

C 42462

(Pages : 3)

Name.....

Reg. No.....

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS—PG)

Mathematics

MAT 2 07—ALGEBRA—II

(2022 Admission onwards)

Time : Three Hours

Maximum : 50 Marks

Part A*Answer all questions.**Each question carries 1 mark.*

1. Show that the ring \mathbb{Z} of integers is Noetherian.
2. Give a Euclidean norm on the field \mathbb{Q} of rational numbers.
3. Let \mathbb{R} be the field of reals and $p(x) = x^2 + 1$. Let $E = \mathbb{R}[x]/\langle p(x) \rangle$. Show that if $\alpha = x + \langle p(x) \rangle$ then $\alpha^2 + 1 = 0$ in E .
4. Find the degree of the extension $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over \mathbb{Q} .
5. Find all zeros of $x^3 + x$ in \mathbb{Z}_5 .
6. Verify whether $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are isomorphic fields.
7. Verify whether $y_1 y_2 + y_1 + y_2$ is a symmetric function on y_1 and y_2 .
8. Find the fourth cyclotomic polynomial over the rationals \mathbb{Q} .

(8 × 1 = 8 marks)

Turn over

Part B

*Answer any six questions.
Each question carries 3 marks.*

9. Let D be an integral domain and $a, b \in D$. Let $\langle a \rangle$ and $\langle b \rangle$ denote the principal ideals generated by a and b respectively. Show that $\langle a \rangle = \langle b \rangle$ if and only if a and b are associates.
10. Let D be a Euclidean domain with Euclidean norm v . Show that if $v(u) = v(1)$ then u is a unit in D .
11. Let N be a multiplicative norm on $\mathbb{Z}[i]$ given by $N(\alpha) = a^2 + b^2$ where $\alpha = a + ib$. Show that if $\pi \in \mathbb{Z}[i]$ is such that $N(\pi)$ is a prime number then π is irreducible in $\mathbb{Z}[i]$.
12. Let E be an extension of a field F and let $\phi_\alpha : F[x] \rightarrow E$ be the evaluation homomorphism. Show that α is algebraic over F if and only if ϕ_α is not one to one.
13. Show that every finite extension of a field F is an algebraic extension of F .
14. Let E be an extension of degree 2 over the field \mathbb{Z}_3 . Find the number of elements in E .
15. Find all automorphisms of the field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
16. Let K be the splitting field of $x^3 - 2$ over \mathbb{Q} . Find all intermediate fields of this extension.
17. Show that the Galois group of the p^{th} cyclotomic extension of \mathbb{Q} is isomorphic to the cyclic group of order $p - 1$ where p is a prime.

(6 × 3 = 18 marks)

Part C

*Answer any three questions.
Each question carries 8 marks.*

18. (a) Define principal ideal domain and maximal ideal.
(b) Show that an ideal $\langle p \rangle$ in a principal ideal domain is a maximal ideal if and only if p is irreducible.

19. (a) Show that every finite extension of a field F is an algebraic extension of F .
(b) Show that the field \mathbb{C} of complex numbers is algebraically closed.
20. (a) Show that if α and β are constructible real numbers then so are $\alpha\beta$ and $\alpha + \beta$.
(b) Show that if γ is constructible and $\gamma \notin \mathbb{Q}$ then $[\mathbb{Q}(\gamma) : \mathbb{Q}] = 2^n$ for some positive integer n .
21. (a) Let F be a finite field of characteristic p . Show that the number of elements in F is p^n for some positive integer n .
(b) Describe a field of 8 elements.
22. (a) Define normal extension of a field.
(b) Let K be a normal extension of a field F and E be an intermediate field. Let $\lambda(E)$ be the set of all automorphisms of K that leave all elements of E fixed. Show that the fixed field of $\lambda(E)$ is E .

(3 × 8 = 24 marks)

C 42457

(Pages : 3)

Name.....

Reg. No.....

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS—PG)

Mathematics

MAT 2C 06—ALGEBRA—II

(2019 Admission onwards)

Time : Three Hours

Maximum : 80 Marks

Part A*Answer all questions.**Each question is of 2 marks.*

1. Discuss the statement : \mathcal{Z} has no composition series.
2. Give a principal series of $S_n, n \geq 5$.
3. Find all composition series of Z_{60} and show that they are isomorphic.
4. Prove that no group of order 20 is simple.
5. Prove that the set of all automorphisms of a field E is a group under function composition.
6. Is the polynomial $x^5 - 1$ solvable by radicals over \mathcal{Q} ? Justify your answer.

(6 × 2 = 12 marks)

Part B*Answer any five questions.**Each question is of 4 marks.*

7. Let H and K be normal subgroups of a group G with $K \leq H$. Then prove that $G/H \cong (G/K)/(H/K)$.
8. Find a composition series of $S_3 \times S_3$. Is $S_3 \times S_3$ solvable?
9. Prove that for a prime number p , every group G of order p^2 is abelian.
10. Find the splitting field of the polynomial $x^4 - 5x^2 + 6$ in $\mathcal{Q}[x]$.
11. If E is a finite extension of F , then the index $\{E:F\}$ of E divides $[E:F]$.

Turn over

12. Prove that every field of characteristic zero is perfect.
13. Describe the group of the polynomial $(x^3 - 1) \in \mathbb{Q}[x]$ over \mathbb{Q} .
14. Show that the regular 7-gon is not constructible.

(5 × 4 = 20 marks)

Part C*Answer Part A or Part B of each question.**Each question is of 16 marks.*

15. A) If G is a finite group and p divides $|G|$, Then show that :
- the number of Sylow p -subgroups is congruent to 1 modulo p and divides $|G|$ and,
 - show that no group of order 15 is simple.
- B) Let P_1 and P_2 be Sylow p -subgroups of a finite group G . Then show that,
- P_1 and P_2 are conjugate subgroups of G and,
 - Show that the Sylow 2-subgroups of S_3 have order 2.
16. A) (a) Let F be a finite field of characteristic p . Then the map $\sigma_p : F \rightarrow F$ defined by $\sigma_p(a) = a^p$ for $a \in F$ is an automorphism, of F . Also, prove that $F_{\sigma_p} \cong Z_p$.
- (b) Prove that two elements, α and β , of an algebraic extension E of a field F are conjugate over F if and only if they are both zeros of the same polynomial $f(x) \in F[x]$.
- B) Let F be a field, and let α and β be algebraic over F with $\deg(\alpha, F) = n$. Then prove that,
- the map $\Psi_{\alpha, \beta} : F(\alpha) \rightarrow F(\beta)$ defined by

$$\Psi_{\alpha, \beta} (c_0 + c_1\alpha + \dots + c_{n-1}\alpha^{n-1}) = c_0 + c_1\beta + \dots + c_{n-1}\beta^{n-1} \text{ for } c_i \in F$$
 is an isomorphism $F(\alpha)$ onto $F(\beta)$ if and only if α and β are conjugate over F .

- (b) Let $f(x) \in \mathbb{R}(x)$. If $f(a + bi) = 0$ for $(a + bi) \in \mathbb{C}$, where $a, b \in \mathbb{R}$, then apply part (a) to show that $f(a - bi) = 0$.

17. A) Prove that,

- (a) a field E , where $F \leq E \leq \bar{F}$, is a splitting field over F if and only if every automorphism of \bar{F} leaving F fixed maps E onto itself and thus induces an automorphism of E leaving F fixed.
- (b) Find the splitting field of the polynomial $x^4 - 5x^2 + 6$ in $\mathbb{Q}[x]$.

B) Prove that,

- (a) let F be a field of characteristic zero, and let $F \leq E \leq K \leq \bar{F}$, where E is a normal extension of F and K is an extension of F by radicals. Then prove that $G(E/F)$ is a solvable group.
- (c) Establish the insolvability of the Quintic by showing that there is a subfield F of the real numbers and a polynomial $f(x) \in F[x]$ of degree 5 such that the splitting field E of $f(x)$ over F has a Galois group isomorphic to symmetric group S_5 .

(3 × 16 = 48 marks)

C 42466

(Pages : 3)

Name.....

Reg. No.....

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS—PG)

Mathematics

MAT 211—MULTIVARIABLE CALCULUS AND GEOMETRY

(2022 Admission onwards)

Time : Three Hours

Maximum : 50 Marks

Part A*Answer all questions.**Each question carries 1 mark.*

- Let A and B be linear operators on \mathbb{R}^2 given as follows. $A(x, y) = (x + y, x + y)$ and $B(x, y) = (x, x)$.
Verify whether $AB = BA$.
- Let $f(x, y) = \frac{x}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Find $D_1 f$.
- Give a parametrization of the curve $y = x^2 + x$.
- Find the curvature of the curve $\gamma(t) = (\cos t, \sin t)$.
- Let $U = \left\{ (\theta, \phi) : -\frac{\pi}{2} < \theta < \frac{\pi}{2}; 0 < \phi < 2\pi \right\}$. Describe the image of U by the map
 $\sigma(\theta, \phi) = (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta)$.
- Verify whether $S = \left\{ (x, y, x^2 + y^2) \in \mathbb{R}^3 \right\}$ is a smooth surface.
- Find the unit normal N corresponding to the surface patch $\sigma(u, v) = (\cos v, \sin v, u)$ at a chosen point.
- Find the normal curvature of a curve on the sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 2\}$.

(8 × 1 = 8 marks)

Turn over

Part B

*Answer any six questions.
Each question carries 3 marks.*

9. Find an invertible linear operator A on \mathbb{R}^2 such that $\|A^{-1}\| \neq \frac{1}{\|A\|}$.
10. Let f be a \mathbb{C}^1 -mapping on an open set $E \subseteq \mathbb{R}^n$. Show that f is an open mapping on E .
11. Let A be an $n \times n$ matrix and A_1 be obtained by interchanging two columns of A . Show that $\det A_1 = -\det A$.
12. Give an example of a unit speed curve in \mathbb{R}^3 with constant curvature. Verify the details.
13. Find the torsion of the circular helix $\gamma(\theta) = (\cos \theta, \sin \theta, 2\theta)$.
14. Verify whether $S = \{(x, y, z) : x^2 + y^2 = 1\}$ is a surface in \mathbb{R}^3 .
15. Calculate the first fundamental form of the surface given by $\sigma(u, v) = (u - v, u + v, u^2 + v^2)$.
16. Show that the derivative $D_p G$ of the Gauss map G is a linear map on the tangent space at p .
17. Describe using an example the normal section of a surface. (6 × 3 = 18 marks)

Part C

*Answer any three questions.
Each question carries 8 marks.*

18. (a) Define norm of a linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$.
(b) Show that if $\|A\|$ is finite then A is a uniformly continuous mapping of \mathbb{R}^n to \mathbb{R}^m .
19. (a) Define torsion of a regular curve.
(b) Let γ be a unit speed curve in \mathbb{R}^3 with constant curvature and zero torsion. Show that γ is part of a circle.

20. (a) Define tangent space of a surface S at a point $p \in S$.
- (b) Show that if $\sigma : U \rightarrow \mathbb{R}^3$ is a patch of a surface S containing a point p then the tangent space of S at p is the space spanned by σ_u and σ_v where u and v denotes co-ordinates of points in U .
21. (a) Describe the standard unit normal N_σ corresponding to a patch σ of a surface S at a point on S .
- (b) Show that if $\bar{\sigma}$ is another surface patch for S then $N_{\bar{\sigma}} = \pm N_\sigma$.
22. (a) Define principal curvatures k_1 and k_2 of a surface.
- (b) Show that

$$K = k_1 k_2 \text{ and } H = \frac{k_1 + k_2}{2}$$

where K is the Gaussian curvature and H is the mean curvature.

(3 × 8 = 24 marks)

C 42464

(Pages : 4)

Name.....

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SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS-PG)

Mathematics

MAT2C09—ORDINARY DIFFERENTIAL EQUATIONS

(2022 Admission onwards)

Time : Three Hours

Maximum : 50 Marks

Part A

*Answer all questions.
Each question carries 1 mark.*

1. Compute the Wronskian of the two functions :

$$\phi_1(x) = x^2, \phi_2(x) = x|x| \text{ for } -\infty < x < \infty.$$

2. Find a second independent solution of the equation $xy'' - (x+1)y' + y = 0$, given that $\phi_1(x) = e^x$ ($x > 0$) is a solution.

3. Compute the indicial polynomial and its roots for the equation $4x^2y'' + (4x^4 - 5x)y' + (x^2 + 2)y = 0$.

4. Show that for a positive integer n , $J_{-n}(x) = (-1)^n J_n(x)$.

5. Verify whether the equation $\cos x \cos^2 y dx - \sin x \sin 2y dy = 0$ is exact.

6. Show that the function $f(x, y) = 4x^2 + y^2$ satisfies Lipschitz condition on $\mathbb{R} : |x| \leq 1, |y| \leq 1$.

7. Describe the relation between the phase portraits of the system :

$$\begin{cases} \frac{dx}{dt} = F(x, y) \\ \frac{dy}{dt} = G(x, y) \end{cases} \text{ and } \begin{cases} \frac{dx}{dt} = -F(x, y) \\ \frac{dy}{dt} = -G(x, y) \end{cases}$$

8. Construct a Lyapunov function of the form $E(x, y)$ for the system :

$$\frac{dx}{dt} = -2x + xy^3, \frac{dy}{dt} = x^2y^2 - y^3.$$

(8 × 1 = 8 marks)

Turn over

Part B

Answer any **six** questions.
Each question carries 3 marks.

9. Find all solutions of the equation $y'' - 7y' + 6y = \sin x$.
10. One solution of the equation $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$ for $x > 0$ is $\phi_1(x) = x$. Find a basis for the solutions for $x > 0$.
11. Find two linearly independent power series solutions of the equation $y'' - xy' + y = 0$.
12. Show that $J_{\alpha+1}(x) - J_{\alpha-1}(x) = 2J'_\alpha(x)$.
13. Compute the first four successive approximations $\phi_0, \phi_1, \phi_2, \phi_3$ for the initial value problem $y' = y^2, y(0) = 0$.
14. If y is a nontrivial solution of $y'' + q(x)y = 0$, then show that y has an infinite number of positive zeros if $q(x) > \frac{k}{x^2}$ for all $x > 0$ and for some $k > \frac{1}{4}$.
15. Determine the nature and stability properties of the critical point $(0, 0)$ for the system :
- $$\frac{dx}{dt} = -4x - y, \frac{dy}{dt} = x - 2y.$$
16. Find the critical points and the differential equation of the paths of the system :
- $$\frac{dx}{dt} = -x, \frac{dy}{dt} = 2x^2 y^2.$$
17. Verify that $(0, 0)$ is a simple critical point for the system $\frac{dx}{dt} = x + y - 2xy, \frac{dy}{dt} = -2x + y + 3y^2$.

(6 × 3 = 18 marks)

Part C

Answer any **three** questions.
Each question carries 8 marks.

18. Let ϕ be any solution of the equation $y'' + a_1 y' + a_2 y = 0$ on an interval I containing a point x_0 .

Show that $\|\phi(x_0)\| e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{k|x-x_0|}$ for all x in I , where :

$$\|\phi(x)\| = \left[|\phi(x)|^2 + |\phi^1(x)|^2 \right]^{1/2}, k = 1 + |a_1| + |a_2|.$$

19. (a) Let a_1, \dots, a_n be continuous functions on an interval I , containing x_0 . Show that there exist

n linearly independent solutions of the equation $y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = 0$ on I .

- (b) Show that $P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2 - 1)^n$, where $P_n(x)$ is the n^{th} Legendre polynomial.

20. (a) Find all solutions ϕ of the form $\phi(x) = |x|^r \sum_{k=0}^{\infty} c_k x^k$, ($|x| > 0$), for the equation :

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0.$$

- (b) Show that $x^{1/2} J_{1/2}(x) = \frac{\sqrt{2}}{\Gamma(1/2)} \sin x$, $\Gamma(z)$ is the gamma function.

21. (a) Let M, N be two real valued functions which have continuous first partial derivatives on some rectangle $R: |x - x_0| \leq a, |y - y_0| \leq b$. Show that the equation $M(x, y) + N(x, y)y' = 0$ is exact

$$\text{in } R \text{ iff } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ in } R.$$

- (b) Let y be any nontrivial solution of the equation $y'' + q \cdot y = 0$, where $q(x) > 0$ for all $x > 0$. Show

that if $\int_1^{\infty} q(x) dx = +\infty$, then y has infinitely many zeros on the positive x -axis.

Turn over

22. (a) Assume that the system $\frac{dx}{dt} = F(x, y), \frac{dy}{dt} = G(x, y)$ has an isolated critical point $(0, 0)$. Show that if there exists a Lyapunov function $E(x, y)$ for the system, then the critical point $(0, 0)$ is stable.
- (b) Sketch the phase portrait of the equation $\frac{d^2x}{dt^2} = 2x^3$, and show that it has an unstable isolated critical point at the origin.

(3 × 8 = 24 marks)

C 42459

(Pages : 4)

Name.....

Reg. No.....

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS—PG)

Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

(2019 Admission onwards)

Time : Three Hours

Maximum : 80 Marks

Part A*Answer all questions.**Each question carries 2 marks.*

1. Show that e^x and e^{-2x} are linearly independent solutions of the equation $y'' + y' - 2y = 0$ on any interval $[0, 2]$.
2. Find the indicial equation and its roots of the equation $4x^2y'' + (2x^4 - 5x)y' + (3x^2 + 2)y = 0$.
3. Show that $\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!} \sqrt{\pi}$.
4. Find the external for the integral $I = \int_{x_1}^{x_2} [y^2 - (y')^2] dx$.
5. Describe the phase portrait of the system $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2$.
6. Determine the nature of the quadratic form $-x^2 - 4xy - 5y^2$.

(6 × 2 = 12 marks)

Turn over

Part B

*Answer any five questions.
Each question carries 4 marks.*

7. Consider the two functions $f(x) = x^2$ and $g(x) = x^2 |x|$ on the interval $[-1, 1]$. Find the Wronskian $W(f, g)$. Whether f and g are linearly dependent or independent? justify.
8. Show that $\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$.
9. Find the general solution of the differential equation $(x^2 - x - 6)y'' + (5 + 3x)y' + y = 0$ at the singular point $x = 3$.
10. Find the curve of fixed length L that joins the points $(0, 0)$ and $(1, 0)$, lies above the x -axis, and encloses the maximum area between itself and the x -axis.
11. Determine the nature of the point $x = \infty$ for Legendre's equation $x^2 y'' + xy' + (x^2 - p^2)y = 0$.
12. Find the stationary function of $\int_0^4 (xy' - (y')^2) dx$, which is determined by the boundary conditions $y(0) = 0$ and $y(4) = 3$.
13. Find the critical points, the differential equation of the paths and its solution of the system
- $$\frac{dx}{dt} = y(x^2 + 1), \quad \frac{dy}{dt} = -x(x^2 + 1).$$
14. Show that if there exists a Liapunov function $E(x, y)$ for the system
- $$\frac{dx}{dt} = F(x, y) \quad \frac{dy}{dt} = G(x, y)$$
- such that $\frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G$ negative definite then the critical point $(0, 0)$ is asymptotically stable.

(5 × 4 = 20 marks)

Part C

Answer either A or B of each of the following four questions.

Each question carries 16 marks.

15. (A) (a) If n is a positive integer, find two linearly independent solutions of

$$xy'' - (x+n)y' + ny = 0.$$

Also find the general solution for the cases $n = 1$ and 2 .

- (b) Find the particular solution of $y'' + 2y' + 5y = e^{-x} \sec(2x)$.

- (B) (a) Find the power series solution of the form $\sum a_n x^n$ of the differential equation $y' = 2xy$, try to recognize the resulting series as the expansion of familiar function and verify conclusion by solving the equation directly.

- (b) Find two independent Frobenius series solutions of the equation

$$2x^2 y'' + x(2x+1)y' - y = 0.$$

16. (A) (a) Derive Euler's differential equation for an extremal.

- (b) Find the point on the plane $ax + by + cz = d$ that is nearest the origin.

- (B) (a) Explain Picard's method of successive approximation to solve the initial value problem.

- (b) Let $f(x)$ be a function defined on the interval $-1 \leq x \leq 1$. Determine the polynomial

$$p(n) \text{ of degree } \leq n \text{ which minimize the value of the integral } I = \int_{-1}^1 [f(x) - p(x)]^2 dx.$$

17. (A) (a) Find the general solution of the system :

$$\frac{dx}{dt} = 4x - 2y, \quad \frac{dy}{dt} = 5x + 2y.$$

Turn over

- (b) Let $(0, 0)$ is the only critical point of the linear system

$$\frac{dx}{dt} = a_1x + b_1y, \frac{dy}{dt} = a_2x + b_2y$$

and $m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0$ is the auxiliary equation of the system at $(0, 0)$.

Show that if the roots of the equation are conjugate complex but not purely imaginary then the critical point $(0, 0)$ is a spiral.

- (B) (a) Show that $(0, 0)$ is an asymptotically stable critical point for the system :

$$\frac{dx}{dt} = -3x^3 - y, \frac{dy}{dt} = x^5 - 2y^3.$$

- (b) Consider the equation of motion for the damped vibration of a pendulum is

$$\frac{d^2x}{dt^2} + \frac{c}{m} \frac{dx}{dt} + \frac{g}{a} \sin(x) = 0, \text{ where } a \text{ is the length of the pendulum, } m \text{ is a mass and } c \text{ is a}$$

constant. Find the critical point and its nature, also give physical

(3 × 16 = 48 marks)

C 42463

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Name.....

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SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS—PG)

Mathematics

MAT 2C 08—REAL ANALYSIS—II

(2022 Admission onwards)

Time : Three Hours

Maximum : 50 Marks

Part A*Answer all questions.**Each question carries 1 mark.*

1. Show that every continuous function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable.
2. Show that, if $E \subseteq A$ and $m^*(E) = 0$, then $m^*(A \setminus E) = m^*(A)$.
3. Define a measure space. Give an example.
4. Let (X, \mathcal{A}) be a measurable space and $f : X \rightarrow [-\infty, \infty]$. If f is measurable, then prove that $\{x \in X : f(x) \geq a\} \in \mathcal{A}$, for every a in \mathbb{R} .
5. Let f be an extended real valued non-negative measurable function defined on a measure space (X, \mathcal{A}, μ) and let $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$. Prove that

$$\int_{A \cup B} f = \int_A f + \int_B f.$$
6. Let (X, \mathcal{A}, μ) be a measure space and $f \in \mathcal{L}(\mu)$ be such that $\int_E f = 0$ for all $E \in \mathcal{A}$. Prove that $f = 0$ a.e.
7. Give an example of a function which is not of bounded variation.

Turn over

8. Suppose that $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be measure spaces. Let $E = A \times B$, with $A \in \mathcal{A}_1$ and

$B \in \mathcal{A}_2$, be a measurable rectangle. Prove that the x -section of the set E is $E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$.

(8 × 1 = 8 marks)

Part B

Answer any **six** questions.

Each question carries 3 marks.

9. Let S be any countable subset of $[a, b]$. Construct a function $f : [a, b] \rightarrow \mathbb{R}$ such that S is exactly the set of points in $[a, b]$ at which f is discontinuous.

10. Suppose $A_k \subseteq \mathbb{R}$ for $k \in \mathbb{N}$. Prove that

$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k).$$

11. For any $a \in \mathbb{R}$, prove that the interval (a, ∞) is Lebesgue measurable.

12. Suppose (X, \mathcal{A}) is a measurable space and $X_0 \in \mathcal{A}$. Prove that $\mathcal{A}_0 = \{A \cap X_0 : A \in \mathcal{A}\}$ is a σ -algebra on X_0 and it is the largest σ -algebra on X_0 contained in \mathcal{A} .

13. Let X be a measurable space and let f, g be measurable functions on X . Prove that the function $h : X \rightarrow \mathbb{R}^2$ defined by $h(x) = (f(x), g(x))$, $x \in X$, is measurable.

14. Let (X, \mathcal{A}, μ) be a measure space. Let $f : X \rightarrow [0, \infty]$ be measurable and μ be a finite measure. Let

$A_n = \{x \in X : f(x) \geq n\}$ for $n \in \mathbb{N}$. Prove that

$$\int_X f d\mu < \infty \Leftrightarrow \sum_{n=1}^{\infty} \mu(A_n) < \infty.$$

15. If f and g are in $\mathcal{L}(\mu)$, the set of all integrable complex measurable functions on the measure space (X, \mathcal{A}, μ) , then prove that $f + g$ is also in $\mathcal{L}(\mu)$ and

$$\int_X (f + g) = \int_X f + \int_X g.$$

16. Suppose $\varphi : [a, b] \rightarrow \mathbb{K}$ be a function of bounded variation. For any $x, y, z \in [a, b]$ with $x < y < z$, prove that

$$V_{x,y}(\varphi) + V_{y,z}(\varphi) = V_{x,z}(\varphi).$$

17. Suppose that $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ are σ -finite measure spaces. Let $f_1 : X_1 \rightarrow \mathbb{R}$ and $f_2 : X_2 \rightarrow \mathbb{R}$ be measurable functions and Let $f(x, y) = f_1(x)f_2(y)$ where $(x, y) \in X_1 \times X_2$. Prove that f is measurable on the product space $(X_1 \times X_2, \mathcal{A}_1 \times \mathcal{A}_2)$.

(6 × 3 = 18 marks)

Part C

Answer any **three** questions.

Each question carries 8 marks.

18. (a) Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Prove that f is Riemann integrable if and only if for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.
- (b) Prove that the Cantor set is uncountable but the measure of Cantor set is 0.
19. (a) Let μ^* be an outer measure on a set X and let \mathcal{A} be the family of all μ^* -measurable subsets of X . Prove that \mathcal{A} is a σ -algebra, and that $\mu = \mu^*|_{\mathcal{A}}$, the restriction of μ^* to \mathcal{A} , is a complete measure on \mathcal{A} .
- (b) Let (X, \mathcal{A}, μ) be a measurable space (possibly incomplete) and let $(\tilde{X}, \tilde{\mathcal{A}}, \tilde{\mu})$ be its completion. Let $f : X \rightarrow \mathbb{R}$ be a measurable function with respect to $\tilde{\mathcal{A}}$. Prove that there exists a measurable function $g : X \rightarrow \mathbb{R}$ with respect to \mathcal{A} such that $f = g$ a.e.

Turn over

20. (a) Let $f : X \rightarrow [0, \infty)$ be a measurable function and (φ_n) be an increasing sequence of non-negative simple measurable functions on a measure space (X, \mathcal{A}, μ) which converges to f pointwise. Prove that

$$\lim_{n \rightarrow \infty} \int_X \varphi_n d\mu = \sup_{\varphi \in S_f} \int_X \varphi d\mu,$$

where S_f is the set of all non-negative simple measurable functions φ satisfying $\varphi \leq f$.

- (b) Let (f_n) be a sequence of extended real valued non-negative measurable functions defined on a measure space (X, \mathcal{A}, μ) such that $f_n \leq f_{n+1}$ for every $n \in \mathbb{N}$ and (f_n) converges pointwise on X . Let $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for $x \in X$. Prove that f is measurable and

$$\int_X f_n \rightarrow \int_X f \text{ as } n \rightarrow \infty.$$

21. (a) Let φ be a non-negative simple measurable function on a measure space (X, \mathcal{A}, μ) . Prove that

$$\nu(E) = \int_E \varphi d\mu, E \in \mathcal{A}, \text{ defines a measure on } \mathcal{A}.$$

- (b) Suppose (f_n) is a sequence of complex measurable functions such that (f_n) converges pointwise on a measure space (X, \mathcal{A}, μ) and there exists $g \in \mathcal{L}(\mu)$ satisfying $|f_n| \leq |g|$ for all $n \in \mathbb{N}$. Let $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for $x \in X$. Prove that f_n and f are integrable,

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| = 0 \text{ and } \lim_{n \rightarrow \infty} \int_X f_n = \int_X f.$$

22. (a) Prove that every absolutely continuous function on $[a, b]$ is of bounded variation.

- (b) Suppose that $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ are σ -finite measure spaces. For $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$, let

$$\mu(E) = \int_{X_1} \mu_2(E_x) d\mu_1(x) = \int_{X_2} \mu_1(E^y) d\mu_2(y).$$

Prove that μ is a measure on $\mathcal{A}_1 \otimes \mathcal{A}_2$.

(3 × 8 = 24 marks)

C 42458

(Pages : 4)

Name.....

Reg. No.....

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS-PG)

Mathematics

MAT2C07—REAL ANALYSIS – II

(2019 Admission onwards)

Time : Three Hours

Maximum : 80 Marks

Part A

*Answer all questions.
Each question carries 2 marks.*

1. Show that if there is a set A in \mathcal{M} such that $m A < \infty$, then $m\emptyset = 0$.
2. Let f and g two measurable real-valued functions defined on the same domain. Show that $f + g, fg$ are also measurable.
3. Let f and g be bounded measurable functions defined on a set E of finite measure. If $f = g$ a.e., then prove that :

$$\int_E f = \int_E g.$$

4. State and prove Monotone convergence theorem for non-negative measurable functions.
5. Find $D^+ f(x), D^- f(x)$ at $x = 0$, when $f(x) = |x|$.
6. Show that the product of two absolutely continuous functions are absolutely continuous.
7. Show that $\mu(E_1 \Delta E_2) = 0$ implies $\mu E_1 = \mu E_2$ provided $E_1, E_2 \in \mathcal{B}$.
8. Prove the uniqueness of the function f in the Radon-Nikodym theorem.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** questions.
Each question carries 4 marks.

9. Show that for any given collection \mathcal{C} of subsets of X , there is a smallest algebra which contains \mathcal{C}
10. Let $\langle E_n \rangle$ be an infinite decreasing sequence of measurable sets with $mE_1 < \infty$. Show that

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \rightarrow \infty} mE_n.$$

11. Let f and g be integrable over a measurable set E . Then prove that $f + g$ is integrable over E and

$$\int_E f + g = \int_E f + \int_E g.$$

12. Show that every function f is of bounded variation on $[a, b]$ if and only if it is the difference of two monotone real valued functions.

13. State and prove Lebesgue decomposition theorem.

14. Let (X, A, μ) and (Y, B, γ) be two complete measure space. Let E be a measurable subset of $X \times Y$ such that $\mu \times \gamma(E)$ is finite. Show that for almost all x the set E_x is a measurable subset of Y . Also show that the function g defined by $g(x) = \gamma(E_x)$ is a measurable function and evaluate

$$\int g d\mu.$$

(4 × 4 = 16 marks)

Part C

Answer (A) or (B) of the following questions.
Each question carries 12 marks.

Unit I

15. (A) (a) Prove that the outer measure of an interval is its length.
(b) Show that there exist a non measurable set.
- (B) (a) Show that the following five statements are equivalent for any given set E :
- E is measurable.
 - Given $\epsilon > 0$, there is an open set $O \supset E$ with $m^*(O \sim E) < \epsilon$.
 - Given $\epsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \sim F) < \epsilon$.

- iv) There is a G in G_δ with $E \subset G$, $m^*(G \sim E) = 0$.
- v) There is a F in F_σ with $F \subset E$, $m^*(E \sim F) = 0$.
- (b) State and Prove Lusin's theorem.

Unit II

16. (A) (a) Let f be a bounded real valued function defined on a measurable set E of finite measure. Prove that f is measurable if and only if :

$$\inf_{f \rightarrow \psi} \int_E \psi(x) dx = \inf_{f \rightarrow \varphi} \int_E \varphi(x) dx$$

for all simple functions φ and ψ .

- (b) Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then prove that f is a measurable function.
- (B) (a) State and prove generalized Lebesgue Convergence Theorem.
- (b) Let f be integrable over $(-\infty, \infty)$ and g be a bounded measurable function. Show that :
- (i) $\int f(x) dx = \int f(x+t) dx$.
- (ii) $\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} |g(x)[f(x) - f(x+t)]| dx = 0$.

Unit III

17. (A) (a) State and prove Vitali covering principle.

- (b) Show that if f is bounded and measurable on $[a, b]$ and $F(x) = \int_a^x f(t) dt + F(a)$ then $F'(x) = f(x)$ for almost all x in $[a, b]$.

Turn over

- (B) (a) Show that a function F is an indefinite integrable if and only if it is absolutely continuous.
 (b) Let g be a monotone increasing absolutely continuous function on $[a, b]$ with $g(a) = c, g(b) = d$. Show that :
- i) for any open set $O \subset [c, d]$

$$mO = \int_{g^{-1}[O]} g'(x) dx$$

- ii) Let $H = \{x : g'(x) \neq 0\}$. If E is a subset of $[c, d]$ with $mE = 0$, then $g^{-1}(E) \cap H$ has measure zero.
 iii) If E is a measurable subset of $[c, d]$, then $F = g^{-1}(E) \cap H$ is measurable and

$$mE = \int_F g' = \int_a^b \chi_E(g(x)) g'(x) dx.$$

Unit IV

18. (A) (a) State and prove Radon-Nikodym Theorem.
 (b) Show that the set function μ^* is an outer measure.
- (B) (a) Show that the class B of μ^* -measurable set is a σ algebra. Also $\bar{\mu}$ is μ^* restricted to B is a complete measure on B .
 (b) Let (X, A, μ) and (Y, B, γ) be two complete measure space. Let x be a point of X and E a set in $R_{\sigma\delta}$. Show that E_x is a measurable subset of Y .

(4 × 12 = 48 marks)

C 42465

(Pages : 3)

Name.....

Reg. No.....

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS-PG)

Mathematics

MAT2C10—TOPOLOGY

(2022 Admission onwards)

Time : Three Hours

Maximum : 50 Marks

Part A

*Answer all the questions.
Each question carries 1 mark.*

1. Show that the set of rational numbers is countably infinite.
2. Show that there is bijective correspondence of $A \times B$ with $B \times A$.
3. Show that in a topological space, finite unions of closed sets are closed.
4. Show that if a subset A of a topological space is closed, then A contains all its limit points.
5. Prove that boundedness of a set is not a topological property.
6. Let $f : X \rightarrow Y$. If the function f is continuous, then show that for every convergent sequence $x_n \rightarrow x$, the sequence $f(x_n) \rightarrow f(x)$.
7. Prove that every closed subspace of a compact space is compact.
8. Prove that every closed interval in \mathbb{R} with standard topology is compact.

(8 × 1 = 8 marks)

Part B

*Answer any six questions.
Each question carries 3 marks.*

9. Prove that finite product of countable sets is countable.
10. Let X be a set, let B be a basis for a topology T on X . Then prove that T equals the collection of all unions of elements of B .
11. If B is a basis for the topology of X and C is a basis for the topology of Y , then prove that the collection $D = \{B \times C \mid B \in B \text{ and } C \in C\}$ is a basis for the topology of $X \times Y$.

Turn over

12. Let A be a subset of the topological space X . Then prove that $x \in \bar{A}$ if and only if every open set U containing x intersects A .
13. Prove that the topologies on \mathbb{R}^n induced by the Euclidean metric d and the square metric p are the same as the product topology on \mathbb{R}^n .
14. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.
15. Is the union of a collection of connected subspaces of X that have a point in common connected? Justify your answer.
16. Prove that every compact subspace of a Hausdorff space is closed.
17. Let $f : X \rightarrow Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y) . Then prove that f is uniformly continuous.

(6 × 3 = 18 marks)

Part C

*Answer any three questions.
Each question carries 8 marks.*

18. (a) Show that in a well-ordered set, every element except the largest (if one exists) has an immediate successor.
- (b) Show that if A is a subbasis for a topology on X , then the topology generated by A equals the intersection of all topologies on X that contain A .
19. (a) Let Y be a subspace of X . Then prove that a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y .
- (b) Let $X = A \cup B$ where A and B are closed in X . Let $f : A \rightarrow Y$ and $g : B \rightarrow Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then prove that f and g combine to give a continuous function $h : X \rightarrow Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.
20. (a) Let $f_n : X \rightarrow Y$ be a sequence of continuous functions from the topological space X to the metric space Y . If (f_n) converges uniformly to f , then prove that f is continuous.
- (b) Let $p : X \rightarrow Y$ be a quotient map let A be a subspace of X that is saturated with respect to p let $q : A \rightarrow p(A)$ be the map obtained by restricting p . Then prove the following :
 - (i) If A is either open or closed in X , then q is a quotient map.
 - (ii) If p is either an open map or a closed map, then q is a quotient map.

21. (a) Show that every locally compact Hausdorff space is completely regular.
(b) Prove that a product of regular spaces is regular.
22. (a) Prove that every metrizable space is normal.
(b) Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.

(3 × 8 = 24)

C 42460

(Pages : 4)

Name.....

Reg. No.....

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2023

(CCSS—PG)

Mathematics

MAT 2C 09—TOPOLOGY

(2019 Admission onwards)

Time : Three Hours

Maximum : 80 Marks

Part A*Answer all the questions.**Each question carries 2 marks.*

1. Give an example to show that two topologies on a set need not be comparable.
2. Differentiate between co-finite topology and co-countable topology.
3. Define semi-open interval topology. Why is it known as upper limit topology ?
4. What is meant by dense subset of a topological space ? Give an example.
5. Prove that every second countable space is first countable.
6. Prove that the intersection of any family of boxes is a box.
7. What is a Tychonoff space ? Under what condition a normal space can be a Tychonoff space ?
8. Consider the topology $T = \{X, \phi, \{a\}, \{b, c\}\}$ on the set $X = \{a, b, c\}$. Check whether $\{X, T\}$ is a T_1 space.

(8 × 2 = 16 marks)

Part B*Answer any four questions.**Each question carries 4 marks.*

9. Let T_1, T_2 be two topologies for a set having bases $\mathcal{B}_1, \mathcal{B}_2$ respectively. Then prove that T_1 is weaker than T_2 if and only if every member of \mathcal{B}_1 can be expressed as a union of some members of \mathcal{B}_2 .

Turn over

10. Prove that for a subset A of a space X , $\bar{A} = A \cup A'$, where A' denotes the derived set of A and \bar{A} , the closure of A .
11. Prove that every open, surjective map is a quotient map.
12. Prove that every continuous real-valued function on a compact space is bounded and attains its extrema.
13. What is component of a space? Prove that every nonempty connected subset is contained in a unique component.
14. Prove that the projection functions are open.

(4 × 4 = 16 marks)

Part C*Answer either A or B part of the following questions.**Each question carries 12 marks.*

15. A (a) Given any family \mathcal{S} of subsets of X , Prove that there is a unique topology T on X having \mathcal{S} as sub-base. Further, prove that every member of T can be expressed as the union of sets each of which can be expressed as the intersection of finitely many members of \mathcal{S} .
- (b) Let (X, d) be a metric space. Then show that :
- the empty set ϕ and the entire set X are open,
 - the union of any family of open sets is open,
 - the intersection of any finite number of open sets is open.
- B (a) Let X be a set, T a topology on X and \mathcal{S} be a family of subsets of X . Prove that \mathcal{S} is a sub-base for T if and only if \mathcal{S} generates T .
- (b) Let \mathcal{B} be a base for a topology T on X and let $Y \subset X$. Let $\mathcal{B}/Y = \{B \cap Y : B \in \mathcal{B}\}$. Prove that \mathcal{B}/Y is a base for the topology T/Y on Y .
16. A (a) Let $\{(Y_i, T_i) : i \in I\}$ be an indexed family of topological spaces, X any set and $\{f_i : i \in I\}$ an indexed family of functions from Y_i into X . Prove that there exists a unique largest topology on X which makes each f_i continuous.
- (b) Let (X, T) be a topological space and $A \subset X$. Prove that A is a Lindeloff subset of X if and only if the subspace $(A, T/A)$ is Lindeloff.

- B (a) Prove that every second countable space is Lindeloff.
- (b) Show that separability is not a hereditary property.
17. A (a) Prove that for a subset A of a space X ,
- $$\bar{A} = \{y \in X : \text{every neighbourhood of } y \text{ meets } A \text{ non-vacuously}\}.$$
- (b) Let $\{(X_i, T_i) : i = 1, 2, \dots, n\}$ be a collection of topological spaces and (X, T) their topological product. Prove that each projection π_i is continuous.
- B (a) Prove that there is a one-to-one correspondence between the set of topologies on a set and the set of all nearness relations on that set.
- (b) Consider the topology $T = \{X, \phi, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$ on
- $$X = \{a, b, c, d, e\}.$$
- (i) List the closed subsets of X .
- (ii) Determine the closure of the sets $\{a\}$, $\{b\}$ and $\{c, e\}$.
- (iii) Which sets in (ii) are dense in X ?
18. A (a) Show that for a topological space (X, T) the following are equivalent:
- (i) The space X is a T_1 space
- (ii) For any $x \in X$ the singleton set $\{x\}$ is closed.
- (iii) Every finite subset of X is closed.
- (iv) The topology T is stronger than the cofinite topology on X .
- (b) Let $X = \prod X_i, i \in I$, each X_i being a topological space. Suppose $\{x_n\}$ is a sequence in X and that $x \in X$. Then prove that $\{x_n\}$ converges to x in X if and only if for each $i \in I$, the sequence $\{\pi_i(x_n)\}$ converges to $\pi_i(x)$ in X_i , where π_i 's are projection functions.

Turn over

B (a) Prove that for any sets Y, I and J , $(Y^I)^J = Y^{I \times J}$ upto a bijection.

(b) Prove that the axioms T_0, T_1, T_2, T_3 and T_4 form a hierarchy of progressively stronger conditions.

(4 × 12 = 48 marks)