A STUDY ON COUPON COLORING AND LOCALLY IDENTIFYING COLORING OF GRAPHS

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Notations

$V(G)$	: vertex set of a graph G
$E(G)$	: edge set of a graph G
$d_G(u, v)$	: distance between the vertices u and v in G
$N(u)$	: open neighborhood of the vertex u
$N[u]$	: closed neighborhood of the vertex u
$d(u)$	: degree of the vertex u
$diam(G)$	: diameter of a graph G
$\delta(G)$	: minimum degree of a graph G
$\Delta(G)$	: maximum degree of a graph G
$\gamma(G)$	: domination number
$\gamma_t(G)$	: total domination number
TDS	: total dominating set
$c(u)$	: color of the vertex u
$\chi(G)$	: chromatic number of G
$\chi_c(G)$	: coupon coloring number of G
$\chi_{lid}(G)$	: lid-chromatic number of G
$G \square H$	: Cartesian product of graphs G and H
$G \times H$	: direct product of graphs G and H
$G[H]$	: lexicographic product of graphs G and H
$G \boxtimes H$	: strong product of graphs G and H
$G \circ_v H$	: rooted product of graphs G and H with root vertex v
$G \odot H$	: corona product of graphs G and H
$A \times B$	: direct product of sets A and B
$H - u$	: subgraph obtained by the removal of the vertex u from G

P_n	: path with n vertices
C_n	: cycle with n vertices
K_n	: complete graph with n vertices
$K_{n_1,n_2,\ldots,n_k}$	: complete k-partite graph $n_1 n_2 \cdots n_k$ vertices
$\lfloor k \rfloor$	: floor function
$\lceil k \rceil$	: ceiling function
$[k]$	: set of integers $\{1, 2, \ldots, k\}$
$\|A\|$	: number of elements in the set A
$c\|_B$	: restriction of a function c on the set A to $B \subseteq A$
R	: commutative ring with identity
$\mathbb{F}$	: field
R^+	: additive group of the ring R
$\mathbb{Z}_\mathrm{n}$	: integers modulo n
$Z(R)$	: zero-divisors of R
$Z^*(R)$	: non-zero zero-divisors of R
$U(R)$	: units of R
$Ann(A)$	: annihilator ideal of subset A of R
(R, M)	: local ring with maximal ideal M
$\mathbb{C}ay(\Gamma, C)$	: Cayley graph of Γ with connection set C
$\mathbb{C}ay(\mathbb{Z}_n, C)$	: circulant graph of order n
$\mathbb{CAY}(R)$	: Cayley graph of R
$\overline{Cay}(H, S)$	: undirected Cayley graph of a semigroup H
Γ_R^n	: generalized Cayley graph of R
$\Gamma(R)$	: zero-divisor graph of R
$\Gamma_I(R)$	: ideal-based zero-divisor graph of R
A^T	: transpose of the matrix A

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Abstract

A study on coupon coloring and locally identifying coloring of graphs

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Graph theory is a branch of mathematics which deals with many real-life problems. The field of graph coloring is one of the most important branches of graph theory having various applications. This thesis deals with two vertex coloring problems, coupon coloring and locally identifying coloring. A k-coupon coloring of a graph G is a vertex coloring $c : V(G) \to [k] = \{1, 2, \ldots, k\}$ in which every vertex must be adjacent to vertices of all the k colors. Maximum k for which such a k-coupon coloring exists is called the coupon coloring number. It is an improper vertex coloring in which adjacent vertices may have the same color. Coupon coloring number is also referred as the total domatic number, which is the maximum number of disjoint total dominating sets.

To begin with, we have investigated the coupon coloring of rooted product of two graphs and some sharp bounds for coupon coloring number were obtained. In addition, the coupon coloring of lexicographic product of graphs have been studied. If graph G has a Hamilton path, then a sharp lower bound were obtained for the coupon coloring number lexicographic product of G and any graph H. Furthermore, an investigation into the coupon coloring of some Cayley graphs and zero-divisor graphs were conducted.

Unlike coupon coloring, the second vertex coloring, locally identifying coloring is a proper coloring. A vertex coloring $c : V(G) \to \mathbb{N}$ of a graph G is said to be locally identifying if c is a proper coloring and the set of colors in the closed neighborhood of any two adjacent vertices with distinct closed neighborhoods must be distinct. The smallest integer k for which G admits a lid-coloring with k colors is called the lid-chromatic number of G. Finding the exact lid-chromatic number is a complicated task, when the number of colors is more than 4. Because the set of colors in the neighborhoods of adjacent vertices can have many different choices.

Locally identifying coloring (lid-coloring) of rooted product and corona product of two graphs have been studied. Some bounds for the lid-chromatic number of rooted product of graphs were obtained. In addition, sharp bounds for the lid-chromatic number of corona product of graphs have been found. Furthermore, the lid-coloring of strong product of two graphs have been studied. It is proved that if G and H are 3-lid-colorable bipartite graphs, then lid-chromatic number of strong product of G and H is exactly 6.

Keywords: Vertex coloring; Coupon coloring; Locally identifying coloring; Graph products; Cayley graphs; Zero-divisor graphs.

Chapter 1

Introduction

Graph theory is a prominent branch of Mathematics with various applications. In particular, vertex coloring problems imparts a special role in scheduling problems, frequency assignment etc. There are two types of vertex colorings according to the rule regarding the colors of adjacent vertices. If adjacent vertices cannot have the same colors, then it is a proper coloring and if a pair of adjacent vertices can have same color, then the coloring is said to be improper. The coupon coloring is an improper coloring and the locally identifying coloring is a proper vertex coloring.

1.1 Motivation

Graph coloring is an interesting topic in discrete mathematics and it is emerged from the celebrated Four Color Problem. The first known reference to the Four Color Problem can be found in 1852, when Augustus De Morgan wrote a letter to Sir William Rowan Hamilton about a question asked by one his students, Frederick Guthrie, that whether every map can be colored with only four colors in such a way that countries sharing a common border receive different colors. Actually, the question was posed by Frederick Guthrie's brother Francis Guthrie who observed this problem while coloring the counties of a map of England. De Morgan communicated this problem to several other mathematicians and it became a part of mathematical folklore.

The first printed reference to the Four Color Problem was in a book review by De Morgan in a scientific and literacy journal, *Athenaeum* (No. 1694, 14 April 1860, pp. 501–503)[67]. Even though some mathematicians and logicians tried to solve the problem, there was no progress in solving the Four Color Problem until after the death of De Morgan in 1871. On 13 June 1978 Arthur Cayley inquired whether the problem had been solved at a meeting of London Mathematical Society (cf. [68]) and he explained where the difficulties lie in his short paper [24] for Royal Geographical Society. In 1879 Alfred Bray Kempe published a fallacious proof for the Four Color Problem in American Journal of Mathematics [52]. For a decade his proof believed to be true and two papers with simplifications were published. But Percy Heawood found out Kempe's error and published in 1890 [44]. Finally, in 1976, K. Appel, W. Haken and J. Koch [10, 11] published a time consuming algorithm to solve the Four Color Problem.

The Four Color Problem is one of the examples in Mathematics that is very easy to state and visualize, but very hard to solve. More than one hundred years were taken to solve the Four Color Problem and it was solved with the collective efforts of some of the brilliant mathematicians. Even though many mathematicians failed to solve the problem, their attempts contributed to the growth of the field of graph coloring.

Graph coloring is literally assigning labels (called colors) to vertices or edges or faces under some constraints. Vertex coloring is important, because an edge coloring or a face coloring can be transformed into a vertex coloring. There are enormous graph coloring problems in literature. Some books that deals with deals with graph colorings are [26, 49, 66].

Vertex coloring is an extensively studied area in graph theory. Most of the vertex coloring problems that we have found in literature are proper vertex coloring, where adjacent vertices have distinct colors. Several interesting improper vertex coloring problems are also introduced by different authors and coupon coloring is one among them. In an improper vertex coloring, same colors can be assigned to adjacent vertices. Coupon coloring is closely related to the concept of total domination. Even though total domination is a renowned notion in graph theory, it requires much effort to find the coupon coloring number of graphs. So far, coupon coloring number is known only for very limited families of graphs. Unlike coupon coloring the second vertex coloring, locally identifying coloring is a proper coloring. So the adjacent vertices must have different colors. There are only a few articles published on locally identifying coloring. Finding the exact lid-chromatic number is a complicated task, when the number of colors is more than 4. Because the set of colors in the neighborhoods of adjacent vertices can have many different choices. If the clique number is of considerable size, then the task is relatively simple.

1.2 Organization of the thesis

This thesis includes seven chapters. **Chapter 1** is the introductory chapter which deals with motivation for the study and in **Chapter 2**, basic definitions and terminologies in graph theory and abstract algebra have been included.

In **chapter 3**, the coupon coloring of rooted product of graphs and the lexicographic product of graphs were studied. In second section, exact coupon coloring number of the rooted product $G \circ_v H$ is determined, when G is any graph and H is either a cycle or a complete graph. Also, some sharp bounds for the coupon coloring number of rooted product graphs were found. If G and H are two graphs without isolated vertices and if $\delta(G) = 1$, then the coupon coloring number of the rooted product $G \circ_v H$ is either k or $k+1$, where k is the coupon coloring number of H. Third section discusses the coupon coloring of lexicographic product of two graphs G and H. If G has a Hamilton path, it is proved that $|V(H)|+1$ is a lower bound for $\chi_c(G[H])$ and showed that for some graphs $\chi_c(G[H]) = |V(H)| + 1$. A sharp bound for the coupon coloring number of lexicographic product of connected graphs has been obtained.

Chapter 4 focuses on the coupon coloring of Cayley graphs, generalized Cayley graphs, zero-divisor graphs and ideal-based zero-divisor graphs. In the second section, coupon coloring of some Cayley graphs have been studied. To begin with, the coupon coloring of circulant graphs have been found. In addition, the coupon coloring of $\mathbb{CAY}(R)$ has been studied. Moreover, the exact coupon coloring number of $\mathbb{CAY}(R)$ were found, when R is a finite commutative ring with identity. Furthermore, some sharp bounds for coupon coloring number of the generalized Cayley graph Γ_R^n have been obtained. In third section, coupon coloring of $\Gamma(R)$ and $\Gamma_I(R)$ have been discussed. A sharp bound for the coupon coloring number of $\Gamma(R)$ were found. It is observed that the total domination number of $\Gamma_I(R)$ is same as that of $\Gamma(R)$. As a consequence, the exact coupon coloring number of $\Gamma_I(R)$ in terms of the coupon coloring number of zero-divisor graph $\Gamma(R)$ has been found.

In **Chapter 5**, the locally identifying coloring (lid-coloring) of rooted product and corona product of two graphs have been studied. Some sharp bounds for the lid-chromatic number of $G \circ_v H$ were obtained. The exact lid-chromatic number of rooted product of paths and cycles have been found. In third section, some sharp bounds for the lid-chromatic number of corona product of graphs have been found. It is found that for any two connected graphs

$\chi_{lid}(G \odot H) \leq \chi_{lid}(G) + \chi_{lid}(H)$ and $\chi_{lid}(G \odot H) \leq 2\chi(G) + \chi_{lid}(H) - 1$. In addition, if $\log_2(\chi(G)) \leq \chi_{lid}(H)$ and H has a good lid-coloring, then $\chi_{lid}(G \odot H) \leq \chi(G) + \chi_{lid}(H)$. It is also proved that the two conditions in this statement cannot be omitted. When G is bipartite, the exact lid-chromatic number of $G \odot H$ depends only on the lid-chromatic number of H. That is, if G is a bipartite graph, then $\chi_{lid}(G \odot H)$ is either $\chi_{lid}(H) + 2$ or $\chi_{lid}(H) + 3$. Furthermore, the exact lid-chromatic number of $P_m \odot P_n$, $P_m \odot C_n$, $C_m \odot P_n$ and $C_m \odot C_n$ have been found.

In **Chapter 6**, the lid-coloring of strong product of two graphs have been studied. It is found that for any two connected graphs $\chi_{lid}(G \boxtimes H) \leq \chi_{lid}(G)\chi_{lid}(H)$. In addition, it is proved that $\chi_{lid}(G \boxtimes H)$ cannot be 4 if G and H are not K_2. Furthermore, if G and H are 3-lid-colorable bipartite graphs, then $\chi_{lid}(G \boxtimes H)$ is exactly 6. In the last section, the lid-chromatic number of $P_m \boxtimes P_n$ and $P_m \boxtimes C_n$ were studied. In strong product of graphs, lid-chromatic number is greater than 4. So, it is difficult to compute the exact lid-chromatic number of strong product of two graphs with small clique number.

In conclusion, **Chapter 7** gives some directions for future work. A list of presented and published papers, a bibliography and an index are also provided at the end of the thesis.

Chapter 2

Preliminaries

2.1 Basic concepts in graph theory

In this section, basic definitions and terminologies in graph theory have been included. For the definitions in graph theory, refer to [19, 30, 66].

Definition 2.1.1. *[19] A graph G is an ordered triple $G = (V(G), E(G), \psi_G)$ of a nonempty set $V(G)$ of vertices, the set $E(G)$, disjoint from $V(G)$, of edges and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.*

If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to join u and v; the vertices u and v are called the ends of e. Sometimes we denote the edge joining vertices u and v as uv.

Definition 2.1.2. *[30] The number of vertices of a graph G is the order of G and the number of edges of G is the size of G.*

A graph of order 1 is called a *trivial graph*. A graph of size 0 is called an *empty graph*.

Definition 2.1.3. *[19] A graph G is finite, if both its vertex set and the edge set are finite.*

An edge with identical ends is called a *loop*. If any two vertices of a graph are joined by more than one edge, then such edges are the *multiple edges*. A graph G is *simple*, if G has no loops and no multiple edges.

Definition 2.1.4. *[19] The degree $d_G(v)$ or $d(v)$ of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges.*

The *minimum degree* of a graph G denoted by $\delta(G)$ is defined to be $\min\{d_G(x) : x \in V(G)\}$ and the *maximum degree* $\Delta(G)$ is the $\max\{d_G(x) : x \in V(G)\}$. An *isolated vertex* is a vertex with degree zero. The vertex with degree one is called a *pendant vertex* or a *leaf* and the edge incident on a pendant vertex is known as the *pendant edge*.

Definition 2.1.5. *[19] A graph G is k-regular if $d(v) = k$ for all $v \in V$; a regular graph is one that is k-regular for some k.*

For a vertex v in graph G, the open neighborhood or simply the neighborhood $N(v)$ or $N_G(v)$ is the set of vertices in G that are adjacent to v. The closed neighborhood $N[v]$ or $N_G[v]$ of the vertex v is $N(v) \cup \{v\}$.

Definition 2.1.6. *[19] A graph H is a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$.*

Definition 2.1.7. *[19] Let $G = (V, E)$ be a graph. Suppose that V' is a nonempty subset of V. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G induced by V'.*

The induced subgraph obtained from G by deleting the vertices in V' together with their incident edges is denoted by $G - V'$. If $V' = \{v\}$, we write $G - v$ for $G - \{v\}$.

Definition 2.1.8. *[19] Two graphs G and H are isomorphic (written $G \cong H$) if there is a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$; such a mapping ϕ is an isomorphism between G and H.*

Definition 2.1.9. *[19] A walk in G is a finite nonempty sequence $W = v_0e_1v_1e_2 \ldots e_kv_k$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. W is a walk from v_0 to v_k, or a (v_0, v_k)-walk. The integer k is the length of W. A walk W is called a path if the vertices $v_0, v_1, \ldots, v_k$ of a walk W are distinct.*

Definition 2.1.10. *[19] Two vertices u and v of G are said to be connected if there is a $(u, v) - path$ in G. A graph G is connected if any two vertices in G is connected; otherwise G is disconnected.*

Definition 2.1.11. *[19] If vertices u and v are connected in G, then the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G; if there is no path connecting u and v we define $d_G(u, v)$ to be infinite.*

The diameter of G is the maximum distance between two vertices of G and it is denoted by $diam(G)$.

Definition 2.1.12. *[30] A path $P_n = (V, E)$ is a graph with $V = \{v_0, v_1, \dots, v_n\}$ and $E = \{v_0v_1, v_1v_2, \dots, v_{n-1}v_n\}$, where v_i's are all distinct.*

Definition 2.1.13. *[30] A cycle $C_n = (V, E)$ is a graph with $V = \{v_0, v_1, \dots, v_n\}$ and $E = \{v_0v_1, v_1v_2, \dots, v_{n-1}v_n, v_nv_0\}$, where v_i's are all distinct.*

Definition 2.1.14. *[19] A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge. A complete graph on n vertices is denoted by K_n. An empty graph is one with no edges.*

Definition 2.1.15. *[19] A bipartite graph is a graph whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y.*

A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y. If $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.

Definition 2.1.16. *A k-partite graph is one whose vertex set can be partitioned into k subsets so that no edge has both ends in anyone subset; a complete k-partite graph is one that is simple and in which each vertex is joined to every vertex that is not in the same subset.*

If the k subsets of the vertex set of a complete k-partite graph are $X_1, X_2, \dots, X_k$ and $|X_1| = n_1, |X_2| = n_2, \dots, |X_k| = n_k$, then the graph is denoted as $K_{n_1,n_2,\dots,n_k}$.

Definition 2.1.17. *[19] A path that contains every vertex of G is called a Hamilton path of G; similarly, a Hamilton cycle of G is a cycle that contains every vertex of G. A graph is hamiltonian if it contains a Hamilton cycle.*

Definition 2.1.18. *[43] A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.*

Definition 2.1.19. *[43] Let $G = (V, E)$ be a graph without isolated vertices. $T \subseteq V$ is a total dominating set if for every vertex $v \in V$ there is a vertex $u \in T$, $u \neq v$ such that u is adjacent to v.*

Thus, S is a dominating set if every vertex in $V \setminus S$ is adjacent to at least one vertex in S and $T \subseteq V$ is a total dominating set (TDS) if every vertex of G is adjacent to at least one vertex in T. The minimum cardinality among all the dominating sets in G is called the *domination number* denoted by $\gamma(G)$ and the minimum cardinality among all the total dominating sets in G is called the *total domination number* denoted as $\gamma_t(G)$.

Definition 2.1.20. *a directed graph D is an ordered triple $(V(D), A(D), \psi_D)$ consisting of a nonempty set $V(D)$ of vertices, a set $A(D)$, disjoint from $V(D)$, of arcs, and an incidence function ψ_D that associates with each arc of D an ordered pair of (not necessarily distinct) vertices of D.*

If a is an arc and u and v are vertices such that $\psi_D(a) = (u, v)$, then a is said to join u to v; u is the tail of a, and v is its head.

All graphs in this thesis are finite, simple and undirected unless otherwise specified.

2.1.1 Graph products

Products of graphs plays an important role in literature. It gives new classes of graphs from known graphs. Some of the common graph products are Cartesian product, direct product, strong product, lexicographic product, corona product, rooted product etc. Cartesian product, direct product, strong product and lexicographic product are well studied in [42]. The most fundamental graph product is the Cartesian product.

Definition 2.1.21. *[42] The Cartesian product of two graphs G and H, denoted $G \Box H$, is a graph whose vertex set is $V(G) \times V(H) = \{(x, y) : x \in V(G)$ and $y \in V(H)\}$ and two vertices (x_1, y_1) and (x_2, y_2) of $G \Box H$ are adjacent if and only if either $x_1 = x_2$ and $y_1 y_2 \in E(H)$ or $y_1 = y_2$ and $x_1 x_2 \in E(G)$.*

Definition 2.1.22. *[42] The strong product $G \boxtimes H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H) = \{(x, y) : x \in V(G)$ and $y \in V(H)\}$, and two vertices (x_1, y_1), (x_2, y_2) are adjacent if*

$x_1 = x_2$ and $y_1y_2 \in E(H)$, or

$y_1 = y_2$ and $x_1x_2 \in E(G)$, or

$x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$.

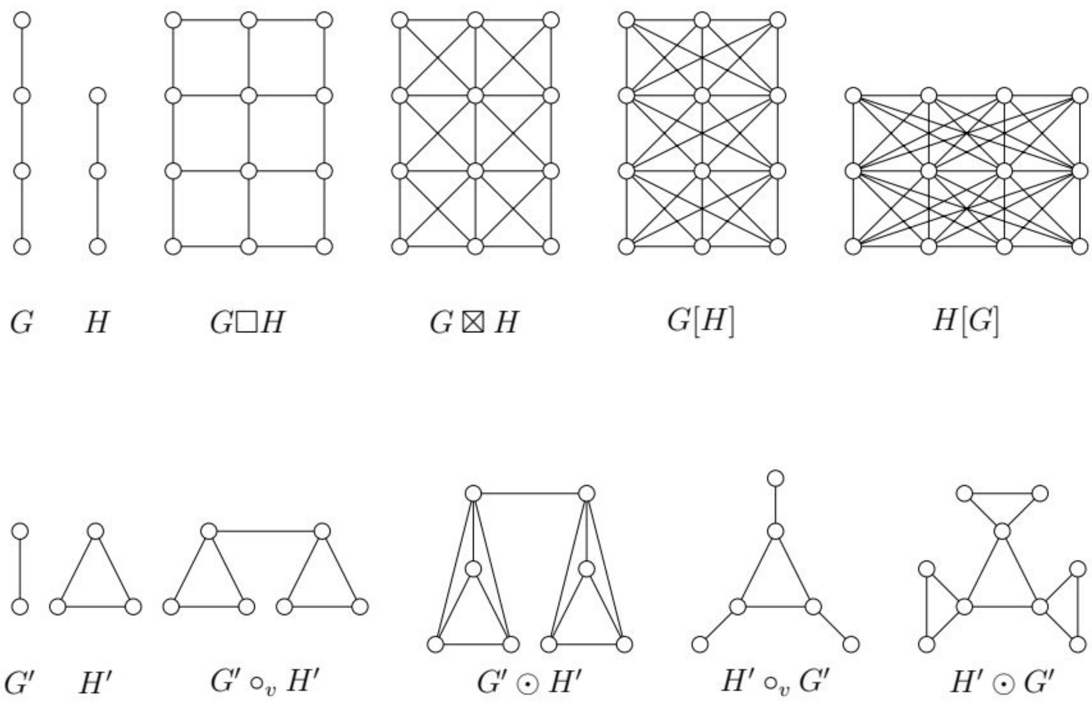


Figure 2.1: Different products of two graphs

Definition 2.1.23. *[42] The lexicographic product of two graphs G and H, denoted $G[H]$, is a graph whose vertex set is $V(G) \times V(H) = \{(x, y) : x \in V(G) \text{ and } y \in V(H)\}$ and two vertices (x_1, y_1) and (x_2, y_2) of $G[H]$ are adjacent if and only if either $x_1x_2 \in E(G)$ or $x_1 = x_2$ and $y_1y_2 \in E(H)$.*

The rooted product of two graphs was introduced by Godsil and McKay in 1978.

Definition 2.1.24. *[39] The rooted product $G \circ_v H$ of two graphs G and H is defined as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and identifying the i^{th} vertex of G with the root vertex v in the i^{th} copy of H for every $i = 1, 2, \ldots, |V(G)|$.*

Definition 2.1.25. *[36] The corona product of two graphs G and H denoted by $G \odot H$, is obtained by taking one copy of the graph G and $|V(G)|$ copies of the graph H and joining the j^{th} vertex of G to every vertex in the j^{th} copy of H with an edge.*

Note that, lexicographic product, rooted product and the corona product are non-commutative graph products. See Figure 2.1. Studying vertex coloring problems in a graph product is an interesting problem in discrete mathematics. This thesis focuses on coupon coloring and locally identifying coloring on these graph products.

2.2 Basic algebraic concepts

For basic definitions and terminology refer to [37, 46, 48].

Let G be a group and H be a subgroup of G. The relation $a \sim b$ if $ab^{-1} \in H$ $(a, b \in G)$, is an equivalence relation on G. Then the equivalence class of an element is called the *coset* of H in G. Note that, $a \sim b$ if $ab^{-1} = h$ for some $h \in H$. That is $a \sim b$ implies $a = hb$. On the other hand, if $a = kb$ where $k \in H$, then $ab^{-1} = (kb)b^{-1} = k \in H$, so $a \sim b$ if and only if $a \in Hb = \{hb \mid h \in H\}$, which is the equivalence class of b. The set Hb is called a right coset of H in G. The set $bH = \{bh \mid h \in H\}$ is called the left coset of H in G. If G is an abelian group(commutative group), then $Hb = bH$. The set of all left(right) cosets of H in G forms a group under the operation $(aH)(bH) = (ab)H$. This group is called the *quotient group* of G by H and it is denoted by G/H.

Definition 2.2.1. *[46] Let R be a ring. A nonempty subset I of R is called an ideal of R if:*

1. *I is an additive subgroup of R.*
2. *Given $r \in R$, $a \in I$, then $ra \in I$ and $ar \in I$*

Let R be a commutative ring and K be an ideal of R. Since K is an additive subgroup of R, the *quotient group* R/K exists. Moreover, R/K is a ring of all cosets $a + K$ as a runs over R under the multiplication operation $(a+K)(b+K) = ab+K$. Then R/K is called the *quotient ring* of R by K.

Definition 2.2.2. *[46] A proper ideal M of R is a maximal ideal of R if the only ideals of R that contain M are M itself and R.*

Definition 2.2.3. *[37] Let R be a commutative ring and let A be any subset of R. The annihilator ideal of A is defined as*

$$Ann(A) = \{r \in R \mid ra = 0 \textit{ for all } a \in A\}.$$

Definition 2.2.4. *[48] A local ring is a commutative ring with identity which has a unique maximal ideal.*

Theorem 2.2.1. *[18] A finite commutative ring with identity can be expressed as a direct sum of local rings. This decomposition is unique up to permutation of direct summands.*

In Chapter 4, we denote the local ring decomposition of the finite commutative ring R with identity as $R_1 \times R_2 \times \cdots \times R_n$.

Definition 2.2.5. *[46] An element $a \neq 0$ in a ring R is a zero-divisor in R if $ab = 0$ for some $b \neq 0$ in R.*

The set of all zero-divisors in the ring R is denoted by $Z(R)$. Nicoals Bourbaki[20] defined the zero-divisor of a ring in a different way.

Definition 2.2.6. *[20] An element a in a commutative ring R is a zero-divisor if there exists $b \neq 0$ in R such that $ab = 0$.*

By the definition 2.2.5, 0 is not a zero-divisor in a non-zero ring R, but 0 can be a zero-divisor by definition 2.2.6. For avoiding this ambiguity, we always use the term '*non-zero zero-divisors*' in this thesis and it will be denoted by $Z^*(R)$.

Definition 2.2.7. *[18] A unit x in R is an element for which there exists an element y in R such that $xy = 1$, 1 being the multiplicative identity of R. The subset*

$$U(R) = \{x \in R \mid \exists\, y \in R \text{ s.t. } xy = yx = 1\}$$

of R is a multiplicative group (with respect to the multiplication in R) and its elements are called the units of R.

2.3 Cayley graphs and zero-divisor graphs

The Cayley graphs and zero-divisor graphs are well studied in literature. This section includes some variants of Cayley graphs and zero-divisor graphs. The basic properties of these graphs were also discussed.

2.3.1 Cayley graphs

The concept of Cayley graphs was introduced by Arthur Cayley [23] in 1878 to explain the concept of abstract groups which are described by a set of generators. Let Γ be a group and let C be a subset of Γ that is closed under taking inverses and does not contain the identity. Then the *Cayley graph*, $Cay(\Gamma, C)$, (cf. [40]) is a graph with vertex set Γ and edge set

$$E(Cay(\Gamma, C)) = \{gh : hg^{-1} \in C\}.$$

Let $\mathbb{Z}_n$ denote the additive group of integers modulo n. If C is a subset of $\mathbb{Z}_n \backslash \{0\}$, then construct a directed graph $Cay(\mathbb{Z}_n, C)$ as follows. The vertices of $Cay(\mathbb{Z}_n, C)$ are elements of $\mathbb{Z}_n$ and (i, j) is an arc of $Cay(\mathbb{Z}_n, C)$ if and only if $j - i \in C$. The graph $Cay(\mathbb{Z}_n, C)$ is called a circulant graph of order n, and C is called its connection set. If the set C is symmetric, that is $C = -C = \{-x : x \in C\}$, then X will be an undirected graph.

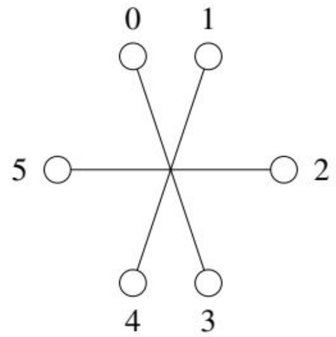


Figure 2.2: $Cay(\mathbb{Z}_6, C)$, where $C = \{3\}$

Let R be a commutative ring with identity, $Z(R)$ be the set of zero-divisors of R and $U(R)$ be the set of units of R. Then the *Cayley graph* of R with respect to its non-zero zero-divisors is the graph $Cay(R^+, Z^*(R))$ denoted by $\mathbb{CAY}(R)$, where $Z^*(R) = Z(R) \setminus \{0\}$. This is the Cayley graph whose vertices are all elements of the additive group R^+ and in which two distinct vertices x and y are joined by an edge if and only if $x - y \in Z^*(R)$.

The following result will be useful for the upcoming sections.

Theorem 2.3.1. *[1] Let R be a ring. Then the following statements hold:*

1. *$\mathbb{CAY}(R)$ has no edge if and only if R is an integral domain.*

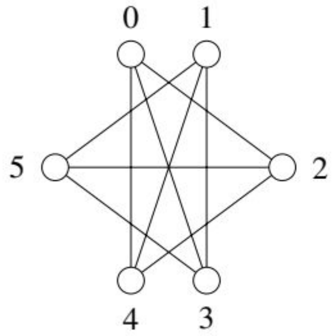


Figure 2.3: $\mathbb{CAY}(\mathbb{Z}_6)$, where $Z^*(\mathbb{Z}_6) = \{2, 3, 4\}$

2. *If (R, M) is an Artinian local ring, then $\mathbb{CAY}(R)$ is a disjoint union of $|\frac{R}{M}|$ copies of the complete graph $K_{|M|}$.*

3. *$\mathbb{CAY}(R)$ cannot be a complete graph.*

4. *$\mathbb{CAY}(R)$ is a regular graph of degree $|Z(R)| - 1$ with isomorphic components.*

Let R be a commutative ring with identity element. For a natural number n, Afkhami et al. [4] defined the *generalized Cayley graph* Γ_R^n in 2012. It is a simple graph with vertex set $R^n \setminus \{0\}$ and two distinct vertices X and Y are adjacent if and only if there is a lower triangular matrix A over R whose entries on the main diagonal are non-zero and such that $AX^T = Y^T$ or $AY^T = X^T$, where B^T is the transpose of the matrix B.

Note that, an undirected Cayley graph $\overline{Cay}(H, S)$ of a semigroup H is the graph with vertex set H and x is adjacent to y if $sx = y$ or $sy = x$ for some $s \in S$, were S is a subset of H (cf. [50,51]). For $n = 1$, generalized Cayley graph Γ_R^1 is $\overline{Cay}(R\setminus\{0\}, R\setminus\{0\})$. The main objective of this section is to introduce some basic results of generalized Cayley graphs when $n > 1$.

Let $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$, $x_1 \in U(R)$ and $y_1 \neq 0$. Then

$$\begin{pmatrix} x_1^{-1}y_1 & 0 & 0 & \cdots & 0 \\ x_1^{-1}(-x_2+y_2) & 1 & 0 & \cdots & 0 \\ x_1^{-1}(-x_3+y_3) & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ x_1^{-1}(-x_n+y_n) & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix},$$

and so, X is adjacent to Y.

Let C_i be the set of all vertices whose first non-zero components are in the i^{th} place for $i = 1, 2, \ldots, n$. If $X = (x_1, x_2, \ldots, x_n) \in C_i$ and $x_i \in U(R)$, then

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ & & \vdots & & & & \\ 0 & 0 & \cdots & x_i^{-1}y_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & x_i^{-1}(-x_{i+1}+y_{i+1}) & 1 & \cdots & 0 \\ & & \vdots & & & & \\ 0 & 0 & \cdots & x_i^{-1}(-x_n+y_n) & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_i \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ y_i \\ y_{i+1} \\ \vdots \\ y_n \end{pmatrix}.$$

These observations are summarized in the following theorems.

Theorem 2.3.2. *[4] Let R be a commutative ring with unity. If $X = (x_1, x_2, \ldots, x_n)$, $x_1 \in U(R)$ and $Y = (y_1, y_2, \ldots, y_n), y_1 \neq 0$ are two vertices of Γ_R^n, then X and Y are adjacent in Γ_R^n.*

Corollary 2.3.1. *[4] The induced subgraph of all vertices whose first components are units is a complete graph.*

Corollary 2.3.2. *[4] Assume that $X \in C_i$ such that its i^{th} component is unit. Then X is adjacent to Y for all $Y \in C_i$.*

Suppose that R is an integral domain. If $X_i \in C_i$ and $Y_j \in C_j$, then X is not adjacent to Y for all $i \neq j$. In addition, there is no path between them.

Let E_i be the n-tuple with i^{th} component is 1 and all other components are 0. Then E_i is adjacent to $X_i \in C_i$ for each i, since

$$AE_i^T = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ & & \vdots & & & & \\ 0 & 0 & \cdots & x_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & x_{i+1} & 1 & \cdots & 0 \\ & & \vdots & & & & \\ 0 & 0 & \cdots & x_n & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_i \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix} = X_i^T$$

Each subgraph induced by C_i is a refinement of a star graph with center E_i and each of these subgraphs are components of Γ_R^n.

Theorem 2.3.3. *[4] If R is an integral domain, then Γ_R^n is disconnected. Moreover, Γ_R^n has n components and every component is a refinement of a star graph.*

Corollary 2.3.3. *[4] Let $\mathbb{F}$ be a field. Then $\Gamma_{\mathbb{F}}^n$ is a union of n complete graphs.*

Suppose that R is not an integral domain. Then there exists $x, z \neq 0$ such that $xz = 0$. Let $X = (x_1, x_2, \ldots, x_n) \notin C_1$ and let $Z = (z, 1, \ldots, 1)$,

$$\begin{pmatrix} x & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix} \begin{pmatrix} z \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \text{ if } x_i \neq 0 \, \forall \, i \geq 2,$$

and

$$\begin{pmatrix} x & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ -1 & 0 & z & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix} \begin{pmatrix} z \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ 0 \\ \vdots \\ x_n \end{pmatrix}, \text{ if } x_3 = 0.$$

Thus, $Z \in C_1$ is adjacent to X.

Theorem 2.3.4. *[4]If R is not an integral domain, then Γ_R^n is connected and $diam(\Gamma_R^n) \in \{2, 3\}$.*

2.3.2 Zero-divisor graphs

Anderson and Livingston introduced the zero-divisor graph of a commutative ring in 1999.

Definition 2.3.1. *[9] For a commutative ring R, the zero-divisor graph, $\Gamma(R)$, of R is the graph whose vertices are the non-zero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $xy = 0$.*

The concept of zero-divisor graph of a commutative ring has been extensively studied by several authors. In 2003, Redmond [58] introduced a generalized concept of zero-divisor graph, called the ideal-based zero-divisor graph denoted as $\Gamma_I(R)$.

Definition 2.3.2. *[58] Let R be a commutative ring and I be a proper ideal of R. An undirected graph $\Gamma_I(R)$ with vertices $\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.*

If I is the trivial ideal $\{0\}$, then $\Gamma_I(R) = \Gamma(R)$. Also note that, if I is a prime ideal of R, then the graph $\Gamma_I(R)$ is empty. Various authors investigated the properties of ideal-based zero-divisor graphs.

In [58], Redmond studied various parameters such as connectivity, clique, diameter, girth etc. of the ideal-based zero-divisor graph. Also, there are several results which connects the values of $\Gamma_I(R)$ and $\Gamma(R/I)$. The basic properties of ideal-based zero-divisor graphs are well studied in [8].

Theorem 2.3.5. *[8, 58] Let I be an ideal of a ring R, and let $x, y \in R \setminus I$. Then:*

1. *If $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$, then x is adjacent to y in $\Gamma_I(R)$.*

2. *If x is adjacent to y in $\Gamma_I(R)$ and $x+I \neq y+I$, then $x+I$ is adjacent to $y+I$ in $\Gamma(R/I)$.*

3. *If x is adjacent to y in $\Gamma_I(R)$ and $x + I = y + I$, then $x^2, y^2 \in I$.*

Corollary 2.3.4. *[8, 58] If x and y are (distinct) adjacent vertices in $\Gamma_I(R)$, then all (distinct) elements of $x + I$ and $y + I$ are adjacent in $\Gamma_I(R)$. If $x^2 \in I$, then all the distinct elements of $x + I$ are adjacent in $\Gamma_I(R)$.*

Remark 2.3.1. *[8,58] Redmond introduced the following method to construct the graph $\Gamma_I(R)$: Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq R$ be a set of coset representatives of the vertices of $\Gamma(R/I)$. For each $i \in I$, define a graph G_i with vertices $\{a_\lambda + i : \lambda \in \Lambda\}$, where edges are defined by the relationship $a_\lambda + i$ is adjacent to $a_\delta + i$ in G_i if and only if $a_\lambda + I$ is adjacent to $a_\delta + I$ in $\Gamma(R/I)$ (i.e., $a_\lambda a_\delta \in I$). Define the graph G to have as its vertex set $V = \cup_{i\in I} G_i$. We define the edge set of G to be:*

1. *all edges contained in G_i for each $i \in I$,*

2. *for distinct $\lambda, \delta \in \Lambda$ and for any $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\delta + j$ if and only if $a_\lambda + I$ is adjacent to $a_\delta + I$ in $\Gamma(R/I)$ (i.e., $a_\lambda a_\delta \in I$),*

3. *for $\lambda \in \Lambda$ and distinct $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\lambda + j$ if and only if $a_\lambda^2 \in I$.*

Remark 2.3.2. *[8, 58] Let I be an ideal of a ring R. Then $\Gamma_I(R)$ is a graph on a finite number of vertices if and only if either R is finite or I is a prime ideal. Moreover, if $\Gamma(R/I)$ is a graph on N vertices, then $\Gamma_I(R)$ is a graph on $N \cdot |I|$ vertices.*

2.4 Vertex coloring problems

Vertex coloring of a graph G is the partitioning the vertices of G under some constraints. In this thesis only the vertex coloring problems have been discussed.

Definition 2.4.1. *[66] A k-coloring of a loopless graph G is a labeling $f : V(G) \to S$, where $|S| = k$ (often we use $S = [k] = \{1, 2, \ldots, k\}$). The labels are called the colors. The coloring is proper if no two adjacent vertices have the same color. A loopless graph is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G)$ of a loopless graph G is the least k for which G is k-colorable.*

In a vertex coloring some labels are assigned to the vertices. These labels are called the *colors* and the set of vertices with same color is called a *color class*.

The two vertex coloring problems discussed in this thesis are coupon coloring and locally identifying coloring.

2.4.1 Coupon coloring

The concept of coupon coloring number was introduced by Chen et al. in 2015.

Definition 2.4.2. *[27] Let G be a graph without isolated vertices. A k-coupon coloring of a graph G is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the (open)neighborhood of every vertex of G contains vertices of all colors from $[k]$.*

The maximum k for which a k-coupon coloring exists is called the *coupon coloring number* of G and it is denoted by $\chi_c(G)$. Note that, coupon coloring is an improper coloring and $\chi_c(G) \leq \delta(G)$.

Suppose that G is a graph without isolated vertices and $\chi_c(G) = 4$. Let c and c' are two vertex colorings of G with 4 colors and $c(u) = 1 = c'(u)$ for some $u \in V(G)$. If u is adjacent to 5 vertices and the colors of the neighbors of u with respect to the coloring c and c' are given in Figure 2.4. Then the coloring c cannot be coupon coloring of G, since the neighborhood of u does not contains a vertex with color 1. Note that, each color must appear at least twice in any coupon coloring. So, $\chi_c(G) \le \frac{|V(G)|}{2}$.

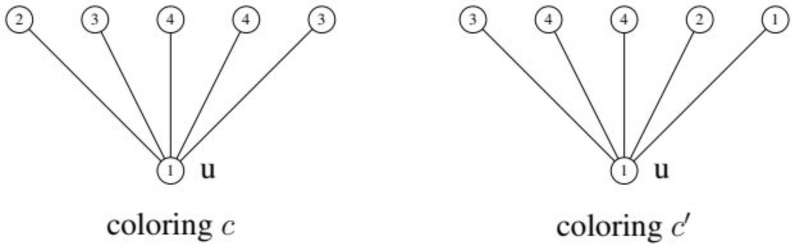


Figure 2.4: Coupon coloring

Coupon coloring is interesting because it is theoretically significant and it has relevant applications in practical problems. Colors can be imagined as coupons of different types. The idea of coupon coloring is to receive all different tokens from the neighbors of each vertex. The concept can be applied to different practical problems. Suppose an information or message is separated into mutually exhaustive parts and is assigned to different members of a group. Each member is assigned only one part of these decomposed parts. Members can be considered as vertices of a graph. Then the conditions of coupon coloring ensures that each member gathers the whole piece of information or message from her neighbors and the maximum k in coupon coloring is then associated with maximizing the length of the original information.

This concept can be applied into problems in network science and one such application in large multi-robot network is given in [2]. Note that, if a network of robots is very large, then they must take measures based on locally accessible information. Suppose a group of robots is arranged for environmental monitoring and they have to observe many different statistics like temperature, barometric pressure, humidity, etc. of that environment. But there may be power limitations, so a robot may equipped with a single sensor (thermometer, barometer, etc.). This network can be modeled as a graph with robots in the network as vertices and there is an edge between vertices if the corresponding robots are able to convey information with each other. Thus, each robot must communicate with their neighbors to collect the complete data about the environment. So, the maximum number of sensors that can be made accessible to a robot in the network is the coupon coloring number of the corresponding graph.

Chen et al. in [27] found some bounds for coupon coloring number and they have proved that $\chi_c(G) \sim \frac{d}{\log d}$.

Theorem 2.4.1. *[27] For every $\delta > 0$, there exists a $d_0(\delta)$ such that if $d \geq d_0(\delta)$, then every d-regular graph G has*

$$\chi_c(G) \geq (1-\delta)\frac{d}{\log d}.$$

For every $\epsilon > 0$, there exists a $d_1(\epsilon)$ such that if $d \geq d_1(\epsilon)$, then as $n \to \infty$, almost every d-regular n-vertex graph has

$$\chi_c(G) \geq (1+\epsilon)\frac{d}{\log d}.$$

The coupon coloring number is also referred to as the total domatic number, introduced in [12], which is the maximum number of disjoint total dominating sets.

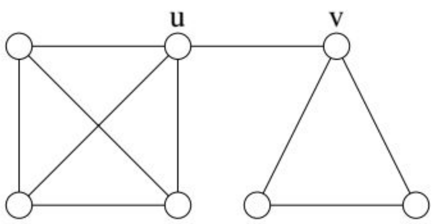


Figure 2.5: Graph G with $\chi_c(G) \neq \frac{|V(G)|}{\gamma_t(G)}$

Note that, every color class in a coupon coloring must be a total dominating set. But it may not be true that $\chi_c(G) = \frac{|V(G)|}{\gamma_t(G)}$. For example, consider the Figure 2.5. Here $\gamma_t(G) = 2$, since $\{u, v\}$ is a total dominating set of the graph G. But there is no other total dominating sets with 2 vertices. In fact, this graph cannot have a total dominating set with 3 vertices disjoin from $\{u, v\}$. Here, $|V(G)| = 7, \gamma_t(G) = 2$ and $\chi_c(G) = 2$.

Coupon coloring was studied by many authors [28,55,59,60]. Observe that coupon coloring number and the total domatic number are the same. Some authors investigated coupon coloring number seeing as the total domatic number [7, 33, 35, 38, 45]. In [60] Yongtang Shi et al. determined coupon coloring number of complete graphs, complete k-partite graphs, wheels, cycles, unicyclic graphs and bicyclic graphs. The coupon coloring number of complete graphs, complete k-partite graphs and cycles are given below.

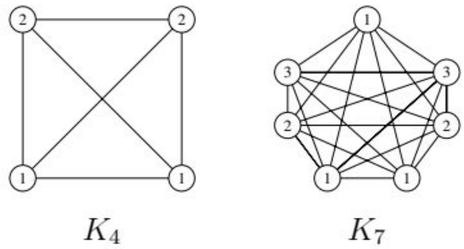


Figure 2.6: Coupon coloring of complete graph

Theorem 2.4.2. *[60] Let K_n be a complete graph with n vertices and $K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph where $k \geq 3$.*

(1) Then $\chi_c(K_n) = \lfloor \frac{n}{2} \rfloor$.

(2) Let $n_1 \leq n_2 \leq, \ldots, \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $n = \sum_{i=1}^{k} n_i$. Then

$$\chi_c(K_{n_1,n_2,\ldots,n_k}) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \textit{if } s \geq \frac{n}{2} \\ s & \textit{otherwise.} \end{cases}$$

Theorem 2.4.3. *[60] Let C_n be the cycle with n vertices. Then*

$$\chi_c(C_n) = \begin{cases} 2 & \textit{if } n \equiv 0 \pmod 4 \\ 1 & \textit{otherwise.} \end{cases}$$

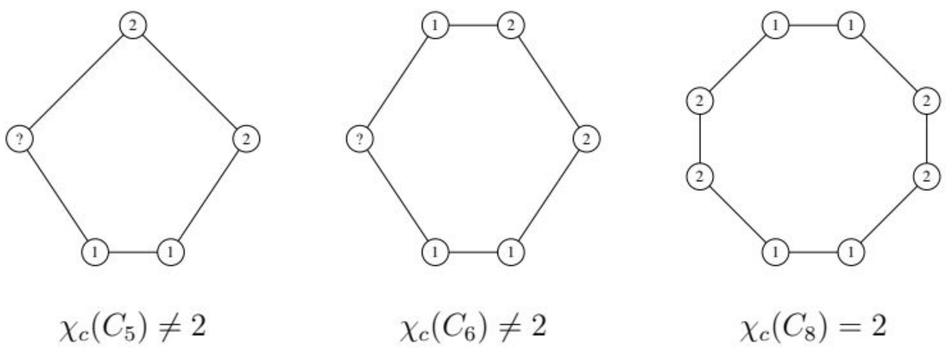


Figure 2.7: Coupon coloring of cycles

Theorem 2.4.4. *[28] If H is an induced subgraph of G, then $\chi_c(G) \geq \chi_c(H)$.*

The coupon coloring of Cartesian product of some graphs have been studied in [34].

Theorem 2.4.5. *[34] For any two graphs G and H with no isolated vertices, we have*

$$2 \leq \chi_c(G \Box H) \leq max\{|V(G)|, |V(H)|\}.$$

Theorem 2.4.6. *[34] If G and H are bipartite graphs without isolated vertex, then*

$$\chi_c(G \Box H) \geq 2min\{\chi_c(G), \chi_c(H)\}$$

2.4.2 Locally identifying coloring

A *k-edge-coloring* (cf. [66]) of a loopless graph $G = (V, E)$ is a labeling $f : E(G) \to S$, where S is a k-element set. It is proper if incident edges have different colors. The vertex-distinguishing proper edge-coloring [21] was defined as follows: A proper edge-coloring c of G is a vertex-distinguishing proper edge-coloring if for any two distinct vertices u and v, the set of colors assigned to the set of edges incident to u and the set of colors assigned to the set of edges incident to v are different. Inspired by this Esperet et al.[31] introduced locally identifying coloring (lid-coloring) in 2010.

Definition 2.4.3. *[31] A vertex coloring $c : V(G) \to \mathbb{N}$ of a graph G is said to be locally identifying if*

1. *c is a proper coloring and*

2. *$c(N[u]) \neq c(N[v])$ for all adjacent vertices u and v with $N[u] \neq N[v]$.*

The smallest integer k for which G admits a lid-coloring is called the lid-chromatic number of G, denoted $\chi_{lid}(G)$. A k-lid-coloring $c : V(G) \to [k]$ of a graph G is called 'bad' if there exists a vertex v in G such that $c(N[v]) = [k]$. Otherwise, c is said to be a 'good' lid-coloring.

In figure 2.8, the proper coloring of G with 3 colors cannot be a lid-coloring, since $c(N[u]) = c(N[v]) = \{1, 2, 3\}$ for all vertices u and v. So it is not a lid-coloring. The proper coloring of G with 4 colors is a lid-coloring, since the colors in closed neighborhoods of all vertices with distinct closed neighborhoods are different. Note that it is a bad 4-lid-coloring.

Figure 2.8: (a) Proper coloring of G (b) Lid-coloring of G.

Locally identifying colorings are related to distinguishing colorings [14, 21, 25] and identifying codes [47]. Esperet et al. found several bounds on $\chi_{lid}(G)$ for different classes of graphs. For bipartite graphs they proved that deciding whether $\chi_{lid}(G) = 3$ or 4 is an NP-complete problem. It was also studied in [15, 17, 32, 41, 54, 56]. If the coloring is not necessarily proper, then it is called relaxed locally identifying coloring [5].

If a connected graph G satisfies $\chi_{lid}(G) \leq 3$, then Esperet et al. [31] showed that G is either a triangle or a bipartite graph. In [17], authors showed that for biconvex bipartite graphs $\chi_{lid}(G)$ can be computed in polynomial time. They also studied the lid-coloring of Cartesian product and lexicographic product of graphs.

The following results are useful for the upcoming sections.

Theorem 2.4.7. *[31] A connected graph G is 2-lid-colorable if and only if G has at most two vertices.*

Theorem 2.4.8. *[31] If G is a bipartite graph, then $\chi_{lid}(G) \leq 4$.*

Theorem 2.4.9. *[31] A tree T with at least 3 vertices is 3-lid-colorable if and only if the distance between every two leaves is even.*

Esperet et al. [31] showed that for a 3-lid-colorable connected bipartite graph, all the vertices of a partite set must have the same color.

Theorem 2.4.10. *[31] Let G be a 3-lid-colorable connected bipartite graph on at least three vertices, with bipartition $\{U, V\}$, and let c be a 3-lid-coloring of G with colors $1, 2, 3$. Then G has a vertex u with $c(N[u]) = \{1, 2, 3\}$ and if $u \in U$, then $c(U) = \{c(u)\}$ and $c(V) = \{1, 2, 3\} \setminus \{c(u)\}$.*

Theorem 2.4.11. *[32] If $n \geq 2$, then*

$$\chi_{lid}(P_n) = \begin{cases} 2 & \textit{if } n = 2 \\ 3 & \textit{if } n \textit{ is odd} \\ 4 & \textit{if } n \textit{ is even and } n \neq 2. \end{cases}$$

Theorem 2.4.12. *[32] If $n \geq 3$,*

$$\chi_{lid}(C_n) = \begin{cases} 3 & \textit{if } n \equiv 0 \pmod 4 \textit{ or } n = 3 \\ 5 & \textit{if } n = 5 \textit{ or } 7 \\ 4 & \textit{otherwise.} \end{cases}$$

Chapter 3

Coupon Coloring of Rooted Product and Lexicographic Product of Graphs

3.1 Introduction

In this chapter, the coupon coloring of rooted product of graphs and the lexicographic product of graphs were studied. A large number of research papers are published in different domination parameters of rooted product graphs. The total domination of rooted product graphs were studied in [22].

3.2 Coupon coloring of rooted product graphs

In this section, coupon coloring of rooted product of two graphs have been studied. The following observation about the root vertex in $G \circ_v H$ will be useful in the upcoming sections.

Lemma 3.2.1. *Let G and H be two graphs without isolated vertices and let $\chi_c(H) = k$. If v is the root vertex of H and H^i be the i^{th} copy of H in $G \circ_v H$, then the set of vertices in H^i adjacent to v can have at most k colors in any coupon coloring of $G \circ_v H$.*

Proof. Suppose that c is a coupon coloring of $G \circ_v H$ with at least $k+1$ colors. If the adjacent vertices of v that are in H^i have $k+1$ colors, then consider the coloring c restricted to the vertices

[1]The section 3.2 of this chapter has been published in *Palestine Journal of Mathematics*, Vol. 12(Special Issue II), 7–12 (2023) [61].

[2]The section 3.3 of this chapter has been published in *The Art of Discrete and Applied Mathematics*, Vol. 6 No. 1 (2023) #P1.03 [64].

of H^i. Define c' in H by $c'(x) = c(x^i)$, if $c(x^i) \leq k+1$ and $c'(x) = 1$, if $c(x^i) > k+1$, for all $x^i \in H^i$. Then c' is a $(k+1)$-coupon coloring of H, because c is the coupon coloring of $G \circ_v H$ and v is the only vertex having neighbors from other copy of H. Therefore, $\chi_c(H) \geq k+1$, which is a contradiction. □

3.2.1 Coupon coloring of rooted product of paths and cycles

Let $G \circ_v H$ denote the rooted product of two graphs G and H with root vertex v. If H is the path P_n with n vertices and v be any vertex of P_n, then $\delta(G \circ_v P_n) = 1$ and so the coupon coloring number of $G \circ_v P_n$ is 1.

Theorem 3.2.1. *Let $n \geq 2$. If $m \geq 2$, then $\chi_c(P_m \circ_v P_n) = 1$. Similarly, if $m \geq 3$, then $\chi_c(C_m \circ_v P_n) = 1$.*

The next result gives the exact coupon coloring number of rooted product of a graph and cycle C_n.

Theorem 3.2.2. *Let G be a graph without isolated vertices. Then*

$$\chi_c(G \circ_v C_n) = \begin{cases} \chi_c(C_n) + 1, & \text{if } n \equiv 1, 3 \pmod 4 \\ \chi_c(C_n), & \text{otherwise.} \end{cases}$$

Proof. Note that $\delta(G \circ_v C_n) = 2$. So, $\chi_c(G \circ_v C_n) \leq 2$. Imagine the vertices of $G \circ_v C_n$ as $v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$, where $v_{11}, v_{21}, \ldots, v_{m1}$ are the vertices of the copy of G in $G \circ_v C_n$ and $v_{i1} - v_{i2} - \cdots - v_{in} - v_{i1}$ is the i^th copy C_n^i of cycle C_n in $G \circ_v C_n$ corresponding to the root vertex v_{i1} for all $i = 1, 2, \ldots, m$.

If $n \equiv 0 \pmod 4$, then $\chi_c(C_n) = 2$ and C_n has a 2-coupon coloring. Color all the vertices of each copy of C_n in $G \circ_v C_n$ with this 2-coupon coloring. Clearly it is a coupon coloring of $G \circ_v C_n$. So, $\chi_c(G \circ_v C_n) = 2 = \chi_c(C_n)$.

If $n \equiv 1 \pmod 4$, then $\chi_c(C_n) = 1$. Define the coloring c_1 of $G \circ_v H$ as follows. color the vertices $v_{11}, v_{21}, \ldots, v_{m1}$ of the copy of G in such a way that a vertex with color 1 is adjacent to at least one vertex with color 2 and vice versa. This is possible since G has no isolated vertices.

If $c_1(v_{i1}) = 1$, then define

$$c_1(v_{ij}) = \begin{cases} 1, & \text{if } j \equiv 1,2 \pmod 4 \\ 2, & \text{otherwise.} \end{cases}$$

If $c_1(v_{i1}) = 2$, then define

$$c_1(v_{ij}) = \begin{cases} 1, & \text{if } j \equiv 0,3 \pmod 4 \\ 2, & \text{otherwise.} \end{cases}$$

Since v_{i1}, v_{i2} and v_{in} have the same color, neighbors of v_{i1} in C_n^i contains only one color. But v_{i1} has a neighbor in the copy of G in $G \circ_v C_n$ with a different color. So, c_1 is a 2-coupon coloring and $\chi_c(G \circ_v C_n) = 2 = \chi_c(C_n) + 1$.

If $n \equiv 2 \pmod 4$, then $\chi_c(C_n) = 1$. It is enough to show that there does not exist a 2-coupon coloring of $G \circ_v C_n$. Suppose that c_2 is any 2-coupon coloring of $G \circ_v C_n$. Then by Lemma 3.2.1, v_{i2} and v_{in} must have same color. Without loss of generality assume that $c_2(v_{i2}) = 1 = c_2(v_{in})$. If $c_2(v_{i1}) = 1$ and define for all $j = 2, 3, \ldots, n$,

$$c_2(v_{ij}) = \begin{cases} 1, & \text{if } j \equiv 1,2 \pmod 4 \\ 2, & \text{otherwise,} \end{cases}$$

then $c_2(v_{i(n-1)}) = 1$ and v_{in} does not have a neighbor with color 2. If $c_2(v_{i1}) = 2$ and define for all $j = 2, 3, \ldots, n$,

$$c_2(v_{ij}) = \begin{cases} 1, & \text{if } j \equiv 2,3 \pmod 4 \\ 2, & \text{otherwise,} \end{cases}$$

then $c_2(v_{i(n-1)}) = 2$ and v_{in} does not have a neighbor with color 1. Hence, c_2 cannot be a 2-coupon coloring. Therefore, $\chi_c(G \circ_v C_n) = 1 = \chi_c(C_n)$.

If $n \equiv 3 \pmod 4$, then $\chi_c(C_n) = 1$. Define the coloring c_3 of $G \circ_v H$ by

$$c_3(v_{ij}) = \begin{cases} 1, & \text{if } j \equiv 2,3 \pmod 4 \\ 2, & \text{otherwise.} \end{cases}$$

Here, v_{i1} is adjacent to v_{i2} and v_{in} with color 1 and v_{s1} with color 2 for some $s \in \{1, 2, \ldots, m\}$, since G has no isolated vertices. Hence, c_3 is a 2-coupon coloring of $G \circ_v C_n$ and $\chi_c(G \circ_v C_n) = 2 = \chi_c(C_n) + 1$. □

3.2.2 Bounds for coupon coloring number of $G \circ_v H$

In this section, some sharp bounds were established for the coupon coloring number of $G \circ_v H$. To show that k is a lower bound for the coupon coloring number, it is enough to show that there exist a k-coupon coloring.

Theorem 3.2.3. *Let G and H be two graphs without isolated vertices. Then*

$$\chi_c(G \circ_v H) \geq \chi_c(H).$$

Proof. Let $\chi_c(H) = k$ and c be a k-coupon coloring of H. Note that, $G \circ_v H$ is obtained by taking $|V(G)|$ copies of H and one copy of G and identifying the i^{th} vertex of G with the root vertex v in the i^{th} copy of H for every $i = 1, 2, \ldots, |V(G)|$. So, the coloring c given to the $|V(G)|$ copies of H in $G \circ_v H$ is a k-coupon coloring of $G \circ_v H$. Thus, $\chi_c(G \circ_v H) \geq \chi_c(H)$. □

A trivial upper bound for the coupon coloring number of a graph without isolated vertices is $\delta(G)$. So, if $G \circ_v H$ is a graph without isolated vertices, then $\chi_c(G \circ_v H) \leq \delta(G \circ_v H)$. Following theorem gives another upper bound for $\chi_c(G \circ_v H)$.

Theorem 3.2.4. *Let G and H be two graphs without isolated vertices. Then*

$$\chi_c(G \circ_v H) \leq \chi_c(H) + \delta(G).$$

Proof. Suppose that $\chi_c(H) = k$. By Lemma 3.2.1, the root vertex can be adjacent to vertices with at most k colors in the corresponding copy of H in any coupon coloring of $G \circ_v H$. Note that, the root vertex v identifies a vertex with degree $\delta(G)$ in G. In that case, v can be adjacent to at most $\delta(G)$ vertices in the copy of G. Thus, in $G \circ_v H$, v can be adjacent to vertices with at most $k + \delta(G)$ colors. Hence, $\chi_c(G \circ_v H) \leq k + \delta(G)$. □

Theorem 3.2.2 shows that the lower bound in Theorem 3.2.3 is sharp. The upper bound in Theorem 3.2.4 is also sharp. For, let $G = K_n$ and H be the graph obtained by adjoining

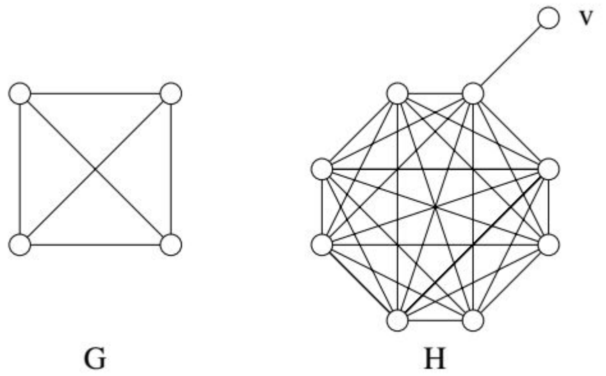


Figure 3.1: Graphs with $\chi_c(G \circ_v H) = \chi_c(H) + \delta(G)$

a vertex v with an edge to a vertex of K_{2n}. The above figure 3.1 gives the the graphs G and H with $n = 4$. Since the degree of the vertex v is one, $\chi_c(H) = 1$. Define a coloring c of $G \circ_v H$ as follows: color all the vertices of G with n different colors. If the root vertex has color s in a copy of H, then color the vertex in H which is adjacent to v with s and color the remaining vertices with appropriate colors so that K_{2n} has an n-coupon coloring. Clearly, c is an n-coupon coloring of $G \circ_v H$. Hence, $\chi_c(G \circ_v H) \geq n = 1 + (n-1) = \chi_c(H) + \delta(G)$. So, $\chi_c(G \circ_v H) = \chi_c(H) + \delta(G)$.

Corollary 3.2.1. *Let G and H be two graphs without isolated vertices and let $\delta(G) = 1$. Then $\chi_c(G \circ_v H) \leq \chi_c(H) + 1$, and so $\chi_c(G \circ_v H) \in \{\chi_c(H), \chi_c(H) + 1\}$.*

Theorem 3.2.5. *Suppose that G and H be two graphs without isolated vertices. If v is a vertex in H such that the graph $H - v$ has no isolated vertices, then*

$$\chi_c(G \circ_v H) \leq \chi_c(H - v) + 1.$$

Proof. Let $\chi_c(H - v) = l$ and c be an $(l+2)$-coupon coloring of $G \circ_v H$. Then we claim that every vertex in $H - v$ must be adjacent to vertices with at least $l + 1$ colors with respect to the coloring c. Otherwise, there exists a vertex u in $H - v$ such that u is not adjacent to vertices with $l + 1$ colors. That is, u is adjacent to vertices with at most l colors in H. Note that, in $G \circ_v H$, u is a vertex which is adjacent only to the vertices of that copy of H. So in $G \circ_v H$, u can be adjacent to vertices with at most $l+1$ colors (since u can be adjacent to v). Thus, $G \circ_v H$ cannot have an $(l+2)$-coupon coloring, a contradiction. Hence the claim.

By the above claim, $H - v$ has an $(l + 1)$-coupon coloring. But it is a contradiction, since $\chi_c(H - v) = l$. Thus, $G \circ_v H$ cannot have an $(l + 2)$-coupon coloring. Therefore, $\chi_c(G \circ_v H) \leq l + 1$. □

The following corollary shows that the upper bound in Theorem 3.2.5 is sharp.

Corollary 3.2.2. *Let G be a graph without isolated vertices. Then*

$$\chi_c(G \circ_v K_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. Suppose that n is odd. Then $\chi_c(K_n - v) = \chi_c(K_{n-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n-1}{2}$. By Theorem 3.2.5, $\chi_c(G \circ_v K_n) \leq \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$. Define the coloring $c : V(G \circ_v K_n) \to [\frac{n+1}{2}]$ as follows: color the $n - 1$ vertices of each copy of $K_n - v$ with the colors $1, 2, \ldots, \frac{n-1}{2}$ such that each color appears twice and color the vertex v of each copy of K_n with the color $\frac{n+1}{2}$. Clearly, c is a coupon coloring of $G \circ_v K_n$ and so $\chi_c(G \circ_v K_n) \geq \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$.

If n is even, then $\chi_c(K_n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$. By Theorem 3.2.3, $\chi_c(G \circ_v K_n) \geq \frac{n}{2} = \left\lceil \frac{n}{2} \right\rceil$. By Theorem 3.2.5, $\chi_c(G \circ_v K_n) \leq \chi_c(K_{n-1}) + 1 = \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \frac{n}{2} = \left\lceil \frac{n}{2} \right\rceil$. □

In Corollary 3.2.1, it is proved that $\chi_c(G \circ_v H)$ is either $\chi_c(H)$ or $\chi_c(H) + 1$ whenever $\delta(G) = 1$. The following corollary, which follows from Theorem 3.2.3 and Theorem 3.2.5, gives a class of rooted product graphs for which $\chi_c(G \circ_v H) = \chi_c(H)$ holds.

Corollary 3.2.3. *Suppose that the graphs G, H and $H - v$ (where v is a vertex in H) have no isolated vertices and $\delta(G) = 1$. If $\chi_c(H) = k$ and $H - v$ cannot have a k-coupon coloring, then*

$$\chi_c(G \circ_v H) = k.$$

Let G and H be two graphs without isolated vertices and $\delta(G) = 1$. Assume that the root vertex $v \in V(H)$ is adjacent to all the other vertices of H. If $\chi_c(H) = k$ and $H - v$ have a k-coupon coloring, then $\chi_c(G \circ_v H) = k + 1$.

3.3 Coupon coloring of lexicographic product of graphs

This section discusses the coupon coloring number of lexicographic product of two graphs G and H. If G has a Hamilton path, it is proved that $|V(H)| + 1$ is a lower bound for $\chi_c(G[H])$ and showed that for some graphs $\chi_c(G[H]) = |V(H)| + 1$. A sharp bound for the coupon coloring number of lexicographic product of connected graphs has been obtained.

3.3.1 Some bounds for the coupon coloring number of $G[H]$

The following are some observations on the degree of the lexicographic product of G and H.

Observation 3.3.1. *Let $G[H]$ be the lexicographic product of G and H. Then*

1. $\deg((x, y)) = \deg(y) + |V(H)| \deg(x)$ *for any* $(x, y) \in G[H]$,
2. $\delta(G[H]) = \delta(G) + |V(H)|\delta(H)$,
3. $\delta(G[H]) \geq 1 + |V(H)|$,
4. *If G is a disconnected graph with components $G_1, \dots, G_k$, then the coupon coloring number,* $\chi_c(G) = \min\{\chi_c(G_1), \dots, \chi_c(G_k)\}$.

One can imagine $G[H]$ as $|V(G)|$ copies of H such that each vertex of a copy of H is adjacent to every vertices of another copy of H if the corresponding vertices of G are adjacent. Let G and H be any two graphs without isolated vertices. Enumerate vertices of H as $h_1, h_2, \dots, h_{|V(H)|}$ and define $c(g, h_i) = i$ for all $g \in G$. Then clearly c is a coupon coloring of $G[H]$. For let $(x, y) \in G[H]$. Then there exists $x \neq g \in G$ such that x is adjacent to g, since G has no isolated vertices. So (x, y) is adjacent to (g, h_i) for all $i = 1, 2, \dots, |V(H)|$. Hence, $\chi_c(G[H]) \geq |V(H)|$.

Theorem 3.3.1. *Let G and H be two graphs without isolated vertices with $|V(G)| \geq 4$ and $|V(H)| \geq 3$. If G contains a Hamilton path and K_2 is not a component of the graph H, then $\chi_c(G[H]) \geq |V(H)| + 1$.*

Proof. Let G and H be two graphs with n and m vertices respectively. Let the Hamilton path in G be $x_1 - x_2 - \cdots - x_n$.

Case 1. $n \equiv 0 \pmod 4$

Define $c_1 : V(G[H]) \to [m + 1]$ by

$$c_1(x_i, y_j) = \begin{cases} m + 1, & \text{if } i \equiv 0, 1 \pmod 4 \\ j, & \text{if } i \equiv 2, 3 \pmod 4. \end{cases}$$

For proving that c_1 is a coupon coloring, let $(x_i, y_j) \in G[H]$. Since x_i is adjacent to x_{i-1} and x_{i+1} in G, (x_i, y_j) is adjacent to vertices with colors $1, 2, \ldots, m$ and also (x_i, y_j) is adjacent to vertices with color $m + 1$ except when $i \equiv 0, 1 \pmod 4$. But in that case, since H has no isolated vertices, y_j is adjacent to y_k in H and so (x_i, y_j) is adjacent to vertex (x_i, y_k) with color $m + 1$.

Case 2. $n \equiv 1 \pmod 4$

Now $n - 5 \equiv 0 \pmod 4$ and so define the coloring c_2 as in case 1 for the vertices (x_i, y_j) for $i \geq 6$. That is, $c_2(x_i, y_j) = c_1(x_{i-1}, y_j)$ for $i \geq 6$. For $i = 1, 2, \ldots, 5$, define the coloring

$$c_2(x_i, y_j) = \begin{cases} m+1, & \text{if } (x_i, y_j) = (x_4, y_1) \text{ or } (x_5, y_1) \text{ or } (x_1, y_j) \text{ for all } j \\ 1, & \text{if } (x_i, y_j) = (x_5, y_j) \text{ for all } j \geq 2 \\ j, & \text{otherwise.} \end{cases}$$

It is enough to show that c_2 is an $(m + 1)$-coupon coloring of the graph induced by the vertices (x_i, y_j) of $G[H]$ with $i = 1, 2, \ldots, 5$ and $j = 1, 2, \ldots, m$. The color j is given to the vertices $(x_2, y_j), (x_3, y_j)$ for all j and (x_4, y_j) for all $j \geq 2$. The following representation gives the coloring of the $5m$ vertices of $G[H]$ described above.

$$\begin{array}{ccccc} m+1 & m+1 & m+1 & \cdots & m+1 \\ 1 & 2 & 3 & \cdots & m \\ 1 & 2 & 3 & \cdots & m \\ m+1 & 2 & 3 & \cdots & m \\ m+1 & 1 & 1 & \cdots & 1 \end{array}$$

The first row gives the coloring of the vertices (x_1, y_j), $j = 1, 2, \ldots, m$, second row gives the coloring of the vertices (x_2, y_j), $j = 1, 2, \ldots, m$ and so on. Since x_1 is adjacent to x_2, all the vertices (x_1, y_j) are adjacent to (x_2, y_j). Similarly, Since x_2 is adjacent to x_3, all the vertices (x_2, y_j) are adjacent to (x_3, y_j). Hence, the vertices (x_2, y_j), (x_3, y_j) and (x_4, y_j) are adjacent to vertices with all the $m + 1$ colors, the vertices (x_1, y_j) are adjacent to vertices with colors $1, 2, \ldots, m$ and the vertices (x_5, y_j) are adjacent to vertices with colors $2, 3, \ldots, m + 1$. Since H has no isolated vertices, (x_1, y_j) is adjacent to (x_1, y_k) for some $j \neq k$ with color $m + 1$. Also H does not contain K_2 as a component and thus we can choose y_1 as a vertex which is not adjacent to any of the leaves in H. This implies that (x_5, y_j) is adjacent to a vertex with color 1.

Case 3. $n \equiv 2 \pmod 4$

Now $n-6 \equiv 0 \pmod 4$. Define the coloring c_3 as $c_3(x_i, y_j) = c_1(x_{i-2}, y_j)$ for the vertices (x_i, y_j) for $i \geq 7$. For $i = 1, 2, \ldots, 6$, define the coloring

$$c_3(x_i, y_j) = \begin{cases} m+1, & \text{if } (x_i, y_j) = (x_1, y_1) \text{ or } (x_2, y_1) \text{ or } (x_5, y_1) \text{ or } (x_6, y_1) \\ 1, & \text{if } (x_i, y_j) = (x_1, y_j) \text{ or } (x_6, y_j) \text{ for all } j \geq 2 \\ j, & \text{otherwise.} \end{cases}$$

It is enough to show that c_3 is an $(m+1)$-coupon coloring of the graph induced by the vertices (x_i, y_j) of $G[H]$ with $i = 1, 2, \ldots, 6$ and $j = 1, 2, \ldots, m$. The following representation gives the coloring of the $6m$ vertices of $G[H]$ described above.

$$\begin{matrix} m+1 & 1 & 1 & \cdots & 1 \\ m+1 & 2 & 3 & \cdots & m \\ 1 & 2 & 3 & \cdots & m \\ 1 & 2 & 3 & \cdots & m \\ m+1 & 2 & 3 & \cdots & m \\ m+1 & 1 & 1 & \cdots & 1 \end{matrix}$$

The proof is similar as in case 2. The vertices (x_2, y_j), (x_3, y_j), (x_4, y_j) and (x_5, y_j) are adjacent to vertices with all the $m+1$ colors, the vertices (x_1, y_j) and (x_6, y_j) are adjacent to vertices with colors $2, 3, \ldots, m+1$. Since H does not contain K_2 as a component and thus we can choose y_1 as a vertex which is not adjacent to any of the leaves in H. This implies that the vertices (x_1, y_j) and (x_6, y_j) are adjacent to a vertex with color 1.

Case 4. $n \equiv 3 \pmod 4$

Now $n-7 \equiv 0 \pmod 4$ and so define the coloring c_4 as in case 1 for the vertices (x_i, y_j) for $i \geq 8$. That is, $c_4(x_i, y_j) = c_1(x_{i-3}, y_j)$. For $i = 1, 2, \ldots, 7$, define the coloring

$$c_4(x_i, y_j) = \begin{cases} m+1, & \text{if } i = 1, 4, 7 \\ j, & \text{otherwise.} \end{cases}$$

To show that c_4 is an $(m+1)$-coupon coloring of the vertices (x_i, y_j) of $G[H]$ with $i = 1, 2, \ldots, 7$ and $j = 1, 2, \ldots, m$, consider the following representation of the coloring c_4 of these $7m$ vertices of $G[H]$.

$$\begin{array}{ccccc} m+1 & m+1 & m+1 & \cdots & m+1 \\ 1 & 2 & 3 & \cdots & m \\ 1 & 2 & 3 & \cdots & m \\ m+1 & m+1 & m+1 & \cdots & m+1 \\ 1 & 2 & 3 & \cdots & m \\ 1 & 2 & 3 & \cdots & m \\ m+1 & m+1 & m+1 & \cdots & m+1 \end{array}$$

In all the above cases, there is a coupon coloring with $m+1$ colors. Thus $\chi_c(G[H]) \geq m+1$. □

The bound proved in the above theorem is sharp. The graph that attains this lower bound is given in Corollary 3.3.1.

Corollary 3.3.1. *Let G and H be two graphs without isolated vertices with $|V(G)| \geq 4$ and let K_2 is not a component of H. If G contains a Hamilton path and if both G and H has a leaf, then $\chi_c(G[H]) = |V(H)| + 1$. Moreover, if $n \geq 4$ and $m \geq 3$, then $\chi_c(P_n[P_m]) = m + 1$.*

Proof. By Theorem 3.3.1, $\chi_c(G[H]) \geq |V(H)| + 1$. If u and v are the leaves of G and H respectively, then $\deg((u, v)) = 1 + |V(H)|$ and so $\delta(G[H]) = 1 + |V(H)|$, by Observation 3.3.1. Hence $\chi_c(G[H]) \leq \delta(G[H]) = |V(H)| + 1$. □

Corollary 3.3.2. *Let H be a graph without isolated vertices and K_2 is not a component of H. If H has a leaf, then $\chi_c(P_n[H]) = |V(H)| + 1$ for all $n \geq 4$.*

Corollary 3.3.3. *Let H be a graph without isolated vertices and K_2 is not a component of H. Then $\chi_c(C_n[H]) \geq |V(H)| + 1$ for all $n \geq 4$.*

Theorem 3.3.2. *Let G and H be two graphs without isolated vertices with $n \geq 4$ and $m \geq 3$ vertices respectively. If G contains a Hamilton path and $n \equiv 0, 3 \pmod 4$, then $\chi_c(G[H]) \geq |V(H)| + \chi_c(H)$.*

Proof. Let G and H be two graphs with n and m vertices respectively and let $\chi_c(H) = k$ and c be a k-coupon coloring of H. Let the Hamilton path in G be $x_1 - x_2 - \cdots - x_n$.

If $n \equiv 0 \pmod 4$, define $c_1 : V(G[H]) \to [m + k]$ by

$$c_1(x_i, y_j) = \begin{cases} c(y_j), & \text{if } i \equiv 0, 1 \pmod 4 \\ k + j, & \text{if } i \equiv 2, 3 \pmod 4. \end{cases}$$

The coloring c_1 is a coupon coloring of $G[H]$. For, let $(x_r, y_s) \in V(G[H])$ and $r \equiv 0, 1 \pmod 4$. Since $\chi_c(H) = k$, the vertex y_s is adjacent to vertices with k colors in H and so, (x_r, y_s) is adjacent to vertices with colors $1, 2, \ldots, k$ in $G[H]$. Note that, x_r is adjacent to at least one of the vertices of $\{x_{r-1}, x_{r+1}\}$. Therefore, (x_r, y_s) is adjacent to vertices (x_{r-1}, y_j) or (x_{r+1}, y_j) ($1 \leq j \leq m$) with colors $k+1, k+2, \ldots, k+m$ in $G[H]$. If $r \equiv 2, 3 \pmod 4$, then clearly, (x_r, y_s) is adjacent to vertices with all the $k+m$ colors.

If $n \equiv 3 \pmod 4$, then $n - 7 \equiv 0 \pmod 4$ and so define the coloring c_1 for the vertices (x_i, y_j) for $i \geq 8$. For $i = 1, 2, \ldots, 7$, define the coloring define $c_2 : V(G[H]) \to [m+k]$ by

$$c_2(x_i, y_j) = \begin{cases} c(y_j), & \text{if } i = 1, 4, 7 \\ k + j, & \text{otherwise.} \end{cases}$$

Clearly, c_2 are coupon coloring of $G[H]$. □

3.3.2 Coupon coloring of lexicographic product of connected graphs

Let G and H be connected nontrivial graphs. Any subset C of $V(G) \times V(H)$ can be written as $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x = \{a \in V(H) : (x, a) \in C\}$ for all $x \in S$. Cris L. Armada et. al. [13], proved a necessary and sufficient condition for a subset of $V(G) \times V(H)$ to be a total dominating set of $G[H]$.

Theorem 3.3.3. *[13] Let G and H be both nontrivial connected graphs. Then $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$ is a total dominating set of $G[H]$ if and only if either (i) S is a total dominating set of G or (ii) S is a dominating set of G and T_x is a total dominating set of H for every $x \in S \setminus N_G(S)$.*

Corollary 3.3.4. *[13] If G and H be both nontrivial connected graphs, then $\gamma_t(G[H]) = \gamma_t(G)$.*

Using Theorem 3.3.3, the following theorem has been proved. Corollary 3.3.5, shows that the upper bound in Theorem 3.3.4 is sharp.

Theorem 3.3.4. *Let G and H be two graphs without isolated vertices. If H is connected, then*

$$\chi_c(G[H]) \geq |V(H)|\chi_c(G).$$

Proof. Let $|V(G)| = n, |V(H)| = m$ and $\chi_c(G) = k$. Suppose that G is connected. Put $C_{ij} = \cup_{x \in S_i}\{(x, h_j)\}$, for all $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$, where S_i is a total dominating set of G. Then by Theorem 3.3.3, C_{ij} is a total dominating set of $G[H]$ for all i and j. Define $c : V(G[H]) \to [mk]$ by

$$c(x, y) = \begin{cases} (i-1)m + j, & \text{if } x \in C_{ij}, \text{ for some } i \text{ and } j \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, c is a coupon coloring of $G[H]$. For let (x, y) be any vertex of $G[H]$. Since each C_{ij} is a total dominating set of $G[H]$, (x, y) is adjacent to at least one vertex of each C_{ij}, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$. Hence, $\chi_c(G[H]) \geq mk$.

If G is a disconnected graph with components $G_1, \ldots, G_k$, then $G[H]$ is a disconnected graph with components $G_1[H], \ldots, G_k[H]$. Since $G_1, \ldots, G_k$ and H are connected graphs, by Observation 3.3.1,

$$\begin{aligned} \chi_c(G[H]) &= \min\{\chi_c(G_1[H]), \ldots, \chi_c(G_k[H])\} \geq \min\{|V(H)|\chi_c(G_1), \ldots, |V(H)|\chi_c(G_2)\} \\ &\geq |V(H)|\chi_c(G). \end{aligned}$$

This completes the proof. □

Corollary 3.3.5. *If n is an even positive integer and H is a connected graph with $|V(H)| = m$, then $\chi_c(K_n[H]) = \frac{nm}{2}$.*

Proof. By Theorem 3.3.4 and Theorem 2.4.2, $\chi_c(K_n[H]) \geq m(\frac{n}{2}) = \frac{nm}{2}$. Note that, in a coupon coloring each color should appear at least two times. So $\chi_c(K_n[H]) \leq \frac{nm}{2}$. □

Chapter 4

Coupon Coloring of Cayley Graphs and Zero-Divisor Graphs

4.1 Introduction

This chapter focuses on the coupon coloring of Cayley graphs, generalized Cayley graphs, zero-divisor graphs and ideal-based zero-divisor graphs. Coupon coloring number is the maximum number of disjoint total dominating sets and some authors studied the total domination of these graphs. But finding coupon coloring number is a different task.

All the algebraic graphs considered in this chapter are on finite commutative rings with identity. Note that, every finite commutative ring with identity can be written as a product of some finite local rings (cf. [18]). So, studying coupon coloring in local rings and extending that idea to finite commutative rings is the method that we have used in this chapter.

4.2 Coupon coloring of Cayley graphs

In this section, coupon coloring of some Cayley graphs such as circulant graphs, $\mathbb{CAY}(R)$ and Γ_R^n, were discussed.

[1]The subsection 4.2.1 and subsection 4.2.2 of this chapter have been published in *Palestine Journal Mathematics*, Vol. 12(1), 2023 [62].

[2]The subsection 4.2.2 and subsection 4.2.3 of this chapter have been published in Amitabha Bagchi, Rahul Muthu(eds) Algorithms and Discrete Applied Mathematics. CALDAM 2023, *Lecture Notes in Computer Science*, Vol. 13947 (2023) Springer, Cham. [65].

4.2.1 Coupon coloring of some circulant graphs

Theorem 4.2.1. *Let* $C = \{1, -1, a = -a\}$. *Then*

$$\chi_c(Cay(\mathbb{Z}_n, C)) = \begin{cases} 3, & \text{if } n \equiv 0 \ (mod\ 3) \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Clearly $Cay(\mathbb{Z}_n, C)$ is a 3-regular graph. Let $a \in \mathbb{Z}_n$ and let $a = -a$. Then n must be even, since $2a = a + a = 0 = n$.

Case 1: $n \equiv 0 \pmod 3$

Define $c : V(Cay(\mathbb{Z}_n, C)) \to [3]$ by

$$c(i) = \begin{cases} 1, & \text{if } i \equiv 0 (\text{mod } 3) \\ 2, & \text{if } i \equiv 1 (\text{mod } 3) \\ 3, & \text{if } i \equiv 2 (\text{mod } 3). \end{cases}$$

Then c is a coupon coloring on $Cay(\mathbb{Z}_n, C)$. For, let $i \in V(Cay(\mathbb{Z}_n, C))$ such that $i \equiv 0 \pmod 3$. Then $c(i) = 1$ and the neighbors of i are $i - 1, i + 1$ and $i + a$. Since $i \equiv 0 \pmod 3$ and $a = \frac{n}{2} \equiv 0 \pmod 3$, $i+a \equiv a \equiv 0 \pmod 3$, $i-1 \equiv -1 \equiv 2 \pmod 3$ and $i+1 \equiv 1 \pmod 3$. So $c(i + a) = 1$, $c(i - 1) = 3$ and $c(i + 1) = 2$. All other possibilities can be proved similarly.

Case 2: $n \equiv 1 \pmod 3$

Suppose that $Cay(\mathbb{Z}_n, C)$ has a 3-coupon coloring. Then at least one color should be given to at most $\lfloor \frac{n}{3} \rfloor$ vertices. This color class D must be a total dominating set. But $\lfloor \frac{n}{3} \rfloor = \frac{n-1}{3}$ and $Cay(\mathbb{Z}_n, C)$ is a 3-regular graph. So, D can dominate at most $n - 1$ vertices and D cannot be a TDS.

So, $\chi_c(Cay(\mathbb{Z}_n, C)) \leq 2$. Now define $c : V(Cay(\mathbb{Z}_n, C)) \to [2]$ by

$$c(i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 2 \\ 2, & \text{if } i \equiv 1 \pmod 2. \end{cases}$$

Then clearly c is a 2-coupon coloring of $Cay(\mathbb{Z}_n, C)$, and so $\chi_c(Cay(\mathbb{Z}_n, C)) \geq 2$. Hence, $\chi_c(Cay(\mathbb{Z}_n, C)) = 2$.

Case 3: $n \equiv 2 \pmod 3$

Proof is similar as in Case 2. □

Theorem 4.2.2. *Let $Cay(\mathbb{Z}_n, C)$ be the circulant graph with $C = \{1, -1, 2, -2\}$, $2 \neq -2$. Then $\chi_c(Cay(\mathbb{Z}_n, C)) \leq 3$ and equality holds if $n \equiv 0 \pmod 6$.*

Proof. Since $2 \neq -2$, $Cay(\mathbb{Z}_n, C)$ is a 4-regular graph. Let a be any vertex of $Cay(\mathbb{Z}_n, C)$. Then neighbors of a are $a-1, a-2, a+1$ and $a+2$. Since $(a+1)-(a-1) = 2 \in C$, there is an edge between $a-1$ and $a+1$. Similarly, $(a-1)-(a-2) = 1 \in C$; $(a+1)-(a+2) = -1 \in C$ and so $a-1$ and $a-2$ are adjacent; $a+1$ and $a+2$ are adjacent.

If c is a 4-coupon coloring of $Cay(\mathbb{Z}_n, C)$, then without loss of generality, assume that $c(a-1) = 1$, $c(a+1) = 2$, $c(a-2) = 3$ and $c(a+2) = 4$. Then $c(a)$ cannot be 1,2,3 or 4.

(i) If $c(a) = 1$, then the vertex $a+1$ will have two neighbors with color 1.

(ii) If $c(a) = 2$, then the vertex $a-1$ will have two neighbors with color 2.

(iii) If $c(a) = 3$, then the vertex $a-1$ will have two neighbors with color 3.

(iv) If $c(a) = 4$, then the vertex $a+1$ will have two neighbors with color 4.

Since $\Delta(Cay(\mathbb{Z}_n, C)) = 4$, all the neighbors of a vertex must be different. Hence, 4-coupon coloring is not possible and so $\chi_c(Cay(\mathbb{Z}_n, C)) \leq 3$.

Claim : $\chi_c(Cay(\mathbb{Z}_n, C)) = 3$ if $n \equiv 0 \pmod 6$

Define $c : V(Cay(\mathbb{Z}_n, C)) \to [3]$ by

$$c(i) = \begin{cases} 1, & \text{if } i \equiv 0, 1 \pmod 6 \\ 2, & \text{if } i \equiv 2, 3 \pmod 6 \\ 3, & \text{if } i \equiv 4, 5 \pmod 6. \end{cases}$$

Then c is a coupon coloring of $Cay(\mathbb{Z}_n, C)$. For, let $i \in V(Cay(\mathbb{Z}_n, C))$ such that $i \equiv 0 \pmod 6$. Then neighbors of i are $i-1, i-2, i+1$ and $i+2$.

Since $i - 1 \equiv -1 \equiv 5 \pmod 6$, so $c(i-1) = 3$. Similarly since $i - 2 \equiv -2 \equiv 4 \pmod 6$, so $c(i-2) = 3$, since $i + 1 \equiv 1 \pmod 6$, so $c(i+1) = 1$ and since $i + 2 \equiv 2 \pmod 6$, so $c(i+2) = 2$. Therefore, the four neighbors of i colored with all the three colors. Other cases can be proved similarly. Hence the claim holds. □

4.2.2 Coupon coloring of $\mathbb{CAY}(R)$

By Theorem 2.3.1, $\mathbb{CAY}(R)$ has no edge if and only if R is an integral domain. So in this section consider only $\mathbb{Z}_n$ with n composite.

Theorem 4.2.3. *If $Z(\mathbb{Z}_n)$ is an ideal of $\mathbb{Z}_n$, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) = \left\lfloor \frac{|Z(\mathbb{Z}_n)|}{2} \right\rfloor$.*

Proof. Let $Z(\mathbb{Z}_n)$ be an ideal of $\mathbb{Z}_n$. Then $n = p^k$ for some prime p and $Z(\mathbb{Z}_n)$ is the maximal ideal of $\mathbb{Z}_n$. By Theorem 2.3.1, $\mathbb{CAY}(\mathbb{Z}_n)$ is the disjoint union of $\left|\frac{\mathbb{Z}_n}{Z(\mathbb{Z}_n)}\right| = p$ copies of the complete graph $K_{|Z(\mathbb{Z}_n)|} = K_{p^{k-1}}$ since, $|Z(\mathbb{Z}_n)| = p^{k-1}$. Therefore, using Theorem 2.4.2

$$\begin{aligned} \chi_c(\mathbb{CAY}(\mathbb{Z}_n)) &= \chi_c(K_{p^{k-1}}) \\ &= \left\lfloor \frac{p^{k-1}}{2} \right\rfloor \\ &= \left\lfloor \frac{|Z(\mathbb{Z}_n)|}{2} \right\rfloor. \end{aligned}$$

The proof is complete. □

The above theorem gives the coupon coloring number of $\mathbb{CAY}(\mathbb{Z}_n)$ when $Z(\mathbb{Z}_n)$ is an ideal of $\mathbb{Z}_n$. Next, consider the case that $Z(\mathbb{Z}_n)$ is not an ideal of $\mathbb{Z}_n$. So there exists at least two prime divisors for n and $n = p_1^{r_1} p_2^{r_2} \dots p_m^{r_m}$, where $p_1 < p_2 < \cdots < p_m$, $m \geq 2$.

Let $n = p_1 p_2$ where $p_1 < p_2$. Then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$. For all $y \in \mathbb{Z}_{p_2}$, define the set

$$D_y = \Big\{(0, y), (1, y), (2, y), \dots, ((p_1 - 1), y)\Big\}.$$

Clearly, there are p_2 disjoint sets and $|D_y| = p_1$. Define the coloring $c : V(\mathbb{CAY}(\mathbb{Z}_n)) \to [p_2]$ by

$$c((x, y)) = y + 1, \text{ if } y \in D_y.$$

Let $(x, y) \in \mathbb{Z}_n$. Then $(x, y) \in D_y$ and (x, y) is adjacent to the vertices $(x, 0)$, $(x, 1)$, $\dots$, $(x, y-1)$, $(x+1, y)$, $(x, y+1)$, $\dots$, $(x, p_2 - 1)$ with colors $1, 2, \dots, p_2$, since the zero-divisors of $\mathbb{Z}_n$ are (a, b) with either $a = 0$ or $b = 0$. Thus, c is a coupon coloring of $\mathbb{CAY}(\mathbb{Z}_n)$ and so $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) \geq p_2 = \frac{n}{p_1}$.

Now consider $n = p_1^{r_1} p_2^{r_2}$ where $p_1 < p_2$. Then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}}$. For all $y \in \mathbb{Z}_{p_2^{r_2}}$, define the set

$$D_{(sp_1,y)} = \Big\{(sp_1, y), (sp_1 + 1, y), (sp_1 + 2, y), \ldots, (sp_1 + (p_1 - 1), y)\Big\}$$

where $s = 0, 1, 2, \ldots, p_1^{r_1-1} - 1$. Clearly, there are $p_1^{r_1-1} p_2^{r_2}$ disjoint sets and $|D_{(sp_1,y)}| = p_1$. Let these $p_1^{r_1-1} p_2^{r_2}$ sets be D_k, $k = 1, 2, \ldots, p_1^{r_1-1} p_2^{r_2} = \frac{n}{p_1}$. Define the coloring $c : V(\mathbb{CAY}(\mathbb{Z}_n)) \to \left[\frac{n}{p_1}\right]$ by

$$c(x) = k, \text{ if } x \in D_k.$$

Let $x \in \mathbb{Z}_n$. Then $x \in D_l$ for some $l \in \Big\{1, 2, \ldots, \frac{n}{p_1}\Big\}$. So, $x = (tp_1 + i, y')$, where $t \in \big\{0, 1, 2, \ldots, p_1^{r_1-1} - 1\big\}$, $i \in \big\{0, 1, \ldots, p_1 - 1\big\}$ and $y' \in \mathbb{Z}_{p_2^{r_2}}$. Note that, the zero-divisors of $\mathbb{Z}_n$ are (a, b) with either $a \in Z(\mathbb{Z}_{p_1^{r_1}})$ or $b \in Z(\mathbb{Z}_{p_2^{r_2}})$. Thus, x is adjacent to the vertices of the form $(sp_1 + i, y) \neq x$, for all $s = 0, 1, 2, \ldots, p_1^{r_1-1} - 1$ and $y \in \mathbb{Z}_{p_2^{r_2}}$. That is, x is adjacent to a vertex in D_k, for all $k \neq l$. Also, x is adjacent to the vertex $(tp_1 + (i + 1), y') \in D_l$. Thus x is adjacent to the vertices with colors $1, 2, \ldots, \frac{n}{p_1}$ and so, c is a coupon coloring of $\mathbb{CAY}(\mathbb{Z}_n)$. Hence, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) \geq \frac{n}{p_1}$.

Generalizing the above ideas, the exact coupon coloring number of Cayley graph of a finite commutative ring can be found.

Coupon coloring is defined for graphs without isolated vertices. So, in this section, $\mathbb{CAY}(R)$, where R is a finite commutative ring which is not an integral domain were considered. If R is a finite local ring, then by Theorem 2.3.1, $\mathbb{CAY}(R)$ is the disjoint union of $|\frac{R}{M}|$ copies of the complete graph $K_{|M|}$. So, $\chi_c(\mathbb{CAY}(R)) = \left\lfloor \frac{|M|}{2} \right\rfloor$. The following theorem gives the exact coupon coloring number of $\mathbb{CAY}(R)$, when R is a finite commutative ring with identity.

Theorem 4.2.4. *Let R be a finite commutative ring with identity and $R \cong R_1 \times R_2 \times \cdots \times R_n$, $n \geq 2$ be the local ring decomposition. If $k = \min\{|R_i/m_i| : i = 1, 2, \ldots, n\}$, m_i be the maximal ideal of R_i, then*

$$\chi_c(\mathbb{CAY}(R)) = \frac{|R|}{k}.$$

Proof. Assume that $k = |R_1/m_1|$. Let

$$R_1/m_1 = \{r_j + m_1 : j = 1, 2, \ldots, k\}$$

Define D_t for each $(n-1)$-tuple $y^{(t)} = (y_2^{(t)}, y_3^{(t)}, \dots y_n^{(t)}) \in R_2 \times \cdots \times R_n$,

$$D_t = \left\{ \left(y_j, y_2^{(t)}, y_3^{(t)}, \dots, y_n^{(t)} \right) : y_j \in r_j + m_1, j = 1, 2, \dots, k \right\}.$$

Note that, one can choose distinct y_j's from each coset $r_j + m_1$ in $|m_1|$ ways. So, $|D_t| = k$ and there are $\frac{|R|}{k}$ disjoint sets.

Define $c : R \to \left[\frac{|R|}{k}\right]$ by $c(x) = t$ if $x \in D_t$. We claim that c is a coupon coloring of $\mathbb{CAY}(R)$. For, let $x \in R$. Then $x = (x_1, x_2 \dots, x_n)$ and so $x_1 \in r_l + m_1$, for some $l \in \{1, 2, \dots, k\}$, since cosets of m_1 forms a partition of R_1. So, there exists $y = (y_1, y_2^{(t)}, y_3^{(t)}, \dots, y_n^{(t)}) \in D_t$ such that $y_1 \in r_l + m_1$. Then $x_1 - y_1 \in m_1$ and x is adjacent to y. If $x = y \in D_t$, then x is adjacent to all other elements of D_t, since $x - y' = (x_1 - y_j, 0, 0, \dots, 0)$ for all $y' = (y_j, y_2^{(t)}, y_3^{(t)}, \dots, y_n^{(t)}) \in D_t$. Hence, x is adjacent to at least one vertex in every D_t. So,

$$\chi_c(\mathbb{CAY}(R)) \geq \frac{|R|}{k}.$$

Let c be a k-coupon coloring of $\mathbb{CAY}(R)$ such that the color 1 is given to at most $k-1$ vertices. Suppose that S is the set of vertices with color 1. Since m_i has at least k cosets, for each $i = 1, 2, \dots, n$, there exists a coset $r_{l_i} + m_i$ of m_i such that none of the elements of S has i^{th} co-ordinate from $r_{l_i} + m_i$. That is, if $T_i = \{(x_1, x_2, \dots, x_n) \in R : x_i \in r_{l_i} + m_i\}$, then $S \cap T_i = \phi$, for $i = 1, 2, \dots, n$. Consider the vertex $z = (z_1, z_2, \dots, z_n), z_i \in r_{l_i} + m_i$ for all $i = 1, 2, \dots, n$. Since c is a k-coupon coloring of $\mathbb{CAY}(R)$, there exists $y \in S$ such that $y - z \in Z^*(R)$. That is, $y_j - z_j \in Z(R_j)$ for some j and $y_j \in r_{l_j} + m_j$, so $y \in T_j$, a contradiction. Hence, in a k-coupon coloring of $\mathbb{CAY}(R)$ each color must be given to at least k vertices and

$$\chi_c(\mathbb{CAY}(R)) \leq \frac{|R|}{k}.$$

Therefore, $\chi_c(\mathbb{CAY}(R)) = \frac{|R|}{k}$. □

4.2.3 Coupon coloring number of Γ_R^n

In this section, the coupon coloring of generalized Cayley graphs of finite commutative rings were discussed. Suppose that $n = 1$. Then the generalized Cayley graph is the undirected Cayley graph $\overline{Cay}(R \setminus \{0\}, R \setminus \{0\})$.

Theorem 4.2.5. *Let R be a finite commutative ring with identity. If $|U(R)| \geq |Z^*(R)|$, then*

$$\chi_c\left(\overline{Cay}(R \setminus \{0\}, R \setminus \{0\})\right) = \left\lfloor \frac{|R|-1}{2} \right\rfloor.$$

Proof. Let u be a unit in R and $x \neq u$ be a non-zero element in R. Since $u(u^{-1}x) = x$, u is adjacent to x in $\overline{Cay}(R \setminus \{0\}, R \setminus \{0\})$. Therefore, $\{u, x\}$ is a total dominating set in $\overline{Cay}(R \setminus \{0\}, R \setminus \{0\})$. Note that, $|U(R)| \geq |Z^*(R)|$. So, there can be $|Z^*(R)|$ disjoint total dominating sets of the form $\{u, x : u \in U(R), x \in Z^*(R)\}$ and $\left\lfloor \frac{|U(R)|-|Z^*(R)|}{2} \right\rfloor$ disjoint total dominating sets of the form $\{u, u' : u, u' \in U(R)\}$. Hence,

$$\chi_c\left(\overline{Cay}(R \setminus \{0\}, R \setminus \{0\})\right) \geq |Z^*(R)| + \left\lfloor \frac{|U(R)|-|Z^*(R)|}{2} \right\rfloor = \left\lfloor \frac{|R|-1}{2} \right\rfloor.$$

Since any total dominating set must have at least two vertices,

$$\chi_c\left(\overline{Cay}(R \setminus \{0\}, R \setminus \{0\})\right) \leq \left\lfloor \frac{|R|-1}{2} \right\rfloor.$$

So, $\chi_c\left(\overline{Cay}(R \setminus \{0\}, R \setminus \{0\})\right) = \left\lfloor \frac{|R|-1}{2} \right\rfloor$. □

Assume that $n > 1$.

If $R \not\cong \mathbb{Z}_2$ is a field, then Γ_R^n is a union of n complete graphs. These n complete graphs are the graphs induced by C_i, for $i = 1, 2, \ldots, n$, where C_i is the set of all vertices whose first non-zero components are in the i^{th} place. Therefore,

$$\begin{aligned} \chi_c(\Gamma_R^n) &= \min\left\{\chi_c(K_{|C_i|}) : i = 1, 2, \ldots, n\right\} \\ &= \min\left\{\left\lfloor \frac{|C_i|}{2} \right\rfloor : i = 1, 2, \ldots, n\right\} \\ &= \left\lfloor \frac{|C_n|}{2} \right\rfloor = \left\lfloor \frac{|R|-1}{2} \right\rfloor, \end{aligned}$$

since $|C_i| = (|R|-1)|R|^{n-i}$.

For the coupon coloring number, the maximum number of disjoint total dominating sets of Γ_R^n has to be found. If R is not an integral domain, Selvakumar [53] proved that $\{X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n) : x_1 \in U(R), y_1 \in Z^*(R), y_2 \in U(R)\}$ is a total dominating set of Γ_R^n. Hereafter assume that R is a finite commutative ring which is not an integral domain. So, there exists at least one non-zero zero-divisor in R.

Theorem 4.2.6. *Let R be a finite commutative ring and $n > 1$ be a positive integer. Then*

$$\chi_c(\Gamma_R^n) \geq (|Z(R)| - 1)|U(R)||R|^{n-2}.$$

Proof. Let $X = (x_1, x_2, \dots, x_n), Y = (y_1, y_2, \dots, y_n)$, and let $k = (|Z(R)| - 1)|U(R)||R|^{n-2}$. Consider the two element sets of the form

$$A = \Big\{X, Y : x_1 \in U(R), x_2 \in Z^*(R), y_1 \in Z^*(R), y_2 \in U(R)\Big\}$$

Note that, for each pair $(a, b) \in U(R) \times Z^*(R)$,

$$\Big\{X = (a, b, x_3, x_4, \dots, x_n), Y = (b, a, x_3, x_4, \dots, x_n)\Big\}$$

are disjoint sets of the form A, for all $(x_3, x_4, \dots, x_n) \in R^{n-2}$. So, there are $|R|^{n-2}$ disjoint sets of the form A, for each $(a, b) \in U(R) \times Z^*(R)$. Hence, there are k sets of the form A. Name these k disjoint sets as $A_1, A_2, \dots, A_k$. Define $c : V(\Gamma_R^n) \to [k]$ by

$$c(X) = \begin{cases} i & \text{if } X \in A_i \\ 1 & \text{otherwise.} \end{cases}$$

Then c is a coupon coloring on Γ_R^n, since each A_i is a total dominating set of Γ_R^n. Hence, $\chi_c(\Gamma_R^n) \geq k$. □

Lemma 4.2.1. *Let R be a finite commutative ring and $n > 1$ be a positive integer. In any k-coupon coloring of Γ_R^n, each color should be given to at least a vertex of the form $X = (x_1, x_2, \dots, x_n)$ with $x_1 \in Z(R)$.*

Proof. Let H be the set of all vertices with first co-ordinate is a zero divisor. Suppose that there is a k-coupon coloring with none of the vertices of H has color 1. Then the vertex $Z = (0, z_2, \dots, z_n)$ has no neighbor with color 1. □

Suppose that Γ_R^n has a k-coupon coloring. Denote H_t as the set of all vertices with color $t \in \{1, 2, \dots, k\}$. Then H_t must be a total dominating set of Γ_R^n for all $t = 1, 2, \dots, k$.

Lemma 4.2.2. *Suppose that Γ_R^n has a k-coupon coloring. If $X = (0, x_2, \ldots, x_n) \in H_t$, then there is a $Y = (y_1, y_2, \ldots, y_n) \neq X$ with $y_1 \in Z(R)$ in H_t.*

Proof. Suppose that $X = (0, x_2, \ldots, x_n) \in H_t$ and $y_1 \notin Z(R)$ for all $Y = (y_1, y_2, \ldots, y_n) \neq X$ in H_t. That is, $y_1 \in U(R)$ for all $Y = (y_1, y_2, \ldots, y_n) \neq X$ in H_t. Then any vertex $Z = (z_1, z_2, \ldots, z_n)$ with $z_1 \neq 0$ in Γ_R^n is adjacent to a vertex Y with color t. But the vertex X has no neighbor with the color t, a contradiction. □

Theorem 4.2.7. *Let R be a finite commutative ring and $n > 1$ be a positive integer. Then*

$$\chi_c(\Gamma_R^n) \leq (|Z(R)| - 1)|R|^{n-1} + \left\lfloor \frac{|R|^{n-1}}{2} \right\rfloor.$$

Proof. Suppose that Γ_R^n has a k-coupon coloring. From Lemma 4.2.1, there is an $X = (x_1, x_2, \ldots, x_n) \in H_t$, with $x_1 \in Z(R)$ in every H_t. So, there can be at most $|Z(R)||R|^{n-1}$ such H_t's. By Lemma 4.2.2, if $X = (0, x_2, \ldots, x_n) \in H_t$, then there is a $Y = (y_1, y_2, \ldots, y_n) \neq X$ with $y_1 \in Z(R)$ in H_t. Thus to get the maximum number of color classes H_t, each H_t must contain either a vertex with non-zero zero-divisor in first co-ordinate or two vertices with 0 in their first co-ordinate, but not both. Hence, there are at most $(|Z(R)| - 1)|R|^{n-1} + \left\lfloor \frac{|R|^{n-1}}{2} \right\rfloor$ distinct color classes. □

Theorem 4.2.8. *Suppose that $Z(R)$ is an ideal of the finite commutative ring R and $n > 1$ is a positive integer. If Γ_R^n has a k-coupon coloring, then there exist a vertex $X = (x_1, x_2, \ldots, x_n)$, $x_1 \in Z^*(R), x_2 \in U(R)$ or a vertex $X' = (0, x'_2, \ldots, x'_n)$ in every H_t.*

Proof. By Lemma 4.2.1, H_t must have a vertex with zero-divisor in its first co-ordinate. Assume that $y_1 \neq 0$ for every $Y = (y_1, y_2, \ldots, y_n)$ in H_t. Suppose that the second co-ordinate is also a zero-divisor for every $Y = (y_1, y_2, \ldots, y_n)$ in H_t with $y_1 \in Z^*(R)$. Consider $Z = (z_1, z_2, \ldots, z_n), z_1 = 0, z_2 \in U(R)$. Then there must be an $X = (x_1, x_2, \ldots, x_n) \in H_t$ with

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ b & c & \cdots & 0 \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \tag{4.1}$$

or

$$\begin{pmatrix} a' & 0 & \cdots & 0 \\ b' & c' & \cdots & 0 \\ & & \vdots & \\ a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{4.2}$$

Equation (2) is not possible, since x_1 is non-zero. Since $a \neq 0$, $x_1 \in Z^*(R)$. Then there exists $a \in Z^*(R)$ such that $ax_1 = 0$. But $bx_1 + cx_2 = z_2$ cannot be a unit, since $x_1, x_2 \in Z(R)$ and $Z(R)$ is an ideal of R, $z_2 = bx_1 + cx_2 \in Z(R)$, a contradiction. □

Next corollary follows from the proof of Theorem 4.2.7 and Theorem 4.2.8.

Corollary 4.2.1. *Let R be a finite commutative ring and $n > 1$ be a positive integer. If $Z(R)$ is an ideal of R, then*

$$\chi_c(\Gamma_R^n) \leq (|Z(R)| - 1)|U(R)||R|^{n-2} + \left\lfloor \frac{|R|^{n-1}}{2} \right\rfloor.$$

Example 1. *Consider a finite commutative ring R with $Z(R)$ is an ideal of R and $n = 2$. Let $r = (|Z(R)| - 1)|U(R)|$, $s = \left\lfloor \frac{|U(R)|}{2} \right\rfloor$.*

$$A = \left\{ X = (x_1, x_2), Y = (y_1, y_2) : x_1, y_2 \in Z^*(R), y_1, x_2 \in U(R) \right\}$$

$$B = \left\{ X = (0, x_2), Y = (0, y_2), Z = (x_2, 0), V = (y_2, 0) : x_2, y_2 \in U(R), x_2 \neq y_2 \right\}$$

Note that, there can be r disjoint sets of the form A and name them as A_i, $i = 1, 2, \ldots, r$. Similarly, there can be s disjoint sets of the form B and name them as B_j for $j = 1, 2, \ldots, s$. Clearly, A_i is a total dominating set for all $i = 1, 2, \ldots, r$. We claim that B_j is also a total dominating set of Γ_R^n. For, let $(a, b) \in V(\Gamma_R^n)$. If $a \neq 0$, then $(a, b) \neq Z$ is adjacent to Z in B_j and Z is adjacent to V. If $a = 0$, then $b \neq 0$ and

$$\begin{pmatrix} b & 0 \\ 0 & x_2^{-1}b \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

So, if $b \neq x_2$, (a,b) is adjacent to $(0,x_2)$ and if $b = x_2$, then (a,b) is adjacent to $(0,y_2)$,

$$\begin{pmatrix} x_2 & 0 \\ 0 & x_2^{-1}y_2 \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_2 \end{pmatrix}.$$

Thus B_j is a total dominating set for all j. Note that $A_i \cap B_j = \phi$ for all i and j. So there are $r+s$ disjoint total dominating sets and

$$\chi_c(\Gamma_R^2) \geq (|Z(R)|-1)|U(R)| + \left\lfloor \frac{|U(R)|}{2} \right\rfloor.$$

By the above corollary,

$$(|Z(R)|-1)|U(R)| + \left\lfloor \frac{|U(R)|}{2} \right\rfloor \leq \chi_c(\Gamma_R^2) \leq (|Z(R)|-1)|U(R)| + \left\lfloor \frac{|R|}{2} \right\rfloor,$$

since $n = 2$.

If $R = \mathbb{Z}_4$, then $Z(R) = \{0,2\}$, $U(R) = \{1,3\}$ and $3 \leq \chi_c(\Gamma_{\mathbb{Z}_4}^2) \leq 4$. Here, $A_1 = \{(2,1),(1,2)\}$, $A_2 = \{(2,3),(3,2)\}$, $B_1 = \{(0,1),(0,3),(1,0),(3,0)\}$. Note that, $C = \{(0,2),(2,2),(1,1)\}$ is also a total dominating set. Therefore, $\chi_c(\Gamma_{\mathbb{Z}_4}^2) = 4$. Hence, the bound in Corollary 4.2.1 is sharp.

4.3 Coupon coloring of zero-divisor graphs

In this section, coupon coloring of the two zero-divisor graphs, $\Gamma(R)$ and $\Gamma_I(R)$, were discussed. An upper found for coupon coloring number of $\Gamma(R)$ were found and the upper bound is strict in some special cases. The coupon coloring number of $\Gamma_I(R)$ were found in terms of coupon coloring number of $\Gamma(R)$.

4.3.1 Coupon Coloring of $\Gamma(R)$

Suppose that p_i denote a prime number for all i and $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, where $n \neq p_1^{\alpha_1}$, $n \neq 2p_1$. In 2008, AbdAlJawad [3] found the domination number of $\Gamma(\mathbb{Z}_n)$ and the idea of his

[1]The sections 4.3.1 and 4.3.2 of this chapter have been published in *South East Asian Journal of Mathematics and Mathematical Sciences*, Vol. 21, Proceedings (2022) 183 – 190, [63].

proof can be used to found the coupon coloring number of $\Gamma(\mathbb{Z}_n)$.

Theorem 4.3.1. *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, $p_1 < p_2 < \cdots < p_k$. *Then*

$$\chi_c(\Gamma(\mathbb{Z}_n)) = (p_1 - 1).$$

Proof. Let $D_i = \{ip_1^{\alpha_1} p_2^{\alpha_2} \dots p_j^{\alpha_j - 1} \dots p_k^{\alpha_k} : j = 1, 2, \dots, k\}$, $i = 1, 2, \dots, p_1 - 1$. Then D_i's are $p_1 - 1$ disjoint TDSs. If x is any zero-divisor of $\mathbb{Z}_n$, then x is divisible by p_j for some $j \in \{1, 2, \dots, k\}$ and so x is adjacent to $ip_1^{\alpha_1} p_2^{\alpha_2} \dots p_j^{\alpha_j - 1} \dots p_k^{\alpha_k}$. Since $p_1 \in V(\Gamma(\mathbb{Z}_n))$, every TDS must contain an element which is a multiple of $\frac{n}{p_1}$. So, there are exactly $p_1 - 1$ disjoint TDSs. Hence, $\chi_c(\Gamma(\mathbb{Z}_n)) = (p_1 - 1)$. □

Theorem 4.3.2. *Let R be a finite commutative ring with identity $1 \neq 0$ and $R \cong R_1 \times \cdots \times R_n$ be the local ring decomposition of R. Then*

$$\chi_c(\Gamma(R)) \geq \min_j \left\{ |Ann(Z(R_j))| - 1 \right\}.$$

Proof. Suppose that, for $j = 1, 2, \dots, n$,

$$Ann(Z(R_j)) = \{0, r_{1,j}, r_{2,j}, \dots, r_{k,j}\},$$

where $k = \min_j \left\{ |Ann(Z(R_j))| - 1 \right\}$. Note that, $|Ann(Z(R_j))| > k$ for all j.

Define $D_i = \{(0, \dots, 0, r_{i,j}, 0, \dots, 0) : j = 1, 2, \dots n\}$. Then D_i is a total dominating set of $\Gamma(R)$. For let $x \neq 0$ be any zero divisor of R. Then $x = (x_1, \dots, x_n)$ such that $x_j \in Z(R_j) = Ann(r_{i,j})$ for some $j \in \{1, 2, \dots, n\}$ and for all $i = 1, 2, \dots, k$. Therefore, for all $i = 1, 2, \dots, k$, $x_j \cdot r_{i,j} = 0$ and

$$(x_1 \dots, x_n) \cdot (0, \dots, 0, r_{i,j}, 0, \dots, 0) = (0, \dots, 0).$$

Thus x is adjacent to a vertex in D_i for all $i = 1, 2, \dots, k$. Hence, D_i is a total dominating set for all $i = 1, 2, \dots, k$ and $\chi_c(\Gamma(R)) \geq k$. □

Lemma 4.3.1. *Let R be a finite commutative ring with identity $1 \neq 0$ and $R \cong R_1 \times \cdots \times R_n$ be the local ring decomposition of R. Then every total dominating set of $\Gamma(R)$ must contain at least one element from $Ann(Z(R_i))$ in the i^{th} co-ordinate.*

Proof. Suppose that there is a total dominating set D of $\Gamma(R)$ does not have an element from $Ann(Z(R_i))$ in the i^{th} co-ordinate. That is, for all $y = (y_1, y_2, \ldots, y_n)$ in D, $y_i \notin Ann(Z(R_i))$. Then there exists $r_i \in Z(R_i)$ such that $y_i \cdot r_i \neq 0$ for all $y \in D$. Let $(1, \ldots, 1, r_i, 1, \ldots, 1) \in D$. Then $y \cdot (1, \ldots, 1, r_i, 1, \ldots, 1) \neq 0$ for all $y \in D$. So, D cannot be a total dominating set, a contradiction. □

Theorem 4.3.3. *Suppose that R is a finite commutative ring with identity $1 \neq 0$ and $R \cong R_1 \times \cdots \times R_n$ is the local ring decomposition of R. If $Ann(Z(R_i)) = Ann(x_i)$ for some $x_i \in R_i$, for all $i = 1, 2, \ldots, n$, then*

$$\chi_c(\Gamma(R)) = \min_j \left\{|Ann(Z(R_j))| - 1\right\}.$$

Proof. Suppose that $Ann(Z(R_i))$ is an annihilator ideal of an element x_i for all $i = 1, 2, \ldots, n$. Consider $(1, \ldots, 1, x_i, 1, \ldots, 1) \in Z(R)^*$. Then, there must exist a vertex $(0, \ldots, 0, r, 0, \ldots, 0)$, where $r \in Ann(Z(R_i)) \setminus \{0\}$, in every total dominating set of $\Gamma(R)$ so that

$$(1, \ldots, 1, x_i, 1, \ldots, 1) \cdot (0, \ldots, 0, r, 0, \ldots, 0) = (0, 0, \ldots, 0).$$

Thus, $\chi_c(\Gamma(R)) \leq \min_j \left\{|Ann(Z(R_j))| - 1\right\}$. By Theorem 4.3.2,

$$\chi_c(\Gamma(R)) = \min_j \left\{|Ann(Z(R_j))| - 1\right\}.$$

This completes the proof. □

Observe that, from the above theorem, $\chi_c(\Gamma(\mathbb{Z}_n)) = (p_1 - 1)$.

4.3.2 Coupon Coloring of $\Gamma_I(R)$

Throughout this section, R denotes a commutative ring with identity and I denotes a proper ideal of R. The graph $\Gamma_I(R)$ can be constructed using the adjacency relation in $\Gamma(R/I)$. In this section, exact coupon coloring number of $\Gamma_I(R)$ were found in terms of coupon coloring number of $\Gamma(R/I)$.

Note that,

$$\begin{aligned} x \in V(\Gamma_I(R)) &\iff \exists y \in R \setminus I \text{ such that } xy \in I \\ &\iff \exists (y+I) \in (R/I) \setminus \{I\} \text{ such that} \\ &\qquad (x+I)(y+I) = xy + I = I \\ &\iff x + I \in V(\Gamma(R/I)) \end{aligned}$$

Lemma 4.3.2. *If $D = \{x_1, x_2, \ldots, x_t\}$ is a TDS of $\Gamma_I(R)$, then $D' = \{x_1+I, x_2+I, \ldots, x_t+I\}$ is a TDS of $\Gamma(R/I)$.*

Proof. Let $D = \{x_1, x_2, \ldots, x_t\}$ be a TDS of $\Gamma_I(R)$. If possible, suppose that $D' = \{x_1 + I, x_2 + I, \ldots, x_t + I\}$ is not a TDS of $\Gamma(R/I)$. Then there is a $y + I \in V(\Gamma(R/I))$ such that $y+I$ is not adjacent to x_i+I for all $i = 1, 2, \ldots, t$. So, $yx_i \notin I$ for all $i = 1, 2, \ldots, t$. Therefore, there exists $y \in V(\Gamma_I(R))$ such that y is not adjacent to any of the vertices of D. Hence, D cannot be a TDS of $\Gamma_I(R)$, a contradiction. □

Theorem 4.3.4. *Let $\Gamma_I(R)$ be a graph without isolated vertices. Then*

$$\gamma_t(\Gamma_I(R)) = \gamma_t(\Gamma(R/I)).$$

Proof. Suppose that $\gamma_t(\Gamma(R/I)) = k$. Let $T = \{x_1 + I, x_2 + I, \ldots, x_k + I\}$ be a TDS of $\Gamma(R/I)$. Claim that $S = \{x_1, x_2, \ldots, x_k\}$ is a TDS of $\Gamma_I(R)$. For, let $y \in V(\Gamma_I(R))$. Then $y \in y + I \in V(\Gamma(R/I))$ there exists $x_i + I \in T$ such that $(y + I)$ is adjacent to $(x_i + I)$ in $\Gamma(R/I)$. So, there exists $x_i \in S$ such that y is adjacent to x_i in $\Gamma_I(R)$. Hence the claim. Thus $\gamma_t(\Gamma_I(R)) \leq k$.

Now, let $S' = \{x_1, x_2, \ldots, x_l\}$ be any TDS of $\Gamma(R/I)$. Then by Lemma 4.3.2, $T' = \{x_1 + I, x_2 + I, \ldots, x_l + I\}$ is a TDS of $\Gamma(R/I)$. Since $\gamma_t(\Gamma(R/I)) = k$, $l \geq k$. That is, $\gamma_t(\Gamma_I(R)) \geq k$. □

Theorem 4.3.5. *Let $\Gamma(R/I)$ be a graph without isolated vertices. Then*

$$\chi_c(\Gamma_I(R)) = \chi_c(\Gamma(R/I))|I|.$$

Proof. Suppose that $I = \{r_1, r_2, \ldots, r_m\}$, $\chi_c(\Gamma(R/I)) = k$ and c be a k-coupon coloring of

$\Gamma(R/I)$. Let $x \in V(\Gamma_I(R))$. Then $x = a_\lambda + r_i$, for some $i \in \{1, 2, \ldots, m\}$ and a_λ is a coset representative of the vertices of $\Gamma(R/I)$. Define $c' : V(\Gamma_I(R)) \to [km]$ by

$$c'(x) = c'(a_\lambda + r_i) = [c(a_\lambda + I) - 1] + i.$$

Let $y \in V(\Gamma_I(R))$. To show that c' is a coupon coloring, it is enough to prove that y is adjacent to a vertex, say x, with color $c(x) = [c(a_\lambda + I) - 1] + i$. Note that, $y = a_\omega + r_j$, for some $j \in \{1, 2, \ldots, m\}$ and a_ω is a coset representative of the vertices of $\Gamma(R/I)$. Since $\Gamma(R/I)$ has the k-coupon coloring c and $a_\omega + I \in V(\Gamma(R/I))$, there exists $a_\delta + I \in V(\Gamma(R/I))$ such that $a_\omega + I$ is adjacent to $a_\delta + I$ and $c(a_\delta + I) = c(a_\lambda + I)$. Then $z = a_\delta + r_i \in V(\Gamma_I(R))$ and

$$yz = (a_\omega + r_j)(a_\delta + r_i) = a_\omega a_\delta + a_\omega r_i + r_j a_\delta + r_j r_i \in I,$$

since I is an ideal and $a_\omega + I$ is adjacent to $a_\delta + I$ in $\Gamma(R/I)$. So, y is adjacent to z and

$$c'(z) = c'(a_\delta + r_i) = [c(a_\delta + I) - 1] + i = [c(a_\lambda + I) - 1] + i.$$

Hence c' is a coupon coloring of $\Gamma_I(R)$ and $\chi_c(\Gamma_I(R)) \geq \chi_c(\Gamma(R/I))|I|$.

By Lemma 4.3.2 and Theorem 4.3.4, $D = \{x_1, x_2, \ldots, x_t\}$ is a TDS of $\Gamma_I(R)$ if and only if $D' = \{x_1 + I, x_2 + I, \ldots, x_t + I\}$ is a TDS of $\Gamma(R/I)$. So, there are at most $\chi_c(\Gamma(R/I))|I|$ disjoint TDSs for $\Gamma_I(R)$. Hence, $\chi_c(\Gamma_I(R)) \leq \chi_c(\Gamma(R/I))|I|$. □

From Theorem 4.3.5, it is clear that the coupon coloring number of $\Gamma_I(R)$ can be found if the coupon coloring number of $\Gamma(R/I)$ is known. Using Theorem 4.3.3, the following corollary is obtained.

Corollary 4.3.1. *Suppose that R is a finite commutative ring with identity $1 \neq 0$ and $R/I \cong R_1 \times \cdots \times R_n$ is the local ring decomposition of R/I. If $Ann(Z(R_i)) = Ann(x_i)$ for some $x_i \in R_i$, for all $i = 1, 2, \ldots, n$, then*

$$\chi_c(\Gamma_I(R)) = \min_j \{|Ann(Z(R_j))| - 1\}|I|.$$

Note that, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, $p_1 < p_2 < \cdots < p_k$ and $I = < p_1 > \times < p_2 > \times \cdots \times < p_k >$, then $\mathbb{Z}_n/I$ is isomorphic to the ring $\mathbb{Z}_{p_1 p_2 \ldots p_k}$ and $\chi_c(\Gamma(\mathbb{Z}_n/I)) = p_1 - 1$. The following corollary can be obtained using Theorem 4.3.5.

Corollary 4.3.2. *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ *and* $< p_1 > \times < p_2 > \times \cdots \times < p_k >$ *be the ideal* I *of* $\mathbb{Z}_n$. *Then*

$$\chi_c(\Gamma_I(\mathbb{Z}_n)) = (p_1 - 1) p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \dots p_k^{\alpha_k - 1}.$$

Corollary 4.3.3. *Let* I *be an ideal of* $\mathbb{Z}_n$ *and* p *be the least prime divisor of* m, *where* $\mathbb{Z}_m \cong \mathbb{Z}_n / I$. *Then*

$$\chi_c(\Gamma_I(\mathbb{Z}_n)) = (p - 1)|I|.$$

Chapter 5

Locally Identifying Coloring of Rooted Product and Corona Product of Graphs

5.1 Introduction

Recall the definition of locally identifying coloring.

A vertex coloring $c : V(G) \to \mathbb{N}$ of a graph G is said to be locally identifying if c is a proper coloring and $c(N[u]) \neq c(N[v])$ for all adjacent vertices u and v with $N[u] \neq N[v]$. The smallest integer k for which G admits a lid-coloring is called the lid-chromatic number of G, denoted $\chi_{lid}(G)$.

In [17], authors showed that for biconvex bipartite graphs $\chi_{lid}(G)$ can be computed in polynomial time. They also studied the lid-coloring of Cartesian product and lexicographic product of graphs. In this section, the lid-coloring of rooted product of graphs have been studied. The rooted product of two graphs was introduced by Godsil and McKay in 1978 [39]. The rooted product of two graphs G and H is defined as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and identifying the i^{th} vertex of G with the root vertex v in the i^{th} copy of H for every $i = 1, 2, \ldots, |V(G)|$. It is denoted by $G \circ_v H$.

[1]The section 5.3 has been included in *Discrete Mathematics, Algorithms and Applications*, 2450032, 2024, [57].

5.2 Lid-coloring of rooted product graphs

To begin with locally identifying coloring of rooted product of paths and cycles were discussed. Later, some connections of the lid-chromatic number of $G \circ_v H$ with $\chi_{lid}(G)$ and $\chi_{lid}(H)$ have been found.

5.2.1 Lid-coloring of rooted product of paths and cycles

In this section, the lid-chromatic numbers of $P_m \circ_v P_n$, $C_n \circ_v P_m$, $P_n \circ_v C_m$ and $C_n \circ_v C_m$ have been obtained. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. Note that, in $G \circ_v H$, there are n copies of H. The vertex set of $G \circ_v H$ is defined as $V(G \circ_v H) = V(G) \times V(H)$ and the edge set

$$E(G \circ_v H) = \bigcup_{i=1}^{n} \{(u_i, x)(u_i, y) \mid xy \in E(H)\} \cup \{(u_i, v)(u_j, v) \mid u_i u_j \in E(G)\}.$$

Remark 5.2.1. *Let G and H be two connected graphs. Then $N[(u_i, v)] \neq N[(u_j, v)]$ in $G \circ_v H$, for all adjacent vertices u_i and u_j in G.*

Theorem 5.2.1. *For any pair of positive integers m and n, $\chi_{lid}(P_m \circ_v P_n) = 4$.*

Proof. If (u_i, x) and (u_j, x) are two leaves in $P_m \circ_v P_n$, then

$$\begin{aligned} dist((u_i, x), (u_j, x)) &= dist((u_i, x), (u_i, v)) + dist((u_j, x), (u_j, v)) + dist((u_i, v), (u_j, v)) \\ &= 2dist((u_i, x), (u_i, v)) + dist((u_i, v), (u_j, v)). \end{aligned}$$

So, $dist((u_i, x), (u_j, x))$ can be odd for some leaves (u_i, x) and (u_j, x). By Theorem 2.4.9, a tree T with at least 3 vertices is 3-lid-colorable if and only if the distance between every two leaves is even. Therefore, $\chi_{lid}(P_m \circ_v P_n) \geq 4$. Since $P_m \circ_v P_n$ is bipartite, by Theorem 2.4.8 $\chi_{lid}(P_m \circ_v P_n) = 4$. □

In Theorem 5.2.2 and Theorem 5.2.3, we use number sequences to define the lid-coloring with the following properties: A sequence M in bracket, $[M]$, means that we can take the sequence M once, the sequence $(M)^*$ means that we can repeat sequence M as many times as we need (or not use it at all).

Theorem 5.2.2. *For any pair of positive integers m and $n \geq 4$,*

$$\chi_{lid}(C_n \circ_v P_m) = 4.$$

Proof. Suppose that $\chi_{lid}(C_n \circ_v P_m) = 3$. Then for any adjacent vertices x and y, $|c(N[x])| = 2$ and $|c(N[y])| = 3$. Let $C_n : u_1 - u_2 - \cdots - u_n - u_1$, $P_m : v_1 - v_2 - v_3 \cdots - v_m$ and $v = v_1$ be the root vertex. If $|c(N[(u_1, v)])| = 2$ and $|c(N[(u_2, v)])| = 3$, then the copies of P_m in $C_n \circ_v P_m$ having root vertices (u_1, v) and (u_2, v) cannot have a 3-lid-coloring simultaneously, since m is a fixed number. Thus $\chi_{lid}(C_n \circ_v P_m) \geq 4$.

If n is even, then $C_n \circ_v P_m$ is a bipartite graph and so $\chi_{lid}(C_n \circ_v P_m) \leq 4$. If n is odd, then color the copy of C_n with $(1234)^*[2]$ or $(1234)^*[124]$, and color the vertices of copy of P_m with $(u_i, v), i \neq n$ in such a way that if $c((u_i, v)) = k$, then $c(u_i, v_j) = k + j - 1 \pmod 4$ for all $j = 2, 3, \ldots, m-1$. Note that, $c((u_n, v))$ is either 2 or 4. If $c((u_n, v)) = 2$, then $c((u_n, v_2)) = 1$, $c((u_n, v_3)) = 3$, $c((u_n, v_4)) = 4$, $c((u_n, v_5)) = 2$ and so on. If $c((u_n, v)) = 4$, then $c((u_n, v_2)) = 3$, $c((u_n, v_3)) = 1$, $c((u_n, v_4)) = 2$, $c((u_n, v)) = 4$ and so on. Therefore, $\chi_{lid}(C_n \circ_v P_m) \leq 4$. □

Theorem 5.2.3. *For any pair of positive integers $m \geq 2$ and $n \geq 3$,*

$$\chi_{lid}(P_m \circ_v C_n) = \begin{cases} 4, & \text{if } n = 3 \\ \chi_{lid}(C_n), & \text{otherwise.} \end{cases}$$

Proof. Let $P_m : u_1 - u_2 - \cdots - u_m$, $C_n : v_1 - v_2 - v_3 \cdots - v_n - v_1$ and c be a lid-coloring of $P_m \circ_v C_n$, where $v = v_1$. If $n = 3$, then $|c(N[(u_i, v)])| \geq 3$ for all $i = 1, 2, \ldots, m$. So, by Remark 5.2.1, $\chi_{lid}(P_m \circ_v C_3) \geq 4$. Coloring the copies of C_3 in $P_m \circ_v C_3$ with colors $\{1, 2, 3\}$ and $\{2, 3, 4\}$ such that root vertices have colors 1 and 2 alternatively, then $\chi_{lid}(P_m \circ_v C_3) = 4$.

Suppose that $n \geq 4$.

Case 1: $|c(N[(u_i, v)])| = 2$

Without loss of generality assume that $c((u_i, v)) = 1$, $c((u_i, v_2)) = 2$ and $c((u_{i+1}, v)) = 2$.

Then the only possible 3-lid-coloring is defined as follows:

$$c((u_i, v_j)) = \begin{cases} 1, \text{ if } & j \equiv 1 \pmod 4 \\ 2, \text{ if } & j \equiv 0, 2 \pmod 4 \\ 3, \text{ if } & j \equiv 3 \pmod 4. \end{cases}$$

If $n \equiv 1 \pmod 4$, then $c((u_i, v_n)) = 1$. So, c cannot be a proper coloring, since (u_i, v) is adjacent to (u_i, v_n) and $c((u_i, v)) = 1$.

If $n \equiv 2 \pmod 4$, then $n - 1 \equiv 1 \pmod 4$ and so, $c((u_i, v_n)) = 2$ and $c((u_i, v_{n-1})) = 1$. Hence, $c(N[(u_i, v)]) = \{1, 2\} = c(N[(u_i, v_{n-1})])$, which is a contradiction, since $N[(u_i, v)] \neq N[(u_i, v_{n-1})]$. If $n \equiv 3 \pmod 4$, then $c((u_i, v_n)) = 3$ and $c(N[(u_i, v)]) = \{1, 2, 3\}$, which is not possible. Therefore $\chi_{lid}(P_m \circ_v C_n) \geq 4$, if $n \not\equiv 0 \pmod 4$.

Case 2: $|c(N[(u_i, v)])| = 3$

Let c be a 3-lid coloring of $P_m \circ_v C_n$. Then $|c(N[(u_{i+1}, v)])| = 2$, which is same as case 1.

From Case 1 and Case 2, $\chi_{lid}(P_m \circ_v C_n) \geq 3$, if $n \equiv 0 \pmod 4$ and $\chi_{lid}(P_m \circ_v C_n) \geq 4$, otherwise.

Let $n \equiv 0 \pmod 4$. Color the i^{th} copy of C_n with $(1232)^*$ and the $(i+1)^{th}$ copy of C_n with $(2321)^*$ in such a way that the coloring starts from the root vertex. Then it is a 3-lid-coloring of $P_m \circ_v C_n$. Therefore, $\chi_{lid}(P_m \circ_v C_n) = 3$, if $n \equiv 0 \pmod 4$.

Since $\chi_{lid}(C_5) = \chi_{lid}(C_7) = 5$, the copy of C_n in $P_m \circ_v C_5$ and $P_m \circ_v C_7$ can have a 4-lid coloring, if the root vertex (u_i, v) satisfies $c(N[(u_i, v)]) = \{1, 2, 3, 4\}$. Then $c(N[(u_{i+1}, v)]) = \{1, 2, 3, 4\} = c(N[(u_i, v)])$, which is not possible.

If $n \equiv 1 \pmod 4$, $n \neq 5$, then color the i^{th} copy of C_n with $[124341232](1232)^*$ and the $(i+1)^{th}$ copy of C_n with $[243412321](2321)^*$, starting from the root vertex. So, there is a 4-lid-coloring of $P_m \circ_v C_n$ and $\chi_{lid}(P_m \circ_v C_n) = 4$.

If $n \equiv 2 \pmod 4$, then color the i^{th} copy of C_n with $[42](1232)^*$ and the $(i+1)^{th}$ copy of C_n with $[24](2321)^*$. Similarly, if $n \equiv 3 \pmod 4$, $n \neq 7$ color the i^{th} copy of C_n with $[12434123242](1232)^*$ and the $(i+1)^{th}$ copy of C_n with $[24341232421](2321)^*$. Note that the coloring starts from the root vertex. Hence, there is a 4-lid-coloring for $P_m \circ_v C_n$ and so, $\chi_{lid}(P_m \circ_v C_n) = 4$. □

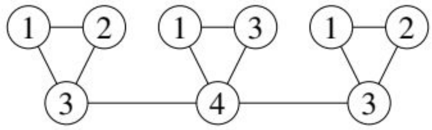


Figure 5.1: A 4-lid-coloring of $P_3 \circ_v C_3$

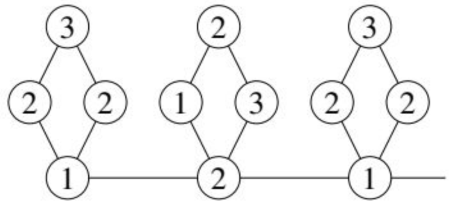


Figure 5.2: A 3-lid-coloring of $P_m \circ_v C_4$

$C_m \circ_v C_n$ is the graph obtained by adding exactly one edge to $P_m \circ_v C_n$. But, the lid-chromatic number of $C_m \circ_v C_n$ is different from $\chi_{lid}(P_m \circ_v C_n)$ when $m = 3$ and $m \neq 3$ is odd, $n \equiv 0 \pmod 4$.

Theorem 5.2.4. *For any pair of positive integers $m \geq 3$ and $n \geq 3$,*

$$\chi_{lid}(C_m \circ_v C_n) = \begin{cases} 5, & \text{if } m = 3 \\ 4, & \text{if } m \neq 3, n = 3 \\ & \quad \text{or } m \neq 3 \text{ is odd, } n \equiv 0 \pmod 4 \\ \chi_{lid}(C_n), & \text{otherwise.} \end{cases}$$

Proof. Let $C_m : u_1 - u_2 - \cdots - u_m - u_1$, $C_n : v - v_2 - v_3 \cdots - v_n - v$ and c be a lid-coloring of $C_m \circ_v C_n$. If m is an even positive integer, define the same colorings defined in the proof of Theorem 5.2.3. So, $\chi_{lid}(C_m \circ_v C_n) = \chi_{lid}(C_n)$, if m is an even positive integer.

If $m = 3$, then for the three vertices $(u_1, v), (u_2, v), (u_3, v)$ in the copy of C_3, $c((u_1, v)) = 1, c((u_2, v)) = 2, c((u_3, v)) = 3$ and $\{1, 2, 3\} \subseteq c(N[(u_i, v)])$. By Remark 5.2.1, $c(N[(u_1, v)])$, $c(N[(u_2, v)])$ and $c(N[(u_3, v)])$ are mutually disjoint. So, there must be at least two colors different from 1,2 and 3.

Note that, if $\chi_{lid}(G) = 3$, then for any adjacent vertices x and y in graph G, $|c(N[x])| = 2$ and $|c(N[y])| = 3$. If m is odd, then the number of colors in the neighborhood of first and last vertices of the copy of C_m will contain either all the three colors or two similar colors. So,

$\chi_{lid}(C_m \circ_v C_n) \geq 4$ if m is odd. □

5.2.2 Relation between $\chi_{lid}(G \circ_v H)$, $\chi_{lid}(G)$ and $\chi_{lid}(H)$

In this section, some bounds for $\chi_{lid}(G \circ_v H)$ have been obtained. It cannot true that $\chi_{lid}(G \circ_v H) \geq \chi_{lid}(G)$. eg: $\chi_{lid}(P_4 \circ_v C_4) = 3$, but $\chi_{lid}(P_4) = 4$. Similarly, it is not true that $\chi_{lid}(G \circ_v H) \leq \chi_{lid}(G)$ or $\chi_{lid}(G \circ_v H) \leq \chi_{lid}(H)$. eg: $\chi_{lid}(C_3 \circ C_4) = 5$, but $\chi_{lid}(C_3) = 3 = \chi_{lid}(C_4)$.

Note that the lid-chromatic number of a graph G is the maximum of lid-chromatic numbers of its connected components. So, if G is totally disconnected graph, then $\chi_{lid}(G \circ_v H) = \chi_{lid}(H)$. If H is totally disconnected graph, then $\chi_{lid}(G \circ_v H) = \chi_{lid}(G)$. Therefore, consider the graphs with at least one edge.

Theorem 5.2.5. *Let G and H be two graphs. Then* $\chi_{lid}(G \circ_v H) \leq \chi_{lid}(G) + \chi_{lid}(H) - 1$.

Proof. Suppose that $\chi_{lid}(G) = r$ and $\chi_{lid}(H) = s$. Let f be an r-lid-coloring of G and g be an s-lid-coloring of H. Define $c : V(G \circ_v H) \to [r + s - 1]$ by

$$c((x,y)) = \begin{cases} g(y), & \text{if } y \neq v \\ f(x) + s - 1, & \text{if } y = v. \end{cases}$$

Then c is a lid-coloring of $G \circ_v H$. □

Theorem 5.2.6. *Suppose that G and H be two connected graphs and H has at least 3 vertices. If* $\chi_{lid}(G \circ_v H) = 3$*, then* $\chi_{lid}(H) = 3$.

Proof. Suppose that $\chi_{lid}(G \circ_v H) = 3$ and c be a 3-lid-coloring of $G \circ_v H$. Let u_i and u_j be two adjacent vertices in G. By Remark 5.2.1, $c(N[(u_i, v)]) \neq c(N[(u_j, v)])$ in $G \circ_v H$. Note that, $|c(N[(u_i, v)])|$ and $|c(N[(u_j, v)])|$ can be either 2 or 3. WLOG $c((u_i, v)) = 1, c((u_j, v)) = 2, c(N[(u_i, v)]) = \{1, 2\}$, and $c(N[(u_j, v)]) = \{1, 2, 3\}$. Then, $c(N[(u_i, x_t)]) = \{1, 2, 3\}$ for all vertices x_t in H which are adjacent to v, and $c((u_i, x_t)) = 2$ for all t. Define the coloring f on H by $f(x) = c(u_i, x)$. Then f is a 3-lid-coloring of H. Therefore, $\chi_{lid}(H) \leq 3$. By Theorem 2.4.7, $\chi_{lid}(H) = 3$. □

If $\chi_{lid}(G \circ_v H) = 3$, then there is no relation between $\chi_{lid}(G \circ_v H)$ and $\chi_{lid}(G)$. eg: $\chi_{lid}(P_2 \circ_v C_4) = \chi_{lid}(P_3 \circ_v C_4) = \chi_{lid}(P_4 \circ_v C_4) = 3$, but $\chi_{lid}(P_2) = 2$, $\chi_{lid}(P_3) = 3$, $\chi_{lid}(P_4) = 4$.

The converse of the above theorem does not holds in general. For example, $\chi_{lid}(C_3) = 3$ but $\chi_{lid}(P_3 \circ_v C_3) = 4$.

Theorem 5.2.7. *If G and H are connected bipartite graphs, then $\chi_{lid}(G \circ_v H) \geq \chi_{lid}(H)$.*

Proof. Suppose that G and H be bipartite graphs. In $G \circ_v H$, there are n copies of H. Let $\{A, B\}$ be the partite sets of G and let $\{U_i, V_i\}$ be the partite sets of H_i, for $i = 1, 2, \ldots, n$. Then $\{X, Y\}$ is the partite sets of $G \circ_v H$, where

$$\begin{aligned} X &= \{(x, y) \mid x \in A \text{ and } y \in U_i, x \in B \text{ and } y \in V_i\} \\ Y &= \{(x, y) \mid x \in A \text{ and } y \in V_i, x \in B \text{ and } y \in U_i\}. \end{aligned}$$

Hence, $G \circ_v H$ is bipartite and $\chi_{lid}(G \circ_v H) \leq 4$. So, by Theorem 5.2.6, $\chi_{lid}(G \circ_v H) \geq \chi_{lid}(H)$. □

Remark 5.2.2. *The converse of Theorem 5.2.6 holds when G is bipartite graph and $H = C_n, n \equiv 0 \pmod 4$.*

Proof. Let G be a bipartite graph with partite sets $\{A, B\}$ and $H = C_n, n \equiv 0 \pmod 4$. Since $\chi_{lid}(H) = 3$, H has a 3-lid-coloring. Let c_1 and c_2 be the 3-lid-colorings $(1232)^*$ and $(2321)^*$ of C_n respectively. Note that, both colorings starts from the root vertex v. Now define $c : V(G \circ_v H) \to \{1, 2, 3\}$ on $G \circ_v H$ by

$$c((x, y)) = \begin{cases} c_1(y), & \text{if } x \in A \\ c_2(y), & \text{if } x \in B. \end{cases}$$

Since c is a 3-lid-coloring of $G \circ_v H$, $\chi_{lid}(G \circ_v H) \leq 3$. If $\chi_{lid}(H) = 3$, then $\chi_{lid}(G \circ_v H) \geq 3$ and so $\chi_{lid}(G \circ_v H) = 3$. □

Theorem 5.2.8. *Let G and H be two connected graphs. If the root vertex is a universal vertex in H, then $\chi_{lid}(G \circ_v H) \geq \chi_{lid}(H) + 1$.*

Proof. Let $\chi_{lid}(H) = k$ and c be a k-lid-coloring of $G \circ_v H$. If u_i and u_j are adjacent vertices in G, then $c(N[(u_i, v)]) = [k] = c(N[(u_j, v)])$, which is not possible, since (u_i, v) is adjacent to (u_j, v) in $G \circ_v H$. So, $\chi_{lid}(G \circ_v H) \geq k + 1$. □

Corollary 5.2.1. *Let G be a connected graph. Then* $\chi_{lid}(G \circ_v K_n) \geq n+1$.

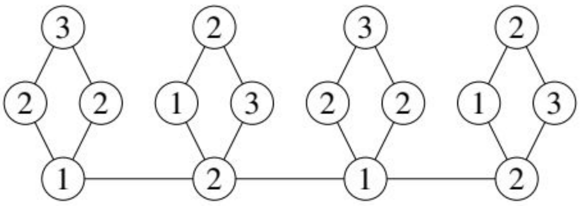


Figure 5.3: A 3-lid-coloring of $P_4 \circ_v C_4$

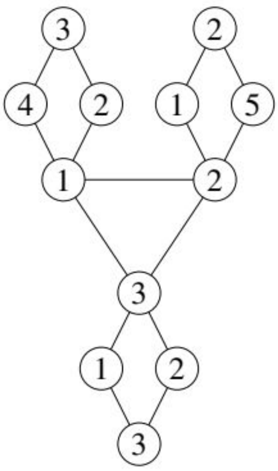


Figure 5.4: A 5-lid-coloring of $C_3 \circ_v C_4$

5.3 Locally identifying coloring of corona product of graphs

5.3.1 Lid-coloring of corona product of graphs

Assume that $V(G) = \{u_1, u_2, \ldots, u_n\}$, $V(H) = \{v_1, v_2, \ldots, v_m\}$. In $G \odot H$, denote the vertices of the copy of G as $u_1^0, u_2^0, \ldots, u_n^0$ and the vertices of the copy H^{u_i} of H in $G \odot H$ as $v_1^i, v_2^i, \ldots, v_m^i$ for all $i = 1, 2, \ldots, n$. See figure 5.5.

In this section, some sharp bounds for the lid-chromatic number of corona product of two graphs have been found. When G is bipartite, the exact lid-chromatic number of $G \odot H$ depends only on the lid-chromatic number of H.

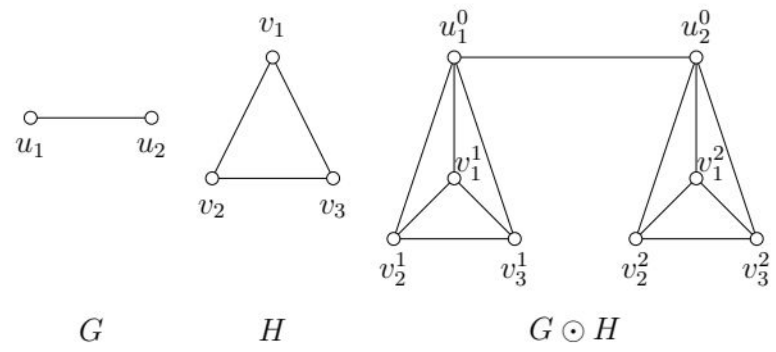


Figure 5.5: Labeling the vertices of corona product of two graphs.

Theorem 5.3.1. *Let G and H be two connected graphs and $N_G[u] \neq N_G[v]$ for all adjacent vertices u and v in G. Then*

$$\chi_{lid}(G \odot H) \leq \chi_{lid}(G) + \chi_{lid}(H).$$

Proof. Assume that $\chi_{lid}(G) = r$ and $\chi_{lid}(H) = s$. Let $f : V(G) \to [r]$ be an r-lid-coloring of G and $g : V(H) \to [s]$ be an s-lid-coloring of H. Define $c : V(G \odot H) \to [r + s]$ by

$$c(x) = \begin{cases} f(u_i) & \text{if } x = u_i^0, 1 \leq i \leq n \\ g(v_j) + r & \text{if } x = v_j^i, 1 \leq i \leq n, 1 \leq j \leq m. \end{cases}$$

Let x and y be two adjacent vertices in $G \odot H$. If $x = u_i^0$ and $y = u_t^0$, then $c(x) = f(u_i) \neq f(u_t) = c(y)$, since f is a lid-coloring.

$$\begin{aligned} c(N[x]) &= f(N_G[u_i]) \cup \{r+1, r+2, \ldots, r+s\} \\ c(N[y]) &= f(N_G[u_t]) \cup \{r+1, r+2, \ldots, r+s\}. \end{aligned}$$

Since $f(N_G[u_i]) \neq f(N_G[u_t])$, $c(N[x]) \neq c(N[y])$.

Let $x = u_i^0$ and $y = v_j^i$. Then $c(x) = f(u_i) \leq r$ but $c(y) = g(v_j) + r > r$. So, $c(x) \neq c(y)$. Note that, $x = u_i^0$ is adjacent to u_t^0 for some $t \in \{1, 2, \ldots, n\}$ and $c(u_t^0) = f(u_t) \in c(N[u_i^0])$ but $c(u_t^0) \notin c(N[v_j^i])$, since $c(u_t^0) = f(u_t) \leq r$ and $f(u_t) \neq f(u_i)$. Therefore, $c(N[x]) \neq c(N[y])$.

If $x = v_j^i$ and $y = v_t^i$, then

$$c(N[x]) = \{f(u_i)\} \cup \{g(z) + r : z \in N_H[v_j]\},$$
$$c(N[y]) = \{f(u_i)\} \cup \{g(z) + r : z \in N_H[v_t]\}.$$

Since g is a lid-coloring of H, $g(N_H[v_j]) \neq g(N_H[v_t])$ and so, $c(N[x]) \neq c(N[y])$. Hence, c is a lid-coloring of $G \odot H$. □

Note that, for a graph G, $\chi(G) \leq \chi_{lid}(G)$. So, there arises a question that can we have an upper bound for $\chi_{lid}(G \odot H)$ in terms of $\chi(G)$. Theorem 5.3.2 gives an upper bound for the lid-chromatic number of $G \odot H$ in terms of the chromatic number of G. If $\chi_{lid}(H) = k$ and H has a good k-lid-coloring, then Theorem 5.3.3 shows that $\chi_{lid}(G \odot H) \leq \chi(G) + \chi_{lid}(H)$.

Theorem 5.3.2. *Let G and H be two connected graphs. Then*

$$\chi_{lid}(G \odot H) \leq 2\chi(G) + \chi_{lid}(H) - 1.$$

Proof. Assume that $\chi(G) = r$ and $\chi_{lid}(H) = s$. Let $f : V(G) \to [r]$ be a proper coloring of G and $g : V(H) \to [s]$ be an s-lid-coloring of H. Define $c : V(G \odot H) \to [2r + s - 1]$ by

$$c(x) = \begin{cases} f(u_i) & \text{if } x = u_i^0, 1 \leq i \leq n \\ f(u_i) + g(v_j) + r - 1 & \text{if } x = v_j^i, 1 \leq i \leq n, 1 \leq j \leq m. \end{cases}$$

Let $x, y \in V(G \odot H)$ be two adjacent vertices. If $x = u_i^0$ and $y = u_t^0$, then $c(x) = f(u_i) \neq f(u_t) = c(y)$, since f is a proper coloring. Without loss of generality assume that $f(u_i) < f(u_t)$. Let $a_i = \min\{c(v_j^i) : 1 \leq j \leq m\}$. Then $a_i \in c(N[u_i^0])$, but

$$c(v_j^t) = f(u_t) + g(v_j) + r - 1 > f(u_i) + g(v_j) + r - 1 \geq a_i,$$

for all v_j^t, $1 \leq j \leq m$. So, $a_i \notin c(N[u_t^0])$ and $c(N[u_i^0]) \neq c(N[u_t^0])$.

Let $x = u_i^0$ and $y = v_j^i$. Then u_i^0 is adjacent to u_t^0 for some $t \in \{1, 2, \ldots, n\}$ and $c(u_t^0) = f(u_t) \in c(N[u_i^0])$ but $c(u_t^0) \notin c(N[v_j^i])$, since $c(u_t^0) = f(u_t) \leq r$ and $f(u_t) \neq f(u_i)$. Therefore, $c(N[u_i^0]) \neq c(N[v_j^i])$.

If $x = v_j^i$ and $y = v_t^i$, then

$$c(N[v_j^i]) = \{f(u_i)\} \cup \{f(u_i) + g(z) + r - 1 : z \in N_H[v_j]\},$$
$$c(N[v_t^i]) = \{f(u_i)\} \cup \{f(u_i) + g(z) + r - 1 : z \in N_H[v_t]\}.$$

Since g is a lid-coloring of H, $g(N_H[v_j]) \neq g(N_H[v_t])$ and so, $c(N[v_j^i]) \neq c(N[v_t^i])$. Thus, c is a lid-coloring of $G \odot H$. □

Theorem 5.3.3. *Suppose that G and H are two connected graphs and $\chi_{lid}(H) = k$. If $\log_2(\chi(G)) \leq \chi_{lid}(H)$ and H has a good k-lid-coloring, then*

$$\chi_{lid}(G \odot H) \leq \chi(G) + \chi_{lid}(H).$$

Proof. Assume that $\chi(G) = r$ and $\chi_{lid}(H) = k$ and $f : V(G) \to [r]$ be a proper coloring of G. Let $\log_2(r) \leq k$. Then we can have at least r subsets of $\{r+1, r+2, \ldots, r+k\}$, since $r \leq 2^k$. If $r \leq k$, then $A_1, A_2, \ldots, A_r$ be distinct $(k-1)$-element subsets of $\{r+1, r+2, \ldots, r+k\}$. This is possible, since there are k such $(k-1)$-element subsets. If $r > k$, then $A_1, A_2, \ldots, A_r$ be distinct subsets of $\{r+1, r+2, \ldots, r+k\}$.

Let $f(u_i) = q$, $B_q = \{k - |A_q|$ colors from $[r] \setminus \{q\}\}$ and g_q be a good k-lid-coloring of H^{u_i} on the k colors from the set $A_q \cup B_q$, $1 \leq q \leq r, 1 \leq i \leq n$. Define $c : V(G \odot H) \to [r+k]$ by

$$c(x) = \begin{cases} f(u_i) & \text{if } x = u_i^0, 1 \leq i \leq n \\ g_q(v_j) & \text{if } x = v_j^i, f(u_i) = q, 1 \leq i \leq n, 1 \leq j \leq m. \end{cases}$$

Let x, and y be two adjacent vertices in $G \odot H$. If $x = u_i^0$ and $y = u_t^0$, then $c(x) = f(u_i) \neq f(u_t) = c(y)$, since f is a proper coloring. Moreover, $A_1, A_2, \ldots, A_r$ are distinct subsets of $\{r+1, r+2, \ldots, r+k\}$, $c(N[x]) \neq c(N[y])$.

Let $x = u_i^0$ and $y = v_j^i$. Then $c(x) = f(u_i) = q$, but $c(y) \in A_q \cup B_q$. So, $c(x) = q \neq c(y)$. Since g_q is a good k-lid-coloring, $c(N[y])$ is a proper subset of $A_q \cup B_q$, but $A_q \cup B_q \subseteq c(N[x])$. Therefore, $c(N[x]) \neq c(N[y])$.

If $x = v_j^i$ and $y = v_t^i$, then $c(x) \neq c(y)$ and $c(N[x]) \neq c(N[y])$, since g_q is a lid-coloring. Hence, c is a lid-coloring of $G \odot H$. □

Note that, the two conditions in the statement of Theorem 5.3.3 cannot be omitted. For example, if $\log_2(\chi(G)) > \chi_{lid}(H)$, then consider $G = K_{17}$ and $H = P_4$. Clearly, H has a good 4-lid coloring, $\chi(G) = 17$ and $\chi_{lid}(H) = 4$. Then, $\chi_{lid}(G \odot H) = t \leq \chi(G) + \chi_{lid}(H) = 21$. Also, let c be a t-lid-coloring of $G \odot H$. Since c is a proper vertex coloring, we require 17 colors on the vertices of G, say all colors in $[17]$. The copies of H can receive colors from the entire set $[21]$. However, any two copies of H corresponding to two adjacent vertices in G, cannot receive all colors from the same subset A of $\{18, 19, 20, 21\}$ (Otherwise, the set of colors in the neighborhood of both the vertices would be $[17] \cup A = [21]$). Thus, every copy of H must correspond to a unique subset of $\{18, 19, 20, 21\}$ and there are at most $2^4 = 16$ subsets. This contradicts the fact that there are 17 copies of H in the graph $G \odot H$.

If H does not have a good-lid-coloring, then Theorem 5.3.3 does not holds. Consider $G = P_3$ and $H = K_3$. Here, H does not have a good lid-coloring, $\chi(G) = 2$ and $\chi_{lid}(H) = 3$. But $G \odot H$ cannot have a 5-lid-coloring.

Note that, every vertex in a copy of H in $G \odot H$ is adjacent to a unique vertex of G. So, $\chi_{lid}(G \odot H) \geq \chi_{lid}(H) + 1$. Following theorem gives a lower bound for the lid-chromatic number of $G \odot H$. If every lid-coloring of H is bad, then Theorem 5.3.5 gives another lower bound for lid-chromatic number of $G \odot H$.

The following lemma is immediate from the observation that $u_i \in V(G \odot H)$ is adjacent to every vertices of the copy H^{u_i} of H.

Lemma 5.3.1. *If $\chi_{lid}(H) = k$ and if c is a lid-coloring of $G \odot H$, then any restriction of c to a copy of H is a k-lid-coloring of H.*

Theorem 5.3.4. *Let G and H be two connected graphs with $|V(G)| \geq 2$. Then*

$$\chi_{lid}(G \odot H) \geq \chi_{lid}(H) + 2.$$

Proof. Note that, any lid-coloring c of $G \odot H$ must have at least $\chi_{lid}(H) + 1$ colors. Let $\chi_{lid}(H) = k$. Without loss of generality assume that H^{u_i} has a k-lid-coloring with colors $1, 2, \ldots, k$ (Lemma 5.3.1) and $c(u_i) = k + 1$. Let u_r be any vertex in G which is adjacent to u_i.

If H^{u_r} is also colored with same colors $1, 2, \ldots, k$ as H^{u_i}, then $c(u_r) \notin [k + 1]$. Therefore, $\chi_{lid}(G \odot H) \geq \chi_{lid}(H) + 2$.

If H^{u_r} is colored with at least one different color from the coloring of H^{u_i}, then it must be a new color, $k+2$. □

Theorem 5.3.5. *Let G and H be two connected graphs with $|V(G)| \geq 2$ and $\chi_{lid}(H) = k$. If every k-lid-coloring of H is bad, then*

$$\chi_{lid}(G \odot H) \geq \chi_{lid}(H) + 3.$$

Proof. Let $\chi_{lid}(H) = k$. Assume that c is a $(k+2)$ lid-coloring of $G \odot H$. By Lemma 5.3.1, c in a copy of H in $G \odot H$ must be a k-lid-coloring. Since $c|_{V(H^{u_i})}$ is bad, there exists $v_j \in H$ such that $c|_{V(H^{u_i})}(N[v_j^i]) = c(V(H^{u_i}))$ for all i.

Let u_1^0 and u_2^0 be two adjacent vertices in the copy of G. If all the copies of H has lid-colorings of the same k colors, $1, 2, \ldots, k$, then $c(u_1^0) = k+1$, $c(u_1^0) = k+2$ and $c(N[u_1^0]) = [k+2] = c(N[u_2^0])$, which is a contradiction.

Assume that, $c(V(H^{u_1})) = \{1, 2, \ldots, k\}$ and $c(V(H^{u_2})) = \{2, 3, \ldots, k+1\}$. Then

$$\begin{aligned} c(N[u_1^0]) = & \quad \{c(u_1^0), 1, 2, \ldots, k\} & = c(N[v_j^1]), \\ c(N[u_2^0]) = & \quad \{c(u_2^0), 2, 3, \ldots, k+1\} & = c(N[v_j^2]). \end{aligned}$$

Note that, $c(u_1^0) \in \{k+1, k+2\}$ and $c(u_1^0) \in \{1, k+2\}$. If $c(u_1^0) = k+1$ and $c(u_2^0) = 1$, then $c(N[u_1^0]) = c(N[v_j^1]) = [k+1] = c(N[v_j^2]) = c(N[u_2^0])$. Clearly, c cannot be a lid-coloring even though color $k+2$ is given to a neighbor of u_1^0 and a neighbor of u_2^0 in the copy of G, since $c(N[u_1^0]) = [k+2] = c(N[u_2^0])$.

If $c(u_1^0) = k+1$ and $c(u_2^0) = k+2$, then $c(N[u_1^0]) = [k+2] = c(N[v_j^1])$. Here also there is no possibility for c to become a lid-coloring. If $c(u_1^0) = k+2$ and $c(u_2^0) = 1$, then $c(N[u_2^0]) = [k+2] = c(N[v_j^2])$ and so, c cannot be a lid-coloring.

If $c(H^{u_1}) = \{1, 2, \ldots, k\}$ and $c(H^{u_2}) = \{3, 4, \ldots, k+1, k+2\}$, then

$$\begin{aligned} c(N[u_1^0]) = & \quad \{c(u_1^0), 1, 2, \ldots, k\} & = c(N[v_j^1]), \\ c(N[u_2^0]) = & \quad \{c(u_2^0), 3, 4, \ldots, k+1, k+2\} & = c(N[v_j^2]). \end{aligned}$$

If $c(u_1^0) = k+1$ and $c(u_2^0) = 1$, then $c(N[u_1^0])) = [k+1] = c(N[v_j^1]$ and $c(N[u_2^0]) =$

$\{1, 3, 4, \ldots, k+1, k+2\} = c(N[v_j^2])$. Clearly, c cannot be a lid-coloring even though $k+2$ is given to a neighbor of u_1^0 and 2 is given to a neighbor of u_2^0 in the copy of G, since $c(N[u_1^0]) = [k+2] = c(N[u_2^0])$. All other cases are similar. Hence, $G \odot H$ cannot have a $(k+2)$-lid-coloring. □

The upper bound in Theorem 5.3.2 and the lower bound in Theorem 5.3.5 are sharp. For example, consider $P_m \odot K_n$. By Theorem 5.3.2,

$$\chi_{lid}(P_m \odot K_n) \leq 2\chi(P_m) + \chi_{lid}(K_n) - 1 = n + 3.$$

Since every lid-coloring of K_n is bad, by Theorem 5.3.5, $\chi_{lid}(P_m \odot K_n) = n + 3$.

If G is a bipartite graph, then the exact lid-chromatic number of $G \odot H$ is given in Corollary 5.3.1.

Corollary 5.3.1. *Let G and H be two connected graphs and $\chi_{lid}(H) = k$. If G is bipartite, then*

$$\chi_{lid}(G \odot H) = \begin{cases} \chi_{lid}(H) + 2 & \text{if } H \text{ has a good } k\text{-lid-coloring} \\ \chi_{lid}(H) + 3 & \text{otherwise.} \end{cases}$$

Proof. Let G be a bipartite graph. If H has a good k-lid-coloring, then by Theorem 5.3.3,

$$\chi_{lid}(G \odot H) \leq \chi(G) + \chi_{lid}(H) = 2 + \chi_{lid}(H).$$

So, Theorem 5.3.4 implies that $\chi_{lid}(G \odot H) = \chi_{lid}(H) + 2$.

By Theorem 5.3.2,

$$\chi_{lid}(G \odot H) \leq 2\chi(G) + \chi_{lid}(H) - 1 = 4 + \chi_{lid}(H) - 1 = 3 + \chi_{lid}(H).$$

Hence, Theorem 5.3.5 implies that $\chi_{lid}(G \odot H) = \chi_{lid}(H) + 3$. □

5.3.2 Lid-coloring of corona product of paths and cycles

Suppose that c is a lid-coloring of P_n. If $n \neq 2$ is even, then $\chi_{lid}(P_n) = 4$. Since $\Delta(P_n) = 2$, $|c(N[u]))| \leq 3$, for all $u \in V(P_n)$ and so P_n has a good 4-lid-coloring. Note that, lid-coloring of P_2 is bad. If n is odd, then $\chi_{lid}(P_n) = 3$ and the vertex u adjacent to a leaf must have

$c(N[u]) = \{1, 2, 3\}$. So, every 3-lid-coloring of P_n is bad. Hence, if $n \neq 2$ is even, P_n has a good 4-lid-coloring and every 3-lid-coloring of P_n is bad if n is odd. Similarly, C_n ($n \neq 3$) has a good k-lid-coloring if $n \not\equiv 0 \pmod 4$, and every k-lid-coloring of C_n is bad if $n \equiv 0 \pmod 4$ or $n = 3$, where $k = \chi_{lid}(C_n)$.

Theorem 5.3.6. *For positive integers $m, n \geq 2$,*

$$\chi_{lid}(P_m \odot P_n) = \begin{cases} 5 & \textit{if } n = 2 \\ 6 & \textit{otherwise.} \end{cases}$$

Proof. By Corollary 5.3.1,

$$\chi_{lid}(P_m \odot P_n) = \begin{cases} \chi_{lid}(P_n) + 2 & \text{if } P_n \text{ has a good } k\text{-lid-coloring} \\ \chi_{lid}(P_n) + 3 & \text{otherwise.} \end{cases}$$

were $k = \chi_{lid}(P_n)$.

If $n \neq 2$ is even, then $\chi_{lid}(P_n) = 4$ and any 4-lid-coloring of P_n is good. So, $\chi_{lid}(P_m \odot P_n) = \chi_{lid}(P_n) + 2 = 6$. If n is odd, then every 3-lid-colorings of P_n are bad. Hence, $\chi_{lid}(P_m \odot P_n) = \chi_{lid}(P_n) + 3 = 6$. If $n = 2$, then $\chi_{lid}(P_m \odot P_n) = \chi_{lid}(P_n) + 3 = 5$. □

Theorem 5.3.7. *For positive integers $m \geq 2$ and $n \geq 3$,*

$$\chi_{lid}(P_m \odot C_n) = \begin{cases} 7 & \textit{if } n = 5 \textit{ or } 7 \\ 6 & \textit{otherwise.} \end{cases}$$

Proof. Let $\chi_{lid}(C_n) = k$. By Corollary 5.3.1,

$$\chi_{lid}(P_m \odot C_n) = \begin{cases} \chi_{lid}(C_n) + 2 & \text{if } C_n \text{ has a good } k\text{-lid-coloring} \\ \chi_{lid}(C_n) + 3 & \text{otherwise.} \end{cases}$$

If $n \equiv 0 \pmod 4$ or $n = 3$, then C_n cannot have a good k-lid-coloring, so, $\chi_{lid}(P_m \odot C_n) = \chi_{lid}(C_n) + 3 = 6$.

If $n \not\equiv 0 \pmod 4$, $n \neq 3$, then every k-lid-colorings of C_n are good. Hence, $\chi_{lid}(P_m \odot C_n) = \chi_{lid}(C_n) + 2$. From Theorem 2.4.12, result follows. □

Theorem 5.3.8. *Let $\chi_{lid}(H) = k$. For a positive integer $m \geq 3$,*

$$\chi_{lid}(C_m \odot H) = \begin{cases} \chi_{lid}(H) + 2 & \text{if } H \text{ has a good } k\text{-lid-coloring} \\ \chi_{lid}(H) + 3 & \text{otherwise.} \end{cases}$$

Proof. If m is even, then C_m is bipartite and the result follows from Corollary 5.3.1. If m is odd, then $\chi(C_m) = 3$. Let f be the proper coloring of $C_m : u_1 - u_2 - \cdots - u_m - u_1$ defined by

$$f(u_i) = \begin{cases} 1 & \text{if } i \neq m \text{ is odd} \\ 2 & \text{if } i \text{ is even} \\ 3 & \text{if } i = m. \end{cases}$$

Case 1: H has a good k-lid-coloring.

Let g be a good k-lid-coloring of H. Define $c : V(C_m \odot H) \to [k+2]$ by

$$c(u_i^0) = f(u_i),$$

and color the copies of H as follows: If $c(u_i^0) = 1$, then color the copy H^{u_i} by the good k-lid-coloring g with colors $\{2, 3, \ldots, k, k+1\}$. Similarly, if $c(u_i^0) = 2$, then color H^{u_i} by the good k-lid-coloring g with colors $\{1, 3, 4, \ldots, k, k+2\}$ and if $c(u_i^0) = 3$, then color the copy H^{u_i} by the good k-lid-coloring g with colors $\{1, 4, 5, \ldots, k, k+1, k+2\}$. Clearly, c is a $(k+2)$-lid-coloring of $C_m \odot H$. Hence, from Theorem 5.3.4, $\chi_{lid}(C_m \odot H) = \chi_{lid}(H) + 2$.

Case 2: Every k-lid-coloring of H is bad.

Let g be any k-lid-coloring of H. Define $c : V(C_m \odot H) \to [k+3]$ by

$$c(u_i^0) = f(u_i),$$

and color the copies of H as follows: If $c(u_i^0) = 1$, then color the copy H^{u_i} by g with colors $\{3, 4, \ldots, k+1, k+2\}$. Similarly, if $c(u_i^0) = 2$, then color H^{u_i} by the k-lid-coloring g with colors $\{3, 4, \ldots, k, k+1, k+3\}$ and if $c(u_i^0) = 3$, then color the copy H^{u_i} by g with colors $\{1, 4, 5, \ldots, k, k+2, k+3\}$. It is easy to verify that c is a $(k+3)$-lid-coloring of $C_m \odot H$. Therefore, from Theorem 5.3.5, $\chi_{lid}(C_m \odot H) = \chi_{lid}(H) + 3$. □

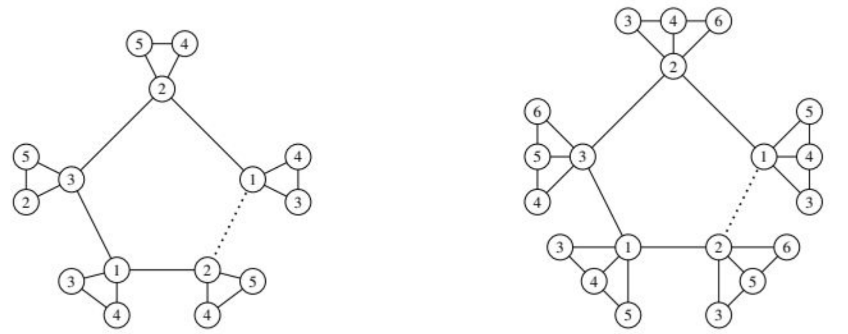


Figure 5.6: A 5-lid-coloring of $C_m \odot P_2$ and a 6-lid-coloring of $C_m \odot P_3$ when m is odd.

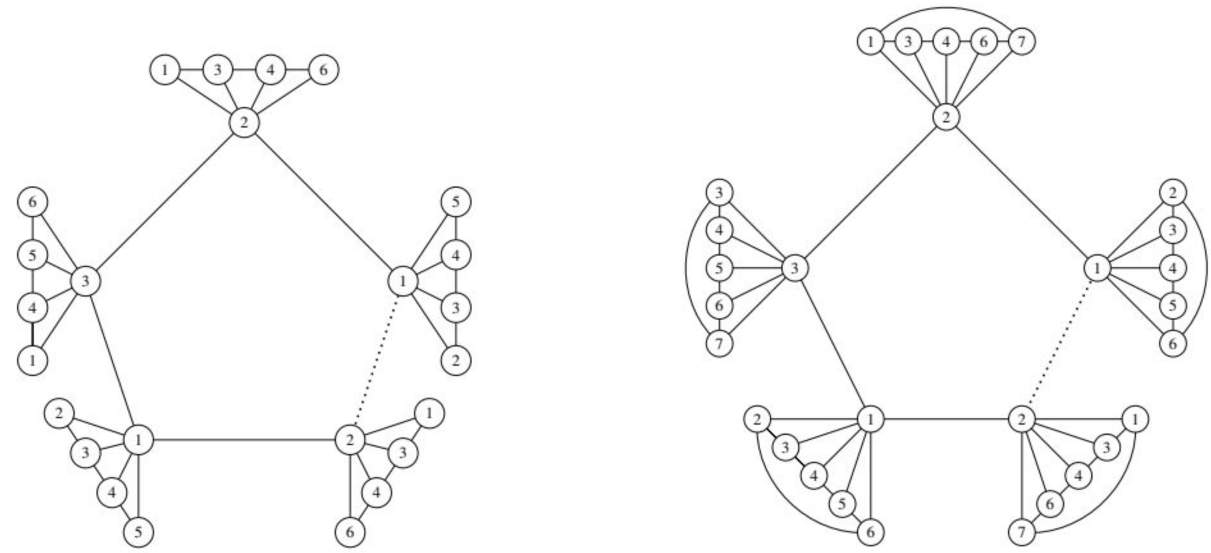


Figure 5.7: A 6-lid-coloring of $C_m \odot P_4$, and a 7-lid-coloring of $C_m \odot C_5$ when m is odd.

The Corollary 5.3.2 and the Corollary 5.3.3 follows immediately from Theorem 5.3.8.

Corollary 5.3.2. *For positive integers $m \geq 3, n \geq 2$,*

$$\chi_{lid}(C_m \odot P_n) = \begin{cases} 5 & \textit{if } n = 2 \\ 6 & \textit{otherwise.} \end{cases}$$

Corollary 5.3.3. *For positive integers $m, n \geq 3$,*

$$\chi_{lid}(C_m \odot C_n) = \begin{cases} 7 & \textit{if } n = 5 \textit{ or } 7 \\ 6 & \textit{otherwise.} \end{cases}$$

Chapter 6

Locally identifying coloring of strong product of graphs

6.1 Introduction

In this chapter, the lid-coloring of strong product of two graphs have been studied. It is found that for any two connected graphs $\chi_{lid}(G \boxtimes H) \leq \chi_{lid}(G)\chi_{lid}(H)$. In addition, it is proved that $\chi_{lid}(G \boxtimes H)$ cannot be 4 if G and H are not K_2. Furthermore, if G and H are 3-lid-colorable bipartite graphs, then $\chi_{lid}(G \boxtimes H)$ is exactly 6. In the last section, the lid-chromatic number of $P_m \boxtimes P_n$ and $P_m \boxtimes C_n$ were studied. Finding the exact lid-chromatic number is a complicated task, when the number of colors is more than 4. Because the set of colors in the neighborhoods of adjacent vertices can have many different choices. If the clique number is of considerable size, then the task is relatively simple. In strong product of graphs, lid-chromatic number is greater than 4. So, it is difficult to compute the exact lid-chromatic number of strong product of two graphs with small clique number.

6.2 Lid-coloring of strong product of graphs

For any vertex (u, v) in $G \boxtimes H$,

$$N[(u, v)] = N[u] \times N[v] = \{(x, y) : x \in N[u], y \in N[v]\}.$$

In particular, the strong product of two complete graphs is a complete graph and thus $N[(u,v)] = N[(x,y)]$ for all (u,v) and (x,y). So, it is not generally true that $N[(u,v)] \neq N[(x,y)]$ for adjacent vertices (u,v) and (x,y) in $G \boxtimes H$.

Using this idea we have the following lemma.

Lemma 6.2.1. *Let G and H be two connected graphs. Then $N[(u_1,v_1)] \neq N[(u_2,v_2)]$ for two adjacent vertices (u_1,v_1) and (u_2,v_2) in $G \boxtimes H$ if and only if either $N[u_1] \neq N[u_2]$ in G or $N[v_1] \neq N[v_2]$ in H.*

6.2.1 Some bounds for $\chi_{lid}(G \boxtimes H)$

Let G and H be two connected graphs. Then

$$\chi_{lid}(G \boxtimes H) \geq \chi(G \boxtimes H) \geq \omega(G)\omega(H),$$

since, $\omega(G \boxtimes H) = \omega(G)\omega(H)$. From this observation, it is clear that $\chi_{lid}(G \boxtimes H) \geq 4$. The following theorem gives a necessary and sufficient condition to have the lid-chromatic number equal to 4.

Theorem 6.2.1. *Let G and H be two graphs without isolated vertices. Then $G \boxtimes H$ is 4-lid-colorable if and only if $G \cong K_2 \cong H$.*

Proof. If $G \cong K_2$ and $H \cong K_2$, then $G \boxtimes H \cong K_4$ and $\chi_{lid}(G \boxtimes H) = 4$. Without loss of generality suppose that G is not K_2. Then G has a vertex w with at least two neighbors u and v. Let x and y be adjacent vertices in H. Then the graph induced by $\{(u,x),(u,y),(w,x),(w,y)\}$ is K_4. So, any proper coloring of $G \boxtimes H$ must have at least 4 colors. Assume that, c is a 4-lid-coloring of $G \boxtimes H$ and $c((u,x)) = 1, c((u,y)) = 2, c((w,x)) = 3, c((w,y)) = 4$. Since (v,x) and (v,y) are adjacent to both (w,x) and (w,y), the vertices (v,x) and (v,y) must be colored with 1 and 2. Then $c(N[(w,x)]) = [4] = c(N[(v,x)])$ and so, $N[(w,x)] = N[(v,x)]$. By Lemma 6.2.1, $N[w] = N[v]$ in G. This implies that u is adjacent to v. Then the graph induced by $\{(u,x),(u,y),(w,x),(w,y),(v,x),(v,y)\}$ is K_6 and $\chi_{lid}(G \boxtimes H) \geq 6$, a contradiction. □

Theorem 6.2.2. *Let G and H be two connected graphs. Then*

$$\chi_{lid}(G \boxtimes H) \leq \chi_{lid}(G)\chi_{lid}(H)$$

Proof. Suppose that $\chi_{lid}(G) = k_1, \chi_{lid}(H) = k_2$. Let g be a k_1-lid-coloring of G and h be a k_2-lid-coloring of H. Define $c : V(G \boxtimes H) \to [k_1k_2]$ by

$$c(u, v) = (g(u), h(v)).$$

Since

$$c(N[(u, v)]) = c(N[u] \times N[v]) = g(N[u]) \times h(N[v]),$$

$c(N[(u, v)]) \neq c(N[(x, y)])$ whenever $g(N[u]) \neq g(N[x])$ or $h(N[v]) \neq h(N[y])$. If $c(N[(u, v)]) = c(N[(x, y)])$, then $g(N[u]) = g(N[x])$ and $h(N[v]) = h(N[y])$. So, $N[u] = N[x]$ and $N[v] = N[y]$. This implies that $N[(u, v)] = N[(x, y)]$. Hence, c is a lid-coloring of $G \boxtimes H$. □

Lemma 6.2.2. *Let G and H be connected bipartite graphs. $N[(u_1, v_1)] \neq N[(u_2, v_2)]$ for all adjacent vertices (u_1, v_1) and (u_2, v_2) in $G \boxtimes H$ if and only if $G \ncong K_2$ and $H \ncong K_2$.*

Proof. Note that, in a bipartite graph G, $N[u] = N[v]$ for adjacent vertices u and v if and only if $deg(u) = 1 = deg(v)$. That is, K_2 is the only connected bipartite graph with $N[u] = N[v]$ for adjacent vertices u and v. If $G \ncong K_2$ and $H \ncong K_2$ are connected bipartite graphs, then by Lemma 6.2.1, $N[(u_1, v_1)] \neq N[(u_2, v_2)]$ for all adjacent vertices (u_1, v_1) and (u_2, v_2) in $G \boxtimes H$. Conversely, if $G \cong K_2 : u - v$, then $N[(u, x)] = N[(v, x)]$ for all $x \in V(H)$. This completes the proof. □

If G and H are bipartite graphs without isolated vertices, then Esperet et al. [31] showed that the Cartesian product $G \square H$ is 3-lid-colorable. But for the strong product of G and H, Theorem 6.2.1 implies that $\chi_{lid}(G \boxtimes H) \geq 5$ if either G or H is not K_2.

Lemma 6.2.3. *If $G \ncong K_2$ and $H \ncong K_2$ are connected graphs, then $\chi_{lid}(G \boxtimes H) \geq 6$.*

Proof. Suppose that $G \ncong K_2$ and $H \ncong K_2$ are connected graphs. Without loss of generality suppose that G has a vertex w with at least two neighbors u and v and H has a vertex z with at least two neighbors x and y. Since G and H are bipartite graphs, u and v cannot be adjacent in G and x and y cannot be adjacent in H. The induced subgraph of $A = \{(u, x), (w, x), (v, x), (u, z), (w, z), (v, z), (u, y), (w, y), (v, y)\}$ in $G \boxtimes H$ is given in Figure 6.1. If c is a 5-lid-coloring of $G \boxtimes H$ and $c((u, x)) = 1, c((u, z)) = 2, c((w, x)) = 3$ and $c((w, z)) = 4$, then the closed neighborhoods of the vertices $(u, x), (u, z), (w, x)$ and (w, z) contains all the four colors $1, 2, 3$

and 4. Since all the pairs of adjacent vertices have different closed neighborhoods by Lemma 6.2.2, the set of colors in the closed neighborhoods of the vertices $(u,x), (u,z), (w,x)$ and (w,z) must be different. So, there must be at least two more colors different from 1,2,3 and 4.Then $\chi_{lid}(G \boxtimes H) \geq 6$. □

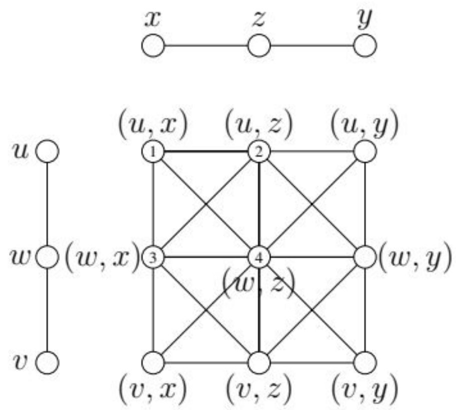


Figure 6.1: Induced subgraph of A.

6.2.2 Lid-coloring of strong product of bipartite graphs

Esperet et al. [31] showed that the lid-chromatic number of a bipartite graph is at most 4. A connected graph is 2-lid-colorable if and only if it has at most two vertices (cf. Theorem 2.4.7 and Theorem 2.4.8). So, any connected bipartite graph with at least 3 vertices have lid-chromatic number is either 3 or 4. Here, only 3-lid-colorable connected bipartite graphs are considered.

Theorem 6.2.3. *If G and H are 3-lid-colorable connected bipartite graphs, then*

$$\chi_{lid}(G \boxtimes H) = 6.$$

Proof. Let G and H be connected bipartite graphs with bipartition $\{U_G, V_G\}$ and $\{U_H, V_H\}$ respectively. If g and h are 3-lid-colorings of G and H respectively, then by Theorem 2.4.10, the vertex sets of G and H can be partitioned as follows:

1. $V(G) = X_1 \cup X_2 \cup X_3$, $U_G = X_1$, $V_G = X_2 \cup X_3$ and $g(X_1) = \{1\}, g(X_2) = \{2\}, g(X_3) = \{3\}$,

2. $V(H) = Y_1 \cup Y_2 \cup Y_3$, $U_H = Y_1$, $V_H = Y_2 \cup Y_3$ and $h(Y_1) = \{1\}, h(Y_2) = \{2\}, h(Y_3) = \{3\}$.

Note that, every $x \in X_1$ must have a neighbor in both X_2 and X_3. If there is an $x \in X_1$ which has no neighbors in X_3, then c($N[x]$) $= \{1, 2\} = c(N[x'])$ for every $x' \in X_2$, a contradiction. Similarly, every $y \in Y_1$ must have a neighbor in both Y_2 and Y_3.

Define $c : V(G \boxtimes H) \to [6]$ by

$$c(u,v) = \begin{cases} 1 & \text{if } (u,v) \in X_2 \times Y_1 \\ 2 & \text{if } (u,v) \in (X_2 \times Y_2) \cup (X_3 \times Y_2) \\ 3 & \text{if } (u,v) \in (X_1 \times Y_2) \cup (X_2 \times Y_3) \cup (X_3 \times Y_3) \\ 4 & \text{if } (u,v) \in X_1 \times Y_1 \\ 5 & \text{if } (u,v) \in X_1 \times Y_3 \\ 6 & \text{if } (u,v) \in X_3 \times Y_1. \end{cases}$$

	Y_2	Y_1	Y_3
X_2	2	1	3
X_1	3	4	5
X_3	2	6	3

Figure 6.2: 6-lid-coloring of $G \boxtimes H$.

A representation of the coloring c is given in Figure 6.2. Each cell $X_i \times Y_j$ represents set of vertices in $G \boxtimes H$ with same color. Note that, no two vertices of the same cell are adjacent, since no two vertices of X_i or Y_j are adjacent. i.e., $(x_1, y_1), (x_2, y_2) \in X_i \times Y_j$ are not adjacent in $G \boxtimes H$, since x_1 is not adjacent to x_2 in X_i and y_1 is not adjacent to y_2 in Y_j.

There is at least one pair of adjacent vertices in two different cells if two cells are orthogonally adjacent or diagonally adjacent. For let $X_i \times Y_j$ and $X_r \times Y_s$ be two orthogonally adjacent cells. Then either $X_i = X_r$ or $Y_j = Y_s$. If $X_i = X_r$ then one of $\{Y_j, Y_s\}$ must be Y_1 and the other must be Y_2 or Y_3. So, there exists $y_1 \in Y_j$ and $y_2 \in Y_s$ such that y_1 is adjacent to y_2. That

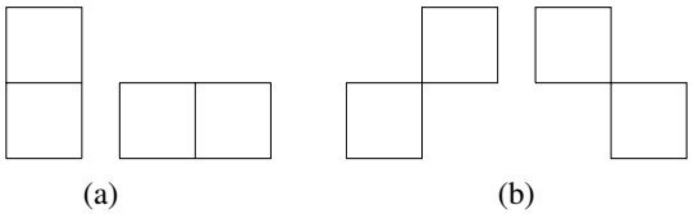


Figure 6.3: (a) Orthogonally adjacent cells (b) Diagonally adjacent cells.

is $(x, y_1) \in X_i \times Y_j$ is adjacent to $(x, y_2) \in X_r \times Y_s$ in $G \boxtimes H$. Similarly, if $Y_j = Y_s$, then there exists $(x_1, y) \in X_i \times Y_j$ which is adjacent to $(x_2, y) \in X_r \times Y_s$ in $G \boxtimes H$.

Let $X_i \times Y_j$ and $X_r \times Y_s$ be two diagonally adjacent cells. Then $i = 1$ or $r = 1$ and $j = 1$ or $s = 1$. So, there exists $x_1 \in X_i$, $x_2 \in X_r$, $y_1 \in Y_j$ and $y_2 \in Y_s$ such that x_1 is adjacent to x_2 in G and y_1 is adjacent to y_2 in H. Thus, $(x_1, y_1) \in X_i \times Y_j$ is adjacent to $(x_2, y_2) \in X_r \times Y_s$ in $G \boxtimes H$.

If two cells $X_i \times Y_j$ and $X_r \times Y_s$ are neither orthogonally adjacent nor diagonally adjacent, then either $i, r \in \{2, 3\}$ such that $i \neq r$ or $j, s \in \{2, 3\}$ such that $j \neq s$. Suppose that $i, r \in \{2, 3\}$ such that $i \neq r$. Then, for every $x_1 \in X_i$ and $x_2 \in X_r$, neither $x_1 = x_2$ nor x_1 is adjacent to x_2. So, (x_1, y_1) is not adjacent to (x_2, y_2) in $G \boxtimes H$, for all $(x_1, y_1) \in X_i \times Y_j$ and $(x_2, y_2) \in X_r \times Y_s$.

Therefore, two vertices in different cells are adjacent only if the cells are orthogonally adjacent or diagonally adjacent. So, the coloring c is a proper coloring. All the vertices in the same cell have the same colors in their closed neighborhood and the vertices in the nine cells have distinct set of colors in their closed neighborhoods. Hence c is a lid-coloring and $\chi_{lid}(G \boxtimes H) \leq 6$. By Lemma 6.2.3, $\chi_{lid}(G \boxtimes H) = 6$. □

Corollary 6.2.1. *Suppose that G and H are 3-lid-colorable connected bipartite graphs or K_2 (not both G and H are K_2). $\chi_{lid}(G \boxtimes H) = 5$ if and only if $G \cong K_2$ or $H \cong K_2$.*

Proof. If $\chi_{lid}(G \boxtimes H)$ is 5, then by Lemma 6.2.3, either G or H is K_2. Conversely, suppose that $G \cong K_2 : u - v$ and H is a 3-lid-colorable connected bipartite graph. Then $V(H) = Y_1 \cup Y_2 \cup Y_3$, $U_H = Y_1$, $V_H = Y_2 \cup Y_3$ and $h(Y_1) = \{1\}, h(Y_2) = \{2\}, h(Y_3) = \{3\}$, where h is a 3-lid-coloring of H as in the poof of Theorem 6.2.3. Then Figure 6.4 gives a 5-lid-coloring of $K_2 \boxtimes H$. □

	Y_2	Y_1	Y_3
$\{v\}$	2	1	3
$\{u\}$	3	4	5

Figure 6.4: 5-lid-coloring of $K_2 \boxtimes H$.

6.3 Lid-coloring of strong product of paths and cycles

This section focuses on lid-coloring of strong product of paths and cycles. The exact lid-chromatic number of $P_m \boxtimes P_n$ has been found in Theorem 6.3.1.

Theorem 6.3.1. *For positive integers $m, n \geq 2$,*

$$\chi_{lid}(P_m \boxtimes P_n) = \begin{cases} 4 & \text{if } m = 2 \text{ and } n = 2 \\ 5 & \text{if } m = 2 \text{ and } n \text{ is odd} \\ & \text{or } n = 2 \text{ and } m \text{ is odd} \\ 6 & \text{if } m \text{ and } n \text{ are odd} \\ & \text{or } m = 2 \text{ and } n > 2 \text{ is even} \\ & \text{or } n = 2 \text{ and } m > 2 \text{ is even} \\ 7 & \text{otherwise.} \end{cases}$$

Proof. Let $P_m : u_1 - u_2 - \cdots - u_m$ and $P_n : v_1 - v_2 - \cdots - v_n$. The graph $P_m \boxtimes P_n$ contains copies of K_4. Let $X = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_2)\}$. Then the subgraph induced by X is K_4. Without loss of generality, assume that for any lid-coloring c of $P_m \boxtimes P_n$, $c(u_1, v_1) = 1$, $c(u_1, v_2) = 2$, $c(u_2, v_1) = 3$ and $c(u_2, v_2) = 4$. So, $[4] \subseteq c(N[x])$ for all $x \in X$.

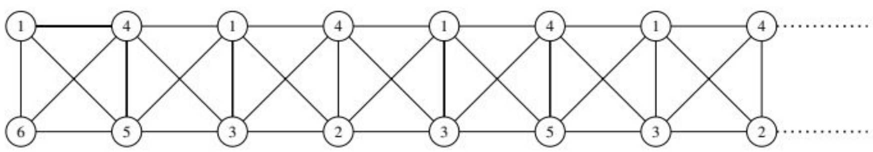


Figure 6.5: A 6-lid-coloring of $P_2 \boxtimes P_n$, when n is even.

By Theorem 6.2.1, $\chi_{lid}(P_2 \boxtimes P_2) = 4$ and $\chi_{lid}(P_m \boxtimes P_n) \geq 5$ if $m \neq 2$ or $n \neq 2$.

Case 1: $m = 2$ and $n > 2$ is odd.

By Lemma 6.2.1, $N[(u_1, v_i)] = N[(u_2, v_i)]$ for all $i = 1, 2, \ldots, n$, since $N[u_1] = N[u_2]$. By Corollary 6.2.1 and Theorem 2.4.11, $\chi_{lid}(P_m \boxtimes P_n) = 5$, if $m = 2$ and n is odd (or $n = 2$ and m is odd).

Case 2: $m = 2$ and $n > 2$ is even.

Suppose that n is even and c is a 5-lid-coloring of $P_2 \boxtimes P_n$. Since $|N[(u_1, v_1)]| = 4$ and $G[X] = K_4$, $|c(N[(u_1, v_1)])| = 4$. Since $N[(u_1, v_1)] \neq N[(u_1, v_2)]$, $c(N[(u_1, v_1)]) \neq c(N[(u_1, v_2)])$ and so $|c(N[(u_1, v_2)])| = 5$. Similarly, $|c(N[(u_1, v_3)])| = 4$. Consequently, $|c(N[(u_1, v_i)])|$ is 4 if i is odd and 5 if i is even. So, $c(N[(u_1, v_n)]) = [5]$, which is not possible, since $|N[(u_1, v_n)]| = 4$. Thus, $\chi_{lid}(P_m \boxtimes P_n) \geq 6$. Figure 6.5 gives a 6-lid-coloring of $P_m \boxtimes P_n$, when $m = 2$ and n is even. Hence, $\chi_{lid}(P_m \boxtimes P_n) = 6$ if $m = 2$ and n is even (or $n = 2$ and m is even).

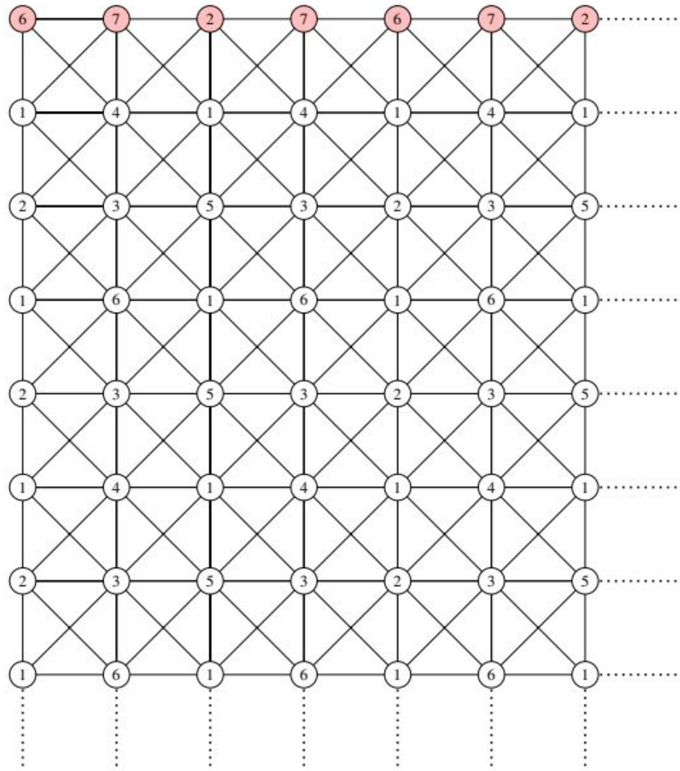
Case 3: $m > 2$ and $n > 2$ are odd.

If m and n are odd integers, by Theorem 2.4.11 and Theorem 6.2.3 $\chi_{lid}(P_m \boxtimes P_n) = 6$.

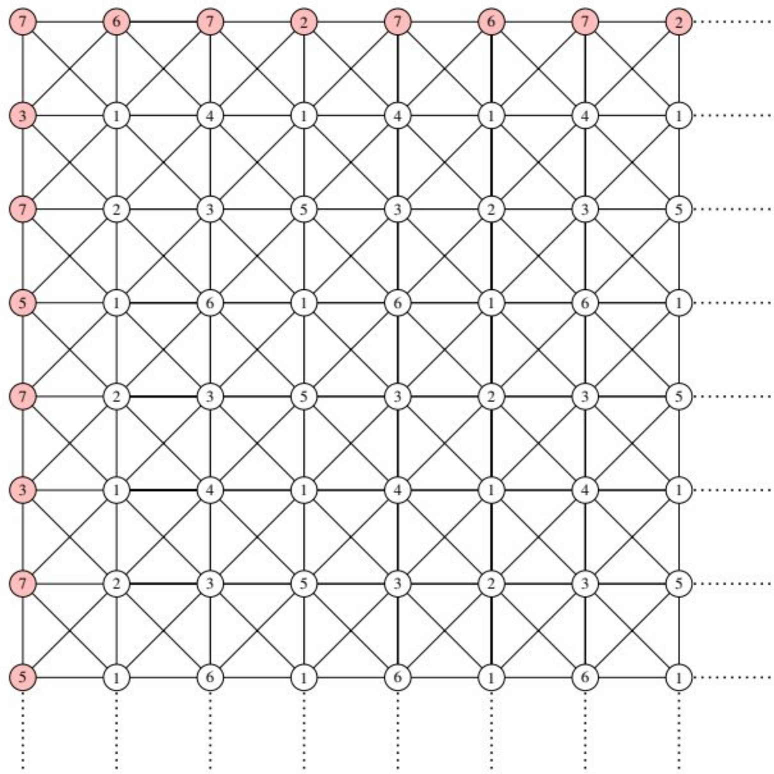
Case 4: $m > 2$ is even and $n > 2$.

In this case, by Lemma 6.2.3, $\chi_{lid}(P_m \boxtimes P_n) \geq 6$. Let c be a 6-lid-coloring of $P_m \boxtimes P_n$. Then $\{1, 2, 3, 4\}$, $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 6\}$ and $\{1, 2, 3, 4, 5, 6\}$ are the only possible set of colors in the closed neighborhoods of the vertices of X. So, a vertex in X must have all the six colors in its closed neighborhood. Since $N[(u_1, v_j)] \subsetneq N[(u_2, v_j)]$, $c(N[(u_1, v_j)]) \neq [6]$ for all j. So, $c(N[(u_2, v_j)])$ is $[6]$ for $j = 1$ or 2. The graph induced by $\{(u_2, v_1), (u_2, v_2), (u_3, v_1), (u_3, v_2)\}$ is K_4. Then $c(N[(u_3, v_j)]) \neq [6]$ for $j = 1, 2$. So, if i is even, then $c(N[(u_i, v_j)]) = [6]$ for $j = 1$ or 2. Since m is even, $c(N[(u_m, v_j)]) = [6]$ for $j = 1$ or 2. Note that, the subgraph induced by $Y = \{(u_m, v_1), (u_m, v_2), (u_{m-1}, v_1), (u_{m-1}, v_2)\}$ is K_4. Since $N[(u_m, v_j)] \subsetneq N[(u_{m-1}, v_j)]$ for all j and $c(N[(u_m, v_j)]) = [6]$ for $j = 1$ or 2, $c(N[(u_{m-1}, v_j)]) = [6]$ for $j = 1$ or 2, a contradiction. So, $\chi_{lid}(P_m \boxtimes P_n) \geq 7$ if m is even.

Since $P_m \boxtimes P_n$ and $P_n \boxtimes P_m$ are isomorphic, $\chi_{lid}(P_m \boxtimes P_n) \geq 7$ if m or n is even. If m or n is even, then we can have a 7-lid-coloring of $P_m \boxtimes P_n$ as in Figure 6.6. □



(a)



(b)

Figure 6.6: (a) A 7-lid-coloring of $P_m \boxtimes P_n$, when m is even and n is odd. (b) A 7-lid-coloring of $P_m \boxtimes P_n$, when m and n are even.

Theorem 6.3.2. *If $m \geq 2$ and $n \geq 4$ is even,*

$$\chi_{lid}(P_m \boxtimes C_n) = \begin{cases} 5 & \text{if } m = 2 \\ 6 & \text{if } m \text{ is odd} \\ 7 & \text{if } m > 2 \text{ is even} \end{cases}$$

Proof. Let $P_m : u_1 - u_2 - \cdots - u_m$, $C_n : v_1 - v_2 - \cdots - v_n - v_1$ and $Y_i = \{(u_i, y) : y \in V(C_n)\}$ and C_n^i be the cycle induced by Y_i. The coloring of the cycles C_n^i can be defined using sequences of numbers: A sequence in bracket, $[M]$, means that we take the sequence M once, the sequence $(M)^*$ means that we can repeat sequence M as many times as we need (or not use it at all). Note that, $\chi_{lid}(P_m \boxtimes C_n) \geq 5$.

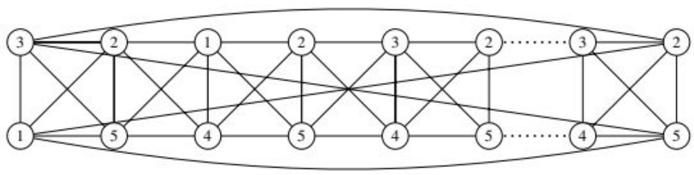


Figure 6.7: 5-lid-coloring of $P_2 \boxtimes C_n$, when $n \equiv 2 \pmod 4$.

Case 1: $m = 2$ and n is even.

If $n \equiv 0 \pmod 4$, then by Theorem 2.4.12 and Corollary 6.2.1 $\chi_{lid}(P_2 \boxtimes C_n) = 5$. If $n \equiv 2 \pmod 4$, then define the coloring c by coloring copies of C_n^i as follows:

- Color C_n^1 with $[32](1232)^*$
- Color C_n^2 with $[15](4545)^*$

This gives a 5-lid-coloring of $P_2 \boxtimes C_n$. See Figure 6.7. So, $\chi_{lid}(P_2 \boxtimes C_n) = 5$, if n is even.

Case 2: $m > 2$ and n is even.

If m is odd, then define the coloring c by coloring copies of C_n^i as follows:

- Color C_n^i with $[32](1232)^*$, if $i \equiv 1 \pmod 4$
- Color C_n^i with $[15](4545)^*$, if $i \equiv 0, 2 \pmod 4$
- Color C_n^i with $[36](1636)^*$, if $i \equiv 3 \pmod 4$

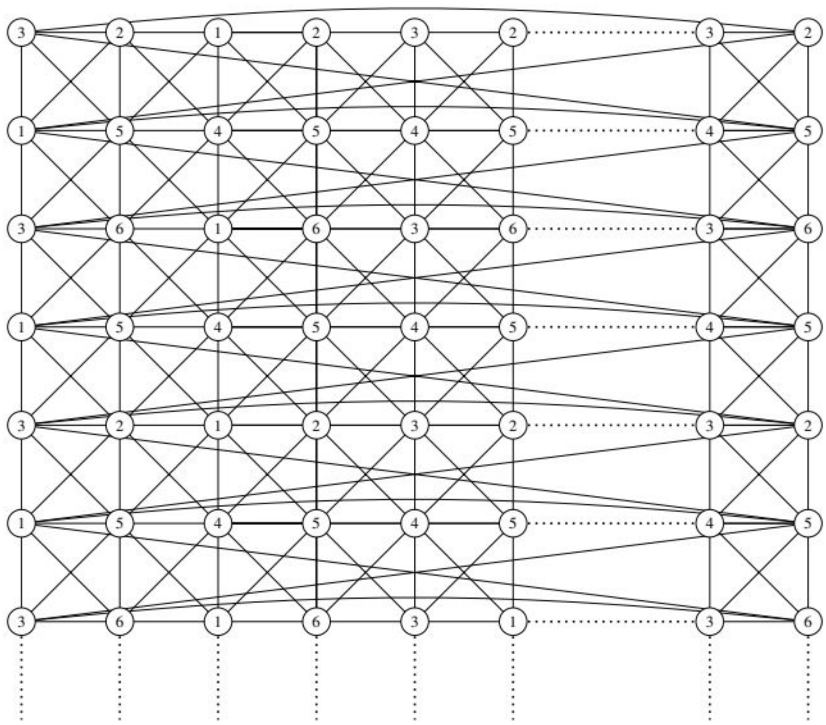


Figure 6.8: A 6-lid-coloring of $P_m \boxtimes C_n$, when $m \geq 3$ is odd and $n \geq 4$ is even.

Then c is a 6-lid-coloring of $P_m \boxtimes C_n$. See Figure 6.8. So, by Lemma 6.2.3, $\chi_{lid}(P_m \boxtimes C_n) = 6$ if $m \geq 3$ is odd and n is even.

If $m > 2$ is even, we proceed similarly as in the Case 4 of the proof of Theorem 6.3.1. Then $\chi_{lid}(P_m \boxtimes C_n) \geq 7$. In this case, Figure 6.9 gives 7-lid-colorings of $P_m \boxtimes C_n$. □

Theorem 6.3.3. *Suppose that $m \geq 2$ and $n \geq 3$ is odd.*

1. $\chi_{lid}(P_2 \boxtimes C_5) = 7$ *and* $\chi_{lid}(P_2 \boxtimes C_n) = 6$.
2. *If m is odd, then* $\chi_{lid}(P_m \boxtimes C_3) = 7$.
3. *If m is even, then* $\chi_{lid}(P_m \boxtimes C_3) = 8$.
4. *If* $n \equiv 1 \pmod 4$, *then* $7 \leq \chi_{lid}(P_m \boxtimes C_n) \leq 8$.
5. *If* $n \equiv 3 \pmod 4$, *then* $7 \leq \chi_{lid}(P_m \boxtimes C_n) \leq 9$.

Proof. Note that, $\chi_{lid}(P_m \boxtimes C_n) \geq 5$.

Case 1: $m = 2$.

If c is a 5-lid-coloring of $P_2 \boxtimes C_n$, then C_n^1 is an odd cycle and $|c(N[z])|$ is 4 or 5 for all

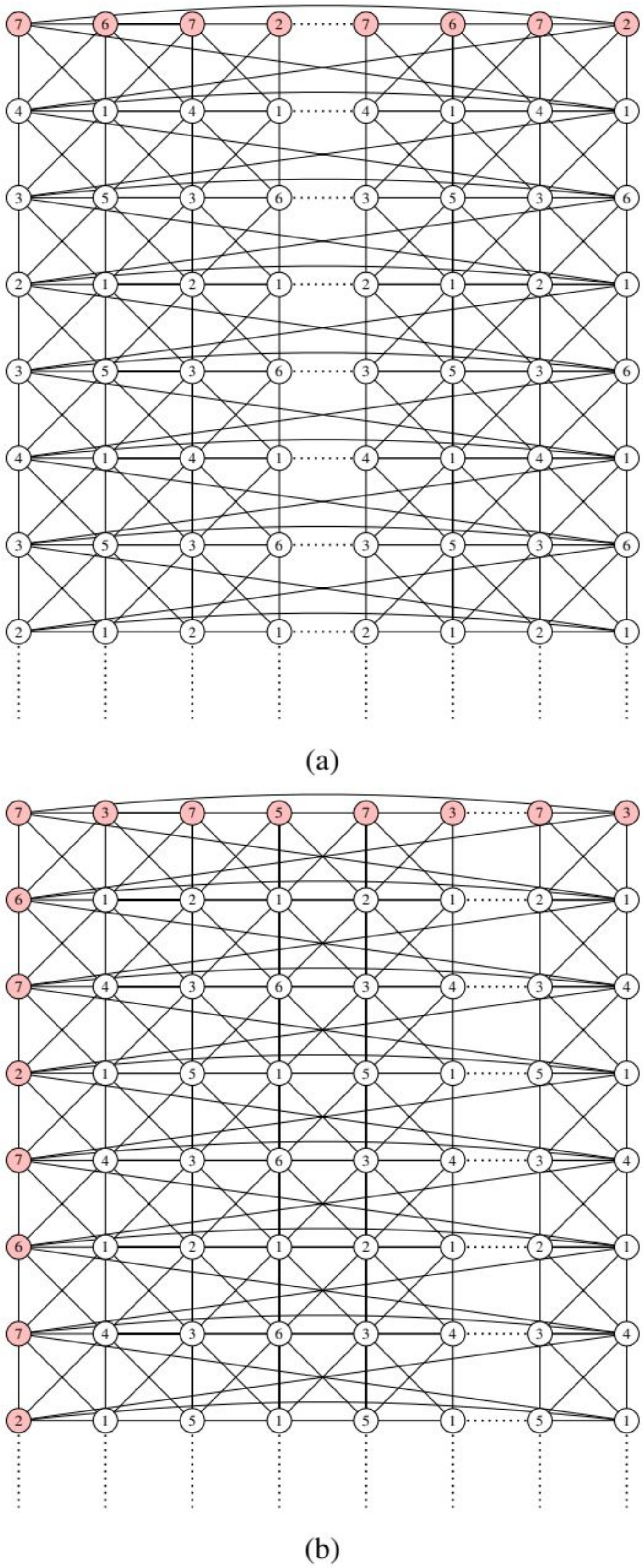


Figure 6.9: (a) A 7-lid-coloring of $P_m \boxtimes C_n$, when $m > 2$ is even and $n \equiv 0 \pmod 4$. (b) A 7-lid-coloring of $P_m \boxtimes C_n$, when $m > 2$ is even and $n \equiv 2 \pmod 4$.

$z \in V(C_n^1)$. Since n is odd, there exists v_j and v_{j+1} such that either $|c(N[(u_1, v_j)])| = 5 = |c(N[(u_1, v_{j+1})])|$ or $|c(N[(u_1, v_j)])| = 4 = |c(N[(u_1, v_{j+1})])|$. If $|c(N[(u_1, v_j)])| = 5 = |c(N[(u_1, v_{j+1})])|$, then $c(N[(u_1, v_j)]) = [5] = c(N[(u_1, v_{j+1})])$, not possible, since $N[(u_1, v_j)] \neq N[(u_1, v_{j+1})]$. Let $|c(N[(u_1, v_j)])| = 4 = |c(N[(u_1, v_{j+1})])|$ and $A = \{(u_1, v_j), (u_1, v_{j+1}), (u_2, v_j), (u_2, v_{j+1})\}$. Then the subgraph induced by A is K_4. Without loss of generality, assume that $c(u_1, v_j) = 1$, $c(u_1, v_{j+1}) = 2$, $c(u_2, v_j) = 3$ and $c(u_2, v_{j+1}) = 4$. So, $[4] \subseteq c(N[z])$ for all $z \in A$. Thus $c(N[(u_1, v_j)]) = [4] = c(N[(u_1, v_{j+1})])$. This is also not possible, since $N[(u_1, v_j)] \neq N[(u_1, v_{j+1})]$. Hence, $\chi_{lid}(P_2 \boxtimes C_n) \geq 6$.

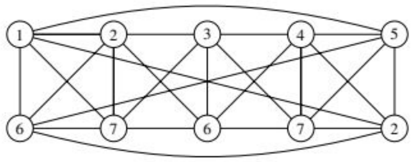


Figure 6.10: 7-lid-coloring of $P_2 \boxtimes C_5$.

Note that, $P_2 \boxtimes C_3 \cong K_6$. If $n = 5$, then Figure 6.10 gives a 7-lid-coloring of $P_2 \boxtimes C_n$. If $n \geq 7$ is odd, define the coloring c by coloring copies of C_n^i as follows:

- ➢ Color C_n^1 with [124341232](1232)* and C_n^2 with [451265656](5656)* if $n \equiv 1 \pmod 4$.
- ➢ Color C_n^1 with [1243412](3242)* and C_n^2 with [5365656](5656)* if $n \equiv 3 \pmod 4$.

Then c is a 6-lid-coloring of $P_2 \boxtimes C_n$ and so, $\chi_{lid}(P_2 \boxtimes C_n) = 6$, if n is odd.

Suppose that $m > 2$. Consider $X_k = \{(u_1, v_k), (u_2, v_k), (u_1, v_{k+1}), (u_2, v_{k+1})\}$, $k = 1, 2, \ldots, n-1$ and $X_n = \{(u_1, v_n), (u_2, v_n), (u_1, v_1), (u_2, v_1)\}$. Then the subgraph induced by X_k is K_4. Consequently, $\{c(u_1, v_k), c(u_2, v_k), c(u_1, v_{k+1}), c(u_2, v_{k+1})\} \subseteq c(N[x])$ for all $x \in X_k$. Since the 4 vertices in X_k must have different set of colors in their closed neighborhoods, there must be at least 6 distinct colors in every lid-coloring. Let c be a 6-lid-coloring of $P_m \boxtimes C_n$. Then a vertex in X_k must have all the 6 colors in its closed neighborhood. Since $N[(u_1, v_j)] \subsetneq N[(u_2, v_j)]$, $c(N[(u_1, v_j)]) \neq [6]$ for all j. So, if $c(N[(u_2, v_1)]) = [6]$, then $c(N[(u_2, v_j)]) = [6]$ for all $j = 1, 3, 5, \ldots$. Since n is odd, $c(N[(u_2, v_n)]) = [6] = c(N[(u_2, v_1)]) = [6]$, a contradiction. Thus $\chi_{lid}(P_m \boxtimes C_n) \geq 7$.

Case 2: $m > 2$ and $n = 3$.

Suppose that $m \geq 3$ is odd. Define the coloring c by coloring copies of C_3^i as follows:

➢ Color C_n^i with $[123]$, if $i \equiv 1 \pmod 4$

➢ Color C_n^i with $[456]$, if $i \equiv 0, 2 \pmod 4$

➢ Color C_n^i with $[173]$, if $i \equiv 3 \pmod 4$

Suppose that c is 7-lid-coloring of $P_m \boxtimes C_n$. If $n = 3$, then $|c(N[(u_i, v_j)])| = 6$ if i is odd and $|c(N[(u_i, v_j)])| = 7$ if i is even, but $|c(N[(u_m, v_j)])| = 6$. So, $\chi_{lid}(P_m \boxtimes C_n) \geq 8$ if m is even. Suppose that $m \geq 4$ is even. Define the coloring c by coloring copies of C_n^i as follows:

➢ Color C_n^i with $[123]$, if $i \equiv 1 \pmod 6$

➢ Color C_n^i with $[456]$, if $i \equiv 0, 2 \pmod 6$

➢ Color C_n^i with $[173]$, if $i \equiv 3, 5 \pmod 6$

➢ Color C_n^i with $[486]$, if $i \equiv 4 \pmod 6$

Case 3: $m > 2$ and $n \equiv 1 \pmod 4$.
Suppose that $m \geq 3$ is odd. Define the coloring c by coloring copies of C_n^i as follows:

➢ Color C_n^i with $[12324](1232)^*$, if $i \equiv 1 \pmod 4$

➢ Color C_n^i with $[54167](5416)^*$, if $i \equiv 0, 2 \pmod 4$

➢ Color C_n^i with $[18284](1828)^*$, if $i \equiv 3 \pmod 4$

Suppose that $m \geq 4$ is even. Define the coloring c by coloring copies of C_n^i as follows:

➢ Color C_n^i with $[12324](1232)^*$, if $i \equiv 1 \pmod 6$

➢ Color C_n^i with $[54167](5416)^*$, if $i \equiv 0, 2 \pmod 6$

➢ Color C_n^i with $[18284](1828)^*$, if $i \equiv 3, 5 \pmod 6$

➢ Color C_n^i with $[64173](6417)^*$, if $i \equiv 4 \pmod 6$

Case 4: $m > 2$ and $n \equiv 3 \pmod 4$.
Suppose that $m \geq 3$ is odd. Define the coloring c by coloring copies of C_n^i as follows:

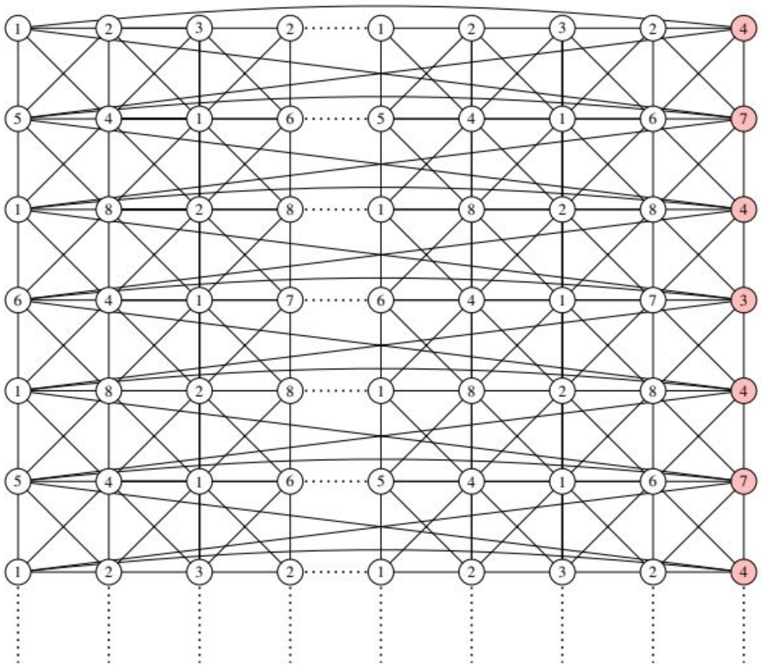
➢ Color C_n^i with $[1232123](1232)^*$, if $i \equiv 1 \pmod 4$

- ➢ Color C_n^i with $[5414657](5414)^*$, if $i \equiv 0, 2 \pmod 4$
- ➢ Color C_n^i with $[1828182](1828)^*$, if $i \equiv 3 \pmod 4$

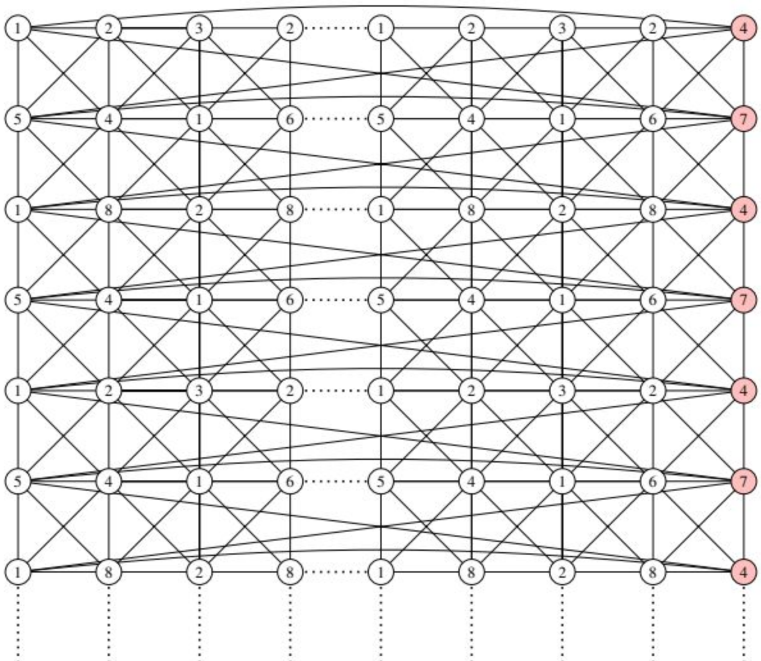
If $m \geq 4$ is even, then define the coloring c by coloring copies of C_n^i as follows:

- ➢ Color C_n^i with $[1243412](3242)^*$, if $i \equiv 1 \pmod 6$
- ➢ Color C_n^i with $[5365656](5656)^*$, if $i \equiv 0, 2 \pmod 6$
- ➢ Color C_n^i with $[1749417](3747)^*$, if $i \equiv 3, 5 \pmod 6$
- ➢ Color C_n^i with $[8368686](8686)^*$, if $i \equiv 4 \pmod 6$

This completes the proof. □



(a)



(b)

Figure 6.11: (a) A 8-lid-coloring of $P_m \boxtimes C_n$, when $m > 2$ is even and $n \equiv 1 \pmod 4$. (b) A 8-lid-coloring of $P_m \boxtimes C_n$, when $m > 2$ is odd and $n \equiv 1 \pmod 4$.

Chapter 7

Recommendations

Coupon coloring

Coupon coloring number of a graph is the maximum number of disjoint total dominating sets in that graph. The concept of total domination is extensively studied in graph theory. Many authors found the total domination number of graphs, but finding the coupon coloring number is a different task. Investigating the coupon coloring of a class of graphs is an interesting topic for research.

In 2017, Wyatt J. Desormeaux et al.[29] posed a question: Is it true that every connected cubic graph containing a 3-cycle has two vertex disjoint total dominating sets? Recently S. Akbari et al.[6] were studied 2-coupon coloring of cubic graphs containing 3-cycle or 4-cycle. They constructed a cubic graph with 60 vertices containing a 3-cycle whose vertex set cannot be partitioned into two total dominating sets. Hence, similar problems can be constructed for research.

In our study, we have found several bounds for the coupon coloring number. An interested researcher can try to improve those bounds.

Locally identifying coloring

Locally identifying coloring is comparatively new vertex coloring in literature. Lid-coloring of Cartesian product and tensor product of two graphs were studied in [16]. We have investigated the lid-coloring of some other graph products like rooted product corona product and the strong product.

We have studied lid-coloring of rooted product of two graphs and proved that $\chi_{lid}(H)$ is a lower bound for $\chi_{lid}(G \circ H)$ only if G and H are connected bipartite graphs. We strongly believe that the lid-chromatic number of H is a lower bound for $\chi_{lid}(G \circ H)$. It is open to prove this lower bound.

In chapter 6, lid-coloring of strong product of graphs were studied. The lid-chromatic number of strong product of two 3-lid-colorable connected bipartite graphs found to be 6. Note that, lid-chromatic number of bipartite graph if at most 4. So, lid-chromatic number of strong product of two connected bipartite graphs can be investigated.

List of publications and papers presented in conferences

List of publications

1. Reji T. and Pavithra R., On Coupon Coloring of Lexicographic Product of Graphs, *The Art of Discrete and Applied Mathematics*, Vol. 6 No. 1 (2023) #P1.03.

2. Reji T. and Pavithra R., Coupon Coloring of Ideal-Based Zero-Divisor Graphs, *South East Asian Journal of Mathematics and Mathematical Sciences*, Vol. 21, Proceedings (2022) 183 – 190.

3. Reji T. and Pavithra R., On Coupon Coloring of Cayley Graphs: In Amitabha Bagchi, Rahul Muthu(eds) Algorithms and Discrete Applied Mathematics. CALDAM 2023, *Lecture Notes in Computer Science*, Vol. 13947 (2023) Springer, Cham.

4. Reji T. and Pavithra R., On Coupon Coloring of Some Cayley Graphs, *Palestine Journal Mathematics*, Vol. 12(1), 2023.

5. Reji T. and Pavithra R., On Coupon Coloring of Rooted Product of Graphs, *Palestine Journal Mathematics*, Vol. 12(Special Issue II) 2023.

6. Pavithra R. and Reji T., On Locally Identifying Coloring of Corona Product of Graphs, *Discrete Mathematics, Algorithms and Applications*, 2450032, 2024.

7. Pavithra R. and Reji T., On Locally Identifying Coloring of Strong Product of Graphs (Communicated).

List of paper presentations

1. Presented a paper entitled "On coupon coloring of Cayley graphs" in the International Conference on Number Theory and Discrete Mathematics (ICNTDM-2020) organised by Ramanujan Mathematical Society and hosted by Rajagiri School of Engineering and Technology during December 11–14, 2020.

2. Presented a paper entitled "Coupon coloring of rooted product graphs" in the International Conference on Discrete Mathematics (ICDM-2021) organized by Dept. of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, in association with Academy of Discrete Mathematics and Applications (ADMA) during October 11–13, 2021.

3. Presented a paper entitled "Coupon coloring of direct product graphs" in the International Conference on Graphs, Combinatorics and Otimization (Online) organized by Dept. of Mathematics, BITS-Pilani, Dubai Campus, during February 6–8, 2022.

4. Presented a paper entitled "Coupon coloring of ideal-based zero-divisor graphs" in the Rev. Dr. Albert Muthumalai SJ Endowment Conference on Mathematical Sciences and Applications(Virtual) ICMSA 2022 organised by Dept. of Mathematics, St. Josephs College (Autonomous), Tiruchirappalli during March 9–11, 2022.

5. Presented a paper entitled "Coupon coloring of Cayley graphs" in the 9th Annual International Conference on Algorithms and Discrete Applied Mathematics (CALDAM 2023) organized by the Dhirubhai Ambani Institute of Information and Communication Technology, Gandhinagar, Gujarat during February 9–11, 2023.

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