242588

C 22988 **(Pages : 3)** **Name..................................**

Reg. No...............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2022

(CCSS)

Mathematics

MAT 2C 10—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Let A be a linear transformation from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$. Prove that A is uniformly continuous.
2. Is every contraction of a metric space X continuous ? Justify your answer.
3. Find the Cartesian equation of the parametrized curve $\gamma(t) = (\cos^2 t, \sin^2 t)$.
4. Find the tangent vector of the curve $\gamma(t) = (\cos^2 t, \sin^2 t)$.
5. Is $\sigma(u,v) = (u + u^2, v, v^2)$ a regular surface patch ? Justify your answer.
6. Show that the curve $\gamma(t) = (\cos^3 t \cos 3t, \cos^3 t \sin 3t), t \in \mathbb{R}$ is a closed curve which has exactly one self intersection.
7. Prove that the second fundamental form of a plane is zero.
8. Prove that mean curvature of a surface $\mathcal{S}$ is a smooth function on $\mathcal{S}$.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let Ω be the set of all invertible linear operators on a vector space $\mathbb{R}^n$. Prove that the map $\psi : \Omega \to \Omega$ defined by $\psi(A) = A^{-1}$ is a continuous function.
10. Prove or disprove. A level curve can have both regular and non-regular parametrization.
11. Compute the curvature of the curve $\gamma(t) = (\cos^3 t, \sin^3 t)$.
12. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the surfaces and let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a diffeomorphism. If σ_1 is an allowable surface patch on $\mathcal{S}_1$, then prove that $f \circ \sigma_1$ is an allowable surface patch on $\mathcal{S}_2$.

Turn over

242588

13. Prove that the ellipsoid $\frac{x^2}{p^2}+\frac{y^2}{q^2}+\frac{z^2}{r^2}=1$ where p, q, r are non-zero constants, is a smooth surface.

14. Prove that on a sphere every point is an umbilic.

(4 × 4 = 16 marks)

Part C

Answer A **or** *B of following questions.*
Each question carries 12 *marks.*

Unit I

15. A (a) Let $E \subset \mathbb{R}^n$ be an open set and the map $f : E \to \mathbb{R}^k$ be differentiable at $x_0 \in E$. If g maps an open set containing $f(E)$ into $\mathbb{R}^m$ and g is differentiable at $f(x_0)$, then prove that the map $F : E \to \mathbb{R}^m$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0))f'(x_0)$.

(b) If X is a complete metric space and if φ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\varphi(x) = x$.

B (a) Prove that a linear transformation A on a finite dimensional vector space X is one to one if and only if it is onto.

(b) Let f map an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Prove that f is continuously differentiable in E if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m, 1 \le j \le n$.

Unit II

16. A (a) Prove that a parametrized curve has a unit-speed reparametrization if and only if it is regular.

(b) Find the curvature of the curve $\gamma(t) = (t, \cosh t)$.

B (a) Prove that any reparametrization of a regular curve is regular.

(b) Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere-vanishing curvature. Prove that its torsion is given by

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2},$$

where × denote the vector product and dot denote $\frac{d}{dt}$.

Unit III

17. A (a) Let U and $\overline{\mathrm{U}}$ be open subsets of $\mathbb{R}^2$ and let $\sigma : \mathrm{U} \to \mathbb{R}^3$ be a regular surface patch. Let $\Phi : \overline{\mathrm{U}} \to \mathrm{U}$ be a bijective smooth map with smooth inverse map $\Phi^{-1} : \mathrm{U} \to \overline{\mathrm{U}}$. Prove that $\overline{\sigma} = \sigma \circ \Phi : \overline{\mathrm{U}} \to \mathbb{R}^3$ is a regular surface patch.

(b) Prove that the quadric $x^2 + y^2 - 2z^2 - \frac{2}{3}xy + 4z = 5$ is a hyperboloid of one sheet.

B (a) Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a diffeomorphism. Prove that the linear map $\mathrm{D}_p f : \mathrm{T}_p\mathcal{S}_1 \to \mathrm{T}_{f(p)}\mathcal{S}_2$ is invertible for all $p \in \mathcal{S}_1$.

(b) Show that every compact surface is orientable.

Unit IV

18. A (a) Prove that the Weingarten map is self adjoint.

(b) Show that the Gaussian curvature of a surface $\mathcal{S}$ is a smooth function on $\mathcal{S}$.

B (a) Prove that the Gaussian curvature of a ruled surface is negative or zero.

(b) Prove that the principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point.

(4 × 12 = 48 marks)

C 22986 **(Pages : 3)** **Name..................................**

Reg. No................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2022

(CCSS)

Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Verify whether $f(x,y) = xy$ satisfies a Lipschitz condition on the entire plane.
2. Find the indicial equation and its roots for the equation $4x^2y'' + \left(2x^4 - 5x\right)y' + \left(3x^2 + 2\right)y = 0$.
3. Find the general solution of the equation $\left(2x^2 + 2x\right)y'' + (1 + 5x)y' + y = 0$ near its singular point $x = 0$.
4. Prove that $J_{-m}(x) = (-1)^m J_m(x)$ when m is a non-negative integer.
5. Show that (0, 0) is an isolated critical point for the system : $\dfrac{dx}{dt} = 4y, \dfrac{dy}{dt} = -x$.
6. Prove that the function $E(x,y) = ax^2 + bxy + cy^2$ is negative definite iff $a < 0$ and $b^2 - 4ac < 0$.

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 marks.

7. Find a power series solution of the equation $y' = x - y,\ y(0) = 0$.
8. Locate and classify the singular points on the x-axis of the equation $x^2\left(x^2 - 1\right)^2 y'' - x(1 - x)y' + 2y = 0$.
9. Determine the nature of the point $x = \infty$ for Bessel's equation $x^2y'' + xy' + \left(x^2 - p^2\right)y = 0$.
10. Prove that $\dfrac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$.

Turn over

11. It $f(x)$ is defined by $f(x)=\begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ ½, & x = \frac{1}{2} \\ 0, & \frac{1}{2} < x \leq 1, \end{cases}$ show that $f(x)=\sum_{n=1}^{\infty} \frac{J_1\left(\frac{\lambda_2}{n}\right)}{\lambda_n J_1\left(\lambda_n\right)^2}, J_0\left(\lambda_n n\right),$ where the λ_n' are the positive zeros of $J_0(x)$.

12. Using the method of Lagrange multipliers, show that the triangle with greatest area A for a given perimeter in equilateral.

13. Determine the nature and stability properties of the critical point (0, 0) for the system $\frac{dx}{dt}=4x-2y, \frac{dy}{dt}=5x+2y.$

14. Show that (0, 0) is an asymptotically stable critical point for the system $\frac{dx}{dt}=-3x^3-y, \frac{dy}{dt}=x^5-2y^3.$

(5 × 4 = 20 marks)

Part C

Answer **either** *A* **or** *B of each of the following three questions.*
Each question carries 16 *marks.*

15. A (a) Explain Picard's method of successive approximations for solving the initial value problem $y'=f(x,y), y\left(x_0\right)=y_0$.

(b) Solve the initial value problem by Picard's method :

$$\begin{cases} \frac{dy}{dx}=z, & y(0)=1 \\ \frac{dz}{dx}=-y, & z(0)=0 \end{cases}$$

B (a) Solve Legendre's equation : $\left(1-x^2\right)y''-2xy'+p(p+1)y=0,$ where p is a constant.

(b) Show that the equation $4x^2y''-8x^2y'+\left(4x^2+1\right)y=0$ has only one Frobenius series solution and find it.

16. A (a) Show that $\int_{-1}^{1} P_m(x)P_n(x)dx=\begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m=n, \end{cases}$ where $P_n(x)$ is the nth degree Legender polynomial.

(b) Find the first three terms of the Legendre series of $f(x) = e^x$.

B (a) Derive Euler's differential equation for an extremal.

(b) Find the extremal $\mathrm{I} = \int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{y}\, dx.$

17\. A (a) Find the general solution of the system :

$$\frac{dx}{dt} = 7x + 6y, \frac{dy}{dt} = 2x + 6y.$$

(b) Describe the relation between the phase portraits of the systems $\frac{dx}{dt} = \mathrm{F}(x, y), \frac{dy}{dt} = \mathrm{G}(x, y)$ and $\frac{dx}{dt} = -\mathrm{F}(x, y), \frac{dy}{dt} = -\mathrm{G}(x, y).$

B (a) State and prove Liapunov's theorem.

(b) Verify that (0, 0) is a simple critical point for the following system $\frac{dx}{dt} = x + y - 2xy, \frac{dy}{dt} = -2x + y + 3y^2$ and determine its nature and stability properties.

(3 × 16 = 48 marks)

C 22985 **(Pages : 3)** **Name..................................**

Reg. No...............................

SECOND SEMESTER P.G. (CCSS) DEGREE EXAMINATION, APRIL 2022

Mathematics

MAT 2C 07—REAL ANALYSIS—II

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Define a measurable set. If $m^*E = 0$, then prove that E is measurable.
2. If A is a measurable set, then prove that the characteristic function X_A is a measurable function.
3. Find the value of $R\int_a^{-b} f(x)\,dx$ and $R\int_{-a}^{b} f(x)\,dx$ for the function :

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational}. \end{cases}$$

4. Let f be an integrable function over E. If A and B are disjoint measurable sets contained in E, then prove that $\int_{A\cup B} f = \int_A f + \int_B f$.
5. Show that $D^+[-f(x)] = -D_+[f(x)]$.
6. If f is integrable on $[a, b]$, then show that the function F defined by $F(x) = \int_a^x f(t)\,dt$ is a continuous function of bounded variation on $[a, b]$.
7. Define a complete measure space. Given an example of a measure space which is not complete.
8. Let $\{(A_i \times B_i)\}$ be a countable collection of measurable rectangles whose union is a measurable rectangle A × B. Prove that $\lambda(A \times B) = \sum \lambda(A_i \times B_i)$.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let $\langle \mathrm{E}_n \rangle$ be an infinite decreasing sequence of measurable sets and let $m\mathrm{E}_1$ be finite. Prove that $m\left(\bigcap_{i=1}^{\infty} \mathrm{E}_i\right) = \lim_{n\to\infty} m\mathrm{E}_n$.

10. Show that there is a measurable set which is not a Borel set.

11. If f and g are bounded measurable functions defined on a set E of finite measure, then prove that $\int_{\mathrm{E}} (af + bg) = a\int_{\mathrm{E}} f + b\int_{\mathrm{E}} g$.

12. Let f be a non-negative measurable function. Show that $\int f = 0$ if and only if $f = a$ a.e.

13. If f is of bounded variation on $[a, b]$, then show that $\mathrm{T}_a^b = \mathrm{P}_a^b + \mathrm{N}_a^b$ and $f(b) - f(a) = \mathrm{P}_a^b - \mathrm{N}_a^b$.

14. If v_1 and v_2 are any two finite signed measures, show that $\alpha v_1 + \beta v_2$ is also a finite signed measure, where α and β are real numbers.

(4 × 4 = 16 marks)

Part C

Answer either A or B of each question.
Each question carries 12 marks.

15. A (a) Prove that the collection of measurable sets is a σ-algebra.

(b) For each $y \in [0,1)$, let the set $\mathrm{E} \dot{+} y$ be defined by $\mathrm{E} \dot{+} y = \{x \dot{+} y : x \in \mathrm{E}\}$. Where $\mathrm{E}C[0,1)$ is a measurable set and $x \dot{+} y$ denotes the sum module 1 of x and y. Prove that $\mathrm{E} \dot{+} y$ is measurable and $m(\mathrm{E} \dot{+} y) = m\mathrm{E}$.

B (a) Let $\{\mathrm{A}_n\}$ be a countable collection of sets of real numbers. Prove that $m^*(\cup \mathrm{A}_n) \leq \sum m^* \mathrm{A}_n$.

(b) Let E be a measurable set of finite measure, and let $\langle f_n \rangle$ a sequence of measurable functions defined on E. Let f be a real valued function such that for each x in E, $f_n(x) \to f(x)$. Prove that for given $\in > 0$ and $\delta > 0$, there is a measurable set $\mathrm{A} \subset \mathrm{E}$ with $m\mathrm{A} < \delta$ and an integer N such that for all $x \notin \mathrm{A}$ and all $n \geq \mathrm{N}$, $|f_n(x) - f(x)| < \in$.

16. A (a) State and prove Bounded convergence theorem.

(b) If $\langle u_n \rangle$ is a sequence of non-negative measurable functions and if $f = \sum_{n=1}^{\infty} u_n$, then prove that $\int f = \sum_{n=1}^{\infty} \int u_n$.

B (a) Let g be integrable over E and let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all x in E, let $f(x) = \lim f_n(x)$. Prove that $\int_E f = \lim \int_E f_n$.

(b) If f is integrable over E, show that $|f|$ is also integrable over E. Also show that $\left|\int_E f\right| \leq \int_E |f|$.

17. A (a) State and prove Vitali's lemma.

(b) If f is of bounded variation on $[a, b]$, then show that $f'(x)$ exists for almost all x in $[a, b]$.

B (a) Let f be an integrable function on $[a, b]$ and suppose that

$$F(x) = F(a) + \int_a^x f(t)\, dt.$$

Prove that $F'(x) = f(x)$ for almost all x in $[a, b]$.

(b) Prove that a function F is an indefinite integral if and only if it is absolutely continuous.

18. A (a) Suppose that for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $\mu\left(B_\alpha \sim B_\beta\right) = 0$ for $\alpha < \beta$. Prove that there is a measurable function f such that $f \leq \alpha$ a.e. on B_α and $f \geq \alpha$ a.e. on $X \sim B_\alpha$. If g is any other function with this property, show that $g = f$ a.e.

(b) State and prove Hahn decomposition theorem.

B (a) Let μ be a σ-finite measure on an algebra Q and let μ^* be the outer measure generated by μ. Prove that a set E is μ^* measurable if and only if E is the proper difference $A \sim B$ of a set E in $Q_{\sigma\delta}$ and a set B with $\mu^* B = 0$.

(b) Let E be a set in $\mathcal{R}_{\sigma\delta}$ with $\mu \times \upsilon(E) < \infty$. Prove that the function g defined by $g(x) = \upsilon E$, is a measurable function of x and $\int g\, d\mu = \mu \times \upsilon(E)$.

(4 × 12 = 48 marks)

C 22984 (Pages : 2) **Name.................................**

Reg. No..............................

SECOND SEMESTER P.G. (CCSS) DEGREE EXAMINATION, APRIL 2022

Mathematics

MAT 2C 06—ALGEBRA—II

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Let G = $\mathbb{Z}_{24}$ and N = <4>. Give all elements in G/N.
2. Verify whether $(0) < 9\mathbb{Z} < 3\mathbb{Z} < \mathbb{Z}$ is a composition series.
3. Find all conjugates of $1+2\sqrt{2}$ over the rationals $\mathbb{Q}$.
4. Verify whether $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ is a splitting field over $\mathbb{Q}$.
5. Let K be an extension of degree 3 over $\mathbb{Z}_2$. Give all intermediate field of the extension K over $\mathbb{Z}_2$.
6. Describe the eighth cyclotomic polynomial over $\mathbb{Q}$.

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 *marks.*

7. Let N be a normal subgroup of a group G and $\gamma : G \to G/N$ be the canonical homomorphism. Show that if H is a normal subgroup of G then $\gamma(H)$ is a normal subgroup of G/N.
8. Show that every group of order 45 has a normal subgroup of order 9.
9. Find the number of Sylow 3-subgroups of the symmetric group S_4.
10. Show that if α and β are algebraic over $\mathbb{Q}$ and if $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are isomorphic then α and β are conjugates over $\mathbb{Q}$.
11. Verify whether $\mathbb{Q}(\omega)$ is a splitting field over $\mathbb{Q}$ where ω is a non-real cube root of 1.
12. Let K be a finite extension of F and F < E < K. Show that if K is separable over F then E is separable over F.
13. Let K be the splitting field of $x^4 + 1$ over $\mathbb{Q}$. Show that $G(K/\mathbb{Q})$ is isomorphic to the Klein 4 group.
14. Show that the regular 7-gon is not constructible.

(5 × 4 = 20 marks)

Turn over

Part C

Answer part A **or** *part B of each question.*
Each question carries 16 *marks.*

15. A (a) Let G be a group and H, K be normal subgroups of G with K < H. Show that :

  (i) H/K is a normal subgroup of G/K.

  (ii) G/H is a isomorphic to (G/K)/(H/K).

 (b) Let $G = \mathbb{Z}$, $H = 2\mathbb{Z}$ and $K = 6\mathbb{Z}$. Find G/H, G/K, H/K and (G/K)/(H/K).

 B (a) Define Sylow *p*-subgroup of a group G.

 (b) Let P_1, P_2 by Sylow *p*-sub-groups of a group G. Show that P_1 and P_2 are conjugates in G.

 (c) Show that if G has only one Sylow *p*-subgroup P then P is a normal subgroup of G.

16. A (a) Let E be an extension of a field F. Describe the group G(E/F) of all automorphisms of E leaving elements of F fixed. Verify the group axioms for G(E/F).

 (b) Describe all elements of the automorphism group $G(K/\mathbb{Q})$ where $K = \mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$.

 B (a) Define splitting field.

 (b) Verify whether $\mathbb{Q}\left(\sqrt{2}\right)$ is a splitting over $\mathbb{Q}$.

 (c) Let $F < E < \overline{F}$. Show that E is a splitting field over F if and only if every automorphism of $\overline{F}$ leaving elements of F fixed maps E onto E.

17. A Let K be a finite normal extension of F with Galois group G(K/F). Let $\lambda(E) = G(K/E)$ for all intermediate fields E. Show that

  (a) the fixed field of G(K/E) in K is E.

  (b) λ is one to one on the set of all intermediate fields.

  (c) If E is a normal extension of F then G(K/E) is a normal subgroup of G(K/F) ?

 B (a) Define the *n*th cyclotonic polynomial over a field F.

 (b) Show that the Galois group of the *n*th cyclotomic extension of $\mathbb{Q}$ is isomorphic to the multiplicative group of units in $\mathbb{Z}_n$.

 (c) Show that if *p* is a prime then the Galois group of the *p*th cyclotomic extension of $\mathbb{Q}$ is cyclic of order $p - 1$.

(3 × 16 = 48 marks)

C 22982 **(Pages : 3)** **Name..................................**

Reg. No...............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2022

(CCSS)

Mathematics

MAT 2 C 09—TOPOLOGY

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 2 *marks.*

1. Using examples prove that any *two* topologies on a set need not be comparable.
2. Define base for a topology. Can the same topology have more than one base ? Justify your claim.
3. Prove that the family of all closed sets of a topological space is closed under arbitrary intersections.
4. Prove that a homeomorphism from one topological space into another is an open map.
5. Define absolute property of a subset of a topological space. Give an example for an absolute property.
6. If the topological space X is connected, then prove that X cannot be written as the disjoint union of two non-empty closed sets.
7. Suppose y is an accumulation point of a subset A of a T_1 space X. Then prove that every neighbourhood of y contains infinitely many points of A.
8. When do we say that a subset S of a topological space is saturated ? If S is saturated, prove that complement of S is also saturated.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Let τ be the cofinite topology on a set X. Suppose $\{x_n\}$ is a sequence in X. Prove that $\{x_n\}$ is convergent if and only if there is at most one term of it which repeats infinitely often.
10. Define hereditary property of a topological space. Prove that second countability is a hereditary property.

Turn over

11. Define derived set of a subset of a topological space. If for any set $A, \bar{A}$ and A' denote respectively the closure and derived sets of A, then prove that $\bar{A} = A \cup A'$.

12. Prove that every second countable space is separable. i

13. Define countable chain condition. Prove that every separable topological space satisfies the countable chain condition.

14. Prove that if a product is non-empty, then each projection function is onto.

(4 × 4 = 16 marks)

Part C

Answer **either** *A* **or** *B part of the following questions.*
Each question carries 12 *marks.*

15. (A) (a) Let X be a set and $\{T_i : i \in I\}$ be an indexed family of topologies on X. Let $T = \bigcap_{i \in I} T_i$. Then prove that T is a topology on X. Also prove that it is weaker than each T_i and if U is any topology on X which is weaker than each T_i, then T is stronger than U.

(b) Prove that a subset A of a metric space X is open if and only if no sequence in X – A converges to a point of A.

(B) (a) If a space is second countable, prove that every open cover of it has a countable subcover.

(b) Let $(X. \tau)$ be a topological space and $\mathbb{B} \subset \tau$. Then prove that $\mathbb{B}$ is a base for τ if and only if for any $x \in X$ and any open G containing x, there exists $B \in \mathbb{B}$ such that $x \in B$ and $B \subset G$.

16. (A) (a) Let (X, τ) be a topological space and $\mathcal{C}$ be the family of all closed subsets of X. Prove or disprove that $\mathcal{C}$ is the complement of τ in P (X), the set of all subsets of X.

(b) Define dense subset of a topological space. State and prove a necessary and sufficient condition for a set to be dense.

(B) Let X and Y be two topological spaces and $f : X \to Y$ be a function and $x_0 \in X$. Prove that the following are equivalent :

(1) f is continuous at x_0.

(2) The inverse image (under f) of every neighbourhood of $f(x_0)$ in Y is a neighbourhood of x_0 in X.

(3) For every subset $A \subset X$, $x_0 \in \bar{A}$ implies $f(x_0) \in \overline{f(A)}$.

17. (A) (a) Prove that every closed surjective map is a quotient map.

(b) Prove that every continuous real valued function on a compact space is bounded and attains its extrema.

(B) (a) Let (X, d) be a compact metric space and U be an open cover of X. Then prove that there exists a positive real number r such that for any $x \in X$, there exits $V \in U$ such that $B(x, r) \subset V$.

(b) Let $f : X \to Y$ be a continuous surjective map. Then if X is connected, prove that Y is also connected.

18. (A) (a) Prove that the axioms T_0, T_1, T_2, T_3 and T_4 form a hierarchy of progressively stronger conditions.

(b) Prove that complete regularity is a hereditary property in topological spaces.

(B) (a) Prove that a subset of X is a box if and only if it is the intersection of a family of walls.

(b) For any closed set C and any open set G containing C, prove that there exists an open set H and a closed set K such that $C \subset H \subset K \subset G$.

(4 × 12 = 48 marks)