State and prove Morera's Theorem. (3 marks)

15. (A) (a) State and prove the Goursat's Theorem. (4 marks)

(b) State and prove the theorem on Laurent Series Development in an annulus. (4 marks)

(B) (a) State and prove the second version of Cauchy's Theorem. (4 marks)

(b) State and prove the Casorati-Weierstrass Theorem. (4 marks)

16. (A) (a) Show that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = \frac{\pi}{\sqrt{2}}.$$

(3 marks)

(b) State and prove the Argument Principle. (3 marks)

(c) Is the Maximum Modulus Theorem true for unbounded regions. Justify. (2 marks)

(B) (a) State and prove the Residue Theorem. (3 marks)

(b) State Rouché's Theorem and deduce the fundamental theorem of algebra from it. (3 marks)

(c) State and prove the Minimum Modulus Principle for analytic functions. (2 marks)

[4 × 8 = 32 marks]

