PHD THESIS

MATHEMATICS

SUBNORMAL BLOCK TOEPLITZ OPERATORS

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PhD Thesis Mathematics

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CERTIFICATE

I hereby certify that the thesis entitled **"SUBNORMAL BLOCK TOEPLITZ OPERATORS"** is a bonafide record of the original research work carried out by **Mr. Abhinand M.** under my guidance, in partial fulfilment of the requirements for the award of the degree of Ph.D. in Mathematics from the University of Calicut. I further certify that this work or any part of it has not been included in any other thesis submitted previously for the award of any degree at this or any other University or Institution.

Dr. Prasad T.
(Research Supervisor)

DECLARATION

I hereby declare that the work presented in the thesis entitled **"SUBNORMAL BLOCK TOEPLITZ OPERATORS"** is based on the original work done by me under the guidance of Dr. Prasad T. and has not been included in any other thesis submitted previously for the award of any degree. The contents of the thesis are undergone plagiarism check using iThenticate software at C. H. M. K. Library, University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.

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ABSTRACT

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$ and subnormal if it has a normal extension, where T^* denotes the adjoint of T.

P. R. Halmos posed a problem, known as Halmos' Problem 5, concerning the characterization of subnormal Toeplitz operators on the Hardy space. The problem asks: *"Is every subnormal Toeplitz operator either normal or analytic?"* C. Cowen answered this question negatively, leading to the refined problem: *"Which subnormal Toeplitz operators are either normal or analytic?"*

In this thesis, we address the refined version of Halmos' Problem 5, particularly in the context of Toeplitz and block Toeplitz operators with finite rank self-commutator and answer a problem recently posed by R. E. Curto, I. S. Hwang and W. Y. Lee [22, Problem 6.2]. We establish conditions for identifying subnormal block Toeplitz operators whose self-commutators are of finite rank. Furthermore, we investigate the subnormality and hyponormality of Toeplitz operators with operator-valued symbols and present sufficient conditions for hyponormality in this setting. In addition, we provide characterizations of hyponormality and subnormality for analytic Toeplitz operators on the Hardy space of Hilbert space-valued functions.

Keywords: Hardy space, Halmos' Problem 5, Subnormal operators, Hyponormal operators, Block Toeplitz operators.

CONTENTS

PREFACE

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if it satisfies the relation $T^*T = TT^*$ and hyponormal if $T^*T \geq TT^*$, where T^* denotes the adjoint of T. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be subnormal if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace and a normal operator $N \in \mathcal{B}(\mathcal{K})$ such that $T = N|_{\mathcal{H}}$, that is, T is the restriction of N to $\mathcal{H}$.

Let H^2 denote the Hardy space and T_φ be the Toeplitz operator on H^2 with symbol φ. It is well known that every normal operator is subnormal. Similarly, every analytic Toeplitz operator T_φ is subnormal, since the corresponding multiplication operator M_φ acts as a normal extension. The converse of these observations was posed as a problem by P. R. Halmos [35, 36], known as Halmos' Problem 5. The problem asks:

"Is every subnormal Toeplitz operator either normal or analytic?"

C. Cowen and J. Long [18] gave a negative answer to Halmos' Problem 5 by presenting an example of a subnormal Toeplitz operator that is neither normal nor analytic. This naturally gave rise to the following two problems:

"Which Toeplitz operators are subnormal?"

"Which subnormal Toeplitz operators are either normal or analytic?"

In the major part of this thesis, we address the second question in the setting of Toeplitz and block Toeplitz operators whose self-commutators are of finite rank. This thesis is based on three articles [1, 2, 3]. Among these, [1] and [2] are joint works with R. E. Curto, I. S. Hwang, W. Y. Lee and T. Prasad, whereas [3] is a collaboration with R. E. Curto and T. Prasad. The preliminaries are covered in the first chapter.

In the second chapter, we address the refined version of Halmos' Problem 5 in the context of Toeplitz operators with finite rank self-commutators and obtain partial results that extend the classical Nakazi–Takahashi Theorem [44].

In the third chapter, we identify a large class of subnormal block Toeplitz operators whose self-commutators are of finite rank. The results presented provide a partial answer to a conjecture recently posed by R. E. Curto, I. S. Hwang and W. Y. Lee [22].

The fourth chapter discusses the refined version of Halmos' Problem 5 in the context of block Toeplitz operators with finite rank self-commutators. The results we obtain form an analogue of the Nakazi–Takahashi Theorem in the setting of block Toeplitz operators and answer a problem recently posed by R. E. Curto, I. S. Hwang and W. Y. Lee [22]. Furthermore, we present the block Toeplitz analogue of the result obtained in Chapter 2.

Chapter 5 explores the hyponormality and subnormality of Toeplitz operators on the Hardy space of Hilbert space-valued functions, a topic of significant current interest in operator theory. We present a sufficient condition for the hyponormality of Toeplitz operators with operator-valued symbols. In addition, we provide characterizations of the hyponormality and subnormality of analytic Toeplitz operators on the Hardy space of Hilbert space-valued functions. Finally, we conclude the thesis with recommendations and an outline of future research directions.

Chapter 1

INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, we denote by $\ker T$, $\operatorname{Ran} T$ and $\operatorname{Rank} T$ the kernel, range and rank of T, respectively. If $\mathcal{K} \subseteq \mathcal{H}$ is a subspace, then $\dim \mathcal{K}$ denotes its dimension. The adjoint of T is denoted by T^* and its *self-commutator* is defined as $[T^*, T] := T^*T - TT^*$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *self-adjoint* if $T = T^*$, *unitary* if $T^*T = TT^* = I$ and *normal* if $[T^*, T] = 0$. The class of subnormal operators introduced by P. R. Halmos [33] is an interesting extension of normal operators. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be *subnormal* if it has a normal extention, that is, if there exists a normal operator N on a Hilbert space $\mathcal{K}$, where $\mathcal{H}$ is the closed subspace of $\mathcal{K}$, $N_{|\mathcal{H}} = T$ and $N\mathcal{H} \subseteq \mathcal{H}$. It can be observed that the theories of subnormal operators found in the literature are not easy and highly nontrivial (see [14]). A more general class that of hyponormal operators includes both the normal and subnormal operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is *hyponormal* if $[T^*, T]$ is positive.

Research on normal and hyponormal operators has an extensive range of applications across various fields including mathematics, mathematical physics and machine learning (see [39], for instance). In particular, the self-adjoint operators play a key role in problems related to electrophoretic transport, enzyme reaction dynamics and microwave heating in composite media [41]. The classical commutation relation naturally leads to the concept of subnormal operators, which has been explored by F. H. Szafraniec [53] in connection with the quantum harmonic oscillator. Furthermore, the theory of subnormal operators made significant contributions to fileds like functional analysis, operator theory, mathematical physics, etc [37]. Thus, the following question arises naturally

"Which operators are subnormal?"

P. R. Halmos [33] provided a characterization of subnormal operators in terms of their action on a finite set of vectors in their domain, while J. Bram [10] further refined this result.

Bram-Halmos criterion for subnormality. [10] An operator $S \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if $\sum_{j,k=0}^{n} \langle S^j f_k, S^k f_j \rangle \geq 0$ for every finite set $f_0, f_1, \ldots, f_n \in \mathcal{H}$.

In other words, Bram-Halmos criterion on subnormality says that $S \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if

$$\begin{bmatrix} I & S^* & \cdots & S^{*k} \\ S & S^*S & \cdots & S^{*k}S \\ \vdots & \vdots & \ddots & \vdots \\ S^k & S^*S^k & \cdots & S^{*k}S^k \end{bmatrix} \geq 0 \quad \text{for all } k \geq 1. \tag{1.1}$$

Subnormal operators have also been characterized by several authors including J. Agler, J. W. Bunce - J. A. Deddens (jointly), M. R. Embry and W. Szymanski (see [5, 12, 29, 54]). These methods generally require an infinite number of trials, which makes addressing the question challenging. Finding feasible methods to check whether an operator is subnormal is one of the important problems in the theory of subnormal operators. P. R. Halmos [35, 36] addressed this question particularly in case of Toeplitz operators. To set the stage, we begin with a brief overview of Hardy space and Toeplitz operators.

Hardy space

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle in the complex plane $\mathbb{C}$. The *Hardy space* H^2, also known as *Hardy-Hilbert space* is the space of all analytic functions on $\mathbb{D}$ having power series representation with square-summable complex coefficients. Equivalently, an analytic function f on $\mathbb{D}$ belongs to H^2 if and only if $\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$ is finite. The Hardy space is a Hilbert space equipped with an inner product defined by $\langle f, g\rangle := \sum_{n=0}^{\infty} a_n \overline{b_n}$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. The mapping $\sum_{n=0}^{\infty} a_n z^n \mapsto (a_0, a_1, a_2, \ldots)$ defines an isometric isomorphism between the Hardy space H^2 and the Hilbert space $\ell^2(\mathbb{W})$, the space of square-summable sequences indexed by the set $\mathbb{W} = \mathbb{N} \cup \{0\}$ of non-negative integers.

For any measurable set $E \subseteq [0, 2\pi]$, the normalized Lebesgue measure m is defined by $m(E) := \frac{1}{2\pi} \int_0^{2\pi} \chi_E(\theta)\, d\theta$, where $d\theta$ is the ordinary Lebesgue measure on $[0, 2\pi]$ and χ_E denotes the characteristic function of E. Let $L^2 := L^2(\mathbb{T})$ denote the Hilbert space of all equivalence classes of square integrable complex-valued functions which are equal almost everywhere with respect to the normalized Lebesgue measure m on $[0, 2\pi]$. It is well known that the set of functions $\{e_n(e^{i\theta}) = e^{in\theta} : n \in \mathbb{Z}\}$ defines an orthonormal basis for L^2. Furthermore, the mapping $\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_n e^{in\theta}$ defines an isometric isomorphism from H^2 into L^2. This mapping allows us to identify H^2 as the closed subspace of L^2 consisting of functions whose negative Fourier coefficients vanish, that is, $H^2 \equiv \widetilde{H}^2 := \{f \in L^2 : \langle f, e_n\rangle = 0 \text{ for all } n < 0\}$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$, the corresponding image in $\widetilde{H}^2$ is denoted by $\check{f}$ and is given by $\check{f}(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$.

Let H^∞ denote the space of all bounded analytic functions on the unit disc $\mathbb{D}$ and let $L^\infty := L^\infty(\mathbb{T})$ be the space of essentially bounded measurable functions on the unit circle $\mathbb{T}$, defined modulo equality almost everywhere with respect to the normalized Lebesgue measure on $[0, 2\pi]$. The space H^∞, equipped with the supremum norm $\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{D}\}$, is a Banach algebra under pointwise operations. Moreover, H^∞ embeds into the Hardy space and is isometrically isomorphic to $\widetilde{H}^\infty := L^\infty \cap \widetilde{H}^2$.

A function $\varphi \in H^\infty$ is called an *inner function* if it satisfies $|\breve{\varphi}(e^{i\theta})| = 1$ for almost every $\theta \in [0, 2\pi]$. A function $\varphi \in L^\infty$ is said to be of *bounded type* if there exist $\varphi_1, \varphi_2 \in \widetilde{H}^\infty$ such that $\varphi = \frac{\varphi_1}{\varphi_2}$.

From now onwards, we will use the notation H^2 to refer to both H^2 and $\widetilde{H}^2$ and similarly, H^∞ to denote both H^∞ and $\widetilde{H}^\infty$.

Toeplitz operators

For $\varphi \in L^\infty$, the *multiplication operator* M_φ on L^2 is defined by $M_\varphi f := \varphi f$, for all $f \in L^2$. Let P denote the orthogonal projection of L^2 onto H^2. For $\varphi \in L^\infty$, the *Toeplitz operator* T_φ with symbol φ on H^2 is defined by $T_\varphi f := (PM_\varphi)f = P(\varphi f)$, $f \in H^2$. Equivalently, an operator $T \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $S^*TS = T$, where S is the unilateral shift operator on $\ell^2(\mathbb{W})$. If $\varphi \in H^\infty$, then the Toeplitz operator T_φ is called an *analytic Toeplitz operator.* It is referred to as a *co-analytic Toeplitz operator* if its adjoint $T_\varphi^* = T_{\overline{\varphi}}$ is analytic. For $\varphi \in H^\infty$, the *Hankel operator* H_φ on H^2 is defined by $H_\varphi f := JP^\perp M_\varphi f$ for $f \in H^2$, where $P^\perp$ denotes the orthogonal projection of L^2 onto $L^2 \ominus H^2$ and J is an operator on L^2 defined by $J(g)(e^{i\theta}) := e^{-i\theta} g(e^{-i\theta})$, $g \in L^2$.

The following lemma by M. B. Abrahamse [4], related to the bounded type symbols, plays a crucial role in the subsequent development.

Lemma 1.1. [4] Let $\varphi \in L^\infty$, then the following statements are equivalent:

(i) φ is of bounded type;

(ii) $P(\overline{\varphi})$ is noncyclic for S^*;

(iii) $\ker H_\varphi \neq \{0\}$.

For more details regarding Hardy space and Toeplitz operators, see [4, 27, 42, 46].

Halmos' Problem 5

P. R. Halmos studied the characterization of subnormal Toeplitz operators by the properties of their symbol. He observed that every normal operator is subnormal and that every analytic Toeplitz operator is subnormal. Also, he thought about the converse and posed a problem in 1970, which is the well-celebrated Halmos' Problem 5 [35]. The problem states that

"Is every subnormal Toeplitz operator either normal or analytic?"

A Toeplitz operator is self-adjoint if and only if its symbol is real-valued almost everywhere [42, Theorem 3.2.15]. The class of normal Toeplitz operators was characterized by A. Brown and P. R. Halmos [11, Corollary], who showed that the only normal Toeplitz operators are those that are linear functions of self-adjoint ones. Later, a complete characterization of hyponormal Toeplitz operators was given by C. Cowen [16].

Cowen's Theorem. [16, 44] For each $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty, where

$$\mathcal{E}(\varphi) = \{k \in H^\infty(\mathbb{T}) : \|k\|_\infty \leq 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty\}.$$

In 1976, M. B. Abrahamse [4] gave a partial answer to Halmos' Problem 5 as follows:

Abrahamse's Theorem. [4, Corollary A] If $\varphi \in L^\infty$ and φ or $\overline{\varphi}$ is of bounded type, then the subnormal Toeplitz operator T_φ is either normal or analytic.

In [4], M. B. Abrahamse raised several questions, including the following:

"Is the Bergmann shift unitarily equivalent to a Toeplitz operator?"

S. Shunhua [52] provided a negative answer to this question. Motivated by Shunhua's approach, C. Cowen and J. Long [18] presented a negative solution to Halmos' Problem 5 in 1984 by constructing an example for a subnormal Toeplitz operator which is neither normal nor analytic.

Example 1.2. [18, Theorem] For $0 < \alpha < 1$, let ψ be a conformal map of the unit disk onto the interior of the ellipse with vertices $\pm i(1-\alpha)^{-1}$ and passing through the points $\pm i(1+\alpha)^{-1}$. Then $T_{\psi+\alpha\overline{\psi}}$ is a subnormal weighted shift operator that is neither normal nor analytic.

The negative solution to Halmos' Problem 5 gives rise to the following two natural questions:

"Which Toeplitz operators are subnormal?"

"Which subnormal Toeplitz operators are either normal or analytic?"

In recent years, similar questions have also been explored in the context of block Toeplitz operators [22, 23, 32]. Before proceeding, we first present a brief overview of the theory of Banach-valued functions.

Banach-valued functions

Let $(\Omega, \mathcal{F}, \mu)$ be a positive σ-finite measure space and X be a Banach space. A Banach-valued function $f : \Omega \to X$ is called *countable-valued* if it can be expressed in the form $f = \sum_{k=1}^{\infty} x_k \chi_{E_k}$, where $x_k \in X$, $E_k \in \mathcal{F}$ and the sets $\{E_k\}$ are pairwise disjoint, that is, $E_i \cap E_j = \emptyset$ for $i \neq j$. A function $f : \Omega \to X$ is called *weakly measurable* if for every $x^* \in X^*$, the scalar-valued function $x^* \circ f : \Omega \to \mathbb{C}$ is measurable, where X^* denotes the dual of X. The function f is said to be *strongly* *measurable* if there exists a sequence of countable-valued functions $\{f_n\}$ on Ω such that $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ for almost every $\omega \in \Omega$.

A Banach-valued function $f : \mathbb{T} \to X$ is said to be *essentially separably valued* if there exists a measurable subset $\mathbb{T}'$ of $\mathbb{T}$ such that the image $f(\mathbb{T}')$ is separable and $m(\mathbb{T} \setminus \mathbb{T}') = 0$. The following theorem, known as the Pettis measurability Theorem, addresses the connection between weak measurability and strong measurability.

Pettis measurability Theorem. [38, 47] For a function $f : \mathbb{T} \to X$, the following are equivalent:

(i) f is strongly measurable;

(ii) f is essentially separably valued and weakly measurable.

In particular, if X is a separable Banach space, then f is strongly measurable if and only if it is weakly measurable.

A countable-valued function $f : \Omega \to X$ is said to be (*Bochner*) *integrable* if $\int_\Omega \|f(s)\|\, d\mu(s) < \infty$ and its integral is defined by $\int_\Omega f d\mu = \sum_{k=1}^{\infty} x_k\, \mu(E_k)$. A function $g : \Omega \to X$ is said to be (*Bochner*) *integrable* if there exists a sequence $\{f_n\}$ of countable-valued (Bochner) integrable functions f_n satisfying $\lim_{n\to\infty} f_n(s) = g(s)$ for almost all $s \in \Omega$ and $\lim_{n\to\infty} \int_\Omega \|g - f_n\|\, d\mu = 0$. Then, $\int_\Omega g\, d\mu = \lim_{n\to\infty} \int_\Omega f_n\, d\mu$ exists and $\int_\Omega g\, d\mu$ is called the (*Bochner*) *integral* of g. If $f : \Omega \to X$ is integrable, then we see that

$$T\left(\int_\Omega f d\mu\right) = \int_\Omega \big(Tf\big) d\mu \quad \text{for each } T \in \mathcal{B}(X, Y), \tag{1.2}$$

where Y is a Banach space and $\mathcal{B}(X, Y)$ denotes the space of all bounded linear operators from X to Y. For a complex Banach space X and $1 \le p \le \infty$, define the Banach space $L^p(\mathbb{T}, X)$ by

$$L^p(\mathbb{T}, X) := \{f : \mathbb{T} \to X : f \text{ is strongly measurable and } \|f\|_p < \infty\}, \tag{1.3}$$

where the norm is given by

$$\|f\|_p \equiv \|f\|_{L^p(\mathbb{T},X)} := \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(e^{i\theta})\|_X^p d\theta\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty; \\ \operatorname*{ess\,sup}_{\theta\in[0,2\pi]} \|f(e^{i\theta})\|_X & \text{if } p = \infty. \end{cases} \tag{1.4}$$

Given $f \in L^1(\mathbb{T}, X)$ and $n \in \mathbb{Z}$, the n-th *Fourier coefficient* of f, denoted by $\hat{f}(n)$, is defined by $\hat{f}(n) := \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(e^{i\theta})\, d\theta$. For $1 \leq p \leq \infty$, the Hardy space $H^p(\mathbb{T}, X)$ is defined as the subspace of $L^p(\mathbb{T}, X)$ consisting of those functions f for which $\hat{f}(n) = 0$ for all $n < 0$.

A function $f : \mathbb{D} \to X$ is said to be *analytic* if it admits a power series representation of the form $f(z) = \sum_{n=0}^{\infty} x_n z^n$, where $x_n \in X$ and $z \in \mathbb{D}$. Let $\mathrm{Hol}(\mathbb{D}, X)$ denote the set of all such analytic functions. Also, we write $H^2(\mathbb{D}, X)$ for the set of all $f \in \mathrm{Hol}(\mathbb{D}, X)$ satisfying $\|f\|_{H^2(\mathbb{D},X)} := \sup_{0<r<1}\big(\int_0^{2\pi} \|f(re^{i\theta})\|_X^2\, d\theta\big)^{\frac{1}{2}} < \infty$.

For a separable complex Hilbert space $\mathcal{H}$, if $f \in H^2(\mathbb{D}, \mathcal{H})$, then there exists a boundary function $\check{f} \in H^2(\mathbb{T}, \mathcal{H})$ such that $f(z) = \frac{1}{2\pi}\int_0^{2\pi} P_z(e^{i\theta})\check{f}(e^{i\theta})\, d\theta$, where $P_z(e^{i\theta}) = \frac{1-|z|^2}{|1-\bar{z}e^{i\theta}|^2}$ is the *Poisson kernel* and θ ranges from 0 to 2π. It can be observed that $\check{f}(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ nontangentially almost everywhere on $\mathbb{T}$. Moreover, the mapping $f \mapsto \check{f}$ defines an isometric isomorphism between $H^2(\mathbb{D}, \mathcal{H})$ and $H^2(\mathbb{T}, \mathcal{H})$ [47, Theorem 3.11.7]. Consequently, we identify $H^2(\mathbb{D}, \mathcal{H})$ with $H^2(\mathbb{T}, \mathcal{H})$. For f and g in $L^2(\mathbb{T}, \mathcal{H})$, the inner product $\langle f, g\rangle$ is defined by

$$\langle f, g\rangle \equiv \big\langle f(e^{i\theta}), g(e^{i\theta})\big\rangle_{L^2(\mathbb{T},\mathcal{H})} := \frac{1}{2\pi}\int_0^{2\pi} \langle f(e^{i\theta}), g(e^{i\theta})\rangle_{\mathcal{H}}\, d\theta. \tag{1.5}$$

A function $\Phi : \mathbb{T} \to \mathcal{B}(X, Y)$ is said to be *strong operator topology (SOT) measurable* if for every $x \in X$, the function $\Phi x : \mathbb{T} \to Y$ defined by $\Phi x(e^{i\theta}) = \Phi(e^{i\theta})x$ is strongly measurable. Similarly, Φ is said to be *weak operator topology (WOT) measurable* if the same funtion is weakly measurable for every $x \in X$. Let D and E be separable complex Hilbert spaces. Then a function $\Phi : \mathbb{T} \to \mathcal{B}(D, E)$ is SOT measurable if and only if it is WOT measurable (see [47], page 46).

For $1 \leq p \leq \infty$, a function $\Phi : \mathbb{T} \to \mathcal{B}(D, E)$ is called a *strong L^p-function* if for each $x \in D$, the map $\Phi x : \mathbb{T} \to E$ belong to $L^p(\mathbb{T}, E)$. The space of all such functions is denoted by $L^p_s(\mathbb{T}, \mathcal{B}(D, E))$. It is clear that $L^p(\mathbb{T}, \mathcal{B}(D, E)) \subseteq L^p_s(\mathbb{T}, \mathcal{B}(D, E))$. The notion of strong L^2-function was introduced by V. Peller [50] and its formal theory was further developed in [26].

If $\Phi \in L^1_s(\mathbb{T}, \mathcal{B}(D, E))$ and $x \in D$, then the function Φx belongs to $L^1(\mathbb{T}, E)$. For each $n \in \mathbb{Z}$, the n-th *Fourier coefficient* of Φ, denoted by $\hat{\Phi}(n)$, is defined by $\hat{\Phi}(n)x := \widehat{\Phi x}(n)$. We define

$$H^p_s(\mathbb{T}, \mathcal{B}(D, E)) := \{\Phi \in L^p_s(\mathbb{T}, \mathcal{B}(D, E)) : \hat{\Phi}(n) = 0 \text{ for } n < 0\}.$$

Let $L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(D, E))$ denote the space of all bounded SOT-measurable functions from $\mathbb{T}$ into $\mathcal{B}(D, E)$ and let

$$H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(D, E)) := \big\{\Phi \in L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(D, E)) : \Phi x \in H^\infty(\mathbb{T}, E) \text{ for all } x \in D\big\}.$$

Moreover, it can be observed that

$$L^\infty(\mathbb{T}, \mathcal{B}(D, E)) \subseteq L^\infty_s(\mathbb{T}, \mathcal{B}(D, E)) = L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(D, E)).$$

On the other hand, we define $H^\infty(\mathbb{D}, \mathcal{B}(D, E))$ as the set of all analytic functions $\Phi : \mathbb{D} \to \mathcal{B}(D, E)$ such that $\|\Phi\|_{H^\infty(\mathbb{D}, \mathcal{B}(D,E))} := \sup_{z \in \mathbb{D}} \|\Phi(z)\| < \infty$. By convention, we identify $H^\infty(\mathbb{D}, \mathcal{B}(D, E))$ with $H^\infty_s(\mathbb{T}, \mathcal{B}(D, E))$ [47, Theorem 3.11.10]. If $\mathcal{B}(D, E)$ is separable, then we see that $L^\infty(\mathbb{T}, \mathcal{B}(D, E)) = L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(D, E))$ (see [47], page 46).

Let $\Phi \in L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$. Then the *Toeplitz operator* T_Φ on $H^2(\mathbb{T}, \mathcal{H})$ with symbol Φ is defined by $T_\Phi f := P_+(\Phi f)$, where P_+ denotes the orthogonal projection of $L^2(\mathbb{T}, \mathcal{H})$ onto $H^2(\mathbb{T}, \mathcal{H})$ and $f \in H^2(\mathbb{T}, \mathcal{H})$. Similarly, the *Hankel operator* H_Φ with symbol Φ is defined by $H_\Phi f := JP_- f$, where P_- denotes the orthogonal projection of $L^2(\mathbb{T}, \mathcal{H})$ onto $L^2(\mathbb{T}, \mathcal{H}) \ominus H^2(\mathbb{T}, \mathcal{H})$ and J is the unitary operator on $L^2(\mathbb{T}, \mathcal{H})$ given by $Jg(e^{i\theta}) := e^{-i\theta} I_{\mathcal{H}}\, g(e^{-i\theta})$ ($I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$).

For a function $\Phi : \mathbb{T} \to \mathcal{B}(\mathcal{H})$, we denote $\breve{\Phi}(e^{i\theta}) := \Phi(e^{-i\theta})$ and $\widetilde{\Phi} := \breve{\Phi}^*$. Let $\Phi \in L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$ and $\Psi \in H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$. The following identities are established in [26]:

$$T_\Phi^* = T_{\Phi^*} \text{ and } H_\Phi^* = H_{\widetilde{\Phi}}; \tag{1.6}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi} \text{ and } H_{\Psi\Phi} = T_{\widetilde{\Psi}}^* H_\Phi. \tag{1.7}$$

Extensive studies on operator-valued Toeplitz and Hankel operators are presented in [9, 26, 27, 28, 45, 47].

Toeplitz operators with matrix-valued symbols $\Phi \in L^\infty(\mathbb{T}, M_n)$ are often referred to as *block Toeplitz operators,* where M_n denotes the set of all $n \times n$ complex matrices. For a fixed natural number n, let P_n and $P_n^\perp$ denote the orthogonal projection of $L^2(\mathbb{T}, \mathbb{C}^n)$ onto $H^2(\mathbb{T}, \mathbb{C}^n)$ and $L^2(\mathbb{T}, \mathbb{C}^n) \ominus H^2(\mathbb{T}, \mathbb{C}^n)$, respectively. By identifying the vector-valued Hardy space $H^2(\mathbb{T}, \mathbb{C}^n)$ with the direct sum

$$H^2(\mathbb{T}, \mathbb{C}^n) = H^2 \oplus H^2 \oplus \cdots \oplus H^2 \quad (n \text{ times}),$$

the Toeplitz and Hankel operators with matrix-valued symbol Φ take the block matrix form

$$T_\Phi = \begin{bmatrix} T_{\Phi_{11}} & \cdots & T_{\Phi_{1n}} \\ \vdots & \ddots & \vdots \\ T_{\Phi_{n1}} & \cdots & T_{\Phi_{nn}} \end{bmatrix}, \quad H_\Phi = \begin{bmatrix} H_{\Phi_{11}} & \cdots & H_{\Phi_{1n}} \\ \vdots & \ddots & \vdots \\ H_{\Phi_{n1}} & \cdots & H_{\Phi_{nn}} \end{bmatrix}, \tag{1.8}$$

where $\Phi = [\phi_{ij}]_{1 \le i,j \le n} \in L^\infty(\mathbb{T}, M_n)$ and each $\phi_{ij} \in L^\infty$. A function $\Theta \in H^\infty(\mathbb{T}, M_n)$ is called an *inner function* if $\Theta^*\Theta = I_n$ almost everywhere on $\mathbb{T}$. For notational convenience, we denote by $\mathcal{H}(\Theta) := H^2(\mathbb{T}, \mathbb{C}^n) \ominus \Theta H^2(\mathbb{T}, \mathbb{C}^n)$ the model space generated by the inner function $\Theta \in H^\infty(\mathbb{T}, M_n)$. The following basic properties of Toeplitz and Hankel operators, involving the symbols $\Phi, \Psi \in L^\infty(\mathbb{T}, M_n)$ and $\Theta \in H^\infty(\mathbb{T}, M_n)$, are used implicitly in the sequel:

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi; \tag{1.9}$$

$$H^*_\Phi H_\Phi - H^*_{\Theta\Phi} H_{\Theta\Phi} = H^*_\Phi H_{\Theta^*} H^*_{\Theta^*} H_\Phi. \tag{1.10}$$

For $m, n \in \mathbb{N}$, let $M_{m\times n}$ denote the set of all $m \times n$ complex matrices. Given a function $\Phi \in H^2(\mathbb{T}, M_{n\times r})$, an inner function $\Delta \in H^2(\mathbb{T}, M_{n\times m})$ is a called a *left inner divisor* of Φ if $\Phi = \Delta A$ for some $A \in H^2(\mathbb{T}, M_{m\times r})$ $(m \le n)$. Note that if $\Phi \in L^\infty(\mathbb{T}, M_n)$ and $\det \Phi$ is not identically zero, then any left inner divisor Δ of Φ belongs to $H^2(\mathbb{T}, M_n)$ [22]. Two functions $\Phi \in H^2(\mathbb{T}, M_{n\times r})$ and $\Psi \in H^2(\mathbb{T}, M_{n\times m})$ are said to be *left coprime* if the only common left inner divisor of both Φ and Ψ is a constant unitary matrix. They are said to be *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Equivalently, two functions Φ and Ψ in $H^2(\mathbb{T}, M_n)$ with $\det \Phi \not\equiv 0$ and $\det \Psi \not\equiv 0$ are called right coprime if there does not exists a nonconstant inner function $\Delta \in H^2(\mathbb{T}, M_n)$ and functions $A, B \in H^2(\mathbb{T}, M_n)$ such that $\Phi = A\Delta$ and $\Psi = B\Delta$. We say that functions Φ and Ψ in $H^2(\mathbb{T}, M_n)$ are *coprime* if they are both left and right coprime.

A function $\Phi \in L^\infty(\mathbb{T}, M_n)$ is said to be of *bounded type* if each of its entries is of bounded type. Let $\Phi = [\phi_{ij}] \in L^\infty(\mathbb{T}, M_n)$ be such that Φ^* is of bounded type. Then by Lemma 1.1, each ϕ_{ij} can be written as $\phi_{ij} = \theta_{ij}\overline{b_{ij}}$, where θ_{ij} is inner and $b_{ij} \in H^2$. If θ denotes the least common inner multiple of $\theta_{ij}{}'$s, then we have

$$\Phi = [\theta\overline{a_{ij}}] = \Theta A^*, \tag{1.11}$$

where $\Theta = \theta I_n$ and $A = [a_{ij}] \in H^2(\mathbb{T}, M_n)$.

For a function $\Phi \in L^2(\mathbb{T}, M_n)$, we denote $\Phi_+ := \mathbb{P}_n(\Phi)$ and $\Phi_- := [\mathbb{P}_n^\perp(\Phi)]^*$, where $\mathbb{P}_n$ and $\mathbb{P}_n^\perp$ denote the orthogonal projections from $L^2(\mathbb{T}, M_n)$ onto $H^2(\mathbb{T}, M_n)$ and $L^2(\mathbb{T}, M_n) \ominus H^2(\mathbb{T}, M_n)$, respectively. It is clear that $\Phi_+ \in H^2(\mathbb{T}, M_n)$ and $\Phi_- \in zI_nH^2(\mathbb{T}, M_n)$. With this notation, we may write $\Phi = \Phi_-^* + \Phi_+$. Suppose that $\Phi \in L^\infty(\mathbb{T}, M_n)$ is such that both Φ and Φ^* are of bounded type. Then by Lemma

1.1, it follows that both Φ_+^* and Φ_-^* are of bounded type. In view of Equation (1.11), we write

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*, \tag{1.12}$$

where $\Theta_i = \theta_i I_n$ with an inner function θ_i and $A, B \in H^2(\mathbb{T}, M_n)$, for $i = 1, 2$. In this case, if the block Toeplitz operator T_Φ is hyponormal, then Θ_2 is a right inner divisor of Θ_1 [23, Lemma 3.1]. Hence, we can write

$$\Phi_+ = A^* \Theta_0 \Theta_2 \quad \text{and} \quad \Phi_- = B^* \Theta_2, \tag{1.13}$$

where $\Theta_0 = \theta_0 I_n$ for some inner function θ_0.

An $n \times n$ matrix-valued function Q is called a *finite Blaschke-Potapov product* if it can be expressed in the form

$$Q(z) = v \prod_{m=1}^{M} (b_m(z)\mathcal{P}_m + (I - \mathcal{P}_m)), \tag{1.14}$$

where v is an $n \times n$ constant unitary matrix, each b_m is a scalar *Blaschke factor* given by

$$b_m(z) = \frac{z - \alpha_m}{1 - \overline{\alpha}_m z}, \quad \alpha_m \in \mathbb{D}$$

and $\mathcal{P}_m$ is an orthogonal projection on $\mathbb{C}^n$. An $n \times n$ matrix-valued function Q is rational and inner if and only if it admits a representation as a finite Blaschke-Potapov product [51].

Chapter 2

SUBNORMAL TOEPLITZ OPERATORS WITH FINITE RANK SELF COMMUTATOR

The theory of subnormal operators has found extensive applications in various fields, including analytic function theory, differential geometry, approximation theory and quantum mechanics [53]. In particular, subnormal operators with finite rank self-commutator have been one of the interesting topics in functional analysis and operator theory. The finite rank condition suggests that the operator is nearly normal, thereby simplifying the spectral analysis. In 1973, B. B. Morrel [43] established that any subnormal operator whose self-commutator is of rank one can be written as a linear combination of the unilateral shift operator and the identity operator. Later in 1987, D. X. Xia [55, 56] made an attempt to classify all subnormal operators with finite rank self-commutator. In 1998, J. B. Conway and L. Yang [15] posed the following open problem:

Classify all subnormal operators whose self-commutator has finite rank.

In this chapter, we address this problem within the framework of Toeplitz operators in conjunction with Halmos' Problem 5 and establish a sufficient condition for subnormal Toeplitz operators with finite rank self-commutator to be classified as either normal or analytic, building on and modifying the methods of T. Nakazi and K. Takahashi [44].

2.1 Introduction

In 1993, T. Nakazi and K. Takahashi [44] studied the self-commutator of hyponormal Toeplitz operators and characterized the hyponormality of Toeplitz operators with finite rank self-commutator.

Theorem 2.1.1. [44, Theorem 10] The operator T_φ is hyponormal and the self-commutator $[T_\varphi^*, T_\varphi]$ is of finite rank if and only if there exists a finite Blaschke product b in $\mathcal{E}(\varphi)$. In this case, the Blaschke product b can be chosen such that deg $(b) = \text{Rank}\ [T_\varphi^*, T_\varphi]$.

In this chapter, we examine the following question in detail.

Problem 2.1.2. Which subnormal Toeplitz operators with finite rank self commutator are either normal or analytic?

We begin by discussing the historical developments related to Problem 2.1.2. The following result was proved by T. Nakazi and K. Takahashi [44] in 1993.

Nakazi-Takahashi Theorem. [44, Theorem 15] If T_φ is subnormal and $\varphi = q\overline{\varphi}$, where q is a finite Blaschke product, then T_φ is either normal or analytic.

Later, in 2012, R. E. Curto, I. S. Hwang and W. Y. Lee [23] posed a conjecture on the subnormality of Toeplitz operators whose self-commutator has finite rank.

Conjecture 2.1.3. [23, Conjecture 4.1] If T_φ is a subnormal Toeplitz operator with finite rank self commutator, then T_φ is either normal or analytic.

Furthermore, they provided positive evidence for Conjecture 2.1.3 by demonstrating its validity under specific conditions [23]. The following Theorem 2.1.4 offers significant insights into the subnormality of Toeplitz operators.

Theorem 2.1.4. [23, Corollary 4.3] Suppose T_φ is a subnormal Toeplitz operator with finite rank self-commutator. If $b \ker[T_\varphi^*, T_\varphi]$ is invariant under T_φ for some $b \in \mathcal{E}(\varphi)$, then T_φ is normal or analytic.

With these historical foundations, we now proceed to a rigorous formulation of Problem 2.1.2 and a detailed presentation of our key findings.

2.2 Subnormality of Toeplitz operators with finite rank self commutator

In this section, we provide further supporting evidence for Conjecture 2.1.3, thereby contributing to the ongoing investigation of its validity. Our results provide a partial answer to Problem 2.1.2, offering deep insights into a broader aspect of the conjecture. Moreover, Theorem 2.2.3 is a significant extension of the classical Nakazi-Takahashi Theorem to the setting of subnormal Toeplitz operators with finite rank self-commutator. We begin by presenting a couple of auxiliary lemmas, each of which contributes to a different aspect of the proof of Theorem 2.2.3.

Lemma 2.2.1. [44, Lemma 5] If $\varphi = q\overline{\varphi} + g$, where q is inner and $g \in H^\infty(\mathbb{T})$, then the closure of the range of $[T_\varphi^*, T_\varphi]$ equal to the closure of $T_{\varphi\overline{q}}(\mathcal{H}(q))$.

Lemma 2.2.2. Suppose $\varphi = q\overline{\varphi} + g \in L^\infty$, where q is an inner function, $g \in H^\infty$ and the Toeplitz operator T_φ is subnormal. If φ is not of bounded type, then $T_\varphi^*(\mathcal{H}(q)) \subseteq \overline{\text{Ran } [T_\varphi^*, T_\varphi]}$.

Proof. Initially, we establish that

$$T_\varphi^*(\mathcal{H}(q)) \bigcap \mathcal{H}(q) = \{0\}. \tag{2.1}$$

Since $\{0\} \subseteq T_\varphi^*(\mathcal{H}(q)) \bigcap \mathcal{H}(q)$, it suffices to show that

$$T_\varphi^*(\mathcal{H}(q)) \bigcap \mathcal{H}(q) \subseteq \{0\}.$$

Recall that if $z \in \mathcal{H}(q)$, then $z \perp qH^2$ which implies that $T_q^* z = 0$ and hence $q\overline{z} \in H^2$.

Let $y \in T_\varphi^*(\mathcal{H}(q)) \bigcap \mathcal{H}(q)$. Then $y \in \mathcal{H}(q)$ and there exists some $x \in \mathcal{H}(q)$ such that $y = T_\varphi^* x$. Therefore, we see that

$$0 = T_q^* y = T_q^* T_\varphi^* x = T_{\overline{q}} T_{\overline{\varphi}} x = T_{\varphi q}^* x$$

and hence $\varphi q\overline{x} \in H^2$. Since x belongs to $\mathcal{H}(q)$, we have $q\overline{x} \in H^2$ and this implies that $q\overline{x} \in \ker H_\varphi$. It is easy to see that H_φ is an injective map, since φ is not of bounded type (see Lemma 1.1). Thus, we get $q\overline{x} = 0$ and therefore, $x = 0$. This proves (2.1), because $y = T_\varphi^* x = 0$.

Since $\varphi = q\overline{\varphi} + g$, it follows that

$$T_{\overline{\varphi}}(\mathcal{H}(q)) \subseteq T_{\overline{q}\varphi}(\mathcal{H}(q)) + T_{\overline{g}}(\mathcal{H}(q)). \tag{2.2}$$

Furthermore, observe that for any $x \in \mathcal{H}(q)$,

$$\langle T_{\overline{g}} x, qH^2 \rangle = \langle \overline{g} x, qH^2 \rangle = \langle x, qH^2 \rangle = 0,$$

since $x \perp qH^2$ and g is bounded. Thus, $T_{\overline{g}}(\mathcal{H}(q)) \subseteq \mathcal{H}(q)$. Applying Lemma 2.2.1 and Equation (2.1) to Equation (2.2), we conclude that:

$$T_{\overline{\varphi}}(\mathcal{H}(q)) \subseteq \overline{\text{Ran}[T_\varphi^*, T_\varphi]} + \mathcal{H}(q) \subseteq \overline{\text{Ran}[T_\varphi^*, T_\varphi]}.$$

This completes the proof. □

Having established the necessary background, we are now in a position to extend the Nakazi-Takahashi Theorem to the case of scalar-valued Toeplitz operators with finite rank self-commutator.

Theorem 2.2.3. Let T_φ be a subnormal Toeplitz operator and assume that $\varphi = q\overline{\varphi} + g \in L^\infty(\mathbb{T})$, where q is a finite Blaschke product and $g \in H^\infty$. If $gH^2 \subseteq \text{Ran}\ T_\varphi$, then T_φ is normal or analytic.

Proof. If either φ or $\overline{\varphi}$ is of bounded type, then by Abrahamse's Theorem, the subnormal Toeplitz operator T_φ is either normal or analytic. Therefore, we may assume that φ is not of bounded type. It then follows from [4, Lemma 6] that $\overline{\varphi}$ is also not of bounded type.

We begin by showing that

$$\dim \mathcal{H}(q) \leq \dim \ker T_\varphi^*.$$

Since q is a finite Blaschke product and by Theorem 2.1.1, we see that $[T_\varphi^*, T_\varphi]$ is a finite rank operator. Moreover, $\operatorname{Ran}[T_\varphi^*, T_\varphi]$ is invariant under T_φ^*, because T_φ is subnormal. By applying Lemma 2.2.2, we obtain

$$T_\varphi^*\Big(\operatorname{Ran}[T_\varphi^*, T_\varphi] + \mathcal{H}(q)\Big) \subseteq \operatorname{Ran}[T_\varphi^*, T_\varphi]. \tag{2.3}$$

Also, by [44, Lemma 6], we have $\operatorname{Ran}[T_\varphi^*, T_\varphi] \bigcap \mathcal{H}(q) = \{0\}$. This implies that

$$\dim\Big(\operatorname{Ran}[T_\varphi^*, T_\varphi] + \mathcal{H}(q)\Big) = \operatorname{Rank}[T_\varphi^*, T_\varphi] + \dim(\mathcal{H}(q)). \tag{2.4}$$

Now, by the Rank Theorem and Equation 2.3, we see that

$$\begin{aligned}
\dim\Big(\operatorname{Ran}[T_\varphi^*, T_\varphi] + \mathcal{H}(q)\Big) &= \dim\Big(\ker T^*_{\varphi|(\operatorname{Ran}[T_\varphi^*, T_\varphi]+\mathcal{H}(q))}\Big) \\
&\quad + \dim\Big(T_\varphi^*\Big(\operatorname{Ran}[T_\varphi^*, T_\varphi] + \mathcal{H}(q)\Big)\Big) \\
&\leq \dim\Big(\ker T^*_{\varphi|(\operatorname{Ran}[T_\varphi^*, T_\varphi]+\mathcal{H}(q))}\Big) + \operatorname{Rank}[T_\varphi^*, T_\varphi].
\end{aligned}$$

Comparing with (2.4), we obtain

$$\dim(\mathcal{H}(q)) \leq \dim\Big(\ker T^*_{\varphi|(\operatorname{Ran}[T_\varphi^*, T_\varphi]+\mathcal{H}(q))}\Big) \leq \dim\ker T_\varphi^*. \tag{2.5}$$

Next, we establish that

$$T_\varphi(\ker T_\varphi^*) \subseteq \mathcal{H}(q). \tag{2.6}$$

Since $gH^2 \subseteq \operatorname{Ran} T_\varphi$, it follows that

$$\ker T_\varphi^* = \ker T_{\overline{\varphi}} \subseteq H^2 \ominus gH^2.$$

That is, $x \perp gH^2$ whenever $x \in \ker T_\varphi^*$. Moreover, if $x \in \ker T_\varphi^*$, we see that

$$T_{\overline{q}} T_\varphi x = T_{\overline{q}\varphi} x = T_{\overline{\varphi}} x - T_{\overline{g}} x = 0.$$

This implies that $T_\varphi x \perp qH^2$ for all $x \in \ker T_\varphi^*$. Hence, the conclusion (2.6) is established.

Proceeding further, we show that $\mathcal{H}(q) \subseteq \text{Ran } T_\varphi$. Recall that either T_φ or T_φ^* is injective by Coburn's Theorem [42, Theorem 3.3.10]. Since T_φ is hyponormal, we obtain that $\ker T_\varphi$ is contained in $\ker T_\varphi^*$. These two facts together imply that T_φ is injective. Thus, we have $\dim \ker T_\varphi^* \leq \dim \mathcal{H}(q)$ by Equation (2.6). In conjunction with (2.5), we get $\dim \ker T_\varphi^* = \dim \mathcal{H}(q)$ and $T_\varphi(\ker T_\varphi^*) = \mathcal{H}(q)$. Thus, we obtain $\mathcal{H}(q) \subseteq \text{Ran}\, T_\varphi$, as desired.

We now move on to show that

$$q^n \mathcal{H}(q) \subseteq \text{Ran}\, T_\varphi \quad \text{for all } n = 0, 1, 2, \ldots. \tag{2.7}$$

Recall, $\overline{q}x \in H^2$ is equivalent to $x \in qH^2$. Therefore, we have $\ker H_{\overline{q}} = qH^2$, which implies that $\text{Ran}\, H_{\overline{q}}^* = \mathcal{H}(q)$. Since $\mathcal{H}(q) \subseteq \text{Ran}\, T_\varphi$, it follows that

$$\text{Ran}\, H_{\overline{q}}^* \subseteq \text{Ran}\, T_\varphi.$$

We now use the identity

$$T_\varphi T_q - T_q T_\varphi = H_{\overline{q}}^* H_\varphi$$

and the fact that $T_\varphi(\ker T_\varphi^*) = \mathcal{H}(q)$ to compute:

$$q\mathcal{H}(q) = T_q T_\varphi(\ker T_\varphi^*) = (T_\varphi T_q - H_{\overline{q}}^* H_\varphi)(\ker T_\varphi^*) \subseteq \text{Ran}\, T_\varphi. \tag{2.8}$$

Since $q\mathcal{H}(q) \subseteq \text{Ran}\, T_\varphi$, we obtain

$$q^2 \mathcal{H}(q) \subseteq T_q T_\varphi H^2 = (T_\varphi T_q - H_{\overline{q}}^* H_\varphi) H^2 \subseteq \text{Ran}\, T_\varphi.$$

By proceeding inductively, we conclude that (2.7) holds.

We now proceed by considering two distinct cases, depending on whether q is constant or non-constant. If q is constant, then by Theorem 2.1.1, $\text{Rank}[T_\varphi^*, T_\varphi] = 0$. That is, T_φ is normal.

Now, assume that q is non-constant. We will show that this assumption leads to a contradiction. First, we claim that

$$H^2 = \bigoplus_{n=0}^{\infty} q^n \mathcal{H}(q).$$

Observe that for each $n = 0, 1, 2, \ldots$, we have

$$q^n \mathcal{H}(q) = (q^n H^2) \ominus (q^{n+1} H^2).$$

Then, by the Projection Theorem, it follows that

$$q^n H^2 = (q^{n+1} H^2) \oplus (q^n \mathcal{H}(q)) \quad \text{for all } n = 0, 1, 2, \ldots.$$

Repeated application of this decomposition yields:

$$H^2 = qH^2 \oplus \mathcal{H}(q) = \mathcal{H}(q) \oplus q\mathcal{H}(q) \oplus q^2 H^2 = \cdots = \bigoplus_{j=0}^{n} q^j \mathcal{H}(q) \oplus q^{n+1} H^2.$$

Since this decomposition does not terminate in a finite number of steps, we have the equality

$$H^2 = \bigoplus_{n=0}^{\infty} q^n \mathcal{H}(q).$$

Therefore by Equation (2.7), we have $H^2 \subseteq \text{Ran } T_\varphi$ and hence T_φ^* is injective. However, this contradicts the fact that

$$0 = \dim \ker T_\varphi^* = \dim \mathcal{H}(q) \geq 1,$$

since $\mathcal{H}(q)$ is nontrivial when q is non-constant. This completes the proof. □

Remark 2.2.4. It is important to observe that if we set $g = 0$ in Theorem 2.2.3, the result reduces to the classical Nakazi-Takahashi Theorem. In this case, the given decomposition of φ reduces to $\varphi = q\overline{\varphi}$, where q is a finite Blaschke product. This special case corresponds precisely with the setting of the Nakazi–Takahashi Theorem, thus establishing Theorem 2.2.3 as a natural extension of the classical result.

Remark 2.2.5. In the proof of Theorem 2.2.3, we do not use any properties of subnormal operators beyond the invariance of $\ker[T_\varphi^*, T_\varphi]$ under T_φ. As a consequence, we obtain the following result:

Suppose that T_φ *is hyponormal and that* $\ker[T_\varphi^*, T_\varphi]$ *is invariant under* T_φ*. Further, assume that* $\varphi \in L^\infty$ *satisfies the representation* $\varphi = q\overline{\varphi} + g$*, where* q *is a finite Blaschke product,* $g \in H^\infty$ *and* $gH^2 \subseteq Ran\ T_\varphi$*. Then* T_φ *is normal or analytic.*

Remark 2.2.6. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *2-hyponormal* if Equation (1.1) holds for $k = 2$. It was established by R. E. Curto and W. Y. Lee [21] that the kernel of the self-commutator is invariant under any 2-hyponormal operator. Therefore, Theorem 2.2.3 continues to hold if the subnormality condition on T_φ is replaced with 2-hyponormality.

The following example establishes that the condition $gH^2 \subseteq$ Ran T_ϕ in Theorem 2.2.3 is not necessary.

Example 2.2.7. Consider the unilateral shift operator T_z. It is clear that T_z is subnormal and that the self-commutator $[T_z^*, T_z] = T_z^* T_z - T_z T_z^*$ is of finite rank. Moreover, observe that

$$z = z\overline{z} + z - 1$$

on the boundary $\mathbb{T}$. Here, $z - 1$ belongs to $(z-1)H^2$, but doesn't belongs to zH^2. Hence, the condition $gH^2 \subseteq$ Ran T_z fails in this case, even though the conclusions of Theorem 2.2.3 still hold. This shows that the condition is not necessary.

Chapter 3

HYPONORMAL BLOCK TOEPLITZ OPERATORS WITH FINITE RANK SELF COMMUTATOR

In this chapter, we introduce the problem of characterizing subnormal operators with finite rank self-commutator in the setting of block Toeplitz operators, in conjunction with Halmos' Problem 5. The primary objective of this investigation is to identify subnormal block Toeplitz operators whose self-commutator has finite rank. Significant progress in the scalar case was made by T. Nakazi and K. Takahashi [44], who characterized hyponormal Toeplitz operators with finite rank self-commutator in terms of the Cowen set of the symbol (see Theorem 2.1.1). This naturally leads to the question of whether similar characterizations extend to the framework of block Toeplitz operators. Motivated by this question and building on a conjecture proposed by R. E. Curto, I. S. Hwang and W. Y. Lee [22], this chapter aims to investigate the validity of analogous results for block Toeplitz operators. In addition, we explore the underlying relationship between the Cowen set of the symbol and various operator-theoretic properties associated with hyponormal block Toeplitz operators.

We begin by examining certain properties of hyponormal block Toeplitz operators whose symbols are trigonometric polynomials.

3.1 Block Toeplitz operators with trigonometric polynomial symbol and hyponormality

Let $\Phi \in L^\infty(\mathbb{T}, M_n)$. The function Φ is said to be *normal*, *subnormal* or *hyponormal* if $\Phi(e^{i\theta})$ is normal, subnormal or hyponormal, respectively, for almost every $e^{i\theta} \in \mathbb{T}$. A characterization of hyponormal block Toeplitz operators was given by C. Gu, J. Hendricks and D. Rutherford [32], as stated below.

Hyponormality of block Toeplitz operators. [32, Theorem 3.3] For each Φ in $L^\infty(\mathbb{T}, M_n)$, let

$$\mathcal{E}(\Phi) := \Big\{K \in H^\infty(\mathbb{T}, M_n) : \|K\|_\infty \leq 1 \quad \text{and} \quad \Phi - K\Phi^* \in H^\infty(\mathbb{T}, M_n)\Big\}.$$

Then, the block Toeplitz operator T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

This characterization of hyponormality for block Toeplitz operators is analogous to Cowen's Theorem with the notable addition of a normality condition on the symbol.

In the case of scalar-valued symbols, it has been observed that if the symbol $\varphi \in L^\infty$ of a Toeplitz operator T_φ is a trigonometric polynomial, then the self-commutator $[T_\varphi^*, T_\varphi]$ is of finite rank. To see this, let $\varphi(e^{i\theta}) = \sum_{n=-k}^{m} a_n e^{in\theta} \in L^\infty$ be a trigonometric polynomial, where $k, m \in \mathbb{W}$. Recall that for $\varphi, \psi \in L^\infty$, the identity

$$T_{\varphi\psi} - T_\varphi T_\psi = H^*_{\overline{\varphi}} H_\psi$$

holds. Using this, we obtain

$$T_\varphi^* T_\varphi = T_{\overline{\varphi}\varphi} - H_\varphi^* H_\varphi \quad \text{and} \quad T_\varphi T_\varphi^* = T_{\varphi\overline{\varphi}} - H_{\overline{\varphi}}^* H_{\overline{\varphi}},$$

which shows that

$$[T_\varphi^*, T_\varphi] = T_{\overline{\varphi}\varphi} - T_{\varphi\overline{\varphi}} + H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi = H_{\overline{\varphi}}^* H_{\overline{\varphi}} - H_\varphi^* H_\varphi.$$

Since φ is a trigonometric polynomial, the Hankel operators $H_{\overline{\varphi}}^*, H_{\overline{\varphi}}, H_\varphi^*, H_\varphi$ are all of finite rank. Consequently, the self-commutator $[T_\varphi^*, T_\varphi]$ is a finite rank operator.

However, this result does not generally extend to the case of block Toeplitz operators. To understand the limitations of such an extension, we begin by presenting the following lemma, which plays a key role in the subsequent discussion.

Lemma 3.1.1. Let $\Phi \in L^\infty(\mathbb{T}, M_n)$ be such that T_Φ is a hyponormal operator and let $K \in \mathcal{E}(\Phi)$. Then the self-commutator of T_Φ satisfies

$$[T_\Phi^*, T_\Phi] = H_{\Phi^*}^* (I - T_{\widetilde{K}} T_{\widetilde{K}^*}) H_{\Phi^*},$$

where $\widetilde{K}(e^{i\theta}) = K^*(e^{-i\theta})$.

Proof. Recall from Equation (1.9) that

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H_{\Phi^*}^* H_\Psi \quad \text{for all } \Phi, \Psi \in L^\infty(\mathbb{T}, M_n).$$

Using this identity, we observe the following:

$$T_\Phi^* T_\Phi = T_{\Phi^*\Phi} - H_{\Phi^*} H_\Phi \quad \text{and} \quad T_\Phi T_\Phi^* = T_{\Phi\Phi^*} - H_{\Phi^*}^* H_{\Phi^*}.$$

This implies that

$$[T_\Phi^*, T_\Phi] = T_{\Phi^*\Phi - \Phi\Phi^*} + H_{\Phi^*}^* H_{\Phi^*} - H_{\Phi^*} H_\Phi.$$

Since T_Φ is hyponormal, it follows that Φ is normal. Consequently, we obtain

$$[T_\Phi^*, T_\Phi] = H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi.$$

Given $K \in \mathcal{E}(\Phi)$, then there exists $G \in H^\infty(\mathbb{T}, M_n)$ such that

$$\Phi = K\Phi^* + G.$$

Thus, we see that $H_\Phi = H_{K\Phi^* + G} = H_{K\Phi^*}$ and hence $H_\Phi^* = H_{K\Phi^*}^*$. Therefore, the self-commutator of T_Φ becomes

$$[T_\Phi^*, T_\Phi] = H_{\Phi^*}^* H_{\Phi^*} - H_{K\Phi^*}^* H_{K\Phi^*}.$$

From Equation (1.7), it follows that

$$H_{K\Phi^*} = T_{\widetilde{K}}^* H_{\Phi^*} \quad \text{and} \quad H_{K\Phi^*}^* = (T_{\widetilde{K}}^* H_{\Phi^*})^* = H_{\Phi^*}^* T_{\widetilde{K}}.$$

By the above identities, we obtain

$$\begin{aligned}[T_\Phi^*, T_\Phi] &= H_{\Phi^*}^* H_{\Phi^*} - H_{K\Phi^*}^* H_{K\Phi^*} \\ &= H_{\Phi^*}^* H_{\Phi^*} - H_{\Phi^*}^* T_{\widetilde{K}} T_{\widetilde{K}}^* H_{\Phi^*} \\ &= H_{\Phi^*}^* (I - T_{\widetilde{K}} T_{\widetilde{K}}^*) H_{\Phi^*}.\end{aligned}$$

This completes the proof. □

Now, consider the matrix-valued trigonometric polynomial symbol

$$\Phi(e^{i\theta}) := \begin{bmatrix} e^{i\theta} & 1 \\ 0 & e^{-i\theta} \end{bmatrix} \in L^\infty(\mathbb{T}, M_2).$$

In this case, the self-commutator $[T_\Phi^*, T_\Phi]$ of the block Toeplitz operator T_Φ is infinite dimensional. To illustrate this, observe the following calculation:

$$\begin{aligned}[T_\Phi^*, T_\Phi] &= \begin{bmatrix} T_{e^{-i\theta}} & 0 \\ I & T_{e^{i\theta}} \end{bmatrix} \begin{bmatrix} T_{e^{i\theta}} & I \\ 0 & T_{e^{-i\theta}} \end{bmatrix} - \begin{bmatrix} T_{e^{i\theta}} & I \\ 0 & T_{e^{-i\theta}} \end{bmatrix} \begin{bmatrix} T_{e^{-i\theta}} & 0 \\ I & T_{e^{i\theta}} \end{bmatrix} \\ &= \begin{bmatrix} -T_{e^{i\theta}} T_{e^{-i\theta}} & T_{e^{-i\theta}} - T_{e^{i\theta}} \\ T_{e^{i\theta}} - T_{e^{-i\theta}} & T_{e^{i\theta}} T_{e^{-i\theta}} \end{bmatrix}.\end{aligned}$$

Therefore, the range of $[T_\Phi^*, T_\Phi]$ is infinite dimensional.

In general, the following identity holds for a trigonometric polynomial symbol $\Phi \in L^\infty(\mathbb{T}, M_n)$:

$$[T_\Phi^*, T_\Phi] = T_{\Phi^*\Phi - \Phi\Phi^*} + H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi.$$

Hence, for a normal matrix-valued symbol Φ, we obtain

$$[T_\Phi^*, T_\Phi] = H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi.$$

Since the Hankel operators $H_{\Phi^*}^*, H_{\Phi^*}, H_\Phi^*, H_\Phi$ are of finite rank, it follows that for a normal matrix-valued trigonometric polynomial Φ, the self-commutator $[T_\Phi^*, T_\Phi]$ is a finite rank operator. In particular, if T_Φ is hyponormal, then $[T_\Phi^*, T_\Phi]$ is a positive semi-definite finite rank operator. This observation motivates a detailed examination of hyponormal block Toeplitz operators with trigonometric polynomial symbols.

Building on the work of C. Gu, J. Hendricks and D. Rutherford [32], who examined hyponormal block Toeplitz operators with matrix-valued trigonometric polynomial symbols and analyzed their structural properties, we further investigate such operators by extending the analytical techniques developed by R. E. Curto and W. Y. Lee [20].

Theorem 3.1.2. For $m, N \in \mathbb{W}$, suppose that $\Phi \in L^\infty(\mathbb{T}, M_n)$ is a trigonometric polynomial of the form $\Phi(e^{i\theta}) = \sum_{j=-m}^{N} A_j e^{ij\theta} I_n$, with A_N is invertible. Then, the hyponormality of T_Φ is independent of the matrices $A_0, A_1, \ldots A_{N-m}$.

Proof. Since T_Φ is hyponormal, it follows from [32, Corollary 5.2] that $m \le N$. Moreover, there exists a function $K \in H^\infty(\mathbb{T}, M_n)$ with $\|K\|_\infty \le 1$ such that

$$\Phi - K\Phi^* \in H^\infty(\mathbb{T}, M_n).$$

Consequently, we obtain

$$(K\Phi_+^* - \Phi_-^*)(e^{i\theta}) = K \sum_{j=1}^{N} A_j^* e^{-ij\theta} I_n - \sum_{j=1}^{m} A_{-j} e^{-ij\theta} I_n \in H^\infty(\mathbb{T}, M_n).$$

Let $K(e^{i\theta}) = \sum_{j=0}^{\infty} K_j e^{ij\theta} I_n$. Then, we get

$$K_0 = K_1 = \cdots = K_{N-m-1} = 0 I_n \quad \text{and} \quad K_{N-m} = A_{-m}(A_N^*)^{-1}.$$

Moreover, for $n = N-m+1, N-m+2, \ldots, N-1$, we have

$$K_n = \left(A_{-N+n} - \sum_{j=N-m}^{n-1} K_j A^*_{N-n+j} \right) (A^*_N)^{-1}.$$

To see this, observe that

$$K_j e^{ij\theta} I_n \cdot A^*_N e^{-iN\theta} I_n = K_j A^*_N e^{i(j-N)\theta} I_n.$$

If $j < N-m$, then K_j must be zero; otherwise, the term $K_j A^*_N e^{i(j-N)\theta} I_n$ will appear in the expansion of $(K\Phi^*_+ - \Phi^*_-)$, which contradicts its analyticity. In the case where $j = N - m$, we observe that

$$(K_{N-m} A^*_N - A_{-m}) e^{-im\theta} I_n = 0 \quad \text{if and only if} \quad K_{N-m} = A_{-m} (A^*_N)^{-1}.$$

For $j > N - m$, we obtain the recurrence relation

$$K_j A^*_N + \sum_{i=N-m}^{j-1} K_i A^*_{N-j+i} - A_{-N+j} = 0.$$

This relation is obtained by equating the corresponding Fourier coefficients in the expansion of $(K\Phi^*_+ - \Phi^*_-)$.

Since $N - n + j > N - m$ for $n = N - m + 1, N - m + 2, \ldots, N - 1$ and $N - m \leq j \leq n - 1$, the recurrence is well-defined and the proof is complete. □

Theorem 3.1.3. Suppose that $\Phi = F^* + G = \sum_{j=-m}^{N} A_j e^{ij\theta} I_n \in L^\infty(\mathbb{T}, M_n)$, where F and G are analytic polynomials of degree m and N, respectively and the leading coefficient A_N is invertible. Now, define $\Psi = F^* + \mathbb{P}(e^{-ir\theta} I_n G)$, where $r \leq N - m$ and $\mathbb{P}(A)$ denotes the analytic part of the matrix-valued function $A \in L^\infty(\mathbb{T}, M_n)$. Then, T_Ψ is hyponormal if and only if T_Φ is hyponormal.

Proof. Let $G_0(e^{i\theta}) = \sum_{j=r}^{N} A_j e^{ij\theta}$ and define $\Phi_0 := F^* + G_0$. Consider the function $H(e^{i\theta}) = \mathbb{P}(e^{-ir\theta} I_n G(e^{i\theta}))$. Then, $H(e^{i\theta}) = e^{-ir\theta} I_n G_0(e^{i\theta})$ and $\Psi = F^* + H$. By Theorem 3.1.2, T_Φ is hyponormal if and only if T_{Φ_0} is hyponormal.

Let us first assume that the operator T_Φ is hyponormal. Since T_{Φ_0} is also hyponormal, there exists $K \in H^\infty(\mathbb{T}, M_n)$ such that

$$\Phi_0 - K\Phi_0^* \in H^\infty(\mathbb{T}, M_n).$$

Equivalently, this condition can be expressed as

$$\Phi_{0-}^* - K\Phi_{0+}^* = F^* - KG_0^* \in H^\infty(\mathbb{T}, M_n).$$

Furthermore, by Theorem 3.1.2, the function K admits a Fourier expansion of the form

$$K(e^{i\theta}) = \sum_{j=N-m}^{\infty} C_j e^{ij\theta} I_n.$$

Since $G_0^*(e^{i\theta}) = e^{-ir\theta} I_n H^*(e^{i\theta})$, it follows that

$$F^* - Ke^{-ir\theta} I_n H^* \in H^\infty(\mathbb{T}, M_n).$$

Under the assumption $r \leq N - m$, we define $K' := e^{-ir\theta} I_n K$. Then, K' belongs to $H^\infty(\mathbb{T}, M_n)$ and we see that

$$\|K'\|_\infty = \|Ke^{-ir\theta} I_n\|_\infty \leq \|K\|_\infty \leq 1.$$

Hence,

$$F^* - K'H^* = \Psi_-^* - K'\Psi_+^* \in H^\infty(\mathbb{T}, M_n),$$

which implies that

$$\Psi - K'\Psi^* \in H^\infty(\mathbb{T}, M_n).$$

Therefore, T_Ψ is hyponormal.

Conversely, assume that T_Ψ is hyponormal. Then, there exists a matrix-valued function $K' \in H^\infty(\mathbb{T}, M_n)$ with $\|K'\|_\infty \leq 1$ such that

$$\Psi - K'\Psi^* \in H^\infty(\mathbb{T}, M_n).$$

Equivalently, since $\Psi_- = F$ and $\Psi_+ = H$, it follows that

$$\Psi_-^* - K'\Psi_+^* = F^* - K'H^* \in H^\infty(\mathbb{T}, M_n). \tag{3.1}$$

Now, we define $K = K'e^{ir\theta}I_n$. Clearly, $K \in H^\infty(\mathbb{T}, M_n)$. Moreover, $\|e^{ir\theta}\|_\infty \leq 1$ implies that $\|K\|_\infty \leq 1$. From Equation (3.1), it follows that

$$F^* - Ke^{-ir\theta}I_nH^* \in H^\infty(\mathbb{T}, M_n).$$

Since $G_0^* = e^{-ir\theta}I_nH^*$, we immediately obtain $F^* - KG_0^* \in H^\infty(\mathbb{T}, M_n)$. The representation $\Phi_0 = F^* + G_0$ implies that ${\Phi_0}_-^* - K{\Phi_0}_+^* \in H^\infty(\mathbb{T}, M_n)$. Equivalently,

$$\Phi_0 - K\Phi_0^* \in H^\infty(\mathbb{T}, M_n).$$

Hence T_{Φ_0} is hyponormal and therefore, T_Φ is hyponormal. □

3.2 Hyponormality of block Toeplitz operators with finite rank self commutator

In this section, we identify certain hyponormal block Toeplitz operators with finite rank self-commutator by analyzing the Cowen set of the symbol. T. Nakazi and K. Takahashi [44] proved that if the set $\mathcal{E}(\varphi)$ contains at least two elements for some $\varphi \in L^\infty(\mathbb{T})$, then φ must be of bounded type. However, this result does not extend to the matrix-valued case, where $\Phi \in L^\infty(\mathbb{T}, M_n)$. For example, consider the matrix-valued function

$$\Phi(e^{i\theta}) := \begin{bmatrix} (f+\overline{f})(e^{i\theta}) & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in L^\infty(\mathbb{T}, M_2),$$

where $\overline{f}$ is not of bounded type. Since

$$\Phi(e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} (f+\overline{f})(e^{i\theta}) & 0 \\ 0 & e^{-i\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & e^{i\theta}-1 \end{bmatrix}$$

and

$$\Phi(e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{bmatrix} \begin{bmatrix} (f+\overline{f})(e^{i\theta}) & 0 \\ 0 & e^{-i\theta} \end{bmatrix},$$

it follows that

$$B(e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \quad \text{and} \quad B_0(e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{bmatrix}$$

are elements of $\mathcal{E}(\Phi)$. Nevertheless, Φ is not of bounded type.

We now turn our attention to the following Conjecture 3.2.1 formulated by R. E. Curto, I. S. Hwang and W. Y. Lee [22], which plays a significant role in the study of hyponormal block Toeplitz operators with finite rank self-commutator.

Conjecture 3.2.1. [22, Conjecture 6.1] If $\Phi \in L^\infty(\mathbb{T}, M_n)$ is such that T_Φ is a hyponormal operator whose self-commutator $[T_\Phi^*, T_\Phi]$ is of finite rank, then there exists a finite Blaschke-Potapov product $B \in \mathcal{E}(\Phi)$ such that Rank $[T_\Phi^*, T_\Phi] = \deg(\det B)$.

We now present a sufficient condition under which a hyponormal block Toeplitz operator has a finite rank self-commutator. The following theorem asserts that if the Cowen set $\mathcal{E}(\Phi)$ contains a finite Blaschke-Potapov product B for a normal symbol Φ, then the associated hyponormal block Toeplitz operator T_Φ has a finite rank self-commutator.

Theorem 3.2.2. If the block Toeplitz operator T_Φ is hyponormal and the Cowen set $\mathcal{E}(\Phi)$ contains a finite Blaschke-Potapov product Q, then the self-commutator $[T_\Phi^*, T_\Phi]$ is a finite rank operator.

Proof. By Lemma 3.1.1, we have $[T_\Phi^*, T_\Phi] = H_{\Phi^*}^*(I - T_{\widetilde{Q}} T_{\widetilde{Q}}^*) H_{\Phi^*}$. To analyze the range of the operator $I - T_{\widetilde{Q}} T_{\widetilde{Q}}^*$, consider the following computation:

$$\begin{aligned}\big\langle (I - T_{\widetilde{Q}}T^*_{\widetilde{Q}})x, \widetilde{Q}H^2 \big\rangle &= \langle x, \widetilde{Q}H^2\rangle - \langle T_{\widetilde{Q}}T^*_{\widetilde{Q}}x, \widetilde{Q}H^2\rangle \\ &= \langle x, \widetilde{Q}H^2\rangle - \langle x, \widetilde{Q}H^2\rangle \\ &= 0.\end{aligned}$$

This implies that $\text{Ran}(I - T_{\widetilde{Q}}T^*_{\widetilde{Q}}) \subseteq \mathcal{H}(\widetilde{Q})$. Since Q is a finite Blaschke-Potapov product, it follows that $\widetilde{Q}$ is also a finite Blaschke-Potapov product. Therefore, by [22, Lemma 2.4], we obtain

$$\text{Rank } [T^*_\Phi, T_\Phi] \leq \text{Rank } (I - T_{\widetilde{Q}}T^*_{\widetilde{Q}}) \leq \dim \mathcal{H}(\widetilde{Q}) < \infty.$$

This completes the proof. □

We begin our investigation of Conjecture 3.2.1 by verifying its validity in the special case of normal block Toeplitz operators. We recall a core result by C. Gu, J. Hendricks and D. Rutherford [32] that characterizes normal block Toeplitz operators.

Lemma 3.2.3. [32, Theorem 4.3] Let $G = G_{+'} + G_0 + G_-^* \in L^\infty(\mathbb{T}, M_n)$, where G_0 is a constant matrix and $G_{+'} = G_+ - G_0$. Suppose that $\det G_{+'}$ is not identically zero. Then T_G is normal if and only if the following two conditions are satisfied:

(i) $G^*G = GG^*$ almost everywhere on $\mathbb{T}$ and

(ii) there exists a constant unitary matrix $U \in M_n$ such that $G_{+'} = G_-U$.

Next, we observe that if the Cowen set $\mathcal{E}(\Phi)$ contains a constant unitary matrix U, then the block Toeplitz operator T_Φ is normal. Our approach follows a modification of the techniques developed by M. Abhinand, R. E. Curto, I. S. Hwang, W. Y. Lee and T. Prasad [1].

Theorem 3.2.4. If $\Phi \in L^\infty(\mathbb{T}, M_n)$ and $\mathcal{E}(\Phi)$ contains a constant unitary matrix U, then the block Toeplitz operator T_Φ is normal.

Proof. Suppose that $U \in \mathcal{E}(\Phi)$ is a constant unitary matrix, i.e., $U^*U = UU^* = I_n$. Then, it follows that

$$[T_\Phi^*, T_\Phi] = H_{\Phi^*}^*(I - T_{\widetilde{U}} T_{\widetilde{U}^*}) H_{\Phi^*} = 0.$$

This establishes the normality of T_Φ and thus completes the proof. □

The following theorem addresses the converse direction of Theorem 3.2.4 by providing conditions under which the normality of the block Toeplitz operator T_Φ implies the existence of a constant unitary matrix in $\mathcal{E}(\Phi)$.

Theorem 3.2.5. Let $\Phi = \Phi_{+'} + \Phi_0 + \Phi_-^* \in L^\infty(\mathbb{T}, M_n)$ be such that $\det \Phi_{+'}$ is not identically zero. If T_Φ is normal, then $\mathcal{E}(\Phi)$ contains a constant unitary matrix $U \in M_n$.

Proof. Recall that every normal operator is hyponormal. Therefore, the normality of T_Φ ensures that it is hyponormal. This, in turn, guarantees the existence of a function $K \in H^\infty(\mathbb{T}, M_n)$ with $\|K\|_\infty \leq 1$ such that

$$\Phi - K\Phi^* \in H^\infty(\mathbb{T}, M_n).$$

Equivalently, this condition holds if and only if

$$\Phi_-^* - K\Phi_+^* \in H^\infty(\mathbb{T}, M_n).$$

By applying Lemma 3.2.3, we conclude that the normality of T_Φ implies two conditions:

$$\Phi^*\Phi = \Phi\Phi^* \quad \text{and} \quad \Phi_{+'} = \Phi_- U$$

for some constant unitary matrix $U \in M_n$. To verify that $U \in \mathcal{E}(\Phi)$, observe that

$$\begin{aligned} \Phi_-^* - U\Phi_+^* &= \Phi_-^* - U({\Phi_{+'}}^* + \Phi_0^*) \\ &= \Phi_-^* - U(U^*\Phi_-^* + \Phi_0^*) \\ &= -U\Phi_0^*, \end{aligned}$$

where the last equality holds because U is unitary. Hence, $\Phi_-^* - U\Phi_+^*$ belongs to $H^\infty(\mathbb{T}, M_n)$, which completes the proof. □

The following example reveals an important feature of the Cowen set $\mathcal{E}(\Phi)$. Specifically, it illustrates that the set $\mathcal{E}(\Phi)$ can contain a constant unitary matrix even in the case when the block Toeplitz operator T_Φ is normal but the determinant of $\Phi_{+'}$ vanishes identically. This example highlights that the assumption $\det \Phi_{+'} = 0$ in the previous result is sufficient, but not necessary.

Example 3.2.6. Consider the function

$$\Phi(e^{i\theta}) := \begin{bmatrix} e^{i\theta} + e^{-i\theta} & e^{i\theta} + e^{-i\theta} \\ e^{i\theta} + e^{-i\theta} & e^{i\theta} + e^{-i\theta} \end{bmatrix} \in L^\infty(\mathbb{T}, M_2).$$

The associated block Toeplitz operator is given by

$$T_\Phi = \begin{bmatrix} T_{e^{i\theta}+e^{-i\theta}} & T_{e^{i\theta}+e^{-i\theta}} \\ T_{e^{i\theta}+e^{-i\theta}} & T_{e^{i\theta}+e^{-i\theta}} \end{bmatrix}.$$

Note that T_Φ is self-adjoint and hence normal. Observe that the symbol Φ satisfies the relation $\Phi = I_2\Phi^*$, where I_2 is the 2×2 identity matrix. Consequently, the constant unitary matrix $U = I_2$ belongs to $\mathcal{E}(\Phi)$. However, a straightforward computation shows that

$$\Phi_{+'} = \begin{bmatrix} e^{i\theta} & e^{i\theta} \\ e^{i\theta} & e^{i\theta} \end{bmatrix} \quad \text{and hence } \det \Phi_{+'} \equiv 0.$$

Now, we give a partial answer to Conjecture 3.2.1.

Theorem 3.2.7. Let $\Phi \in L^\infty(\mathbb{T}, M_n)$ be such that each row of Φ^* contains at least one scalar-valued function that is not of bounded type. If T_Φ is hyponormal and has a self-commutator of finite rank, then the set $\mathcal{E}(\Phi)$ contains a finite Blaschke-Potapov product.

Proof. Assume that each row of Φ^* contains at least one scalar-valued function that is not of bounded type. Under this condition, it follows that the Hankel operator H_{Φ^*} has dense range. Consequently, the adjoint operator $H^*_{\Phi^*}$ is injective. Therefore, by Lemma 3.1.1, we obtain the identity

$$\text{Rank}[T^*_\Phi, T_\Phi] = \text{Rank}(I - T_{\widetilde{K}} T_{\widetilde{K}^*}),$$

for some $K \in \mathcal{E}(\Phi)$. Since the commutator $[T^*_\Phi, T_\Phi]$ is assumed to be of finite rank, it follows that the operator $I - T_{\widetilde{K}} T_{\widetilde{K}^*}$ is also of finite rank. This implies that $\widetilde{K}$ must be a finite Blaschke-Potapov product and hence so is K. This completes the proof. □

The following example demonstrates that the Cowen set $\mathcal{E}(\Phi)$ can contain a finite Blaschke-Potapov product even in the presence of a row in Φ^* consisting entirely of scalar-valued functions that are of bounded type. This shows that the assumption on the presence of non-bounded type functions in each row of Φ^* is not necessary for $\mathcal{E}(\Phi)$ to include a finite Blaschke-Potapov product.

Example 3.2.8. Consider the matrix-valued function

$$\Phi(e^{i\theta}) := \begin{bmatrix} (f + \overline{f})(e^{i\theta}) & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in L^\infty(\mathbb{T}, M_2),$$

where $f \in H^2$. In this case, the block Toeplitz operator T_Φ is hyponormal and the self-commutator is given by

$$[T^*_\Phi, T_\Phi] = \begin{bmatrix} 0 & 0 \\ 0 & [T^*_{e^{i\theta}}, T_{e^{i\theta}}] \end{bmatrix}.$$

Since $[T^*_{e^{i\theta}}, T_{e^{i\theta}}]$ has rank one, it follows that the self-commutator $[T^*_\Phi, T_\Phi]$ is a finite rank operator . However, note that the second row of Φ does not contain a scalar-valued function that fails to be of bounded type. Despite this, the matrix-valued function

$$B(e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in \mathcal{E}(\Phi).$$

Although a complete characterization of subnormal block Toeplitz operators with finite rank self-commutator remains elusive, we have successfully identified a significant class of such operators. The unresolved components of this conjecture are posed as problems in the final recommendations, presented as refined versions of the original conjecture. Answers to these problems would contribute significantly toward a complete characterization of block Toeplitz operators with finite rank self-commutator. In light of this, we devote the next chapter to the study of Problem 2.1.2 in the setting of block Toeplitz operators.

Chapter 4

SUBNORMAL BLOCK TOEPLITZ OPERATORS

4.1 Introduction

The study of subnormal operators with finite rank self-commutator has gained significant attention from many researchers. In 2006, under certain assumptions on the normal extensions of operators, D. V. Yakubovich [57] developed an insightful characterization of subnormal operators with finite rank self-commutator.

Yakubovich's Theorem. [19, 57] Let $T \in \mathcal{B}(\mathcal{H})$ be a pure subnormal operator with finite rank self-commutator and no point masses (i.e., T has a normal extension N that has no nonzero eigenvectors). Then, T is unitarily equivalent to an analytic block Toeplitz operator T_Φ with rational symbol Φ.

Yakubovich's Theorem served as a motivation for exploring the subnormality of block Toeplitz operators. C. Gu, J. Hendricks and D. Rutherford [32] characterized the hyponormality of block Toeplitz operators, as discussed in Chapter 3, while R. E. Curto, I. S. Hwang and W. Y. Lee [23] developed a matrix-valued extension of Abrahamse's Theorem.

Abrahamse's Theorem for matrix-valued symbols. [23, Theorem 3.5] Suppose $\Phi = \Phi_+ + \Phi_-^* \in L^\infty(\mathbb{T}, M_n)$ is such that Φ and Φ^* are of bounded type of the form $\Phi_+ = A^*\Theta_0\Theta_2$ and $\Phi_- = B^*\Theta_2$, where $\Theta_i = \theta_i I_n$ with an inner function $\theta_i (i = 0, 2)$ and $A, B \in H^2(\mathbb{T}, M_n)$. Assume that A, B and Θ_2 are left coprime. If

(i) T_Φ is hyponormal and

(ii) ker $[T_\Phi^*, T_\Phi]$ is invariant under T_Φ,

then T_Φ is normal or analytic. In particular, if T_Φ is subnormal then it is normal or analytic.

In the scalar-valued case, every analytic Toeplitz operator is subnormal. However, this property does not generally extend to the case of block Toeplitz operators. For instance, consider the analytic matrix-valued symbol

$$\Phi = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in H^\infty(\mathbb{T}, M_2).$$

The self-commutator of the corresponding block Toeplitz operator T_Φ is given by

$$[T_\Phi^*, T_\Phi] = T_\Phi^* T_\Phi - T_\Phi T_\Phi^* = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix},$$

where I denotes the identity operator on H^2. Now observe that

$$\left\langle [T_\Phi^*, T_\Phi] \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = -1.$$

Since the self-commutator is not positive semi-definite, T_Φ is not hyponormal and hence not subnormal.

However, it is evident that any analytic block Toeplitz operator with a normal symbol Φ is subnormal, as the associated multiplication operator M_Φ serves as a normal extension of T_Φ. Furthermore, C. Gu, J. Hendricks and D. Rutherford [32, Corollary 3.4] proved that an analytic block Toeplitz operator T_Φ is hyponormal if and only if $\Phi(e^{i\theta})$ is normal for almost every $e^{i\theta} \in \mathbb{T}$. Consequently, if T_Φ is subnormal, then $\Phi(e^{i\theta})$ must be normal for almost every $e^{i\theta} \in \mathbb{T}$.

In this chapter, we discuss under what conditions a subnormal block Toeplitz operator with finite rank self-commutator can be classified as either normal or analytic, by developing the ideas of T. Nakazi and K. Takahashi [44].

4.2 Nakazi-Takahashi Theorem for matrix-valued symbols

From this point onward, we focus on the study of subnormality in the context of block Toeplitz operators with finite rank self-commutator. A significant open question in this area was posed by R. E. Curto, I. S. Hwang and W. Y. Lee [22], who formulated Problem 4.2.1. This problem investigates whether the Nakazi-Takahashi Theorem, originally established for scalar-valued Toeplitz operators, extends to the framework of block Toeplitz operators.

Our analysis reveals that, in general, the answer to Problem 4.2.1 is negative, as evidenced by Example 4.2.2. However, by considering specific subclasses of block Toeplitz operators, we obtain affirmative results under certain conditions. These findings are formally presented in Theorem 4.2.3 and Theorem 4.2.10, where we derive sufficient criteria ensuring the validity of the Nakazi-Takahashi Theorem within these restricted settings. The subsequent discussion presents a detailed examination of these results along with the mathematical techniques and insights that support our conclusions.

Problem 4.2.1. [22, Problem 6.2] If $\Phi \in L^\infty(\mathbb{T}, M_n)$ is such that T_Φ is subnormal and $\Phi = Q\Phi^*$, where Q is a finite Blaschke-Potapov product, does it follow that T_Φ is normal or analytic?

We are now in a position to provide a negative answer to Problem 4.2.1. In particular, we construct a counterexample that reveals the invalidity of the anticipated conclusion. A careful examination of Example 4.2.2 establishes that the assumptions of Problem 4.2.1 do not necessarily lead to the expected outcome.

Example 4.2.2. Let

$$\Phi(e^{i\theta}) := \begin{bmatrix} e^{i\theta} + e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in L^\infty(\mathbb{T}, M_2).$$

We observe that

$$\begin{bmatrix} e^{i\theta} + e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{bmatrix} \begin{bmatrix} e^{i\theta} + e^{-i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}, \tag{4.1}$$

which implies $\Phi(e^{i\theta}) = Q(e^{i\theta})\Phi^*(e^{i\theta})$, where $\Phi^*(e^{i\theta}) = \Phi(e^{i\theta})^*$ and

$$Q(e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{bmatrix}.$$

Since Q is a rational inner function, it must be a finite-Blaschke Potapov product. Furthermore, the corresponding block Toeplitz operator is given by

$$T_\Phi = \begin{bmatrix} T_{e^{i\theta}+e^{-i\theta}} & 0 \\ 0 & T_{e^{i\theta}} \end{bmatrix},$$

which is the direct sum of a self-adjoint Toeplitz operator $T_{e^{i\theta}+e^{-i\theta}}$ and an analytic

Toeplitz operator $T_{e^{i\theta}}$. Consequently, T_Φ is subnormal. However, since Φ is not analytic and $T_{e^{i\theta}}$ is not normal, we conclude that T_Φ is neither normal nor analytic.

Now, we examine the approach used to analyze Problem 4.2.1 and establish affirmative results. By investigating the structural properties of the symbol, we identify specific conditions that ensure the validity of the expected outcome. The following discussion provides a detailed account of the techniques used in our analysis, leading to the affirmative answers presented in Theorem 4.2.3 and Theorem 4.2.10.

Let $\Phi \in L^\infty(\mathbb{T}, M_n)$ be such that $\Phi = Q\Phi^*$, where Q is a finite Blaschke-Potapov product. There are three cases to consider:

Case 1: Φ^* is of bounded type of the form

$$\Phi_- = B^*\Theta \quad \text{(where } B \in H^2(\mathbb{T}, M_n)\text{)},\ \Theta = \theta I_n \text{ for } \theta \text{ an inner function}),$$

where B and Θ are left coprime.

In this case, we obtain an affirmative result as presented below.

Theorem 4.2.3. Suppose $\Phi = \Phi_+ + \Phi_-^* \in L^\infty(\mathbb{T}, M_n)$ is such that both Φ and Φ^* are of bounded type of the form $\Phi_- = B^*\Theta_2$, where $\Theta_2 = \theta_2 I_n$ with an inner function θ_2 and $B \in H^2(\mathbb{T}, M_n)$. Moreover, assume that B and Θ_2 are left coprime. If

(i) T_Φ is hyponormal and

(ii) $\ker\,[T_\Phi^*, T_\Phi]$ is invariant under T_Φ,

then T_Φ is normal or analytic. In particular, if T_Φ is subnormal, then it is either normal or analytic.

Before presenting the proof of Theorem 4.2.3, we discuss a series of intermediate results that play a crucial role in establishing the main argument. These results offer valuable insights into the structure and behavior of the operators involved.

Beurling-Lax Theorem [8, 34, 40]. A nonzero closed subspace M of $H^2(\mathbb{T}, \mathbb{C}^n)$ is invariant under the shift operator $\mathcal{S}$ on $H^2(\mathbb{T}, \mathbb{C}^n)$ if and only if $M = \Theta H^2(\mathbb{T}, \mathbb{C}^m)$, where Θ is an inner matrix function in $H^\infty(\mathbb{T}, M_{n\times m})$ $(m \leq n)$. Moreover, the function Θ is unique up to a unitary constant right factor. That is, if $M = \Delta H^2(\mathbb{T}, \mathbb{C}^r)$ for another inner function $\Delta \in H^\infty(\mathbb{T}, M_{n\times r})$, then $m = r$ and there exists a constant unitary matrix $W \in M_{m\times m}$ such that $\Theta = \Delta W$.

Lemma 4.2.4. [23, Lemma 3.2] Let $\Phi = \Phi_-^* + \Phi_+ \in L^\infty(\mathbb{T}, M_n)$ be such that both Φ and Φ^* are of bounded type. Suppose Φ_+ and Φ_- admit right coprime factorizations of the form

$$\Phi_+ = \Delta_1 A_r^*, \quad \Phi_- = \Delta_2 B_r^*,$$

where $\Delta_1, \Delta_2 \in H^\infty(\mathbb{T}, M_n)$ are inner matrix functions. If T_Φ is hyponormal, then Δ_2 is a left inner divisor of Δ_1, that is, there exists an inner function $\Delta_0 \in H^\infty(\mathbb{T}, M_n)$ such that $\Delta_1 = \Delta_2 \Delta_0$.

Lemma 4.2.5. [23, Lemma 3.3] Suppose that $B \in H^2(\mathbb{T}, M_n)$ and $\Theta := \theta I_n$, where θ is a finite Blaschke product. Let $\mathcal{Z}(\theta)$ denote the set of zeros of the function θ. Then the following statements are equivalent:

(a) $B(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$;

(b) B and Θ are right coprime;

(c) B and Θ are left coprime.

We are now ready for the proof of Theorem 4.2.3.

Proof of Theorem 4.2.3. Let $\Phi = [\phi_{ij}] = \Phi_-^* + \Phi_+ \in L^\infty(\mathbb{T}, M_n)$ and suppose that Φ and Φ^* are of bounded type. Assume further that the Toeplitz operator T_Φ is hyponormal and $\ker[T_\Phi^*, T_\Phi]$ is invariant under T_Φ. Since Φ and Φ^* are of bounded type, it follows that each entry ϕ_{ij} and its complex conjugate $\overline{\phi_{ij}}$ are of bounded type for all $i, j \in \{1, 2, \ldots n\}$. This implies that both $\overline{\phi_{ij_-}}$ and $\overline{\phi_{ij_+}}$ are of bounded type (see Lemma 1.1). Consequently, Φ_+^* and Φ_-^* are also of bounded type. By

Equation (1.12), we obtain the following representations:

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*,$$

where $\Theta_i = \theta_i I_n$ with an inner function θ_i and $A, B \in H^2(\mathbb{T}, M_n)$, for $i = 1, 2$.

In view of Equation (1.13), we write

$$\Phi_+ = A^*\Theta_0\Theta_2 \quad \text{and} \quad \Phi_- = B^*\Theta_2,$$

where $\Theta_i = \theta_i I_n$ for $i = 0, 2$ with each θ_i being an inner function. Furthermore, according to Abrahamse's Theorem for matrix-valued symbols, it suffices to prove that if B and Θ_2 are left coprime, then A and Θ_2 are also left coprime.

By Lemma 4.2.5, the notions of left coprimeness and right coprimeness coincide for matrix-valued functions $A \in H^2(\mathbb{T}, M_n)$ and $\Theta_2 = \theta_2 I_n$, where θ_2 is an inner function. In light of Lemma 4.2.4, we can express

$$\Phi_+ = A^*\Theta_0\Theta_2 = \Theta_2\Delta_1 A_r^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*,$$

where $\Delta_1 \in H^\infty(\mathbb{T}, M_n)$ is an inner function and $A_r \in H^2(\mathbb{T}, M_n)$. Furthermore, the functions A_r and $\Theta_2\Delta_1$ are right coprime.

First of all, a careful analysis of STEP 1 in the proof of [23, Theorem 3.5] reveals that

$$\Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \subseteq \ker[T_\Phi^*, T_\Phi]. \tag{4.2}$$

(It is important to note that the assumption that "A and Θ_2 are left coprime" was not used in establishing (4.2) within this part of the proof.)

We begin by showing that

$$\Theta_0 \quad \text{and} \quad \Theta_2 \quad \text{are coprime.} \tag{4.3}$$

To see this, assume for the sake of contradiction that Θ_0 and Θ_2 are not coprime. Then there exists a non-constant inner function Ω_1 such that

$$\Theta_0 = \Omega_1\Theta_0' \quad \text{and} \quad \Theta_2 = \Omega_1\Theta_2'. \tag{4.4}$$

Suppose further that there exists an inner function ω such that $\Omega = \omega I_n$ satisfying

Equation (4.4). Then, using (4.4) and $\Theta_i' = \overline{\omega}\,\theta_i I_n$ for $i = 0, 2$, we obtain

$$\Theta_2\Theta_0' = \Omega\Theta_2'\Theta_0' = \Omega\Theta_0'\Theta_2' = \Theta_0\Theta_2',$$

since $\Theta_0'\Theta_2' = \Theta_2'\Theta_0'$. It then follows from Equation (4.2) that

$$\Theta_2\Theta_0' H^2(\mathbb{T}, \mathbb{C}^n) = \Theta_0\Theta_2' H^2(\mathbb{T}, \mathbb{C}^n) \subseteq \Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \subseteq \ker[T_\Phi^*, T_\Phi]. \tag{4.5}$$

We now recall that the hyponormality of T_Φ implies that the symbol Φ is normal. Consequently, the self-commutator of T_Φ can be written as

$$[T_\Phi^*, T_\Phi] = H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi = H_{\Phi_+^*}^* H_{\Phi_+^*} - H_{\Phi_-^*}^* H_{\Phi_-^*}. \tag{4.6}$$

Moreover, since $\Phi_+ = A^*\Theta_0\Theta_2$ and $\Phi_- = \Theta_2 B^*$, we obtain

$$[T_\Phi^*, T_\Phi] = H_{A\Theta_2^*\Theta_0^*}^* H_{A\Theta_2^*\Theta_0^*} - H_{B\Theta_2^*}^* H_{B\Theta_2^*}. \tag{4.7}$$

From Equation (4.5), it follows that

$$[T_\Phi^*, T_\Phi]\Theta_2\Theta_0' H^2(\mathbb{T}, \mathbb{C}^n) = \{0\},$$

which in turn implies

$$\big(H_{A\Theta_2^*\Theta_0^*}^* H_{A\Theta_2^*\Theta_0^*} - H_{B\Theta_2^*}^* H_{B\Theta_2^*}\big)\Theta_2\Theta_0' H^2(\mathbb{T}, \mathbb{C}^n) = \{0\}.$$

Since

$$H_{B\Theta_2^*}\Theta_2\Theta_0' H^2(\mathbb{T}, \mathbb{C}^n) = \{0\},$$

we conclude that

$$H_{A\Theta_2^*\Theta_0^*}(\Theta_2\Theta_0' H^2(\mathbb{T}, \mathbb{C}^n)) = \{0\}.$$

Now, using the identity $\Theta_0 = \Omega\Theta_0'$, we obtain

$$A\Theta_2^*\Theta_0^*\Theta_2\Theta_0' = A\Omega^*{\Theta_0'}^*\Theta_0' = A\Omega^*.$$

Hence,

$$\{0\} = H_{A\Theta_2^*\Theta_0^*}\big(\Theta_2\Theta_0' H^2(\mathbb{T}, \mathbb{C}^n)\big) = H_{A\Omega^*} H^2\big(\mathbb{T}, \mathbb{C}^n\big).$$

This implies that $G = A\Omega^* \in H^2(\mathbb{T}, M_n)$. Replacing Ω' with Ω in Equation (4.4) and applying the identity $G = A\Omega^*$, we get

$$\Phi_+ = \Theta_0\Theta_2 A^* = \Theta_0\Theta_2'\Omega A^* = \Theta_0\Theta_2' G^*, \tag{4.8}$$

which leads to a contradiction, since the representation $\Phi_+ = \Theta_0\Theta_2 A^*$ is assumed to be in minimal form, as noted in (1.13).

To establish the claim in (4.3), it suffices to show the existence of a non-constant inner function ω such that $\Omega = \omega I_n$ satisfying Equation (4.4). By the Beurling-Lax Theorem, the subspace

$$\mathcal{M} = \Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \vee \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n)$$

is invariant under the shift operator $\mathcal{S}$. Furthermore, we observe that

$$\mathcal{M} = \Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \vee \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n) = \bigoplus_{j=1}^{n} (\theta_0 H^2 \vee \theta_2 H^2) = \bigoplus_{j=1}^{n} \omega H^2.$$

This follows from the fact that each subspace $\theta_0 H^2 \vee \theta_2 H^2$ is invariant under the (scalar) right shift operator, where ω is a common inner divisor for both θ_0 and θ_2. Since $\Theta_i H^2(\mathbb{T}, \mathbb{C}^n) \subseteq \mathcal{M}$ for $i = 0, 2$, it follows from [30, Corollary 2.2, Chapter IX] that $\Omega = \omega I_n$ is a left inner divisor of each Θ_i $(i = 0, 2)$. Furthermore, by applying [30, Corollary 2.2, Chapter IX] to Equation (4.4), we obtain

$$\mathcal{M} \subseteq \Omega_1 H^2(\mathbb{T}, \mathbb{C}^n).$$

Since Ω_1 is non-constant, the space $\Omega_1 H^2(\mathbb{T}, \mathbb{C}^n)$ is a proper subspace of $H^2(\mathbb{T}, \mathbb{C}^n)$, which in turn implies ω is a non-constant inner function.

We next show that A and Θ_2 are right coprime. To see this, let Δ' be a common right inner divisor of A and Θ_2. Then we can write

$$A = A_1\Delta' \quad \text{and} \quad \Theta_2 = \Delta\Delta', \tag{4.9}$$

where $\Delta \in H^\infty(\mathbb{T}, M_n)$ is an inner function and $A_1 \in H^2(\mathbb{T}, M_n)$. It follows from Equation (4.9) that

$$\Phi_+ = \Theta_0\Theta_2 A^* = \Theta_0\Delta\Delta'\Delta'^* A_1^* = \Theta_0\Delta A_1^* = \Delta\Theta_0 A_1^*.$$

On the other hand, since

$$\Phi_+ = \Theta_2\Delta_1 A_r^* \quad \text{(where } \Theta_2\Delta_1 \text{ and } A_r \text{ are right coprime)},$$

we have $\ker H_{\Phi_+^*} = \Theta_2\Delta_1 H^2(\mathbb{T}, \mathbb{C}^n)$ (see [22, Remark 2.2]). In addition, we observe that

$$H_{\Phi_+^*}\Delta\Theta_0 f = J_n P_n^\perp\big(A_1\Theta_0^*\Delta^*\Delta\Theta_0 f\big) = J_n P_n^\perp\big(A_1 f\big) = 0,$$

which shows that

$$\Delta\Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \subseteq \Theta_2\Delta_1 H^2(\mathbb{T}, \mathbb{C}^n).$$

Hence, by [30, Corollary 2.2, Chapter IX], it follows that $\Theta_2\Delta_1$ is a left inner divisor of $\Delta\Theta_0$. In particular, this implies that Θ_2 is a left inner divisor of $\Delta\Theta_0$.

We now claim

$$\Theta_2 \text{ is a left inner divisor of } \Delta. \tag{4.10}$$

Since the intersection $\Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \cap \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n)$ is an ivariant subspace for the shift operator $\mathcal{S}$, the Beurling-Lax Theorem implies that

$$\Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \cap \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n) = \Omega H^2(\mathbb{T}, \mathbb{C}^n),$$

for some inner function $\Omega \in H^\infty(\mathbb{T}, M_n)$.

We now show that $\Omega H^2(\mathbb{T}, \mathbb{C}^n) = \Theta_0\Theta_2 H^2(\mathbb{T}, \mathbb{C}^n)$. Observe that

$$\Theta_0\Theta_2 H^2(\mathbb{T}, \mathbb{C}^n) \subseteq \Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \cap \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n) = \Omega H^2(\mathbb{T}, \mathbb{C}^n),$$

which implies that Ω is a left inner divisor of $\Theta_0\Theta_2$. Furthermore, since

$$\Omega H^2(\mathbb{T}, \mathbb{C}^n) \subseteq \Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \quad \text{and} \quad \Omega H^2(\mathbb{T}, \mathbb{C}^n) \subseteq \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n),$$

it follows that both Θ_0 and Θ_2 are left inner divisors of Ω. Therefore, there exists inner functions $\Omega', \Delta_0, \Delta_2 \in H^2(\mathbb{T}, M_n)$ such that

$$\Theta_0\Theta_2 = \Omega\Omega' \quad \text{and} \quad \Omega = \Theta_0\Delta_0 = \Theta_2\Delta_2.$$

From these identities, we obtain

$$\Theta_0\Theta_2 = \Omega\Omega' = \Theta_0\Delta_0\Omega' \quad \text{and} \quad \Theta_0\Theta_2 = \Omega\Omega' = \Theta_2\Delta_2\Omega',$$

which implies that

$$\Theta_2 = \Delta_0 \Omega' \quad \text{and} \quad \Theta_0 = \Delta_2 \Omega'.$$

Thus, Ω' is a common right inner divisor for both Θ_0 and Θ_2. Since Θ_0 and Θ_2 are coprime by Equation (4.3), it follows that Ω' must be a constant unitary matrix. Hence,

$$\Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \cap \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n) = \Omega H^2(\mathbb{T}, \mathbb{C}^n) = \Theta_0 \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n).$$

Since Θ_2 is a left inner divisor of $\Delta\Theta_0$, it follows that

$$\Delta\Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \subseteq \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n).$$

Consequently, we have

$$\begin{aligned}
\Delta\Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) &= \Delta\Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \cap \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n) \\
&\subseteq \Theta_0 H^2(\mathbb{T}, \mathbb{C}^n) \cap \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n) \\
&= \Theta_0 \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n),
\end{aligned}$$

which implies

$$\Delta H^2(\mathbb{T}, \mathbb{C}^n) \subseteq \Theta_2 H^2(\mathbb{T}, \mathbb{C}^n).$$

This proves that Θ_2 is a left inner divisor for Δ. Hence, there exists an inner function Δ'' such that $\Delta = \Theta_2 \Delta''$. Combining with Equation (4.9), we obtain

$$\Theta_2 = \Delta\Delta' = \Theta_2 \Delta'' \Delta'.$$

This identity implies that both Δ' and Δ'' must be constant unitary matrices. Therefore, A and Θ_2 are right coprime. This completes the proof. □

Having established the result for the first case, we now proceed to analyze the second case.

Case 2: Φ^* is of bounded type of the form

$$\Phi_- = B^* \Theta \quad (\text{where } B \in H^2(\mathbb{T}, M_n),\ \Theta = \theta I_n \text{ for } \theta \text{ an inner function}),$$

where B and Θ are not left coprime.

In this case, we present a counterexample to demonstrate that the anticipated result does not hold. The following discussion provides a detailed analysis, highlighting the key factors that lead to this negative outcome.

Example 4.2.6. Let

$$\Phi(e^{i\theta}) := \begin{bmatrix} e^{i\theta}+e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in L^\infty(\mathbb{T}, M_2).$$

Then by Example 4.2.2, we have $\Phi = Q\Phi^*$ and the block Toeplitz operator T_Φ is subnormal, but neither normal nor analytic.

We now show that $\Phi_- = B^*\Theta$, where $B \in H^2(\mathbb{T}, M_2)$ and $\Theta = \theta I_2$ for some inner function θ, and that B and Θ are not left coprime. Observe that

$$\Phi_-(e^{i\theta}) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e^{i2\theta} & 0 \\ 0 & e^{i2\theta} \end{bmatrix} = B^*(e^{i\theta})\Theta(e^{i\theta}),$$

where

$$B(e^{i\theta}) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Theta(e^{i\theta}) = \begin{bmatrix} e^{i2\theta} & 0 \\ 0 & e^{i2\theta} \end{bmatrix}.$$

Since

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} e^{i2\theta} & 0 \\ 0 & e^{i2\theta} \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix},$$

it follows that both B and Θ admit a common left inner divisor. This implies that they are not left coprime.

Now we turn our attention to the third case.

Case 3: Φ^* is not of bounded type.

Similar to the first case, we establish a negative answer to Problem 4.2.1 through the following example, which illustrates the failure of the anticipated conclusion under the given assumptions.

Example 4.2.7. Let

$$\Phi(e^{i\theta}) := \begin{bmatrix} (f+\overline{f})(e^{i\theta}) & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in L^\infty(\mathbb{T}, M_2),$$

where $f \in L^\infty$ is a function that is not of bounded type. Then Φ^* is also not of bounded type. Moreover, by following the same approach as in Example 4.2.2, we find that the block Toeplitz operator T_Φ is a subnormal operator but neither normal nor analytic. In addition, we observe that

$$\begin{aligned} \Phi(e^{i\theta}) &= \begin{bmatrix} (f+\overline{f})(e^{i\theta}) & 0 \\ 0 & e^{i\theta} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{bmatrix} \begin{bmatrix} (f+\overline{f})(e^{i\theta}) & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \\ &= Q(e^{i\theta})\Phi^*(e^{i\theta}), \end{aligned}$$

where

$$Q(e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{bmatrix}$$

is a finite Blaschke-Potapov product.

Nevertheless, in Case 3, we establish a sufficient condition that ensures an affirmative answer. This result is formally stated as Theorem 4.2.10. To proceed with the proof of Theorem 4.2.10, we first present a couple of auxiliary lemmas.

Lemma 4.2.8. Let Φ be a normal matrix-valued function in $L^\infty(\mathbb{T}, M_n)$ such that $\Phi = \Theta\Phi^* + G$, where Θ is an inner function and $G \in H^\infty(\mathbb{T}, M_n)$. Then the following holds.

(i) $H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})(H^2(\mathbb{T}, \mathbb{C}^n)) = H^*_{\Phi^*}(\mathcal{H}(\widetilde{\Theta}))$

(ii) $H^*_{\Phi^*}(\mathcal{H}(\widetilde{\Theta})) = T_{\Phi\Theta^*}(\mathcal{H}(\Theta))$

(iii) $\big((I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})H_{\Phi^*}\big)^*(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})H_{\Phi^*} = H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})H_{\Phi^*}$.

(iv) $H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})(H^2(\mathbb{T}, \mathbb{C}^n)) \subseteq (\ker[T^*_\Phi, T_\Phi])^\perp = \overline{\operatorname{Ran}\,[T^*_\Phi, T_\Phi]}$.

Proof. Since $\Phi = \Theta\Phi^* + G$, it follows that $\Theta \in \mathcal{E}(\Phi)$, implying that the set $\mathcal{E}(\Phi)$ contains an inner function. Moreover, by Equation (1.9), we observe that

$$I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}} = T^*_{\widetilde{\Theta}}T_{\widetilde{\Theta}} - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}} = H_{\Theta^*}H^*_{\Theta^*}.$$

Since $\ker H^*_{\Theta^*} = \ker H_{\widetilde{\Theta}^*} = \widetilde{\Theta}H^2(\mathbb{T}, \mathbb{C}^n)$ and recalling that $\ker S^*S = \ker S$ for any $S \in \mathcal{B}(\mathcal{H})$, it follows that

$$\ker(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}}) = \ker H_{\Theta^*}H^*_{\Theta^*} = \ker H^*_{\Theta^*} = \widetilde{\Theta}H^2(\mathbb{T}, \mathbb{C}^n).$$

Furthermore, since $I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}}$ is self adjoint, we conclude that

$$\operatorname{Ran}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}}) = \mathcal{H}(\widetilde{\Theta}).$$

Consequently, we obtain

$$H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})(H^2(\mathbb{T}, \mathbb{C}^n)) = H^*_{\Phi^*}(\mathcal{H}(\widetilde{\Theta})),$$

which completes the proof of (i).

Let $f \in H^*_{\Phi^*}(\mathcal{H}(\widetilde{\Theta}))$. Then there exists $g \in \mathcal{H}(\widetilde{\Theta})$ such that $f = H^*_{\Phi^*}g$. We observe that

$$f = H^*_{\Phi^*}g = H_{\breve{\Phi}}g = J_nP_n^\perp M_{\breve{\Phi}}g = P_nM_\Phi J_ng.$$

Now, let $g' := \Theta(J_ng)$. Since $g \in \mathcal{H}(\widetilde{\Theta})$, it follows that $\breve{\Theta}g \perp H^2(\mathbb{T}, \mathbb{C}^n)$. Therefore,

$$g' = \Theta(J_n g) = J_n(\breve{\Theta} g) \in H^2(\mathbb{T}, \mathbb{C}^n) \quad \text{and} \quad \Theta^* g' = J_n g \perp H^2(\mathbb{T}, \mathbb{C}^n),$$

which implies that $g' \in \mathcal{H}(\Theta)$. Next, observe that

$$T_{\Phi\Theta^*} g' = P_n(\Phi\Theta^*\Theta(J_n g)) = P_n(\Phi(J_n g)) = P_n M_\Phi J_n g = f.$$

Hence, we conclude that

$$H^*_{\Phi^*}(\mathcal{H}(\widetilde{\Theta})) \subseteq T_{\Phi\Theta^*}(\mathcal{H}(\Theta)).$$

Conversely, let $f \in T_{\Phi\Theta^*}(\mathcal{H}(\Theta))$, so that $f = T_{\Phi\Theta^*} g$ for some $g \in \mathcal{H}(\Theta)$. Let $g' := J_n(\Theta^* g)$. Since $g \perp \Theta H^2(\mathbb{T}, \mathbb{C}^n)$, it follows that $\Theta^* g \perp H^2(\mathbb{T}, \mathbb{C}^n)$ and hence $g' \in H^2(\mathbb{T}, \mathbb{C}^n)$. Furthermore, we obtain

$$\widetilde{\Theta}^* g' = \breve{\Theta} J_n(\Theta^* g) = J_n g \perp H^2(\mathbb{T}, \mathbb{C}^n),$$

which implies that $g' \in \mathcal{H}(\widetilde{\Theta})$. Now observe that

$$H^*_{\Phi^*} g' = H^*_{\Phi^*} J_n(\Theta^* g) = J_n P_n^\perp M_{\breve{\Phi}} J_n(\Theta^* g).$$

Since $J_n J_n = I$, we have

$$P_n M_\Phi J_n J_n(\Theta^* g) = P_n(\Phi\Theta^* g) = T_{\Phi\Theta^*} g$$

and therefore, $H^*_{\Phi^*} g' = T_{\Phi\Theta^*} g = f$. This shows that

$$T_{\Phi\Theta^*}(\mathcal{H}(\Theta)) \subseteq H^*_{\Phi^*}(\mathcal{H}(\widetilde{\Theta})),$$

which proves (ii).

First, observe that

$$\begin{aligned}(I - T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}})(I - T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}}) &= I - T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}} - T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}} + T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}} T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}} \\ &= I - 2T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}} + T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}} \\ &= I - T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}}.\end{aligned}$$

That is, $I - T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}}$ is an idempotent operator. Therefore, we compute

$$\big((I - T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}}) H_{\Phi^*}\big)^* (I - T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}}) H_{\Phi^*} = H^*_{\Phi^*}(I - T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}}) H_{\Phi^*},$$

which establishes the validity of (iii).

Let $f \in H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})(H^2(\mathbb{T}, \mathbb{C}^n))$ and $g \in \ker[T^*_\Phi, T_\Phi]$. Then

$$f = H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})h$$

for some $h \in H^2(\mathbb{T}, \mathbb{C}^n)$. From (iii), it follows that

$$\ker[T^*_\Phi, T_\Phi] = \ker H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})H_{\Phi^*} = \ker(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})H_{\Phi^*}.$$

Therefore, we compute

$$\langle f, g\rangle = \langle H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})h, g\rangle = \langle h, (I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})H_{\Phi^*}g\rangle = 0.$$

Thus, $f \in (\ker[T^*_\Phi, T_\Phi])^\perp$. Recall that for any bounded self-adjoint operator A on a Hilbert space $\mathcal{H}$, we have

$$\ker A = (\operatorname{Ran} A)^\perp \quad \text{and} \quad (\ker A)^\perp = \overline{\operatorname{Ran} A}.$$

Consequently,

$$H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})(H^2(\mathbb{T}, \mathbb{C}^n)) \subseteq (\ker[T^*_\Phi, T_\Phi])^\perp = \overline{\operatorname{Ran}\,[T^*_\Phi, T_\Phi]},$$

which establishes (iv). □

As a direct consequence of Lemma 4.2.8, we obtain the following result.

Lemma 4.2.9. Let Φ be a normal matrix-valued function in $L^\infty(\mathbb{T}, M_n)$. If there is an inner function Θ in $\mathcal{E}(\Phi)$, then the closure of $\operatorname{Ran}[T^*_\Phi, T_\Phi]$ equals the closure of $T_{\Phi\Theta^*}(\mathcal{H}(\Theta))$.

Proof. By combining parts (i) and (ii) of Lemma 4.2.8, we obtain

$$H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})\Big(H^2(\mathbb{T}, \mathbb{C}^n)\Big) = T_{\Phi\Theta^*}(\mathcal{H}(\Theta)).$$

Now by applying Lemma 3.1.1, it follows that

$$\begin{aligned}\operatorname{Ran}\,[T^*_\Phi, T_\Phi] &= H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})H_{\Phi^*}\big(H^2(\mathbb{T}, \mathbb{C}^n)\big)\\ &\subseteq H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})\big(H^2(\mathbb{T}, \mathbb{C}^n)\big)\\ &= T_{\Phi\Theta^*}(\mathcal{H}(\Theta)).\end{aligned}$$

It then follows from part (iv) of Lemma 4.2.8 that

$$T_{\Phi\Theta^*}(\mathcal{H}(\Theta)) = H^*_{\Phi^*}(I - T_{\widetilde{\Theta}}T^*_{\widetilde{\Theta}})\big(H^2(\mathbb{T}, \mathbb{C}^n)\big) \subseteq \overline{\mathrm{Ran}\,[T^*_\Phi, T_\Phi]}. \tag{4.11}$$

Therefore, the closure of Ran $[T^*_\Phi, T_\Phi]$ coincides with the closure of $T_{\Phi\Theta^*}(\mathcal{H}(\Theta))$. This completes the proof. □

Theorem 4.2.10 gives a sufficient condition for the normality of a subnormal block Toeplitz operator T_Φ under the assumption that $\Phi = Q\Phi^*$, where Q is a finite Blaschke-Potapov product and Φ^* is not of bounded type.

For $\Phi = [\phi_{ij}] \in L^\infty(\mathbb{T}, M_n)$, we define $\overline{\Phi} := [\overline{\phi_{ij}}]$. Then we have:

Theorem 4.2.10. Let $\Phi \in L^\infty(\mathbb{T}, M_n)$ be a matrix-valued function of the form $\Phi = Q\Phi^*$, where Q is a finite Blaschke-Potapov product and Φ^* is not of bounded type. Suppose that both the adjoint Toeplitz operator T^*_Φ and the Hankel operator $H_{\overline{\Phi}}$ are injective. Then the subnormality of T_Φ implies that it is normal.

Proof. Suppose that $\Phi = Q\Phi^*$, where Q is a finite Blaschke-Potapov product. Then by Lemma 3.2.2, it follows that Ran$[T^*_\Phi, T_\Phi]$ is finite dimensional. In particular, Ran$[T^*_\Phi, T_\Phi]$ is a closed subspace. Therefore, applying Lemma 4.2.9, we obtain

$$\mathrm{Ran}[T^*_\Phi, T_\Phi] = T_{\Phi Q^*}(\mathcal{H}(Q)) = T_{\Phi^*}(\mathcal{H}(Q)). \tag{4.12}$$

We now consider two cases. First, suppose that Q is a unitary matrix. In this case, we observe that

$$QH^2(\mathbb{T}, \mathbb{C}^n) = H^2(\mathbb{T}, \mathbb{C}^n),$$

which implies $\mathcal{H}(Q) = \{0\}$. Therefore, from Equation (4.12), it follows that

$$\mathrm{Ran}[T^*_\Phi, T_\Phi] = T_{\Phi^*}(\mathcal{H}(Q)) = \{0\}.$$

Hence, the self-commutator of T_Φ vanishes and we conclude that T_Φ is a normal operator.

Now, suppose that Q is not a unitary matrix. Then $\dim \mathcal{H}(Q) \geq 1$. Since the self-commutator $[T^*_\Phi, T_\Phi]$ is a finite rank self-adjoint operator, it follows that

$$\text{Ran}[T_\Phi^*, T_\Phi] = (\ker[T_\Phi^*, T_\Phi])^\perp.$$

Recall that if T_Φ is subnormal, then $\text{Ran}[T_\Phi^*, T_\Phi]$ is invariant under T_Φ^*. Therefore, from Equation (4.12), we deduce that

$$\begin{aligned} T_\Phi^*\Big(\text{Ran}[T_\Phi^*, T_\Phi] + \mathcal{H}(Q)\Big) &= T_\Phi^*\big(\text{Ran}[T_\Phi^*, T_\Phi]\big) + T_\Phi^*(\mathcal{H}(Q)) \\ &= \text{Ran}[T_\Phi^*, T_\Phi]. \end{aligned} \tag{4.13}$$

We now proceed to show that

$$\text{Ran}[T_\Phi^*, T_\Phi] \bigcap \mathcal{H}(Q) = \{0\}. \tag{4.14}$$

To prove this, let $f \in \text{Ran}[T_\Phi^*, T_\Phi] \bigcap \mathcal{H}(Q)$. Then,

$$f \in \text{Ran}[T_\Phi^*, T_\Phi] \quad \text{and} \quad f \perp QH^2(\mathbb{T}, \mathbb{C}^n),$$

which implies that $Q^* f \perp H^2(\mathbb{T}, \mathbb{C}^n)$ and hence $T_{Q^*} f = 0$. By Equation (4.12), there exists $g \in \mathcal{H}(Q)$ such that $f = T_{\Phi Q^*} g$. In addition, observe that $Q^t \overline{g} \in H^2(\mathbb{T}, \mathbb{C}^n)$, where Q^t denotes the transpose of Q. Since Φ and Q^* commute, it follows that

$$0 = T_{Q^*} f = T_{Q^*} T_{\Phi Q^*} g = T_{Q^{*2}\Phi} g.$$

This implies that

$$Q^{*2} \Phi g = \Phi Q^{*2} g \perp H^2(\mathbb{T}, \mathbb{C}^n).$$

Taking the transpose and using the identity $(g^* Q^2 \Phi^*)^t = \overline{\Phi}(Q^2)^t \overline{g}$, we obtain

$$(g^* Q^2 \Phi^*)^t = \overline{\Phi}(Q^2)^t \overline{g} \in H^2(\mathbb{T}, \mathbb{C}^n),$$

which implies that $(Q^2)^t \overline{g} \in \ker H_{\overline{\Phi}}$. Moreover, by hypothesis, the Hankel operator $H_{\overline{\Phi}}$ is injective and thus $(Q^2)^t \overline{g} = 0$. Since Q is invertible, it follows that $\overline{g} = 0$ and hence $g = 0$. This shows that

$$f = T_{\Phi Q^*} g = 0. \tag{4.15}$$

This completes the proof of Equation (4.14).

Using Equation (4.15), we obtain

$$\dim(\text{Ran}[T_\Phi^*, T_\Phi] + \mathcal{H}(Q)) = \dim \text{Ran}[T_\Phi^*, T_\Phi] + \dim \mathcal{H}(Q). \tag{4.16}$$

Moreover, by the Rank Theorem and Equation (4.13), we have

$$\begin{aligned}
&\dim(\text{Ran}[T_\Phi^*, T_\Phi] + \mathcal{H}(Q)) \\
&\qquad = \dim\ker T_{\Phi^*|(\text{Ran}[T_\Phi^*,T_\Phi]+\mathcal{H}(Q))} + \dim \text{Ran} T_{\Phi^*|(\text{Ran}[T_\Phi^*,T_\Phi]+\mathcal{H}(Q))} \\
&\qquad = \dim\ker T_{\Phi^*|(\text{Ran}[T_\Phi^*,T_\Phi]+\mathcal{H}(Q))} + \dim \text{Ran}[T_\Phi^*, T_\Phi]. \qquad (4.17)
\end{aligned}$$

Comparing Equations (4.16) and (4.17), we obtain

$$\dim \mathcal{H}(Q) = \dim\ker T_{\Phi^*|(\text{Ran}[T_\Phi^*,T_\Phi]+\mathcal{H}(Q))},$$

which implies that

$$\dim\ker T_{\Phi^*} \geq \dim \mathcal{H}(Q) \geq 1.$$

This contradicts the assumption that T_{Φ^*} is injective. This concludes the proof. □

Remark 4.2.11. The preceding Theorem 4.2.10 relied on the injectivity of the Hankel operator $H_{\overline{\Phi}}$ to conclude that the intersection $\text{Ran}[T_\Phi^*, T_\Phi] \bigcap \mathcal{H}(Q)$ is trivial. However, one may naturally ask whether this conclusion remains valid if $H_{\overline{\Phi}}$ fails to be injective. The following example shows that the result may fail if the operator $H_{\overline{\Phi}}$ is not injective.

Let $f \in H^\infty$ be such that $\overline{f}$ is not of bounded type and define $\varphi := f + \overline{f}$. Set

$$\Phi(e^{i\theta}) := \begin{bmatrix} \varphi(e^{i\theta}) & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in L^\infty(\mathbb{T}, M_2).$$

Then Φ^* is not of bounded type and observe that

$$\Phi = Q\Phi^*, \quad \text{where} \quad Q(e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{bmatrix}$$

is a finite Blaschke-Potapov product. We now compute

$$H_{\overline{\Phi}} \begin{bmatrix} 0 \\ e^{i\theta} \end{bmatrix} = \begin{bmatrix} H_{\varphi} & 0 \\ 0 & H_{e^{-i\theta}} \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which shows that $H_{\overline{\Phi}}$ is not injective.

Next, consider

$$[T_{\Phi}^*, T_{\Phi}] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & [T_{e^{i\theta}}^*, T_{e^{i\theta}}] \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For any $f, g \in H^2$, we compute

$$\left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Q \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} f \\ e^{i2\theta} g \end{bmatrix} \right\rangle = 0.$$

Thus, we conclude that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \operatorname{Ran}[T_{\Phi}^*, T_{\Phi}] \bigcap \mathcal{H}(Q),$$

that is,

$$\operatorname{Ran}[T_{\Phi}^*, T_{\Phi}] \bigcap \mathcal{H}(Q) \neq \{0\}.$$

Remark 4.2.12. In Theorem 4.2.10, the injectivity of T_{Φ}^* and $H_{\overline{\Phi}}$ is not a necessary condition, even in the case where T_{Φ} is self-adjoint. To illustrate this, consider a function φ as described in Remark 4.2.11. Since φ is real-valued, it follows that the Toeplitz operator T_{φ} is self-adjoint. Define the matrix-valued function

$$\Phi := \begin{bmatrix} \varphi & 0 \\ 0 & 0 \end{bmatrix} \in L^{\infty}(\mathbb{T}, M_2).$$

Then, Φ^* is not of bounded type and T_{Φ} is self-adjoint. Furthermore, observe that

$$\Phi = Q\Phi^*, \quad \text{where} \quad Q(e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

and note that Q is a finite Blaschke-Potapov product. Now consider the following:

$$T_{\Phi^*} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T_\varphi & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and similarly,

$$H_{\overline{\Phi}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} H_{\overline{\varphi}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

These computations demonstrate that both T_{Φ^*} and $H_{\overline{\Phi}}$ fail to be injective, even though T_Φ is self-adjoint.

4.3 Extension of Nakazi-Takahashi Theorem for matrix valued symbols

In this section, we develop the matrix-valued counterpart of Theorem 2.2.3. We begin by showing with Example 4.3.1 that Theorem 2.1.4, which holds in the scalar-valued case, does not extend to the setting of block Toeplitz operators. This example highlights the additional complexities of matrix-valued symbols and the need for a refined approach in the setting of block Toeplitz operators.

Example 4.3.1. Let

$$\Phi(e^{i\theta}) := \begin{bmatrix} e^{i\theta} + e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in L^\infty(\mathbb{T}, M_2).$$

Then, it is easy to see that

$$\Phi(e^{i\theta}) = \begin{bmatrix} e^{i\theta} + e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \tag{4.18}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} e^{i\theta} + e^{-i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & e^{i\theta} - 1 \end{bmatrix}$$

for all $e^{i\theta} \in \mathbb{T}$. This shows that

$$B(e^{i\theta}) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in \mathcal{E}(\Phi).$$

Next, observe that

$$[T_\Phi^*, T_\Phi] = \begin{bmatrix} 0 & 0 \\ 0 & [T_{e^{i\theta}}^*, T_{e^{i\theta}}] \end{bmatrix} \quad \text{and} \quad \ker[T_\Phi^*, T_\Phi] = \begin{bmatrix} H^2 \\ e^{i\theta} H^2 \end{bmatrix}.$$

Moreover, we compute

$$T_\Phi\big(B \ker[T_\Phi^*, T_\Phi]\big) = \begin{bmatrix} T_{e^{i\theta}+e^{-i\theta}} & 0 \\ 0 & T_{e^{i\theta}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} H^2 \\ e^{i\theta} H^2 \end{bmatrix}$$

$$= \begin{bmatrix} T_{e^{i\theta}+e^{-i\theta}} & 0 \\ 0 & T_{e^{i\theta}} \end{bmatrix} \begin{bmatrix} H^2 \\ e^{i2\theta} H^2 \end{bmatrix}$$

$$\subseteq B \ker[T_\Phi^*, T_\Phi].$$

However, T_Φ is a subnormal block Toeplitz operator which is neither normal nor analytic.

We now turn our attention to the validity of Theorem 2.2.3 in the setting of block Toeplitz operators. Unfortunately, Example 4.2.2 shows that Theorem 2.2.3

does not extend directly to the case of block Toeplitz operators. To address this limitation, we investigate what additional conditions may ensure the validity of the theorem. Our analysis proceeds by considering three distinct cases.

Case 1: Suppose that Φ^* is of bounded type and can be written in the form

$$\Phi_+ = A^*\Theta_0\Theta_2 \quad \text{and} \quad \Phi_- = B^*\Theta_2,$$

where $\Theta_0 = \theta_0 I_n$ and $\Theta_2 = \theta_2 I_n$ for some scalar inner functions θ_0 and θ_2, and $A, B \in H^2(\mathbb{T}, M_n)$. Further assume that A, B and Θ_2 are left coprime.

In this case, we have an affirmative answer by Abrahamse's Theorem for matrix-valued symbols [23].

Case 2: Suppose that Φ^* is of bounded type and can be written in the form

$$\Phi_+ = A^*\Theta_0\Theta_2 \quad \text{and} \quad \Phi_- = B^*\Theta_2,$$

where $\Theta_0 = \theta_0 I_n$ and $\Theta_2 = \theta_2 I_n$ for some scalar inner functions θ_0 and θ_2, and $A, B \in H^2(\mathbb{T}, M_n)$. Further assume that A, B and Θ_2 are not left coprime.

A careful analysis of Example 4.2.6 reveals that the conditions under consideration in this case are insufficient to guarantee a positive outcome.

Case 3: Φ^* is not of bounded type.

In this case as well, Example 4.2.7 yields a negative outcome, indicating that the desired conclusion does not hold under the given assumptions. Nevertheless, the following theorem provides a partial positive result by identifying a subclass where the conclusion does hold.

Theorem 4.3.2. Let $\Phi = B\Phi^* + G \in L^\infty(\mathbb{T}, M_n)$, where B is a finite Blaschke-Potapov product, $G \in H^\infty(\mathbb{T}, M_n)$ and Φ^* is not bounded type. Suppose the following conditions hold:

(i) $B^*\Phi = \Phi B^*$

(ii) $GH^2(\mathbb{T}, \mathbb{C}^n) \subseteq BH^2(\mathbb{T}, \mathbb{C}^n)$

(iii) T_Φ^* and $H_{\overline{\Phi}}$ are injective.

Then if the Toeplitz operator T_Φ is subnormal, it must be normal.

Proof. Suppose that T_Φ is subnormal. Then by [32, Theorem 3.3], it follows that the symbol Φ is a normal matrix-valued function. Since $B \in \mathcal{E}(\Phi)$, Theorem 3.2.2 ensures that the self-commutator $[T_\Phi^*, T_\Phi]$ is of finite rank. Consequently, applying Lemma 4.2.9, we obtain

$$\text{Ran}[T_\Phi^*, T_\Phi] = T_{\Phi B^*}(\mathcal{H}(B)).$$

We now claim that B is a unitary matrix. This immediately yields

$$\mathcal{H}(B) = H^2(\mathbb{T}, \mathbb{C}^n) \ominus BH^2(\mathbb{T}, \mathbb{C}^n) = \{0\}$$

and therefore,

$$\text{Ran}[T_\Phi^*, T_\Phi] = T_{\Phi B^*}(\mathcal{H}(B)) = \{0\}.$$

It follows that the self-commutator vanishes and thus, T_Φ is normal.

To establish that B is unitary, we proceed by contradiction. Suppose that B is not unitary. Then the model space $\mathcal{H}(B)$ is nontrivial, that is, $\dim \mathcal{H}(B) \geq 1$. Given that

$$GH^2(\mathbb{T}, \mathbb{C}^n) \subseteq BH^2(\mathbb{T}, \mathbb{C}^n),$$

it follows that $\mathcal{H}(B) \subseteq H^2(\mathbb{T}, \mathbb{C}^n) \ominus GH^2(\mathbb{T}, \mathbb{C}^n)$. Consequently, for any $f \in \mathcal{H}(B)$, we have $T_{G^*}f = 0$ and hence,

$$T_{G^*}\big(\mathcal{H}(B)\big) = \{0\}.$$

Given that $\Phi = B\Phi^* + G$, we immediately obtain the identities

$$B\Phi^* = \Phi - G \quad \text{and} \quad \Phi B^* = \Phi^* - G^*.$$

Using these identities, the range of the self-commutator can be written as

$$\text{Ran}[T_\Phi^*, T_\Phi] = T_{\Phi B^*}\big(\mathcal{H}(B)\big) = T_{\Phi^* - G^*}\big(\mathcal{H}(B)\big).$$

However, since we have already observed that $T_{G^*}\big(\mathcal{H}(B)\big) = \{0\}$, the term involving G^* vanishes, and we obtain

$$\operatorname{Ran}[T_\Phi^*, T_\Phi] = T_{\Phi^*}\big(\mathcal{H}(B)\big).$$

By employing the method used in the proof of Theorem 4.2.10 and noting that B^* and Φ commute, we get

$$\operatorname{Ran}[T_\Phi^*, T_\Phi] \bigcap \mathcal{H}(B) = \{0\}.$$

Consequently, the sum of these subspaces is direct and we have

$$\dim \Big(\operatorname{Ran}[T_\Phi^*, T_\Phi] + \mathcal{H}(B)\Big) = \dim \operatorname{Ran}[T_\Phi^*, T_\Phi] + \dim \mathcal{H}(B).$$

Now, by the Rank Theorem, we see that

$$\dim \operatorname{Ran}[T_\Phi^*, T_\Phi] + \dim \mathcal{H}(B) = \dim \ker T^*_{\Phi|\{\operatorname{Ran}[T_\Phi^*, T_\Phi] + \dim \mathcal{H}(B)\}} + \dim \operatorname{Ran}[T_\Phi^*, T_\Phi].$$

This yields

$$\dim \ker T_\Phi^* \geq \dim \mathcal{H}(B) \geq 1,$$

which contradicts the assumption that T_Φ^* is injective. Hence, the claim is established. □

We conclude this chapter by presenting an extension of [44, Lemma 6(1)] to the setting of matrix-valued symbols. This result may serve as a useful tool in the further investigation of subnormal block Toeplitz operators.

Theorem 4.3.3. Let $\Phi = \Theta\Phi^* + G \in L^\infty(\mathbb{T}, M_n)$, where $G \in L^\infty(\mathbb{T}, M_n)$, Θ is an inner function and suppose that the block Toeplitz operator T_Φ is subnormal. Assume further that $\Phi\Theta^* = \Theta^*\Phi$ and $G\Theta = \Theta G$. Define the subspace $\mathcal{M}$ by

$$\mathcal{M} := \overline{\operatorname{Ran}\ [T_\Phi^*, T_\Phi] + \mathcal{H}(\Theta)}.$$

Then $\mathcal{M}$ is invariant under the adjoint operator T_Φ^*.

Proof. Since $\Phi = \Theta\Phi^* + G \in L^\infty(\mathbb{T}, M_n)$, it follows that

$$\Phi^* = \Phi\Theta^* + G^*.$$

Moreover, the commutativity of Φ and Θ^* implies that

$$T_\Phi^*(\mathcal{H}(\Theta)) \subseteq T_{\Theta^*\Phi}(\mathcal{H}(\Theta)) + T_G^*(\mathcal{H}(\Theta)).$$

We claim that $T_G^*(\mathcal{H}(\Theta)) \subseteq \mathcal{H}(\Theta)$. To see this, let $f \in \mathcal{H}(\Theta)$ and define $g := T_G^* f$. For $h \in H^2(\mathbb{T}, \mathbb{C}^n)$, we compute:

$$\langle g, \Theta h \rangle = \langle T_G^* f, \Theta h \rangle = \langle f, G\Theta h \rangle.$$

Since $G\Theta = \Theta G$, it follows that

$$\langle g, \Theta h \rangle = \langle f, G\Theta h \rangle = \langle f, \Theta G h \rangle = 0$$

for $f \in \mathcal{H}(\Theta)$. Hence, $g = T_G^* f \in \mathcal{H}(\Theta)$ and so

$$T_G^*(\mathcal{H}(\Theta)) \subseteq \mathcal{H}(\Theta).$$

Now, by Lemma 4.2.9, we obtain

$$T_\Phi^*(\mathcal{H}(\Theta)) \subseteq \overline{\text{Ran } [T_\Phi^*, T_\Phi] + \mathcal{H}(\Theta)}.$$

Finally, since $\text{Ran}[T_\Phi^*, T_\Phi]$ is invariant under T_Φ^* due to the subnormality of T_Φ, it follows that $\mathcal{M}$ is invariant under T_Φ^*. $\square$

Chapter 5

TOEPLITZ OPERATORS WITH OPERATOR VALUED SYMBOLS

The study of subnormal and hyponormal Toeplitz operators, along with their block matrix analogues, has sparked significant curiosity and interest among operator theorists. This interest naturally extends to the more general framework of Toeplitz operators with operator-valued symbols, where the symbol is a function taking values in the algebra of bounded linear operators on a Hilbert space. Investigating these properties offers deeper insights into the intricate interplay between operator theory and function theory, particularly in understanding how the structural and spectral characteristics of Toeplitz operators evolve when their symbols are generalized to take operator values.

In this chapter, we present a sufficient condition for the hyponormality of Toeplitz operators whose symbols are bounded operator-valued functions. This result is formally presented in Theorem 5.2.3. Furthermore, by extending and refining the methods developed by C. Gu, J. Hendricks and D. Rutherford [32], we provide characterizations of both hyponormality and subnormality of analytic Toeplitz operators with operator-valued symbols. These characterizations, which reveal the intricate relationship between analyticity and operator-theoretic behavior in the operator-valued setting, are presented in Corollary 5.2.5 and Theorem 5.3.1.

5.1 Introduction

The characterization of hyponormality for block Toeplitz operators established by C. Gu, J. Hendricks and D. Rutherford [32] plays a foundational role in this area. However, when the symbol is operator-valued, neither of the two conditions identified in their characterization remains necessary for hyponormality. This significant observation was first made by C. Gu [31] and the following example demonstrate it explicitly.

Example 5.1.1. Consider the function $\Phi(e^{i\theta}) := \varphi(e^{i\theta})S \in L^\infty_{SOT}(\mathbb{T}, B(\ell^2))$, where $\varphi(e^{i\theta}) = |e^{i\theta} + e^{-i\theta}|$ is a scalar-valued function and S is the right shift operator on ℓ^2. Then Φ defines a bounded operator-valued function on the unit circle $\mathbb{T}$ and the associated Toeplitz operator T_Φ acts on the Hilbert space-valued Hardy space $H^2(\mathbb{T}, \ell^2)$. The function $\Phi(e^{i\theta})$ and the corresponding Toeplitz operator T_Φ admit the following infinite matrix representations:

$$\Phi(e^{i\theta}) = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \varphi(e^{i\theta}) & 0 & 0 & \cdots \\ 0 & \varphi(e^{i\theta}) & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad T_\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ T_\varphi & 0 & 0 & 0 & \cdots \\ 0 & T_\varphi & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where T_φ denotes the scalar Toeplitz operator with symbol φ.

Now observe that

$$T_\Phi^* T_\Phi = \begin{bmatrix} T_\varphi^2 & 0 & 0 & \cdots \\ 0 & T_\varphi^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad T_\Phi T_\Phi^* = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & T_\varphi^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Thus, the self-commutator of T_Φ is given by

$$[T_\Phi^*, T_\Phi] = T_\Phi^* T_\Phi - T_\Phi T_\Phi^* = \begin{bmatrix} T_\varphi^2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Finally, since for any $f \in H^2$,

$$\langle T_\varphi^2 f, f \rangle = \langle T_\varphi f, T_\varphi f \rangle = \|T_\varphi f\|^2 \geq 0$$

it follows that the self-commutator $[T_\Phi^*, T_\Phi]$ is positive semidefinite. Therefore, T_Φ is hyponormal.

In the framework of block Toeplitz operators, the hyponormality of the operator T_Ψ is characterized by the following two conditions (see [32, Theorem 3.3]):

(i) Ψ is normal;

(ii) $\mathcal{E}(\Psi)$ is nonempty,

where condition (ii) is equivalent to the positivity of the operator

$$H_{\Psi^*}^* H_{\Psi^*} - H_\Psi^* H_\Psi,$$

as established in [31, Corollary 2].

We begin by observing that

$$\Phi^*(e^{i\theta})\Phi(e^{i\theta}) = \begin{bmatrix} |\varphi(e^{i\theta})|^2 & 0 & \cdots \\ 0 & |\varphi(e^{i\theta})|^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\Phi(e^{i\theta})\Phi^*(e^{i\theta}) = \begin{bmatrix} 0 & 0 & \cdots \\ 0 & |\varphi(e^{i\theta})|^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

which shows that $\Phi(e^{i\theta})$ does not commute with its adjoint. Hence, Φ is not normal.

Next, we examine the associated Hankel operators, which take the following matrix forms:

$$H_\Phi = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ H_\varphi & 0 & 0 & \cdots \\ 0 & H_\varphi & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad H_{\Phi^*} = \begin{bmatrix} 0 & H_\varphi & 0 & \cdots \\ 0 & 0 & H_\varphi & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now, we compute

$$H_{\Phi^*}^* H_{\Phi^*} = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & H_\varphi^* H_\varphi & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ and } H_\Phi^* H_\Phi = \begin{bmatrix} H_\varphi^* H_\varphi & 0 & 0 & \cdots \\ 0 & H_\varphi^* H_\varphi & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore,

$$H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi = \begin{bmatrix} -H_\varphi^* H_\varphi & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which shows that the positivity condition fails even though T_Φ is hyponormal.

5.2 Hyponormality of Toeplitz operators with operator valued symbols

We now turn our attention to the study of the hyponormality of Toeplitz operators with operator-valued symbols. As a preliminary step, we investigate the Poisson integral, which will be essential in establishing the auxiliary results needed for our analysis.

Let X be a complex Banach space. For $f \in L^1(\mathbb{T}, X)$, the *Poisson integral* of f, denoted by $P[f]$, is defined by

$$P[f](z) := \frac{1}{2\pi} \int_0^{2\pi} P_z(e^{i\theta}) f(e^{i\theta})\, d\theta, \tag{5.1}$$

where $z \in \mathbb{D}$ and P_z is the *Poisson kernel* given by

$$P_z(e^{i\theta}) := \frac{1 - |z|^2}{|e^{i\theta} - z|^2}.$$

The following result by N. K. Nikolski [47] is important and will be used in the subsequent development.

Lemma 5.2.1. [47, Lemma 3.11.6] Let X be a Banach space and $h \in L^p(\mathbb{T}, X)$, where $1 \le p \le \infty$. For $0 \le r < 1$, we denote $(P[h])_r(e^{i\theta}) := P[h](re^{i\theta})$. Then

(a) $\|(P[h])_r\|_{L^p(\mathbb{T},X)} \le \|h\|_{L^p(\mathbb{T},X)}$ for all $0 \le r < 1$;

(b) if $p < \infty$, then $\lim_{r\to 1} \|(P[h])_r - h\|_{L^p(\mathbb{T},X)} = 0$;

(c) $\lim_{r\to 1} \|(P[h])_r(e^{i\theta}) - h(e^{i\theta})\|_X = 0$ for almost all $\theta \in [0, 2\pi]$.

Let $\mathcal{H}$ be a separable complex Hilbert space. For each pair (G, x), where G belongs to $L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$ and $x \in \mathcal{H}$, consider the function $Gx : \mathbb{T} \to \mathcal{H}$ defined by

$$Gx(e^{i\theta}) := G(e^{i\theta})x, \quad \theta \in [0, 2\pi].$$

Since $G \in L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$, it follows that Gx is strongly measurable and essentially bounded. Hence, $Gx \in L^\infty(\mathbb{T}, \mathcal{H})$.

We now define a function L on $L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$ by setting

$$L[G](z)\, x := P[Gx](z) = \frac{1}{2\pi} \int_0^{2\pi} P_z(e^{i\theta})\, G(e^{i\theta})\, x d\theta, \quad z \in \mathbb{D}. \tag{5.2}$$

Observe that for each $z \in \mathbb{D}$, the mapping $x \mapsto L[G](z)\,x$ defines a linear operator on $\mathcal{H}$. Moreover, using the inequality

$$0 < P_z(e^{i\theta}) \leq \frac{1+|z|}{1-|z|},$$

we obtain

$$\begin{aligned}\|L[G](z)\,x\|_{\mathcal{H}} &\leq \frac{1}{2\pi}\int_0^{2\pi} |P_z(e^{i\theta})|\;\|G(e^{i\theta})\,x\|_{\mathcal{H}} d\theta \\ &\leq \frac{1+|z|}{1-|z|}\,\|G\|_{L^\infty_{SOT}(\mathbb{T},\mathcal{B}(\mathcal{H}))}\,\|x\|_{\mathcal{H}}.\end{aligned}$$

This shows that $L[G](z)$ belongs to $\mathcal{B}(\mathcal{H})$ for each $z \in \mathbb{D}$.

The following lemma, which examines the connection between the positivity of the Toeplitz operator T_G with an operator-valued symbol G and the positivity of the symbol G itself, plays a crucial role in the study of hyponormal Toeplitz operators with operator-valued symbols.

Lemma 5.2.2. Let $\mathcal{H}$ be a separable complex Hilbert space and $G \in L^\infty_{SOT}(\mathbb{T},\mathcal{B}(\mathcal{H}))$. Then the following statements are equivalent:

(i) T_G is a positive operator on $H^2(\mathbb{T},\mathcal{H})$;

(ii) $L[G](z)$ is a positive operator on $\mathcal{H}$ for all $z \in \mathbb{D}$;

(iii) $G(e^{i\theta})$ is a positive operator on $\mathcal{H}$ for almost all $e^{i\theta} \in \mathbb{T}$.

Proof. Suppose that T_G is a positive operator on $H^2(\mathbb{T},\mathcal{H})$. For each $z \in \mathbb{D}$, let k_z denote the normalized reproducing kernel of H^2 defined by

$$k_z(e^{i\theta}) := \frac{(1-|z|^2)^{\frac{1}{2}}}{(1-\overline{z}e^{i\theta})}, \quad e^{i\theta} \in \mathbb{T}.$$

Then for each $x \in \mathcal{H}$ and $z \in \mathbb{D}$, the function $e^{i\theta} \mapsto k_z(e^{i\theta})x$ belongs to $H^2(\mathbb{T},\mathcal{H})$ and we have:

$$\begin{aligned}\big\langle T_G k_z x, k_z x\big\rangle_{H^2(\mathbb{T},\mathcal{H})} &= \big\langle G k_z x, k_z x\big\rangle_{H^2(\mathbb{T},\mathcal{H})} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \big\langle G(e^{i\theta})k_z(e^{i\theta})x, k_z(e^{i\theta})x\big\rangle_{\mathcal{H}}\, d\theta.\end{aligned}$$

Since $k_z(e^{i\theta})$ is scalar-valued, it factors out of the inner product, yielding

$$\big\langle T_G k_z x, k_z x\big\rangle_{H^2(\mathbb{T},\mathcal{H})} = \frac{1}{2\pi}\int_0^{2\pi} |k_z(e^{i\theta})|^2 \langle G(e^{i\theta})x, x\rangle_{\mathcal{H}}\, d\theta.$$

Recall that

$$|k_z(e^{i\theta})|^2 = \frac{1-|z|^2}{|e^{i\theta}-z|^2} = P_z(e^{i\theta}).$$

Moreover, P_z is scalar valued. Therefore, by the properties of the inner product, we have

$$\langle T_G k_z x, k_z x\rangle_{H^2(\mathbb{T},\mathcal{H})} = \frac{1}{2\pi}\int_0^{2\pi} \langle P_z(e^{i\theta})G(e^{i\theta})x, x\rangle_{\mathcal{H}}\, d\theta.$$

By the Riesz representation theorem, we obtain

$$\langle T_G k_z x, k_z x\rangle_{H^2(\mathbb{T},\mathcal{H})} = \int_0^{2\pi} x^*\Big(P_z(e^{i\theta})G(e^{i\theta})x\Big)d\theta.$$

Using Equations (1.2) and (5.2), this can be rewritten as

$$\langle T_G k_z x, k_z x\rangle_{H^2(\mathbb{T},\mathcal{H})} = x^*\left(\frac{1}{2\pi}\int_0^{2\pi} P_z(e^{i\theta})G(e^{i\theta})x\, d\theta\right) = x^*\Big(L[G](z)x\Big).$$

Again by the Riesz representation theorem, we conclude that

$$\langle T_G k_z x, k_z x\rangle_{H^2(\mathbb{T},\mathcal{H})} = \langle L[G](z)x, x\rangle_{\mathcal{H}}.$$

Since T_G is a positive operator, it follows that the left-hand side is nonnegative for every $x \in \mathcal{H}$. Hence, $L[G](z)$ is a positive operator on $\mathcal{H}$ for every $z \in \mathbb{D}$. This establishes that (i) $\implies$ (ii).

Assume that $L[G](z)$ is a positive operator on $\mathcal{H}$ for all $z \in \mathbb{D}$. Then, by Lemma 5.2.1, we obtain

$$\langle G(e^{i\theta})x, x\rangle_{\mathcal{H}} = \left\langle \lim_{r\to 1_-} P[Gx](re^{i\theta}), x\right\rangle_{\mathcal{H}} = \lim_{r\to 1_-} \langle L[G](re^{i\theta})x, x\rangle_{\mathcal{H}} \geq 0$$

for all $x \in \mathcal{H}$ and for almost every $e^{i\theta} \in \mathbb{T}$. Hence, (ii) $\implies$ (iii).

Let $G(e^{i\theta})$ is a positive operator on $\mathcal{H}$ for almost all $e^{i\theta} \in \mathbb{T}$. Since G belongs to $L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$, we may assume without loss of generality that $\|G(e^{i\theta})\|_{\mathcal{B}(\mathcal{H})} < 1$ for all $e^{i\theta} \in \mathbb{T}$. Then there exists a sequence $(p_n)_{n\geq 0}$ of polynomials such that $Q_n(e^{i\theta}) := p_n(G(e^{i\theta})) \geq 0$ and the sequence $(Q_n)_{n\geq 0}$ converges strongly for almost every $e^{i\theta} \in \mathbb{T}$. Moreover, for every $x \in \mathcal{H}$, we have

$$\lim_{n\to\infty} Q_n(e^{i\theta})^2 x = G(e^{i\theta})x \quad \text{for all } x \in \mathcal{H} \quad (\text{see [36, Problem 121]}).$$

Let $Q(e^{i\theta}) := \mathrm{SOT}-\lim_{n\to\infty} Q_n(e^{i\theta})$ and $\{e_n : n = 1, 2, 3, \ldots\}$ be an orthonormal basis for $\mathcal{H}$. Then for any $x, y \in \mathcal{H}$, we see that

$$\begin{aligned}\langle G^2(e^{i\theta})x, y\rangle = \langle G(e^{i\theta})x, G(e^{i\theta})y\rangle &= \sum_{n=1}^{\infty} \langle G(e^{i\theta})x, e_n\rangle\langle e_n, G(e^{i\theta})y\rangle \\ &= \sum_{n=1}^{\infty} \langle G(e^{i\theta})x, e_n\rangle\langle G(e^{i\theta})e_n, y\rangle.\end{aligned}$$

This expression shows that G^2 is SOT-measurable by the Pettis measurability theorem. By induction, it follows that G^n is SOT-measurable for all $n \in \mathbb{N}$. Consequently, each $Q_n = p_n(G)$ is SOT-measurable and so is the pointwise SOT-limit Q.

For any polynomial $p = \sum_{k=0}^n \hat{p}(k)e^{in\theta}$ with coefficients in $\mathcal{H}$ and for almost every $e^{i\theta} \in \mathbb{T}$, we have

$$\begin{aligned}\langle G(e^{i\theta})p(e^{i\theta}), p(e^{i\theta})\rangle_{\mathcal{H}} &= \sum_{j,k} \langle G(e^{i\theta})\hat{p}(j)e^{ij\theta}, \hat{p}(k)e^{ik\theta}\rangle_{\mathcal{H}} \\ &= \sum_{j,k} \langle Q(e^{i\theta})\hat{p}(j)e^{ij\theta}, Q(e^{i\theta})\hat{p}(k)e^{ik\theta}\rangle_{\mathcal{H}} \\ &= \langle Q(e^{i\theta})p(e^{i\theta}), Q(e^{i\theta})p(e^{i\theta})\rangle_{\mathcal{H}}.\end{aligned}$$

Hence, integrating over the circle, we obtain

$$\langle T_G p, p\rangle_{H^2(\mathbb{T},\mathcal{H})} = \langle Gp, p\rangle_{H^2(\mathbb{T},\mathcal{H})} = \|Qp\|^2_{H^2(\mathbb{T},\mathcal{H})} \geq 0,$$

which shows that T_G is a positive operator. This completes the proof. □

We now proceed to present a sufficient condition for the hyponormality of Toeplitz operators with operator-valued symbols.

Theorem 5.2.3. Let $\mathcal{H}$ be a separable complex Hilbert space and suppose that $G \in L^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$. Assume that

(i) G is hyponormal and

(ii) there exists a function $K \in H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$ such that $\|K\|_\infty \leq 1$ and $G - KG^*$ belongs to $H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$.

Then the Toeplitz operator T_G is hyponormal.

Proof. Since the identities $JP_- = P_+J$ and $JM_\Phi = M_{\breve{\Phi}}J$ hold, we obtain the following identity:

$$T_G^* T_G - T_G T_G^* = H_{G^*}^* H_{G^*} - H_G^* H_G + T_{G^*G-GG^*}.$$

Now, if the function $G - KG^*$ belongs to $H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$, then the corresponding Hankel operator vanishes, that is, $H_{G-KG^*} = 0$. Consequently, using Equation (1.7), we obtain

$$H_G = H_{KG^*} = T_{\widetilde{K}^*} H_{G^*}.$$

Since $K \in H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$ and

$$\|K\|_\infty = \|\widetilde{K}\|_\infty = \|T^*_{\widetilde{K}}\| \leq 1,$$

it follows that $T^*_{\widetilde{K}}$ is co-analytic and contractive. Therefore by [31, Corollary 2], we have

$$H_{G^*}^* H_{G^*} - H_G^* H_G \geq 0.$$

Moreover, Lemma 5.2.2 ensures that $T_{G^*G-GG^*} \geq 0$. Combining these two inequalities, we conclude that the Toeplitz operator T_G is hyponormal. □

The following example demonstrates that the hyponormality assumption on G is essential and cannot be omitted in the statement of Theorem 5.2.3.

Example 5.2.4. Consider the function $G(e^{i\theta}) = S^*$, where S denotes the unilateral shift operator on H^2 and let T_G be the corresponding Toeplitz operator acting on $H^2(\mathbb{T}, H^2)$. Then for each $\theta \in [0, 2\pi]$, we have

$$[G(e^{i\theta})^*, G(e^{i\theta})] = SS^* - S^*S.$$

This shows that $G(e^{i\theta})$ is not hyponormal for any $\theta \in [0, 2\pi]$. On the other hand, choosing $K = 0$, we obtain

$$G - KG^* = G \in H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(H^2)).$$

However, T_G is not hyponormal, as demonstrated below. For

$$f = (f_1, f_2, f_3, \cdots) \in H^2(\mathbb{T}, H^2),$$

we compute:

$$\big\langle [T_G^*, T_G]f, f \big\rangle \ = \big\langle T_G^* T_G f, f \big\rangle - \big\langle T_G T_G^* f, f \big\rangle.$$

Note that

$$T_G^* T_G f = T_G^*(f_2, f_3, f_4, \ldots) = (0, f_2, f_3, \ldots),$$

$$T_G T_G^* f = T_G(0, f_1, f_2, \ldots) = (f_1, f_2, f_3, \ldots).$$

Therefore,

$$[T_G^*, T_G]f = (0, f_2, f_3, \ldots) - (f_1, f_2, f_3, \ldots) = (-f_1, 0, 0, \ldots),$$

and hence

$$\big\langle [T_G^*, T_G]f, f \big\rangle = \big\langle (-f_1, 0, 0, \ldots), (f_1, f_2, f_3, \ldots) \big\rangle = -|f_1|^2 \leq 0.$$

The following corollary characterizes the hyponormality of analytic Toeplitz operators with operator-valued symbols, showing that the operator is hyponormal if and only if its symbol is hyponormal.

Corollary 5.2.5. Let $\mathcal{H}$ be a separable complex Hilbert space and suppose that $G \in H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$. Then the Toeplitz operator T_G is hyponormal if and only if G is hyponormal.

Proof. Since both G and 0 are analytic, we see that

$$G = G - 0G^* \in H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H})).$$

Assuming that G is hyponormal, it then follows from Theorem 5.2.3 that the associated Toeplitz operator T_G is hyponormal.

Conversely, suppose that T_G is hyponormal. Then

$$[T_G^*, T_G] = H^*_{G^*}H_{G^*} - H^*_G H_G + T_{G^*G-GG^*} \geq 0, \tag{5.3}$$

which shows that

$$\langle H^*_{G^*}H_{G^*}e^{im\theta}h, e^{im\theta}h\rangle - \langle H^*_G H_G e^{im\theta}h, e^{im\theta}h\rangle + \langle T_{G^*G-GG^*}e^{im\theta}h, e^{im\theta}h\rangle \geq 0, \tag{5.4}$$

for all $m \geq 0$ and $h \in H^2(\mathbb{T}, \mathcal{H})$.

It is straightforward to verify that

$$\langle T_{G^*G-GG^*}e^{im\theta}h, e^{im\theta}h\rangle = \langle \mathcal{S}_{\mathcal{H}}^{*\,m} T_{G^*G-GG^*}\mathcal{S}^m_{\mathcal{H}}h, h\rangle = \langle T_{G^*G-GG^*}h, h\rangle, \tag{5.5}$$

where $\mathcal{S}_{\mathcal{H}}$ denotes the right shift operator on $H^2(\mathbb{T}, \mathcal{H})$. Since

$$\lim_{m\to\infty} H_{G^*}e^{im\theta}h = \lim_{m\to\infty} H_G e^{im\theta}h = 0,$$

it then follows from Equations (5.4) and (5.5) that

$$\langle T_{G^*G-GG^*}h, h\rangle \geq 0.$$

By applying Lemma 5.2.2, we obtain

$$G^*(e^{i\theta})G(e^{i\theta}) - G(e^{i\theta})G^*(e^{i\theta}) \geq 0$$

for all $e^{i\theta} \in \mathbb{T}$. That is, G is hyponormal. $\square$

5.3 Subnormality of analytic Toeplitz operators with operator valued symbols

This section presents an in-depth study of the subnormality of analytic Toeplitz operators whose symbols are operator-valued. Our objective is to establish a necessary and sufficient condition for subnormality in terms of the structural properties of the symbol. Particularly, we illustrate that an analytic block Toeplitz operator is subnormal if and only if its symbol is subnormal. We begin with the following characterization.

Theorem 5.3.1. Let $\mathcal{H}$ be a separable complex Hilbert space and suppose that $G \in H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$. Then the Toeplitz operator T_G is subnormal if and only if the symbol G is subnormal.

Proof. In order to prove this result, we use the Agler criterion for subnormality [5, Theorem 3.1]: *a contractive operator T is subnormal if and only if for each $n \geq 1$,*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*j} T^j \geq 0.$$

Since $G \in H^\infty_{SOT}(\mathbb{T}, \mathcal{B}(\mathcal{H}))$, we may assume that $\|G(e^{i\theta})\|_{\mathcal{B}(\mathcal{H})} < 1$ for almost every $e^{i\theta} \in \mathbb{T}$. Under this assumption, we observe that

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T_G^{*j} T_G^j = \sum_{j=0}^{n} (-1)^j \binom{n}{j} T_{G^{*j}} T_{G^j} = T_{\sum\limits_{j=0}^{n} (-1)^j \binom{n}{j} G^{*j} G^j}.$$

Now suppose that T_G is subnormal. Then, by Agler's criterion, it follows that the operator $T_{\sum\limits_{j=0}^{n} (-1)^j \binom{n}{j} G^{*j} G^j}$ is positive for each $n \in \mathbb{N}$. Applying Lemma 5.2.2 together with Agler's criterion, it follows that $G(e^{i\theta})$ is subnormal for almost every $e^{i\theta} \in \mathbb{T}$.

Conversely, assume that G is subnormal. Then, by Agler's Criterion, the expression

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}G^{*j}G^j$$

defines a positive operator on $\mathcal{H}$ for each natural number n. Applying Lemma 5.2.2 once again, we obtain that

$$T_{\sum\limits_{j=0}^{n}(-1)^j\binom{n}{j}G^{*j}G^j}$$

is a positive operator for all $n \in \mathbb{N}$. Hence, T_G is subnormal. □

The following example illustrates that the hyponormality of G is not, in general, a sufficient condition for the subnormality of the corresponding analytic Toeplitz operator T_G.

Example 5.3.2. Consider the Hardy space $H^2(\mathbb{T}, H^2)$ and let T_G be the Toeplitz operator on this space with operator-valued symbol

$$G(e^{i\theta}) = S^* + 2S,$$

with S denoting the unilateral shift on H^2. It is straightforward to verify that $G(e^{i\theta})$ is hyponormal for all $\theta \in [0, 2\pi]$, but not subnormal for any $\theta \in [0, 2\pi]$. Indeed, for any $\theta \in [0, 2\pi]$, we compute the self-commutator of $G(e^{i\theta})$ as

$$\begin{aligned}[G^*(e^{i\theta}), G(e^{i\theta})] &= [(S^* + 2S)^*, (S^* + 2S)] \\ &= (S + 2S^*)(S^* + 2S) - (S^* + 2S)(S + 2S^*) \\ &= 3(I - SS^*).\end{aligned}$$

Hence, the self-commutator of T_G is given by

$$[T_G^*, T_G] = T_{3(I-SS^*)}.$$

Therefore, for any $f \in H^2(\mathbb{T}, H^2)$, we have

$$\begin{aligned}\langle [T_G^*, T_G]f, f\rangle_{H^2(\mathbb{T},H^2)} &= \langle T_{3(I-SS^*)}f, f\rangle_{H^2(\mathbb{T},H^2)} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle 3(I-SS^*)f(e^{i\theta}), f(e^{i\theta})\rangle_{H^2} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} 3|f_0(e^{i\theta})|^2 d\theta \\ &\geq 0,\end{aligned}$$

where $f = \sum_{n=0}^{\infty} f_n e^{in\theta}$ with $f_n \in H^2$. This shows that T_G is hyponormal.

Next, consider the kernel of the self-commutator $[T_G^*, T_G]$, which is given by

$$\ker[T_G^*, T_G] = \left\{ f(e^{i\theta}) = \sum_{n=0}^{\infty} f_n e^{in\theta} \in H^2(\mathbb{T}, H^2) : f_0 = 0 \right\}.$$

For $g_1 \in H^2$, define $g(e^{i\theta}) := g_1 e^{i\theta}$. Clearly, $g \in \ker[T_G^*, T_G]$. Now, observe that

$$T_G(g_1 e^{i\theta}) = g_1 + 2g_1 e^{i2\theta} \notin \ker[T_G^*, T_G]$$

and hence the kernel is not invariant under T_G. Therefore, T_G is not subnormal, even though its symbol G is hyponormal.

CONCLUSIONS AND RECOMMENDATIONS

Halmos' Problem 5

The classical problem of characterizing subnormal Toeplitz operators via the properties of their symbol was initially raised by P. R. Halmos [35]. Significant progress was made through the contributions of M. B. Abrahamse [4] and jointly, T. Nakazi and K. Takahashi [44]. More recently, R. E. Curto, I. S. Hwang and W. Y. Lee [23] contributed further insights. In this thesis, we have presented a partial answer to this question in the form of Theorem 2.2.3, which extends the Nakazi–Takahashi Theorem to the setting of scalar-valued Toeplitz operators whose self-commutator is of finite rank.

Despite the progress described above, a complete characterization of subnormal Toeplitz operators remains unsettled. This leads to the following central problem:

Problem 6.1. Determine the necessary and sufficient conditions on the symbol for a Toeplitz operator to be subnormal.

Furthermore, the result established in Theorem 2.2.3, together with Theorem 2.1.4 due to R. E. Curto, I. S. Hwang and W. Y. Lee [23], provides supporting evidence for Conjecture 2.1.3. Nevertheless, a complete proof of the conjecture remains open.

Block Toeplitz operators

In the context of Toeplitz operators, the hyponormal Toeplitz operators with finite rank self-commutator were characterized by T. Nakazi and K. Takahashi [44]. Building upon this direction of research, R. E. Curto, I. S. Hwang and W. Y. Lee [22] examined the analogous question in the framework of block Toeplitz operators and proposed Conjecture 3.2.1. Theorems 3.2.2, 3.2.4, 3.2.5 and 3.2.7 provide partial progress towards solving this conjecture.

Despite these advancements, a complete solution of Conjecture 3.2.1 remains open. Thus, it is appropriate to reframe the conjecture in light of our partial results. The following problems are proposed as refined versions of the conjecture, whose answers would significantly contribute to a complete characterization of block Toeplitz operators with finite rank self-commutator.

Problem 6.2. Let $\Phi = \Phi_{+'} + \Phi_0 + \Phi_-^* \in L^\infty(\mathbb{T}, M_n)$ be such that $\det \Phi_{+'} \equiv 0$ and T_Φ is normal. Does there exist a finite Blaschke-Potapov product $B \in \mathcal{E}(\Phi)$ such that $\deg(\det B) = 0$?

Problem 6.3. Let $\Phi \in L^\infty(\mathbb{T}, M_n)$ be a normal symbol and suppose that $\mathcal{E}(\Phi)$ contains a finite Blaschke-Potapov product B. Does it follow that

$$\text{Rank}[T_\Phi^*, T_\Phi] = \deg(\det B)?$$

Motivated by Example 3.2.8, we pose the following problem:

Problem 6.4. Suppose Φ^* is not of bounded type such that there exists a row of Φ^* that contains no scalar-valued function which is not of bounded type. If T_Φ is a hyponormal block Toeplitz operator with finite rank self-commutator, does there exist a finite Blaschke-Potapov product $B \in \mathcal{E}(\Phi)$ such that

$$\deg(\det B) = \text{Rank}[T_\Phi^*, T_\Phi]?$$

Problem 6.5. Suppose that $\Phi \in L^\infty(\mathbb{T}, M_n)$ is such that T_Φ is a hyponormal operator with finite rank self-commutator $[T_\Phi^*, T_\Phi]$. If Φ^* is of bounded type, does there exist a finite Blaschke-Potapov product $B \in \mathcal{E}(\Phi)$ such that

$$\text{Rank}[T_\Phi^*, T_\Phi] = \deg(\det B)?$$

Toeplitz operators with operator-valued symbols

Theorem 5.2.3 provides a sufficient condition for the hyponormality of Toeplitz operators with operator-valued symbols. Furthermore, Corollary 5.2.5 and Theorem 5.3.1 offer characterizations for the hyponormality and subnormality of analytic Toeplitz operators in this setting.

Although significant progress has been made, a complete characterization of hyponormal Toeplitz operators with operator-valued symbols remains open. This leads to the following problem:

Problem 6.6. Which Toeplitz operators with operator-valued symbols are hyponormal?

Invariant subspace problem

The Invariant subspace problem is one of the most historically significant and fundamental questions in operator theory. It asks whether every bounded linear operator on a separable complex Hilbert space admits a non-trivial closed invariant subspace. Even after decades of significant effort, this problem remains open in its full generality. Over the years, several strategies have been proposed to attack this problem. One of the more recent and potentially fruitful approaches involves the concept of universal operators.

Let X be a Banach space. An operator $U \in \mathcal{B}(X)$ is said to be *universal* for X

if, for every nonzero operator $T \in \mathcal{B}(X)$, there exists a scalar $\lambda \neq 0$ and an invariant subspace $\mathcal{M}$ of U such that the restriction $U|_{\mathcal{M}}$ is similar to λT. In the setting of complex Hilbert spaces, it is known that every nonzero operator $T \in \mathcal{B}(\mathcal{H})$ has a non-trivial invariant subspace if and only if there exists a universal operator U on $\mathcal{H}$ that has a minimal non-trivial invariant subspace which is one-dimensional (see [13, 17]).

A key example of such a universal operator is the adjoint of the unilateral shift operator $\mathcal{S}_{\mathcal{H}} = T_{e^{i\theta} I_{\mathcal{H}}}$, which is universal for the Hilbert space-valued Hardy space $H^2(\mathbb{T}, \mathcal{H})$. Furthermore, if q is an infinite Blaschke product or an inner function possessing a non-trivial singular inner factor, then the scalar Toeplitz operator $T_{\overline{q}}$ is known to be universal for H^2.

Therefore, the problem of characterizing the lattice of invariant subspaces for such universal Toeplitz operators has attracted considerable attention in recent research. Progress in this direction could yield deeper insights into the structure of universal operators. Furthermore, it could significantly contribute to the broader understanding of the Invariant subspace problem.

Riemann hypothesis

The Riemann hypothesis, one of the most profound and long-standing open problems in mathematics, states that all non-trivial zeros of the Riemann zeta function lie on the critical line in the complex plane where the real part is one-half. This classical problem has been reformulated in the framework of various function spaces. In particular, S. W. Noor [48] provided a reformulation of the Riemann hypothesis within the context of scalar-valued Hardy spaces.

Motivated by this line of research, it is natural to explore analogous formulations in various Hilbert spaces. The following problem outlines a potential direction for future investigation:

Problem 6.7. What is an appropriate formulation of the Riemann hypothesis in the framework of Banach-valued Hardy spaces?

Furthermore, J. Manzur, S. W. Noor and C. F. Santos [49] established that the Riemann hypothesis is equivalent to the shift-invariance of a certain subspace considered in [49, Theorem 11]. Motivated by this characterization, we propose the following question as a potential direction for further exploration:

Problem 6.8. Is the Riemann hypothesis equivalent to the invariance of the subspace considered in [49, Theorem 11], under some subnormal Toeplitz operator T_φ on H^2?

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APPENDIX

Publications

[1] M. Abhinand, R. E. Curto, I. S. Hwang, W. Y. Lee and T. Prasad, *Subnormal block Toeplitz operators*, J. d' Analyse Math. **155** (2025), 485–500. https://doi.org/10.1007/s11854-025-0358-3

Paper presentations

[1] Presented a paper titled *Subnormal Block Toeplitz Operators* in the International Conference on Spectral and Approximation Theory, organized by the Department of Mathematics, CUSAT, November 27–30, 2023.

[2] Delivered an invited talk on *Subnormality of block Toeplitz operators* in the 2024 International Workshop: Function Theory, Operator Theory and Applications, organized by KIAS, Seoul, June 20–22, 2024.

[3] Presented a paper titled *Halmos' problem 5 and block Toeplitz operators* in the Third National Seminar on Glimpses of Analysis and Geometry, organized by the Department of Mathematics, University of Calicut, March 13–14, 2025.

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