

C 43751

(Pages : 3)

Name.....

Reg. No.....

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS)

Mathematics

MAT 4E 02—ADVANCED FUNCTIONAL ANALYSIS

(2019 Admissions)

Time : Three Hours

Maximum : 80 Marks

Part A*Answer all questions.**Each question carries 2 marks.*

1. Show that a normed space X is separable if the dual space X' is separable.
2. Define the spectral radius $r_\sigma(A)$ of a bounded operator on a normed space X . Show by an example that $r_\sigma(A)$ can be strictly less than $\|A\|$.
3. Define compact operators with examples. Show that a compact operator on an infinite dimensional Banach space is never invertible.
4. Give example of a compact operator on an infinite dimensional Banach space with 0 not an eigenvalue of it.
5. Give example for an infinite orthonormal set. Show that every orthonormal set is linearly independent.
6. Let X be an inner product space and $E \subset X$. Define $E^\perp = \{y \in X; \langle x, y \rangle = 0 \text{ for every } x \in E\}$. Show that E^\perp is always a closed subspace.
7. Let $A, B \in BL(X)$ with A self-adjoint. Show that $AB = 0$ if and only if $R(A) \perp R(B)$.
8. Define normal, unitary and self-adjoint operators. Give an example for a normal operator that is not unitary.

(8 × 2 = 16 marks)

Turn over

Part B

*Answer any four questions.
Each question carries 4 marks.*

9. Let X be a Banach space and $A \in BL(X)$. Show that A is invertible if and only if A is bounded below and range of A is dense in X .
10. Show that the spectrum of a compact operator on a Banach space is at most countable.
11. Show that sum of two self-adjoint operators on a Hilbert space is self-adjoint and the product is self-adjoint if and only if they commute. Give examples for two self-adjoint operators do not commute.
12. Show that if A is a normal operator on a Hilbert space, and λ is an eigenvalue of A , then $\bar{\lambda}$ is an eigenvalue of A^* . Show by an example that this need not be true if A is not normal.
13. Show that $x_n \xrightarrow{w} x$ if and only if $x_n \rightarrow x$ in l^1 .
14. Show that sum of two compact operators is compact. Also show that composition of a compact operator and a bounded operator is compact.

(4 × 4 = 16 marks)

Part C

*Answer either Part A or B of each of the following questions.
Each question carries 12 marks.*

15. (a) For $1 \leq p < \infty$, show that the dual $(l^p)'$ of l^p is isometrically isomorphic to l^q , where $\frac{1}{p} + \frac{1}{q} = 1$.

Or

- (b) Introduce the spectrum $\sigma(A)$, eigenspectrum $\sigma_e(A)$, approximate eigenspectrum $\sigma_a(A)$ of a bounded operator A on a normed space X .

Let $A : l^2 \rightarrow l^2$ be defined by

$A(x(1), x(2), \dots) = (0, x(1), x(2), \dots)$; for all $(x(1), x(2), \dots) \in l^2$. Compute the spectrum $\sigma(A)$, eigenspectrum $\sigma_e(A)$ and approximate eigenspectrum $\sigma_a(A)$.

16. (a) Let X be a normed space and A be a compact operator on X :
- Show that every nonzero spectral value of A is an eigenvalue of A .
 - Show that $\sigma_a(A) = \sigma(A)$.

Or

- (b) Show that the class $CL(X)$ of all compact operators on a Banach space X is a closed two sided ideal in $BL(X)$, the class of all bounded operators on X .

17. (a) (i) State and Prove Projection theorem.
(ii) State and Prove Riesz Representation theorem.

Or

- (b) Let $\mathbb{H} = L^2(\mathbb{R})$ and $z \in L^\infty(\mathbb{R})$. Define A on \mathbb{H} by $A(x) = zx, x \in \mathbb{H}$. Show that A is a bounded linear map on \mathbb{H} and $\|A\| = \|z\|_\infty$. Also find A^* .

18. (a) State and prove spectral theorem for compact self-adjoint operators on a Hilbert space.

Or

- (b) Let \mathbb{H} be a Hilbert space and $A \in BL(\mathbb{H})$.

(i) If A is self-adjoint, then show that $\|A\| = \sup \{ |\langle A(x), x \rangle| : x \in \mathbb{H}, \|x\| \leq 1 \}$.

(ii) Show that A is unitary if and only if $\|A(x)\| = \|x\|$, for every $x \in \mathbb{H}$. Also show that in this case, $\|A^{-1}(x)\| = \|x\|$ for every $x \in \mathbb{H}$ and $\|A\| = 1 = \|A^{-1}\|$.

(iii) Show that A is normal if and only if $\|A(x)\| = \|A^*(x)\|$, for every $x \in \mathbb{H}$. Also show that in this case, $\|A^2\| = \|A\|^2 = \|A^*A\|$.

(4 × 12 = 48 marks)

C 43752

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Name.....

Reg. No.....

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS)

Mathematics

MAT 4E 04—ALGEBRAIC GRAPH THEORY

(2019 Admissions)

Time : Three Hours

Maximum : 80 Marks

Part A

*Answer all the questions.
Each question carries 2 marks.*

1. Obtain one spanning subgraph and one induced subgraph of a Petersen graph.
2. Define the terms edge cut set and edge connectivity.
3. Is the edge connectivity of a vertex transitive graph is at most its valency ? Justify your answer.
4. What is an asymmetric graph ? Illustrate with an example.
5. Show that the chromatic number of graph X is the least integer r such that there is a homomorphism from X to K_r .
6. If θ is an automorphism of the group G , then $X(G, C)$ and $X(G, \theta(C))$ are isomorphic. Discuss the statement.

(6 × 2 = 12 marks)

Part B

*Answer any five questions.
Each question carries 4 marks.*

7. Prove that the automorphism group of a group is equal to the automorphism group of its complement.
8. Characterize a line graph in terms of its induced subgraph.
9. State and prove Orbit-Stabilizer Lemma.
10. If X is regular graph with valency k , then prove that $L(X)$ is regular graph with valency $2k - 2$.

Turn over

11. Prove that $|Aut(C_n)| = 2n$.
12. Let G be a transitive permutation group on V . Then G is primitive if and only if each non-diagonal orbit is connected.
13. Prove that if $v \geq k \geq i$, then $J(v, k, i) \cong J(v, v-k, v-2k+i)$.
14. Prove two properties of fragments.

(5 × 4 = 20 marks)

Part C*Answer A or B of each of the following questions.**Each question carries 16 Marks .*

UNIT I

15. A) Prove that the k -cube Q_k is vertex transitive.
- B) Let D be a directed graph such that in-valency and out-valency of any *two* vertices are equal. Then show that D is strongly connected if and only if D is weakly connected.

UNIT II

16. A) Let G be a transitive permutation group on V and let x be a point in V . Prove that G is primitive if and only if G_x is a maximal subgroup of G .
- B) Let X be a connected vertex-transitive graph. Then prove that X has a matching that misses at most one vertex and each edge is contained in a maximum matching.

UNIT III

17. A) (i) If A is an atom and B is a fragment of X , then prove that A is a subset of exactly one of B , $N(B)$ and \bar{B} .
- (ii) Show that X and \bar{X} have the same automorphism group for any graph X .
- B) If the graph X is vertex transitive and edge transitive, but not arc transitive, then prove that its valency is even.

(3 × 16 = 48 marks)

C 43757

(Pages : 2)

Name.....

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FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS)

Mathematics

MAT 4E 13—ADVANCED TOPOLOGY

(2019 Admissions)

Time : Three Hours

Maximum : 80 Marks

Part A*Answer all questions from this part.**Each question carries 2 marks.*

1. Show that a continuous bijection from a compact space into a Hausdorff space is open.
2. Show that every compact Hausdorff space is T_4 .
3. Define local connectedness and path connectedness. Give an example each for locally connected and path connected spaces.
4. Show that a topological product is T_0 if each of the coordinate space is T_0 .
5. Show that the neighbourhood system of a point x in a topological space X is a directed set with the relation defined by $U \geq V$ if $U \subset V$.
6. Show that a family which does not contain empty set and which is closed under finite intersection is a base for a unique filter.
7. Give an example of a metric space which is not complete. Justify your claim.
8. Prove that every totally bounded metric space is bounded.

(8 × 2 = 16 marks)

Part B*Answer any four questions.**Each question carries 4 marks.*

9. Show that every regular, Lindelof space is normal.
10. Show that every path connected space is connected.

Turn over

11. Prove that a topological product of spaces is Tychonoff if and only if each co-ordinate space is so.
12. Show that a subset B of a space X is open if and only if no net in the complement $X-B$ can converge to a point in B .
13. Prove that a topological space is Hausdorff if and only if no filter can converge to more than one point in it.
14. Prove that every compact metric space is complete.

(4 × 4 = 16 marks)

Part C*Answer either A or B of each question.**Each question carries 12 marks.*

15. A. State and prove Urysohn's lemma.
 B. (a) If X be a regular space, C a closed subset of X and F a compact subset of X , such that $C \cap F = \emptyset$. Show that there exist open sets U, V such that $x \in U, F \subset V$ and $U \cap V = \emptyset$.
 (b) If A is a closed subset of a normal space X and if $f : A \rightarrow (-1, 1)$ is continuous, then show that there exist a continuous function $F : X \rightarrow (-1, 1)$ such that $F(x) = f(x)$ for all $x \in A$.
16. A. (a) Show that every quotient space of a locally connected space is locally connected.
 (b) Is local connectedness a hereditary property. Justify your answer.
 B. Prove that countable product of metrizable space is metrizable.
17. A. Show that a topological space is Hausdorff if and only if limits of all nets in it are unique.
 B. Let $S : D \rightarrow X$ be a net in topological space X and let $x \in X$. Prove that x is a cluster point of S if and only if there exist a subnet of S which converges to x in X .
18. A. Show that every metric space can be isometrically embedded as a dense subspace of a complete metric space.
 B. (a) Prove that a metric space is compact if and only if it is complete and totally bounded.
 (b) Prove that every contraction of a complete metric space into itself has a unique fixed point.

(4 × 12 = 48 marks)

C 43755

(Pages : 2)

Name.....

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FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS)

Mathematics

MAT 4E 06—CRYPTOGRAPHY

(2019 Admissions)

Time : Three Hours

Maximum : 80 Marks

Part A*Answer all questions.**Each question carries 2 marks.*

1. Define a Cryptosystem.
2. Find $(-7503) \bmod 81$.
3. Describe perfect secrecy.
4. Define entropy.
5. Define cryptographic hash functions
6. Explain the terms round key mixing and whitening in SPN.

(6 × 2 = 12 marks)

Part B*Answer any five questions.**Each question carries 4 marks.*

7. Explain Autokey Cipher and decrypt the message ZVRQH DUJIM which was produced with $K = 8$.
8. Explain various types of Stream ciphers.
9. Describe the index of coincidence method for cryptanalysis in Vigenere Cipher.
10. State and prove Bayes theorem.
11. Define unicity distance of a cryptosystem. Calculate it for Substitution Cipher with redundancy 0.75.

Turn over

12. State and prove the Piling-up-lemma.
 13. Describe and analyse Data Encryption Standard(DES).
 14. Explain Nested Message Authentication Codes(MAC).

(5 × 4 = 20 marks)

Part C*Answer either A or B of each of the following questions.**Each question carries 16 marks.*

15. A (i) Explain Affine cryptosystem and derive the formula for number of keys in an Affine cipher over Z_m .
 (ii) Prove that $K = (a, b)$ is an involutory key in Affine Cipher over Z_n if and only if $a^{-1} \pmod n = a$ and $b(a + 1) \equiv 0 \pmod n$.

- B (i) Let π be the permutation of $\{1, 2, \dots, 8\}$. Compute π^{-1} where

x	1	2	3	4	5	6	7	8
$\pi(x)$	4	1	6	2	7	3	8	5

- (ii) Decrypt the following cipher text which was encrypted using above key π for a permutation cipher with $m = 8$. TGEEMNELNNTDROEOAAHDOETCSHAEIRLM
16. A (i) Let (P, C, K, E, D) be a cryptosystem. Then prove that $H(K/C) = H(K) + H(P) - H(C)$.
 (ii) Prove that $H(X, Y) \leq H(X) + H(Y)$, with equality if and only if X and Y are independent random variables.
- B (i) Explain the cryptosystem One-time Pad.
 (ii) Prove that the Shift Cipher achieves perfect secrecy if every key is used with equal probability $1/26$.
17. A (i) Explain Random Oracle Model Hash functions.
 (ii) Describe Differential cryptanalysis
- B (i) Define iterated hash functions and explain the generic construction and the Merkle-Damgard Construction methods of the same.
 (ii) Describe the Hash Algorithm SHA-1.

(3 × 16 = 48 marks)

C 43756

(Pages : 2)

Name.....

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FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS)

Mathematics

MAT 4E 08—GRAPH THEORY

(2019 Admissions)

Time : Three Hours

Maximum : 80 Marks

Part A*Answer all the questions.**Each question carries 4 marks.*

1. Prove or disprove : An edge $e = xy$ of a connected graph G is a cut edge of G if and only if e belongs to no cycle of G .
2. Define r -connectivity of a graph G . Prove that if H is a subgraph of G , the $\kappa(H) \leq \kappa(G)$.
3. Define chromatic number $\chi(G)$ of a graph G . Prove that $\chi(K_n) = n$.
4. What is a critical graph ? Prove that every critical graph is connected.

(4 × 4 = 16 marks)

Part B*Answer any two questions.**Each question carries 8 marks.*

5. Prove that a simple cubic connected graph G has a cut vertex if and only if it has a cut edge.
6. Let G be a simple graph. Suppose there exists a homomorphism $f : G \rightarrow K_k$ and let k be the least positive integer with this property. Then prove that $\chi(G) = k$.
7. Define Mycielskian $\mu(G)$ of a graph G . Prove that $\chi(\mu(G)) = \chi(G) + 1$.

(2 × 8 = 16 marks)

Turn over

Part C

*Answer A or B of each of the following two questions
Each question carries 24 marks.*

UNIT I

8. A) A graph G with at least 3 vertices is 2-connected if and only if any *two* vertices of G are connected by at least two internally disjoint paths.
- B) Prove that if a connected graph G is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.

UNIT II

9. A) Define source and sink in a network N . Prove that in a given network N with source and sink t , the maximum value of the flow is equal to the minimum value of the capacities of all the cuts in N .
- B) Prove that for connected graph G and H , $\mu(G) \approx \mu(H)$ if and only if $G \approx H$, where $\mu(G)$ is the Mycielskian of G .

(2 × 24 = 48 marks)

C 43758

(Pages : 4)

Name.....

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FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS)

Mathematics

MAT4E17—REPRESENTATION THEORY AND FINITE GROUP

(2019 Admissions)

Time : Three Hours

Maximum : 80 Marks

Part A

*Answer all questions.
Each question is of 2 marks.*

1. Let $G = \{1, a, a^2 : a^3 = 1\}$ be the cyclic group of order 3 and $\pi(x)$ be the permutation representation given by :

$$\pi(x) = \begin{pmatrix} 1 & a & a^2 \\ x & ax & a^2x \end{pmatrix}.$$

Find $\pi(a^2)$.

2. Let $N(x)$ be the natural representation of the symmetric group S_3 . Find $N(\sigma)$ where $\sigma = (1\ 2)$.
3. Let $G = \{1, a : a^2 = 1\}$ be the cyclic group of order 2 and let \mathbb{R}^2 be the G -module with action given by
- $$(x, y)a = (y, x).$$
- Verify whether $\theta(x, y) = (2x, 2y)$ is a G -homomorphism.
4. Let ν be the natural character of the symmetric group S_3 . Find $\langle \nu, \nu \rangle$.
5. Find the number of simple characters of the cyclic group \mathbb{Z}_3 .

Turn over

6. Let $G = \{1, a, a^2, a^3 : a^4 = 1\}$ be the cyclic group of order 4 and $H = \{1, a^2\}$ be a subgroup. Let $B(x)$ be a representation of H . Describe the induced representation $A(x)$ on G .
7. Let H be a normal subgroup of a finite group G and $B(x)$ be a representation of H . For each $t \in G$ let $B_t(x) = B(txt^{-1})$. Show that $B_t(x)$ is a representation of H .
8. Verify whether the subgroup $H = \{(1), (12), (34), (12)(34)\}$ of S_4 is transitive.

(8 × 2 = 16 marks)

Part B

*Answer any **four** questions.
Each question is of 4 marks.*

9. Let G be a finite group and for each $x \in G$ let $A(x)$ be an $m \times m$ matrix such that $A(x)A(y) = A(xy)$ for all $x, y \in G$ and $A(1) = I$ where 1 is the identity of G and I is the $m \times m$ identity matrix. Show that $A(x)$ is non singular and $(A(x))^{-1} = A(x^{-1})$.
10. Show that if $A(x)$ and $B(x)$ are equivalent representations of a group G then $\chi_A = \chi_B$.
11. Let $A(x)$ and $B(x)$ be equivalent irreducible representations of a group G . Show that the commutant algebras $\mathbb{C}(A)$ and $\mathbb{C}(B)$ are isomorphic.
12. Let M_2 be the ring of all 2×2 matrices over the field \mathbb{C} and let $Z \in M_2$ be such that :
 $ZX = XZ$
for all $Z \in M_2$. Show that Z is a diagonal matrix.
13. Describe the character table of the Klein four group.
14. Show that a permutation group G of degree n is transitive if for each α with $1 \leq \alpha \leq n$ there is a permutation $p_\alpha \in G$ such that $1p_\alpha = \alpha$.

(4 × 4 = 16 marks)

Part C

Answer **either** Part A **or** Part B of each of the four questions.
Each question is of 12 marks.

15. (A) (a) Define reducible representations and irreducible representations.
(b) Let $A(x)$ be a reducible representation of a finite group G over a field of characteristic zero. Show that there exist irreducible representations $A_1(x), A_2(x), \dots, A_k(x)$ such that :

$$A(x) \sim \begin{bmatrix} A_1(x) & 0 & 0 & \dots & 0 \\ A_{21}(x) & A_2(x) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{k1}(x) & A_{k2}(x) & \dots & \dots & A_k(x) \end{bmatrix}.$$

- (B) (a) Define submodule of a G -module.
(b) Let U be a submodule of a G -module V . Show that if $A(x)$ is the representation of G provided by V then :

$$A(x) \sim \begin{pmatrix} C(x) & 0 \\ E(x) & D(x) \end{pmatrix}$$

where $C(x)$ is the representation of G induced by U .

16. (A) (a) Define G -homomorphism between two G -modules.
(b) Let U, V be G -modules and $\theta: U \rightarrow V$ be a G -homomorphism. Let $A(x)$ and $B(x)$ be the matrix representations of G afforded by U and V respectively. Show that there is a matrix T corresponding to θ such that $TA(x) = B(x)T$ for all $x \in G$.

- (B) Let $A(x)$ be an irreducible representation of a group G over an algebraically closed field K and T be a matrix over K such that $TA(x) = A(x)T$ for all $x \in G$. Show that

(a) $T = \lambda I$ for some $\lambda \in K$.

- (b) Dimension of the commutant algebra $\mathbb{C}(A)$ is one.

17. (A) (a) Show that every simple character of a finite abelian group is a linear character.

- (b) Let $\{A_1 = I, A_2, \dots, A_g\}$ be an abelian group of $n \times n$ matrices over the complex field \mathbb{C} . Show that these matrices are simultaneously diagonalizable.

Turn over

(B) Let H be a subgroup of a finite group G and

$$G = Ht_1 \cup Ht_2 \cup \dots \cup Ht_n$$

be a coset decomposition of G . Let $B(x)$ be a representation of H with character ϕ and

$A(x) = B(t_i x t_i^{-1})$ be the induced representation of G and ϕ^G be the induced character.

Show that for $x \in G$.

$$(a) \quad \phi^G(x) = \sum_{i=1}^n \phi(t_i x t_i^{-1}).$$

$$(b) \quad \phi^G(x) = \frac{1}{h} \sum_{y \in G} \phi(yxy^{-1}).$$

where $h = |H|$.

18. (A) (a) For the alternating group A_5 find the number of conjugacy classes and the number of elements in each conjugacy class.

(b) Let ν be the natural character of A_5 and $\chi^{(2)}$ be defined by :

$$\chi^{(2)}(x) = \nu(x) - 1$$

for all $x \in G$. Show that $\chi^{(2)}$ is a simple character of A_5 .

(B) Let G be a transitive permutation group on $1, 2, 3, \dots, n$ and let $p_\alpha \in G$ be such that $1 p_\alpha = \alpha$ for all $\alpha = 1, 2, \dots, n$. Let H be the stabilizer of 1. Show that :

(a) For each $\alpha, p_\alpha^{-1} H p_\alpha$ is the stabilizer of α .

(b) $G = H p_1 \cup H p_2 \cup \dots \cup H p_n$.

(c) $[G : H] = n$.

(4 × 12 = 48 marks)

C 43753

(Pages : 4)

Name.....

Reg. No.....

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS)

Mathematics

MAT 4E 05—ALGEBRAIC TOPOLOGY

(2019 Admissions)

Time : Three Hours

Maximum : 80 Marks

Part A*Answer all questions.**Each question carries 2 marks.*

1. Give a brief description of Mobius strip.
2. When two continuous functions are said to be homotopic? Give an example.
3. Define simplicial approximation.
4. State the Hopf Classification Theorem.
5. Define degree of a loop in S^1 .
6. Define covering path.

(6 × 2 = 12 marks)

Part B*Answer any five questions.**Each question carries 4 marks.*

7. Let M denote the triangulation of the Mobius strip. Then find $B_2(M)$, $Z_2(M)$ and $H_2(M)$.
8. Let K be a complex with r combinatorial components. Then prove that $H_0(K)$ is isomorphic to the direct sum of r copies of the group Z of integers.
9. State and prove Euler's Theorem.
10. Prove that every simplicial mapping $\varphi: |K| \rightarrow |L|$ is continuous.

Turn over

11. If $f: S^n \rightarrow S^n$ and $g: S^n \rightarrow S^n$ are continuous maps, then prove that
- $$\deg(gf) = \deg(g) \cdot \deg(f).$$
12. If σ is a path in S^1 and r is a real number such that $p(r) = \sigma(0)$, then prove that there is a unique covering path $\bar{\sigma}$ of σ with initial point r .
13. If A is a deformation retract of a space X and x_0 is a point of A , then prove that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(A, x_0)$.
14. Let X and Y be spaces with points x_0 in X and y_0 in Y . Then prove that
- $$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \oplus \pi_1(Y, y_0).$$

(5 × 4 = 20 marks)

Part C

Answer **either A or B** of each of the following questions.
Each question carries 16 marks.

15. A. (a) If K is an oriented complex and $p \geq 2$, then prove that the composition $\partial\partial: C_p(K) \rightarrow C_{p-2}(K)$ in the diagram
- $$C_p(K) \xrightarrow{\partial} C_{p-1}(K) \xrightarrow{\partial} C_{p-2}(K)$$
- is a trivial homomorphism.
- (b) Let K be a geometric complex with two orientations, and let K_1 and K_2 denote the resulting oriented geometric complexes. Then prove that the homology groups $H_p(K_1)$ and $H_p(K_2)$ are isomorphic for each dimension p .
- B. (a) Prove that a simplex σ is the smallest convex set which contains all vertices of σ .
- (b) Prove that a set $A = \{a_0, a_1, \dots, a_k\}$ of points in \mathbb{R}^n is geometrically independent if and only if the set of vectors $\{a_1 - a_0, \dots, a_k - a_0\}$ is linearly independent.

(c) Suppose that the components K_1 and K_2 have the same simplexes but different orientations. How are chain groups $C_p(K_1)$ and $C_p(K_2)$ related?

16. A. (a) Consider the projective plane P . Prove the following :

(i) $H_2(P) = \{0\}$.

(ii) $H_1(P) \cong \mathbb{Z}_2$.

(iii) $H_0(P) \cong \mathbb{Z}$.

(iv) $R_2(P) = R_1(P) = 0$.

(iv) $R_0(P) = 1$.

(v) $\chi(P) = 1$.

(b) If $\varphi: K \rightarrow L$ is a simplicial mapping, and let p be a non-negative integer. If $g \cdot \sigma^p$ is an elementary p -chain on K , define

$$\varphi_p(g \cdot \sigma^p) = \begin{cases} 0 & \text{if } \varphi \text{ collapses } \sigma^p \\ g \cdot \varphi(\sigma^p) & \text{if } \varphi \text{ does not collapse } \sigma^p \end{cases}$$

Prove that the sequence $\{\varphi_p\}_0^\infty$ of homomorphisms is a chain mapping.

B. (a) If two continuous maps $f, g: S^n \rightarrow S^n$ are homotopic, then prove that they have the same degree.

(b) Show that homotopy of paths is an equivalence relation on the set of all paths in a topological space X .

17. A. (a) Prove that the set $\pi_1(X, x_0)$ is a group under the \circ operation.

(b) State and prove the Covering Homotopy Property.

Turn over

- B. (a) If $\sigma: I \rightarrow S^1$ is a path in S^1 with initial point 1, then prove that there is a unique covering path $\bar{\sigma}: I \rightarrow \mathbb{R}$ with initial point 0.
- (b) Prove that equivalence of paths is an equivalence relation on the set of paths in a space X .
- (c) Prove that equivalence of loops is an equivalence relation on the set of loops in X with base point x_0 .

(3 × 16 = 48 marks)