C 25232 (Pages : 2) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2017

(CCSS)

Mathematics

MAT 4E 08—GRAPH THEORY

(2010 Admissions)

Time : One Hour and a Half Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 8 *marks.*

1. Let $k \geq 2$ and let G be a k-connected graph. Prove that any set of k vertices of G will lie on a cycle of G.

2. Determine the values of the parameters and α, β, α' and β' for the graph K_n.

3. Define chromatic number of a graph G. Prove that $\chi(G) = 2$ if and only if G is a bipartite graph with atleast one edge.

4. Let C_5 denote the cycle of length 5. Find the chromatic number of C_5.

(4 × 8 = 32 marks)

Part B

Answer A **or** *B of each question.*
Each question carries 24 *marks.*

5. A (a) Prove that the connectivity and edge connectivity of a simple cubic graph are equal.

(b) Show that in any network N, the value of any flow f is less than or equal to the capacity of any cut K.

B (a) Prove that a graph G with $n \geq k+1$ vertices is k-connected if and only if any two vertices of G are connected by at least k internally disjoint paths.

(b) Let G be a graph with n vertices and $\delta > 0$. Prove that $\alpha' + \beta' = n$.

Turn over

6. A (a) Let G be a graph with n vertices and independence number α. Prove that

$$\frac{n}{\alpha} \leq \chi \leq n - \alpha + 1.$$

(b) Let k be a positive integer. Prove that there exists a triangle free graph with chromatic number k.

B (a) If G is a loopless bipartite graph, then prove that $\chi'(G) = \Delta(G)$.

(b) Prove that a simple graph on n vertices is a tree if and only if $f(G;\lambda) = \lambda(\lambda - 1)^{n-1}$.

(2 × 24 = 48 marks)

C 25230 **(Pages : 3)** Name.................................

Reg. No.............................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2017

(CCSS)

Mathematics

MAT 4E 06—FUNCTIONAL ANALYSIS—II

(2010 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Let $\| \ \|'$ be a complete norm on $C([-1,1])$ such that if $\|x_n - x\|' \to 0$, then $x_n(t) \to x(t)$ for every $t \in [-1,1]$, show that $\| \ \|'$ is equivalent to the sup norm on $C([-1,1])$.

2. Let X be a Banach space and $A \in BL(X)$. Show that A is invertible iff A is bounded below and the range of A is dense in X.

3. Let X be a Banach space and $P \in BL(X)$ be a projection. Show that $P \in CL(X)$ iff P is of finite rank.

4. Show that an inner product $<, >$ on a linear space X induces a norm on X.

5. Show that if E and F are closed subsets of a Hilbert space H and $E \perp F$, then E + F is closed in H.

6. Let H be a Hilbert space and $A \in BL(H)$. Show that

$$\|A^*A\| = \|A\|^2 = \|AA^*\|.$$

7. Let H be a Hilbert space and A, $B \in BL(H)$ with A self-adjoint. Show that AB = 0 iff $R(A) \perp R(B)$.

8. Let H be a Hilbert space and $A \in BL(H)$. Show that $\sigma(A)$ is contained in the closure of $W(A)$.

(8 × 4 = 32 marks)

Turn over

Part B

Answer (A) **or** (B) *of each question.*
Each question carries 12 *marks.*

9. (A) (a) Let X be a normed space and $A \varepsilon BL(X)$ be of finite rank. Show that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.

(b) Let X be a normed space and $A \varepsilon BL(X)$. Let (k_n) be a sequence of eigenvalues of A. Show that if $k_n \to k$ in K, then k is an approximate eigenvalue of A.

(B) Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual of K^n with the norm $\| \|_p$ is linearly isometric to K^n with the norm $\| \|_q$.

10. (A) (a) Let X and Y be normed spaces and $F : X \to Y$ be linear. Show that F is a compact map iff for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

(b) Let X and Y be normed spaces and $X \neq \{0\}$. Show that Y is a Banach space iff CL (X, Y) is a Banach space in the operator norm.

(B) (a) Let X be a normed space and $A \varepsilon CL(X)$. show that

$$\{k : k \in \sigma_e(A), k \neq 0\} = \{k : k \in \sigma(A), k \neq 0\}.$$

(b) Show that zero can be a spectral value of a compact operator A without being its eigen value.

11. (A) (a) Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Show that $\{u_\alpha\}$ is an orthonormal basis for H iff $x \varepsilon H$ and $\langle x, u_\alpha \rangle = 0$ for all α, then $x = 0$.

(b) Let $H = L^2([0, 1])$. Show that $\{1, \sqrt{2} \cos \pi t, \sqrt{2} \cos 2\pi t, ...\}$ is an orthonormal basis for H.

(B) (a) State and prove Riesz representation theorem.

(b) Show that the Riesz representation theorem does not hold for an incomplete inner product space.

12. (A) (a) Let A be a Hilbert space and $A \varepsilon BL(H)$. Show that A is normal iff

$$\|A(x)\| = \|A^*(x)\| \text{ for all } x \varepsilon H.$$

(b) State and prove the finite dimensional spectral theorem for self-adjoint or normal operators.

(B) (a) Show that a Hilbert-Schmidt operator on a Hilbert space is compact.

(B) Let H be a Hilbert space and $A \varepsilon BL(H)$. Show that A is compact iff A^*A is compact.

($4 \times 12 = 48$ marks)

C 3715

Name..................................

Reg. No...............................

FOURTH SEMESTER M.Sc. (CCSS) DEGREE EXAMINATION, JUNE 2016

Mathematics

MAT 4E 08—GRAPH THEORY

Time : Three Hours | Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 8 *marks.*

1. Show that if G is k-connected, then $G \vee K_1$ is $(k + 1)$-connected.
2. Determine the connectivity and edge connectivity of the Petersen graph P.
3. For any graph G, prove that $\chi(G) \leq 1 + \Delta$.
4. Show that there exists not graph with chromatic polynomial $\lambda^4 - 3\lambda^3 + \lambda^2$.

(4 × 8 = 32 marks)

Part B

Answer A **or** B *of each question.*
Each question carries 24 *marks.*

5. A (a) Prove that in a 2-connected graph G, any two longest cycles have at least two vertices in common.

   (b) Prove that in any network, the value of any flow f is less than or equal to the capacity of any cut K.

   B (a) Let x and y be two vertices of a graph G. Prove that the maximum number of edge-disjoint (x, y)-paths in G is equal to the minimum number of edges of G whose deletion destroys all (x, y)-paths in G.

   (b) (i) Let G be a graph with n vertices. Prove that $\alpha + \beta = n$.

   (ii) Determine the values of the parameters α, β, α' and β' for the Herschel graph.

6. A (a) Prove that in a critical graph G, no vertex cut is a clique.

   (b) Prove that $\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$.

   B Let G be a simple graph. Prove that $\Delta(G) \leq \chi'(G) \leq 1 + \Delta(G)$.

(2 × 24 = 48 marks)

C 3714 (Pages : 2) Name.....................................

Reg. No.....................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2016

(CCSS)

Mathematics

MAT 4E 07—ALGEBRAIC TOPOLOGY

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Define the r-skeleton of a geometric complex. Verify that r-skeleton of a geometric complex is a geometric complex.
2. Compute the 2-dimensional cycle group $Z_2(p)$ for the projective plane p.
3. If k is an oriented complex and $p \geq 2$, prove that the composition $22 : C_p(k) \to C_{p-2}(k)$ in the diagram $C_p(k) \xrightarrow{\partial} C_{p-1}(k) \xrightarrow{\partial} C_{p-2}(k)$ is the trivial homomorphism.
4. Let k be a 2-pseudomanifold with α_0 vertices, α_1 1-simplexes and α_2 2-simplexes. Prove that :

   $$\alpha_0 \geq \frac{1}{2}\left(7 + \sqrt{49 - 24\chi(k)}\right).$$
5. Prove that every simplicial mapping $\phi : |k| \to |L|$ is continuous.
6. Prove that the mesh of a complex K is the maximum length of its 1-simplexes if K has positive dimension.
7. Prove that $\pi_1(X, x_0)$ has an identify element $[c]$ where c is the constant loop whose only value is x_0.
8. If two loops α and β in S' with base point 1 are equivalent, prove that they have the same degree.

(8 × 4 = 32 marks)

Turn over

Part B

Answer **either** *A* **or** *B of each question.*
Each question carries 16 *marks.*

9. A (a) Define the face of a simplex. How many faces does an n-simplex have ? Give justification to your answer.

(b) Define the p^{th} incidence matrix of an oriented geometric complex K. Compute the 2^{nd} incidence matrix of the closure of the 3-simplex $< a_0 \ a_1 \ a_2 \ a_3 >$ with vertices ordered by $a_0 < a_1 < a_2 < a_3$.

B (a) Let K be an oriented geometric complex. Briefly describe the p-dimensional homology group $H_p(k)$ of K where p is a non-negative integer.

(b) Let K be geometric complex with two orientations, and let K_1 and K_2 denote the oriented geometric complexes. Prove that the homology groups $H_p(k_1)$ and $H_p(k_2)$ are isomorphic for each dimension p.

10. A (a) State and prove Euler-Poincare theorem.

(b) Prove that the n-sphere S^n, $n \geq 1$ is orientable.

B (a) Let $|K|$ and $|L|$ be polyhedra with triangulations K and L respectively and $f : |K| \to |L|$ a continuous function such that K is star related to L relative to f. Prove that f has a simplicial approximation $g : |K| \to |L|$.

(b) Prove that there is a vector field on $S^n, n \geq 1$ if and only if n is odd.

11. A (a) Show that, for each homotopy class $[\alpha]$ in $\pi_1(X, x_0)$, the inverse of $[\alpha]$ with respect to the multiplication 0 and the identity element [c] is the class $[\bar{\alpha}]$ where $\bar{\alpha}(t) = \alpha(1-t), t \in I$.

(b) Let $f : X \to Y$ be a continuous function. Prove that the function :

$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ defined by $f_*([\alpha]) = [f\alpha], [\alpha] \in \pi_1(X, x_0)$ is a homomorphism.

B (a) State and prove the covering Homotopy property.

(b) Let A be a deformation retract of a space X and let x_0 be a point of A. Prove that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(A, x_0)$.

(3 × 16 = 48 marks)

C 3713 (Pages : 3) Name.................................

Reg. No...............................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2016

(CCSS)

Mathematics

MAT 4E 06—FUNCTIONAL ANALYSIS—II

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Let X and Y be Banach spaces and $F \varepsilon BL(X, Y)$. Show that R (F) is linearly homeomorphic to $X/Z(F)$ iff R (F) is closed in Y.

2. Let X be a normed space and $A \varepsilon BL(X)$. Show that if A is invertible then $\lim \inf_{n \to \infty} \|A^n\|^{1/n} > 0$.

3. Show that a compact linear map on a normed space is continuous. Is the converse true ? Justify your answer.

4. Let <, > be an inner product on a linear space X. Show that for all $x, y \varepsilon X$, $|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$, where equality holds iff the set $\{x, y\}$ is linearly dependent.

5. Let $\{x_1, x_2, ...\}$ be an orthogonal set in a Hilbert space H. Show that $\sum_{n=1}^{\infty} x_n$ converges in H iff $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$.

6. Let F be a closed subspace of a Hilbert space H and $A \varepsilon BL(H)$. Show $A(F) \subset F$ iff $A^*(F^\perp) \subset F^\perp$.

Turn over

7. Show that * if x_1 and x_2 are eigen-vectors of a normal operator A on a Hilbert space H corresponding to distinct eigen-values then $x_1 \perp x_2$.

8. Let H be a Hilbert space and $A \varepsilon BL(H)$ be self-adjoint. Show that $A^2 \geq 0$ and $A \leq \|A\| I$.

(8 × 4 = 32 marks)

Part B

Answer (A) **or** (B) *of each question.*
Each question carries 12 *marks.*

9. (A) (a) Let X be a Banarch space, $A \varepsilon BL(X)$ and $\|A^p\| < 1$ for some positive integer p. Show that I – A is invertible.

(b) Let X be a non-zero Banach space over the field of complex numbers and $A \varepsilon BL(X)$. Show that $\sigma(A)$ is non-empty.

(B) (a) Let $1 \leq p,\ q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual of c_{oo} with the norm $\| \ \|_p$ is linearly isometric to l^q.

(b) Let Y be a subspace of a normed space X. For $x' \varepsilon x'$, let $F(x') = x'/_y$. Show that F is a surjective linear map from X' to Y' such that $\|F(x')\| \leq \|x'\|$ for all $x' \varepsilon X'$.

10. (A) (a) Let Y be a Banach space, $F_n \varepsilon \subset L(X, Y)$, $F \varepsilon BL(X, Y)$ and $\|F_n - F\| \to 0$. Show that $F \varepsilon \subset L(X, Y)$.

(b) State and prove Schauder theorem.

(B) (a) Show that if A is a compact linear operator on an infinite dimensional normed space X, then zero is an approximate eigenvalue of A.

(b) Let X be a normed space and $A \varepsilon \subset L(X)$. Show that the eigen spectrum and the spectrum of A are countable sets and have zero as the only possible limit point.

11. (A) (a) State and prove Bessel's inequality.

(b) Let H be a non-zero Hilbert space. Show that H has a countable orthonormal basis iff H is separable.

(B) (a) Let H be a Hilbert space, G be a subspace of H and g be a continuous linear functional on G. Show that there is a unique continuous linear functional f on H such that $f/_G = g$ and $\|f\| = \|g\|$.

(b) Let H be a Hilbert space and $A \varepsilon BL(H)$. Show that there is a unique $B \varepsilon BL(H)$ such that for all $x, y \varepsilon H$, $\langle A(x), y\rangle = \langle x, B(y)\rangle$.

12. (A) (a) Let H be a Hilbert space and $A \varepsilon BL(H)$. Show that A is unitary iff $\|A(x)\| = \|x\|$ for all $x \varepsilon H$ and A is surjective.

(b) Let H be a Hilbert space and $A \varepsilon BL(H)$. Show that if A is unitary, then for every orthonormal basis $\{u_\alpha\}$ of H, $\{A(u_\alpha)\}$ and $\{A^*(u_\alpha)\}$ are both orthonormal bases for H and that if for some orthonormal basis (u_α) for H, both $\{A(u_\alpha)\}$ and $\{A^*(u_\alpha)\}$ are orthonormal basis, then A is unitary.

(B) (a) Let H be a non-zero Hilbert space and $A \varepsilon BL(H)$ be self-adjoint show that $\{m_A, M_A\} \subset \sigma_a(A) = \sigma(A) \subset [m_A, M_A]$.

(b) Let H be a Hilbert space and $A \varepsilon BL(H)$ be compact. Show that A^* is compact.

(4 × 12 = 48 marks)