
Ph.D. THESIS

MATHEMATICS

**RESULTS OF STUDIES ON RESOLVABILITY
AND METRIC DIMENSION IN GRAPHS**

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I hereby certify that the thesis entitled "**Results of Studies on Resolvability and Metric Dimension in Graphs**" is a bonafide work carried out by **Sri. Shamsudheen P. V**, under my guidance for the award of Degree of Ph.D. in Mathematics of the M.E.S Mampad College(Autonomous) and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.



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DECLARATION

I hereby declare that the thesis, entitled “**Results of Studies on Resolvability and Metric Dimension in Graphs**” is based on the original work done by me under the supervision of **Dr. Shahida A. T**, Assistant Professor, Department of Mathematics, Mes Mampad College and has not been included in any other thesis submitted previously for the award of any degree. The contents of the thesis are undergone plagiarism check using iThenticate software at C.H.M.K. Library , University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.



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ABSTRACT

This thesis explores advanced topics in graph theory, focusing on metric dimension, neighborhood resolving sets, and the ϕ_2 -polynomial across various graph families and applications. It begins with the analysis of metric dimension and bases in graph composition products, particularly compositions involving star graphs and empty graphs. Graphs such as $S_t(O_t)$, $S_t[S_t]$, and other variants are examined for their metric dimension, and the properties of their resolving sets are characterized in depth. Study the detailed analysis of neighborhood resolving sets in all connected graphs of order at most six, providing insight into the structure and uniqueness of vertex identification based on their neighborhoods. Further, the neighborhood resolvability of specific graph classes such as the bipartite graph $K_{r,t}$ and graph sums $P_t + P_t$, $P_t + O_t$ are studied extensively. These investigations contribute to understanding the distinct neighborhood structures and resolving capabilities within complex graph constructions. The thesis also delves into the properties of the ϕ_2 -polynomial for various graph families, including $P_m + P_n$, $P_n + O_n$, $K_n - e$, and the Cartesian product $P_m \times P_n$. These analyses offer insights into how vertex degrees, distances, and edge distributions shape the structural identity of graphs.

A novel interdisciplinary approach is presented by modeling the 12 Zodiac signs as graphs and studying their interrelationships using the ϕ_2 -polynomial, merging mathematical structure with symbolic representations. Another impor-

tant aspect involves the study of the ϕ_2 -polynomial and neighborhood resolvability in zero divisor graphs of small finite commutative rings of order at most 10, revealing deep connections between algebraic properties and graph invariants. The practical relevance of neighborhood resolving sets is highlighted through two real-world applications: (1) selecting a nodal center for effective drug distribution networks, and (2) optimizing construction labor management by modeling trade-specific laborers as graph vertices. These case studies demonstrate the applicability of graph-theoretic tools to problems in logistics, resource allocation, and operational efficiency. The thesis concludes with a summary of key findings and recommendations for future research in combinatorial optimization and applied graph theory.

സംഗ്രഹം

ഈ പ്രബന്ധം ഗ്രാഫ് സിദ്ധാന്തത്തിലെ വിപുലമായ വിഷയങ്ങൾ പര്യവേക്ഷണം ചെയ്യുന്നു. വിവിധ ഗ്രാഫ് കുടുംബങ്ങളിലും ആപ്ലിക്കേഷനുകളിലും മെട്രിക് അളവ്, അയൽപക്ക പരിഹാര സെറ്റുകൾ, ഫൈ 2-പോളിനോമിയൽ എന്നിവയിൽ ശ്രദ്ധ കേന്ദ്രീകരിക്കുന്നു. ഗ്രാഫ് കോമ്പോസിഷൻ ഉൽപ്പന്നങ്ങളിലെ മെട്രിക് അളവുകളുടെയും ബേസുകളുടെയും വിശകലനത്തോടൊണ് ഇത് ആരംഭിക്കുന്നത്, പ്രത്യേകിച്ച് നക്ഷത്ര ഗ്രാഫുകൾ, ശൂന്യമായ ഗ്രാഫുകൾ, എന്നിവ ഉൾപ്പെടുന്ന കോമ്പോസിഷനുകൾ. $S_1[O_1], S_2[S_2]$ മറ്റ് വകഭേദങ്ങൾ തുടങ്ങിയ ഗ്രാഫുകൾ അവയുടെ മെട്രിക് അളവിനായി പരിശോധിക്കുന്നു, കൂടാതെ അവയുടെ പരിഹരിക്കൽ സെറ്റുകളുടെ സവിശേഷതകൾ ആഴത്തിൽ ചിത്രീകരിച്ചിരിക്കുന്നു. ബന്ധിപ്പിച്ചിരിക്കുന്ന എല്ലാ ഗ്രാഫുകളിലും അയൽപക്ക പരിഹാര സെറ്റുകളുടെ വിശദമായ വിശകലനം പഠിക്കുക, പരമാവധി ആറ് എണ്ണം, അവയുടെ അയൽപക്കങ്ങളെ അടിസ്ഥാനമാക്കിയുള്ള വെർട്ടെക്സ് ഐഡന്റിഫിക്കേഷന്റെ ഘടനയെയും അതുല്യതയെയും കുറിച്ചുള്ള ഉൾക്കാഴ്ച നൽകുന്നു. കൂടാതെ, ബൈപാർട്ടൈറ്റ് ഗ്രാഫ് $K_{r,s}$, ഗ്രാഫ് സംഗ്രഹങ്ങൾ $P_r + P_s, P_r + O_s$

എന്നിവ പോലുള്ള നിർദ്ദിഷ്ട ഗ്രാഫ് ക്ലാസുകളുടെ അയൽപക്ക പരിഹാരക്ഷമത വിപുലമായി പഠിക്കുന്നു. സങ്കീർണ്ണമായ ഗ്രാഫ് നിർമ്മാണങ്ങൾക്കുള്ളിലെ വ്യത്യസ്തമായ അയൽപക്ക ഘടനകളും പരിഹരിക്കൽ കഴിവുകളും മനസ്സിലാക്കുന്നതിന് ഈ അന്വേഷണങ്ങൾ സംഭാവന ചെയ്യുന്നു. $P_m + P_n, P_n + O_n, K_{n-1}$ കാർട്ടീഷ്യൻ ഉൽപ്പന്നം $P_m \times P_n$ എന്നിവയുൾപ്പെടെ വിവിധ ഗ്രാഫ് കുടുംബങ്ങൾക്കായുള്ള ഫൈ -2 പോളിനോമിയലിന്റെ ഗുണങ്ങളെക്കുറിച്ചും ഈ പ്രബന്ധം ആഴത്തിൽ പരിശോധിക്കുന്നു. ഗ്രാഫുകളുടെ ശീർഷക ഡിഗ്രികൾ, ദൂരങ്ങൾ, എഡ്ജ് ഡിസ്ക്രിബ്യൂഷനുകൾ എന്നിവ എങ്ങനെ ഘടനാപരമായ

ഐഡൻറിറ്റിയെ രൂപപ്പെടുത്തുന്നു എന്നതിനെക്കുറിച്ചുള്ള ഉൾക്കാഴ്ചകൾ ഈ വിശകലനങ്ങൾ നൽകുന്നു.

12 രാശിചിഹ്നങ്ങളെ ഗ്രാഹുകളായി മാതൃകയാക്കിയും ഫൈ -2 പോളിനോമിയൽ ഉപയോഗിച്ച് അവയുടെ പരസ്പരബന്ധങ്ങൾ പഠിച്ചും, ഗണിതശാസ്ത്ര ഘടനയെ പ്രതീകാത്മക പ്രാതിനിധ്യങ്ങളുമായി ലയിപ്പിച്ചും ഒരു നൂതനമായ ഇന്റർ ഡിസിപ്ലിനറി സമീപനം അവതരിപ്പിക്കുന്നു. മറ്റൊരു പ്രധാന വശം, പരമാവധി 10 എന്ന ക്രമത്തിലുള്ള ഫൈനിറ്റ് കമ്മ്യൂട്ടറ്റീവ് റിങ്ങിലെ സീറോ ഡിവൈസർ ഗ്രാഹുകളിലെ ഫൈ -2 പോളിനോമിയലിന്റേയും അയൽപക്ക പരിഹാരത്തിന്റേയും, മെട്രിക് ഡൈമെൻഷൻ ന്റേയും പഠനം ഉൾപ്പെടുന്നു, ഇത് ബീജഗണിത ഗുണങ്ങളും ഗ്രാഫ് മാറ്റങ്ങളും തമ്മിലുള്ള ആഴത്തിലുള്ള ബന്ധങ്ങൾ വെളിപ്പെടുത്തുന്നു. അയൽപക്ക പരിഹാര സെറ്റുകളുടെ പ്രായോഗിക പ്രസക്തി രണ്ട് യഥാർത്ഥ പ്രയോഗങ്ങളിലൂടെ എടുത്തുകാണിക്കുന്നു: (1) ഫലപ്രദമായ ആരോഗ്യ മരുന്ന് വസ്തുക്കളുടെ വിതരണ ശൃംഖലകൾക്കായി ഒരു നോഡൽ കേന്ദ്രം തിരഞ്ഞെടുക്കൽ, (2) വ്യാപാര-നിർദ്ദിഷ്ട തൊഴിലാളികളെ ഗ്രാഫ് വെർട്ടീസുകളായി മാതൃകയാക്കി നിർമ്മാണ തൊഴിൽ മാനേജ്മെന്റ് ഒപ്റ്റിമൈസ് ചെയ്യുക. ലോജിസ്റ്റിക്സ്, റിസോഴ്സ് അലോക്കേഷൻ, പ്രവർത്തന കാര്യക്ഷമത എന്നിവയിലെ പ്രശ്നങ്ങൾക്ക് ഗ്രാഫ്-സൈദ്ധാന്തിക ഉപകരണങ്ങളുടെ പ്രയോഗക്ഷമത ഈ കേസ് പഠനങ്ങൾ തെളിയിക്കുന്നു. കോമ്പിനേറ്റോറിയൽ ഒപ്റ്റിമൈസേഷനിലും പ്രായോഗിക ഗ്രാഫ് സിദ്ധാന്തത്തിലും ഭാവി ഗവേഷണത്തിനായുള്ള പ്രധാന കണ്ടെത്തലുകളുടെയും ശുപാർശകളുടെയും സംഗ്രഹത്തോടെയാണ് പ്രബന്ധം അവസാനിക്കുന്നത്.

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List of symbols

G	A simple finite graph
$E(G)$	Edge set of G
$V(G)$	Vertex set of G
$G_1[G_2]$	Composition product of G_1 and G_2
$d(u, v)$	Distance between two vertices u and v
$r(v/W)$	Resolving set of v w.r.t W
$\beta(G)$	Metric dimension of G
$dim(G)$	Metric dimension of G
$N(v)$	Set of all adjacency vertices of v
$N[v]$	$N(v) \cup v$
r	Resolving set
nr	Neighbourhood resolving set
$ln(G)$	Least cardinality of neighbourhood set
$lnr(G)$	Least cardinality of neighbourhood resolving set
K_n	Complete graph on n vertices
P_n	Path on n vertices

List of symbols

C_n	Cycle on n vertices
$K_{m,n}$	Complete bipartite graph with $m + n$ Vertices
S_t	Star graph
O_t	Empty graph
$G_1 + G_2$	Sum of the graphs G_1 and G_2
$G_1 \times G_2$	Product of the graphs G_1 and G_2
$\lambda_2(x)$	The 2^{th} Neighbour of the vertex x
$\phi_2 - polynomial$	The 2^{th} Degree polynomial
$\phi_2(G, x, y)$	The 2^{th} degree polynomial of G with x and y
$G - e$	An edge removed from the graph G
$\alpha_2(G)$	Minimum number of $\lambda_2(x)$ of G
$\beta_2(G)$	Maximum number of $\lambda_2(x)$ of G
$ \{u : x - u\} $	Cardinality of edges

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Introduction

Graph theory is one of the most well-established branches of mathematics that are in high demand. With its roots in the modeling and negative resolution of the famous Konigsberg bridge problem by Leonard Euler [1], graph theory has strongly established itself as one of the best tools to model network systems involved in complex problems that often face everyday life. The main attraction of graph theory lies in the wide range of fields in which it can be applied, from network theory, chemistry, and operational research, to architecture and morphology, in which it has numerous application possibility. There is no doubt that graph theory is an amazing branch of applied mathematics, which performs the function of translating real-life problems into mathematical models that can be analyzed mathematically. In order to understand the dynamic properties and structural properties of graphs, metric dimensions are one of the best-studied concepts in graph theory as they provide understanding into the structural and active properties of graphs Metric dimension [2] is a fundamental concept in graph theory that estimates the minimum number of landmarks (vertices) required to uniquely determine the distances between all pairs of ver-

tices in a graph. This concept plays a critical role in various network-related problems, such as navigation systems, facility location, and fail-safe analysis. Understanding metric dimension allows us to gain understanding into the structural characteristics of a graph and its navigational properties. Neighbourhood resolvability [4] is a concept closely related to metric dimension. It focuses on the ability of a set of vertices to distinguish between different neighborhoods in a graph. Specifically, it seeks to determine the smallest number of vertices required to uniquely identify any two vertices based on their respective neighborhoods. Neighbourhood resolvability has applications in areas like network security, sensor network design, and community detection in social networks, making it an engaging and practically relevant subject of study. The complicated interplay between metric dimension, neighbourhood resolvability, and ϕ_k -polynomials [6] in connected graphs remains an exciting and evolving area of research. This thesis aims to contribute to the existing body of knowledge by exploring the connections between these concepts, presenting novel understandings, and providing practical applications and algorithms to address real-world problems. In the subsequent chapters of this thesis, we will investigate into each of these topics in greater detail, highlighting their significance, reviewing related literature, and presenting new results and findings that contribute to our understanding of metric dimension, neighbourhood resolvability, and ϕ_k -polynomials in the context of connected graphs. By the end of this research, it is our hope that readers will gain a deeper appreciation for the rich hanging of ideas that support these graph-theoretic concepts and their potential applications in solving complex problems in various domains.

An overview of the thesis

The thesis contains an introduction chapter as well as six chapters that deal with the metric dimension, neighborhood resolvability, and ϕ_k -polynomials of some graph families, and they are investigated in detail in each chapter. There is a compact description provided in the introductory chapter which explains the motivational background behind the introduction of the graph polynomial and the resolvability of graphs. It is also provided with a discription for the upcoming chapters so that participants can anticipate what to expect.

In **chapter 1**, the terminology and notations that will appear in the subsequent chapters are detailed. This chapter shows basic definitions and findings in metric dimension,neighbourhood resolving set, zero divisor graph and polynomial properties which will be further discussed in the following chapters.

chapter 2, A subset S of vertices resolves a graph G if every vertex of G is uniquely determined by its vector of distances to the vertices in S . A resolving set of minimum cardinality for a graph G is called a minimum resolving set. A minimum resolving set is usually called a basis for G and the cardinality of basis is called the metric dimension of G , denoted by $dim(G)$. For the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ its composition product is denoted by $G_1[G_2]$ is the graph and two vertices (u, v) and (x, y) are adjacent in $G_1[G_2]$ whenever $ux \in E_1$ or, $u = x$ and $vy \in E_2$. In this chapter metric dimension of composition product of $S_n[O_n], S_n[S_n], S_n[S_m], S_n[O_m]$ are studied. Additionally discuss about general properties of the bases of graphs.

chapter 3, Studied the neighbourhood resolvability of connected graphs with orders at most 6. Determined the properties of resolvability as well as the metric

dimensions of these graphs. More than hundreds of graphs with order at most 6 are examined in this chapter. The concept of "neighborhood resolving set" is closely related to graph theory, specifically in the study of metric dimension. Before investigating into neighborhood resolving sets, let's establish some fundamental concepts; graphs, distance in graphs, metric dimension. In the context of graph theory, a neighborhood resolving set is a specific type of resolving set that focuses on the neighborhoods of vertices.

chapter 4, Examined the Neighbourhood-resolving sets on Bipartite graphs, Composition of star graphs and empty graph, Composition of star graphs, Sum of path graph and Sum of path graphs and empty graph. Given a graph G , the neighborhood resolving set is a subset of vertices (sets of vertices adjacent to a given vertex) uniquely determines the identity of each vertex in the graph. Formally, a set S of vertices in a graph G is a neighborhood resolving set if, for every pair of distinct vertices u and v in G , the sets of vertices adjacent to u and v are distinct and the union of adjacency vertices of each vertex in the set S is G . The study of neighborhood resolving sets is motivated by applications in network design, communication, and location-based services, where it is preferred to have a small set of tactically placed landmarks that can efficiently guide navigation or information recovery. Understanding the properties and sizes of neighborhood resolving sets contributes to understanding into the structure and behavior of graphs, and it has practical implications in various real-world situation involving network analysis and optimization. Researchers often explore the existence and properties of neighborhood resolving sets in different classes of graphs and investigate algorithmic approaches to finding them efficiently.

chapter 5, Graph theory provides a rich framework for analyzing complex net-

works and structures. Among the various tools available, graph polynomials offer powerful means to summarize and study the properties of graphs. This chapter explores the ϕ_k -polynomial, an advanced graph polynomial that encodes the distribution of edges based on the k^{th} degrees of their endpoints. The k^{th} degree of a vertex extends the traditional concept of vertex degree by considering distances between vertices, thereby providing a more refined understanding of graph structure. This polynomial is particularly useful for characterizing graph families and analyzing their combinatorial properties. The ϕ_k -polynomial is a versatile and powerful tool in graph theory, offering a detailed perspective on the distribution of edges based on distance-based vertex degrees. Its ability to characterize graph families and provide insights into their structural properties makes it a valuable addition to the collection of graph polynomials, with broad implications for both theoretical research and practical applications in various fields.

Chapter 6, Graph theory provides a healthy framework for analyzing and understanding complex networks and structures. In this chapter, we explore the graphical and ϕ_k -polynomial properties of the 12 Zodiac signs, a novel and interdisciplinary approach that combines astrological concepts with mathematical accuracy. By representing the relationships and characteristics of the Zodiac signs as graphs, and subsequently analyzing these graphs through the lens of the ϕ_k -polynomial, we aim to uncover new insights into their interconnectedness and essential properties. To analyze the Zodiac signs using graph theory, we represent each sign as a vertex in a graph. Edges between vertices denote significant relationships or interactions between the signs. The primary objectives of this chapter are to develop graph representations for the 12 Zodiac signs,

capturing their interactions and relationships, examine the structural properties of these graphs using traditional graph invariants and metrics, calculate the ϕ_k -polynomial for the Zodiac graphs to reveal detailed information about their edge distributions and structural refinement, correlate the mathematical findings with astrological interpretations, providing a deeper understanding of Zodiac sign interactions through a graph-theoretical lens. By applying graph theory and the ϕ_k -polynomial, we aim to uncover patterns and properties that are not immediately visible through traditional astrological analysis. This approach not only boost our understanding of the Zodiac but also demonstrates the versatility and applicability of graph theoretical methods in different fields.

Chapter 7, Graph theory serves as a powerful tool for understanding and analyzing the structural properties of various algebraic systems. In this we explore into the metric dimension and neighborhood resolving sets of zero divisor graphs associated with small finite commutative rings of order at most 10. By examining these specific graph properties, we aim to uncover new insights into the complex relationships and characteristics fundamental in these algebraic structures. The primary objectives of this chapter are to: develop and analyze the zero divisor graphs for all small finite commutative rings of order at most 10, compute the metric dimension for these zero divisor graphs, identifying the minimum resolving sets, determine neighborhood resolving sets for these graphs, exploring their implications for local graph structure, Investigate the relationships between the algebraic properties of the rings and the graphical properties of their zero divisor graphs. Understanding the metric dimension and neighborhood resolving sets of zero divisor graphs can provide valuable insights into the underlying algebraic structures. These concepts not only contribute to the theoretical development

of algebraic graph theory but also have practical applications in areas such as network design, coding theory, and cryptography. By focusing on small finite commutative rings, this chapter aims to provide an extensive analysis that balances theoretical accuracy with practical relevance. The results obtained here will serve as a foundation for further studies on larger and more complex algebraic systems, laying the way for new discoveries and applications in both mathematics and computer science. In summary, this chapter offers a detailed exploration of the metric dimension and neighborhood resolving sets of zero divisor graphs for small finite commutative rings. Through this investigation, we seek to improve our understanding of the elaborate connections between algebraic and graphical structures, demonstrating the serious interplay between these two fields.

chapter 8, Graph-based problem solving is a fundamental approach in many real-world applications. One such concept, the *neighborhood resolving set* (NRS), plays a significant role in optimizing network structures by uniquely identifying vertices based on their neighborhoods. This chapter explores two key applications of neighborhood resolving sets: Selecting Nodal Centre for Effective Management of Drug Distribution, Optimizing Construction Labor Management Using Neighborhood Resolving Sets. Using graph theory concepts, we model the geographical layout as a graph where the nodes represent emergency drug warehouses and service center, and edges represent the closeness between these establishments based on the defined distances. The goal is to select a nodal center (Emergency drug warehouses or Service center, or both) for effective drug distribution, which ensures that the distance-based connections between vertices are minimized and the network remains connected. Efficient labor management in construction projects is critical for reducing costs, improving productivity, and ensuring timely project

completion. This paper introduces the application of Neighborhood Resolving Sets (NRS) from graph theory to optimize labor task allocation and wage distribution among workers in a construction project. The construction workforce is represented as a graph, where workers from various trades such as carpenters, plumbers, electricians, and masons are modeled as vertices, and communication or assistance between them forms the edges. We aim to identify key laborers using the NRS to ensure efficient work allocation and fair wage distribution.

Chapter 9 concludes the findings of the study and **Chapter 10** provides recommendations for future research in this area.

Chapter 1

Preliminaries

This chapter studies the notations, basic definitions and findings in metric dimension, neighbourhood resolving set and polynomial properties which will be further discussed in the following chapters.

This chapter includes the basic definitions, notations, terminologies and fundamental results in graph theory and which are helpful in the forthcoming chapters.

1.1 Metric dimension

Definition 1.1.1. [2] Let $G = (V, E)$ be a connected, undirected graph and $x, u, v \in V$. A vertex x is said to resolve the vertices u and v if the distance of x from u is different from distance of x from v . that is $d(u, x) \neq d(v, x)$.

Definition 1.1.2. [9] A vertex $x \in V(G)$ is said to resolve a pair of vertices u, v in G if $d(u, x) \neq d(v, x)$.

Definition 1.1.3. [8] Let $G = (V, E)$ be a graph, For an ordered subset $W = \{w_1, w_2, \dots, w_k\}$ of $V(G)$ and for any vertex $v \in V$, the metric representation of v with respect to W is the k – vector which is denoted and defined as $r(v/W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$.

Definition 1.1.4. [2] The set W is called a resolving set for G , if $r(v_1/W) = r(v_2/W)$ implies that $v_1 = v_2$ for all $v_1, v_2 \in V(G)$.

Definition 1.1.5. [8] [10] The minimum cardinality of a resolving set of G is called the metric dimension of G and is denoted by $dim(G)$.

Definition 1.1.6. [8] [10] A resolving set of minimum cardinality for a graph G is called a minimum resolving set. A minimum resolving set is usually called a bases for G .

Definition 1.1.7. [8] A resolving set W is called a minimal resolving set of a connected graph G if no proper subset of W is a resolving set of G

Definition 1.1.8. [8] A bases W for a graph G is said to connected if the sub-graph induced by W is connected.

Definition 1.1.9. [8] A bases W of G is said to be independent if no two vertices of W are adjacent.

Definition 1.1.10. [8] The join of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is a graph $G = (V; E)$ such that $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, edge $uv : u \in V_1$ and $v \in V_2$

Definition 1.1.11. [7] The composition product of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is the graph G , denoted by $G_1[G_2]$, whose vertex set is $V_1 \times V_2$ and two

vertices (u_i, v_l) and (u_j, v_m) are adjacent in G whenever $u_i u_j \in E_1$ or $u_i = u_j$ and $v_l v_m \in E_2$. The number of vertices in above composition graph is $V_1 \times V_2$, number of edges is $(V_1 \times E_2) + (E_1 \times V_2^2)$

Definition 1.1.12. [9] The Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 is a graph with:

- Vertex set: $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, where each vertex is a pair (v_1, v_2) with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$.
- Edge set: Two vertices (v_1, v_2) and (u_1, u_2) are adjacent if:
 - $v_1 = u_1$ and v_2 is adjacent to u_2 , or
 - $v_2 = u_2$ and v_1 is adjacent to u_1 .

Theorem 1.1.13. [7] The metric dimension of the star graph S_n is $n - 2$ for $n > 1$

Theorem 1.1.14. [5]

The metric dimension of C_n is $\dim(C_n) = 2$

Theorem 1.1.15. [7] A connected graph G of order $n > 2$ has dimension $n - 1$ if and only if $G = K_n$

Theorem 1.1.16. [8] A connected graph G of order n has metric dimension 1 if and only if $G = P_n$.

Theorem 1.1.17. [5] For given positive integers m and n the metric dimension

$$\text{of } \dim((P_n + P_m)) = \begin{cases} 1 & \text{if } n = 2, m = 1 \\ 2 & \text{if } 2 \leq m \leq 3 \\ \lfloor \frac{m}{2} \rfloor + n - 2 & \text{if } n \geq 1, m \geq 4, \end{cases}$$

Theorem 1.1.18. [8] *The metric dimension of $K_{1,n}$ is $\dim(K_{1,n}) = n - 1$*

Example 1.1.19. *Let G be the graph given with 4 vertices*

$W = \{1, 2\}$ form the resolving $r(1/W) = (0, 1)$, $r(2/W) = (1, 0)$, $r(3/W) = (2, 1)$, $r(4/W) = (1, 1)$. W is bases and also connected $\dim(G) = 2$

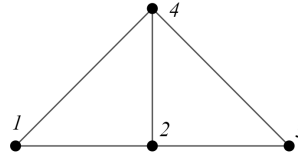


Figure. 2a

1.2 Neighbourhood resolving set

Definition 1.2.1. [12] *Let $G = (V, E)$ be a connected, undirected graph and let $v \in V$, then closed neighborhood of v is denoted by $N[v]$, and defined by $N[v] = \{u, : u \text{ adjacent to } v\} \cup \{v\}$*

Definition 1.2.2. [12] *For a subset $S = \{s_1, s_2, \dots, s_k\}$ of $V(G)$ said to be neighbourhood set, if $N[s_1] \cup N[s_2] \cup \dots \cup N[s_k] = G$. A neighbourhood set S of a connected graph G is called a minimal neighbourhood set if no proper subset of S is a neighbourhood set of G . The least cardinality of a minimal neighbourhood set of G is called the neighbourhood number of G and is denoted by $ln(G)$.*

Definition 1.2.3. [12] *Let $G = (V, E)$ be a connected, undirected graph and $v_1, v_2, v_3 \in V$. A vertex v_2 said to resolve the vertices v_1 and v_3 if the distance of v_1 from v_2 is different from distance of v_3 from v_2 .*

Definition 1.2.4. [12] *For an ordered subset $W = \{w_1, w_2, \dots, w_k\}$ of $V(G)$ and for any vertex $v \in V$, the (metric)representation of v with respect to W is the k -*

vector which is denoted and defined as $r(v/W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$.

The set W is called a resolving set for G if $r(v_1|W) = r(v_2|W)$ implies that $v_1 = v_2$ for all $v_1, v_2 \in V(G)$.

A resolving set of minimum cardinality for a graph G is called a minimum resolving set, A resolving set W is called a minimal resolving set of a connected graph G if no proper subset of W is a resolving set of G , the least cardinality of minimal resolving set of G is denoted by $lr(G)$

Definition 1.2.5. [12] Let G be any graph. Let S is the subset of $V(G)$. Consider the k -tuple (u_1, u_2, \dots, u_k) where $S = \{u_1, u_2, \dots, u_k\}$, $k \geq 1$. Let $v \in V(G)$, define a binary neighbourhood code of v with respect to the k -tuple (u_1, u_2, \dots, u_k) , denoted by $nc_s(v)$ as a k -tuple (r_1, r_2, \dots, r_k)

$$\text{where } r_i = \begin{cases} 1 & \text{if } u \in N[u_i], 1 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

S is called a neighbourhood resolving set or a nr-set if S is neighbourhood set and $nc_s(u) \neq nc_s(v)$ for any $u, v \in V(G)$.

Definition 1.2.6. [12]

Let S be a subset of $V(G)$. If S is both minimal neighborhood and minimal resolving set, then its least cardinality is denoted by $lnr(G)$

Definition 1.2.7. [12]

Let S be a subset of $V(G)$. Let $u \in S$. Then two vertices $x, y \in V$ are said to be privately resolved by u if $nc_s(x)$ and $nc_s(y)$ differ only at the place corresponding to u .

Theorem 1.2.8. [4] Let G be a connected graph of order $n \geq 3$. Then G does not

have any neighbourhood resolving set if and only if there exist two non adjacent vertices u and v in $V(G)$ such that $N[u] = N[v]$.

Theorem 1.2.9. [21] For any graph G , $lnr(G) \leq n - 1$

Theorem 1.2.10. [26] A vertex set S in graph G is an nr -set if every edge in the subgraph induced by $V(G) - S$ is part of a cycle with length 3 that has at least one vertex in S .

Theorem 1.2.11. [12][16] For any positive integer n ,

$$lnr(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{for } n = 1, 3, \\ 2, & \text{for } n = 2, \\ \lfloor \frac{n}{2} \rfloor, & \text{for } n \geq 4. \end{cases}$$

Theorem 1.2.12. [18] For any integer $n \geq 3$, $lnr(K_n) = 0$

Theorem 1.2.13. [16] For any integer $n \geq 3$, the adjacency local metric dimension of the cycle C_n is given by

$$lnr(C_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Theorem 1.2.14. [12] For any integer $n \geq 3$,

$$lnr(C_n) = \begin{cases} 3 & \text{if } n = 4 \\ \lceil \frac{n}{2} \rceil & \text{otherwise} \end{cases}$$

Theorem 1.2.15. [12] For each integer $i \geq 3$ every i element subset S of vertices of a cycle C_n is always an r -set.

Theorem 1.2.16. [16] For any integer $n \geq 3$,

$$lnr(W_{1,n}) = \begin{cases} 3, & \text{for } n = 3, 4, 5, 6, \\ \left\lfloor \frac{2(n+1)}{5} \right\rfloor + 1, & \text{for } n \geq 7. \end{cases}$$

Theorem 1.2.17. [12] For any integer $n \geq 4$, $\text{lnr}(S_n) = n - 2$

Theorem 1.2.18. [12] Let G be a connected graph of order n such that $\text{lnr}(G) = k$. Then $\log_2 n \leq k$

Example 1.2.19. Let G be the connected graph with 5 vertices in the figure. 3a Now $S = \{2, 5, 4\}$ is the neighbourhood resolving set of G . Since $nC_S(1) = (1, 0, 0)$, $nC_S(2) = (1, 1, 0)$, $nC_S(3) = (1, 0, 1)$, $nC_S(4) = (0, 1, 1)$, $nC_S(5) = (1, 1, 1)$ and $N[2] \cup N[5] \cup N[4] = G$ The minimum(least) cardinality of such neighbourhood resolving set is $\text{lnr}(G) = 3$

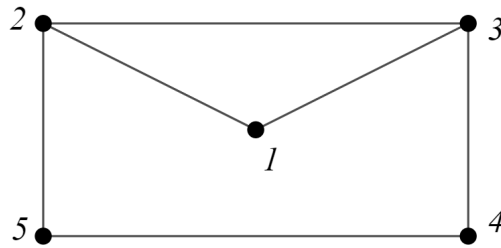


Figure. 3a

1.3 ϕ_k -Polynomial

Definition 1.3.1. [6] Let G be a graph the k^{th} degree of the vertex p is denoted by $\lambda_k(p)$, and define as $\lambda_k(p) = |\{q \in (G) : \lambda(p, q) = k\}|$, where $\lambda(p, q)$ is the distance between the vertices p and q . The minimum and maximum k^{th} degree of the graph G denoted by $\alpha_2(G)$ and $\beta_2(G)$ respectively, then for the graph G , let $\phi_k(i, j)(G)$ be the number of edge, $pq \in E(G)$ such that $\{\lambda_k(p), \lambda_k(q)\} = \{i, j\}$, The ϕ_k -polynomial of G is defined as: $\phi_2(G, x, y) = \sum_{\alpha_k \leq i \leq j \leq \beta_k} \phi_2(i, j)(G)x^i y^j$

Theorem 1.3.2. [6] Let $G = P_n$ be a path with $n \geq 4$ vertices. Then

1.3. ϕ_k -Polynomial

$$\phi_2(G, x, y) = \begin{cases} 3xy & \text{if } n = 4 \\ 2xy + 2xy^2 + (n - 5)x^2y^2 & \text{if } n \geq 5 \end{cases}$$

Theorem 1.3.3. [6] Let $G = C_n$ be a cycle graph with $n \geq 4$ vertices. Then

$$\phi_2(G, x, y) = \begin{cases} 4xy & \text{if } n = 4 \\ nx^2y^2 & \text{if } n \geq 5 \end{cases}$$

Theorem 1.3.4. [6] If G be a connected graph with $n \geq 2$ vertices. Then

$$\phi_2(G, x, y) = cy^r \text{ if } G = S_n. \text{ where } c = (n - 1), \text{ and } r = n - 2.$$

Example 1.3.5. let G be a connected graph in the figure. 5a



Figure. 5a

Table 1.1: 2^{nd} degree and its cardinality of the graphs

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	{3}	1		
$\lambda_2(2)$	{4, 5}	2		
$\lambda_2(3)$	{1}	1	1	2
$\lambda_2(4)$	{2, 5}	2		
$\lambda_2(5)$	{2, 4}	2		

$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $
$\phi_2(1, 1)(G)$	{}	0
$\phi_2(1, 2)(G)$	{1 - 2, 3 - 2, 3 - 4, 3 - 5}	4
$\phi_2(2, 2)(G)$	{}	0

The ϕ_2 - Polynomial of the graph is given by $\phi_2(G, x, y) = 4xy^2$

1.4 Zero divisor graph

Definition 1.4.1. [14] A Commutative Ring $\langle R, +, \cdot \rangle$ is a set R together with two binary operations $+$, and \cdot , which we call addition and multiplication, defined on R such that the following axioms are satisfied

1. $\langle R, + \rangle$ is an abelian group
2. Multiplication is associative
3. For $a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$, hold
4. $a \cdot b = b \cdot a$ for all $a, b \in R$
5. A unity (or identity) in a ring is a nonzero element that is an identity under multiplication.

The set of all zero divisors in R is denoted by $Z(R)$ and the set of all non-zero zero divisors in R is denoted by $Z^*(R)$

Example 1.4.2. Consider the ring $R = Z_6 = \{0, 1, 2, 3, 4, 5\}$

$Z^*(R) = \{2, 3, 4\}$ and corresponding zero divisor graph $G = K_{1,2}$ is given Figure.

7a

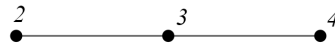


Figure. 7a
 $K_{1,2}$

Chapter 2

Analysis of metric dimension and bases in graph composition products

2.1 Introduction

The concept metric dimension of connected graphs and its related properties are first introduced by by PJ Slater [11] in 1975, independently by Harary and Melter [2] in 1976. The concept of minimum resolving set has a significant role in various context such as organic chemistry[17], Robotic navigation[15], coin weighing problem[7], drug discovery[7], network discovery and verification[13]. The motivation behind the results on this work is due to the large range of application of resolving sets in various fields as mentioned as network discovery and verification. In the field of chemistry, to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. Also provides the chemical structure for the

2.2. Bases and metric dimension of composition product of star graph and empty graphs

compounds. Let $W = \{w_1, w_2, \dots, w_k\}$ be an ordered subset of $V(G)$. The representation of a vertex v of G with respect to W is defined as the k -vector $r(v/W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. The set W is called a resolving set of G if every two distinct vertices $x, y \in V(G)$ satisfy $r(x/W) \neq r(y/W)$. A bases of G is a resolving set of G with the minimum cardinality, and the metric dimension of G refers to its cardinality and is denoted by $\dim(G)$. In this chapter, mainly discussed about the metric dimension and related basic properties of composition products of graphs.

2.2 Bases and metric dimension of composition product of star graph and empty graphs

In this section, we first define the concepts of bases and metric dimension, as well as the composition operation on graphs. Moreover, this chapter derives the bases and the metric dimension of the composition product of certain families of graphs.

Example 2.2.1. *Let G be the graph given with 4 vertices*

$W = \{1, 2\}$ form the resolving $r(1/W) = (0, 1)$, $r(2/W) = (1, 0)$, $r(3/W) = (2, 1)$, $r(4/W) = (1, 1)$. W is bases and also connected $\dim(G) = 2$

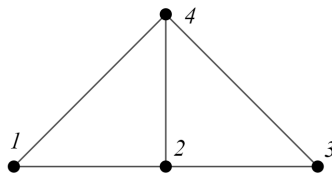


Figure. 2a

2.3 The metric dimension of the graph $S_n[O_m]$

Theorem 2.3.1. *The metric dimension of the composition product of star graph and empty graph is*
$$\dim(S_n[O_m]) = \begin{cases} 1 & \text{if } n = 2, m = 1 \\ nm - 2 & \text{if } n > 2, m > 1 \end{cases}$$

Proof. case.1: For $n = 1$ and for any value of m , the composition product $S_n[O_m]$ is a disconnected graph. Since the metric dimension is defined only for connected graphs, there is no need to determine the metric dimension of the composition product when $n = 1$.

case.2: If $n = 2$ and $m = 1$, the composition product is a star graph with 2 vertices, as shown in Figure 2.1. Therefore, $d(S_2[O_1]) = d(S_2) = 1$



Figure 2.1: $S_2[O_1]$

case.3: If $n \geq 2$ and $m \geq 1$, with $(n, m) \neq (2, 1)$, then the vertices of the composition product $S_n[O_m]$ are as shown in Figure 2.2. This graph has n vertices of degree $(n - 1)m$. Therefore, the resolving set contains $n - 1$ of these vertices and $n(m - 1) - 1$ vertices from the remaining set. Since the graph has $n(m - 1)$ vertices of degree m , the minimum cardinality of the resolving set is $(n - 1) + (n(m - 1) - 1) = nm - 2$.

2.3. The metric dimension of the graph $S_n[O_m]$

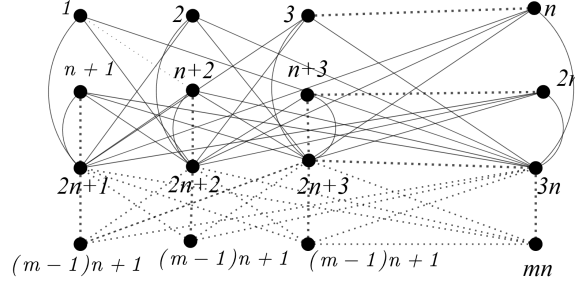


Figure 2.2
 $S_n[O_m]$

Figure 2.2: $S_n[O_m]$

□

Corollary 2.3.2. *From the above theorem, the metric dimension of the composition product $S_n[O_n]$, when $n = m$, is given by*

$$\dim(S_n[O_n]) = n^2 - 2, \quad \text{for all } n \geq 2.$$

Remark: Here, $S_n[O_n]$ represents the composition product of a star graph S_n (with n leaves) and O_n (a set of n isolated vertices). The composition product $S_n[O_n]$ creates a graph structure where each leaf of the star graph S_n is replaced by n isolated vertices.

The result shows that the metric dimension grows quadratically as n increases. Specifically:- For $n = 2$, $\dim(S_2[O_2]) = 2^2 - 2 = 2$. For $n = 3$, $\dim(S_3[O_3]) = 3^2 - 2 = 7$.

This quadratic growth highlights the increasing complexity of determining unique vertex positions in composite graphs.

Corollary 2.3.3. *The lower bound of the metric dimension of the composition*

2.3. The metric dimension of the graph $S_n[O_m]$

product of the graph $S_n[O_m]$ is the metric dimension of S_n . Therefore:

$$\dim(S_n) \leq \dim(S_n[O_m]).$$

In this case, independent bases do not exist; all the resolving bases are connected.

Remark: This corollary establishes an important lower bound for the metric dimension of $S_n[O_m]$, showing that the complexity of the resolving set cannot drop below the metric dimension of the original star graph S_n .

Furthermore, it is noted that:

- **No independent bases exist:** The resolving set cannot be formed by isolated vertices; all bases must involve vertices that are connected to other parts of the graph.
- **Connected bases:** All minimal resolving sets (or bases) involve vertices that are connected within the graph structure, reflecting the influence of the underlying star graph S_n .

Remarks:

1. The results of these corollaries emphasize the importance of understanding the interplay between the base graph S_n and the added vertices O_m in the composition product $S_n[O_m]$.
2. The absence of independent bases suggests that the structure of the graph is inherently connected, and resolving sets must leverage this connectivity to ensure uniqueness.

3. Further studies could explore the upper bounds and tight bounds for $\dim(S_n[S_m])$ as m varies.

2.4 The metric dimension of the graph $S_n[S_m]$

Theorem 2.4.1. *The metric dimension of the composition product of two star*

$$\text{graphs of order } n \geq 3 \text{ and } m \text{ is, } \dim(S_n[S_m]) = \begin{cases} 1 & \text{if } n = 2, 3, m = 1 \\ 3 & \text{if } n = 2, 3, m = 2 \\ m - 2 & \text{if } n = 1, m \geq 3 \\ 2m - 3 & \text{if } n = 2, m \geq 3 \\ 3m - 5 & \text{if } n = 3, m \geq 3 \end{cases}$$

Proof. The composition product of $S_n[S_m]$ as shows in the figure 2.3

case.1: If $2 \leq n \leq 3$ and $m = 1$, then the composition product reduces to $S_2[S_1]$ and $S_3[S_1]$. Here, $S_2[S_1] = P_2$ and $S_3[S_1] = P_3$. Therefore, $\dim(S_2[S_1]) = \dim(P_2) = 1$ and $\dim(S_3[S_1]) = \dim(P_3) = 1$.

case.2: For $2 \leq n \leq 3$ and $m = 2$, the composition product satisfies $S_2[S_2] = K_4$. Hence, $\dim(S_2[S_2]) = \dim(K_4) = 3$.

case.3: If $n = 1$ and $m \geq 3$, then the composition product $S_1[S_m]$ is a star graph with m vertices. Therefore, $\dim(S_1[S_m]) = m - 2$

case.4: If $n = 2$ and $m \geq 3$, then every resolving set containing at least $2m - 3$ vertices is a basis. Therefore, $\dim(S_2[S_m])$ for any $m \geq 3$ is $2m - 3$.

case.5: If $n = 3$ and $m \geq 3$, then every resolving set containing at least $3m - 5$ vertices is a basis. Therefore $\dim(S_3[S_m])$ for any $m \geq 3$ is $3m - 5$.

2.5. Conclusion

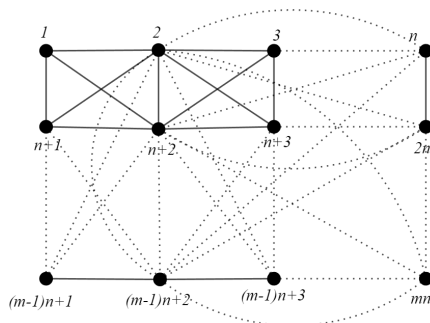


Figure 2.3: $S_n[S_m]$

□

2.5 Conclusion

The study of the metric dimension on the graphs $S_n[O_m]$ (a star graph with isolated vertices) and $S_n[S_m]$ (a star graph with smaller star graphs attached to each leaf) provides valuable insights into graph representation and unique vertex identification.

- To determine the minimum resolving set (or bases) that uniquely identifies every vertex in these graph structures.
- To analyze the complexity and structure of composite graphs derived from basic star graphs, which are widely used in theoretical computer science and network modeling.

Some applications are given by

- **Robotic navigation:** Understanding the metric dimension allows robots to efficiently navigate networks modeled by such graphs. Resolving sets

help minimize excess in position determination or shortest path identification.

- **Network design:** These graph structures are suitable for modeling communication networks with hubs (central nodes) and peripheral devices. Metric dimension helps optimize monitoring or routing strategies.
- **Biological networks:** The graphs $S_n[O_m]$ and $S_n[S_m]$ can model hierarchical biological systems, such as protein interaction networks or neural networks, where central nodes interact with clusters of smaller nodes.
- **Data analysis:** Metric dimension aids in identifying unique nodes in large data networks, such as social or computer networks, enabling network clustering, fault diagnosis, and efficient resource allocation.

Further studies

- **Generalization:** Extend the study to other composite graphs, such as $S_n[K_m]$ (a star graph with complete graphs).
- **Algorithmic applications:** Develop efficient algorithms to compute the metric dimension for large-scale composite graphs.
- **Real-world network analysis:** Apply the findings to real-world networks, where hierarchical structures like $S_n[S_m]$ naturally occur.
- **Dynamic graphs:** Investigate changes in metric dimension when vertices or edges are added or removed dynamically.
- **Optimization problems:** Explore optimization techniques to minimize resolving sets in networks with resource constraints.

Chapter 3

Neighborhood resolving sets in graphs with order atmost six

3.1 Introduction

Let $G(V; E)$ be a graph. For a vertex $v \in V$, $N(v)$ denotes the set of all vertices of G which are adjacent to v and $N[v] = N(v) \cup v$. A subset S of V is called a neighborhood set or n -set of G if $G = \cup_{v \in S} \langle N[v] \rangle$, where for a subset S of V . An nr -set S is called minimal if no proper subset of S is an nr -set [12]. The minimum cardinality of a minimal n -set is called the neighbourhood number of G and is denoted by $ln(G)$. The concept of neighborhood number for a graph was first introduced by E. Sampathkumar[19] A subset S of V is called a neighborhood resolving set or nr -set of G if S is both a neighborhood and a resolving set of G . The minimum cardinality of a minimal nr -set is called the neighborhood resolving number of G and is denoted by $lnr(G)$. The concepts of

resolving set a graph and its related properties such as bases were introduced by P.J.Slater [11] and independently by Harary and Melter [2]

3.2 Neighborhood resolving sets of graphs with order 1, 2, 3, 4, and 5

If $n = 3$ and $m \geq 3$, then every resolving set containing at least $3m - 5$ vertices is a basis.

Example 3.2.1. *Let G be the connected graph with 5 vertices shown in Figure 3a. Consider the set $S = \{2, 5, 4\}$, which is a neighbourhood resolving set of G . Since $nC_S(1) = (1, 0, 0)$, $nC_S(2) = (1, 1, 0)$, $nC_S(3) = (1, 0, 1)$, $nC_S(4) = (0, 1, 1)$, and $nC_S(5) = (1, 1, 1)$, and moreover $N[2] \cup N[5] \cup N[4] = V(G)$, it follows that S is a neighbourhood resolving set of G . Hence, the minimum cardinality of such a neighbourhood resolving set is $lnr(G) = 3$.*

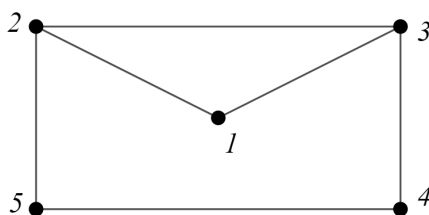


Figure. 3a

Theorem 3.2.2. *If G be a graph with, order less than or equal to 3, then $lnr(G) = 0$*

Proof. case.1: Let G be a graph with a single vertex. Up to isomorphism, there is only one connected graph of order one, which is isomorphic to K_1 (or

equivalently P_1). Therefore, $lnr(G) = 1$. For the case when G is a connected graph with 2 vertices, we have $G \cong P_2$. Hence, $lnr(G) = 0$ by Theorems 1.2.9 and 1.2.11.

case.2: Let G be a connected graph with 2 or 3 vertices. Then the possible graphs are P_2 , P_3 , and C_3 (where $C_3 \cong K_3$). By Theorems 1.2.11 and 1.2.17, we have $nr(P_3) = 0$. Moreover, since $C_3 \cong K_3 \cong P_1 + P_2$, these graphs are isomorphic. Therefore $lnr(C_3) = lnr(K_3) = lnr(P_1 + P_2) = 0$. \square

Theorem 3.2.3. *For a connected graph G with order 4,*

$$lnr(G) = \begin{cases} 3 & \text{if } G = P_4, C_4 \\ 2 & \text{if } G = S_4 \text{ or } K_{3,1} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let G be a connected graph with 4 vertices. Up to isomorphism, the possible degree sequences are $(1, 2, 2, 1)$, $(1, 3, 2, 2)$, $(2, 2, 2, 2)$, $(2, 3, 2, 3)$, $(3, 3, 3, 3)$, and $(3, 1, 1, 1)$. The graph with degree sequence $(1, 2, 2, 1)$ is isomorphic to P_4 and hence $lnr(P_4) = 3$. The graph corresponding to $(3, 1, 1, 1)$ is $K_{3,1}$, and therefore $lnr(K_{3,1}) = 2$ by Theorem 1.2.11. The degree sequence $(2, 2, 2, 2)$ corresponds to C_4 , and thus $lnr(C_4) = 3$ by Theorem 1.2.17. The only graph with degree sequence $(3, 3, 3, 3)$ is K_4 , and hence $lnr(K_4) = 0$ by Theorem 1.2.9. The graph with degree sequence $(2, 3, 2, 3)$ is $K_4 - \{e\}$, so $lnr(K_4 - \{e\}) = 0$. Finally, the graph with degree sequence $(1, 3, 2, 2)$ is as shown in Figure 3.1.

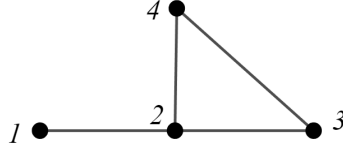


Figure 3.1: $lnr(G) = 0$

From Figure 3.1, there is no neighbourhood resolving set. Hence, $lnr(G) = 0$. Therefore, for any connected graph with 4 vertices, the cardinality of the neighbourhood resolving set is determined. \square

Theorem 3.2.4. For a connected graph G with order 5,

$$lnr(G) = \begin{cases} 3 & \text{if } G = P_5, C_5, K_{1,4}, K_{3,2} \text{ or } K_{2,3} \\ 0 \text{ or } 4 & \text{otherwise} \end{cases}$$

Proof. case.1: Let G be a connected graph with 5 vertices. Then G may be isomorphic to P_5 , C_5 , $K_{1,4}$, or $K_{3,2}$ (note that $K_{3,2} \cong K_{2,3}$). These graphs have degree sequences $(1, 2, 2, 2, 1)$, $(2, 2, 2, 2, 2)$, $(4, 1, 1, 1, 1)$, and $(3, 3, 2, 2, 2)$, respectively. By Theorems 1.2.9, 1.2.11, and 1.2.17, we obtain $lnr(G) = 3$.

The above graphs are shown in the figure 3.2

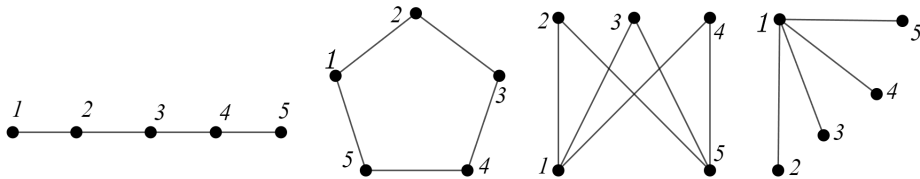


Figure 3.2: $lnr(G) = 3$

case.2: Graphs with $lnr(G) = 4$ are shown in Figure 3.3.

3.2. Neighborhood resolving sets of graphs with order 1, 2, 3, 4, and 5

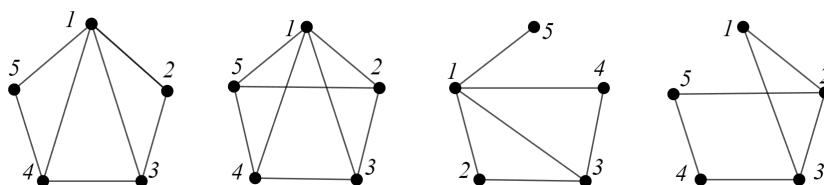


Figure 3.3: $lnr(G) = 4$

case.3: Graphs with $lnr(G) = 0$ are shown in Figure 3.4.

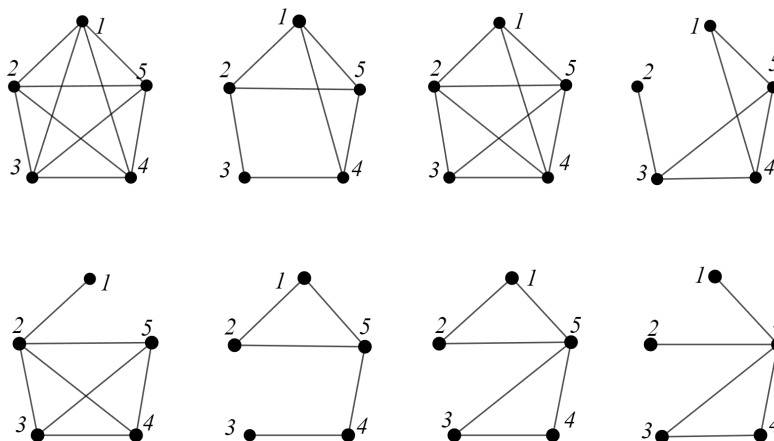


Figure 3.4: $lnr(G) = 0$

□

Corollary 3.2.5. *The graph G with order 5, The minimum cardinality of neighbourhood resolving set is neither 1 or 2.*

Proof. From the theorem 3.2.4 we shows that the minimum cardinality of neighbourhood resolving set assumes the values 4, 3, 0 □

3.3 Neighbourhood resolving set of graph of order 6

Theorem 3.3.1. *Let G be a graph of order 6, then the neighbourhood resolvability of G satisfies $lnr(G) \in \{0, 3, 4, 5\}$*

Proof. There are 110 graphs of order 6. The value of $lnr(G)$ for each of these graphs is analyzed under four distinct cases.

case.1: The graphs satisfying $lnr(G) = 5$ are illustrated in Figure 3.5. A total of 4 graphs fall under this case.

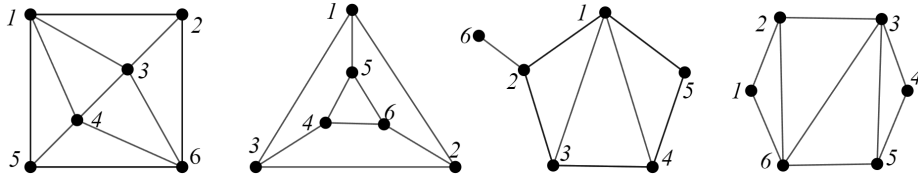
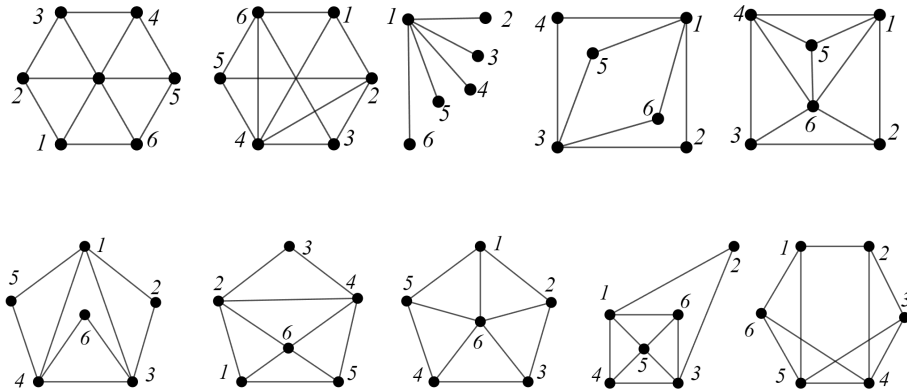


Figure 3.5: $lnr(G) = 5$

case.2: There are 29 graphs with $lnr(G) = 4$, as shown in Figure 3.6.



3.3. Neighbourhood resolving set of graph of order 6

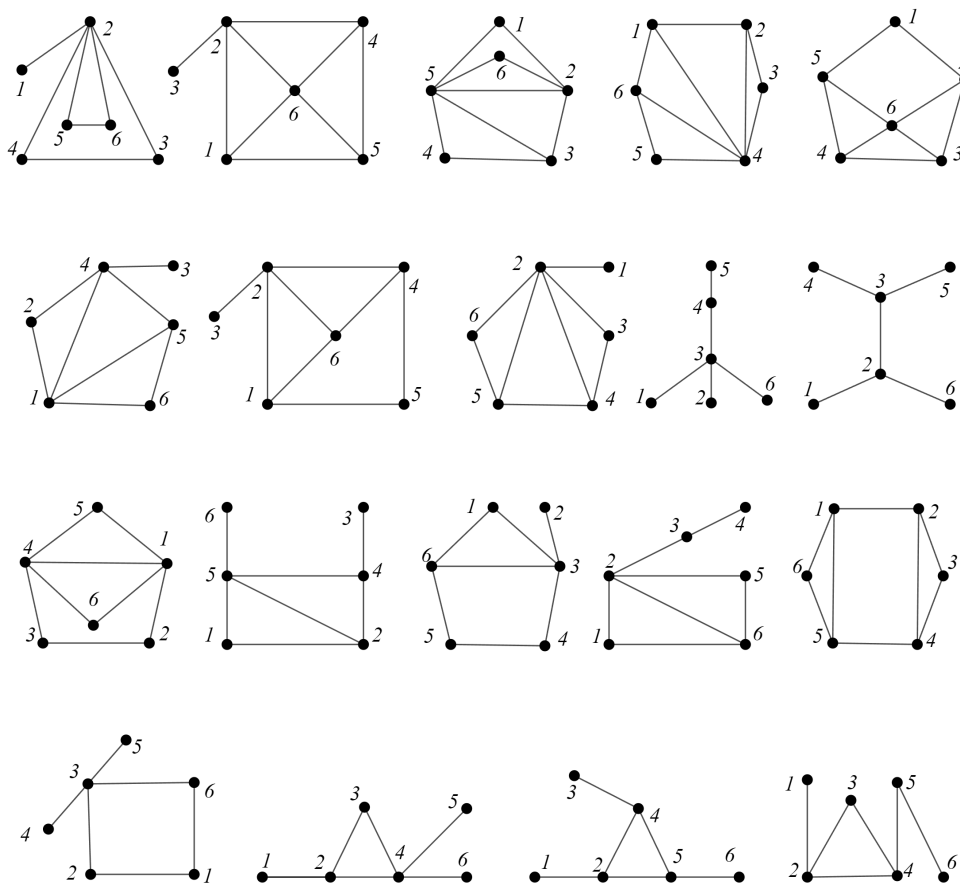
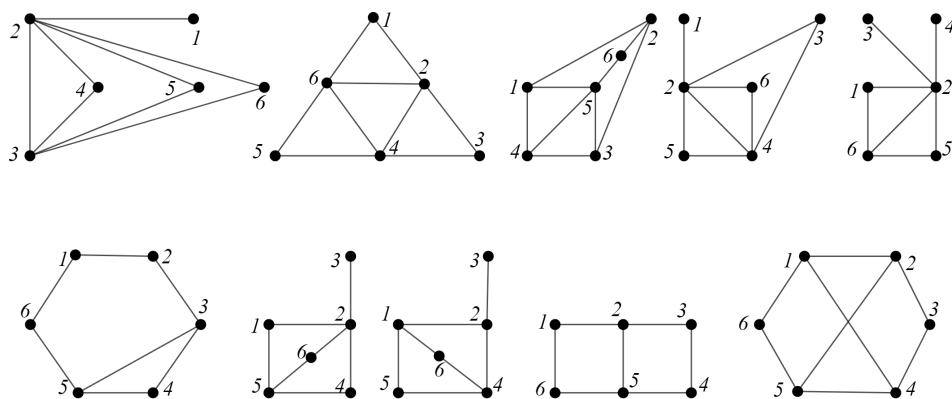


Figure 3.6: $lnr(G) = 4$

case.3: Graphs with $lnr(G) = 3$ are shown in Figure 3.7.



3.3. Neighbourhood resolving set of graph of order 6

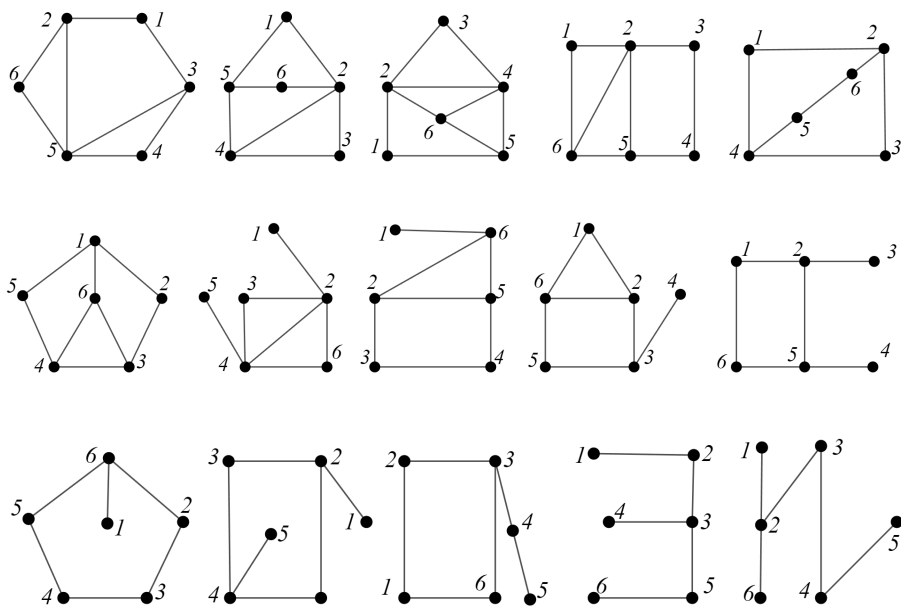
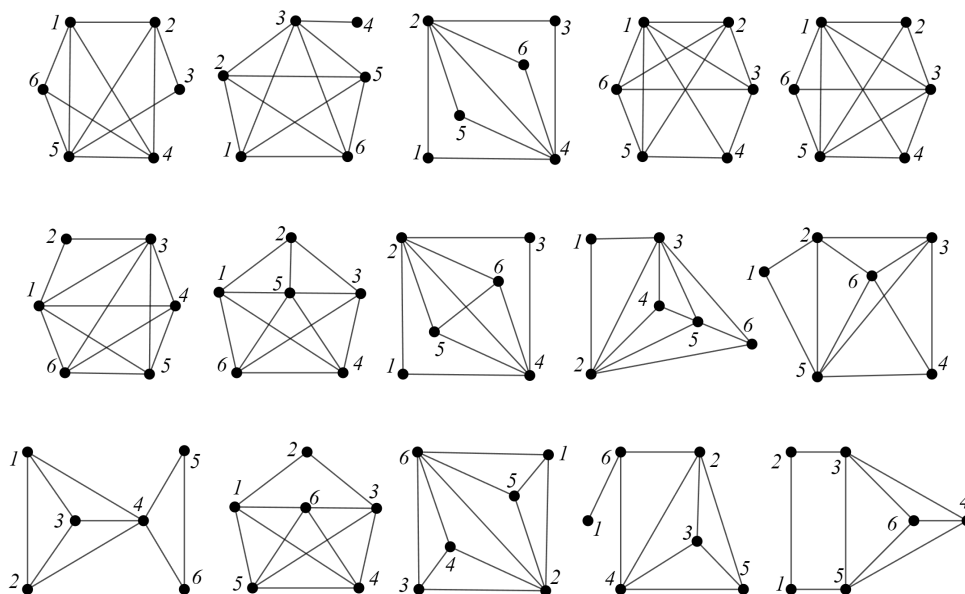


Figure 3.7: $lnr(G) = 3$

case.4: Graphs with $lnr(G) = 0$ are described as follows. The complete graph K_6 has no neighbourhood resolving set. Similarly, the graph $K_6 - e$ also has no neighbourhood resolving set. The remaining graphs on 6 vertices with $lnr(G) = 0$ are shown in Figure 3.8.



3.3. Neighbourhood resolving set of graph of order 6

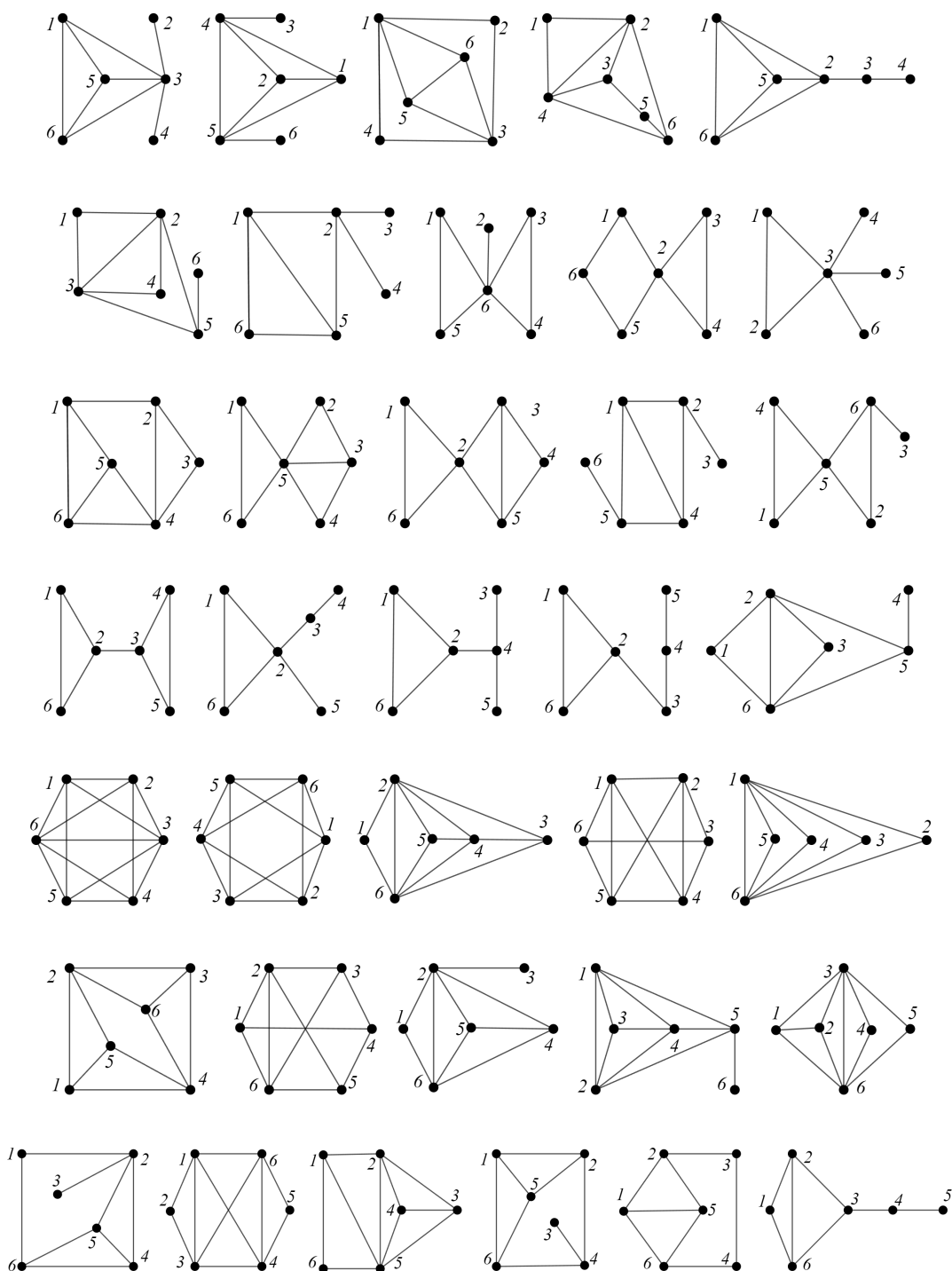


Figure 3.8: $lnr(G) = 0$

□

Conclusion

In this chapter, we have explored the concept of neighborhood resolving sets in graphs with an order of at most six. The primary focus was to determine minimal resolving sets for small graphs and analyze their structural properties. Key findings demonstrate that the structure of a graph, including its connectivity and symmetry, significantly influences the size and nature of resolving sets. The results of this study have various practical applications, particularly in:

- **Network design:** Resolving sets can aid in the optimal placement of sensors or controllers in communication networks, ensuring efficient monitoring and fault detection.
- **Robotics and navigation:** Minimal resolving sets are crucial for robotic path planning and environment mapping, where localization of robots trust on distinguishing points in the graph-based representation of space.
- **Biological networks:** In biological studies, neighborhood resolving sets can be applied to analyze protein interaction networks or metabolic pathways, aiding in identifying critical nodes.
- **Chemical graph theory:** Resolving sets are useful in identifying unique molecular structures based on graph invariants.

The study presented here opens several channel for further research:

- **Generalization to larger graphs:** Extending the analysis to graphs of higher order and investigating the flexibility of methods used for smaller graphs.

- **Algorithmic development:** Designing efficient algorithms for finding minimal resolving sets in large and complex graphs.
- **Dynamic graphs:** Studying neighborhood resolving sets in dynamic graphs, where edges and vertices may change over time, to model real-world systems more accurately.
- **Graph families:** Investigating specific families of graphs, such as hypercubes, planar graphs, or random graphs, for their resolving set characteristics.
- **Weighted graphs:** Extending the concept to weighted graphs, where edge weights represent distances or costs, adding complexity and practical significance.

The study of neighborhood resolving sets provides deep insights into graph theory and its applications in various scientific and engineering domains. Future work should aim to bridge theoretical advancements with practical implementations for solving real-world problems.

Chapter 4

Neighborhood resolving sets on bipartite graphs, graph compositions and sum of graphs

4.1 Introduction

Let $G(V, E)$ be a graph. For any element $v \in V$, the collection $\bar{N}[v]$ of all elements which are adjacent to v and also v itself. A subcollection N is known as \bar{n} - set of G , if the graph G is the finite union of $\bar{N}[v]$ for each $v \in N$. The least number of elements in \bar{n} set is called the neighbourhood number of G and is denoted by $n(G)$. The concept of neighbourhood number is introduced by E. Sampathkumar[19]. The least number of elements in \bar{r} - set is known as resolving number of G and is represented by $r(G)$. A neighbourhood resolving number $nr(G)$ is the least number of elements in the $\bar{n}r$ set. The concepts of

resolving set introduced by P.J.Slater [11] and independently by Harary and Melter [10]

4.2 Neighbourhood resolving set of the bipartite graph $K_{r,t}$

In this section, discusses the properties of neighbourhood resolving sets for certain graph families, such as complete bipartite graphs, composition of graphs, and sum of graphs and obtain the following results.

Theorem 4.2.1. *If $G = K_{r,t}$ be the bipartite graph then*

$$nr(K_{r,t}) = \begin{cases} 0 & \text{if } r = 1, t = 1 \\ 2 & \text{if } r = 1, t = 2 \\ 2 & \text{if } r = 2, t = 1 \\ 3 & \text{if } r, t = 2 \\ t - 1 & \text{if } r = 1, t \geq 3 \\ t & \text{if } r = 2, t \geq 3 \\ r + t - 2 & \text{if } r \geq 3, t \geq 1 \end{cases}$$

Proof. The graph of $K_{r,t}$ is shown in the figure. 4.1

case.1: If $t = 1, r = 1$, then $K_{r,t} = P_2$, $nr(K_{r,t}) = nr(P_2) = 0$

case.2: If $r, t = 1, 2$, then $nr(K_{r,t}) = nr(S_3) = 2$

case.3: If $r, t = 2$, then $nr(K_{r,t}) = nr(C_4) = 3$

case.4: If $r, t \geq 3$, then $nr(K_{r,t}) = nr(S_t) = t - 1$

case.5: If $r = 2$ and $t \geq 3$, then the graph has $t + 2$ vertices. Among these, exactly two vertices have maximum degree t , while the remaining t vertices have

degree 2. We may choose any one vertex of degree t together with any $t - 1$ vertices from the set of vertices of degree 2. These t vertices form a neighbourhood resolving set of the graph, and hence t is the least cardinality of a neighbourhood resolving set of the graph.

case.6: If $r \geq 3$ and $t \geq 1$, then from Figure 4.1 we observe that the graph has $r + t$ vertices. Among these, t vertices have degree r and r vertices have degree t . A neighbourhood resolving set consists of $t - 1$ vertices from the first set and $r - 1$ vertices from the second set. Hence, the number of elements in the neighbourhood resolving set is $(t - 1) + (r - 1) = r + t - 2$.

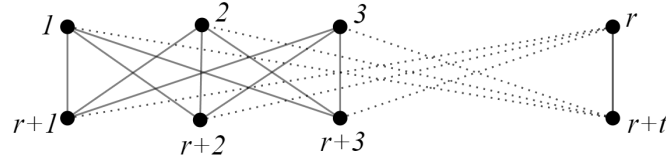


Figure 4.1: $K_{r,t}$

□

4.3 Neighbourhood resolving set of $S_t(O_t)$

Theorem 4.3.1. If $G = S_t(O_t)$, then $nr(G) = \begin{cases} 0 & \text{if } t = 1 \\ 3 & \text{if } t = 2 \\ t^2 - 2 & \text{if } t \geq 3 \end{cases}$

Proof. **case:1** If $t=1$, then $nr(S_1[O_1]) = nr(S_1) = 0$

case:2 If $t=2$, then $nr(S_2[O_2])$ is same as the $nr(C_2)$. Therefore $nr(S_2[O_2]) = 3$

case:3 If $t \geq 3$, then the vertices of $S_t[O_t]$ are as shown in Figure 4.2. In this graph G , there are t vertices of degree $t^2 - t$ and $t^2 - t$ vertices of degree t . Hence, a resolving set contains $t - 1$ vertices of degree $t^2 - t$ and $t(t - 1) - 1$ vertices of degree t . Therefore, the number of elements in the resolving set is $(t - 1) + (t(t - 1) - 1) = t^2 - 2$.

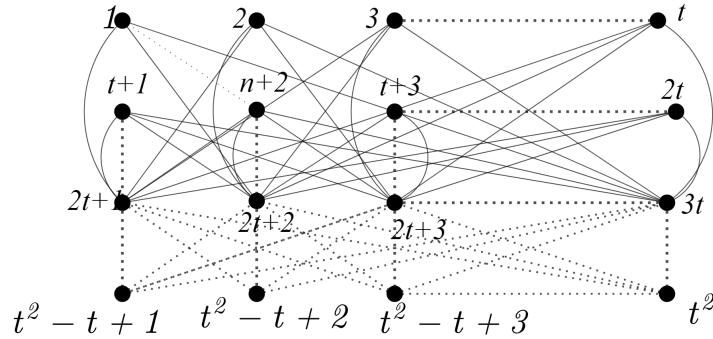


Figure 4.2: $S_t[O_t]$

□

4.4 Neighbourhood resolving set of $S_t[S_t]$

Theorem 4.4.1. If $G = S_t[S_t]$, $nr(G) = \begin{cases} 0 & \text{if } t = 1, 2 \\ t^2 - 3 & \text{if } t \geq 3 \end{cases}$

Proof. The composition product of graph family $S_t[S_t]$ shows in the figure. 4.3

case.1: If $t = 1$, then $nr(S_t[S_t]) = 0$.

If $t = 2$, then the graph of $S_t[S_t]$ is same as the complete graph of K_4 so $nr(S_t[S_t]) = nr(K_4) = 0$

case.2: If $t \geq 3$, Here every neighbourhood resolving set having minimum $t^2 - 3$

elements. Therefore $nr(S_t[S_t]) = t^2 - 3$.

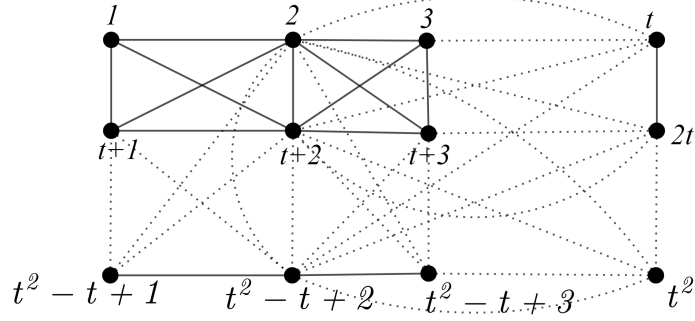


Figure 4.3: $S_t[S_t]$

□

4.5 Neighbourhood resolving set of $P_t + P_t$

Theorem 4.5.1. *If $G = P_t + P_t$, $nr(G) = \begin{cases} 0 & \text{if } t = 1, 2, 3 \\ 7 & \text{if } t = 4 \\ (1 + t) & \text{if } t \text{ is odd } \geq 5 \\ 2(t - 2) & \text{if } t \text{ is even } \geq 6 \end{cases}$*

Proof. case.1: If $t = 1$, then $P_t + P_t$ is same as the graph S_2 . Therefore $nr(P_t + P_t) = lnr(S_2) = 0$

case.2: If $t = 2$, then $P_t + P_t$ is same as the graph K_4 . Therefore $nr(P_t + P_t) = lnr(K_4) = 0$

case.3: If $t = 3$, then $P_3 + P_3$ is shown in the figure. 4.4 given below. Therefore $nr(P_t + P_t) = 0$

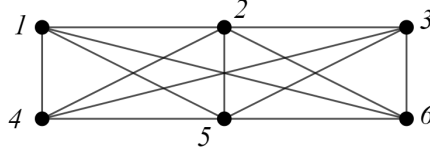


Figure 4.4: $P_3 + P_3$

case.4: If $t = 4$, then the graph $P_4 + P_4$ is shown in the figure. 4.5 given below. Therefore $nr(P_t + P_t) = 7$.

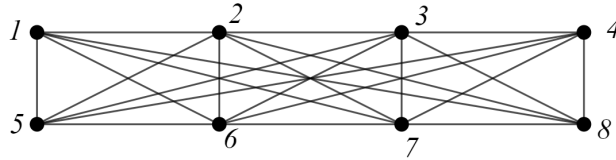


Figure 4.5: $P_4 + P_4$

case.5: In this case, if $t \geq 5$ is an odd integer, then the graph $P_t + P_t$ has $2t$ vertices, as shown in Figure 4.6. Among these, exactly 4 vertices have degree $t + 1$, and the remaining $2t - 4$ vertices have degree $t + 2$. A neighbourhood resolving set consists of all 4 vertices of degree $t + 1$ together with $t - 3$ vertices of degree $t + 2$. Hence, the least cardinality of the resolving set is $4 + (t - 3) = t + 1$. Therefore, $lnr(P_t + P_t) = t + 1$ for all odd integers $t \geq 5$.

case.6: In this case, if $t \geq 6$ is an even integer, then the graph $P_t + P_t$ has $2t$ vertices, as shown in Figure 4.6. Among these, exactly 4 vertices have degree $t + 1$, and the remaining $2(t - 2)$ vertices have degree $t + 2$. Therefore, the resolving set contains vertices of both degrees $t + 1$ and $t + 2$. Hence, the number of elements in the resolving set is $2 + (t - 3) + (t - 3) = 2t - 4$, which gives the least cardinality of the neighbourhood resolving set.

Therefore the $nr(P_t + P_t) = 2(t - 2)$, $t \geq 6$, where t is an even integer.

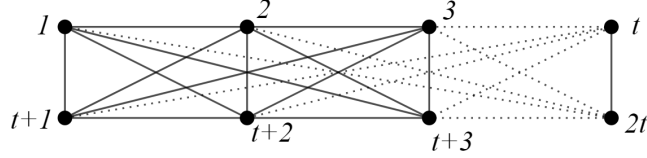


Figure 4.6: $P_t + P_t$

□

4.6 Neighbourhood resolving set of $P_t + O_t$

Theorem 4.6.1. *If $G = P_t + O_t$, $nr(G) =$*

$$\begin{cases} 0 & \text{if } t = 1, 2 \\ 4 & \text{if } t = 3 \\ 6 & \text{if } t = 4 \\ 2\lceil \frac{t}{2} \rceil + (t - 1) & \text{if } t \text{ is odd } \geq 5 \\ 2t - 3 & \text{if } t \text{ is even } \geq 6 \end{cases}$$

Proof. The graph of $P_t + O_t$ is shown in the figure. 4.7

case.1: If $t = 1$ then the graph is S_2 , and if $t = 2$ then the graph is $K_4 - e$, therefore $nr(P_t + O_t) = 0$

case.2 : If $t = 3, 4$ then the graph is same as $(P_t + P_t - 2e)$ and $(P_t + P_t - 3e)$, therefore the $nr(P_t + O_t) = 4$, if $t = 3$ and $nr(P_t + O_t) = 6$, if $t = 4$

case.3 : If $t \geq 5$ is an odd integer or $t \geq 6$ is an even integer, then the graph contains $2t$ vertices. Among these, t vertices have degree t , 2 vertices have degree $t + 1$, and the remaining $t - 2$ vertices have degree $t + 2$.

subcase.3.1 : If $t \geq 5$ is an odd integer, then a neighbourhood resolving set

contains $t - 1$ vertices of degree t , 2 vertices of degree $t + 1$, and $\lfloor \frac{t}{2} \rfloor$ vertices of degree $t + 2$. This set represents the least cardinality of a neighbourhood resolving set for the graph. Therefore $nr(P_t + O_t) = \begin{cases} 2\lfloor \frac{t}{2} \rfloor + (t - 1) & \text{if } t \geq 5, \end{cases}$ where t is an odd integer

subcase.3.2 : If even integer $t \geq 6$, then the neighbourhood resolving set contains $t - 1$ points of t degree, $t - 3$ points of $t + 2$ degree and 1 point of $t + 1$ degree. These neighbourhood resolving set has least cardinality for this graph.

Therefore $nr(P_t + O_t) = \begin{cases} 2t - 3 & \text{if } t \geq 6, \end{cases}$ where t is an even integer.

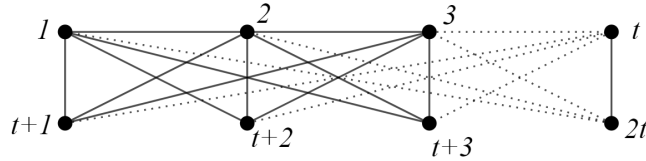


Figure 4.7: $P_t + O_t$

□

Conclusion

In this chapter, we explored the concept of neighbourhood-resolving sets in the context of several graph classes, including bipartite graphs $K_{r,t}$, the composition product of star graphs and empty graphs $S_t(O_t)$, the composition product of star graphs $S_t[S_t]$, the sum of path graphs $P_t + P_t$, and the sum of path graphs and empty graphs $P_t + O_t$. For each of these graph classes:

- We established the neighbourhood-resolving set and its cardinality.

- We analyzed the structural properties that influence resolving sets and their practical implications.
- Results for each class of graphs were presented with proofs and relevant examples to demonstrate the concepts.

This study contributes to the broader understanding of graph theory and its applications in resolving sets and metric dimension analysis. The concept of neighbourhood-resolving sets finds applications in various domains:

- **Network design:** Resolving sets are used in designing efficient network topologies for fault detection and localization.
- **Robotics:** Metric dimensions aid in navigation systems, where robots can determine their positions based on resolving sets within graph-modeled environments.
- **Biology:** Analyzing biological networks, such as protein-protein interaction networks, to identify unique nodes or pathways.
- **Chemistry:** Studying molecular structures represented as graphs to understand chemical properties and reactions.
- **Data science:** Resolving sets can be applied in clustering and data retrieval to efficiently identify or classify datasets.

Future research in neighbourhood-resolving sets and their applications may include:

- **Extension to other graph classes:** Investigating neighbourhood-resolving sets in more complex graph classes such as hypergraphs, directed graphs, and weighted graphs.
- **Algorithm development:** Designing efficient algorithms to compute neighbourhood-resolving sets for large-scale graphs.
- **Dynamic graphs:** Analyzing resolving sets in dynamic or evolving graphs where the topology changes over time.
- **Optimization problems:** Exploring optimization problems related to minimizing the size of resolving sets in constrained environments.
- **Interdisciplinary applications:** Applying neighbourhood-resolving set concepts to interdisciplinary fields such as neuroscience, social networks, and transportation systems.

Chapter 5

Analysis on ϕ_k -polynomial properties of $P_m + P_n$, $P_n + O_n$, $K_n - e$ and $P_m \times P_n$

5.1 Introduction

Study of graphs plays a vital role in various fields such as computer science, chemistry, social network analysis, etc. Graph polynomials are powerful tools used to capture essential graph properties and aid in solving complex problems. One intriguing and versatile graph polynomial is the ϕ_k - *polynomial* [6], which offers unique insights into the structural and combinatorial characteristics of graphs. The ϕ_k - *polynomial* is a polynomial associated with a graph, and it is defined in terms of the second degree of the vertices of the graph. There are a great number of graph polynomials available in the recent literature, and several of them are relevant to mathematical chemistry in some way [6]. A graph $G = (V, E)$ is connected, if there exists a path between any pair of vertices in

G . A network graph is simple connected graph. If two vertices x and y of the graph G are adjacent, then the edge connecting those vertices will be denoted by xy . If $x, y \in V(G)$, then the distance $\lambda(x, y)$ between x and y is defined as the length of a shortest path in G connecting them. In the graph G , the first degree of a vertex x , denoted $\lambda_1(x)$, is the number of first neighbours (the number of edges incident with x). The second degree of a vertex x , denoted $\lambda_2(x)$, is the number of vertices in G which are connected by x with exactly 2 vertices. The minimum and maximum degrees among the vertices of G , are denoted by $\alpha_2(G)$ and $\beta_2(G)$ [6]. Here, we delve into the fascinating world of the $\phi_k - polynomial$ of graphs, aiming to uncover its significance and applications in various domains. Our aim to investigate its connection with graph theory, explore how it relates to important graph parameters, and demonstrate its utility in solving problems ranging from network analysis to quantum chemistry. By understanding the $\phi_k - polynomial$, we gain valuable tools to challenge real-world problems, analyze complex networks, and solve the mysteries of graph structures. The $\phi_k - polynomial$, unlocking its potential to provide deeper insights into the world of graphs and their different applications. The $\phi_k - polynomial$ is a polynomial associated with a graph, and it is defined in terms of the second degree of the vertices of the graph. In essence, the $\phi_k - polynomial$ provides a way to analyze the distribution of edges between the vertices with different second degree.

5.2 Polynomial properties of the graph families

$P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

In this section, discusses the ϕ_k -Polynomial properties of graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$. Obtain the following results

Example 5.2.1. Consider the connected graph G as shown in the figure. 5a, The ϕ_2 - Polynomial of the graph is given by $\phi_2(G, x, y) = 4xy^2$



Figure. 5a

Theorem 5.2.2. If $G = P_n + O_n$ be the sum of path graph and empty graph then

ϕ_k - polynomial for $k = 2$ of $G = P_n + O_n$ is given by

$$\phi_2(G, x, y) = \begin{cases} 0 & \text{if } n = 1 \\ 1 + 4y & \text{if } n = 2 \\ (n - 3)x^{n-3}y^{n-3} + 2x^{n-3}y^{n-2} + \\ n(n - 2)x^{n-3}y^{n-1} + 2nx^{n-2}y^{n-1} & \text{if } n \geq 3 \end{cases}$$

Proof. case.1: If $n = 1$, then $P_n + O_n = P_2$, where P_2 is a complete graph, is in figure. 5.1. $\phi_2(G, x, y) = 0$



Figure 5.1: P_2

case.2: If $n = 2$, then then $P_n + O_n = K_4 - e$ as in Figure. 5.2

The 2^{nd} degree ($k = 2$) of each vertices in the graph $K_4 - e$ is given by $\lambda_2(1) =$

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

$|\{\}\rangle = 0, \lambda_2(2) = |\{\}\rangle = 0, \lambda_2(3) = |\{4\}\rangle = 1, \lambda_2(3) = |\{3\}\rangle = 1$ Here $\alpha_2(G) = 0$ and $\beta_2(G) = 1$

Therefore $\phi_2(G, x, y) = \sum_{\alpha_k \leq i \leq j \leq \beta_k} \phi_2(i, j)(G)x^i y^j = \sum_{\alpha_2 \leq 0 \leq 1 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 1 + 4y$

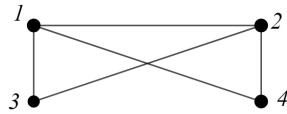


Figure 5.2: $K_4 - e$

case.3: If $n \geq 3$, then $G = P_n + O_n$ is shown in Figure 5.3. From Figure 5.3, the second degree (λ_2) of vertices 1 and n of P_n is $n - 2$, while the second degree of the remaining $n - 2$ vertices of P_n is $n - 3$. The λ_2 of the n vertices of O_n is $n - 1$. Hence, for $n \geq 3$, the second-degree values of the graph satisfy $(n - 3) < (n - 2) < (n - 1)$, and $\alpha_2(G) = n - 3$ and $\beta_2(G) = n - 1$. The number of edges between vertices with second degree $\lambda_2 = n - 3$ is $n - 3$. The number of edges between vertices of second degrees $n - 3$ and $n - 2$ is 2. The number of edges between vertices of second degrees $n - 3$ and $n - 1$ is $n(n - 2)$. There are no edges between vertices with second degree $n - 2$. The number of edges between vertices of second degrees $n - 2$ and $n - 1$ is $2n$, and there are no edges between vertices with second degree $n - 1$.

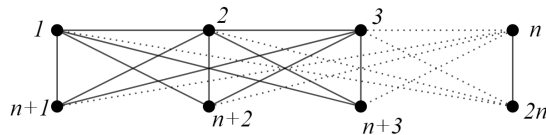


Figure 5.3: $P_n + O_n$

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

Therefore

$$\phi_2(G, x, y) = (n - 3)x^{n-3}y^{n-3} + 2x^{n-3}y^{n-2} + n(n - 2)x^{n-3}y^{n-1} + 2nx^{n-2}y^{n-1}$$

□

Theorem 5.2.3. *If $G = P_m + P_n$ be the sum of path graphs with vertices m and n respectively then $\phi_k -$ polynomial for $k = 2$ of $G = P_m + P_n$ is given by*

$$\phi_2(G, x, y) = \begin{cases} 0 & \text{if } m = 1, 2, n = 1, 2 \\ 1 + 4y & \text{if } m = 1, n = 3 \\ (n - 2)y^{n-3} + 2y^{n-2} + (n - 3)x^{n-3}y^{n-3} + 2x^{n-3}y^{n-2} & \text{if } m = 1, n \geq 4 \\ 3 + 6y & \text{if } m = 2, n = 3 \\ 1 + (2n - 4)y^{n-3} + 4y^{n-2} + (n - 3)x^{n-3}y^{n-3} + 2x^{n-3}y^{n-2} & \text{if } m = 2, n \geq 4 \\ (n^2 - 2n - 2)x^{n-3}y^{n-3} + (4n - 4)x^{n-3}y^{n-2} + 4x^{n-2}y^{n-2} & \text{if } m = n, n \geq 3. \\ (m - 3)x^{m-3}y^{m-3} + (m^2 - 3m + 4)x^{m-3}y^{m-2} + & \\ (2m - 4)x^{m-3}y^{m-1} + (3m - 4)x^{m-2}y^{m-2} + 6x^{m-2}y^{m-1} & \text{if } m \geq 3, n = m + 1. \\ (m - 3)x^{m-3}y^{m-3} + 2x^{m-3}y^{m-2} + & \\ (m - 2)(n - 2)x^{m-3}y^{n-3} + 2(m - 2)x^{m-3}y^{n-2} + & \\ 2(n - 2)x^{m-2}y^{n-3} + 4x^{m-2}y^{n-2} + (n - 3)x^{n-3}y^{n-3} + 2x^{n-3}y^{n-2} & \text{if } m \geq 3, n > m + 1. \end{cases}$$

Proof. case.1(a): If $m = 1$ and $n = 1$, then $P_m + P_n = P_2$. Where P_2 is a complete graph, from Figure. 5.1, $\phi_2(G, x, y) = 0$

(b): If $m = 1$ and $n = 2$, then $P_m + P_n = K_3$. Where K_3 is a complete graph from figure.5.4. Similarly If $m = 2$ and $n = 1$ then $P_m + P_n = K_3$. $\phi_2(G, x, y) = 0$

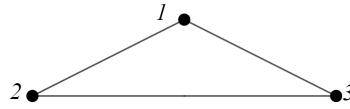


Figure 5.4: K_3

(c): If $m = 2$ and $n = 2$, then $P_m + P_n = K_4$. Where K_4 is a complete graph from figure. 5.5. $\phi_2(G, x, y) = 0$

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

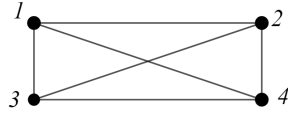


Figure 5.5: K_4

case.2: If $m = 1$ and $n = 3$, then $P_m + P_n = P_1 + P_3$ as the Figure. 5.6 is similar to $K_4 - e$.

Therefore $\phi_2(G, x, y) = 1 + 4y$

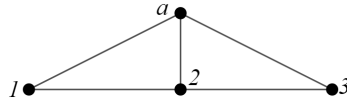


Figure 5.6: $K_4 - e$

case.3: If $m = 1$ and $n \geq 4$, then $G = P_1 + P_n$, as shown in Figure 5.7. The total number of vertices in $P_1 + P_n$ is $n + 1$. Among these, $\lambda_2(a) = 0$, and $\lambda_2(1) = \lambda_2(n) = n - 2$. For the remaining $n - 2$ vertices of P_n , we have $\lambda_2 = n - 3$. Thus, the set of λ_2 -values of the vertices of $P_1 + P_n$ is $\{0, n - 3, n - 2\}$. Consequently, $\alpha_2(G) = 0$ and $\beta_2(G) = n - 2$.

To determine the number of edges between vertices corresponding to each λ_2 -value, we observe the following. There are no edges between vertices with $\lambda_2 = 0$. There are $n - 2$ edges between vertices with $\lambda_2 = 0$ and $\lambda_2 = n - 3$, and exactly 2 edges between vertices with $\lambda_2 = 0$ and $\lambda_2 = n - 2$. Among the vertices with $\lambda_2 = n - 3$, there are $n - 3$ edges. Furthermore, there are 2 edges between vertices with $\lambda_2 = n - 3$ and $\lambda_2 = n - 2$, while there are no edges between vertices with $\lambda_2 = n - 2$.

Hence, the total number of edges in the graph $P_1 + P_n$ is $2n - 1$.

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

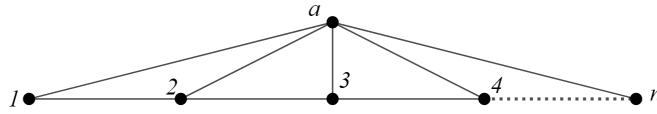


Figure 5.7: $P_1 + P_n$

Therefore, For $m = 1, n \geq 4$

$$\phi_2(G, x, y) = (n - 2)y^{n-3} + 2y^{n-2} + (n - 3)x^{n-3}y^{n-3} + 2x^{n-3}y^{n-2}$$

case.4: If $m = 2$ and $n = 3$, then $G = P_2 + P_3$, as shown in Figure 5.8. From Figure 5.8, the λ_2 -values of vertices 1, 2, and 4 are 0, while the remaining vertices 3 and 5 have $\lambda_2 = 1$. Thus, $\alpha_2(G) = 0$ and $\beta_2(G) = 1$.

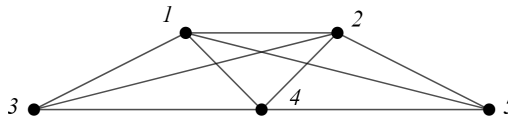


Figure 5.8: $P_2 + P_3$

Therefore $\phi_2(G, x, y) = 3 + 6y$

case.5: If $m = 2$ and $n \geq 4$, then $G = P_2 + P_n$, as shown in Figure 5.9. The total number of vertices in $P_2 + P_n$ is $n + 2$. Among these, $\lambda_2(a) = \lambda_2(b) = 0$, and $\lambda_2(1) = \lambda_2(n) = n - 2$. For the remaining $n - 2$ vertices of P_n , we have $\lambda_2 = n - 3$. Thus, the set of λ_2 -values of the vertices of $P_2 + P_n$ is $\{0, n - 3, n - 2\}$, and consequently $\alpha_2(G) = 0$ and $\beta_2(G) = n - 2$.

To determine the number of edges between vertices corresponding to each λ_2 -value, we observe the following. There is exactly 1 edge among the vertices with $\lambda_2 = 0$. The number of edges between vertices with $\lambda_2 = 0$ and $\lambda_2 = n - 3$ is $2(n - 2)$, and between vertices with $\lambda_2 = 0$ and $\lambda_2 = n - 2$ is 4. Among

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

the vertices with $\lambda_2 = n - 3$, there are $n - 3$ edges. Furthermore, there are 2 edges between vertices with $\lambda_2 = n - 3$ and $\lambda_2 = n - 2$, while there are no edges between vertices with $\lambda_2 = n - 2$.

Hence, the total number of edges in the graph $P_2 + P_n$ is $3n..$

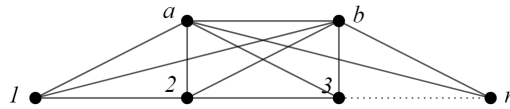


Figure 5.9: $P_2 + P_n$

Therefore, $m = 2, n \geq 4$

$$\phi_2(G, x, y) = 1 + (2n - 4)y^{n-3} + 4y^{n-2} + (n - 3)x^{n-3}y^{n-3} + 2x^{n-3}y^{n-2}$$

case.6: If $m = n$ and $n \geq 3$, then $G = P_n + P_n$, as shown in Figure 5.10.

The total number of vertices in $P_n + P_n$ is $2n$, among which $\lambda_2(1) = \lambda_2(n) = \lambda_2(n+1) = \lambda_2(2n) = n - 2$, and the remaining $2n - 4$ vertices have $\lambda_2 = n - 3$. Thus, the set of λ_2 -values of the vertices of $P_n + P_n$ is $\{n - 3, n - 2\}$, and consequently $\alpha_2(G) = n - 3$ and $\beta_2(G) = n - 2$. The total number of edges in this graph is $n^2 + 2n - 2$. Among the vertices with $\lambda_2 = n - 3$, the number of edges is $(n - 2)^2 + 2(n - 3)$, while the number of edges between vertices with $\lambda_2 = n - 3$ and $\lambda_2 = n - 2$ is $4(n - 2) + 4 = 4n - 4$. There are exactly 4 edges among the vertices with $\lambda_2 = n - 2$.

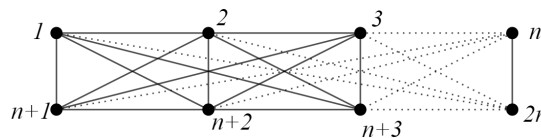


Figure 5.10: $P_n + P_n$

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

Therefore, For $m = n, n \geq 3$

$$\phi_2(G, x, y) = (n^2 - 2n - 2)x^{n-3}y^{n-3} + (4n - 4)x^{n-3}y^{n-2} + 4x^{n-2}y^{n-2}$$

case.7: If $m \geq 3$ and $n = m + 1$, then $G = P_m + P_n = P_m + P_{m+1}$, as shown in Figure 5.11. For the vertices 1 and m of P_m , we have $\lambda_2(1) = \lambda_2(m) = m - 2$, and for the remaining $(m - 2)$ vertices of P_m , $\lambda_2 = m - 3$. Considering the vertices of P_{m+1} , $\lambda_2(m + 1) = \lambda_2(2m + 1) = m - 1$, and the remaining $(m - 1)$ vertices have $\lambda_2 = m - 2$. Thus, the set of λ_2 -values of the vertices of $P_m + P_{m+1}$ is $\{m - 3, m - 2, m - 1\}$. Therefore, $\alpha_2(G) = m - 3$ and $\beta_2(G) = m - 1$.

The number of edges among the vertices with $\lambda_2 = m - 3$ is $(m - 3)$. The number of edges between the vertices with $\lambda_2 = m - 3$ and $\lambda_2 = m - 2$ is $2 + (n - 2)(m - 2) = 4 + m^2 - 3m$. There are $2(m - 2)$ edges between the vertices with $\lambda_2 = m - 3$ and $\lambda_2 = m - 1$. Among the vertices with $\lambda_2 = m - 2$, the number of edges is $3n - 7 = 3m - 4$. There are exactly 6 edges between the vertices with $\lambda_2 = m - 2$ and $\lambda_2 = m - 1$. Hence, the total number of edges in the graph $P_m + P_{m+1}$ is $m^2 + 3m - 1$.

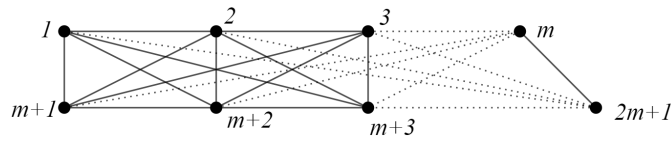


Figure 5.11: $P_m + P_{m+1}$

Therefore, For $m \geq 3$ and $n = m + 1$ then

$$\phi_2(G, x, y) = (m - 3)x^{m-3}y^{m-3} + (m^2 - 3m + 4)x^{m-3}y^{m-2} + (2m - 4)x^{m-3}y^{m-1} + (3m - 4)x^{m-2}y^{m-2} + 6x^{m-2}y^{m-1}.$$

case.8: If $m \geq 3$ and $n \geq m + 1$, then $G = P_m + P_n$, as shown in Figure 5.12. For the vertices a_1 and a_m of P_m , we have $\lambda_2(a_1) = \lambda_2(a_m) = m - 2$, and the

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

remaining $(m - 2)$ vertices of P_m have $\lambda_2 = m - 3$. For the vertices of P_n , $\lambda_2(1) = \lambda_2(n) = n - 2$, and the remaining $(n - 2)$ vertices have $\lambda_2 = n - 3$.

Thus, the set of λ_2 -values of the vertices of $P_m + P_n$ is $\{m - 3, m - 2, n - 3, n - 2\}$. Therefore, $\alpha_2(G) = m - 3$ and $\beta_2(G) = n - 2$. The total number of vertices and edges in $P_m + P_n$ are $m + n$ and $mn + m + n - 2$, respectively.

According to the λ_2 -values of the vertices, we determine the number of edges between the corresponding classes of vertices. From Figure 5.12, the number of edges among the vertices with $\lambda_2 = m - 3$ is $(m - 3)$, and there are exactly 2 edges between the vertices with $\lambda_2 = m - 3$ and $\lambda_2 = m - 2$. The number of edges between the vertices with $\lambda_2 = m - 3$ and $\lambda_2 = n - 3$ is $(m - 2)(n - 2)$, and there are $2(m - 2)$ edges between the vertices with $\lambda_2 = m - 3$ and $\lambda_2 = n - 2$. Similarly, there are $2(n - 2)$ edges between the vertices with $\lambda_2 = m - 2$ and $\lambda_2 = n - 3$. Among the vertices with $\lambda_2 = n - 3$, the number of edges is $(n - 3)$, and there are exactly 2 edges between the vertices with $\lambda_2 = n - 3$ and $\lambda_2 = n - 2$. Finally, there are exactly 4 edges between the vertices with $\lambda_2 = m - 2$ and $\lambda_2 = n - 2$.

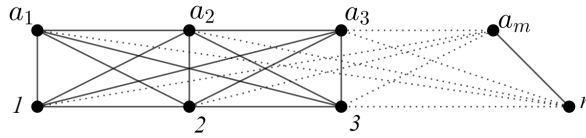


Figure 5.12: $P_m + P_n$

Therefore, For $m \geq 3$ and $n \geq m + 1$ then

$$\phi_2(G, x, y) = (m - 3)x^{m-3}y^{m-3} + 2x^{m-3}y^{m-2} + (m - 2)(n - 2)x^{m-3}y^{n-3} + 2(m - 2)x^{m-3}y^{n-2} + 2(n - 2)x^{m-2}y^{n-3} + 4x^{m-2}y^{n-2} + (n - 3)x^{n-3}y^{n-3} + 2x^{n-3}y^{n-2}$$

□

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

Theorem 5.2.4. *If $G = K_n - e$ be the graph obtained by removing one edge from K_n then $\phi_k -$ polynomial for $k = 2$ of $G = K_n - e$ is given by*

$$\phi_2(G, x, y) = \begin{cases} \frac{(n-2)(n-3)}{2} + 2(n-2)y & \text{if } n \geq 3 \end{cases}$$

Proof. The graph $G = K_n - e$ is shown in Figure 5.13. As previously discussed, in the complete graph K_n , the second degree λ_2 of each vertex is 0. However, in $K_n - e$, the vertices 1 and 2 have $\lambda_2 = 1$, while the remaining $n - 2$ vertices have $\lambda_2 = 0$. Therefore, $\alpha_2(G) = 0$ and $\beta_2(G) = 1$.

The number of edges between vertices with $\lambda_2 = 0$ is $\frac{(n-2)(n-3)}{2}$, since these $n - 2$ vertices form a complete subgraph K_{n-2} . The number of edges between vertices with $\lambda_2 = 0$ and $\lambda_2 = 1$ is $2(n - 2)$, while there are no edges between vertices with $\lambda_2 = 1$. Hence, $\phi_2(0, 0) = \frac{(n-2)(n-3)}{2}$, $\phi_2(0, 1) = 2(n - 2)$, and $\phi_2(1, 1) = 0$.

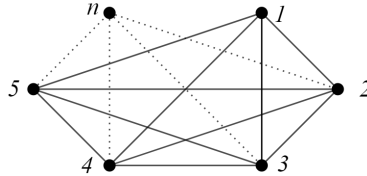


Figure 5.13: $K_n - e$

Therefore for $n \geq 3$, $\phi_2(G, x, y) = \left\{ \frac{(n-2)(n-3)}{2} + 2(n-2)y \right.$ □

Theorem 5.2.5. *If $G = P_m \times P_n$ be the cartecian product of path graphs then $\phi_k -$ polynomial for $k = 2$ of $G = P_m \times P_n$ is given by*

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

$$\phi_2(G, x, y) = \begin{cases} 4xy & \text{if } m = n = 2 \\ 7x^2y^2 & \text{if } m = 2, n = 3 \\ 2x^2y^2 + 4x^2y^3 + (3n - 8)x^3y^3 & \text{if } m = 2, n \geq 4 \\ 2y & \text{if } m = 3, n = 1 \\ 8x^3y^3 + 4x^3y^4 & \text{if } m = n = 3 \\ 4x^3y^3 + 4x^3y^4 + 2x^3y^5 + 2x^4y^4 + 4x^4y^5 + x^5y^5 & \text{if } m = 3, n \geq 4 \\ 4x^3y^3 + 4x^3y^4 + (2n - 10)x^5y^5 + (2n - 6)x^5y^6 + (n - 5)x^6y^6 & \text{if } m = 3, n \geq 5 \\ 3xy & \text{if } m = 4, n = 1 \\ 8x^3y^4 + 4x^4y^4 + 8x^4y^6 + 4x^6y^6 & \text{if } m = n = 4 \\ 8x^3y^4 + 2x^4y^4 + 4x^4y^5 + 8x^4y^6 + \\ (2n - 10)x^5y^5 + (2n - 8)x^5y^7 + 2x^6y^6 + \\ 4x^6y^7 + (3n - 14)x^7y^7 & \text{if } m = 4, n \geq 5 \\ 8x^3y^4 + 8x^4y^5 + 8x^4y^6 + \\ (2(m - 5) + 2(n - 5))x^5y^5 + (2(m - 4) + 2(n - 4))x^5y^7 + \\ 8x^6y^7 + (2(m - 5) + 2(n - 5))x^7y^7 + \\ (2(m - 4) + 2(n - 4))x^7y^8 + \\ (2mn - 9m - 9n + 40)x^8y^8 & \text{if } m \geq 5, n \geq 5 \end{cases}$$

Proof. case.1: If $m = 2, n = 2$, then $P_m \times P_n$ become C_4 . Therefore $\phi_2(G, x, y) = 4xy$

case.2: If $m = 2, n = 3$ then the graph $P_2 \times P_3$ is given by the figure. 5.14.

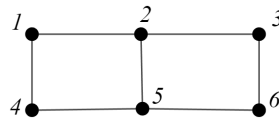


Figure 5.14: $P_2 \times P_3$

The following table details the vertex cardinalities for the second-order ($k = 2$) within the Cartesian product graph $P_2 \times P_3$

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

Table 5.1: 2^{nd} degree and its cardinality of the graphs

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{3, 5\}$	2		
$\lambda_2(2)$	$\{4, 6\}$	2		
$\lambda_2(3)$	$\{1, 5\}$	2	2	2
$\lambda_2(4)$	$\{2, 6\}$	2		
$\lambda_2(5)$	$\{1, 3\}$	2		
$\lambda_2(6)$	$\{2, 4\}$	2		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(2, 2)(G)$	Total edges in this graph	7		

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 2 \leq 2 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 7x^2 y^2$

case.3: If $m = 2, n \geq 4$, then $P_m \times P_n$ is $P_2 \times P_n$ is given in the figure. 5.15.

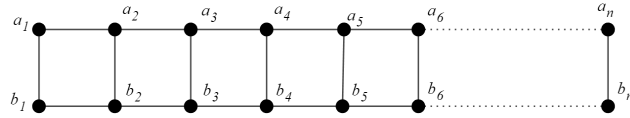


Figure 5.15: $P_2 \times P_n$

From the figure λ_2 value of the vertices a_1, b_1, a_n, b_n is 2. The remaining $2n - 4$ vertices gives the 3 is the λ_2 value, $\alpha_2(G)$ and $\beta_2(G)$ are 2 and 3 respectively.

$$\phi_2(2, 2) = 2, \quad \phi_2(2, 3) = 4, \quad \phi_2(3, 3) = 3n - 8$$

Therefore $\phi_2(G, x, y) = 2x^2 y^2 + 4x^2 y^3 + (3n - 8)x^3 y^3$

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

case.4: If $m = 3, n = 3$, then $P_m \times P_n$ is $P_3 \times P_3$ is given in the figure. 5.16.

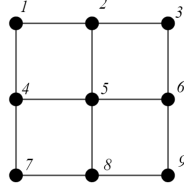


Figure 5.16: $P_3 \times P_3$

The following table details the vertex cardinalities for the second-order ($k = 2$) within the graph $P_3 \times P_3$

Table 5.2: 2^{nd} degree and its cardinality of the graphs

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{3, 5, 7\}$	3	3	4
$\lambda_2(2)$	$\{4, 6, 8\}$	3		
$\lambda_2(3)$	$\{1, 5, 9\}$	3		
$\lambda_2(4)$	$\{2, 6, 8\}$	3		
$\lambda_2(5)$	$\{1, 3, 7, 9\}$	4		
$\lambda_2(6)$	$\{2, 4, 8\}$	3		
$\lambda_2(7)$	$\{1, 5, 9\}$	3		
$\lambda_2(8)$	$\{2, 4, 6\}$	3		
$\lambda_2(9)$	$\{3, 5, 7\}$	3		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(3, 3)(G)$	$\{1 - 2, 2 - 3, 1 - 4, 3 - 6\}$ $\{4 - 7, 4 - 8, 8 - 9, 6 - 9\}$	8		
$\phi_2(3, 4)(G)$	$\{2 - 5, 4 - 5, 6 - 5, 8 - 5\}$	4		

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 3 \leq 4 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 8x^3 y^3 + 4x^3 y^4$

case.5: If $m = 3, n = 4$, then $P_m \times P_n$ is $P_3 \times P_4$ is given in the figure. 5.17.

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

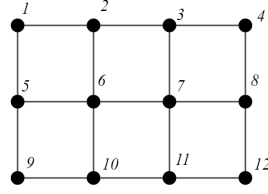


Figure 5.17: $P_3 \times P_4$

The following table details the vertex cardinalities for the second-order ($k = 2$) within the graph $P_3 \times P_4$

Table 5.3: 2^{nd} degree and its cardinality of the graphs

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{3, 9, 6\}$	3	3	5
$\lambda_2(2)$	$\{4, 10, 7\}$	3		
$\lambda_2(3)$	$\{1, 11, 6, 8\}$	4		
$\lambda_2(4)$	$\{2, 12, 7\}$	3		
$\lambda_2(5)$	$\{2, 7, 10\}$	3		
$\lambda_2(6)$	$\{1, 3, 9, 11, 8\}$	5		
$\lambda_2(7)$	$\{2, 4, 10, 12, 5\}$	5		
$\lambda_2(8)$	$\{3, 6, 11\}$	3		
$\lambda_2(9)$	$\{1, 11, 6\}$	3		
$\lambda_2(10)$	$\{2, 5, 7, 12\}$	4		
$\lambda_2(11)$	$\{3, 9, 6, 8\}$	4		
$\lambda_2(12)$	$\{4, 10, 7\}$	3		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(3, 3)(G)$	$\{1 - 5, 5 - 9, 4 - 8, 8 - 12\}$	4		
$\phi_2(3, 4)(G)$	$\{1 - 2, 4 - 3, 9 - 10, 12 - 11\}$	4		
$\phi_2(3, 5)(G)$	$\{5 - 6, 8 - 7\}$	2		
$\phi_2(4, 4)(G)$	$\{2 - 3, 10 - 11\}$	2		
$\phi_2(4, 5)(G)$	$\{2 - 6, 3 - 7, 10 - 6, 11 - 7\}$	4		
$\phi_2(5, 5)(G)$	$\{6 - 7\}$	1		

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 3 \leq 5 \leq \beta_2} \phi_2(i, j)(G) x^i y^j = 4x^3 y^3 + 4x^3 y^4 + 2x^3 y^5 + 2x^4 y^4 + 4x^4 y^5 + x^5 y^5$

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

case.6: If $m = 3, n \geq 5$, then $P_m \times P_n$ is $P_3 \times P_n$ is given in the figure. 5.18.

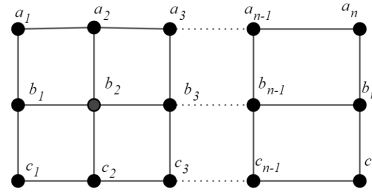


Figure 5.18: $P_3 \times P_n$

From the figure. 5.18, λ_2 value of the vertices $a_1, b_1, c_1, a_n, b_n, c_n$ is 3. λ_2 value of the vertices $a_2, c_{n-1}, c_2, a_{n-1}$ is 4, similarly $a_3, a_4, \dots, a_{n-2}, b_2, b_{n-1}$ and c_3, c_4, \dots, c_{n-2} gives the 5 is the λ_2 value. The λ_2 value of the remaining vertices b_3, b_4, \dots, a_{n-2} is 6. From the collection of λ_2 values of each vertices we get the values of $\alpha_2(G)$ and $\beta_2(G)$ are 2 and 3 respectively.

$$\begin{aligned} \phi_2(3, 3) &= 4 & \phi_2(3, 4) &= 4 & \phi_2(4, 5) &= 8 \\ \phi_2(5, 5) &= 2n - 10 & \phi_2(5, 6) &= 2n - 6 & \phi_2(6, 6) &= n - 5 \end{aligned}$$

Therefore $\phi_2(G, x, y) = 4x^3y^3 + 4x^3y^4 + 8x^4y^5 + (2n - 10)x^5y^5 + (2n - 6)x^5y^6 + (n - 5)x^6y^6$

case.7: If $m = 4, n = 4$, then $P_m \times P_n$ is $P_4 \times P_4$ is given in the figure. 5.19.

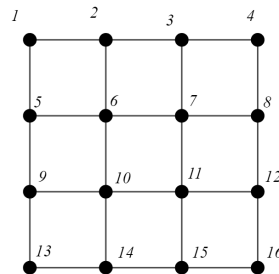


Figure 5.19: $P_4 \times P_4$

The following table details the vertex cardinalities for the second-order ($k =$

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

2) within the graph $P_3 \times P_4$

Table 5.4: 2^{nd} degree and its cardinality of the graphs

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	{3, 9, 6}	3	3	6
$\lambda_2(2)$	{4, 10, 7}	3		
$\lambda_2(3)$	{1, 11, 6, 8}	4		
$\lambda_2(4)$	{2, 12, 7}	3		
$\lambda_2(5)$	{2, 7, 10, 13}	4		
$\lambda_2(6)$	{1, 3, 9, 11, 8, 14}	6		
$\lambda_2(7)$	{2, 4, 10, 12, 5, 15}	6		
$\lambda_2(8)$	{3, 6, 11, 16}	4		
$\lambda_2(9)$	{1, 11, 6, 14}	4		
$\lambda_2(10)$	{2, 5, 7, 12, 13, 15}	6		
$\lambda_2(11)$	{3, 9, 6, 8, 14, 16}	6		
$\lambda_2(12)$	{4, 10, 7, 15}	4		
$\lambda_2(13)$	{5, 10, 15}	3		
$\lambda_2(14)$	{6, 11, 6}	3		
$\lambda_2(15)$	{7, 10, 13}	3		
$\lambda_2(16)$	{8, 11, 14}	3		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(3, 4)(G)$	{1 - 2, 1 - 5, 13 - 9, 13 - 14} {16 - 15, 16 - 12, 4 - 3, 4 - 8}	8		
$\phi_2(4, 4)(G)$	{2 - 3, 5 - 9, 8 - 12, 14 - 15}	4		
$\phi_2(4, 6)(G)$	{2 - 6, 5 - 6, 9 - 10, 14 - 10} {3 - 7, 8 - 7, 12 - 11, 15 - 11}	8		
$\phi_2(6, 6)(G)$	{6 - 7, 6 - 10, 10 - 11, 7 - 11}	4		

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 3 \leq 5 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 8x^3 y^4 + 4x^4 y^4 + 8x^4 y^6 + 4x^6 y^6$

case.8: If $m = 4, n \geq 5$, then $P_m \times P_n$ is $P_4 \times P_n$ is given in the figure. 5.20.

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

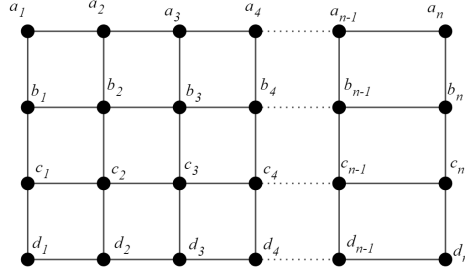


Figure 5.20: $P_4 \times P_n$

From the figure. 5.20, λ_2 value of the vertices a_1, d_1, a_n, d_n , is 3. λ_2 value of the vertices $a_2, b_1, c_1, d_2, a_{n-1}, b_n, c_n, d_{n-1}$ is 4, λ_2 value of the vertices $a_3, a_4, \dots, a_{n-2}, d_3, c_4, \dots, d_{n-2}$ is 5. The λ_2 value of the vertices $b_2, c_2, b_{n-1}, c_{n-1}$ is 6. λ_2 value of the vertices $b_3, b_4, \dots, b_{n-2}, c_3, c_4, \dots, c_{n-2}$ is 7. From the collection of λ_2 values of each vertices we get the values of $\alpha_2(G)$ and $\beta_2(G)$ are 3 and 7 respectively.

$$\begin{aligned} \phi_2(3, 4) &= 8 & \phi_2(4, 4) &= 2 & \phi_2(4, 5) &= 4 \\ \phi_2(4, 6) &= 8 & \phi_2(5, 5) &= 2n - 10 & \phi_2(5, 7) &= 2n - 8 \\ \phi_2(6, 6) &= 2 & \phi_2(6, 7) &= 4 & \phi_2(7, 7) &= 3n - 14 \end{aligned}$$

Therefore $\phi_2(G, x, y) = 8x^3y^4 + 2x^4y^4 + 4x^4y^5 + 8x^4y^6 + (2n - 10)x^5y^5 + (2n - 8)x^5y^7 + 2x^6y^6 + 4x^6y^7 + (3n - 14)x^7y^7$

case.9: If $m \geq 5, n \geq 5$ then $P_m \times P_n$ is given in the figure. 5.21.

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

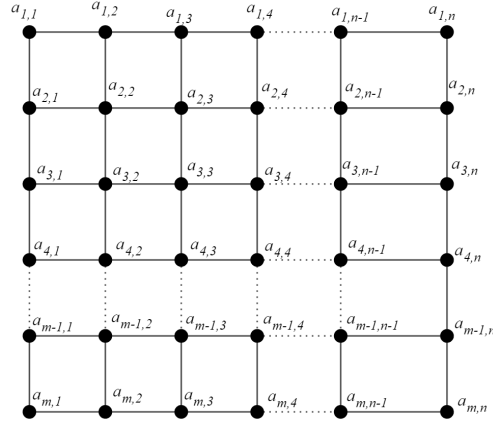


Figure 5.21: $P_m \times P_n$

From the figure. 5.21, λ_2 value of the vertices in the graph $P_m \times P_n$ is given in the table.

Table 5.5: The value of λ_2 and its cardinality of vertices

λ_2	Vertices	No. of Vertices
3	$a_{1,1}, a_{m,1}, a_{1,n}, a_{m,n}$	4
4	$a_{2,1}, a_{1,2}, a_{1,n-1}, a_{2,n}, a_{m-1,1}, a_{m,2}, a_{m-1,n}, a_{m,n-1}$	8
5	$a_{1,3}, a_{1,4}, \dots, a_{1,n-2}$ $a_{m,3}, a_{m,4}, \dots, a_{m,n-2}$ $a_{3,1}, a_{4,1}, \dots, a_{m-2,1}$ $a_{3,n}, a_{4,n}, \dots, a_{m-2,n}$	$2(m+n-8)$
6	$a_{2,2}, a_{m-1,2}, a_{2,n-1}, a_{m-1,n-1}$	4
7	$a_{2,3}, a_{2,4}, \dots, a_{2,n-2}$ $a_{m-1,3}, a_{m-1,4}, \dots, a_{m-1,n-2}$ $a_{3,2}, a_{4,2}, \dots, a_{m-2,2}$ $a_{3,n-1}, a_{4,n-1}, \dots, a_{m-2,n-1}$	$2(m+n-8)$
8	<i>Remaining vertices</i>	$(m-4)(n-4)$

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

From the table above of λ_2 values of each vertices we get the values of $\alpha_2(G)$ and $\beta_2(G)$ are 3 and 8 respectively.

$$\begin{aligned} \phi_2(3, 4) &= 8 & \phi_2(4, 5) &= 8 & \phi_2(4, 6) &= 8 \\ \phi_2(5, 5) &= 2m + 2n - 20 = P, & \phi_2(5, 7) &= 2m + 2n - 16 = Q, & \phi_2(6, 7) &= 8 \\ \phi_2(7, 7) &= 2m + 2n - 20 = P & \phi_2(7, 8) &= 2m + 2n - 16 = Q \\ \phi_2(8, 8) &= 2mn - 9m - 9n + 40 = R \end{aligned}$$

Therefore $\phi_2(G, x, y) = \sum_{3 \leq i \leq j \leq 8} \phi_2(i, j)(G)x^i y^j = 8x^3 y^4 + 8x^4 y^5 + 8x^4 y^6 + Px^5 y^5 + Qx^5 y^7 + 8x^6 y^7 + Px^7 y^7 + Qx^7 y^8 + Rx^8 y^8$ \square

Conclusion

In this chapter, we explored the polynomial properties of the graph families $P_m + P_n$, $P_n + O_n$, $K_n - e$, and $P_m \times P_n$. Our analysis focused on deriving key polynomial invariants, including chromatic polynomials, characteristic polynomials, and matching polynomials, parallel investigating their structural and combinatorial properties. The results provide a deeper understanding of the algebraic and topological behaviors of these graph families. The findings from this study have several practical applications:

- **Network design and optimization:** Understanding polynomial properties aids in designing efficient communication networks and optimizing resource allocation.

5.2. Polynomial properties of the graph families $P_n + O_n$, $P_m + P_n$, $K_n - e$ and $P_m \times P_n$

- **Data structures and algorithms:** These properties can be utilized in algorithmic graph theory, particularly in graph traversal, coloring, and partitioning problems.
- **Mathematical modeling:** Applications extend to mathematical models in chemistry, physics, and social sciences, where graph representations are employed to model molecular structures, physical systems, and social networks.

Future research directions include:

- **Extending to other graph families:** Investigating similar polynomial properties in other graph families, such as hypercubes, bipartite graphs, and wheel graphs.
- **Computational complexity:** Studying the computational aspects of determining these polynomials for large-scale graphs.
- **Dynamic graphs:** Exploring how polynomial properties evolve under dynamic operations like edge deletion, vertex addition, or graph morphing.
- **Interdisciplinary applications:** Applying the results in interdisciplinary domains such as machine learning, quantum computing, and bioinformatics.

This chapter establishes a foundation for further exploration of polynomial graph properties and their applications, contributing significantly to the field of algebraic graph theory.

Chapter 6

Study of 12 zodiac signs in the form of graph and ϕ_k -polynomial

6.1 Introduction

A graph $G = (V, E)$ of order $n = |V(G)|$ and size $m = |E(G)|$ is connected if there exists a path between any pair of vertices in G . A network is also a simple connected graph. If two vertices x and y of the graph G are adjacent, then the edge connecting them will be denoted by xy . If $x, y \in V(G)$, then the distance $\lambda(x, y)$ between x and y is defined as the length of a shortest path in G connecting them. In the graph G , the first degree of a vertex x , denoted $\lambda_1(x)$, is the number of first neighbours (the number of edges incident with x). The second degree of a vertex x , denoted $\lambda_2(x)$, is the number of vertices in G which are connected by x with exactly 2 vertices. The minimum and maximum degrees among the vertices of G , are denoted by $\alpha_2(G)$ and $\beta_2(G)$ respectively. Some of

the star clusters and its corresponding graphs are including in this chapter.

6.2 ϕ_k -polynomial properties on cancer

Cancer is one of the twelve zodiac signs, and its corresponding graph is a connected graph with 6 vertices, as shown in Figure 6.1. The k^{th} degree and its cardinality for each vertex in the graph are given in Table 6.1.

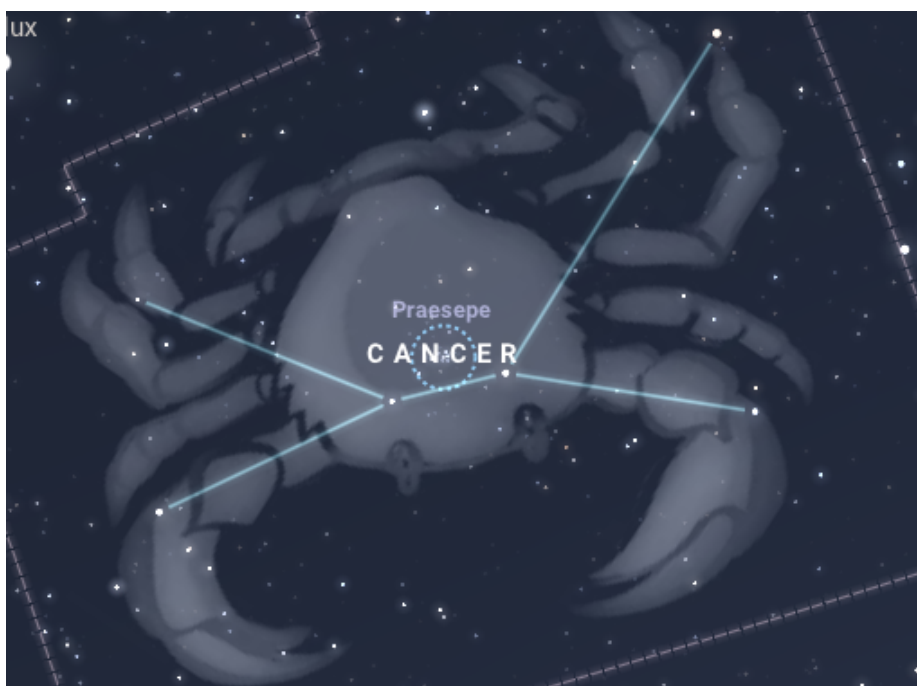


Figure 6.1: *Cancer*

6.3. ϕ_k -polynomial properties on leo

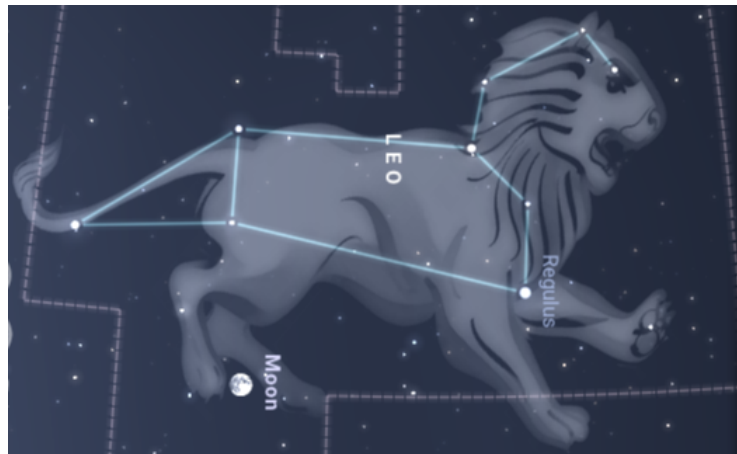
$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$		
$\lambda_2(1)$	$\{3, 6\}$	2				
$\lambda_2(2)$	$\{4, 5\}$	2				
$\lambda_2(3)$	$\{1, 6\}$	2	2	2		
$\lambda_2(4)$	$\{2, 5\}$	2				
$\lambda_2(5)$	$\{4, 2\}$	2				
$\lambda_2(6)$	$\{1, 3\}$	2				
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $				
$\phi_2(2, 2)(G)$	$\{3 - 4, 3 - 5\}$ $\{3 - 2, 2 - 1, 2 - 6\}$	5				

Table 6.1: 2^{nd} degree and its cardinality of the graphs

Therefore $\phi_2(G, x, y) = \sum_{\alpha_k \leq 2 \leq 2 \leq \beta_k} \phi_2(i, j)(G) x^i y^j = 5x^2 y^2$

6.3 ϕ_k -polynomial properties on leo

Leo is a connected graph with 9 vertices, as shown in Figure 6.2. The k^{th} degree and its cardinality for each vertex in the graph are given in Table 6.2.



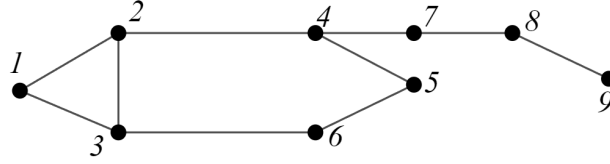


Figure 6.2: *Leo*

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{4, 6\}$	2		
$\lambda_2(2)$	$\{7, 5, 6\}$	3		
$\lambda_2(3)$	$\{4, 5\}$	2		
$\lambda_2(4)$	$\{1, 6, 8\}$	3		
$\lambda_2(5)$	$\{3, 2, 8\}$	3	1	3
$\lambda_2(6)$	$\{1, 4\}$	2		
$\lambda_2(7)$	$\{2, 5, 9\}$	3		
$\lambda_2(8)$	$\{4\}$	1		
$\lambda_2(9)$	$\{7\}$	1		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	$\{8 - 9\}$	1		
$\phi_2(1, 2)(G)$	$\{\}$	0		
$\phi_2(1, 3)(G)$	$\{8 - 7\}$	1		
$\phi_2(2, 2)(G)$	$\{1 - 3, 3 - 6\}$	2		
$\phi_2(2, 3)(G)$	$\{1 - 2, 2 - 3, 6 - 5\}$	3		
$\phi_2(3, 3)(G)$	$\{2 - 4, 4 - 5, 4 - 7\}$	3		

Table 6.2: 2^{nd} degree and its cardinality of the graphs

$$\text{Therefore } \phi_2(G, x, y) = \sum_{\alpha_k \leq i \leq 2 \leq j \leq \beta_k} \phi_2(i, j)(G) x^i y^j = xy + xy^3 + 2x^2y^2 + 3x^2y^3 + 3x^3y^3$$

6.4 ϕ_k -polynomial properties on virgo

Virgo is a connected graph with 12 vertices, as shown in Figure 6.3. The k^{th} degree and its cardinality for each vertex in the graph are given in Table 6.3.

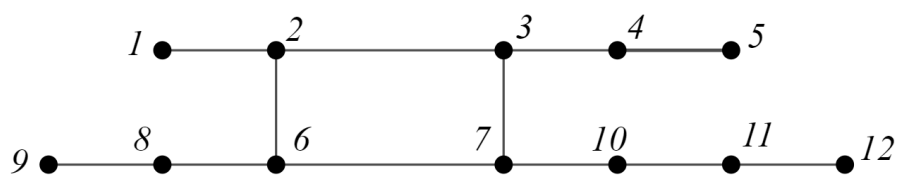
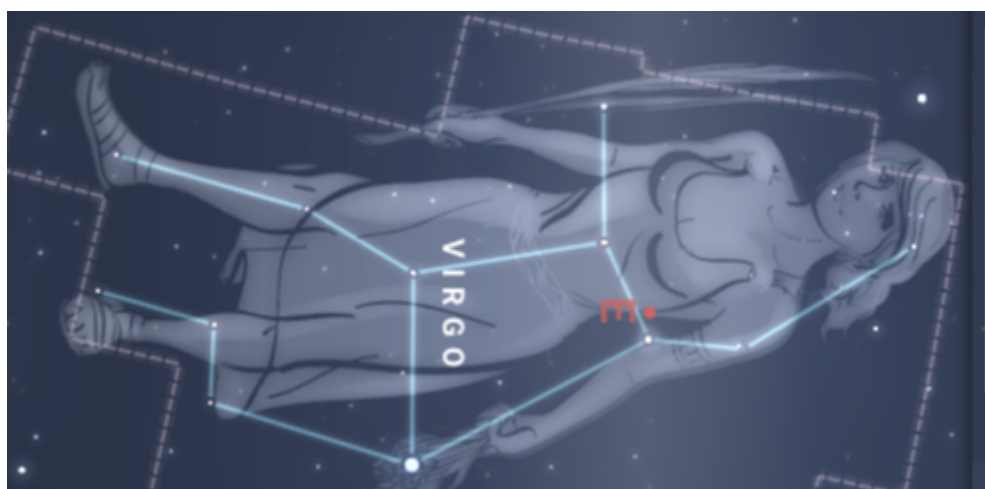


Figure 6.3: *Virgo*

6.4. ϕ_k -polynomial properties on virgo

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{3, 6\}$	2	1	4
$\lambda_2(2)$	$\{4, 8, 7\}$	3		
$\lambda_2(3)$	$\{1, 5, 10, 6\}$	4		
$\lambda_2(4)$	$\{2, 7\}$	2		
$\lambda_2(5)$	$\{3\}$	1		
$\lambda_2(6)$	$\{9, 1, 3, 10\}$	4		
$\lambda_2(7)$	$\{11, 8, 2, 4\}$	4		
$\lambda_2(8)$	$\{2, 7\}$	2		
$\lambda_2(9)$	$\{6\}$	1		
$\lambda_2(10)$	$\{12, 6, 3\}$	3		
$\lambda_2(11)$	$\{7\}$	1		
$\lambda_2(12)$	$\{10\}$	1		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	$\{11 - 12\}$	1		
$\phi_2(1, 2)(G)$	$\{4 - 5, 8 - 9\}$	2		
$\phi_2(1, 3)(G)$	$\{10 - 11\}$	1		
$\phi_2(1, 4)(G)$	$\{\}$	0		
$\phi_2(2, 2)(G)$	$\{\}$	0		
$\phi_2(2, 3)(G)$	$\{1 - 2\}$	1		
$\phi_2(2, 4)(G)$	$\{3 - 4, 8 - 6\}$	2		
$\phi_2(3, 3)(G)$	$\{\}$	0		
$\phi_2(3, 4)(G)$	$\{2 - 3, 2 - 6, 10 - 7\}$	3		
$\phi_2(4, 4)(G)$	$\{3 - 7, 6 - 7\}$	2		

Table 6.3: 2^{nd} degree and its cardinality of the graphs

Therefore $\phi_2(G, x, y) = \sum_{\alpha_k \leq i \leq j \leq \beta_k} \phi_2(i, j)(G)x^i y^j = xy + 2xy^2 + xy^3 + x^2 y^3 + 2x^2 y^4 + 3x^3 y^4 + 2x^4 y^4$

6.5 ϕ_k -polynomial properties on libra

Libra is a connected graph with 5 vertices, as shown in Figure 6.4. The k^{th} degree of each vertex in the graph is given in Table 6.4.

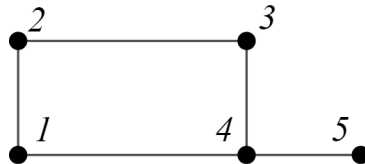
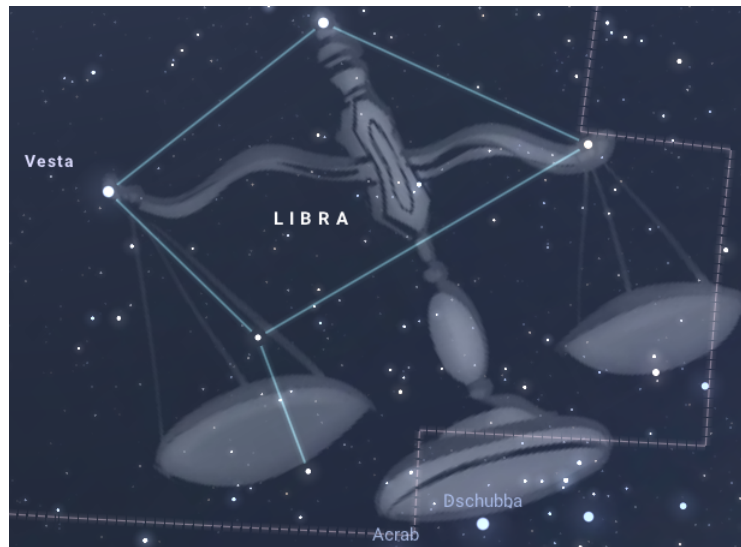


Figure 6.4: *Libra*

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{3, 4\}$	2	1	2
$\lambda_2(2)$	$\{4\}$	1		
$\lambda_2(3)$	$\{1, 5\}$	2		
$\lambda_2(4)$	$\{2\}$	1		
$\lambda_2(5)$	$\{1, 3\}$	2		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	$\{\}$	0		
$\phi_2(1, 2)(G)$	$\{1 - 2, 4 - 1, 4 - 5, 2 - 3, 4 - 3\}$	5		
$\phi_2(2, 2)(G)$	$\{\}$	0		

Table 6.4: 2^{nd} degree and its cardinality of the graphs

Therefore $\phi_2(G, x, y) = \sum_{\alpha_k \leq 1 \leq 2 \leq \beta_k} \phi_2(i, j)(G)x^i y^j = 5xy^2$

6.6 ϕ_k -polynomial properties on scorpius

Scorpius is a connected graph with 13 vertices, as shown in Figure 6.5. The k^{th} degree of each vertex in the graph is given in Table 6.5.

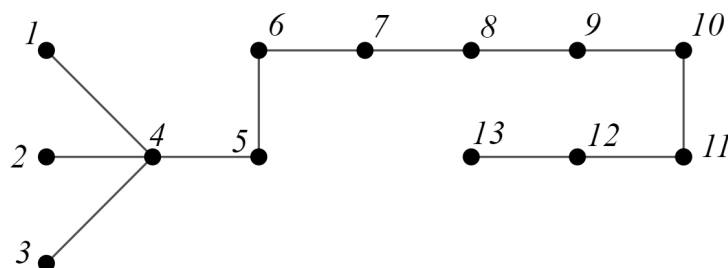
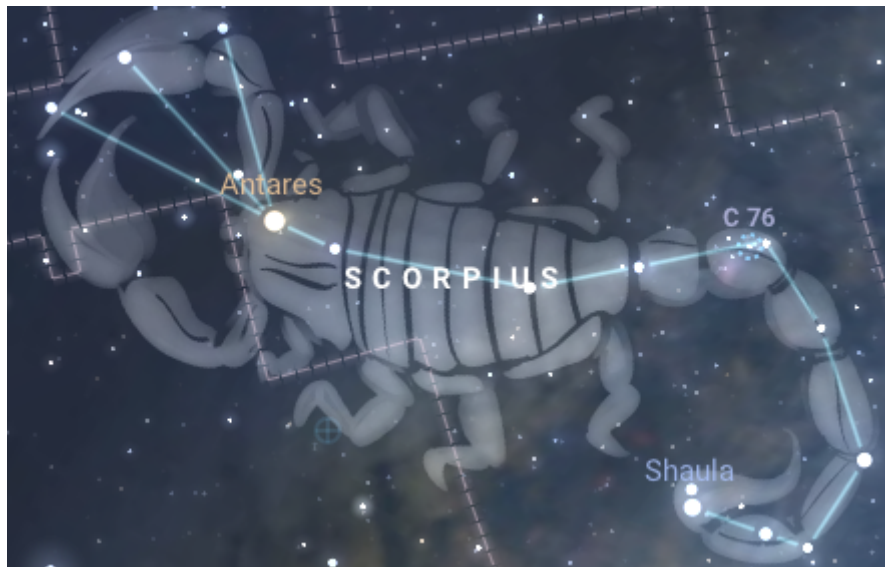


Figure 6.5: *Scorpius*

6.6. ϕ_k -polynomial properties on scorpis

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	{1}	1		
$\lambda_2(2)$	{5}	1		
$\lambda_2(3)$	{5}	1		
$\lambda_2(4)$	{7}	1		
$\lambda_2(5)$	{1, 2, 3, 7}	4		
$\lambda_2(6)$	{4, 8}	2		
$\lambda_2(7)$	{5, 9}	2	1	4
$\lambda_2(8)$	{6, 10}	2		
$\lambda_2(9)$	{7, 11}	2		
$\lambda_2(10)$	{8, 12}	2		
$\lambda_2(11)$	{9, 13}	2		
$\lambda_2(12)$	{10}	1		
$\lambda_2(13)$	{11}	1		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	{1 - 4, 2 - 4, 3 - 4, 10 - 11}	4		
$\phi_2(1, 2)(G)$	{11 - 12}	1		
$\phi_2(1, 3)(G)$	{}	0		
$\phi_2(1, 4)(G)$	{4 - 5}	1		
$\phi_2(2, 2)(G)$	{6 - 7, 7 - 8, 8 - 9, 9 - 10, 10 - 11}	5		
$\phi_2(2, 3)(G)$	{}	0		
$\phi_2(2, 4)(G)$	{5 - 6}	1		
$\phi_2(3, 3)(G)$	{}	0		
$\phi_2(3, 4)(G)$	{}	0		
$\phi_2(4, 4)(G)$	{}	0		

Table 6.5: 2^{nd} degree and its cardinality of the graphs

Therefore $\phi_2(G, x, y) = \sum_{\alpha_k \leq 1 \leq 4 \leq \beta_k} \phi_2(i, j)(G)x^i y^j = 4xy + xy^2 + xy^4 + 5x^2y^2 + x^2y^4$

6.7 ϕ_k -polynomial properties on aries

Aries is a connected graph with 4 vertices given in the Figure. 6.6. The k^{th} degree of each vertices in the graph is given in the table 6.6



Figure 6.6: *Aries*

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{3\}$	1	1	1
$\lambda_2(2)$	$\{4\}$	1		
$\lambda_2(3)$	$\{1\}$	1		
$\lambda_2(4)$	$\{2\}$	1		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	$\{1 - 2, 2 - 3, 3 - 4\}$	3		

Table 6.6: 2^{nd} degree and its cardinality of the graphs

$$\text{Therefore } \phi_2(G, x, y) = \sum_{\alpha_k \leq 1 \leq 1 \leq \beta_k} \phi_2(1, 1)(G)x^1y^1 = 3xy$$

6.8 ϕ_k -polynomial properties on taurus

Taurus is a connected graph with 8 vertices given in the Figure. 6.7. The k^{th} degree of each vertices in the graph is given in the table 6.7

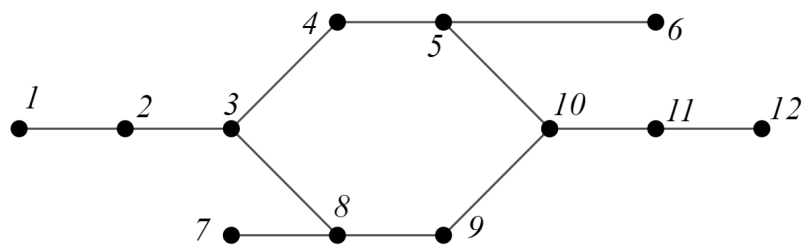
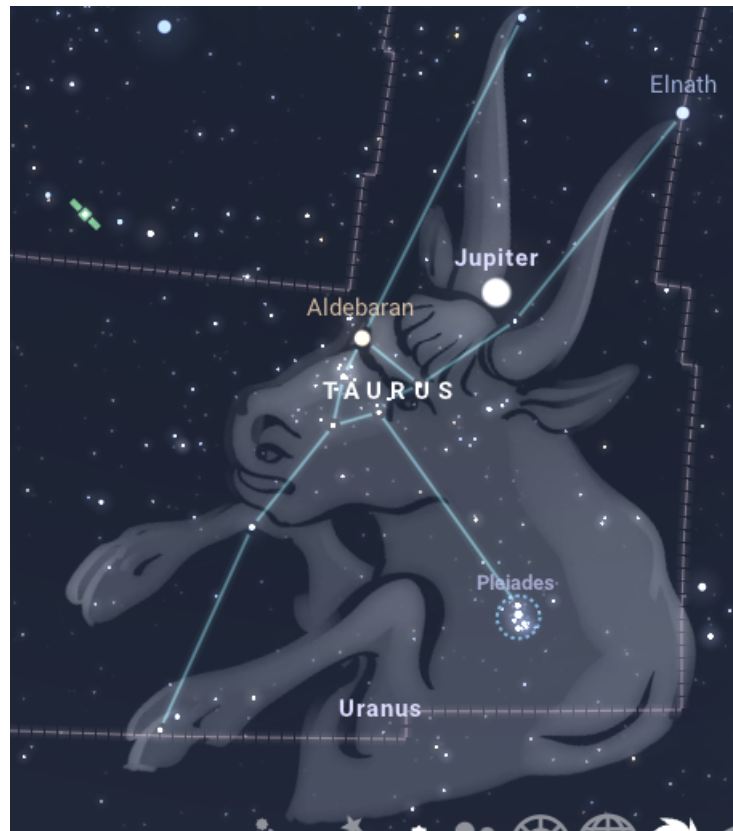


Figure 6.7: *Taurus*

6.8. ϕ_k -polynomial properties on taurus

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{3\}$	1		
$\lambda_2(2)$	$\{4, 8\}$	2		
$\lambda_2(3)$	$\{1, 5, 7, 9\}$	4		
$\lambda_2(4)$	$\{2, 8, 10, 6\}$	4		
$\lambda_2(5)$	$\{3, 9, 11\}$	3		
$\lambda_2(6)$	$\{4, 10\}$	2	1	4
$\lambda_2(7)$	$\{3, 9\}$	2		
$\lambda_2(8)$	$\{2, 4, 10\}$	3		
$\lambda_2(9)$	$\{7, 3, 5, 11\}$	4		
$\lambda_2(10)$	$\{6, 4, 8, 12\}$	4		
$\lambda_2(11)$	$\{5, 9\}$	2		
$\lambda_2(12)$	$\{10\}$	1		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	$\{\}$	0		
$\phi_2(1, 2)(G)$	$\{1 - 2, 12 - 11\}$	2		
$\phi_2(1, 3)(G)$	$\{\}$	0		
$\phi_2(1, 4)(G)$	$\{\}$	0		
$\phi_2(2, 2)(G)$	$\{\}$	0		
$\phi_2(2, 3)(G)$	$\{6 - 5, 7 - 8\}$	2		
$\phi_2(2, 4)(G)$	$\{2 - 3, 11 - 10\}$	2		
$\phi_2(3, 3)(G)$	$\{\}$	0		
$\phi_2(3, 4)(G)$	$\{5 - 4, 5 - 10, 8 - 3, 8 - 9\}$	4		
$\phi_2(4, 4)(G)$	$\{3 - 4, 9 - 10\}$	2		

Table 6.7: 2^{nd} degree and its cardinality of the graphs

$$\text{Therefore } \phi_2(G, x, y) = \sum_{\alpha_k \leq i \leq j \leq \beta_k} \phi_2(i, j)(G) x^i y^j = \sum_{\alpha_k \leq 1 \leq 3 \leq \beta_k} \phi_2(i, j)(G) x^i y^j = 2xy^2 + 2x^2y^3 + 2x^2y^4 + 4x^3y^4 + 2x^4y^4$$

6.9 ϕ_k -polynomial properties on the graph aquarius

Aquarius is a connected graph with 15 vertices given in the Figure. 6.8. The k^{th} degree of each vertices in the graph is given in the table 6.8

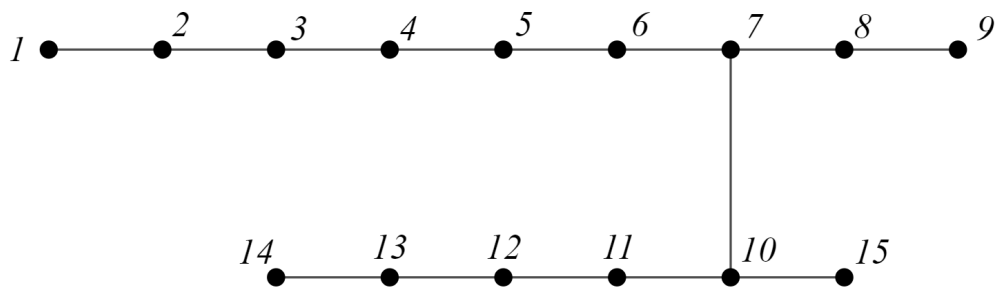
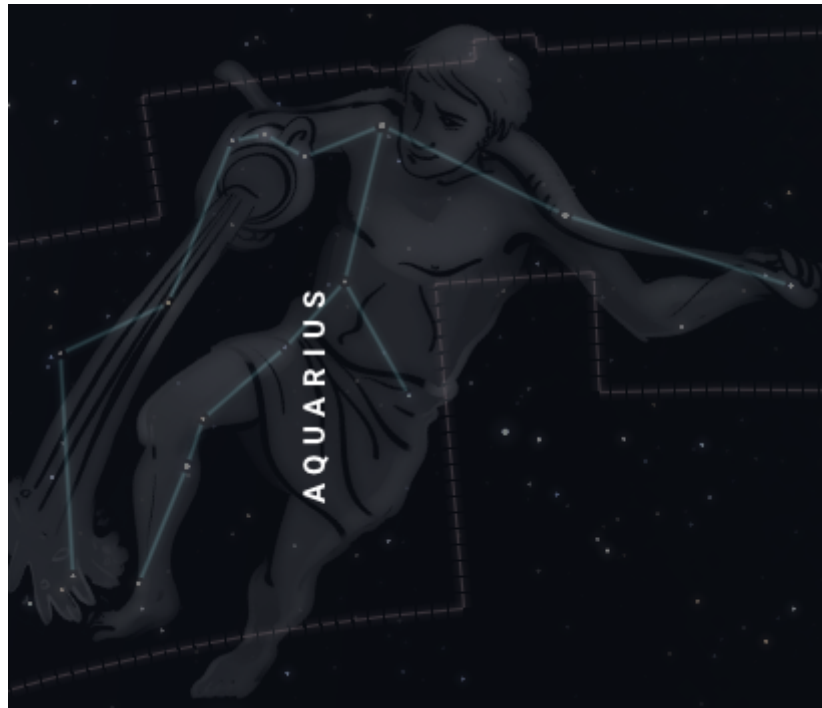


Figure 6.8: *Aquarius*

6.9. ϕ_k -polynomial properties on the graph aquarius

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	{3}	1		
$\lambda_2(2)$	{4}	1		
$\lambda_2(3)$	{1, 5}	2		
$\lambda_2(4)$	{2, 6}	2		
$\lambda_2(5)$	{3, 7}	2		
$\lambda_2(6)$	{4, 8, 10}	3		
$\lambda_2(7)$	{5, 9, 11, 15}	4		
$\lambda_2(8)$	{6, 10}	2	1	4
$\lambda_2(9)$	{7}	1		
$\lambda_2(10)$	{6, 8, 12}	3		
$\lambda_2(11)$	{7, 13, 15}	3		
$\lambda_2(12)$	{14, 10}	2		
$\lambda_2(13)$	{11}	1		
$\lambda_2(14)$	{12}	1		
$\lambda_2(15)$	{11, 7}	2		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	{1 - 2, 13 - 14}	2		
$\phi_2(1, 2)(G)$	{2 - 3, 9 - 8, 13 - 12}	3		
$\phi_2(1, 3)(G)$	{}	0		
$\phi_2(1, 4)(G)$	{}	0		
$\phi_2(2, 2)(G)$	{3 - 4, 4 - 5}	2		
$\phi_2(2, 3)(G)$	{5 - 6, 15 - 10, 12 - 11}	3		
$\phi_2(2, 4)(G)$	{8 - 7, }	1		
$\phi_2(3, 3)(G)$	{10 - 11}	1		
$\phi_2(3, 4)(G)$	{6 - 7, 10 - 7}	2		
$\phi_2(4, 4)(G)$	{}	0		

Table 6.8: 2^{nd} degree and its cardinality of the graphs

Therefore $\phi_2(G, x, y) = \sum_{\alpha_k \leq i \leq j \leq \beta_k} \phi_2(i, j)(G)x^i y^j = 2xy + 3xy^2 + 2x^2y^2 + 3x^2y^3 + x^2y^4 + x^3y^3 + 2x^3y^4$

6.10 ϕ_k -polynomial properties on the graph pisces

Pisces is a connected graph with 15 vertices given in the Figure. 6.9. The k^{th} degree of each vertices in the graph is given in the table 6.9

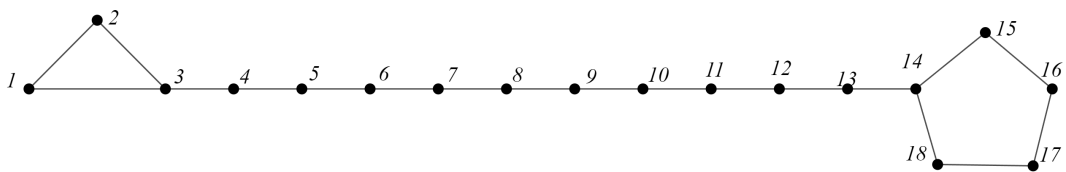


Figure 6.9: *Pisces*

6.10. ϕ_k -polynomial properties on the graph
pisces

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	{4}	1		
$\lambda_2(2)$	{4}	1		
$\lambda_2(3)$	{5}	1		
$\lambda_2(4)$	{1, 2, 6}	3		
$\lambda_2(5)$	{3, 7}	2		
$\lambda_2(6)$	{4, 8}	2		
$\lambda_2(7)$	{5, 9}	2		
$\lambda_2(8)$	{6, 10}	2		
$\lambda_2(9)$	{7, 11}	2		
$\lambda_2(10)$	{8, 12}	2	1	3
$\lambda_2(11)$	{9, 13}	2		
$\lambda_2(12)$	{10, 14}	2		
$\lambda_2(13)$	{11, 15, 18}	3		
$\lambda_2(14)$	{12, 16, 17}	3		
$\lambda_2(15)$	{13, 17, 18}	3		
$\lambda_2(16)$	{14, 18}	2		
$\lambda_2(17)$	{14, 15}	2		
$\lambda_2(18)$	{13, 16, 15}	3		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	{1 - 2, 1 - 3, 2 - 3}	3		
$\phi_2(1, 2)(G)$	{}	0		
$\phi_2(1, 3)(G)$	{3 - 4}	1		
$\phi_2(2, 2)(G)$	{5 - 6, 6 - 7, 7 - 8, 8 - 9}			
	{9 - 10, 10 - 11, 11 - 12, 16 - 17}	8		
$\phi_2(2, 3)(G)$	{5 - 4, 12 - 13, 16 - 15, 17 - 18}	4		
$\phi_2(3, 3)(G)$	{13 - 14, 14 - 15, 14 - 18, }	3		

Table 6.9: 2^{nd} degree and its cardinality of the graphs

Therefore $\phi_2(G, x, y) = \sum_{\alpha_k \leq 1 \leq 3 \leq \beta_k} \phi_2(i, j)(G)x^i y^j = 3xy + xy^3 + 8x^2y^2 + 4x^2y^3 + 3x^3y^3$

6.11 ϕ_k -polynomial properties capricornus

Capricornus is a connected graph with 8 vertices given in the Figure. 6.10. The k^{th} degree of each vertices in the graph is given in the table 6.10

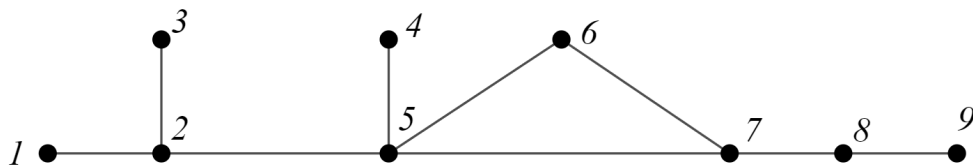


Figure 6.10: *Capricornus*

6.12. ϕ_k -polynomial properties gemini

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	{3, 5}	2		
$\lambda_2(2)$	{4, 6, 7}	3		
$\lambda_2(3)$	{1, 5}	2		
$\lambda_2(4)$	{2, 6, 7}	3		
$\lambda_2(5)$	{1, 3, 8}	3	2	3
$\lambda_2(6)$	{2, 4, 9}	3		
$\lambda_2(7)$	{2, 4, 9}	3		
$\lambda_2(8)$	{6, 5}	2		
$\lambda_2(9)$	{7}	1		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	{}	0		
$\phi_2(1, 2)(G)$	{9 - 8}	1		
$\phi_2(1, 3)(G)$	{}	0		
$\phi_2(2, 2)(G)$	{}	0		
$\phi_2(2, 3)(G)$	{1 - 2, 3 - 2, 8 - 7}	3		
$\phi_2(3, 3)(G)$	{5 - 6, 2 - 5, 4 - 5, 5 - 7, 6 - 7}	5		

Table 6.10: 2^{nd} degree and its cardinality of the graphs

Therefore $\phi_2(G, x, y) = \sum_{\alpha_k \leq 2 \leq 3 \leq \beta_k} \phi_2(i, j)(G)x^i y^j = xy^2 + 3x^2y^3 + 5x^3y^3$

6.12 ϕ_k -polynomial properties gemini

Gemini is a connected graph with 17 vertices given in the Figure. 6.11. The k^{th} degree of each vertices in the graph is given in the table 6.11

6.12. ϕ_k -polynomial properties gemini

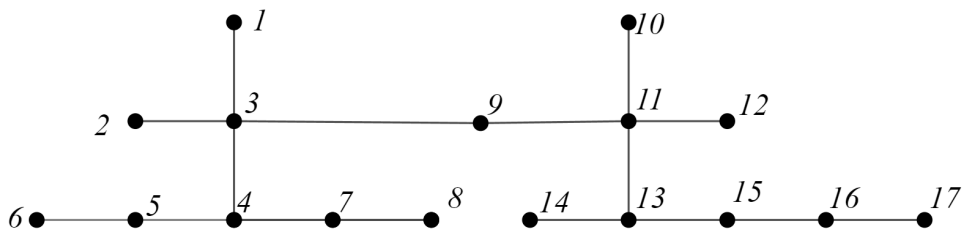
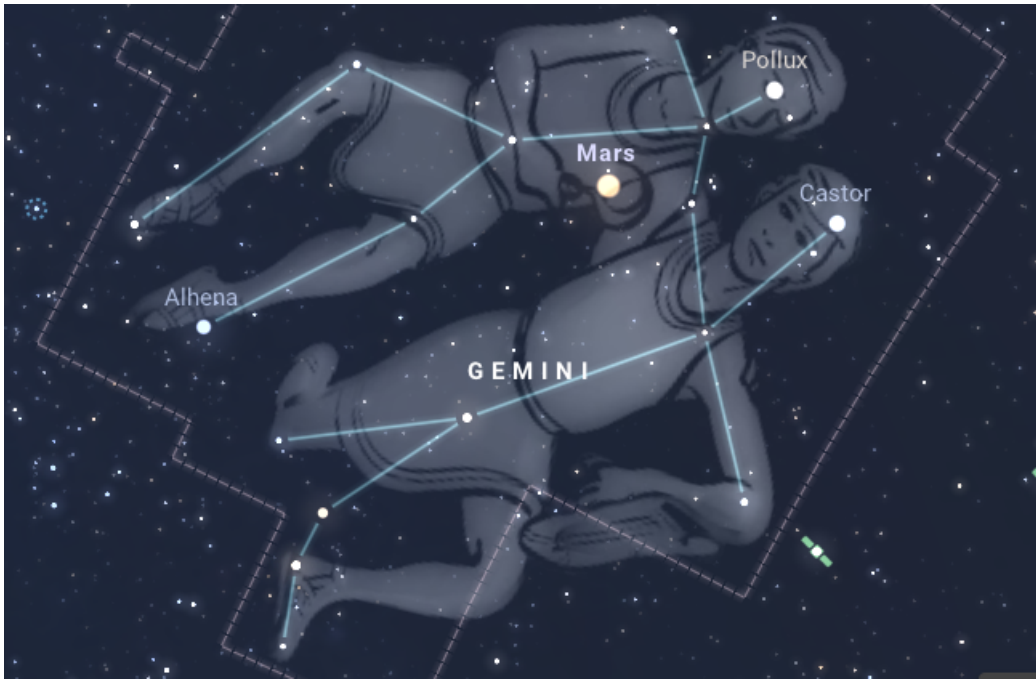


Figure 6.11: *Gemini*

6.12. ϕ_k -polynomial properties gemini

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	{2, 4, 9}	3		
$\lambda_2(2)$	{1, 4, 9}	3		
$\lambda_2(3)$	{5, 7, 11}	3		
$\lambda_2(4)$	{1, 2, 9, 6, 8}	5		
$\lambda_2(5)$	{3, 7}	2		
$\lambda_2(6)$	{4}	1		
$\lambda_2(7)$	{5, 3}	2		
$\lambda_2(8)$	{4}	1		
$\lambda_2(9)$	{1, 2, 4, 10, 12, 13}	6	1	6
$\lambda_2(10)$	{9, 12, 13}	3		
$\lambda_2(11)$	{3, 14, 15}	3		
$\lambda_2(12)$	{9, 10, 13}	3		
$\lambda_2(13)$	{9, 10, 12, 16}	4		
$\lambda_2(14)$	{11, 15}	2		
$\lambda_2(15)$	{11, 14, 17}	3		
$\lambda_2(16)$	{13}	1		
$\lambda_2(17)$	{15}	1		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	{16 - 17}	1		
$\phi_2(1, 2)(G)$	{8 - 7, 6 - 5}	2		
$\phi_2(1, 3)(G)$	{16 - 15}	1		
$\phi_2(1, 4)(G)$	{}	0		
$\phi_2(1, 5)(G)$	{}	0		
$\phi_2(1, 6)(G)$	{}	0		
$\phi_2(2, 2)(G)$	{}	0		
$\phi_2(2, 3)(G)$	{14 - 15}	1		
$\phi_2(2, 4)(G)$	{14 - 13}	1		
$\phi_2(2, 5)(G)$	{5 - 4, 7 - 4}	2		
$\phi_2(2, 6)(G)$	{}	0		
$\phi_2(3, 3)(G)$	{1 - 3, 3 - 2, 10 - 11, 11 - 12}	4		
$\phi_2(3, 4)(G)$	{15 - 13}	1		
$\phi_2(3, 5)(G)$	{3 - 4}	3		
$\phi_2(3, 6)(G)$	{3 - 9, 11 - 9}	2		
$\phi_2(4, 4)(G)$	{}	0		
$\phi_2(4, 5)(G)$	{}	0		
$\phi_2(4, 6)(G)$	{}	0		
$\phi_2(5, 5)(G)$	{}			
$\phi_2(5, 6)(G)$	{}	0		
$\phi_2(6, 6)(G)$	{}	0		

Table 6.11: 2^{nd} degree and its cardinality of the graphs

Therefore $\phi_2(G, x, y) = \sum_{\alpha_k \leq 1 \leq 6 \leq \beta_k} \phi_2(i, j)(G)x^i y^j = xy + 2xy^2 + xy^3 + x^2y^3 + x^2y^4 + 2x^2y^5 + 4x^3y^3 + x^3y^4 + x^3y^5 + 2x^3y^6$

6.13 ϕ_k -polynomial properties on sagitarius

Sagitarius is connected graph with 16 vertices given in the Figure. 6.12. The k^{th} degree of each vertices in the graph is given in the table 6.12

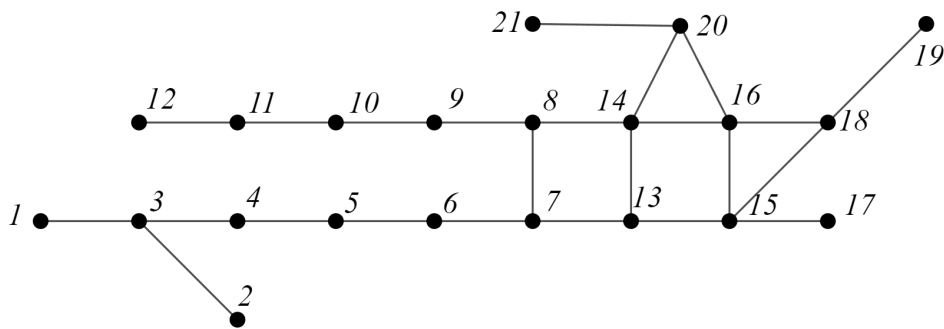


Figure 6.12: *Sagitarius*

6.13. ϕ_k -polynomial properties on sagitarius

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	{2, 4}	2		
$\lambda_2(2)$	{1, 4}	2		
$\lambda_2(3)$	{5}	1		
$\lambda_2(4)$	{1, 2, 6}	3		
$\lambda_2(5)$	{3, 7}	2		
$\lambda_2(6)$	{4, 8, 13}	3		
$\lambda_2(7)$	{5, 9, 14, 15}	4		
$\lambda_2(8)$	{10, 6, 20, 13, 16}	5		
$\lambda_2(9)$	{11, 7, 14}	3		
$\lambda_2(10)$	{12, 8}	2		
$\lambda_2(11)$	{9}	1	1	6
$\lambda_2(12)$	{10}	1		
$\lambda_2(13)$	{6, 8, 20, 16, 18, 17}	6		
$\lambda_2(14)$	{9, 7, 21, 18, 15}	5		
$\lambda_2(15)$	{7, 14, 20, 19}	4		
$\lambda_2(16)$	{8, 21, 19, 17, 13}	5		
$\lambda_2(17)$	{18, 16, 13}	3		
$\lambda_2(18)$	{17, 20, 13, 14}	4		
$\lambda_2(19)$	{15, 16}	2		
$\lambda_2(20)$	{8, 18, 13, 15}	4		
$\lambda_2(21)$	{14, 16}	2		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	{11 - 12}	1		
$\phi_2(1, 2)(G)$	{2 - 1, 3 - 2, 11 - 10}	3		
$\phi_2(1, 3)(G)$	{3 - 4}	1		
$\phi_2(1, 4)(G)$	{}	0		
$\phi_2(1, 5)(G)$	{}	0		
$\phi_2(1, 6)(G)$	{}	0		
$\phi_2(2, 2)(G)$	{}	0		
$\phi_2(2, 3)(G)$	{5 - 4, 5 - 6, 10 - 9}	3		
$\phi_2(2, 4)(G)$	{19 - 18, 21 - 20}	2		
$\phi_2(2, 5)(G)$	{}	0		
$\phi_2(2, 6)(G)$	{}	0		
$\phi_2(3, 3)(G)$	{}	0		
$\phi_2(3, 4)(G)$	{6 - 7, 17 - 15}	2		
$\phi_2(3, 5)(G)$	{9 - 8}	1		
$\phi_2(3, 6)(G)$	{}	0		
$\phi_2(4, 4)(G)$	{15 - 18}	1		
$\phi_2(4, 5)(G)$	{7 - 8, 15 - 16, 18 - 16, 20 - 14, 20 - 16}	5		
$\phi_2(4, 6)(G)$	{7 - 13, 15 - 13}	2		
$\phi_2(5, 5)(G)$	{8 - 14, 14 - 16}	2		
$\phi_2(5, 6)(G)$	{14 - 13}	1		
$\phi_2(6, 6)(G)$	{}	0		

Table 6.12: 2^{nd} degree and its cardinality of the graphs

Therefore $\phi_2(G, x, y) = \sum_{\alpha_k \leq 1 \leq 6 \leq \beta_k} \phi_2(i, j)(G)x^i y^j = xy + 3xy^2 + xy^3 + 3x^2y^3 + 2x^2y^4 + 2x^3y^4 + x^3y^5 + x^4y^4 + 5x^4y^5 + 2x^4y^6 + 2x^5y^5 + x^5y^6$

Conclusion

The study of the 12 zodiac signs through graphic and ϕ_k -polynomial forms has provided a unique perspective on representing astrological characteristics in a mathematical framework. This approach has demonstrated how abstract concepts like zodiac signs can be translated into structured forms, allowing for quantitative analysis and visualization. The graphical representation offers natural understanding of the relationships between zodiac signs, while the ϕ_k -polynomial forms provide an algebraic insight into their interdependencies. By merging these two methodologies, the study bridges the gap between astrology and mathematical modeling, nurturing interdisciplinary exploration. This research contributes to the broader effort of using mathematical tools to analyze traditional and cultural knowledge systems. The integration of graphic and ϕ_k -polynomials extends the scope of analysis to applications in data visualization, pattern recognition, and symbolic computation. Additionally, the approach can inspire new ways of categorizing and interpreting non-linear relationships in various domains, including social sciences, psychology, and network theory. Future studies could expand upon the current work in several directions:

- Extending the analysis to include other astrological systems, such as Vedic or Chinese astrology, and comparing their graphical and polynomial repre-

sentations.

- Investigating the use of machine learning techniques to analyze ϕ_k -polynomial patterns and predict astrological influences.
- Exploring real-world applications, such as using the mathematical models for personality analysis or compatibility studies based on zodiac signs.
- Conducting a statistical evaluation of the correlation between the mathematical representations and real data from psychological or sociological studies.

Chapter 7

A study on ϕ_k -polynomial and
resolvability of small finite commutative
rings of order at most 10

7.1 Introduction

The concept of zero-divisor graph of a commutative ring was introduced by I. Beck in 1988 [24]. He let all elements of the ring be vertices of the graph and was interested mainly in colorings. In [25], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors. This graph turns out to best exhibit the properties of the set of zero-divisors of a commutative ring. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. The concept metric dimension of connected graphs and its related properties are first introduced by by PJ

Slater [11] in 1975, independently by Harary and Melter [10] in 1976. A subset of vertices S resolves a graph G if every vertex of G is uniquely determined by its vector of distances to the vertices in S . A resolving set of minimum cardinality for a graph G is called a minimum resolving set. A minimum resolving set is usually called a bases for G and the cardinality of bases is called the metric dimension of G , denoted by $dim(G)$. Let $G(V, E)$ be a graph. For a vertex $v \in V$, $N(v)$ denotes the set of all vertices of G which are adjacent to v and $N[v] = N(v) \cup v$. A subset S of V is called a neighborhood set or n -set of G if $G = \cup_{v \in S} \langle N[v] \rangle$, where for a subset S of V . An nr -set S is called minimal if no proper subset of S is an nr -set. The minimum cardinality of a minimal n -set is called the neighbourhood number of G and is denoted by $ln(G)$. The concept of neighborhood number for a graph was first introduced by E. Sampathkumar[19]. A subset S of V is called a resolving set or an r -set of G if for each pair $u, v \in V - S, u \neq v$, there is a vertex w in S such that $d(u, w) \neq d(v, w)$. The minimum cardinality of a minimal r -set is called the resolving number of G and is denoted by $lr(G)$. A subset S of V is called a neighborhood resolving set or nr -set of G if S is both a neighborhood and a resolving set of G . The minimum cardinality of a minimal nr -set is called the neighborhood resolving number of G and is denoted by $lnr(G)$.

ϕ_k -Polynomial and resolvability of zero divisor graphs for commutative rings of order at most 10

Example 7.1.1. Consider the ring $R = Z_6 = \{0, 1, 2, 3, 4, 5\}$

$Z^*(R) = \{2, 3, 4\}$ and corresponding zero divisor graph $G = K_{1,2}$ is given Figure.

7a

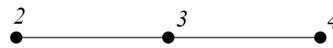


Figure. 7a
 $K_{1,2}$

7.2 Zero divisor graphs with 3 vertices and ϕ_2 -polynomials

The following table gives the list of zero divisor graphs with vertices 3 and the metric dimension and the least cardinality of neighbourhood resolving set for the zero divisor graphs

7.2. Zero divisor graphs with 3 vertices and ϕ_2 - polynomials

No. Vertices	Ring	$ R $	Graph G	$\beta(G)$	$lnr(G)$
3	Z_6	6	$K_{1,2}$	1	2
	Z_8	8			
	$\frac{Z_2[x]}{(x^3)}$	8			
	$Z_4[x]$	8			
	$\frac{Z_4[x]}{(2x, x^2 - 2)}$	8			
3	$\frac{Z_2[x, y]}{(x, y)^2}$	8	K_3	2	0
	$Z_4[x]$	8			
	$\frac{(2, x)^2}{F_4[x]}$	16			
	$\frac{(x^2)}{Z_4[x]}$	16			
	$Z_4[x]$	16			
	$\frac{Z_4[x]}{(x^2 + x + 1)}$	16			

Table 7.1: Resolvability of the graphs with vertices 3

Proof. Case. 1: The zero divisor graphs $K_{1,2}$ is the special cases of bipartite graph $K_{m,n}$, with $m = 1, n = 2$.

Therefore $\beta(K_{1,2}) = 2, lnr(K_{1,2}) = 2$

Case. 2: The zero divisor graphs K_3 is the complete graph K_n with $n = 3$

Therefore $\beta(K_3) = 2$ and $lnr(K_3) = 0$

Case. 3: The $\phi_2 - Polynomials$ of these graphs $K_{1,2}$ and K_3 are in the table

Graph	$\alpha_2(G)$	$\beta_2(G)$	$\phi_2(G, x, y)$
$K_{1,2}$	0	1	$2y$
K_3	0	0	3

Table 7.2: $\phi_2 - Polynomials$ of these graphs $K_{1,2}$ and K_3

□

7.3 Zero divisor graphs with 4 vertices and ϕ_2 -polynomials

The following table gives the list of zero divisor graphs with vertices 4 and the metric dimension($\beta(G)$) and the least cardinality of neighbourhood resolving set ($lnr(G)$)for the zero divisor graphs

No. Vertices	Ring	$ R $	Zero divisor graph	$\beta(G)$	$lnr(G)$
4	$Z_2 \times Z_4$	8	$K_{1,3}$	2	3
	$Z_3 \times Z_3$	9	$K_{2,2}$	2	3
	Z_{25}	25	K_4	3	0
	$\frac{Z_5[x]}{(x^2)}$	25	K_4	3	0

Table 7.3: Resolvability of the graphs with vertices 4

Proof. Case. 1: The zero divisor graphs $K_{1,3}$ and $K_{2,2}$ are the special cases of bipartite graph $K_{m,n}$, with $m = 1, n = 3$ and $m = n = 2$.

Therefore $\beta(K_{1,3}) = 2, lnr(K_{1,3}) = 3$

Case.2: The zero divisor graphs K_4 is the complete graph K_n with $n = 4$

Therefore $\beta(K_4) = 3$ and $lnr(K_4) = 0$

Case.3: The $\phi_2 - Polynomials$ of these graphs $K_{1,3}, K_{2,2}$ and K_4 are given in the table

Graph	$\alpha_2(G)$	$\beta_2(G)$	$\phi_2(G, x, y)$
$K_{1,3}$	0	2	$3y^2$
$K_{2,2}$	1	1	$4xy$
K_4	0	0	6

Table 7.4: The $\phi_2 - Polynomials$ of these graphs $K_{1,3}, K_{2,2}$ and K_4

□

7.4 Zero divisor graphs with 5 vertices and ϕ_2 -polynomials

The given table represents the list of zero divisor graphs with $\beta(G)$ and $lnr(G)$

No. Vertices	Ring	$ R $	Zero divisor graph	$\beta(G)$	$lnr(G)$
5	$Z_2 \times Z_5$	10	$K_{1,4}$	3	4
	$Z_3 \times F_4$	12	$K_{2,3}$	3	3
	$Z_2 \times Z_4$	8	Fig. 7.1	2	3
	$Z_2 \times \frac{Z_2[x]}{x^2}$	8	Fig. 7.1	2	3

Table 7.5: Resolvability of zero divisor graphs with 5 vertices

Proof. Case. 1: The zero divisor Graph $K_{1,4}$ and $K_{2,3}$ represents the special cases of the bipartite graph $K_{m,n}$ with $m = 1, n = 4$ and $m = 2, n = 3$ respectively. Here $\beta(K_{1,4}) = 3, lnr(K_{1,4}) = 4, \beta(K_{2,3}) = 3, lnr(K_{2,3}) = 3$

Case. 2: The zero divisor graph for the rings $Z_2 \times Z_4$ and $Z_2 \times \frac{Z_2[x]}{x^2}$ is in Figure. 7.1 From this Figure three vertices have the degree one and the remaining two vertices have degree more than one. The set $\{1, 4\}$ is the resolving set for the graph. Therefore $\beta(G) = 2$. Also set $\{2, 3, 4\}$ act as a neighbourhood resolving set for this graph. Therefore $lnr(G) = 3$

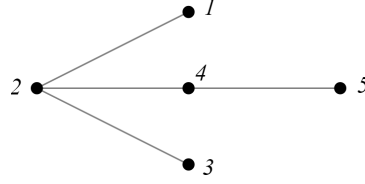


Figure 7.1: Graph of order 5

Case .3: The $\phi_2 - \text{Polynomials}$ of these graphs $K_{1,4}$, $K_{2,3}$ are given in the table

Graph	$\alpha_2(G)$	$\beta_2(G)$	$\phi_2(G, x, y)$
$K_{1,4}$	0	3	$4y^3$
$K_{2,3}$	1	2	$6xy^2$

Table 7.6: $\phi_2 - \text{Polynomials}$ of these graphs $K_{1,4}$, $K_{2,3}$

Case. 4 $\phi_2 - \text{Polynomials}$ of Figure. 7.1 The 2^{nd} degree and its cardinality of the graph in the Figure. 7.1 is given the table by

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	{3, 4}	2		
$\lambda_2(2)$	{5}	1		
$\lambda_2(3)$	{1, 4}	2	1	2
$\lambda_2(4)$	{1, 3}	2		
$\lambda_2(5)$	{2}	1		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(1, 1)(G)$	{}	0		
$\phi_2(1, 2)(G)$	{2 - 3, 2 - 1, 2 - 4, 5 - 4}	4		
$\phi_2(2, 2)(G)$	{}	0		

Table 7.7: 2^{nd} degree and its cardinality of the graphs

$$\text{Therefore } \phi_2(G, x, y) = \sum_{\alpha_2 \leq 1 \leq 2 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 4xy^2$$

□

7.5 Zero divisor graphs with 6 vertices and ϕ_2 -polynomials

The given table represents the list of zero divisor graphs with $\beta(G)$ and $lnr(G)$

No. Vertices	Ring	$ R $	Zero divisor graph	$\beta(G)$	$lnr(G)$
6	$Z_3 \times Z_5$	15	$K_{2,4}$	4	4
	$F_4 \times F_4$	16	$K_{3,3}$	4	4
	$Z_2 \times Z_2 \times Z_2$	8	Fig. 7.2	2	4
	$\frac{Z_7[x]}{x^2}$	49	K_6	5	0
	Z_{49}	49	K_6	5	0

Table 7.8: Resolvability of zero divisor graphs with 6 vertices

Proof. Case. 1: The zero divisor graphs $K_{2,4}$ and $K_{3,3}$ are the special cases of bipartite graph $K_{m,n}$.

Therefore $\beta(K_{2,4}) = 4$, $lnr(K_{2,4}) = 4$ and $\beta(K_{3,3}) = 4$, $lnr(K_{3,3}) = 4$. Also $lnr(K_6) = 0$

Case. 2: The zero divisor graph of the ring $Z_2 \times Z_2 \times Z_2$ is shown in Figure 7.2. From the figure, there are 3 vertices with degree 1, and the remaining 3 vertices have degree 3. Therefore, the set $\{2, 3\}$ is a resolving set, and the metric dimension of the graph is 2. Moreover, the set $\{2, 4, 5, 6\}$ is a least neighbourhood resolving set for the graph shown in Figure 7.2.

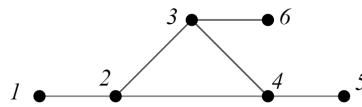


Figure 7.2: Graph of order 6

7.5. Zero divisor graphs with 6 vertices and ϕ_2 - polynomials

Case. 3: The $\phi_2 - Polynomials$ of $K_{2,4}$, $K_{3,3}$, K_6 graphs are in the table below

Graph	$\alpha_2(G)$	$\beta_2(G)$	$\phi_2(G, x, y)$
$K_{2,4}$	1	3	$8xy^3$
$K_{3,3}$	2	2	$9x^2y^2$
K_6	0	0	15

Table 7.9: $\phi_2 - Polynomials$ of $K_{2,4}$, $K_{3,3}$, K_6

Case. 4: The 2^{nd} degree and its cardinality of the graph in the Figure. 7.2 is given the table below

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	{3, 5}	2	2	2
$\lambda_2(2)$	{4, 6}	2		
$\lambda_2(3)$	{1, 6}	2		
$\lambda_2(4)$	{2, 5}	2		
$\lambda_2(5)$	{1, 4}	2		
$\lambda_2(6)$	{2, 3}	2		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(2, 2)(G)$	{1 - 2, 2 - 3, 2 - 5} {3 - 4, 3 - 5, 5 - 6}	6		

Table 7.10: 2^{nd} degree and its cardinality of graph

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 2 \leq 2 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 6x^2 y^2$

□

7.6 Zero divisor graphs with 7 vertices and ϕ_2 -polynomials

The zero divisor graphs with 7 vertices are $K_{1,6}$, $K_{3,4}$, Figure. 7.3, Figure. 7.4, Figure. 7.5, Figure. 7.6, Figure. 7.7 The given table represents the list of Zero divisor Graphs with $\beta(G)$ and $lnr(G)$

No. Vertices	Ring	$ R $	Zero divisor graph	$\beta(G)$	$lnr(G)$
7	$Z_2 \times Z_7$	14	$K_{1,6}$	5	6
	$F_4 \times Z_5$	20	$K_{3,4}$	5	5
	$Z_3 \times Z_4$	12	Fig. 7.3	4	5
	$Z_3 \times \frac{Z_2[x]}{x^2}$	12	Fig. 7.3	4	5
	Z_{16}	16	Fig. 7.4	4	0
	$\frac{Z_2[x]}{x^4}$	16	Fig. 7.4	4	0
	$\frac{Z_4[x]}{x^4+2}$	16	Fig. 7.4	4	0
	$\frac{Z_4[x]}{x^2+3x}$	16	Fig. 7.4	4	0
	$\frac{Z_4[x]}{(x^3-2, 2x^2, 2x)}$	16	Fig. 7.4	4	0
	$\frac{Z_2[x,y]}{(x^3, xy, y^2)}$	16	Fig. 7.5	5	0
	$\frac{Z_8[x]}{(2x, x^2)}$	16	Fig. 7.5	5	0
	$\frac{Z_4[x]}{(x^3, 2x^2, 2x)}$	16	Fig. 7.5	5	0
	$\frac{Z_4[x]}{(x^2+2x)}$	16	Fig. 7.6	3	0
	$\frac{Z_8[x]}{(2x, x^2+4)}$	16	Fig. 7.6	3	0
	$\frac{Z_2[x,y]}{(x^2, y^2-xy)}$	16	Fig. 7.6	3	0
	$\frac{Z_4[x,y]}{(x^2, y^2-xy, xy-2, 2x, 2y)}$	16	Fig. 7.6	3	0
	$\frac{Z_4[x,y]}{(x^2, y^2, xy-2, 2x, 2y)}$	16	Fig. 7.7	3	0
	$\frac{Z_2[x,y]}{(x^2, y^2)}$	16	Fig. 7.7	3	0
	$\frac{Z_4[x]}{(x^2)}$	16	Fig. 7.7	3	0
	$\frac{Z_4[x]}{(x^3-x^2-2, 2x^2, 2x)}$	16	Fig. 7.8	4	5
	$\frac{Z_2[x,y,z]}{(x,y,z)^2}$	16	K_7	6	0
	$\frac{Z_4[x,y]}{(x^2, y^2, 2x, 2y)}$	16	K_7	6	0
	$\frac{F_8[x]}{(x^2)}$	64	K_7	6	0
	$\frac{Z_4[x]}{(x^3+x+1)}$	64	K_7	6	0

Table 7.11: Resolvability of the graphs with 7 vertices

7.6. Zero divisor graphs with 7 vertices and ϕ_2 -polynomials

Proof. Case. 1: The zero divisor graph $K_{1,6}$ and $K_{3,4}$ are the special case of $K_{m,n}$ with $m = 1, n = 6$ and $m = 3, n = 4$. Therefore $\beta(K_{1,6}) = 5, \text{lnr}(K_{1,6}) = 6$.
 $\beta(K_{3,4}) = 5, \text{lnr}(K_{3,4}) = 5$

The ϕ_2 - *Polynomials* of $K_{1,6}, K_{3,4}$, graphs are given in the table by

Graph	$\alpha_2(G)$	$\beta_2(G)$	$\phi_2(G, x, y)$
$K_{1,6}$	0	5	$6y^5$
$K_{3,4}$	2	3	$12x^2y^3$

Table 7.12: ϕ_2 - *Polynomials* of $K_{1,6}, K_{3,4}$

Case. 2: The zero divisor graphs of the rings $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$ are shown in Figure 7.3. From this figure, we can determine the metric dimension and the neighbourhood resolving set for each graph.

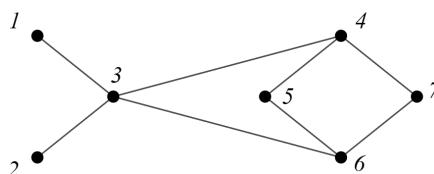


Figure 7.3: Graph of order 7

Therefore, the metric dimension and neighbourhood resolvability of the graph in Figure 7.3 are $\beta(G) = 4$ and $\text{lnr}(G) = 5$, respectively. The second degree of each vertex and its cardinality for the graph in Figure 7.3 are given in the corresponding table.

7.6. Zero divisor graphs with 7 vertices and ϕ_2 - polynomials

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{2, 4, 6\}$	3	2	3
$\lambda_2(2)$	$\{1, 4, 6\}$	3		
$\lambda_2(3)$	$\{5, 7\}$	2		
$\lambda_2(4)$	$\{1, 2, 6\}$	3		
$\lambda_2(5)$	$\{3, 7\}$	2		
$\lambda_2(6)$	$\{1, 2, 4\}$	3		
$\lambda_2(7)$	$\{5, 3\}$	2		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(2, 2)(G)$	$\{\}$	0		
$\phi_2(2, 3)(G)$	$\{3 - 1, 3 - 2, 3 - 4, 3 - 6\}$	8		
	$\{5 - 4, 5 - 6, 7 - 4, 7 - 6\}$			

Table 7.13: 2^{nd} degree and its cardinality of graph

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 2 \leq 3 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 8x^2 y^3$

Case. 3: The zero divisor graphs of the rings \mathbb{Z}_{16} , $\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_4[x]/(x^4 + 2)$, $\mathbb{Z}_4[x]/(x^2 + 3x)$, and $\mathbb{Z}_4[x]/(x^3 - 2, 2x^2, 2x)$ are shown in Figure 7.4. From this figure, we can determine the metric dimension and the neighbourhood resolving set for each graph.

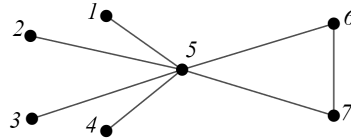


Figure 7.4: Graph of order 7

Here $\beta(G) = 4$, $lnr(G) = 0$. The 2^{nd} degree of the graph in the Figure. 7.4 is given the table by

7.6. Zero divisor graphs with 7 vertices and ϕ_2 - polynomials

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{6, 7, 2, 3, 4, \}$	5		
$\lambda_2(2)$	$\{1, 3, 4, 6, 7\}$	5		
$\lambda_2(3)$	$\{1, 2, 4, 6, 7\}$	5		
$\lambda_2(4)$	$\{1, 2, 3, 6, 7\}$	5	0	5
$\lambda_2(5)$	$\{\}$	0		
$\lambda_2(6)$	$\{1, 2, 3, 4\}$	4		
$\lambda_2(7)$	$\{1, 2, 3, 4\}$	4		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(0, 4)(G)$	$\{5 - 6, 5 - 7\}$	2		
$\phi_2(0, 5)(G)$	$\{5 - 1, 5 - 2, 5 - 3, 5 - 4\}$	4		
$\phi_2(4, 4)(G)$	$\{6 - 7\}$	1		

Table 7.14: 2^{nd} degree and its cardinality of graph

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 0 \leq 5 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 2y^4 + 4y^5 + x^4 y^4$.

Case.4: The zero divisor graph for the rings $Z_2[x, y]/(x^3, xy, y^2)$, $Z_8[x]/(2x, x^2)$, $Z_4[x]/(x^3, 2x^2, 2x)$ are in the Figure. 7.5. The metric dimension and least neighbourhood resolving set for the graph is given by

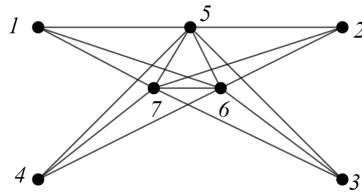


Figure 7.5: Graph of order 7

Here $\beta(G) = 5$, $lnr(G) = 0$. The 2^{nd} degree of the graph in the Figure. 5 is given the table below

7.6. Zero divisor graphs with 7 vertices and ϕ_2 - polynomials

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{2, 3, 4, \}$	3		
$\lambda_2(2)$	$\{1, 3, 4\}$	3		
$\lambda_2(3)$	$\{1, 2, 4\}$	3		
$\lambda_2(4)$	$\{1, 2, 3\}$	3	0	3
$\lambda_2(5)$	$\{\}$	0		
$\lambda_2(6)$	$\{\}$	0		
$\lambda_2(7)$	$\{\}$	0		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(0, 0)(G)$	$\{5 - 6, 5 - 7, 6 - 7\}$	3		
$\phi_2(0, 5)(G)$	$\{5 - 1, 5 - 2, 5 - 3, 5 - 4\}$ $\{6 - 1, 6 - 2, 6 - 3, 6 - 4\}$ $\{7 - 1, 7 - 2, 7 - 3, 7 - 4\}$	12		

Table 7.15: 2^{nd} degree and its cardinality of graph

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 0 \leq 3 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 3 + 12y^5$.

Case. 5: The zero divisor graph for the rings $Z_4[x]/(x^2 + 2x)$, $Z_8[x]/(2x, x^2 + 4)$, $Z_2[x, y]/(x^2, y^2 - xy)$, $Z_4[x, y]/(x^2, y^2 - xy, xy - 2, 2x, 2y)$ are in the Figure. 7.6. The metric dimension and least cardinality of neighbourhood resolving sets are determined from the figure for the above graphs.

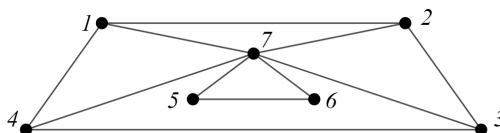


Figure 7.6: Graph of order 7

Here $\beta(G) = 3$, $l nr(G) = 0$. The 2^{nd} degree of the graph in the Figure. 7.6 is given the table by

7.6. Zero divisor graphs with 7 vertices and ϕ_2 - polynomials

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{3, 5, 6\}$	3	0	4
$\lambda_2(2)$	$\{4, 5, 6\}$	3		
$\lambda_2(3)$	$\{1, 5, 6\}$	3		
$\lambda_2(4)$	$\{2, 5, 6\}$	3		
$\lambda_2(5)$	$\{1, 2, 3, 4\}$	4		
$\lambda_2(6)$	$\{1, 2, 3, 4\}$	4		
$\lambda_2(7)$	$\{\}$	0		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(0, 3)(G)$	$\{7 - 1, 7 - 2, 7 - 3, 7 - 4\}$	4		
$\phi_2(0, 4)(G)$	$\{7 - 5, 7 - 6\}$	2		
$\phi_2(3, 3)(G)$	$\{1 - 2, 1 - 4, 2 - 3, 3 - 4\}$	4		

Table 7.16: 2^{nd} degree and its cardinality of graph

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 0 \leq 4 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 4y^3 + 2y^4 + 4x^3 y^3 + x^4 y^4$.

Case. 6: The zero divisor graph for the rings $Z_4[x, y]/(x^2, y^2, xy - 2, 2x, 2y)$, $Z_2[x, y]/(x^2, y^2)$, $Z_4[x]/(x^2)$ are in the Figure. 7.7. The metric dimension and least cardinality of neighbourhood resolving set of above graphs are given below.

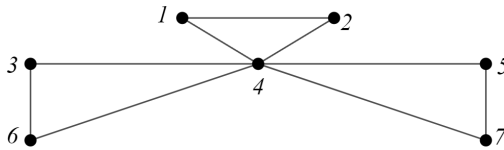


Figure 7.7: Graph of order 7

Here $\beta(G) = 3$, $lnr(G) = 0$. The 2^{nd} degree of the graph in the Figure. 7.7 is given the table by

7.6. Zero divisor graphs with 7 vertices and ϕ_2 - polynomials

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{3, 5, 6, 7\}$	4	0	4
$\lambda_2(2)$	$\{3, 5, 6, 7\}$	4		
$\lambda_2(3)$	$\{1, 2, 5, 7\}$	4		
$\lambda_2(4)$	$\{\}$	0		
$\lambda_2(5)$	$\{1, 2, 3, 6\}$	4		
$\lambda_2(6)$	$\{1, 2, 5, 7\}$	4		
$\lambda_2(7)$	$\{1, 2, 3, 6\}$	4		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(0, 4)(G)$	$\{4 - 1, 4 - 2, 4 - 3, 4 - 5, 4 - 6, 4 - 7\}$	6		
$\phi_2(4, 4)(G)$	$\{1 - 2, 3 - 6, 5 - 7\}$	3		

Table 7.17: 2^{nd} degree and its cardinality of graph

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 0 \leq 4 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 6 + 3x^4 y^4$

Case. 7: The zero divisor graph for the ring $Z_4[x]/(x^3 - x^2 - 2, 2x^2, 2x)$ is in the Figure. 7.8. Also determined the metric dimension and least cardinality of this graph.

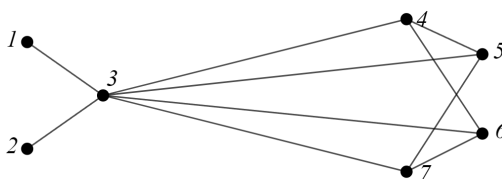


Figure 7.8: Graph of order 7

Here $\beta(G) = 4$, $l nr(G) = 5$.

The 2^{nd} degree of the graph in the Figure. 7.8 is given the table by

7.7. Zero divisor graphs with 8 vertices and ϕ_2 -polynomials

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{4, 5, 6, 7\}$	4		
$\lambda_2(2)$	$\{4, 5, 6, 7\}$	4		
$\lambda_2(3)$	$\{\}$	0		
$\lambda_2(4)$	$\{1, 2, 7\}$	3	0	4
$\lambda_2(5)$	$\{1, 2, 6\}$	3		
$\lambda_2(6)$	$\{1, 2, 5\}$	3		
$\lambda_2(7)$	$\{1, 2, 4\}$	3		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(0, 3)(G)$	$\{3 - 4, 3 - 5, 3 - 6, 3 - 7\}$	4		
$\phi_2(0, 4)(G)$	$\{3 - 1, 3 - 2\}$	2		
$\phi_2(3, 3)(G)$	$\{4 - 5, 4 - 6, 5 - 7, 6 - 7\}$	4		

Table 7.18: 2^{nd} degree and its cardinality of graph

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 0 \leq 4 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 4y^3 + 2y^4 + 4x^3 y^3$.

Case. 8: The zero divisor graph of K_7 is the case of K_n with $n = 7$, $\beta(K_7) = 6$, $lnr(K_7) = 0$. Therefore $\phi_2(G, x, y) = 21$ □

7.7 Zero divisor graphs with 8 vertices and ϕ_2 -polynomials

The zero divisor graphs with 8 vertices include $K_{1,7}$, $K_{2,6}$, $K_{4,4}$, K_8 , as well as the graph shown in Figure 7.9. The table below lists these zero divisor graphs along with their metric dimension, $\beta(G)$, and least neighbourhood resolving set cardinality, $lnr(G)$.

7.7. Zero divisor graphs with 8 vertices and ϕ_2 -polynomials

No. Vertices	Ring	$ R $	Zero divisor graph	$\beta(G)$	$lnr(G)$
8	$Z_2 \times F_8$	16	$K_{1,7}$	6	7
	$Z_3 \times Z_7$	21	$K_{2,6}$	6	6
	$Z_5 \times Z_5$	25	$K_{4,4}$	6	6
	Z_{27}	27	Fig. 7.9	6	0
	$Z_9[x]/(3x, x^2 - 3)$	27	Fig. 7.9	6	0
	$Z_9[x]/(3x, x^2 - 6)$	27	Fig. 7.9	6	0
	$Z_3[x]/(x^3)$	27	Fig. 7.9	6	0
	$Z_3[x, y]/(x, y)^2$	27	K_8	7	0
	$Z_9[x]/(3, x)^2$	27	K_8	7	0
	$F_9[x]/(x^2)$	81	K_8	7	0
	$Z_9[x]/(x^2 + 1)$	81	K_8	7	0

Table 7.19: Resolvability of the zero divisor graphs

Proof. Case. 1: The Zero divisor graph $K_{1,7}$, $K_{2,6}$ and $K_{4,4}$ are the special case of $K_{m,n}$ with $m = 1, n = 7$, $m = 2, n = 6$ and $m = 4, n = 4$. Therefore $\beta(K_{1,7}) = 6$, $lnr(K_{1,7}) = 7$.

$$\beta(K_{2,6}) = 6, lnr(K_{2,6}) = 6, \beta(K_{4,4}) = 6, lnr(K_{4,4}) = 6.$$

The ϕ_2 - *Polynomials* of $K_{1,7}$, $K_{2,6}$, $K_{4,4}$, K_8 graphs are given in the table by

Graph	$\alpha_2(G)$	$\beta_2(G)$	$\phi_2(G, x, y)$
$K_{1,7}$	0	6	$7y^6$
$K_{2,6}$	1	5	$12xy^5$
$K_{4,4}$	3	3	$16x^3y^3$
K_8	0	0	28

Table 7.20: ϕ_2 - *Polynomials* of $K_{1,7}$, $K_{2,6}$, $K_{4,4}$, K_8

Case. 2: The zero divisor graphs of the rings Z_{27} , $Z_9[x]/(3x, x^2 - 3)$, $Z_9[x]/(3x, x^2 - 6)$, and $Z_3[x]/(x^3)$ are shown in Figure 7.9. From this figure,

7.7. Zero divisor graphs with 8 vertices and ϕ_2 - polynomials

there are 2 vertices with degree 7 and the remaining 6 vertices have degree 2. The resolving set contains 5 vertices of degree 2 and one vertex of degree 7, giving a least cardinality of the resolving set equal to 6. Furthermore, there are two vertices whose neighbourhoods contain all vertices of the graph; therefore, the neighbourhood resolving set of this graph has cardinality zero.

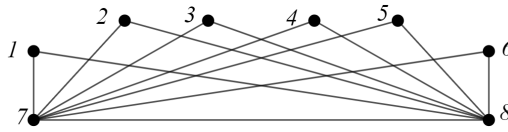


Figure 7.9: Graph of order 8

Here $\beta(G) = 6$, $lnr(G) = 0$ The 2^{nd} degree of the graph in the Figure. 7.9 is given the table by

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{2, 3, 4, 5, 6\}$	5	2	3
$\lambda_2(2)$	$\{1, 3, 4, 5, 6\}$	5		
$\lambda_2(3)$	$\{1, 2, 4, 5, 6\}$	5		
$\lambda_2(4)$	$\{1, 2, 3, 5, 6\}$	5		
$\lambda_2(5)$	$\{1, 2, 3, 4, 6\}$	5		
$\lambda_2(6)$	$\{1, 2, 3, 4, 5\}$	5		
$\lambda_2(7)$	$\{\}$	0		
$\lambda_2(8)$	$\{\}$	0		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(0, 0)(G)$	$\{7 - 8\}$	1		
$\phi_2(0, 5)(G)$	$\{7 - 1, 7 - 2, 7 - 3, 7 - 4\}$	12		
	$\{7 - 5, 7 - 6, 8 - 1, 8 - 2\}$			
	$\{8 - 3, 8 - 4, 8 - 5, 8 - 6\}$			

Table 7.21: 2^{nd} degree and its cardinality of graph

$$\text{Therefore } \phi_2(G, x, y) = \sum_{\alpha_2 \leq 0 \leq 5 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 1 + 12y^5$$

Case. 3: The zero divisor graph of K_8 is the case of K_n with $n = 8$, so the $\beta(K_n) = n - 1$, $lnr(K_n) = 0$
 Therefore $\beta(K_8) = 7$, $lnr(K_8) = 0$. □

7.8 Zero divisor graphs with 9 vertices and ϕ_2 -polynomials

The zero divisor graphs with 9 vertices include $K_{1,8}$, $K_{2,7}$, $K_{3,6}$, as well as the graphs shown in Figures 7.10 and 7.11. The table below lists these zero divisor graphs along with their metric dimension, $\beta(G)$, and least neighbourhood resolving set cardinality, $lnr(G)$.

No. Vertices	ring	$ R $	Zero divisor graph	$\beta(G)$	$lnr(G)$
9	$Z_2 \times F_9$	18	$K_{1,8}$	7	8
	$Z_3 \times F_8$	24	$K_{2,7}$	7	7
	$F_4 \times Z_7$	28	$K_{3,6}$	7	7
	$Z_2 \times Z_2 \times Z_3$	12	Fig. 7.10	3	5
	$Z_4 \times F_4$	16	Fig. 7.11	5	6
	$Z_2[x]/(x^2) \times F_4$	16	Fig. 7.11	5	6

Table 7.22: Resolvability of the zero divisor graph with 9 vertices

Proof. Case.1: The zero divisor graph $K_{1,8}$, $K_{2,7}$ and $K_{3,6}$ are the special case of $K_{m,n}$ with $m = 1, n = 8$, $m = 2, n = 7$ and $m = 3, n = 6$. Therefore $\beta(K_{1,8}) = 7$, $lnr(K_{1,8}) = 8$.

$$\beta(K_{2,7}) = 7, lnr(K_{2,7}) = 7, \beta(K_{3,6}) = 7, lnr(K_{3,6}) = 7.$$

The $\phi_2 - Polynomials$ of $K_{1,8}$, $K_{2,7}$, $K_{3,6}$ graphs are given in the table by

Graph	$\alpha_2(G)$	$\beta_2(G)$	$\phi_2(G, x, y)$
$K_{1,8}$	0	7	$8y^7$
$K_{2,7}$	1	5	$14xy^6$
$K_{3,6}$	3	3	$18x^2y^5$

Table 7.23: ϕ_2 – Polynomials of $K_{1,8}$, $K_{2,7}$, $K_{3,6}$

Case. 2: The zero divisor graph of the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ is shown in Figure 7.10. This graph has 9 vertices, among which four vertices have degree 1, another four vertices have degree 3, and the remaining vertex has degree 2. Therefore, the resolving set contains three vertices, and the least cardinality of the neighbourhood resolving set is 5. Hence, $\beta(G) = 3$ and $lnr(G) = 5$.

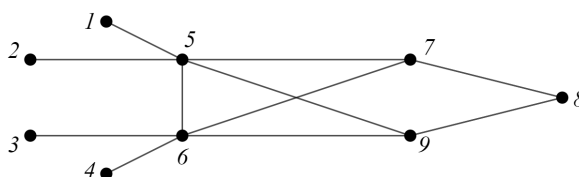


Figure 7.10: Graph of order 9

The 2^{nd} degree of the graph in the Figure. 7.10 is given the table below

7.8. Zero divisor graphs with 9 vertices and ϕ_2 - polynomials

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{2, 6, 7, 9\}$	4		
$\lambda_2(2)$	$\{1, 6, 7, 9\}$	4		
$\lambda_2(3)$	$\{4, 5, 7, 9\}$	4		
$\lambda_2(4)$	$\{3, 5, 7, 9\}$	4		
$\lambda_2(5)$	$\{4, 8, 3\}$	3	2	5
$\lambda_2(6)$	$\{1, 2, 8\}$	3		
$\lambda_2(7)$	$\{1, 2, 3, 4, 9\}$	5		
$\lambda_2(8)$	$\{5, 6\}$	2		
$\lambda_2(9)$	$\{1, 2, 3, 4, 7\}$	5		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(2, 5)(G)$	$\{8 - 7, 8 - 9\}$	2		
$\phi_2(3, 3)(G)$	$\{5 - 6\}$	1		
$\phi_2(3, 4)(G)$	$\{5 - 1, 5 - 2, 6 - 3, 6 - 4\}$	4		
$\phi_2(3, 5)(G)$	$\{5 - 7, 5 - 9, 6 - 7, 6 - 9\}$	4		

Table 7.24: 2^{nd} degree and its cardinality of graph

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 2 \leq 5 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 2x^2 y^5 + x^3 y^3 + 4x^3 y^4 + 4x^3 y^5$

Case. 3: The zero divisor graphs of the rings $\mathbb{Z}_4 \times F_4$ and $\mathbb{Z}_2[x]/(x^2) \times F_4$ are shown in Figure 7.11. The graph has 9 vertices, among which five vertices have degree 3, three vertices have degree 1, and the remaining vertex has degree 6. The resolving set contains five vertices, and the least cardinality of the neighbourhood resolving set is 6. Hence, $\beta(G) = 5$ and $lnr(G) = 6$.

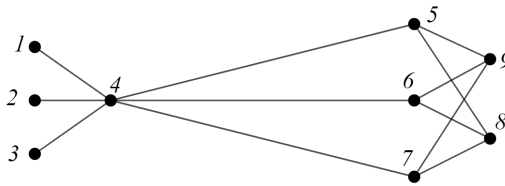


Figure 7.11: Graph of order 9

The 2^{nd} degree of the graph in the Figure. 7.11 is given the table by

$\lambda_2(x)$	$\{u : x - u\}$	$ \{u : x - u\} $	$\alpha_2(G)$	$\beta_2(G)$
$\lambda_2(1)$	$\{2, 3, 5, 6, 7\}$	5	2	5
$\lambda_2(2)$	$\{1, 3, 5, 6, 7\}$	5		
$\lambda_2(3)$	$\{1, 2, 5, 6, 7\}$	5		
$\lambda_2(4)$	$\{8, 9\}$	2		
$\lambda_2(5)$	$\{1, 2, 3, 6, 7\}$	5		
$\lambda_2(6)$	$\{1, 2, 3, 5, 7\}$	5		
$\lambda_2(7)$	$\{1, 2, 3, 6, 5\}$	5		
$\lambda_2(8)$	$\{4, 9\}$	2		
$\lambda_2(9)$	$\{4, 8\}$	2		
$\alpha_k \leq i \leq j \leq \beta_k : \phi_2(i, j)(G)$	$\{u - v\}$	$ \{u - v\} $		
$\phi_2(2, 5)(G)$	$\{4 - 1, 4 - 2, 4 - 3, 4 - 5\}$ $\{4 - 6, 4 - 7, 8 - 5, 8 - 6\}$ $\{8 - 7, 9 - 5, 9 - 6, 9 - 7\}$	12		

Table 7.25: 2^{nd} degree and its cardinality of graph

Therefore $\phi_2(G, x, y) = \sum_{\alpha_2 \leq 2 \leq 5 \leq \beta_2} \phi_2(i, j)(G)x^i y^j = 12x^2 y^5$

□

7.9 Zero divisor graphs with 10 vertices and ϕ_2 -polynomials

The zero divisor graphs with 10 vertices include $K_{2,8}$, $K_{3,7}$, $K_{4,6}$, and K_{10} . The table below lists these zero divisor graphs along with their metric dimension, $\beta(G)$, and least neighbourhood resolving set cardinality, $lnr(G)$.

7.9. Zero divisor graphs with 10 vertices and ϕ_2 - polynomials

No. Vertices	ring	$ R $	Zero divisor graph	$\beta(G)$	$lnr(G)$
10	$Z_3 \times F_9$	27	$K_{2,8}$	8	8
	$F_4 \times F_8$	32	$K_{3,7}$	8	8
	$Z_5 \times Z_7$	35	$K_{4,6}$	8	8
	Z_{121}	121	K_{10}	9	0
	$Z_{11}[x]/(x^2)$	121	K_{10}	9	0

Table 7.26: Resolvability of graphs with 10 vertices

Proof. Case. 1: The Zero divisor graph $K_{2,8}$, $K_{3,7}$ and $K_{4,6}$ are the special case of $K_{m,n}$. Therefore $\beta(K_{2,8}) = 8$, $lnr(K_{2,8}) = 8$. $\beta(K_{3,7}) = 8$, $lnr(K_{3,7}) = 8$, $\beta(K_{4,6}) = 8$, $lnr(K_{4,6}) = 8$.

Case.2: The zero divisor graph K_{10} is a case of K_n complete graph with $n = 10$, so the $\beta(K_n) = n - 1$, $lnr(K_n) = 0$

Therefore $\beta(K_{10}) = 9$, $lnr(K_{10}) = 0$. The $\phi_2 - Polynomials$ of $K_{2,8}$, $K_{3,7}$, $K_{4,6}$, K_{10} graphs are given in the table by

Graph	$\alpha_2(G)$	$\beta_2(G)$	$\phi_2(G, x, y)$
$K_{2,8}$	1	7	$16xy^7$
$K_{3,7}$	2	6	$21x^2y^6$
$K_{4,6}$	3	3	$24x^3y^5$
K_{10}	0	0	45

Table 7.27: $\phi_2 - Polynomials$ of $K_{2,8}$, $K_{3,7}$, $K_{4,6}$, K_{10}

□

Conclusion

In this study, we have explored the ϕ_k -polynomial, metric dimension, and neighborhood resolvability of small finite commutative rings of order at most 10. By analyzing the algebraic and combinatorial properties of these rings, we gained deeper insights into their structural characteristics and relationships. The following are the main conclusions and future directions from this work: This chapter provides the foundation for further explorations in the interaction between algebraic structures and combinatorial graph theory, opening avenues for interdisciplinary research with theoretical and applied significance.

- We systematically computed the ϕ_k -polynomial for small finite commutative rings and demonstrated its advantage in understanding the underlying algebraic structure.
- The metric dimension of graphs associated with these rings was determined, highlighting the importance of resolving sets in graph-theoretic contexts related to ring structures.
- Neighbourhood resolvability was investigated, providing a new perspective on the interplay between ring theory and graph theory.
- The results highlight the significance of small finite commutative rings as a rich domain for studying algebraic combinatorics.

Future scope

- Extending the study to finite commutative rings of larger orders, potentially revealing more elaborate relationships between algebraic properties and graph invariants.
- Investigating the applicability of ϕ_k -polynomials and metric dimension in coding theory, cryptography, and network science.
- Developing algorithms for efficiently computing the ϕ_k -polynomial and resolving sets for larger rings.
- Analyzing the impact of different graph representations of rings, such as zero-divisor graphs and unit graphs, on metric dimension and resolvability.
- Exploring connections with other combinatorial parameters, such as chromatic number, domination number, and independence number, in ring-related graphs.

Applications of neighbourhood resolving set

8.1 Introduction

Efficient distribution of medicines in a specific geographical region is crucial for ensuring timely delivery, especially during emergencies. The supply chain involves Emergency Drug Warehouses (EDWs), denoted as W_i , and Service Centers (SCs), denoted as S_r . To optimize drug transportation, it is essential to identify one or more nodal centers within the system. These nodal centers can be either EDWs, SCs, or a combination of both, ensuring effective management of logistics.

To model this problem, we represent the distribution network as a connected graph, where vertices correspond to EDWs and SCs, and edges represent the distances between them. We utilize the *Neighborhood Resolving Set* (NRS) to

select key vertices as nodal centers, ensuring an efficient drug distribution system. These nodal centers play a critical role during emergency situations or sudden disease outbreaks by minimizing delays in medicine delivery.

Beyond healthcare logistics, efficient labor management is a critical aspect of modern workplaces, particularly in construction projects. These projects require coordinated collaboration among workers from various trades, such as carpenters, plumbers, electricians, and masons. Effective task allocation and fair wage distribution directly impact project success. By applying graph-theoretical principles, specifically *Neighborhood Resolving Sets* (NRS), we model workforce interactions and optimize labor management.

In this context, the objective is to determine a subset of vertices that uniquely resolve the entire graph. The neighborhood resolving set [19] is a subset of vertices ensuring that every other vertex can be uniquely identified based on adjacency relationships. Identifying the minimum NRS [12] is a computationally challenging problem due to its combinatorial nature.

This paper explores the application of NRS to optimize construction labor management by formalizing the problem in graph-theoretic terms. We propose an approach to compute the minimum resolving set and validate its effectiveness through theoretical analysis and practical case studies. The concept of NRS [12] has significant implications not only in workforce management but also in broader optimization, logistics, and robotic navigation problems.

8.2 Selecting nodal centre for effective management of drug distribution

8.2.1 Problem statement

In a particular geographic region, there are multiple emergency drug warehouses (EDW) and service center(SC) distributed across various locations. The goal is to optimize the drugs transportation system between these individuals to ensure timely and efficient delivery of drug supplies.

8.2.2 Graph structure and design

Consider a set of medical stores $W = \{W_1, W_2, \dots, W_k\}$ represent emergency drug warehouses, and $S = \{S_1, S_2, \dots, S_r\}$ represent service center. which are represented as vertices in a graph. The distances between these entities are characterized as follows:

- d_{S_i} is the minimum distance from Service center S_i to its nearest Service center.
- $d_\alpha = \max\{d_{S_i}\}$ (maximum minimum distance for all Service center).
- d_{W_j} is the minimum distance from emergency drug warehouse W_j to its nearest Emergency drug warehouse.
- $d_\beta = \max\{d_{W_j}\}$ (maximum minimum distance for all Emergency drug warehouses).

Finally, the distance from a emergency drug warehouses W_j to its nearest Service center is denoted as $d_{W_j S_i}$, and the maximum of these distances is:

- $d_\gamma = \max\{d_{W_j S_i}\}$, where $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, r$.

The overall distance d is defined as:

$$d = \max(d_\alpha, d_\beta, d_\gamma).$$

An edge is created between two vertices (Emergency drug warehouses or Service center) if their distance is less than or equal to d . Thus, the resulting graph is a **connected graph**.

8.2.3 Graph features

- By the above design will get components(clusters)
- The graph is connected by design, meaning every service center and emergency drug warehouse is connected either directly or indirectly.
- Edges represent proximity, where two nodes are connected if their distance is less than d .
- The maximum distance d ensures the entire network is accessible, and no node (Service center/Emergency drug warehouse) is isolated.
- According to the geographical condition, if the graph is may be more than one component, then we apply Neighborhood Resolving Set on each components and find out the nodal centers.

8.2.4 Justification for graph connectivity

The graph is connected by construction because the edges between vertices are based on the distances between the emergency drug warehouses and service centers. Since we defined the maximum distance d as the largest of the individual distances $d_\alpha, d_\beta, d_\gamma$, all vertices that are sufficiently close to each other (within the distance d) are connected by edges. As a result, the entire set of emergency drug warehouses and service centers forms a connected network, ensuring that there is a path between any two vertices.

8.2.5 Neighborhood resolving set

In the context of this problem, the neighborhood resolving set (NRS) of the graph consists of a set of nodes (Emergency drug warehouse, Service center, or both) such that for any other node in the graph, the neighborhood (i.e., the set of adjacent nodes) of the nodes in the NRS uniquely identifies that node.

The selection of the nodal center (either a Emergency drug warehouse or a Service center) for efficient transportation depends on the structure of this NRS. The goal is to choose a subset of vertices from the graph that will provide a clear distinction between all other vertices. The neighborhood of each node in the NRS should be distinct from that of every other node, thus ensuring effective management of the transportation system.

8.2.6 Optimising drug distribution network: applications of NRS

- NRS identifies key nodes (Service center or Emergency drug warehouses) that will serve as the cores for drugs distribution.
- These cores are strategically selected to minimize transportation time and ensure responsiveness.
- The transportation network is optimized, ensuring that all Emergency drug warehouses and Service center are within manageable adjacency.
- The network remains connected, providing continuous drugs flow even in emergencies.

8.2.7 Example

In a metropolitan region, there exist **three service centers** and **four emergency drug warehouses** strategically located to support healthcare services. These facilities serve as crucial hubs for medical supplies and emergency response. A new viral outbreak, **Zeta Virus**, has rapidly spread across the region, causing severe respiratory infections. Hospitals are overwhelmed, and there is an urgent need to distribute antiviral drugs, oxygen supplies, and protective gear to control the spread and treat affected individuals. To ensure efficient drug circulation and avoid delays in reaching affected populations, the authorities decide to designate **nodal centers** from the available service centers and emergency drug warehouses. The selection of these nodal centers is based on the

concept of the **neighborhood resolving set**, which ensures that every location in the region is effectively covered with minimal need. By identifying an optimal set of **nodal centers**, authorities can:

- **Minimize response time:** Ensure quick accessibility of essential drugs.
- **Reduce redundancy:** Avoid unnecessary tights at multiple locations.
- **Maximize coverage:** Reach all affected areas without gaps.
- **Improve resource allocation:** Allocate medical staff and logistics efficiently.

This method of selecting nodal centers using the **neighborhood resolving set** ensures an organized and strategic response to the **Zeta Virus outbreak**, improving pandemic management in the region. We are given a city with the following locations:

- 3 Service center: S_1, S_2, S_3
- 4 Emergency drug warehouses: W_1, W_2, W_3, W_4

The goal is to optimize the transportation of drug supplies between Service center and Emergency drug warehouses based on the following distance constraints:

- $d_\alpha = 3$: The maximum distance between any two Service center.
- $d_\beta = 4$: The maximum distance between any two Emergency drug warehouses.

8.2. Selecting nodal centre for effective management of drug distribution

- $d_\gamma = 5$: The maximum distance from any Emergency drug warehouse to its nearest Service center.

Let us break down the distance constraints:

- $d_\alpha = 3$: The maximum distance between any two Service centers is 3 units. This means that Service centers S_1, S_2, S_3 are located within 3 units of each other.
- $d_\beta = 4$: The maximum distance between any two Emergency drug warehouses is 4 units. This means that Emergency drug warehouses W_1, W_2, W_3, W_4 are located within 4 units of each other.
- $d_\gamma = 5$: The maximum distance from any Emergency drug warehouse to its nearest Service center is 5 units. This means that for each Emergency drug warehouse, there is a Service center located within 5 units of distance.

These constraints are essential for planning an efficient drugs transportation system. We must ensure that transportation between the Service centers and Emergency drug warehouses meets these distance limits.

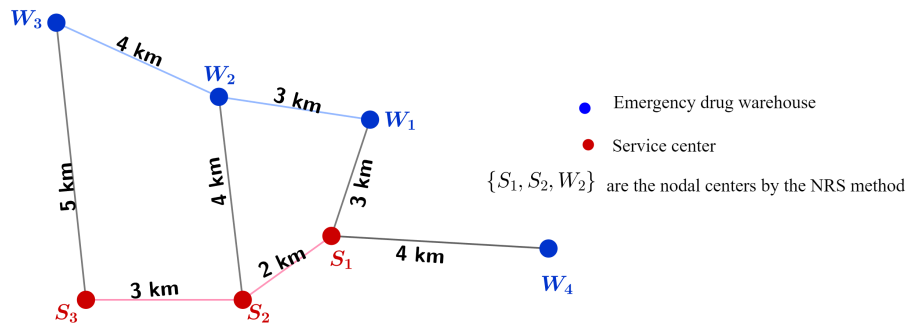


Figure 8.1: Warehouses-service center network

8.2. Selecting nodal centre for effective management of drug distribution

The optimization problem aims to minimize the transportation time and cost while binding to the following goals:

- Ensure that the maximum distance between any two Service centers does not exceed 3 units.
- Ensure that the maximum distance between any two Emergency drug warehouses does not exceed 4 units.
- Ensure that the distance between any Emergency drug warehouse and its nearest Service center does not exceed 5 units.

The solution involves planning the locations of Service centers and Emergency drug warehouses in such a way that these constraints are satisfied, while optimizing the transportation system for timely and efficient delivery of drug supplies.

8.2.8 Real-world benefits

- The authority ensures that an adequate supply of drugs is consistently maintained at the nodal centers (Service centers and Emergency drug warehouses).
- These nodal centers are always prepared to quickly distribute drugs to all other centers in the event of a sudden disease outbreak or emergency situation
- Optimized delivery path based on nearness.

- Faster response time in case of emergencies due to the uniquely identified hubs.
- Ensures timely availability of drugs in critical situations.
- According to geographical condition, can choose another nodal centres by NRS

8.3 Optimizing construction labor management using neighborhood resolving sets

8.3.1 Problem statement

In a construction project, workers from different trades (e.g., carpenters, plumbers, electricians) are distributed across various work zones and collaborate on tasks. However, the labor management system often faces challenges such as inefficiencies, unnecessary tasks, and unfair wage distribution due to the lack of a structured coordination framework. The goal is to optimize task allocation and wage distribution by utilizing the concept of Neighborhood Resolving Sets (NRS) from graph theory

8.3.2 Graph representation of the workforce

In this approach, each worker is represented as a vertex in a graph. The types of workers in a construction project (carpenters, plumbers, electricians, etc.) can be grouped into separate subgraphs, or they can all be part of a single unified

graph, depending on the complexity of the problem.

Each worker in the project is represented as a vertex. For example:

- Carpenter 1, Carpenter 2, ...
- Plumber 1, Plumber 2, ...
- Electrician 1, Electrician 2, ...
- Painter 1, Painter 2, ...

An edge is drawn between two vertices if the corresponding workers communicate or assist each other. The edges can be weighted to represent the strength of the interaction, such as the number of hours worked together or the frequency of communication.

8.3.3 Neighborhood resolving set (NRS)

The concept of Neighborhood Resolving Sets (NRS) is central to this approach. A set of vertices $S \subseteq V$ is called a neighborhood resolving set if for every vertex $v \in V$, there is a vertex in S whose neighborhood (the set of adjacent vertices) uniquely identifies v . The key workers are those in the NRS, as their position in the network makes them critical for task allocation and coordination. These workers will be responsible for managing the communication and tasks for other workers in the construction project. Once the NRS is identified, workers in this set are responsible for task assignment and direct the work done by other workers. Task allocation is based on the workers' skills and the communication network structure. Tasks should be allocated to workers based on the structure

of the graph and the workers' roles within the NRS. Workers who are directly connected to key workers will be assigned specific tasks, ensuring that there is no unnecessary or overlap in their work. Wages can be distributed based on the worker's involvement in the communication network. Key workers in the NRS will likely receive higher wages due to their central role in managing tasks and communication. Other workers will receive wages based on their contribution to the project, with adjustments made for the level of interaction with key workers.

8.3.4 Graph example

In a large-scale construction project, effective workforce distribution, wage and communication are essential for maintaining efficiency. The workforce is structured in a hierarchical yet interconnected network, ensuring smooth task authorization and management. The workforce is organized as follows:

8.3.5 Workforce hierarchy

- **Main contractor:** Oversees the entire project and communicates directly with the master workers.
- **Five Master Workers:** Each specializes in a specific area:
 - **Structural supervisor** Manages foundation and framing.
 - **Masonry supervisor** Oversees brickwork and concrete.
 - **Electrical supervisor** Handles wiring and electrical systems.
 - **Plumbing supervisor** Manages water and drainage systems.

8.3. Optimizing construction labor management using neighborhood resolving sets

- **Finishing supervisor** Responsible for painting, flooring, and interiors.
- **Assistants:** Each master worker has more than two assistants who act as intermediaries between master workers and helpers.
- **Helpers:** Each assistant has more than four helpers performing manual labor such as carrying materials, mixing cement, and installing components. Helpers may or may not communicate directly with each other, forming a partially connected network.
- **Total workforce:** The total number of employees in this project is denoted as n , including all levels of workers.

8.3.6 Network representation and optimization

The workforce structure can be modeled as a graph with:

- The **main contractor as the root node**.
- The **master workers as second-level nodes**.
- The **assistants as third-level nodes**.
- The **helpers as fourth-level nodes**, where some helpers may or may not be directly connected.

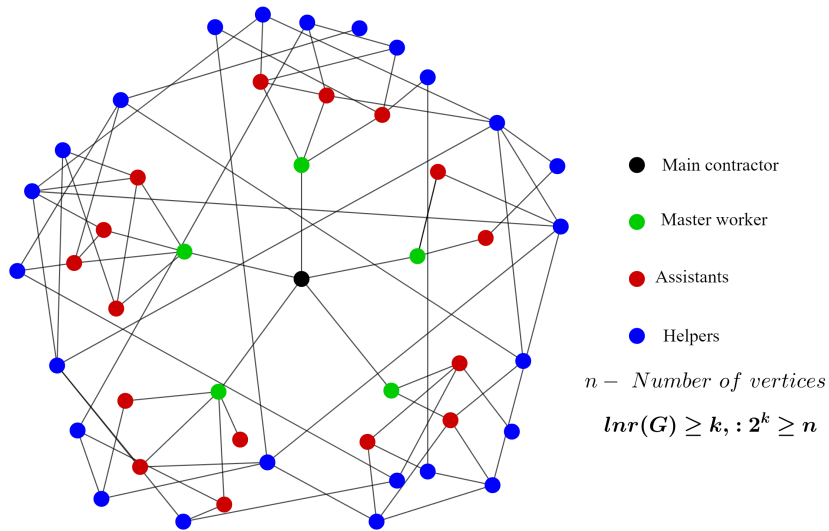


Figure 8.2: Employees network

8.3.7 Result analysis

In this section, we analyze the efficiency of labor management before and after the application of the NRS-based approach. We compare the optimal task allocation and wage distribution to traditional methods, showing how this approach minimizes redundancy and maximizes efficiency. We show that the use of the NRS leads to more efficient task allocation, reducing downtime and ensuring that tasks are assigned to the right workers. We demonstrate how the NRS-based wage distribution method ensures that workers are compensated fairly based on their role and contributions to the project.

Conclusion

By utilizing neighborhood resolvability, this chapter provides a method for selecting the nodal center(s) in a connected graph of Service centers and Emergency drug warehouses. The chosen nodal center will optimize the transportation of drugs, ensuring that all Service centers and Emergency drug warehouses are within a manageable distance from one another. This approach not only addresses the geographical challenges but also improves the efficiency of the overall healthcare logistics system. The application of Neighborhood Resolving Sets in construction labor management has the potential to significantly improve efficiency and fairness in task allocation and wage distribution. By identifying key workers who serve as coordinators, construction projects can benefit from better communication, reduced redundancy, and optimal labor utilization. Further research can explore how this approach can be adapted to larger-scale projects and other industries.

Chapter 9

Conclusion

The thesis concludes that significant advancements were made in understanding the metric dimension, neighborhood resolving sets, and related properties in various graph families. By exploring these concepts in the context of composition products and zero-divisor graphs, the research provided new theoretical insights and practical methodologies. These findings contribute to applications in network design, navigation systems, and algebraic structures. The study highlights the versatile utility of graph theory in solving real-world problems and suggests drive for future research in more complex graph systems and interdisciplinary areas. Also the thesis provides a comprehensive study of the ϕ_k -polynomial for a diverse set of graph families, contributing to a deeper understanding of their combinatorial properties. The analysis of the families $P_m + P_n$, $P_n + O_n$, $K_n - e$, and $P_m \times P_n$, alongside the star clusters representing the 12 zodiac signs, demonstrates the versatility of the ϕ_k -polynomial as a tool for characterizing graph structures.

The behavior of the ϕ_k -polynomial for standard graph families reveals pat-

terns in their connectivity and component structure. For example, the polynomial for $P_m + P_n$ and other connected graphs consistently reflects their edge expansion properties, while the ϕ_k -polynomial of $K_n - e$ and other modified graphs provides insight into how edge removal affects the polynomial's roots and coefficient relationships.

Analyzing the star clusters modeled after the 12 zodiac signs introduced unique topological features that influenced the ϕ_k -polynomial. These clusters, with their outward symmetry and central connectivity, provided a distinct perspective on the interaction of local structure with global properties. The comparison across these families demonstrated how specific structural properties, such as edge density, cycles, and node connections, impact the polynomial's terms. The results offer a clear understanding of how graph families relate to each other in the context of combinatorial enumeration and polynomial invariants. The ϕ_k -polynomial, as explored in this study, has significant applications in network theory, where such polynomials can help model the flexibility and connectivity of various network topologies. Additionally, it provides a valuable approach for studying spectral graph theory, graph invariants, and the study of graph automorphisms. The ϕ_k -polynomial is an essential tool for understanding the underlying structure of a wide range of graphs. This study places the groundwork for future research on the ϕ_k -polynomial, particularly in the context of more complex graph families and their potential applications in theoretical and practical domains. Also presents a method for selecting nodal centers in a connected graph representing service centers and emergency drug warehouses using neighborhood resolvability. The proposed approach optimizes drug transportation by ensuring that all Service Centers and Emergency Drug Warehouses remain within

a manageable distance, thereby enhancing the efficiency of healthcare logistics and addressing geographical challenges. Additionally, the application of Neighborhood Resolving Sets in construction labor management offers a systematic framework for improving efficiency and fairness in task allocation and wage distribution. By identifying key workers as coordinators, construction projects can achieve better communication, minimize unnecessary, and optimize labor utilization. This methodology has significant implications for workforce management in large-scale projects and other industries, paving the way for future research and practical implementations.

Chapter 10

Recommendations

Scope of the thesis

This thesis explores the heterogeneous and complex aspects of graph theory and its applications through several focused studies, including metric dimensions, graph composition products, and neighborhood-resolving sets. By analyzing the properties of ϕ_k -polynomials and their applications in various graph structures, the thesis contributes to a deeper understanding of graph invariants, resolving sets, and their computational implications.

The scope of the thesis extends to:

- Analysis of metric dimensions in graph composition products.
- Study of neighborhood-resolving sets in graphs of order at most six.
- Exploration of ϕ_k -polynomial properties in various graph compositions such as $P_m + P_n$, $P_n + O_n$, $K_n - e$, and $P_m \times P_n$.

-
- Investigation of zodiac signs represented graphically and analyzed through ϕ_k -polynomials.
 - Examination of resolvability and ϕ_k -polynomial properties in finite commutative rings with orders up to ten.
 - Selection of nodal centers using graph theory.

These studies provide valuable theoretical insights and practical applications of graph theory.

Possible future developments

Building on the findings of this thesis, several channel for future research emerge:

- Extending the analysis of metric dimension to dynamic and temporal graphs, where vertex and edge properties evolve over time.
- Investigating neighborhood-resolving sets in higher-order graphs or graphs with additional constraints, such as weighted or directed graphs.
- Developing algorithms for efficient computation of ϕ_k -polynomials in complex graph structures, such as hypergraphs and random graphs.
- Analyzing the interplay between ϕ_k -polynomials and other graph invariants like chromatic number or pathwidth.
- Exploring applications of graph compositions in modeling biological networks, social networks, and communication systems.

-
- Incorporating machine learning techniques to predict resolvability and other graph-theoretic properties for large-scale datasets.
 - The application of NRS can be extended to larger geographical regions with complex logistics networks. Efficient algorithms need to be developed to handle large datasets and optimize the selection of nodal centers.
 - Integrating real-time data sources such as traffic conditions, warehouse stock levels, and emergency demand fluctuations could enhance the adaptability of the proposed system.
 - Future developments could incorporate additional factors such as transportation cost, storage capacity, and delivery time to optimize logistics beyond mere distance-based considerations.
 - The approach developed in this study can be extended to emergency response, supply chain logistics, disaster relief coordination, and other industries requiring efficient resource allocation.

Further studies

Further studies can broaden the theoretical and applied dimensions of this thesis:

- An in-depth examination of graph properties and their connections to algebraic structures in larger finite commutative rings.
- Exploring the potential of bipartite graph theory in solving real-world problems, such as supply chain optimization and network design.

-
- Extending the study of ϕ_k -polynomial properties to non-commutative rings and their graph representations.
 - Generalizing the graphical representation of zodiac signs to include other cultural or symbolic systems and analyzing their structural properties.
 - Investigating the role of graph theory in emerging fields like quantum computing and cryptography.
 - Investigating more efficient algorithms and approximation for computing the minimum Neighborhood Resolving Set in large and complex graphs.
 - Extending the concept of neighborhood resolvability to weighted graphs, directed graphs, and dynamic graphs to explore broader applications.
 - Conducting real-world case studies in healthcare logistics and construction projects to validate theoretical findings and evaluate practical feasibility.
 - Collaborating with experts in optimization, operations research, and artificial intelligence to develop hybrid models incorporating graph-theoretic and machine-learning approaches.
 - Exploring how governments, health organizations, and construction firms can adopt and implement these findings in real-world logistics and workforce management systems.

By addressing these potential developments and further studies, the research can continue to advance the understanding and application of graph theory in both theoretical and practical domains.

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APPENDIX I

List of Publications

1. Shamsudheen P V and Shahida A T, Bases and Metric Dimension of Composition Product of Some Graph Families, Malaya Journal of Matematik, Vol. S, No. 1, 587-589, (UGC)(2020).
2. Shamsudheen P V and Shahida A T, Metric dimension and neighbourhood resolving set for the zero divisor graphs of order at most 10 of a small finite commutative ring, International Journal of Nonlinear Analysis and Applications, Vol. 14, No. 11 , pp. 365- 373.(UGC, Scopus),(2023)
3. Shamsudheen P V and Shahida A T, Studies on Neighbourhood resolving sets-graph with order atmost 6(Communicated)
4. Shamsudheen P V and Shahida A T, Neighbourhood resolving set for the family of graphs.(Communicated)
5. Shamsudheen P V and Shahida A T, Graphical Properties of 12 Zodiac

Signs.(Communicated)

6. Shamsudheen P V and Shahida A T, ϕ_k -polynomial properties of $P_m + P_n$, $P_n + O_n$, $K_n - e$ and $P_m \times P_n$.(Communicated)
7. Shamsudheen P V and Shahida A T, Effective Drug Distribution Management: A Framework for Selecting Ideal Nodal Centers .(Communicated)