Register No. of the Candidate | | | | | | |

UNIVERSITY OF CALICUT
DEPARTMENT OF MATHEMATICS
Ph.D. Entrance Examination, March 2017

Time: 2 hours Max. Marks: 100

PART - A

Answer **all** questions. Each question carries **2** marks

1. Which one of the following is not true about a group of order 120?

   (a) G has an element of order 5 (b) G has a subgroup of order 5
   (c) G has an element of order 8 (d) G has a subgroup of order 8

2. Which one of the following is not an ideal in $\mathbb{Q}[x]$?

   (a) $\{f(x) \in \mathbb{Q}[x] : f(0) = 0\}$ (b) $\{f(x) \in \mathbb{Q}[x] : f(0) = 1\}$
   (c) $\{f(x) \in \mathbb{Q}[x] : f(1) = 0\}$ (d) $\{f(x) \in \mathbb{Q}[x] : f(0) = 0 = f(1)\}$

3. Which one of the following fields has a proper subfield which is algebraically closed?

   (a) $\mathbb{C}$ (b) The field of algebraic numbers
   (c) $\mathbb{R}$ (d) The field of algebraic real numbers

4. Let W_1 and W_2 be two 2-dimensional subspaces of the vectorspace $\mathbb{R}^3$. Consider the following three statements.

   (i) $W_1 + W_2 = \mathbb{R}^3$ (ii) $W_1 = W_2$ (iii) $W_1 \cap W_2 = \{0\}$

   Then

   (a) either (i) or (ii) is true but not both (b) both (i) and (ii) are true
   (c) either (ii) or (iii) is true (d) both (i) and (iii) are true

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a one-one linear operator and $U : \mathbb{R}^3 \to \mathbb{R}^3$ be an onto linear operator. Then

   (a) $UT : \mathbb{R}^3 \to \mathbb{R}^3$ is onto but not one one
   (b) $UT : \mathbb{R}^3 \to \mathbb{R}^3$ is one one but not onto
   (c) $UT : \mathbb{R}^3 \to \mathbb{R}^3$ is both one one and onto
   (d) $UT : \mathbb{R}^3 \to \mathbb{R}^3$ need not be one one or onto

6. The characteristic polynomial of a diagonalizable linear operator on $\mathbb{R}^5$ is $x(x-a)^2(x-b)^2$, where a and b are real numbers with $a \neq b$. If $p(x)$ is its minimal polynomial, then

   (a) $x(x-a)(x-b)$ divides $p(x)$ (b) $x(x-a)$ divides $p(x)$ (c) $(x-a)(x-b)$ divides $p(x)$
   (d) either $(x-a)$ or $(x-b)$ divides $p(x)$ but x does not divide $p(x)$

7. Let μ be the möbius function. Which of the following is not true?

   (a) $\sum_{d|n} \mu(d) = 1$ if $n = 1$.

   (b) $\sum_{d|n} \mu(d) = 0$ if $n > 1$.

   (c) $\sum_{d|n} \mu(d) = \frac{1}{n}$.

   (d) none of the above.

8. Which of the following subset of $\mathbb{R}^2$ is compact?

   (a) $\{(x, y) \in \mathbb{R}^2 : |x| \geq 2, |y| \leq 2\}$.
   (b) $\{(x, y) \in \mathbb{R}^2 : x^2 \leq y^2 + 2\}$.
   (c) $\{(x, y) \in \mathbb{R}^2 : x^2 + 5y^2 \leq 2\}$.
   (d) $\{(x, y) \in \mathbb{R}^2 : y^2 \leq 4x\}$.

9. Which of the following is true?

   (a) The continuous image of a separable space is separable.
   (b) Subspaces of separable spaces are separable.
   (c) The continuous image of a second countable space is second countable.
   (d) none of the above.

10. The normed linear space for which the Hahn-Banach extension is unique

    (a) $\mathbb{K}^n$ with the $\|\|_1$.
    (b) $\mathbb{K}^n$ with the $\|\|_2$.
    (c) $\mathbb{K}^n$ with the $\|\|_\infty$.
    (d) none of these.

11. Let A and B be subsets of a normed space X and $A + B = \{x + y : x \in A, y \in B\}$. Which of the following is not true?

    (a) If A or B is open, then $A + B$ is open.
    (b) If A and B are compact, then $A + B$ is compact.
    (c) If A and B are closed, then $A + B$ is closed.
    (d) If A is compact and B is closed, then $A + B$ is closed.

12. Consider the following statements:

    (P): $x^2y'' + xy' + (x^2 - 1/4)y = 0$ has two linearly independent Frobenious series solutions near $x = 0$.
    (Q): $x^2y'' + 3(\sin x)y' + y = 0$ has two linearly independent Frobenious series solutions near $x = 0$.

    Which of the above statements hold TRUE?

    (a) both **P** and **Q**.
    (b) only **P**.
    (c) only **Q**.
    (d) neither **P** nor **Q**.

13. Let y be the solution of the equation

$$(1 - x^2)y'' - 2xy + 6y = 0$$

    If $y(1) = 2$, then the value of the integral $\int_{-1}^{1} y^2 dx$ is equal to

    (a) 0
    (b) 1/5
    (c) 2/5
    (d) none of these

14. Let $a, b, c, d \in \mathbb{R}$ such that $c^2 + d^2 \neq 0$. Then the Cauchy problem

$$au_x + bu_y = e^{2x+y}, x, y \in \mathbb{R},\ u(x, y) = 0 \text{ on } cx + dy = 0$$

has a unique solution if

(a) $ac + bd \neq 0$

(b) $ad - bc \neq 0$

(c) $ac - bd \neq 0$

(d) $ad + bc \neq 0$

15. For $0 \leq x \leq 1$, let $f_n(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational} \end{cases}$ and $f(x) = \lim_{x\to\infty} f_n(x)$. Then on the interval $[0, 1]$

(a) f is measurable and Riemann Integrable

(b) f is measurable and Lebesgue integrable

(c) f is measurable and not Lebesgue integrable

(d) f is not Lebesgue integrable

16. $f : [0, 1] \to [0, 1]$ is called a shrinking map if $|f(x) - f(y)| < |x - y|$ for all $x, y \in [0, 1]$ and a contraction if there exists an $\alpha < 1$ such that $|f(x) - f(y)| \leq \alpha|x - y|$ for all $x, y \in [0, 1]$.
Which of the following statements is TRUE for the function $f(x) = x - x^2/2$?

(a) f is both a shrinking map and a contraction

(b) f is a shrinking map but NOT a contraction

(c) f is NOT a shrinking map but a contraction

(d) f is Neither a shrinking map nor a contraction

17. Let $S_n = \sum_{k=1}^{n} \frac{1}{n}$ and $I_n = \int_1^n \frac{x-[x]}{x^2}$. Then $S_{100} + I_{100}$ is equal to

(a) 0

(b) $2 \ln 10 + 1$

(c) $2 \ln 10 - 1$

(d) none of these

18. Which of the following is the negation of the following statement?
For every x, there exists a y such that $y.y = x$.

(a) For every x, there exists no y such that $y.y = x$.

(b) For no x, there exists a y such that $y.y = x$.

(c) There exists an x such that for every y, $y.y \neq x$.

(d) none of the above.

19. If $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is onto, then

(a) f is onto.

(b) g is onto.

(c) both f and g are onto.

(d) none of the above.

20. Let f be a multiplicative arithmetical function and f^{-1} be the Dirichlet inverse of f. Which of the following is not true?

    (a) $f(1) = 1$.

    (b) $f(p^a) = f(p)^a$ for all primes p and all integers $a \geq 1$.

    (c) f^{-1} is multiplicative.

    (d) $f^{-1}(p^2) = f(p)^2 - f(p^2)$ for every prime p.

21. A language on an alphabet is

    (a) finite.

    (b) infinite.

    (c) finite or infinite.

    (d) uncountable.

22. Which of the following is not true in a Boolean algebra?

    (a) commutative law

    (b) distributive law

    (c) cancellation law

    (d) none of the above

23. Which of the following is true?

    (a) There exists a connected graph with at least two vertices in which every edge is a cut edge.

    (b) There exists a connected graph with at least two vertices in which every vertex is a cut vertex.

    (c) If there is a cut vertex in a graph, there is also a cut edge.

    (d) none of the above.

24. Which of the following is true.

    (a) There exists an analytic function on the complex plane with values in the open unit disc.

    (b) There exists an analytic function on the complex plane which is zero at $\frac{1}{n}$ for every positive integer n which is not identically zero.

    (c) There exists an analytic function f on the complex plane such that $f(z) = 0$ if and only if z is an integer.

    (d) none of these

25. The singularity of the function $f(z) = e^{\frac{1}{z}}$ at $z = 0$ is a

    (a) pole.

    (b) removable singularity.

    (c) essential singularity.

    (d) none of the above.

PART - B

Answer any **five** questions. Each question carries **10** marks

1. (a) Describe all subgroups of $\mathbb{Z}_{100}$

   (b) Find all intermediate fields in between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2}, i\sqrt{3})$

2. (a) Give an example of a vectorspace V consisting of four elements. Also find its dimension.

   (b) T represents the linear operator on $\mathbb{R}^2$ whose action on (a, b) is rotating (a, b) about the origin at an angle θ, $0 \leq \theta \leq 90°$ in the anticlockwise direction. Find the matrix representation of T in the standard basis.

3. (A) Let $f_n(x) = n^c x(1 - x^2)^n$ for x real and $n \geq 1$. Prove that $\{f_n\}$ converges pointwsie on $[0, 1]$ for every real c. Determine those c for which the convergence is uniform on $[0, 1]$ and those for which term-by-term integration on $[0, 1]$ leads to a correct result.

   (B) In a metric space (S, d), let A be a nonempty subset of S. Define a function $f_A(x) : S \to \mathbb{R}$ by the equation

   $$f_A(x) = \inf\{d(x, y) : y \in A\}$$

   for every $x \in S$. The value $f_A(x)$ is called the distance from x to A.

   (a) Prove that f_A is uniformly continuous on S.

   (b) Prove $\overline{A} = \{x \in S : f_A(x) = 0\}$.

4. (A) Find the general solution of the ODE

   $$(1 - e^x)y'' + \frac{1}{2}y' + e^x y = 0$$

   near the singular point $x = 0$.

   (B) Solve the integral equation $y(x) = \lambda \int_0^{2\pi} \sin(x + t)y(t)dt$.

5. (A) Prove that the cofinite topological space is not metrizable.

   (B) Let X be a compact topological space. Prove that every infinite subset of X has at least one limit point in X.

6. Prove or disprove.

   (A) A Banach space can have a countably infinite basis.

   (B) Let X be a normed space and Y be a non-empty open subspace of X. Then $Y = X$.

7. Let $d(n)$ denotes the number of positive divisors of n. Prove the following.

   (A) $\Pi_{m|n}\, m = n^{d(n)/2}$.

   (B) $d(n)$ is odd if and only if n is a square.

8. (A) If $\sim$ is an equivalence relation on a set X, prove that the equivalence classes form a partition of X.

   (B) Give an example of a bijection from the set $\mathbb{R}$ of real numbers to the open interval $(-1, 1)$ and substantiate your claim.

9. (A) Find all analytic functions with constant real part.

   (B) Determine the value of the integral $\int_C z^n dz$ where C is the unit circle oriented positively and n is any integer.

10. (A) Give an example of a non trivial graph which is self complementary.

    (B) Give an example of a partially ordered set which is not a lattice.

    (C) Give an example of a lattice which is not complete.

    (D) Give an example of a bounded lattice which is not complemented.

    (E) Give an example of a non deterministic finite accepter.