

**PROBABILISTIC ANALYSIS OF SOME
SPARRE ANDERSEN MODELS**

THESIS

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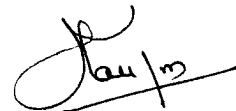
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CERTIFICATE

*Certified that this thesis, entitled "**PROBABILISTIC ANALYSIS OF SOME SPARRE ANDERSEN MODELS**" is a record of research work done independently by Sri. K.K.THAMPI under our guidance and supervision and that it has not previously formed the basis for the award of any degree, diploma, fellowship or associateship to him.*



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Introduction

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Chapter 1

Introduction

1.1 Introduction

People seek security. A sense of security may be the next basic goal after food, clothing and shelter. Only an individual with economic security can be fairly certain that he will be able to satisfy his needs(food, shelter, medical care, and so on) in the present and the future . Economic risk(which we will refer to as risk) is the possibility of losing economic security. Here comes the importance of security due to insurance. Insurance is an agreement where, for a stipulated payment called the premium, one party(the insurer) agrees to pay the other(the policy holder or his designated beneficiary) a defined amount(the claim payment or benefit) upon the occurrence of a specific loss. This defined claim payment can be a fixed amount or can reimburse all or part of his loss that occurred. The insurer

considers the losses expected for the insurance pool and the potential for variation in order to charge a premium that, in total, will be sufficient to cover all of the projected claim payment for the insurance pool. Only a small percentage of policy holders suffer losses and their losses are paid out of the premium collected from the pool of policyholders. Each policy holder exchanges an unknown loss for the payment of a known premium. A *insurance system* is a mechanism for reducing the adverse financial impact of random events that prevent the fulfillment of reasonable expectations. The stochastic nature of both the incidence and severity of claims are fundamental components of a realistic model. In this thesis we discuss different insurance risk models identifying the stochastic nature of the number of claims and amounts of claims.

Modeling the development in time of an insurer's portfolio of risk is not an easy task since such models naturally involve various stochastic processes, especially in non-life insurance, when the claim arrival times and claim severities are random. The claim arrival process is determined by the number of claims occurring in any time interval. The claim numbers are integer valued random variables whereas, in the continuous time model, claim arrival times are real valued. As a consequence of the equivalence of the claim arrival process and the claim number process, the risk process is determined by the claim number process and the claim size process.

The collective point of view in risk theory considers only the arrival time and the severity of a claim produced by the portfolio but neglects the individual risk causing the claim. We assume that the claim severities in the portfolio are independent

and identically distributed so that their distribution can be estimated from the observations. Kupper[50] noticed that the claim number process is much more interesting than the claim size process. Helten and Sterk[46] pointed out that the separate analysis of claim number process and the claim size process, leads to better estimates of aggregate claims amount, that is, the random sum of all claim severities occurring in some time interval.

The use of these random sum models received a large boost with technological advances in terms of numerical evaluation of compound distributions which were noticed by the actuarial profession in early 1980's. The most common and best known of the compound distribution is the compound Poisson distribution, where the claim count component of the model is assumed to follow the Poisson law. This model is the most tractable analytically of all the compound models. While the compound Poisson model is normally appropriate in connection with life insurance modeling, it often suffers from the disadvantage of providing an adequate fit to insurance data in other coverage. In particular, it tends to understate the true variability inherent in these situations. The Poisson model assumes that the individual risks within the portfolio are homogeneous from the standpoint of risk characteristics, presumably as a result some sort of underwriting mechanism. But the risks are normally heterogeneous, which cannot be captured by the Poisson model, resulting in the potentially dangerous situation, whereby the probabilities associated with large numbers of losses are understated. In this situation, we replace the Poisson claim count model with other best fit compound distributions

which are tractable both analytically and numerically.

The study of ruin probabilities, referred to as *risk theory*, was initiated in Sweden in the first half of 20th century. Some of the general ideas and mathematically substantial results are given in Lundberg[54, 55] and Cramer[14]. The classical risk theory is elaborately discussed in the books Buhlmann[11], Daykin et al.[16], De Vylder[18], Bowers et al.[10], Gerber[36], Grandell[42], Embrechts et al.[31], Rolski et al.[61] and Dickson[28]. A book on life insurance, (see Gerber[38] with exercise contributed by Sam Cox) rather a different flavor, gave a concise introduction to life contingencies with the support of probability theory.

In the risk process, a real life insurance business can be simplified by assuming that the insurer starts with some non-negative amount of money, collects premium and pays claims as they occur. An insurance surplus process is thus deemed to have four components: the initial surplus (amount of money at time zero), the amount of premium income received up to time t , the random epochs at which the claim occur and the corresponding claim amounts. The general surplus or risk process of the portfolio is defined by specifying

- 1) the initial surplus $u \geq 0$
- 2) the premiums which are collected at a rate $c > 0$, so that the premium income is a linear function of the time t
- 3) random epochs $\sigma_1, \sigma_2, \dots$ (with $0 < \sigma_1 < \sigma_2 < \dots$) at which the claim occur where $\{\sigma_n\}$ can be discrete or continuous.
- 4) the corresponding positive (individual or aggregate) claim sizes X_1, X_2, \dots

In the sequel, the following notations are important to the study of risk theory.

- Let $T_0 = 0$ and for $n \geq 1$, $T_n = \sigma_n - \sigma_{n-1}$, $n \geq 1$, are the *interarrival times* between successive claims.

- The *number of claims* upto time t is denoted by $N(t)$ where

$N(t) = \sum_{i=1}^{\infty} I(\sigma_i \leq t)$, $t \geq 0$. The relation between the *claim counting process* $\{N(t), t \geq 0\}$ and the sequence of claim arrivals $\{\sigma_1, \sigma_2, \dots\}$ is given

by

$$\{N(t) = n\} = \{\sigma_n \leq t < \sigma_{n+1}\}.$$

- The sequence $\{X_i, i = 1, 2, \dots\}$ of consecutive *claim sizes* are often assumed to be i.i.d random variables having common non-lattice distribution function F , finite mean $E(X_i) = \mu$ and variance $\sigma^2 = V(X_i)$.

- The *aggregate claim amount* upto time t is denoted by $X(t) = \sum_{i=1}^{N(t)} X_i$. Hence the aggregate claim amount is a *random sum* of random variables.

- The *surplus process*, denoted by $\{U(t), t \geq 0\}$, is defined as

$$U(t) = u + ct - X(t), \quad t \geq 0. \quad (1.1)$$

- The *claim process* is $S(t) = X(t) - ct$.

1.2 The Claim Number Process

The *claim number process* $\{N(t), t \geq 0\}$ is a *counting process* such that

* $N(t)$ is a non-negative integer.

* $N(0) = 0$.

* $N(t) \leq N(t + h)$.

$N(t) - N(t + h)$ is the number of claims occurring in the time interval $(t, t + h]$.

Usually we assume that the jumps are of the sizes one so that multiple claims are excluded. But there are cases where this assumption is not valid. We discuss such a situation in Chapter 3.

Let us denote $\{p_n(t), n = 0, 1, 2, \dots\}$ with

$$p_n(t) = P\{N(t) = n\} = P\{\sigma_n \leq t < \sigma_{n+1}\}$$

for the probability function of the counting process $N(t)$.

In some cases we assume that $\{N(t), t \geq 0\}$ is a *renewal process*, ie, the random variables T_n 's are non-negative, independent and have the same distribution as a *generic random variable* T .

Mathematically the simplest renewal process is the *Poisson process* where the inter-arrival times are independent, exponentially distributed. When the process $N(t)$ is a Poisson process with intensity $\lambda > 0$, we have

$$P\{N(t) = n\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n = 0, 1, 2, \dots$$

This model is known as the *Classical* (Cramer-Lundberg) model. The *renewal* (Sparre Andersen) model is the *generalization* of the classical model which allows a general renewal counting process.

1.3 The Aggregate Claim Amount

Let us write the distribution function of $X(t)$ as

$$F_{X(t)}(x) = P\{X(t) \leq x\} = P\left\{\sum_{i=1}^{N(t)} X_i \leq x\right\}.$$

The derivation of an explicit formula for $F_{X(t)}(x)$ in the general case is almost impossible. It is often assumed that the process $N(t)$ and $\{X_i\}$ are stochastically independent, and we follow this in our *study* also. But there are situations where the process $\{N(t)\}$ and $\{X_i\}$ will not be independent.

When $\{N(t)\}$ and $\{X_i\}$ are independent, by an application of the law of total probability, we can see that

$$F_{X(t)}(x) = P\{X(t) \leq x\} = \sum_{n=1}^{\infty} p_n(t) F^{*n}(x) \quad (1.2)$$

where $F^{*n}(x)$ denotes the n -fold *convolution* of $F(x)$, ie, $F^{*n}(x) = P\{X_1 + X_2 + \dots + X_n \leq x\}$, the distribution function of the sum of n independent random variables. Usually we rely on approximation or numerical algorithms for a solution of (1.2).

In some cases, *Laplace-Stieljes transform* helps us to find the n -fold convolution of $F^{*n}(x)$. Then inversion techniques can be used to find a solution of (1.2). We discuss this method to find finite time ruin probability in Chapter 3.

1.4 Transforms

The definitions and notations of the following transforms play crucial roles.

Moment Generating Function

Let $I = \{s \in R : E\{e^{sx}\} < \infty\}$. The moment generating function $\hat{m} : I \rightarrow R$ of X is defined by

$$\hat{m}(s) = E\{e^{sx}\} = \int_{-\infty}^{\infty} e^{sx} f(x) dx$$

The distributions dealt within this thesis, it is assumed that $\hat{m}(s)$ exist in some neighborhood of the origin, ie, for $|s| < 0$, for $s_0 > 0$.

Laplace-Stieltjes transform

Let F be a distribution function concentrated on $(0, \infty)$, then

$$\hat{f}(s) = \int_0^{\infty} e^{-sx} dF(x), \quad s \in R.$$

denotes the Laplace-Stieltjes transform of F . It can be seen that \hat{f} is unique, ie, if two distributions have the same Laplace-Stieltjes transform, they are identical.

Besides Laplace-Stieltjes transform, we sometimes consider simply the Laplace transform, $\hat{f}(s) = \int_0^\infty e^{-sx} f(x) dx$ of a function $f : R \rightarrow R^+$. Clearly, if the distribution function F is a continuous with density f , then Laplace-Stieltjes transform is equal to the Laplace transform of f .

Probability Generating Function

Consider the lattice random variables on N with probability function $\{p_k, k \in N\}$. Then the probability generating function $\hat{k} : [-1, 1] \rightarrow R$ defined as

$$\hat{k}(s) = \sum_{k=0}^{\infty} p_k s^k.$$

If $\{a_k, k \in N\}$ is an arbitrary sequence of real numbers, we also define the probability generating function $\hat{k}(s)$ of a_k by $\hat{k}(s) = \sum_{k=0}^{\infty} a_k s^k$, provided the sum is convergent.

1.5 Ruin Probabilities

Ruin theory is an important part of actuarial mathematics. The calculation of ruin probabilities (either approximation or exact) has been a constant source of inspiration and technique development in actuarial mathematics.

The event that $U(t)$ ever falls below zero is called *ruin*.

$$\text{ie, } \text{Ruin} = \{U(t) < 0, \text{ for some } t > 0\}. \quad (1.3)$$

The *time of ruin*

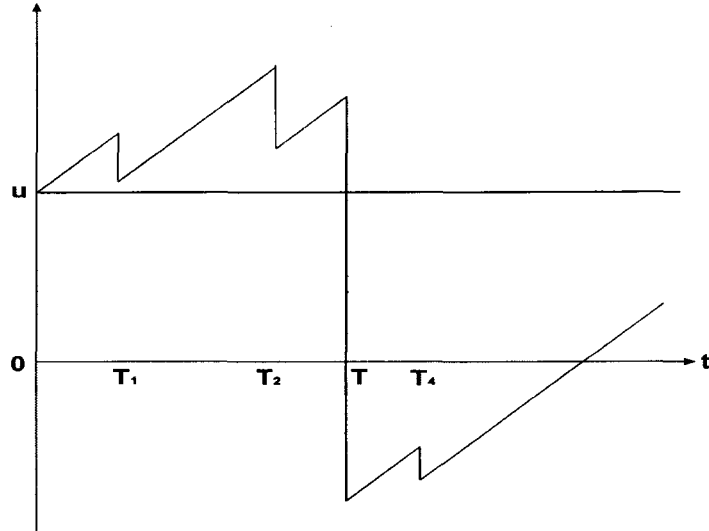


Figure 1.1: A realization of surplus process $U(t)$, ruin at $T(= T_3)$.

$$T = \inf\{t > 0 : U(t) < 0\} \quad (1.4)$$

is the first epoch when the surplus process becomes negative.

The *probability of ultimate ruin*, denoted by $\psi(u)$, is defined as

$$\psi(u) = P\{\text{Ruin} \mid U(0) = u\} = P\{T < \infty\}. \quad (1.5)$$

The *survival probability* is denoted by $\varphi(u) = 1 - \psi(u)$ to be the probability that ruin never occurs starting from the initial surplus u .

The *finite horizon ruin probability* is then given by

$$\begin{aligned} \psi(u, \tau) &= P\{U(t) < 0, \text{ for some } t \leq \tau\} \quad 0 < \tau < \infty \\ &= P\{T < \tau\}. \end{aligned} \quad (1.6)$$

It can be seen that T may be a defective random variable with defective density

$$\omega(u, \tau) = \frac{\partial}{\partial \tau} \psi(u, \tau).$$

Then we can consider the proper random variable T_k such that $T_k = T | \{T < \infty\}$ with proper density function

$$\omega_k(u, \tau) = \frac{\omega(u, \tau)}{\psi(u)}.$$

For practical point of view, $\psi(u, \tau)$, where τ is related to the planning horizon of the company, perhaps be regarded as more interesting than $\psi(u)$. Most insurance managers will closely follow the development of the risk business and increase the premium if the risk business behaves badly. The planning may be thought of as the sum of the following: the time until the risk business is found to behave *badly*, the time for a premium increase takes effect. Therefore, in non-life insurance, it may be natural to regard τ equal to 4 or 5 years as reasonable (Garandell[42]).

1.6 Net Profit Condition

A major problem in insurance industry is to determine a *suitable* insurance premium rate. The actuary has to make some decisions, for instance, what premium should be charged and which type of reinsurance to take. Often the premium is determined by company policies and the tariff of rivals. A possible criterion for optimizing the reinsurance treaty would be to minimize the probability that the claim process ever exceeds the level u . A vast amount of literature has been devoted to

study on insurance premium for general models, *see*, for instance, Goovaerts et al.[41]. It depends on the measure of solvency we want to optimize over a given period of time. The premium rate should be chosen so that a small $\psi(u, \tau)$ results for given u and τ . The first step in this direction would be to require that $\psi(u) < 1$, for all $u \geq 0$, which is equivalent to $P\{T = \infty\} > 0$, the company is given strictly positive probability of infinitely long survival. Therefore, adjustment to this strategy have to be made before real premium has been fixed.

Lemma 1.6.1 *For a renewal(Sparre Andersen) model,*

$$E\{U(t)\} = u + ct - \mu E\{N(t)\}.$$

For a Classical model,

$$E\{U(t)\} = u + ct - \mu\lambda t.$$

Proof

Since $E\{U(t)\} = u + ct - E\{X(t)\}$, and

$$\begin{aligned} E\{X(t)\} &= \sum_{k=0}^{\infty} E\{X(t)/N(t) = k\}P\{N(t) = k\} \\ &= \sum_{k=0}^{\infty} E\left\{\sum_{i=1}^{N(t)} X_i/N(t) = k\right\}P\{N(t) = k\} \\ &= \sum_{k=0}^{\infty} E\left\{\sum_{i=1}^k X_i/N(t) = k\right\}P\{N(t) = k\} \\ &= \mu \sum_{k=0}^{\infty} kP\{N(t) = k\} \\ &= \mu E\{N(t)\}. \end{aligned}$$

Because $E\{N(t)\} = \lambda t$ for the homogeneous Poisson process, the latter holds.

Proposition

For a renewal counting process, the following relations hold, see - Gut[45].

$$(a) E\{N(t)\} = (\lambda + o(1))t \text{ as } t \rightarrow \infty.$$

$$(b) \text{ Suppose that } V(T_i) < \infty. \text{ Then } V(N(t)) = V(T_i)\lambda^3 t + o(1) \text{ as } t \rightarrow \infty.$$

Using the above lemma and proposition it follows that for a renewal risk model, as $t \rightarrow \infty$

$$\begin{aligned} E\{U(t)\} &= u + (c - \lambda\mu)t(1 + o(1)) \\ &= u + \left(\frac{c}{\lambda\mu} - 1\right)\lambda\mu t(1 + o(t)). \end{aligned}$$

Therefore, $E\{U(t)\}/t \rightarrow c - \lambda\mu$ and an obvious condition towards solvency is $c - \lambda\mu > 0$, implying that $\{U(t)\}$ has a positive drift for large values of t . In other words, the average cash flow in the portfolio is on the positive side: on average, more premium flows into the portfolio than the claim amounts flow out. This leads to *net profit condition* in this model.

$$\rho = \frac{c}{\lambda\mu} - 1 > 0. \tag{1.7}$$

The premium income over the period $(0, t]$ is $ct = (1 + \rho)\lambda\mu t$.

By the definition of the risk process, ruin can occur only at the claim times T_i , hence for $u \geq 0$

$$\begin{aligned}\psi(u) &= P\{u + ct - X(t) < 0 \text{ for some } t \geq 0\} \\ &= P\{u + c\sigma_n - X(\sigma_n) < 0 \text{ for some } n \geq 1\} \\ &= P\{u + \sum_{k=1}^n (cT_k - X_k) < 0 \text{ for some } n \geq 1\} \\ &= P\{\sup_{n \geq 1} \sum_{k=1}^n (X_k - cT_k) > u\}.\end{aligned}$$

Therefore $\psi(u) < 1$ implies $\varphi(u) > 0$. That is,

$$1 - \psi(u) = P\{\sup_{n \geq 1} \sum_{k=1}^n (X_k - cT_k) \leq u\} > 0, \quad u \geq 0. \quad (1.8)$$

From (1.8), it can be seen that $\varphi(u)$ is the distribution function of the maximum of a random walk.

Consider the sequence $\{Z_k\}$ such that $Z_k = X_k - cT_k$, $k \geq 1$ and the corresponding random walk $S_0 = 0$, $S_n = \sum_{k=1}^n Z_k$, $n \geq 1$. $E\{Z_i\} = E\{X_i\} - cE\{T_i\} = \mu - \frac{c}{\lambda} < 0$ is the *net profit* condition.

$$\varphi(u) = P\{\sup_{n \geq 1} S_n \leq u\}. \quad (1.9)$$

Later we have shown that $\varphi(u)$ can be expressed as a compound geometric distribution function. The method of determination of *non-ruin* probability can be found in Chung[13], Karlin & Taylor[48], Feller[32] or any standard book on Stochastic Processes.

We say that the renewal risk model satisfies the *net profit* condition if

$$E\{Z_i\} = E\{X_i\} - cE\{T_i\} < 0. \quad (1.10)$$

1.7 The Adjustment Coefficient

The *adjustment coefficient* (also called the Lundberg exponent) plays a key role in calculating the ruin probability in the case of a light tailed claims. The adjustment coefficient, we denote it by R , gives a measure of risk for a surplus process.

Definition

Assume that the moment generating function of Z_1 exist in some neighborhood $(-s_0, s_0)$, $s_0 > 0$, of the origin. If a unique positive solution R to the equation

$$\hat{m}_{z_1}(s) = E\{e^{s(X_1 - cT_1)}\} = 1 \quad (1.11)$$

exists it is called the *adjustment coefficient*.

Clearly, $s = 0$ satisfies (1.11), but there exist a unique positive solution as well. The uniqueness can be seen as follows. The function $h(s) = \hat{m}_{z_1}(s)$ has derivatives on all orders in $(-s_0, s_0)$, $s_0 > 0$. Note that, $h'(0) = E(Z_1) < 0$ by the net profit condition and $h''(s) = E\{Z_1^2 e^{sZ_1}\} > 0$, since $Z_1 \neq 0$. The condition $h'(0) < 0$ and the continuity of h imply that h decreases in some neighborhood of zero and $h''(0) > 0$ implies that h is convex. This means, if there exist some $s_e \in (0, s_0)$ such that $h'(s_e) = 0$, then if h changes its monotonicity behavior from decreasing to increasing at s_e . For $s > s_e$, h increases, see fig (1.2) for illustration. Therefore

the solution R of the equation $h(s) = 1$ is unique, provided the moment generating function exist in a sufficiently large neighborhood of the origin.

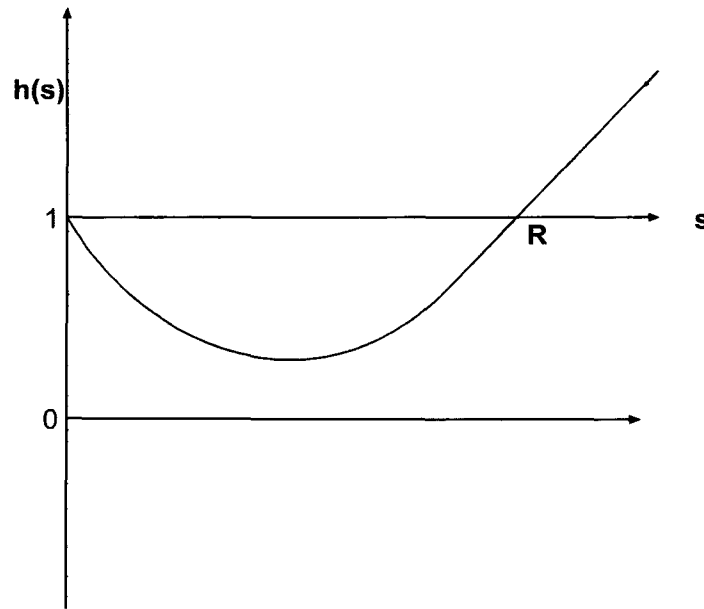


Figure 1.2: Illustration of the existence of Lundberg exponent R .

Now we are ready to discuss one of the classical results in the insurance mathematics.

1.8 The Lundberg inequality

Theorem 1.8.1 *Consider the renewal model with net profit condition (1.10). Also assume that the adjustment coefficient R exist. Then the following inequality holds*

for all $u > 0$,

$$\psi(u) \leq e^{-Ru}. \quad (1.12)$$

Proof

Let $\psi_n(u)$ to be the probability that ruin occurs on or before the n^{th} claim. Therefore it is sufficient to show that

$$\psi_n(u) \leq e^{-Ru}, \text{ for all } n \geq 1, u > 0. \quad (1.13)$$

We have

$$\psi_n(u) = P\left\{\max_{1 \leq k \leq n} S_k > u\right\} = P\{S_k > u \text{ for some } k \in \{1, 2, \dots, n\}\}$$

and note that $\psi_n(u) \uparrow \psi(u)$ as $n \rightarrow \infty$ for $u > 0$. We will prove this by induction.

Let $n = 1$,

$$\psi_1(u) \leq e^{-Ru} \hat{m}_{z_1}(R) = e^{-Ru}$$

Hence the inequality is true for $n = 1$. Assume that (1.13) is true for $n = k > 1$.

Let $F_{z_1}(x)$ denotes the distribution function of Z_1 . Then

$$\begin{aligned} \psi_{k+1}(u) &= P\left\{\max_{1 \leq n \leq k+1} S_n > u\right\} \\ &= P\{Z_1 > u\} + P\left\{\max_{2 \leq n \leq k+1} (Z_1 + (S_n - Z_1)) > u, Z_1 \leq u\right\} \\ &= \int_u^\infty dF_{Z_1}(x) + \int_{-\infty}^u P\left\{\max_{1 \leq n \leq k} [x + S_n] > u\right\} dF_{Z_1}(x) \\ &= p_1 + p_2 \end{aligned}$$

Let us consider p_2 first, where

$$\begin{aligned} p_2 &= \int_{-\infty}^u P\{\max_{1 \leq n \leq k} S_n > u - x\} dF_{Z_1}(x) \\ &= \int_{-\infty}^u \psi_k(u - x) dF_{Z_1}(x) \end{aligned} \quad (1.14)$$

Therefore,

$$p_2 \leq \int_{-\infty}^u e^{-R(u-x)} dF_{Z_1}(x) \quad (1.15)$$

and

$$p_1 \leq \int_u^{\infty} e^{-R(u-x)} dF_{Z_1}(x)$$

Hence, by the definition of the adjustment coefficient R ,

$$p_1 + p_2 \leq e^{-Ru} \hat{m}_{z_1}(R) = e^{-Ru}$$

which proves (1.13) for $n = k + 1$ and hence the theorem.

The exponential bound of the Lundberg inequality ensures that the probability of ruin is very small if one starts with a large initial capital u . Also, the bounds depends on the magnitude of the adjustment coefficient. The smaller the R , more risky is the portfolio. This result tells us that, in the case of light tailed claim distributions(eg: Exponential, Gamma, Hyper-Exponential etc) and with a sufficiently large u , there is no danger of ruin in the portfolio. But in the case of heavy tailed claim distributions (Weibull, Lognormal, Pareto etc) this statement may not be true. A detailed study on heavy tailed distribution is available in Rolski et al[61], Kluppelberg[31].

The onesided bounds of the type $\psi(u) \leq e^{-Ru}$ including the asymptotic relations $\psi(u) \sim ce^{-Ru}$ for large values of u have been studied by F.Lundberg[55] and Cramer [15]. Gerber[35] used martingale techniques to derive one sided inequality for $\psi(u)$ and Kingman[49] obtained similar results in queueing theory. Taylor[64] obtained two sided bounds for the $\psi(u)$. Mammitzsch[56] has estimated the adjustment coefficient graphically and numerically. An approach to identify the adjustment coefficient as the abscissa of convergence of a Laplace transform is done in Deheuvels and Steinebach[17].

Risk Processes

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Chapter 2

Risk Processes

2.1 Classical Risk Model

This is the most simplified risk model. In this model the claims occur according to a compound Poisson process. The most frequently used properties of the surplus process $\{U(t)\}$ are the independence and stationarity of its increments. We assume that the claims are settled in full as soon as they occur, there is no allowance of interest on the insurer's surplus, and there is no mention of expenses that an insurer would incur. This is a useful model which can give us some characteristics of insurance business.

2.1.1 Compound Poisson Process

A counting process is a Poisson process with parameter λ if the distribution of times between events is exponential with mean $1/\lambda$. In our case, an event is the occurrence of a claim. Define T_i as the time between $(i-1)^{th}$ and i^{th} claim with T_1 being the time until the first claim, then $\{T_i\}_{i=1}^{\infty}$ is a sequence of independent, exponentially distributed random variables.

Let $N(t)$ be the number of events up to time t , $t > 0$. Then for $n = 0, 1, 2, \dots$

$$N(t) \geq n + 1 \Leftrightarrow \sum_{i=1}^{n+1} T_i \leq t$$

When $N(t)$ is a Poisson process, T_1, T_2, \dots, T_{n+1} are exponentially distributed with mean $1/\lambda$, ie, $T_i \sim Exp(\lambda)$ then

$$\sum_{i=1}^{n+1} T_i \sim \Gamma(n+1, \lambda),$$

where $\Gamma(n+1, \lambda)$ denotes the Gamma distribution with parameters $(n+1, \lambda)$.

Therefore,

$$\begin{aligned} P\{N(t) \geq n+1\} &= P\left\{\sum_{i=1}^{n+1} T_i \leq t\right\} = 1 - \sum_{j=0}^n \frac{e^{-\lambda t} (\lambda t)^j}{j!} \\ P\{N(t) \leq n\} &= \sum_{j=0}^n \frac{e^{-\lambda t} (\lambda t)^j}{j!}, \text{ giving} \\ P\{N(t) = n\} &= \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \text{ for } n = 0, 1, 2, \dots \end{aligned}$$

Thus the distribution of $N(t)$ is Poisson with parameter λt .

Now let $\{N(t)\}_{t \geq 0}$ be a Poisson process with parameter λ , and let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables, each with distribution function F , independent of $N(t)$ for all $t > 0$. Then the process $\{X(t)\}_{t \geq 0}$ is said to be a compound Poisson process with parameter λ . The random variable $X(t)$ has a compound Poisson distribution with parameter λt .

An important property of compound Poisson process is that they have stationary and independent increments. A stochastic process $\{Y(t)\}_{t \geq 0}$ is said to have stationary increments, if for $0 < s < t$, the distribution of $Y(t) - Y(s)$ depends only on $t - s$ and not on the values of s and t .

A stochastic process $\{Y(t)\}$, $t \geq 0$ is said to have independent increments if for $0 < s < t \leq u < v$, $Y(t) - Y(s)$ is independent of $Y(v) - Y(u)$, ie, the increment over non-overlapping time intervals are independent. Hence the process 'starts over' in a probabilistic sense at any point of time. The idea of 'starting over' holds for a compound Poisson process because of the memoryless property of the exponential distribution.

In the context of a compound Poisson process, representing an aggregate claim process, from any fixed time $t > 0$, the distribution of time until the next claim is exponential with parameter λ and the distribution function of the next claim amount is F . This is exactly the same situation as at time 0.

2.1.2 An Integro-Differential Equation

Recall that $\varphi(u)$ can be expressed in terms of the random walk $\sum_i (X_i - cT_i)$ and considering the time and amount of the first claim, we have

$$\begin{aligned}\varphi(u) &= P\{X(t) - ct \leq u, \text{ for all } t > 0\} \\ &= P\left\{\sum_{k=1}^n (X_k - cT_k) \leq u, \text{ for all } n \geq 1\right\} \\ &= P\left\{\sum_{k=2}^n (X_k - cT_k) \leq u + cT_1 - X_1, \text{ for all } n \geq 2, X_1 - cT_1 \leq u\right\} \\ &= P\{X_1(t) - ct \leq u + cT_1 - X_1, \text{ for all } t > 0, X_1 - cT_1 \leq u\}\end{aligned}$$

where $X_1(t)$ is an independent copy of $X(t)$. Therefore

$$\begin{aligned}\varphi(u) &= E\{P\{X_1(t) - ct \leq u + cT_1 - X_1 \text{ for all } t > 0, X_1 - cT_1 \leq u / (X_1, T_1)\}\} \\ &= \int_0^\infty \int_0^{u+ct} P\{X_1(t) - ct \leq u + ct - x \text{ for all } t > 0\} dF(x) \lambda e^{-\lambda t} dt \\ &= \int_0^\infty \lambda e^{-\lambda t} \left\{ \int_0^{u+ct} \varphi(u + ct - x) dF(x) \right\} dt.\end{aligned}\tag{2.1}$$

Substituting $s = u + ct$ in (2.1), we get

$$\varphi(u) = \frac{\lambda}{c} e^{\lambda u/c} \int_u^\infty e^{-\lambda s/c} \int_0^s \varphi(s - x) dF(x) ds.$$

Differentiating the above integral equation gives

$$c\varphi'(u) = \lambda(\varphi(u) - \int_0^u \varphi(s - x) dF(x)).\tag{2.2}$$

In general the equation (2.2) cannot be solved analytically. However, one can eliminate the integral term, a differential equation can be created, and solved.

2.1.3 An integral Equation

Result:

The ruin function $\psi(u)$ satisfies the integral equation

$$c\psi(u) = \lambda \left(\int_u^\infty \bar{F}(x)dx + \int_0^u \psi(u-x)\bar{F}(x)dx \right) \quad (2.3)$$

Proof:

Integrating (2.2) over the interval $(0, u]$ and simplifying we get

$$\frac{c}{\lambda}(\varphi(u) - \varphi(0)) = \int_0^u \varphi(u-x)\bar{F}(x)dx \quad (2.4)$$

Now letting $u \rightarrow \infty$, (2.4) reduces to

$$c(1 - \varphi(0)) = \lambda \int_0^\infty \bar{F}(x)dx = \lambda\mu.$$

Then

$$\varphi(0) = 1 - \frac{\lambda\mu}{c}, \quad \psi(0) = \frac{\lambda\mu}{c}. \quad (2.5)$$

Therefore, using (2.4) and (2.5), we have

$$c\psi(u) = \lambda\mu - \lambda \int_0^u (1 - \psi(u-x))\bar{F}(x)dx$$

which is equivalent to (2.3).

The equation (2.3) is called a defective renewal equation with respect to the unknown ruin function $\psi(u)$.

Now, assume that the claim sizes are $Exp(\beta)$. Then the net profit condition (1.10) becomes $c\beta > \lambda$. The survival function $\varphi(u)$ is differentiable everywhere and satisfies the integral equation,

$$c\varphi'(u) = \lambda(\varphi(u) - e^{-\beta u} \int_0^u \varphi(x)\beta e^{\beta x} dx). \quad (2.6)$$

Differentiating (2.6) gets,

$$\begin{aligned} c\varphi''(u) &= \lambda(\varphi'(u) + \beta e^{-\beta u} \int_0^u \varphi(x)\beta e^{\beta x} dx - \beta\varphi(u)) \\ &= (\lambda - \beta c)\varphi'(u) \\ &= -(\beta - \frac{\lambda}{c})\varphi'(u). \end{aligned}$$

The general solution to this differential equation is

$$\varphi(u) = a_0 + a_1 e^{-(\beta - \frac{\lambda}{c})u} \quad \text{where } a_0, a_1 \in R. \quad (2.7)$$

But $\lim_{u \rightarrow \infty} \varphi(u) = 1$ implies $a_0 = 1$, $a_1 = -\psi(0)$. Therefore,

$$\psi(u) = \psi(0)e^{-(\beta - \frac{\lambda}{c})u}. \quad (2.8)$$

Note that $R = \beta - \frac{\lambda}{c}$. This method of solution can be used for some other claim size distributions also.

2.1.4 Laplace Transform of $\varphi(u)$

Taking Laplace transform of (2.2), we get

$$c(s\hat{\varphi}(s) - \varphi(0)) = \lambda(\hat{\varphi}(s) - \hat{\varphi}(s)\hat{f}(s)).$$

Hence

$$\hat{\varphi}(s) = \frac{c\varphi(0)}{cs - \lambda(1 - \hat{f}(s))} \quad (2.9)$$

Let $F_t(x)$ denotes the integrated tail distribution of F , then

$$F_t(x) = \frac{1}{\mu} \int_0^x \bar{F}(y) dy \quad (2.10)$$

Let $\hat{f}_t(s)$ and $\hat{f}(s)$ are the Laplace transform of $F_t(x)$ and $\bar{F}(x)$ respectively, then

$$\hat{f}_t(s) = \int_0^\infty e^{-sx} dF_t(x) = \frac{1}{\mu} \hat{f}(s).$$

Now it follows from (2.9) that

$$\hat{\varphi}(s) = \frac{\varphi(0)}{1 - \frac{\lambda\mu}{c} \hat{f}_t(s)}$$

Using (2.5), we get

$$\hat{\varphi}(s) = \left(1 - \frac{\lambda\mu}{c}\right) \sum_{n=0}^{\infty} \left(\frac{\lambda\mu}{c}\right)^n \hat{f}_t^n(s).$$

Therefore

$$\psi(u) = \left(1 - \frac{\lambda\mu}{c}\right) \sum_{n=0}^{\infty} \left(\frac{\lambda\mu}{c}\right)^n \bar{F}_t^{n*}(u). \quad (2.11)$$

The formula (2.11) is known as *Pollaczek-Khinchine* formula. We can use this formula to find the closed form solutions for $\psi(u)$ in some claim size distributions other than exponentially distributed claim sizes. The infinite series representation given in (2.11) is particularly useful for theoretical considerations. It is also useful for numerical approximations to the ruin probability $\psi(u)$ since it shows that $\varphi(u)$ is the distribution function of a geometric compound.

The *Pollaczek-Khinchine* has a natural probabilistic interpretation in terms of *ascending ladder heights of random walk*. The *Pollaczek-Khinchine* formula can be applied to much more general situations. In risk theory literature, the *Pollaczek-Khinchine* formula is often referred to as *Beekman's* convolution formula, see, Beekman[7, 8].

The classical compound Poisson risk model was introduced by Filip Lundberg[54] and extensively studied by Harold Cramer[14, 15]. The integro-differential equation (2.2) and the integral equation (2.3) goes back to these two authors. From the mathematical point of view, the ruin function of the stationary distribution of virtual waiting time in an M/GI/1 queue. Hence the formula (2.11) is same as the *Pollaczek-Khinchine* formula of queueing theory, see, for instance Asmussen[2], Baccelli& Bremaund[6], Fraken et al.[33] and Prabhu[59].

2.2 Renewal Risk Model

Consider the surplus process $\{U(t), t \geq 0\}$ in continuous time. Assume that $N(t)$ is a renewal counting process and is governed by a sequence of i.i.d. inter-occurrence times $\{T_n\}$ with a common distribution function P . The sequence $\{X_n\}$ of claim sizes consist of i.i.d. random variables with distribution function F and is independent of $\{T_n\}$. This is called Sparre Andersen Model.

Let $\{T_i\}$ be a sequence of non-negative i.i.d. random variables. The sequence

$\{\sigma_n, n \in N\}$ with $\sigma_0 = 0$ and $\sigma_n = T_1 + T_2 + \dots + T_n$ for $n = 1, 2, \dots$ is called *renewal point process* and σ_n is the n^{th} *renewal epoch*.

Assume that T_1, T_2, \dots are inter-occurrence times of claims, then the renewal counting process $\{N(t), t \geq 0\}$ is given by

$$N(t) = \sum_{n=1}^{\infty} I(\sigma_n \leq t) \quad (2.12)$$

is the number of renewal epochs in the interval $(0, t]$.

The motivation for introducing renewal process is that the Poisson process does not always describe claim arrivals in an adequate way. The lack of memory property of the exponential distribution may not be a realistic assumption. There can be large gaps between arrival of claims. It may be natural to assume that the inter-arrival times have a distribution which allows for modelling large time intervals. The log-normal or the Pareto distribution would do this job since their tails are much heavier than those of the exponential distribution. The Poisson process is not considered to be a realistic model for real life claim arrivals, if one considers long periods of time.

If we give up the hypothesis of a Poisson process we lose most of the nice properties enjoyed by this process. However, we can see that renewal process and the homogeneous Poisson process have various asymptotic properties in common.

The renewal processes are elaborately discussed in Feller[32], Asmussen[2] and Resnick[60].

Suppose that the inter renewal times T_1, T_2, \dots has distribution P and also $E(T_i) = \frac{1}{\lambda}$. In some cases P may be a *defective distribution*, in the sense that $\lim_{t \rightarrow \infty} P(t) < 1$, ie, T can be *infinity* with positive probability $1 - P(\infty)$. The resulting renewal process is called *terminating*.

Theorem 2.2.1 (a) If $P(\infty) = 1$, non-defective P , then $N(\infty) = \infty$ with probability 1. Therefore, with probability 1

$$\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \lambda.$$

(b) For a terminating renewal process, $N(\infty)$ is finite with probability 1.

see, Rolski et al.[61] and Mikosch[58].

The distribution of the aggregate claim amount $\sum_{i=1}^{N(t)} X_i$ is very important in risk theory. But in many cases, it is impossible to determine the distribution of $N(t)$ explicitly. The following *central limit theorem* gives an approximation to the claim number distribution, provided that t is large enough.

Theorem 2.2.2 Assume that $V(T_i) < \infty$, then for each $x \in R$

$$\lim_{t \rightarrow \infty} P\left\{\frac{N(t) - t\lambda}{\sqrt{bt}} \leq x\right\} = \Phi(x)$$

where $b = \lambda^3 V(T_i)$ and Φ is the distribution function of the standard normal.

2.2.1 Renewal Function

The number of renewals $H(t) = E\{N(t)\}$ as a function of t is called the renewal function of $N(t)$.

Now,

$$\begin{aligned} H(t) &= \sum_{n=1}^{\infty} E\{I(\sigma_n \leq t)\} \\ &= \sum_{n=1}^{\infty} P\left\{\sum_{i=1}^n T_i \leq t\right\} \\ &= \sum_{n=1}^{\infty} P^{*n}(t) \end{aligned}$$

If the inter-renewal times T_1, T_2, \dots are exponentially distributed with parameter λ , then the corresponding renewal counting process $\{N(t), t \geq 0\}$ is a Poisson process. In this case the random variable $N(t)$ are Poisson distributed with parameter λt . Hence $H(t) = \lambda t$, for all $t \geq 0$.

Usually, the renewal function $H(t)$ cannot be obtained in simple form. In such cases Laplace-Stieltjes transforms can be used to find a solution. Consider the Laplace-Stieltjes transform of $H(t)$ given by

$$\begin{aligned} \hat{H}(s) &= \int_0^{\infty} e^{-st} dH(t) \\ &= \int_0^{\infty} e^{-st} d\left\{\sum_{n=1}^{\infty} P^{*n}(t)\right\} \\ &= \sum_{n=1}^{\infty} (\hat{p}(s))^n \\ &= \frac{\hat{p}(s)}{1 - \hat{p}(s)}. \end{aligned}$$

2.2.2 Delayed Renewal Process

Assume that T_1, T_2, \dots is a sequence of independent non-negative random variables and T_2, T_3, \dots are identically distributed with distribution P . Assume that T_1 has an arbitrary distribution P_1 , which may be different from P . Then $\sigma_n = T_1 + T_2 + \dots + T_n$ is called *delayed renewal process*. Defining $N(t)$ as in (2.12), $\{N(t), t \geq 0\}$ is called *delayed renewal counting process*. Assume P is non-defective and has finite mean μ , then $P_1(x) = \frac{1}{\mu} \int_0^x \bar{P}(y) dy$, the integrated tail distribution P_t of P , is of particular interest in risk theory.

2.2.3 Renewal Equation

A function g satisfies the equation

$$g(t) = z(t) + \int_0^t g(t-x) dP(x)$$

where $z : R \rightarrow R^+$ is a locally bounded function vanishing on $(-\infty, 0)$ and P is a distribution on R^+ . If P is a (defective) probability distribution, then the equation is a (defective) renewal equation. There are a number of choices for the function $z(t)$.

1. If $z(t) = P(t)$, then

$$H(t) = P(t) + \int_0^t H(t-x) dP(x)$$

which is the usual renewal equation.

2. Assume that P is non-defective and has finite mean μ . Take $z(t) = P_t$, the

integrated tail distribution of P , then

$$g(t) = \frac{t}{\mu}.$$

2.2.4 Random walks

Let $\{Y_i\}$ be a sequence of i.i.d. random variables, with distribution function F , which can take both positive and negative values. The sequence $\{S_n, n \in N\}$ with $S_0 = 0$, and $S_n = Y_1 + Y_2 + \dots + Y_n$, for $n = 1, 2, \dots$ is called a *random walk*. Assume that $E(Y)$ exists and $P\{Y = 0\} < 1$.

2.2.5 Ladder Epochs

Consider the first entrance time of the random walk $\{S_n\}$ in to the positive half line $(0, \infty)$.

Define $\nu^+ = \min\{n > 0; S_n > 0\}$

and $\nu^+ = \infty$ if $S_n \leq 0$, for all $n \in N$.

Then ν^+ is called the *ascending ladder epoch* of $\{S_n\}$.

Similarly $\nu^- = \min\{n > 0; S_n \leq 0\}$ is the first entrance time to the non-positive half line $(-\infty, 0]$.

$\nu^- = \infty$ if $S_n > 0$ for all $n = 1, 2, \dots$, we call ν^- the *descending ladder epoch* of $\{S_n\}$.

Clearly, for each $k = 1, 2, \dots$, the events

$$\{\nu^+ = k\} = \{S_1 \leq 0, S_2 \leq 0, \dots, S_{k-1} \leq 0, S_k > 0\}$$

and

$$\{\nu^- = k\} = \{S_1 > 0, S_2 > 0, \dots, S_{k-1} > 0, S_k \leq 0\}$$

are determined by the first k values of $\{S_n\}$.

2.2.6 Random Walk with Drift

Theorem 2.2.3 (a) If $E(Y) > 0$, then $\lim_{n \rightarrow \infty} S_n = \infty$ (implies positive drift).

(b) If $E(Y) < 0$, then $\lim_{n \rightarrow \infty} S_n = -\infty$ (implies negative drift).

(c) If $E(Y) = 0$, then $\liminf_{n \rightarrow \infty} S_n = -\infty$, $\limsup_{n \rightarrow \infty} S_n = \infty$ (implies oscillating).

see, Resnick[60].

Let $M = \max\{0, S_1, S_2, \dots\}$ of a random walk. Then theorem (2.2.3) implies that M is finite with probability one for a random walk with *negative drift*, and infinite otherwise.

2.2.7 Ladder Height: Negative Drift

Assume that $\{S_n\}$ has a negative drift. At epoch ν^+ , $\{S_n\}$ has the first ascending height. The overshoot Y^+ above zero level is defined by

$$Y^+ = \begin{cases} S_{\nu^+} & \text{if } \nu^+ < \infty \\ \infty & \text{otherwise} \end{cases}$$

where $N = \max\{n : \nu_n^+ < \infty\}$ is the number of finite ladder epochs.

Now, define $H_0(x) = \frac{H^+(x)}{H^+(\infty)}$, then $H_0(x)$ is a proper distribution function. We can say that M has a compound geometric distribution.

Theorem 2.2.4 *If $E(Y) < 0$, then for all $x \geq 0$, and for $q = H^+(\infty)$.*

$$P\{M \leq x\} = (1 - q) \sum_{k=0}^{\infty} (H^+)^{*k}(x) = (1 - q) \sum_{k=0}^{\infty} q^k H_0^{*k}(x).$$

Note:- We have $\psi(u) = P\{M > u\}$. Therefore, for any $u \geq 0$,

$$\psi(u) = (1 - q) \sum_{k=1}^{\infty} q^k \overline{H_0^{*k}}(u).$$

Corollary 2.2.1 : *In the Sparre Andersen model with exponential claim size distribution, $\text{Exp}(\beta)$,*

$$\psi(u) = \left(1 - \frac{R}{\beta}\right) e^{-Ru}$$

for all $u \geq 0$, where R is the unique positive root of (1.11).

see, Rolski et al.[61] and Asmussen[4].

2.3 Ruin Probabilities: Sparre Andersen Model

Consider the *claim process* $\{S(t)\}$ with $\{S(t)\} = X(t) - ct$, for $t \geq 0$. Then $\psi(u)$ can be written as $\psi(u) = P\{T < \infty\}$ where $T = \min\{t : S(t) > u\}$ is the time of ruin with initial surplus u . The main objective is to obtain a closed form expression

for $\psi(u)$. However, most often this is impossible, as the expression turns out to be too complicated. In such cases, various approximations are considered.

In this work we have considered renewal risk processes in which the claim inter-occurrence times are Generalized Exponential distribution. Our main concern is to find an explicit expression for the ruin probability function $\psi(u)$ and hence analyze various problems related to ruin theory.

2.4 Generalized Exponential Distribution

In 1999, a three parameter distribution, named as Generalized exponential distribution(GE) has been introduced by Gupta and Kundu[43]. A random variable X has Generalized exponential distribution with parameter v , α and θ , if it has the distribution function

$$F(x; \alpha, \theta, v) = (1 - e^{-(x-v)/\theta})^\alpha, \quad x > v, \quad \alpha > 0, \quad \theta > 0 \quad (2.14)$$

If X has distribution function (2.14), then the corresponding density function is given by

$$f(x; \alpha, \theta, v) = \frac{\alpha}{\theta} (1 - e^{-(x-v)/\theta})^{\alpha-1} e^{-(x-v)/\theta}, \quad x > v, \quad \alpha > 0, \quad \theta > 0$$

Here α is a shape parameter, θ is a scale parameter and v is a location parameter and is denoted by $GE(\alpha, \theta, v)$.

The GE distribution has many nice properties and it can be used as an alternative to Gamma and Weibull distributions in many real life situations, for instance, see, Gupta and Kundu[44].

In this thesis, we consider renewal risk processes in which claim inter arrival times have a $GE(\alpha, 1/\lambda)$ distribution (taking $\theta = 1/\lambda$ and $v=0$ in (2.14)). Let T_1, T_2, \dots be a sequence of i.i.d. random variables where T_1 denotes the time until the first claim and for $n > 1$, T_n is the time between the $(n - 1)^{th}$ and n^{th} claim. We assume that T_n has a GE distribution with distribution function

$$G(t) = (1 - e^{-\lambda t})^\alpha, \quad \alpha > 0, \quad \lambda > 0, \quad t > 0$$

with corresponding probability density function

$$g(t) = \alpha \lambda (1 - e^{-\lambda t})^{\alpha-1} e^{-\lambda t}, \quad \alpha > 0, \quad \lambda > 0, \quad t > 0 \quad (2.15)$$

Sparre Andersen[1] considered the situation in which claims occur as a more general renewal process, and he derived an explicit result for the ultimate ruin probability for a particular case. Since then, much of the study of these models is concentrated on numerical procedures for calculating ruin probabilities. An account of such methods is given in a paper by Thorin[66].

Asmussen and Rolski[3] considered the probability of ultimate ruin for the Sparre Andersen model when individual claim amounts are distributed as phase-type, and they present a method of evaluating this probability. Dickson and Hipp[23, 24] considered a risk process in which claim interarrival times have an

Erlang(2) distribution. Further they have obtained expressions for moments of time to ruin, *see*, Dickson & Hipp[25]. Various problems related to Erlang(2) risk processes were considered by Cheng and Tang[12], Tsai and Sun[67], Yuen et al.[68] and Sun[62]. The techniques used in this thesis can lead in certain cases to numerical evaluation of ruin probabilities when the claim amount distribution is not phase-type.

This work mainly concentrate on deriving some explicit results for ruin probabilities and solving related problems for a particular class of renewal risk model. We assume that claims occur according to Generalized Exponential distribution which is not a phase-type distribution. We show that the technique that can be applied to produce explicit solutions to ruin problems in Classical risk model can be extended to more general frame work including models with inter-claims times are Generalized exponential.

A review of insurance risk and definitions of various terms involved in risk theory are presented in Chapter 1. Classical results on risk models are discussed in detail in Chapter 2. Chapter 3 focuses classical risk model where batch arrival of claims occur and numerical evaluation of finite time ruin/survival probabilities are provided. Chapter 4 deals with the ruin probabilities in the renewal risk model in which inter-claim times are Generalized exponential. Chapter 5 obtains the severity of ruin in this model. In Chapter 6, we consider the barrier probabilities and there by obtain the maximum severity of ruin. And Chapter 7 deals with the moments of time to ruin where penalty is imposed at ruin.

Chapters 3 to 7 represent independent research articles by themselves but all together shed new insight on the potential of renewal risk model.

Chapter 3

Batch Arrival of Claims*

In recent years, research in risk theory has concentrated much on the time to ruin, surplus before ruin and deficit at ruin in classical risk model as well as in renewal risk processes. Lin and Willmot[51, 52] found explicit solution for the moments of time to ruin by following a more general expression given by Gerber and Shiu[39]. Dickson and Hipp [25] found an expression for the moments of time to ruin, given ruin has occurred, the claim interarrival distribution is Erlang(2) distribution. Dickson and Waters[26] considered the distribution of time to ruin in classical risk model. Dickson and Willmot[26] obtained an expression for the density of the time of ruin in the classical risk model by inverting its Laplace transform. They suggested some method for calculating this distribution, particularly using an algorithm to calculate the finite time ruin probability. Avram and Usabel [5] has

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considered ruin probabilities and deficit at ruin for renewal risk model with phase type interarrival times using the method of double Laplace transform. Dickson and Willmot [28] derived an expression for the density of time to ruin in classical risk model by inverting its Laplace transform. Garcia[34] obtained explicit solutions for survival/ruin probabilities for the classical risk model in finite time horizon.

The insurer is primarily interested in the overall performance of the portfolio. In the case where the portfolio consists of several risks, the insurer does not care which one of that risk in the portfolio causes a particular claim. This is the collective point of view in risk theory. Further, it is assumed that no two claims occur simultaneously. But when the portfolio is large, it depends on the class of insurance under consideration whether this assumption is really acceptable. There are situations, for example, as in the case of fire insurance or (third party) automobile insurance, where in certain countries a single insurance company holds majority of all such policies, one has to take into account of the possibility that two or more claims arising out of the same incident. For instance, one has to take account the possibility that two insures from the same large portfolio produce a car accident for which both are responsible in parts. In the fire insurance, if fire occurred in a particular area, there are chances that many claims reach as a batch to the insurance company in which they are all insured for their risks. When we have only the knowledge of the probability distribution of a single claim size and the time between any two accidents, this leads to the situation where batch arrival of claims occur. Such a situation is the main focus of this chapter. In routine

models, this situation is avoided by slightly changing the point of view, namely, by considering claim events instead of single claims. The number of single claims occurring at a given claim event can then be interpreted as the size of the claim event. But that need not be available in many of the real situations. In this chapter we identify each of these single claims and assume that there is a random number of such single claims in a claim point.

3.1 The Model

The claims occur in batches and the claim arrival process $N(t)$ is assumed to be Poisson with parameter λ . Each batch consists of a random number of claims and let Y_i be the number of claims in the i^{th} claim point. Let

$$p_k = P\{Y_i = k\}, k = 1, 2, 3, \dots \quad (3.1)$$

be the probability distribution of Y_i assumed to be independent and identically distributed random variables.

Let X_{ij} denotes the amount claimed by the j^{th} claimant at the i^{th} claim point. Then $\sum_{j=1}^{Y_i} X_{ij}$ is the aggregate claim amount claimed in the i^{th} point, then

$$S(t) = \sum_{i=1}^{N(t)} \sum_{j=1}^{Y_i} X_{ij}$$

is the aggregate claim process.

The classical risk model with batch arrival of claims, the insurer's surplus at time t is

$$U(t) = u + ct - \sum_{i=1}^{N(t)} \sum_{j=1}^{Y_i} X_{ij} \quad (3.2)$$

Let $X_i = \sum_{j=1}^{Y_i} X_{ij}$, then $\{X_i\}$ is a sequence of i.i.d. random variables with X_i representing the total amount claimed at the i^{th} claim point. Let $F(x)$ be the cumulative distribution function of X_i and the corresponding density function $f(x) = F'(x)$.

Let $\psi(u, t)$ be the finite time ruin probability, then

$$\varphi(u, t) = 1 - \psi(u, t)$$

is the finite horizon survival probability. Let the double Laplace transform of the finite time ruin probability $\psi(u, t)$ is

$$\hat{\psi}(s, \delta) = \int_0^\infty \int_0^\infty e^{-su - \delta t} \psi(u, t) du dt \quad (3.3)$$

Gerber & Shiu[39] and Dickson & Hipp[25] consider the function

$$\begin{aligned} \phi(u) &= E\{e^{-\delta t} I(T < \infty) / U(0) = u\} \\ &= \int_0^\infty e^{-\delta t} \frac{\partial}{\partial t} \psi(u, t) dt \\ &= \delta \hat{\psi}(u, \delta) \end{aligned} \quad (3.4)$$

(3.4) may be viewed in terms of Laplace transform with argument δ .

3.2 Double Laplace Transform of $\varphi(u, t)$

The double m.g.f. (Laplace transform) of the ruin time T (Asmussen[4]),

$$\int_0^{\infty} E\{e^{\delta T}; T < \infty\} du = \frac{-\delta/r(\delta) - \kappa(s)/s}{\kappa(s) + \delta}$$

where $r(\delta)$ is the solution of the equation,

$$-\delta = \kappa(r(\delta)) = \lambda(\hat{f}(r(\delta)) - 1) - cr(\delta), \quad -\delta \geq \kappa(\gamma_0)$$

and

$$\kappa(s) = \lambda(\hat{f}(r(\delta)) - 1) - cs$$

Equivalently, the double Laplace transform of the time to ruin

$$\begin{aligned} \hat{\phi}(s) &= \int_0^{\infty} e^{-su} E\{e^{-\delta T}; T < \infty\} du \\ &= \frac{-\delta/r + \hat{\theta}(s)/s}{\hat{\theta}(s) - \delta}. \end{aligned} \quad (3.5)$$

Here, we denote the solution by r , with $\hat{\theta}(s) = \lambda(\hat{f}(s) - 1) + cs$, and $r = r(\delta)$ is the solution of the Lundberg equation $\lambda + \delta - cs = \lambda f(s)$. Then

$$\hat{\phi}(s) = \frac{-\delta s + r(\lambda(\hat{f}(s) - 1) + cs)}{rs(\lambda(\hat{f}(s) - 1) + cs - \delta)}. \quad (3.6)$$

This expression is already obtained by Gerber and Shiu[39]. Considering (3.3) and (3.4) the double Laplace transform of the ruin probability

$$\hat{\psi}(s, \delta) = \frac{\hat{\phi}(s)}{\delta} = \frac{-\delta s + r(\lambda(\hat{f}(s) - 1) + cs)}{\delta rs(\lambda(\hat{f}(s) - 1) + cs - \delta)}.$$

The double Laplace transform of the Survival probability is

$$\begin{aligned}
 \hat{\varphi}(s, \delta) &= \frac{1}{\delta s} - \hat{\psi}(s, \delta) \\
 &= \frac{1}{\delta s} - \frac{-\delta s + r(\lambda(\hat{f}(s) - 1) + cs)}{\delta rs(\lambda(\hat{f}(s) - 1) + cs - \delta)} \\
 &= \frac{(s - r)}{rs(\lambda(\hat{f}(s) - 1) + cs - \delta)}.
 \end{aligned} \tag{3.7}$$

3.3 Batch Size Distribution

3.3.1 Geometric Distribution

We assume that the number of claims are distributed Geometrically,

ie, $Y_i \sim Geo(\alpha)$; $0 < \alpha < 1$ with

$$P\{Y_i = y\} = f(y) = (1 - \alpha)\alpha^{y-1}, y = 1, 2, 3, \dots$$

The probability generating function of the distribution is

$$\hat{k}_{y_i}(s) = \frac{(1 - \alpha)s}{1 - \alpha s}.$$

Let us assume that the distribution of a single claim, B , to be Exponential, that

is, $X_{ij} \sim Exp(\beta)$

$$\begin{aligned}
 \hat{f}(s) &= \hat{g}_{y_i}(\hat{B}(s)) \\
 &= \frac{\beta(1 - \alpha)}{\beta(1 - \alpha) + s}.
 \end{aligned} \tag{3.8}$$

The Lundberg equation is

$$\lambda + \delta - cs = \lambda \frac{\beta(1 - \alpha)}{\beta(1 - \alpha) + s} \tag{3.9}$$

and

$$\hat{\varphi}(s, \delta) = \frac{(s - r)(\beta(1 - \alpha) + s)}{rs(cs^2 + c\beta(1 - \alpha)s - \lambda s - \delta s - \delta\beta(1 - \alpha))}. \quad (3.10)$$

An explicit expression for $\varphi(u, t)$ is obtained by complex inversion of expression (3.10).

Now, let

$$F(s) = \frac{P(s)}{Q(s)}$$

where $P(s)$ and $Q(s)$ are polynomial of degree m and n respectively with $n > m$.

The inverse Laplace transform, L^{-1} , of $F(s)$ is

$$f(t) = L^{-1}F(s) = \sum \text{Res}(F(s)e^{st}, s_k)$$

where the sum is taken over all the k residues (Res) of the complex function $F(s)e^{st}$. The expression (3.10) has poles at $s = 0$, $s = \rho$, note that $s = r (=r(\delta))$ is a removable singularity and inversion with respect to s is obtained by using Heaviside theorem *see*-John H Mathews[57].

$$f(t) = L^{-1}\{F(s)\} = \sum_{k=1}^n \frac{P(s_k)}{Q'(s_k)} e^{s_k t}$$

Hence,

$$\hat{\varphi}(u, \delta) = \frac{1}{\delta} + \frac{e^{u\rho}(\rho - r)(\beta(1 - \alpha) + \rho)}{3rc\rho^2 + 2rc\beta(1 - \alpha)\rho - 2r\lambda\rho - 2r\delta\rho - \delta r\beta(1 - \alpha)}$$

Now $\varphi(u, t)$ can be found by the complex inversion of $\hat{\varphi}(u, \delta)$ w.r.t. δ . $\varphi(u, t) =$ Sum of the residues of $e^{\delta t}\hat{\varphi}(u, \delta)$ at the poles of $\hat{\varphi}(u, \delta)$ inside the contour of integration C . Again,

$$e^{\delta t}\hat{\varphi}(u, \delta) = \frac{1}{\delta}e^{\delta t} + \frac{e^{\delta t+u\rho}(\rho - r)(\beta(1 - \alpha) + \rho)}{3rc\rho^2 + 2rc\beta(1 - \alpha)\rho - 2r\lambda\rho - 2r\delta\rho - \delta r\beta(1 - \alpha)} \quad (3.11)$$

The coefficient of $\frac{1}{s}$ in the expansion of $\frac{1}{s}e^{\delta t}$ is 1. Hence the Laplace inverse of the first part of (3.11) is 1. To find the inverse of

$$\frac{e^{\delta t + u\rho}(\rho - r)(\beta(1 - \alpha) + \rho)}{3rc\rho^2 + 2rc\beta(1 - \alpha)\rho - 2r\lambda\rho - 2r\delta\rho - \delta r\beta(1 - \alpha)} \quad (3.12)$$

we change the respective variable of integration δ to ρ and find the corresponding limits of integration for ρ .

From the Lundberg equation (3.9)

$$r = \frac{\delta + \lambda - c\beta(1 - \alpha) + \sqrt{(\delta + \lambda - c\beta(1 - \alpha))^2 + 4\delta c\beta(1 - \alpha)}}{2c}$$

$$\rho = \frac{\delta + \lambda - c\beta(1 - \alpha) - \sqrt{(\delta + \lambda - c\beta(1 - \alpha))^2 + 4\delta c\beta(1 - \alpha)}}{2c}$$

Substitution of r and ρ in (3.12) is not suffice to find the Laplace inversion with respect to δ . Since ρ is one of the root, using (3.9), we have

$$\delta = \frac{c\rho^2 + c\rho\beta(1 - \alpha) - \lambda\rho}{\rho + \beta(1 - \alpha)}$$

and r has been substituted by

$$r = -\rho + \frac{\delta + \lambda - c\beta(1 - \alpha)}{c}$$

$$\frac{d\delta}{d\rho} = \frac{c\rho^2 + 2c\rho\beta(1 - \alpha) + c(\beta(1 - \alpha))^2 - \lambda\beta(1 - \alpha)}{(\rho + \beta(1 - \alpha))^2}$$

After changing the variable of integration δ to ρ , the integral function (4.5) becomes

$$-exp\left\{\frac{\rho}{\rho + \beta(1 - \alpha)}(-\lambda t + c\rho t + \rho u + c\beta(1 - \alpha)t + \beta(1 - \alpha)u)\right\} \times$$

$$\frac{c\rho^2 + 2c\rho\beta(1 - \alpha) + c(\beta(1 - \alpha))^2 - \lambda\beta(1 - \alpha)}{\rho\beta(1 - \alpha)(-\lambda + c\rho + c\beta(1 - \alpha))} \quad (3.13)$$

Also note that as δ increases from 0 to ∞ , ρ decreases from $a (= \frac{\lambda - c\beta(1-\alpha)}{c} < 0)$ to $-\beta(1-\alpha)$. Expression (3.13) is analytic; except at the singularities $\rho = \{0, a, -\beta(1-\alpha)\}$ with $-\beta(1-\alpha) < a < 0$. Clearly, using expression (3.13), $\rho = -\beta(1-\alpha)$ is the only essential singularity. Also $\varphi(u, \delta)$ exists, only when $\text{Re}(\delta) > 0$. Hence the singularities $\{0, a\}$, which implies $\delta = 0$ can be omitted from the inversion with respect to δ . In order to put $-\beta(1-\alpha)$ inside the contour, the limit of the contour can precede by the improper integral as $-\beta(1-\alpha)$ corresponds to $\delta = \infty$

The residue at $\rho = -\beta(1-\alpha)$ is obtained by expanding the integral function (3.13) as a Laurent's series, in powers of $z = \rho + \beta(1-\alpha)$. The function reduces to

$$\exp\left\{\frac{(\lambda t - ctz - u)(\beta(1-\alpha) - z)}{z} \frac{\lambda\beta(1-\alpha) - cz^2}{\beta(1-\alpha)(\lambda - cz)(\beta(1-\alpha) - z)}\right\}$$

Applying partial fraction it becomes

$$\frac{1}{\beta(1-\alpha)} \exp\{-\beta(1-\alpha)(u+ct) - \lambda t\} \left\{ \frac{-\beta(1-\alpha)}{z - \beta(1-\alpha)} + \frac{cz}{-cz + \lambda} \right\} e^{(u+ct)z} e^{\frac{\lambda\beta(1-\alpha)t}{z}}$$

Expanding the terms in powers of z , we can obtain the coefficient of $\frac{1}{z}$, which is the residue at $z = 0$ (that is, $\rho = -\beta(1-\alpha)$).

After simplification:

$$\begin{aligned} \varphi(u, t) = & 1 - \frac{1}{\beta(1-\alpha)} \exp\left[-\beta(1-\alpha)(u+ct) - \lambda t\right] \\ & \left\{ \sum_{k=0}^{\infty} \frac{(u+ct)^k (\lambda\beta(1-\alpha)t)^{k+j+1}}{k!(k+1)!} \right. \\ & \left. + \sum_{k=0}^{\infty} (u+ct)^k \sum_{j=1}^{\infty} \left(\left(\frac{1}{\beta(1-\alpha)}\right)^j + \left(\frac{c}{\lambda}\right)^j \right) \frac{(\lambda\beta(1-\alpha)t)^{k+j+1}}{k!(k+j+1)!} \right\}. \end{aligned}$$

The following table shows the survival probabilities for varies values of u and t .

Table: 1
 Survival Probabilities, $\varphi(u, t)$
 $\alpha = 0.6, \lambda = 1, \beta = 3,$ and $c=1.1$

t	u=0	u=1	u=2	u=5	u=10
1	0.563272	0.806753	0.916746	0.994071	0.999943
2	0.563168	0.784874	0.897141	0.990147	0.999853
3	0.511266	0.733408	0.859051	0.982015	0.999588
4	0.467680	0.690594	0.825476	0.972966	0.999157
5	0.435434	0.657501	0.797898	0.963928	0.998574
6	0.411170	0.631520	0.775164	0.955251	0.997859
7	0.392257	0.610585	0.756151	0.947075	0.997036
8	0.377043	0.593326	0.740022	0.939447	0.996130
9	0.364493	0.578828	0.740022	0.932368	0.995161
10	0.353935	0.566462	0.714138	0.925814	0.994150

3.3.2 Bernoulli's Distribution

Assume that the number of claims are Bernoulli's distributed, that is, $Y_i \sim Ber(\alpha)$,

$P\{Y_i = 1\} = \alpha$, if only one claim occurs at a claim point

$P\{Y_i = 2\} = 1 - \alpha$, if two claims occur at a claim point.

As in the earlier case, let us assume that each single claim distribution, B , to be exponential with parameter β . Therefore,

$$\hat{f}(s) = \frac{\beta^2 + \alpha\beta s}{(\beta + s)^2}. \quad (3.14)$$

By the Lundberg equation,

$$\lambda + \delta - cs = \lambda \frac{\beta^2 + \alpha\beta s}{(\beta + s)^2} \quad (3.15)$$

and

$$\varphi(s, \delta) = \frac{(s-r)(\beta+s)^2}{rs(cs^3+2c\beta s^2-\lambda s^2-\delta s^2+\alpha\lambda\beta s+\beta^2 cs-2\lambda\beta s-2\delta\beta s-\delta\beta^2)}. \quad (3.16)$$

As expression (3.16) is analytic w.r.t. s , excluding at the poles $s = 0$, $s = \rho$ and $s = \eta$. Also, $s = r$ is a removable singularity and $s = \eta$ is outside the domain $\hat{f}(s)$ as

$$\eta < -\beta < \rho < 0 < r.$$

As discussed earlier, by Heaviside's theorem

$$\hat{\varphi}(u, \delta) = \frac{1}{\delta} + e^{u\rho} \frac{(\rho-r)(\rho+\beta)^2}{4rc\rho^3+6cr\beta\rho^2-3r\lambda\rho^2-3r\delta\rho^2+2r\lambda\beta\alpha\rho+2rc\beta^2\rho-4r\lambda\beta\rho-4r\delta\beta\rho-r\delta\beta^2}. \quad (3.17)$$

Again, the inverse Laplace transform of $\frac{1}{\delta}$ w.r.t. δ in expression (3.17) is 1. The inversion of the other part, viz,

$$e^{u\rho+\delta t} \frac{(\rho-r)(\rho+\beta)^2}{4rc\rho^3+6cr\beta\rho^2-3r\lambda\rho^2-3r\delta\rho^2+2r\lambda\beta\alpha\rho+2rc\beta^2\rho-4r\lambda\beta\rho-4r\delta\beta\rho-r\delta\beta^2} \quad (3.18)$$

is carried out by changing the variable of integration δ to ρ , with

$$\delta = c\rho - \lambda + \lambda \frac{(\beta^2 + \beta\rho\alpha)}{(\beta + \rho)^2}$$

$$\frac{d\delta}{d\rho} = c + \frac{\lambda\beta\alpha}{(\beta + \rho)^2} - 2\lambda \frac{(\beta^2 + \beta\rho\alpha)}{(\beta + \rho)^3}$$

Note that as δ increases from 0 to ∞ , ρ decreases from $-\frac{\lambda-2c\beta+\sqrt{\lambda^2-4c\lambda\beta-4c\alpha\lambda\beta}}{2c}$, symmetric of the adjustment coefficient, to $-\beta$. Substituting for δ in expression

(3.18) and multiplied by $\frac{d\delta}{d\rho}$, getting,

$$\begin{aligned} & -\frac{1}{\rho} \exp\left\{\frac{\rho}{(\rho+\beta)^2}(-t\lambda\rho + ct\rho^2 + u\rho^2 - 2t\lambda\beta + \alpha\lambda\beta + 2ct\rho\beta + 2u\rho\beta + ct\beta^2 + u\beta^2)\right\} \\ & + \frac{1}{r} \exp\left\{\frac{\rho}{(\rho+\beta)^2}(-t\lambda\rho + ct\rho^2 + u\rho^2 - 2t\lambda\beta + \alpha\lambda\beta + 2ct\rho\beta + 2u\rho\beta + ct\beta^2 + u\beta^2)\right\} \end{aligned} \quad (3.19)$$

We argue that $\rho = -\beta$ is the only essential singularity and we evaluate the residue at $\rho = -\beta$. Let us take $z = \beta + \rho$, therefore,

$$-\frac{1}{\rho} \exp\left\{\frac{\rho}{(\rho+\beta)^2}(-t\lambda\rho + ct\rho^2 + u\rho^2 - 2t\lambda\beta + \alpha\lambda\beta + 2ct\rho\beta + 2u\rho\beta + ct\beta^2 + u\beta^2)\right\}$$

becomes

$$\frac{1}{\beta} e^{-\beta(u+ct)-t\lambda} e^{(u+ct)z} e^{\frac{\alpha\lambda\beta}{z}} e^{\frac{(t-\alpha)\lambda\beta^2}{z^2}} \left(1 - \frac{z}{\beta}\right)^{-1} \quad (3.20)$$

Expanding the series in powers of z and collecting the coefficient of $\frac{1}{z}$, we get

$$\exp\{-\beta(u+ct) - \lambda t\} \sum_{i=0}^{\infty} b_i \sum_{k=0}^{\infty} d_k \left\{ \sum_{j=0}^{i+2k-1} a_j c_{i+2k-j-1} + b_{i+1} \sum_{j=0}^i a_j c_{i-j} \right\} \quad (3.21)$$

where $a_k = \frac{a^k}{k!}$, $b_j = \frac{b^j}{j!}$, $c_m = \left(\frac{1}{\beta}\right)^{m+1}$, $d_n = \frac{d^n}{n!}$ with $a = u + ct$, $b = \alpha\lambda\beta$ and $d = (t - \alpha)\lambda\beta^2$.

To get the residue of the second term of (3.19), we represent r in terms of ρ using Lundberg equation(3.11),

$$cs^3 - s^2\lambda + 2cs^2\beta - 2s\lambda\beta + s\alpha\lambda\beta + cs\beta^2 - s^2\delta - 2s\beta\delta - \beta^2\delta = 0.$$

Collecting the terms in powers of s

$$s^3 + c_1 s^2 + c_2 s + c_3 = 0$$

where $c_1 = -\frac{\lambda}{c} + 2\beta - \frac{\delta}{c}$, $c_2 = -\frac{2\lambda\beta}{c} + \frac{\alpha\lambda\beta}{c} - \frac{2\beta\delta}{c} + \beta^2$, $c_3 = -\frac{\beta^2\delta}{c}$.

But $r + \rho + \eta = -c_1$ and $r\rho\eta = -c_3$. Therefore,

$$r = \frac{1}{r\rho} \left(-\rho^2 r - \frac{\beta^2\delta}{c} - r\rho \left(-\frac{\lambda}{c} + 2\beta - \frac{\delta}{c} \right) \right). \quad (3.22)$$

When δ is represented in terms of z , we get

$$\delta = \frac{cz^3 - z^2\lambda - cz^2\beta + z\alpha\lambda\beta + \lambda\beta^2 - \alpha\lambda\beta^2}{z^2}. \quad (3.23)$$

Replacing δ in (3.22) using (3.23) and substituting $\rho = z - \beta$

$$\frac{1}{r} = \frac{cz^2r}{-cz^2\beta^2 + z\lambda\beta^2 + \lambda\beta^3 - \alpha\lambda\beta^3 - 2cz^2\beta r + z\alpha\lambda\beta r + \lambda\beta^2 r - \alpha\lambda\beta^2 r}. \quad (3.24)$$

Considering $\frac{1}{r} = x$, and solving for x , we get the two roots for $\frac{1}{r}$ and only one of which can represent $\frac{1}{r}$ is that $x = 0$ when $z = 0$ implies $\rho = -\beta$ and r becomes ∞ .

Hence

$$\begin{aligned} x(z) &= \frac{-2cz^2\beta + z\alpha\lambda\beta + \lambda\beta^2 - \alpha\lambda\beta^2 - \sqrt{\lambda\beta^2(-4cz^3(1-\alpha) + \lambda(z\alpha + \beta^2(1-\alpha)^2))}}{2\beta^2(cz^2 - z\lambda - \lambda\beta + \alpha\lambda\beta)} \\ &= \frac{1}{r}. \end{aligned}$$

Using Maclurin's series,

$$\frac{1}{r} = \sum_{k=0}^{\infty} m_k z^k$$

where $m_k = \frac{1}{k!} [D^k x(z)]_{z=0}$.

The second term of (3.19) becomes

$$e^{-\beta(u+ct) - \lambda t} e^{(u+ct)z} e^{\frac{\alpha\lambda\beta}{z}} e^{\frac{(t-\alpha)\lambda\beta^2}{z^2}} \sum_{k=0}^{\infty} m_k z^k. \quad (3.25)$$

Expanding the series in powers of z and collecting the coefficient of $\frac{1}{z}$, we get

$$\exp\{-\beta(u + ct) - \lambda t\} \sum_{i=0}^{\infty} b_i \sum_{k=0}^{\infty} d_k \left\{ \sum_{j=0}^{i+2k-1} a_j m_{i+2k-j-1} + b_{i+1} \sum_{j=0}^i a_j m_{i-j} \right\} \quad (3.26)$$

where a_k, b_j, d_n and m_k are same as earlier.

Combining (3.21) and (3.26) together, we get the residue at $z = 0$, and hence

$$\varphi(u, t) = 1 - \exp\{-\beta(u + ct) - \lambda t\} \sum_{i=0}^{\infty} b_i \sum_{k=1}^{\infty} d_k \left\{ \sum_{j=0}^{i+2k-1} a_j (c_{i+2k-j-1} + m_{i+2k-j-1}) + b_{i+1} \sum_{j=0}^i a_j (c_{i-j} + m_{i-j}) \right\}.$$

Table: 2

Survival Probabilities, $\varphi(u, t)$.

$\alpha = 0.6, \lambda = 1, \beta = 2,$ and $c=1.1$

t	u=0	u=1	u=2	u=5	u=10
1	0.591332	0.848300	0.945483	0.997796	0.999992
2	0.497412	0.778373	0.906101	0.994027	0.999959
3	0.455785	0.738091	0.878547	0.989861	0.999894
4	0.432024	0.711950	0.858585	0.985812	0.999797
5	0.416657	0.693709	0.843627	0.982082	0.999674
6	0.405951	0.680345	0.832108	0.978731	0.999533
7	0.398114	0.670206	0.823039	0.975759	0.999378
8	0.392171	0.662309	0.815772	0.973140	0.999053
9	0.387545	0.656032	0.809863	0.970837	0.999053
10	0.383867	0.650959	0.804999	0.968814	0.998889

Batch Arrival of Claims

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Department of Statistics , University of Calicut, 2007

Chapter 3

Batch Arrival of Claims*

In recent years, research in risk theory has concentrated much on the time to ruin, surplus before ruin and deficit at ruin in classical risk model as well as in renewal risk processes. Lin and Willmot[51, 52] found explicit solution for the moments of time to ruin by following a more general expression given by Gerber and Shiu[39]. Dickson and Hipp [25] found an expression for the moments of time to ruin, given ruin has occurred, the claim interarrival distribution is Erlang(2) distribution. Dickson and Waters[26] considered the distribution of time to ruin in classical risk model. Dickson and Willmot[26] obtained an expression for the density of the time of ruin in the classical risk model by inverting its Laplace transform. They suggested some method for calculating this distribution, particularly using an algorithm to calculate the finite time ruin probability. Avram and Usabel [5] has

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considered ruin probabilities and deficit at ruin for renewal risk model with phase type interarrival times using the method of double Laplace transform. Dickson and Willmot [28] derived an expression for the density of time to ruin in classical risk model by inverting its Laplace transform. Garcia[34] obtained explicit solutions for survival/ruin probabilities for the classical risk model in finite time horizon.

The insurer is primarily interested in the overall performance of the portfolio. In the case where the portfolio consists of several risks, the insurer does not care which one of that risk in the portfolio causes a particular claim. This is the collective point of view in risk theory. Further, it is assumed that no two claims occur simultaneously. But when the portfolio is large, it depends on the class of insurance under consideration whether this assumption is really acceptable. There are situations, for example, as in the case of fire insurance or (third party) automobile insurance, where in certain countries a single insurance company holds majority of all such policies, one has to take into account of the possibility that two or more claims arising out of the same incident. For instance, one has to take account the possibility that two insures from the same large portfolio produce a car accident for which both are responsible in parts. In the fire insurance, if fire occurred in a particular area, there are chances that many claims reach as a batch to the insurance company in which they are all insured for their risks. When we have only the knowledge of the probability distribution of a single claim size and the time between any two accidents, this leads to the situation where batch arrival of claims occur. Such a situation is the main focus of this chapter. In routine

models, this situation is avoided by slightly changing the point of view, namely, by considering claim events instead of single claims. The number of single claims occurring at a given claim event can then be interpreted as the size of the claim event. But that need not be available in many of the real situations. In this chapter we identify each of these single claims and assume that there is a random number of such single claims in a claim point.

3.1 The Model

The claims occur in batches and the claim arrival process $N(t)$ is assumed to be Poisson with parameter λ . Each batch consists of a random number of claims and let Y_i be the number of claims in the i^{th} claim point. Let

$$p_k = P\{Y_i = k\}, k = 1, 2, 3, \dots \quad (3.1)$$

be the probability distribution of Y_i assumed to be independent and identically distributed random variables.

Let X_{ij} denotes the amount claimed by the j^{th} claimant at the i^{th} claim point. Then $\sum_{j=1}^{Y_i} X_{ij}$ is the aggregate claim amount claimed in the i^{th} point, then

$$S(t) = \sum_{i=1}^{N(t)} \sum_{j=1}^{Y_i} X_{ij}$$

is the aggregate claim process.

The classical risk model with batch arrival of claims, the insurer's surplus at time t is

$$U(t) = u + ct - \sum_{i=1}^{N(t)} \sum_{j=1}^{Y_i} X_{ij} \quad (3.2)$$

Let $X_i = \sum_{j=1}^{Y_i} X_{ij}$, then $\{X_i\}$ is a sequence of i.i.d. random variables with X_i representing the total amount claimed at the i^{th} claim point. Let $F(x)$ be the cumulative distribution function of X_i and the corresponding density function $f(x) = F'(x)$.

Let $\psi(u, t)$ be the finite time ruin probability, then

$$\varphi(u, t) = 1 - \psi(u, t)$$

is the finite horizon survival probability. Let the double Laplace transform of the finite time ruin probability $\psi(u, t)$ is

$$\hat{\psi}(s, \delta) = \int_0^\infty \int_0^\infty e^{-su - \delta t} \psi(u, t) du dt \quad (3.3)$$

Gerber & Shiu[39] and Dickson & Hipp[25] consider the function

$$\begin{aligned} \phi(u) &= E\{e^{-\delta t} I(T < \infty) / U(0) = u\} \\ &= \int_0^\infty e^{-\delta t} \frac{\partial}{\partial t} \psi(u, t) dt \\ &= \delta \hat{\psi}(u, \delta) \end{aligned} \quad (3.4)$$

(3.4) may be viewed in terms of Laplace transform with argument δ .

3.2 Double Laplace Transform of $\varphi(u, t)$

The double m.g.f. (Laplace transform) of the ruin time T (Asmussen[4]),

$$\int_0^{\infty} E\{e^{\delta T}; T < \infty\} du = \frac{-\delta/r(\delta) - \kappa(s)/s}{\kappa(s) + \delta}$$

where $r(\delta)$ is the solution of the equation,

$$-\delta = \kappa(r(\delta)) = \lambda(\hat{f}(r(\delta)) - 1) - cr(\delta), \quad -\delta \geq \kappa(\gamma_0)$$

and

$$\kappa(s) = \lambda(\hat{f}(r(\delta)) - 1) - cs$$

Equivalently, the double Laplace transform of the time to ruin

$$\begin{aligned} \hat{\phi}(s) &= \int_0^{\infty} e^{-su} E\{e^{-\delta T}; T < \infty\} du \\ &= \frac{-\delta/r + \hat{\theta}(s)/s}{\hat{\theta}(s) - \delta}. \end{aligned} \quad (3.5)$$

Here, we denote the solution by r , with $\hat{\theta}(s) = \lambda(\hat{f}(s) - 1) + cs$, and $r = r(\delta)$ is the solution of the Lundberg equation $\lambda + \delta - cs = \lambda f(s)$. Then

$$\hat{\phi}(s) = \frac{-\delta s + r(\lambda(\hat{f}(s) - 1) + cs)}{rs(\lambda(\hat{f}(s) - 1) + cs - \delta)}. \quad (3.6)$$

This expression is already obtained by Gerber and Shiu[39]. Considering (3.3) and (3.4) the double Laplace transform of the ruin probability

$$\hat{\psi}(s, \delta) = \frac{\hat{\phi}(s)}{\delta} = \frac{-\delta s + r(\lambda(\hat{f}(s) - 1) + cs)}{\delta rs(\lambda(\hat{f}(s) - 1) + cs - \delta)}.$$

The double Laplace transform of the Survival probability is

$$\begin{aligned}
 \hat{\varphi}(s, \delta) &= \frac{1}{\delta s} - \hat{\psi}(s, \delta) \\
 &= \frac{1}{\delta s} - \frac{-\delta s + r(\lambda(\hat{f}(s) - 1) + cs)}{\delta r s(\lambda(\hat{f}(s) - 1) + cs - \delta)} \\
 &= \frac{(s - r)}{r s(\lambda(\hat{f}(s) - 1) + cs - \delta)}.
 \end{aligned} \tag{3.7}$$

3.3 Batch Size Distribution

3.3.1 Geometric Distribution

We assume that the number of claims are distributed Geometrically,

ie, $Y_i \sim Geo(\alpha)$; $0 < \alpha < 1$ with

$$P\{Y_i = y\} = f(y) = (1 - \alpha)\alpha^{y-1}, y = 1, 2, 3, \dots$$

The probability generating function of the distribution is

$$\hat{k}_{y_i}(s) = \frac{(1 - \alpha)s}{1 - \alpha s}$$

Let us assume that the distribution of a single claim , B , to be Exponential, that

is, $X_{ij} \sim Exp(\beta)$

$$\begin{aligned}
 \hat{f}(s) &= \hat{g}_{y_i}(\hat{B}(s)) \\
 &= \frac{\beta(1 - \alpha)}{\beta(1 - \alpha) + s}.
 \end{aligned} \tag{3.8}$$

The Lundberg equation is

$$\lambda + \delta - cs = \lambda \frac{\beta(1 - \alpha)}{\beta(1 - \alpha) + s} \tag{3.9}$$

and

$$\hat{\varphi}(s, \delta) = \frac{(s - r)(\beta(1 - \alpha) + s)}{rs(cs^2 + c\beta(1 - \alpha)s - \lambda s - \delta s - \delta\beta(1 - \alpha))}. \quad (3.10)$$

An explicit expression for $\varphi(u, t)$ is obtained by complex inversion of expression (3.10).

Now, let

$$F(s) = \frac{P(s)}{Q(s)}$$

where $P(s)$ and $Q(s)$ are polynomial of degree m and n respectively with $n > m$.

The inverse Laplace transform, L^{-1} , of $F(s)$ is

$$f(t) = L^{-1}F(s) = \sum \text{Res}(F(s)e^{st}, s_k)$$

where the sum is taken over all the k residues (Res) of the complex function $F(s)e^{st}$. The expression (3.10) has poles at $s = 0$, $s = \rho$, note that $s = r (=r(\delta))$ is a removable singularity and inversion with respect to s is obtained by using Heaviside theorem *see*-John H Mathews[57].

$$f(t) = L^{-1}\{F(s)\} = \sum_{k=1}^n \frac{P(s_k)}{Q'(s_k)} e^{s_k t}$$

Hence,

$$\hat{\varphi}(u, \delta) = \frac{1}{\delta} + \frac{e^{u\rho}(\rho - r)(\beta(1 - \alpha) + \rho)}{3rc\rho^2 + 2rc\beta(1 - \alpha)\rho - 2r\lambda\rho - 2r\delta\rho - \delta r\beta(1 - \alpha)}$$

Now $\varphi(u, t)$ can be found by the complex inversion of $\hat{\varphi}(u, \delta)$ w.r.t. δ . $\varphi(u, t) =$ Sum of the residues of $e^{\delta t}\hat{\varphi}(u, \delta)$ at the poles of $\hat{\varphi}(u, \delta)$ inside the contour of integration C . Again,

$$e^{\delta t}\hat{\varphi}(u, \delta) = \frac{1}{\delta}e^{\delta t} + \frac{e^{\delta t+u\rho}(\rho - r)(\beta(1 - \alpha) + \rho)}{3rc\rho^2 + 2rc\beta(1 - \alpha)\rho - 2r\lambda\rho - 2r\delta\rho - \delta r\beta(1 - \alpha)} \quad (3.11)$$

The coefficient of $\frac{1}{s}$ in the expansion of $\frac{1}{s}e^{\delta t}$ is 1. Hence the Laplace inverse of the first part of (3.11) is 1. To find the inverse of

$$\frac{e^{\delta t + u\rho}(\rho - r)(\beta(1 - \alpha) + \rho)}{3rc\rho^2 + 2rc\beta(1 - \alpha)\rho - 2r\lambda\rho - 2r\delta\rho - \delta r\beta(1 - \alpha)} \quad (3.12)$$

we change the respective variable of integration δ to ρ and find the corresponding limits of integration for ρ .

From the Lundberg equation (3.9)

$$r = \frac{\delta + \lambda - c\beta(1 - \alpha) + \sqrt{(\delta + \lambda - c\beta(1 - \alpha))^2 + 4\delta c\beta(1 - \alpha)}}{2c}$$

$$\rho = \frac{\delta + \lambda - c\beta(1 - \alpha) - \sqrt{(\delta + \lambda - c\beta(1 - \alpha))^2 + 4\delta c\beta(1 - \alpha)}}{2c}$$

Substitution of r and ρ in (3.12) is not suffice to find the Laplace inversion with respect to δ . Since ρ is one of the root, using (3.9), we have

$$\delta = \frac{c\rho^2 + c\rho\beta(1 - \alpha) - \lambda\rho}{\rho + \beta(1 - \alpha)}$$

and r has been substituted by

$$r = -\rho + \frac{\delta + \lambda - c\beta(1 - \alpha)}{c}$$

$$\frac{d\delta}{d\rho} = \frac{c\rho^2 + 2c\rho\beta(1 - \alpha) + c(\beta(1 - \alpha))^2 - \lambda\beta(1 - \alpha)}{(\rho + \beta(1 - \alpha))^2}$$

After changing the variable of integration δ to ρ , the integral function (4.5) becomes

$$-exp\left\{\frac{\rho}{\rho + \beta(1 - \alpha)}(-\lambda t + c\rho t + \rho u + c\beta(1 - \alpha)t + \beta(1 - \alpha)u)\right\} \times$$

$$\frac{c\rho^2 + 2c\rho\beta(1 - \alpha) + c(\beta(1 - \alpha))^2 - \lambda\beta(1 - \alpha)}{\rho\beta(1 - \alpha)(-\lambda + c\rho + c\beta(1 - \alpha))} \quad (3.13)$$

Also note that as δ increases from 0 to ∞ , ρ decreases from $a (= \frac{\lambda - c\beta(1-\alpha)}{c} < 0)$ to $-\beta(1-\alpha)$. Expression (3.13) is analytic; except at the singularities $\rho = \{0, a, -\beta(1-\alpha)\}$ with $-\beta(1-\alpha) < a < 0$. Clearly, using expression (3.13), $\rho = -\beta(1-\alpha)$ is the only essential singularity. Also $\varphi(u, \delta)$ exists, only when $\text{Re}(\delta) > 0$. Hence the singularities $\{0, a\}$, which implies $\delta = 0$ can be omitted from the inversion with respect to δ . In order to put $-\beta(1-\alpha)$ inside the contour, the limit of the contour can precede by the improper integral as $-\beta(1-\alpha)$ corresponds to $\delta = \infty$

The residue at $\rho = -\beta(1-\alpha)$ is obtained by expanding the integral function (3.13) as a Laurent's series, in powers of $z = \rho + \beta(1-\alpha)$. The function reduces to

$$\exp\left\{\frac{(\lambda t - ctz - u)(\beta(1-\alpha) - z)}{z} \frac{\lambda\beta(1-\alpha) - cz^2}{\beta(1-\alpha)(\lambda - cz)(\beta(1-\alpha) - z)}\right\}$$

Applying partial fraction it becomes

$$\frac{1}{\beta(1-\alpha)} \exp\{-\beta(1-\alpha)(u+ct) - \lambda t\} \left\{ \frac{-\beta(1-\alpha)}{z - \beta(1-\alpha)} + \frac{cz}{-cz + \lambda} \right\} e^{(u+ct)z} e^{\frac{\lambda\beta(1-\alpha)t}{z}}$$

Expanding the terms in powers of z , we can obtain the coefficient of $\frac{1}{z}$, which is the residue at $z = 0$ (that is, $\rho = -\beta(1-\alpha)$).

After simplification:

$$\begin{aligned} \varphi(u, t) = & 1 - \frac{1}{\beta(1-\alpha)} \exp\left[-\beta(1-\alpha)(u+ct) - \lambda t\right] \\ & \left\{ \sum_{k=0}^{\infty} \frac{(u+ct)^k (\lambda\beta(1-\alpha)t)^{k+j+1}}{k!(k+1)!} \right. \\ & \left. + \sum_{k=0}^{\infty} (u+ct)^k \sum_{j=1}^{\infty} \left(\left(\frac{1}{\beta(1-\alpha)}\right)^j + \left(\frac{c}{\lambda}\right)^j \right) \frac{(\lambda\beta(1-\alpha)t)^{k+j+1}}{k!(k+j+1)!} \right\}. \end{aligned}$$

The following table shows the survival probabilities for varies values of u and t .

Table: 1
 Survival Probabilities, $\varphi(u, t)$
 $\alpha = 0.6, \lambda = 1, \beta = 3,$ and $c=1.1$

t	u=0	u=1	u=2	u=5	u=10
1	0.563272	0.806753	0.916746	0.994071	0.999943
2	0.563168	0.784874	0.897141	0.990147	0.999853
3	0.511266	0.733408	0.859051	0.982015	0.999588
4	0.467680	0.690594	0.825476	0.972966	0.999157
5	0.435434	0.657501	0.797898	0.963928	0.998574
6	0.411170	0.631520	0.775164	0.955251	0.997859
7	0.392257	0.610585	0.756151	0.947075	0.997036
8	0.377043	0.593326	0.740022	0.939447	0.996130
9	0.364493	0.578828	0.740022	0.932368	0.995161
10	0.353935	0.566462	0.714138	0.925814	0.994150

3.3.2 Bernoulli's Distribution

Assume that the number of claims are Bernoulli's distributed, that is, $Y_i \sim Ber(\alpha)$,

$P\{Y_i = 1\} = \alpha$, if only one claim occurs at a claim point

$P\{Y_i = 2\} = 1 - \alpha$, if two claims occur at a claim point.

As in the earlier case, let us assume that each single claim distribution, B , to be exponential with parameter β . Therefore,

$$\hat{f}(s) = \frac{\beta^2 + \alpha\beta s}{(\beta + s)^2}. \quad (3.14)$$

By the Lundberg equation,

$$\lambda + \delta - cs = \lambda \frac{\beta^2 + \alpha\beta s}{(\beta + s)^2} \quad (3.15)$$

and

$$\varphi(s, \delta) = \frac{(s-r)(\beta+s)^2}{rs(cs^3 + 2c\beta s^2 - \lambda s^2 - \delta s^2 + \alpha\lambda\beta s + \beta^2 cs - 2\lambda\beta s - 2\delta\beta s - \delta\beta^2)}. \quad (3.16)$$

As expression (3.16) is analytic w.r.t. s , excluding at the poles $s = 0$, $s = \rho$ and $s = \eta$. Also, $s = r$ is a removable singularity and $s = \eta$ is out side the domain $\hat{f}(s)$ as

$$\eta < -\beta < \rho < 0 < r.$$

As discussed earlier, by Heaviside's theorem

$$\hat{\varphi}(u, \delta) = \frac{1}{\delta} + e^{u\rho} \frac{(\rho-r)(\rho+\beta)^2}{4rc\rho^3 + 6cr\beta\rho^2 - 3r\lambda\rho^2 - 3r\delta\rho^2 + 2r\lambda\beta\alpha\rho + 2rc\beta^2\rho - 4r\lambda\beta\rho - 4r\delta\beta\rho - r\delta\beta^2}. \quad (3.17)$$

Again, the inverse Laplace transform of $\frac{1}{\delta}$ w.r.t. δ in expression (3.17) is 1. The inversion of the other part, viz,

$$e^{u\rho + \delta t} \frac{(\rho-r)(\rho+\beta)^2}{4rc\rho^3 + 6cr\beta\rho^2 - 3r\lambda\rho^2 - 3r\delta\rho^2 + 2r\lambda\beta\alpha\rho + 2rc\beta^2\rho - 4r\lambda\beta\rho - 4r\delta\beta\rho - r\delta\beta^2} \quad (3.18)$$

is carried out by changing the variable of integration δ to ρ , with

$$\delta = c\rho - \lambda + \lambda \frac{(\beta^2 + \beta\rho\alpha)}{(\beta + \rho)^2}$$

$$\frac{d\delta}{d\rho} = c + \frac{\lambda\beta\alpha}{(\beta + \rho)^2} - 2\lambda \frac{(\beta^2 + \beta\rho\alpha)}{(\beta + \rho)^3}$$

Note that as δ increases from 0 to ∞ , ρ decreases from $-\frac{\lambda - 2c\beta + \sqrt{\lambda^2 - 4c\lambda\beta - 4c\alpha\lambda\beta}}{2c}$, symmetric of the adjustment coefficient, to $-\beta$. Substituting for δ in expression

(3.18) and multiplied by $\frac{d\delta}{d\rho}$, getting,

$$\begin{aligned} & -\frac{1}{\rho} \exp\left\{\frac{\rho}{(\rho+\beta)^2}(-t\lambda\rho + ct\rho^2 + u\rho^2 - 2t\lambda\beta + \alpha\lambda\beta + 2ct\rho\beta + 2u\rho\beta + ct\beta^2 + u\beta^2)\right\} \\ & + \frac{1}{r} \exp\left\{\frac{\rho}{(\rho+\beta)^2}(-t\lambda\rho + ct\rho^2 + u\rho^2 - 2t\lambda\beta + \alpha\lambda\beta + 2ct\rho\beta + 2u\rho\beta + ct\beta^2 + u\beta^2)\right\} \end{aligned} \quad (3.19)$$

We argue that $\rho = -\beta$ is the only essential singularity and we evaluate the residue at $\rho = -\beta$. Let us take $z = \beta + \rho$, therefore,

$$-\frac{1}{\rho} \exp\left\{\frac{\rho}{(\rho+\beta)^2}(-t\lambda\rho + ct\rho^2 + u\rho^2 - 2t\lambda\beta + \alpha\lambda\beta + 2ct\rho\beta + 2u\rho\beta + ct\beta^2 + u\beta^2)\right\}$$

becomes

$$\frac{1}{\beta} e^{-\beta(u+ct)-t\lambda} e^{(u+ct)z} e^{\frac{\alpha\lambda\beta}{z}} e^{\frac{(t-\alpha)\lambda\beta^2}{z^2}} \left(1 - \frac{z}{\beta}\right)^{-1} \quad (3.20)$$

Expanding the series in powers of z and collecting the coefficient of $\frac{1}{z}$, we get

$$\exp\{-\beta(u+ct) - \lambda t\} \sum_{i=0}^{\infty} b_i \sum_{k=0}^{\infty} d_k \left\{ \sum_{j=0}^{i+2k-1} a_j c_{i+2k-j-1} + b_{i+1} \sum_{j=0}^i a_j c_{i-j} \right\} \quad (3.21)$$

where $a_k = \frac{a^k}{k!}$, $b_j = \frac{b^j}{j!}$, $c_m = \left(\frac{1}{\beta}\right)^{m+1}$, $d_n = \frac{d^n}{n!}$ with $a = u + ct$, $b = \alpha\lambda\beta$ and $d = (t - \alpha)\lambda\beta^2$.

To get the residue of the second term of (3.19), we represent r in terms of ρ using Lundberg equation(3.11),

$$cs^3 - s^2\lambda + 2cs^2\beta - 2s\lambda\beta + s\alpha\lambda\beta + cs\beta^2 - s^2\delta - 2s\beta\delta - \beta^2\delta = 0.$$

Collecting the terms in powers of s

$$s^3 + c_1s^2 + c_2s + c_3 = 0$$

where $c_1 = -\frac{\lambda}{c} + 2\beta - \frac{\delta}{c}$, $c_2 = -\frac{2\lambda\beta}{c} + \frac{\alpha\lambda\beta}{c} - \frac{2\beta\delta}{c} + \beta^2$, $c_3 = -\frac{\beta^2\delta}{c}$.

But $r + \rho + \eta = -c_1$ and $r\rho\eta = -c_3$. Therefore,

$$r = \frac{1}{r\rho} \left(-\rho^2 r - \frac{\beta^2\delta}{c} - r\rho \left(-\frac{\lambda}{c} + 2\beta - \frac{\delta}{c} \right) \right). \quad (3.22)$$

When δ is represented in terms of z , we get

$$\delta = \frac{cz^3 - z^2\lambda - cz^2\beta + z\alpha\lambda\beta + \lambda\beta^2 - \alpha\lambda\beta^2}{z^2}. \quad (3.23)$$

Replacing δ in (3.22) using (3.23) and substituting $\rho = z - \beta$

$$\frac{1}{r} = \frac{cz^2r}{-cz^2\beta^2 + z\lambda\beta^2 + \lambda\beta^3 - \alpha\lambda\beta^3 - 2cz^2\beta r + z\alpha\lambda\beta r + \lambda\beta^2 r - \alpha\lambda\beta^2 r}. \quad (3.24)$$

Considering $\frac{1}{r} = x$, and solving for x , we get the two roots for $\frac{1}{r}$ and only one of which can represent $\frac{1}{r}$ is that $x = 0$ when $z = 0$ implies $\rho = -\beta$ and r becomes ∞ .

Hence

$$\begin{aligned} x(z) &= \frac{-2cz^2\beta + z\alpha\lambda\beta + \lambda\beta^2 - \alpha\lambda\beta^2 - \sqrt{\lambda\beta^2(-4cz^3(1-\alpha) + \lambda(z\alpha + \beta^2(1-\alpha)^2))}}{2\beta^2(cz^2 - z\lambda - \lambda\beta + \alpha\lambda\beta)} \\ &= \frac{1}{r}. \end{aligned}$$

Using Maclurin's series,

$$\frac{1}{r} = \sum_{k=0}^{\infty} m_k z^k$$

where $m_k = \frac{1}{k!} [D^k x(z)]_{z=0}$.

The second term of (3.19) becomes

$$e^{-\beta(u+ct)-\lambda t} e^{(u+ct)z} e^{\frac{\alpha\lambda\beta}{z}} e^{\frac{(t-\alpha)\lambda\beta^2}{z^2}} \sum_{k=0}^{\infty} m_k z^k. \quad (3.25)$$

Expanding the series in powers of z and collecting the coefficient of $\frac{1}{z}$, we get

$$\exp\{-\beta(u + ct) - \lambda t\} \sum_{i=0}^{\infty} b_i \sum_{k=0}^{\infty} d_k \left\{ \sum_{j=0}^{i+2k-1} a_j m_{i+2k-j-1} + b_{i+1} \sum_{j=0}^i a_j m_{i-j} \right\} \quad (3.26)$$

where a_k, b_j, d_n and m_k are same as earlier.

Combining (3.21) and (3.26) together, we get the residue at $z = 0$, and hence

$$\varphi(u, t) = 1 - \exp\{-\beta(u + ct) - \lambda t\} \sum_{i=0}^{\infty} b_i \sum_{k=1}^{\infty} d_k \left\{ \sum_{j=0}^{i+2k-1} a_j (c_{i+2k-j-1} + m_{i+2k-j-1}) + b_{i+1} \sum_{j=0}^i a_j (c_{i-j} + m_{i-j}) \right\}.$$

Table: 2

Survival Probabilities, $\varphi(u, t)$.

$\alpha = 0.6, \lambda = 1, \beta = 2,$ and $c=1.1$

t	u=0	u=1	u=2	u=5	u=10
1	0.591332	0.848300	0.945483	0.997796	0.999992
2	0.497412	0.778373	0.906101	0.994027	0.999959
3	0.455785	0.738091	0.878547	0.989861	0.999894
4	0.432024	0.711950	0.858585	0.985812	0.999797
5	0.416657	0.693709	0.843627	0.982082	0.999674
6	0.405951	0.680345	0.832108	0.978731	0.999533
7	0.398114	0.670206	0.823039	0.975759	0.999378
8	0.392171	0.662309	0.815772	0.973140	0.999053
9	0.387545	0.656032	0.809863	0.970837	0.999053
10	0.383867	0.650959	0.804999	0.968814	0.998889

Ruin Probabilities Under Generalized Exponential Distribution

K.K.Thampi “Probabilistic analysis of some Sparre Andersen models” Thesis.
Department of Statistics , University of Calicut, 2007

Chapter 4

Ruin Probabilities Under Generalized Exponential Distribution *

4.1 Introduction

A lot of research has been done on ruin probabilities concerned with classical risk models in which claims are occurring according to a Poisson process. Sparre Anderson in a paper to the 1957 International Congress of Actuaries in New York, proposed a generalization of the classical risk theory. Instead of assuming exponentially distributed independent interarrival times, he assumed a more general distribution function which may be considered as the first step in to the renewal

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risk processes. In recent years there has been considerable interest in the study of renewal risk models and in the evaluation of their ruin probabilities.

The numerical computation of the probability of ruin in the classical and Sparre Anderson models were considered by Asmussen and Rolski[3] when the claim amount distribution is of phase type. Dickson and Hipp[23, 25] deals with risk model in which claim interarrival times have an Erlang(2) distribution. Avram and Usabel[5] provide generalization of Thorin's formula[65], for the double Laplace transforms of the finite time ruin probabilities of a renewal risk processes and also consider the deficit at ruin. Gerber and Shiu [40] has investigated a Sparre Andersen risk model in which the distribution of interarrival time random variable is the sum of n independent, exponentially distributed random variables with different parameters, by extending their earlier results on classical risk models[39].

In this chapter, we consider a renewal risk process in which claim inter-arrival times have a Generalized Exponential distribution (5.3). Let T_1, T_2, \dots be the sequence of i.i.d. random variables where T_1 denotes the time until the first claim and, for $n > 1$, T_n denotes the time between the $(n - 1)^{th}$ and n^{th} claim.

The amount of n^{th} claim is denoted by X_n and $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables with distribution function $F(x)$ with prob.density function $f(x)$. It is assumed that the claims and inter-arrival times are independent and their respective means, denoted by m and μ , exist and are finite. For the requirement of a positive security loading, we assume the condition (1.10). In order to find an explicit expression for the ultimate ruin probability and for the ease of numerical evaluation of the ruin probabilities, we consider the case where $\alpha=2$.

4.2 Laplace transform of $\varphi(u)$

Suppose that the first claim at epoch t and has magnitude x . For no ruin ever to occur it is necessary that $x \leq u + ct$ and for $\tau > t$, the increment $X(\tau) - x$ should be $\leq u + c\tau - x$. Such increments being independent of the past, the latter event has probability $\varphi(u + ct - x)$. Summing over all possible t and x , we get

$$\varphi(u) = \int_0^\infty p(t) dt \int_0^{u+ct} \varphi(u + ct - x) dF(x) \quad (4.1)$$

(see - Feller[32]).

The change of variable $s = u + ct$ leads to

$$c\varphi(u) = \int_u^\infty p\left(\frac{s-u}{c}\right) ds \int_0^s \varphi(s-x) dF(x)$$

$\varphi(u)$ is differentiable, and using the condition $p(0) = 0$ we get the integro-differential equation:

$$c^2\varphi''(u) = 3\lambda c\varphi'(u) - 2\lambda^2\varphi(u) + 2\lambda^2 \int_0^u \varphi(u-x) dF(x) \quad (4.2)$$

Taking the Laplace transform of (4.2), rearranging we get:

$$\hat{\varphi}(s) = \frac{c^2 s\varphi(0) + c^2\varphi'(0) - 3\lambda c\varphi(0)}{c^2 s^2 - 3\lambda cs + 2\lambda^2(1 - \hat{f}(s))}$$

Using the final value theorem of Laplace transform and:

$$\lim_{u \rightarrow \infty} \varphi(u) = 1,$$

we have:

$$\hat{\varphi}(s) = \frac{c^2 s\varphi(0) + 2\lambda^2 m - 3\lambda c}{c^2 s^2 - 3\lambda cs + 2\lambda^2(1 - \hat{f}(s))}, \quad \text{if } \lim_{s \rightarrow 0} s\hat{\varphi}(s) \text{ exists.} \quad (4.3)$$

But the condition (1.10) stipulates

$$3c > 2\lambda m.$$

Hence the numerator of expression (4.3) has a positive zero $k = \frac{3\lambda c - 2\lambda^2 m}{c^2 \varphi(0)}$. Also the denominator has a unique positive zero as it is the defining equation for the adjustment coefficient. This positive solution k is used to find $\varphi(0)$, that is,

$$\varphi(0) = \frac{3\lambda c - 2\lambda^2 m}{c^2 k}, \quad k > 0.$$

4.3 Ladder Height Distribution

Considering L to be the maximum aggregate loss, then:

$$P\{L \leq u\} = \varphi(u)$$

If $\hat{\gamma}(s)$ is the Laplace transform of L , we have:

$$\begin{aligned} \hat{\gamma}(s) &= s\hat{\varphi}(s) \\ &= \frac{c^2 s^2 \varphi(0) + (2\lambda^2 m - 3\lambda c)s}{c^2 s^2 - 3\lambda c s + 2\lambda^2(1 - \hat{f}(s))} \end{aligned} \quad (4.4)$$

Let $H(x)$ be the ladder height distribution and the Laplace transform is

$$\hat{h}(s) = \int_0^\infty e^{-sx} dH(x).$$

Thus L has a compound Geometric distribution,

$$P\{L \leq u\} = (1 - q) \sum_{n=0}^{\infty} q^n H^{*n}(u) \quad \text{where } q = \psi(0).$$

Thus we have an expression for the distribution function of L which is a Geometric sum, see, Kalashnikov[47].

Therefore, the Laplace transform is

$$\hat{\gamma}(s) = \frac{\varphi(0)}{1 - \psi(0)\hat{h}(s)}.$$

Let $F_t(x)$ denote the integrated tail distribution of $F(x)$, thus

$$F_t(x) = \frac{1}{m} \int_0^x \bar{F}(y) dy$$

and the Laplace transform is

$$\begin{aligned} \hat{f}_t(s) &= \frac{1 - \hat{f}(s)}{ms} \\ &= \frac{1}{m} \int_0^\infty e^{-sx} \bar{F}(x) dx. \end{aligned}$$

Now we find an expression for $\varphi(0)$ and hence the Ladder Height distribution. The expression for $\hat{\gamma}(s)$ in (4.4) becomes:

$$\hat{\gamma}(s) = \frac{c^2 s \varphi(0) + 2\lambda^2 m - 3c\lambda}{c^2 s - 3c\lambda + 2\lambda^2 m \hat{f}_t(s)} \quad (4.5)$$

The denominator of expression (4.5) will have a positive root and it will be unique.

Let us denote it by k . Hence

$$\begin{aligned} \varphi(0) &= \frac{3c\lambda - 2\lambda^2 m}{c^2 k} \\ \hat{\gamma}(s) &= \frac{c^2 (s - k) \varphi(0)}{c^2 (s - k) + 2\lambda^2 m (\hat{f}_t(s) - \hat{f}_t(k))} \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{h}(s) &= \frac{\hat{\gamma}(s) - \varphi(0)}{(1 - \varphi(0)) \hat{\gamma}(s)} \\ &= \frac{2\lambda^2 k}{(c^2 k - 3\lambda c + 2\lambda^2 m)} m \frac{\hat{f}_t(s) - \hat{f}_t(k)}{(k - s)} \end{aligned} \quad (4.6)$$

Now,

$$\begin{aligned} m \frac{\hat{f}_t(s) - \hat{f}_t(k)}{(k - s)} &= \frac{1}{(k - s)} \int_0^\infty (e^{-sx} - e^{-kx}) \bar{F}(x) dx, \\ &= \int_0^\infty e^{-kx} \left(\int_0^x e^{-(s-k)y} dy \right) \bar{F}(x) dx, \\ &= \int_0^\infty e^{-(s-k)y} \left(\int_y^\infty e^{-kx} \bar{F}(x) dx \right) dy, \end{aligned}$$

$$\begin{aligned}
 &= \int_0^\infty e^{-sy} \left(\int_y^\infty e^{-k(x-y)} \bar{F}(x) dx \right) dy \\
 &= \int_0^\infty e^{-sy} Q(y) dy \quad \text{where } Q(y) = \int_y^\infty e^{-k(x-y)} \bar{F}(x) dx, \quad y > 0.
 \end{aligned}$$

Using (4.6) and by the uniqueness of Laplace transform, the probability density function of the Ladder Height distribution is

$$h(y) = \frac{2\lambda^2 k}{c^2 k - 3\lambda c + 2\lambda^2 m} Q(y), \quad y > 0. \tag{4.7}$$

4.4 Bounds for ruin Probabilities

We have obtained the probability distribution of the Ladder Height distribution. So, we can follow the method suggested by Dufresne and Gerber[30] to find the lower and upper bounds of the ruin probabilities.

The maximal aggregate loss L can be expressed as

$$L = L_1 + L_2 + \dots + L_N \text{ where } N \text{ is the number of Ladder heights.}$$

If the interval of discretisation is the unit interval, then

$$LL = [L_1] + [L_2] + \dots + [L_N]$$

rounded nearest lower integer value of L and:

$$UL = [L_1 + 1] + [L_2 + 1] + \dots + [L_N + 1]$$

rounded nearest higher integer value of L .

Therefore

$$1 - P\{LL < u\} \leq \psi(u) \leq 1 - P\{UL \leq u\}$$

If:

$$f_i^L = P\{LL = i\}, \quad i = 0, 1, 2, \dots$$

and:

$$f_i^U = P\{UL = i\}, \quad i = 0, 1, 2, \dots$$

then we have:

$$1 - \sum_{i=0}^{u-1} f_i^L \leq \psi(u) \leq 1 - \sum_{i=0}^u f_i^U$$

where:

$$f_i^L = \frac{\psi(0)}{1 - \psi(0)H(1)} \sum_{k=1}^i (H(k+1) - H(k)) f_{i-k}^L, \quad i = 1, 2, 3, \dots$$

and:

$$f_i^U = \psi(0) \sum_{k=1}^i (H(k) - H(k-1)) f_{i-k}^U, \quad i = 1, 2, 3, \dots$$

4.5 Calculation of Ruin Probabilities

4.5.1 Exponential Distributed Claim

If the single claim amount distribution is exponential with mean $\frac{1}{\beta}$, we have

$$\bar{F}(x) = e^{-\beta x}, \quad x > 0, \quad \beta > 0,$$

then:

$$\begin{aligned} Q(y) &= e^{ky} \int_y^{\infty} e^{-x(\beta+k)} dx \\ &= \frac{1}{\beta+k} e^{-\beta y} \end{aligned}$$

hence:

$$h(y) = \frac{2\lambda^2 k}{(c^2 k - 3\lambda c + 2\lambda^2 m)} \frac{1}{(\beta + k)} e^{-\beta y}.$$

In this case:

$$H(x) = 1 - e^{-\beta x}, \quad x > 0, \quad \beta > 0, \quad (4.8)$$

where k is the unique positive root of the equation:

$$c^2 s^2 + (c^2 \beta - 3\lambda c)s + (2\lambda^2 - 3\lambda \beta c) = 0. \quad (4.9)$$

Table: 1

Ruin Probabilities

$\lambda = 2, \beta = 2,$ and $c = 1.1$

u	d = 0.01		d = 0.005	
	Lower Bounds	Upper Bounds	Lower Bound	Upper Bound
0	0.514470	0.514470	0.514470	0.514470
1	0.194079	0.196027	0.194334	0.195307
2	0.073172	0.074648	0.073407	0.074144
3	0.027587	0.028427	0.027729	0.028147
4	0.010401	0.010825	0.010474	0.010685
5	0.003921	0.004122	0.003956	0.004057
6	0.001478	0.001570	0.001494	0.001540
7	0.000557	0.000598	0.000564	0.000585
8	0.000210	0.000228	0.000213	0.000222
9	0.000079	0.000086	0.000080	0.000084
10	0.000030	0.000033	0.000030	0.000032

Length of the interval of discretisation = d

4.5.2 Gamma Distributed Claim

If a single claim amount has gamma distribution of the type $\Gamma(2, \beta)$, then:

$$Q(y) = e^{ky} \int_y^\infty e^{-kx}(e^{-\beta x} + \beta x e^{-\beta x}) dx$$

$$= \left(\frac{1}{\beta + k} + \frac{\beta}{(\beta + k)^2}\right)e^{-\beta y} + \frac{\beta}{(\beta + k)}ye^{-\beta y}, \quad y > 0.$$

Therefore,

Table: 2
 Ruin Probabilities
 $\lambda = 2, \beta = 3,$ and $c = 1.1$

u	d = 0.01		d = 0.005	
	Lower Bounds	Upper Bounds	Lower Bound	Upper Bound
0	0.747101	0.747104	0.747101	0.747101
1	0.441301	0.444918	0.442209	0.444018
2	0.253495	0.257927	0.254603	0.256820
3	0.145502	0.149416	0.146477	0.148435
4	0.083514	0.086555	0.084269	0.085790
5	0.047934	0.050840	0.048480	0.049583
6	0.027513	0.029045	0.027891	0.028657
7	0.015792	0.016825	0.016046	0.016563
8	0.009064	0.009747	0.0092310	0.009573
9	0.005202	0.005646	0.005311	0.005533
10	0.002986	0.003271	0.003055	0.003251

Length of the interval of discretisation = d

$$h(y) = \rho\left(\frac{1}{\beta + k} + \frac{\beta}{(\beta + k)^2}\right)e^{-\beta y} + \rho\frac{\beta}{(\beta + k)}ye^{-\beta y}, \quad y > 0$$

where

$$\rho = \frac{2\lambda^2 k}{c^2 k - 3\lambda c + 2\lambda^2 m}$$

$$H(x) = 1 - e^{-\beta x} - \frac{\rho}{\beta + k} x e^{-\beta x}, \quad x > 0 \quad (4.10)$$

where k is the unique positive solution of the equation:

$$c^2 s^3 + s^2(2\beta c^2 - 3\lambda c) + s(c^2 \beta^2 - 6\beta \lambda c + 2\lambda^2) + 4\lambda^2 \beta - 3\beta^2 \lambda c = 0. \quad (4.11)$$

4.5.3 GE Distributed Claim

Assuming that a single claim amount has GE distribution with $GE(2, \beta)$, then

$$\bar{F}(x) = 2e^{-\beta x} - e^{-2\beta x}, \quad x > 0$$

$$\begin{aligned} Q(y) &= e^{ky} \int_y^\infty e^{-kx} (2e^{-\beta x} - e^{-2\beta x}) dx \\ &= \frac{2}{\beta + k} e^{-\beta y} - \frac{1}{2\beta + k} e^{-2\beta y}, \quad y > 0. \end{aligned}$$

Therefore,

$$h(y) = \frac{2\rho}{\beta + k} e^{-\beta y} - \frac{\rho}{2\beta + k} e^{-2\beta y}, \quad y > 0$$

$$H(x) = 1 + \frac{\rho}{2\beta(2\beta + k)} e^{-2\beta x} - \frac{2\rho}{\beta(\beta + k)}, \quad y > 0 \quad (4.12)$$

where k is the positive(unique) solution of the equation:

$$c^2 s^3 + s^2(3c^2 \beta - 3\lambda c) + s(c^2 \beta^2 - 9\beta \lambda c + 2\lambda^2) + 6\beta \lambda^2 - 6\beta^2 \lambda c = 0. \quad (4.13)$$

Table: 3
 Ruin Probabilities
 $\lambda = 2, \beta = 2,$ and $c = 1.1$

u	d = 0.01		d = 0.005	
	Lower Bounds	Upper Bounds	Lower Bound	Upper Bound
0	0.879178	0.879178	0.879178	0.879178
1	0.711106	0.713410	0.711686	0.712837
2	0.570815	0.574661	0.571783	0.573704
3	0.458150	0.462846	0.459330	0.461676
4	0.367722	0.372787	0.368993	0.371522
5	0.295142	0.300250	0.296422	0.298972
6	0.236888	0.241826	0.238125	0.240589
7	0.190131	0.194770	0.191292	0.193606
8	0.152604	0.156870	0.153670	0.155797
9	0.122483	0.126343	0.123448	0.125371
10	0.098308	0.101757	0.099169	0.101231

Length of the interval of discretisation = d

4.5.4 Mixed Exponential Distributed Claim

Assuming that a single claim has mixed Exponential distribution with

$$f(x) = p\beta_1 e^{-\beta_1 x} + (1-p)\beta_2 e^{-\beta_2 x}, \quad x > 0, \quad 0 < \beta_1 < \beta_2, \quad 0 \leq p \leq 1$$

$$\text{Then, } \bar{F}(x) = pe^{-\beta_1 x} + (1-p)e^{-\beta_2 x}, \quad x > 0$$

$$Q(y) = e^{ky} \int_y^\infty e^{-kx} (pe^{-\beta_1 x} + (1-p)e^{-\beta_2 x}) dx.$$

$$\text{Therefore, } h(y) = \rho \frac{p}{(\beta_1 + k)} e^{-\beta_1 y} + \rho \frac{(1-p)}{(\beta_2 + k)} e^{-\beta_2 y}, \quad y > 0$$

$$H(x) = 1 - \frac{\rho p}{\beta_1(\beta_1 + k)} e^{-\beta_1 x} - \frac{\rho(1-p)}{\beta_2(\beta_2 + k)} e^{-\beta_2 x}, \quad x > 0 \quad (4.14)$$

where k is the positive root(unique) of the equation:

$$c^2 s^3 + s^2(c^2 \beta_1 + c^2 \beta_2 - 3\lambda c) + s(c^2 \beta_1 \beta_2 - 3c\beta_1 \lambda - 3c\beta_2 \lambda + 2\lambda^2) + 2p\beta_2 \lambda^2 - 2p\beta_1 \lambda^2 + 2\beta_1 \lambda^2 - 3c\beta_1 \beta_2 \lambda = 0. \quad (4.15)$$

Table: 4

Ruin Probabilities

$\lambda = 1, \beta_1 = 0.5, \beta_2 = 2, p = 1/3$ and $\bar{\pi} = 1.1$

u	d = 0.01		d = 0.005	
	Lower Bounds	Upper Bounds	Lower Bound	Upper Bound
0	0.526778	0.526778	0.526778	0.526778
1	0.373358	0.373836	0.373477	0.373717
2	0.280871	0.281458	0.281017	0.281311
3	0.214261	0.214864	0.214411	0.214713
4	0.163958	0.164537	0.164102	0.164342
5	0.125551	0.126085	0.125684	0.125951
6	0.096156	0.096633	0.096275	0.096514
7	0.073645	0.074064	0.073750	0.073959
8	0.056405	0.056766	0.056495	0.056676
9	0.043201	0.043508	0.043277	0.043431
10	0.033088	0.033347	0.033152	0.033341

Length of the interval of discretisation = d

4.6 Inversion of $\hat{\varphi}(s)$

An explicit expression for $\varphi(u)$ is obtained by the inversion of the Laplace transform of expression (4.3). We use Heaviside's theorem to find the inverse Laplace

transform of $\hat{\varphi}(s)$. But it is difficult to obtain $\varphi(u)$ for most of the claim amount distributions, however numerical evaluation is possible in these cases.

Let $P(s)$ and $Q(s)$ be polynomials of degree m and n respectively, where $n > m$. If $Q(s)$ has n distinct simple zeros at points s_1, s_2, \dots, s_n , and if $\frac{P(s)}{Q(s)}$ is the Laplace transform of the function $f(t)$, then

$$f(t) = \sum_{k=1}^n \frac{P(s_k)}{Q'(s_k)} e^{s_k t}.$$

Now we shall consider different claim amount distributions and ruin/survival probabilities are calculated for each of the individual cases.

If the single claim amount distribution is *exponential* with $m = \frac{1}{\beta}$, we have

$$\hat{f}(s) = \frac{\beta}{\beta + s}, \quad \text{Re}(s) > -\beta.$$

Hence

$$\hat{\varphi}(s) = \frac{c^2 s^2 \varphi(0) + c^2 s \beta \varphi(0) - 3\lambda c s - 3c\lambda\beta + 2m\lambda^2 s + 2m\beta\lambda^2}{s(c^2 s^2 + c^2 \beta s - 3c\lambda s - 3c\beta\lambda + 2\lambda^2)} \quad (4.16)$$

We can see that the expression (4.16) is analytic with respect to s , except at the poles $s = 0$, $s = r (= -R, \text{ the Lundberg exponent})$. Note that $s = k$, the positive root of the equation $c^2 s^2 + (c^2 \beta - 3\lambda c)s + (2\lambda^2 - 3\lambda\beta c) = 0$, which is a removable singularity.

Therefore by applying Heaviside's theorem

$$\varphi(u) = 1 - \frac{1}{\beta} \left(\frac{2\lambda^2}{c^2 R + 3\lambda c} \right) e^{-Ru}.$$

Again considering the equation(1.11)

$$\underline{\psi(u) = \left(1 - \frac{R}{\beta}\right)e^{-Ru}.$$

Assume that a single claim amount has *Gamma distribution* of the type $\Gamma(2, \beta)$ with Laplace transform

$$\hat{f}(s) = \frac{\beta^2}{(\beta + s)^2}, \quad \text{Re}(s) > -\beta$$

then

$$\hat{\varphi}(s) = \frac{c^2 s^3 \varphi(0) + s^2(3c^2 \beta \varphi(0) - 3c\lambda + 2m\lambda^2) + s(2c^2 \beta^2 \varphi(0) - 9c\beta\lambda + 6m\beta\lambda^2) + 4m\beta^2 \lambda^2}{s(c^2 s^3 + s^2(3c^2 \beta - 3c\lambda) + s(2c^2 \beta^2 - 9c\beta\lambda + 2\lambda^2))} \quad (4.17)$$

To avoid complicated inverse Laplace transform, we take $\beta=3$, $\lambda=2$, and $c=1.1$, then the roots of the denominator of expression (4.17) become $s=0$, $s=-4.13423$, $s=-0.550549$, and $s=4.13932$. The expression is analytic with respect to s except at the poles $s=0$, $s=-4.13423$ and $s=-0.550549$.

As explained in the earlier section $s=4.13932$ is a removable singularity. Therefore,

$$\underline{\varphi(u) = 1 + 0.01219596e^{-4.13423u} - 0.76906e^{-0.550549u}.$$

Suppose that the individual claim amount distribution is *Generalized Exponential* with $GE(2, \beta)$. The Laplace transform is:

$$\hat{f}(s) = \frac{2\beta}{(\beta + s)} + \frac{2\beta}{(2\beta + s)}$$

$$\hat{\varphi}(s) = \frac{c^2 s^2 \varphi(0) + s^2 (2c^2 \beta \varphi(0) - 3c\lambda + 2m\lambda^2) + s(c^2 \beta^2 \varphi(0) - 6c\beta\lambda + 4m\beta\lambda^2) + 2m\beta^2 \lambda^2 - 3c\beta^2 \lambda}{s(c^2 s^3 + s^2(2c^2 \beta - 3c\lambda) + s(c^2 \beta^2 - 6c\beta\lambda + 2\lambda^2) + 4\beta\lambda^2 - 3c\beta^2)} \quad (4.18)$$

We assume the values for β , λ and c is such that the condition (1.10) is valid. When $\beta = 2$, $\lambda = 2$, $c = 1.1$, the solution of the denominator of expression(4.18) is $s = 0$, $s = -4.43144$, $s = -0.218118$ and $s = 4.10411$.

We can see that $s = 4.10411$ is a removable singularity and all the other solutions are poles of the expression(4.18). Hence the expression is analytic with respect to s , except , at the poles.

Hence, after finding the inverse Laplace transform of $\hat{\varphi}(s)$ we get,

$$\underline{\varphi(u) = 1 + 0.00678832e^{-4.43144u} - 0.885966e^{-0.218118u}.$$

Assuming that a single claim has *Mixed exponential* distribution with

$$f(x) = p\beta_1 e^{-\beta_1 x} + (1-p)\beta_2 e^{-\beta_2 x}, \quad x > 0, \quad 0 < \beta_1 < \beta_2, \quad 0 \leq p \leq 1$$

The Laplace transform is:

$$\hat{f}(s) = p \frac{\beta_1}{\beta_1 + s} + (1-p) \frac{\beta_2}{\beta_2 + s}.$$

Hence:

$$\hat{\varphi}(s) = \frac{(s + \beta_1)(s + \beta_2)(c^2 s \varphi(0) - 3c\lambda + 2m\lambda^2)}{s(c^2 s(s + \beta_1)(s + \beta_2) - 3c(s + \beta_1)(s + \beta_2) + 2(s + \beta_1 - p\beta_1 + p\beta_2)\lambda^2)} \quad (4.19)$$

When $\beta_1 = 0.5$, $\beta_2 = 2$, $\lambda = 1$, $c = 1.1$ and $p = 1/3$, then the roots of the denominator of expression (4.19) are $s = 0$, $s = -1.77674$, $s = -0.266343$, and

$s = 2.273035$. Note that $s = 2.273035$ is the removable singularity and the other solutions are poles.

Hence the survival function

$$\varphi(u) = 1 - 0.0502656e^{-1.77674u} - 0.476513e^{-0.266343u}.$$

The Probability and Severity of Ruin

K.K.Thampi “Probabilistic analysis of some Sparre Andersen models” Thesis.
Department of Statistics , University of Calicut, 2007

Chapter 5

The Probability and Severity of Ruin*

5.1 Introduction

It has been argued that the probability of ruin is a very crude stability criterion. In this chapter, we are not just interested in the probability of ruin, but also the amount of the insurer's deficit at the time of ruin, should ruin occur.

Recently, many researchers are paying attention to the severity of ruin in different risk models. Gerber, Goovaerts and Kass[37] considered the probability and severity of ruin in classical risk model. They introduced the function $G(u, y)$, the probability that ruin occurs with initial surplus u and that the deficit at the time

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of ruin is less than y . Using an integral equation for $G(u, y)$, they have obtained a closed form solution for $G(u, y)$ when the claim amount distribution is either an exponential mixture or a gamma mixture. Beda Chan[9], in a short contribution, obtained an expression for the coefficient of the ruin probability when the claim amount distribution is translated combination of exponentials. An algorithm to calculate the probability and severity of ruin in both finite and infinite time for the discrete time risk model was introduced by Dickson and Waters[22]. Dickson[24] discussed the probability and severity of ruin when the claim arrival is Erlang(2) with claim distribution being Erlang(2). Lin and Willmot[52] studied the joint and marginal moments of time of ruin, the surplus before ruin and deficit at ruin by extending their results[51].

We derive some explicit results for the probability and severity of ruin for a class of renewal risk process in which claims occur as Generalized Exponential distribution.

5.2 Severity of Ruin

Our main concern in this chapter is how serious the situation is when the ruin occurs. The overshoot above the level u by crossing this level for the first time is defined by

$$Y^+(u) = \begin{cases} S(T) - u & \text{if } T < \infty \\ \infty & \text{if } T = \infty \end{cases}$$

In terms of surplus process,

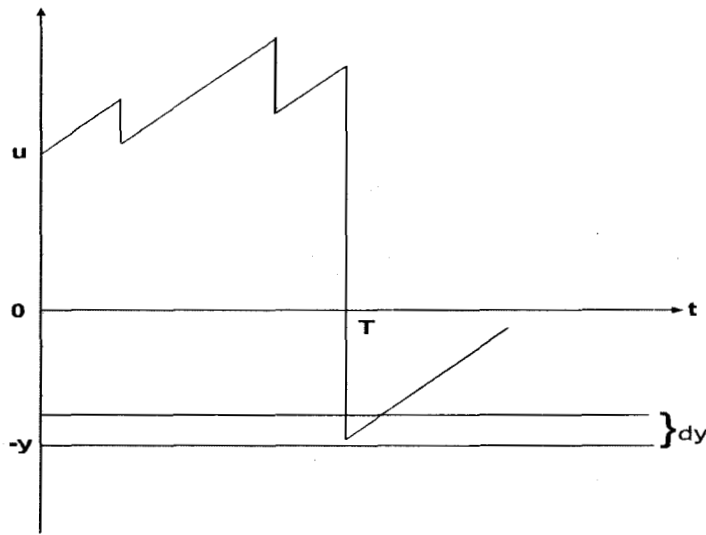


Figure 5.1: The interpretation of $g(u, y)dy$.

$$Y^+(u) = \begin{cases} -U(T) & \text{if } T < \infty, \\ \infty & \text{if } T = \infty \end{cases}$$

$Y^+(u)$ can be considered as the severity of ruin at time T . The probability of ruin with initial surplus u and deficit immediately after claim causing ruin is at most y , denoted by $G(u, y)$, is

$$\begin{aligned} G(u, y) &= P\{T < \infty, Y^+(u) < y\} \\ &= P\{T < \infty, -y \leq U(T) < 0\} \end{aligned} \tag{5.1}$$

in which $u \geq 0$ and $y \geq 0$.

It can be seen that $\psi(u) = \lim_{y \rightarrow \infty} G(u, y)$, with $\frac{G(u, y)}{\psi(u)} = P\{|T| \leq y \mid T < \infty\}$ is a proper distribution function. Therefore $G(u, y)$ is a defective distribution function with corresponding density function

$$g(u, y) = \frac{d}{dy}G(u, y).$$

Hence, we can write:

$$\varphi(u) = \varphi(0) + \int_0^u \varphi(u - y)g(0, y)dy \quad (5.2)$$

Defining $\hat{g}(s) = \int_0^\infty e^{-sy}g(0, y)dy$ and $\hat{\varphi}(s) = \int_0^\infty e^{-sy}\varphi(u)du$, then:

$$\hat{g}(s) = \frac{s\hat{\varphi}(s) - \varphi(0)}{s\hat{\varphi}(s)} \quad (5.3)$$

The inverse Laplace transform of (5.3) gives $g(0, y)$.

An expression for $G(u, y)$ will be obtained by considering the first epoch at which the surplus falls below the initial value u , either the surplus falls to $(u - x) \geq 0$, so that ruin occurs subsequently and that the deficit at the time of ruin is less than y or ruin occurs due to this and deficit at ruin is at most y . Therefore,

$$G(u, y) = \int_0^u g(0, x)G(u - x, y)dy + \int_u^{u+y} g(0, x)dx \quad (5.4)$$

Differentiating w.r.t. y , we have

$$g(u, y) = \int_0^u g(u - x, y)g(0, x)dx + g(0, u + y) \quad (5.5)$$

Taking the Laplace transform of (5.5) w.r.t. u , yields

$$\hat{g}(s, y) = \hat{g}(s, y)\hat{g}(s) + \hat{\zeta}(s, y) \quad (5.6)$$

where $\hat{\zeta}(s, y) = \int_0^\infty g(0, u + y)e^{-su} du$.

Therefore,

$$\hat{g}(s, y) = \frac{\hat{\zeta}(s, y)}{1 - \hat{g}(s)} \quad (5.7)$$

Now the task is to invert $\hat{g}(s, y)$ to obtain $g(u, y)$. In the following sections, we shall look at different claim amount distribution in which this can be done explicitly.

Note that, in the case of GE claim arrivals, using (4.3) and (5.3), we have

$$\hat{g}(s) = \frac{3\lambda cs\varphi(0) + 2\lambda^2 ms - 3\lambda cs - 2\varphi(0)(1 - \hat{f}(s))\lambda^2}{c^2 s^2 \varphi(0) + 2\lambda^2 ms - 3\lambda cs} \quad (5.8)$$

We obtain the inverse Laplace transform of $\hat{g}(s)$, then find $g(u, y)$ for each of the claim amount distributions.

5.3 Explicit Expressions and Numerical Evaluations

5.3.1 Exponential Distribution

Let us assume that the claim amount distribution is exponential with mean $1/\beta$, then

$$\hat{g}(s) = \frac{(s + \beta)(2m\lambda^2 - 3c\lambda + 3c\lambda\varphi(0)) - 2\lambda^2\varphi(0)}{(s + \beta)(2m\lambda^2 - 3c\lambda + c^2\beta\varphi(0))} \quad (5.9)$$

Inverting the Laplace transform of (5.9), and after eliminating the removable singularities, we can write:

$$g(0, y) = ke^{-\beta y} \quad (5.10)$$

$$\text{where } k = \frac{2\lambda^2\varphi(0)}{3c\lambda - 2\lambda^2m + \beta c^2\varphi(0)}.$$

Then:

$$\hat{g}(s) = \frac{k}{s + \beta}$$

and:

$$g(0, u + y) = ke^{-\beta(u+y)}, \quad \text{thus } \hat{\zeta}(s, y) = \frac{k}{\beta + s}e^{-\beta y}$$

Hence:

$$\hat{g}(s, y) = \frac{k}{s + \beta - k}e^{-\beta y} \quad (5.11)$$

Taking the inverse Laplace transform of (5.11), we get

$$g(u, y) = ke^{u(k-\beta)-\beta y}$$

which can be simplified to

$$g(u, y) = (\beta - r)e^{-ru-\beta y}$$

where $r (=R)$ is the adjustment coefficient.

Table: 1
Severity of Ruin $G(u, y)$
 $\beta = 2, \lambda = 2, c = 1.1$

$u \setminus y$	1	2	3	4	5
0	0.444844	0.505048	0.513195	0.514298	0.514447
1	0.168455	0.191252	0.194338	0.194755	0.194812
2	0.063791	0.072424	0.073592	0.073750	0.073772
3	0.024156	0.027425	0.027868	0.027928	0.027936
4	0.009148	0.010385	0.010553	0.010576	0.010579
5	0.003464	0.003933	0.003996	0.004005	0.004006

5.3.2 Gamma Claim Amount Distribution

If a single claim amount has Gamma distribution of the type $\Gamma(2, \beta)$, then we have

$$\hat{g}(s) = \frac{(s + \beta)^2(2m\lambda^2 - 3c\lambda + 3c\lambda\varphi(0)) - 2\lambda^2\varphi(0)(s + 2\beta)}{(s + \beta)^2(2m\lambda^2 - 3c\lambda + c^2\beta\varphi(0))} \quad (5.12)$$

Taking the inverse Laplace transform of (5.12), we get:

$$g(0, y) = k_1e^{-\beta y} + k_2ye^{-\beta y} \quad (5.13)$$

$$\text{where } k_1 = \frac{2\lambda^2\varphi(0)(3c\lambda - 2m\lambda^2 + 2c^2\beta\varphi(0))}{(3c\lambda - 2m\lambda^2 + c^2\beta\varphi(0))^2}$$

$$\text{and } k_2 = \frac{2\lambda^2\beta\varphi(0)}{(3c\lambda - 2m\lambda^2 + c^2\beta\varphi(0))}.$$

Then:

$$\hat{g}(s) = \frac{k_1}{(s + \beta)} + \frac{k_2}{(s + \beta)^2}$$

Also:

$$g(0, u + y) = k_1e^{-\beta(u+y)} + k_2(u + y)e^{-\beta(u+y)}$$

Hence:

$$\hat{\zeta}(s, y) = \frac{k_1}{\beta + s}e^{-\beta y} + \frac{k_2}{\beta + s}ye^{-\beta y} + \frac{k_2}{(\beta + s)^2}e^{-\beta y}$$

Therefore,

$$\hat{g}(s, y) = \frac{1 - \frac{k_1}{(s+\beta)} - \frac{k_2}{(s+\beta)^2}}{\frac{k_1}{\beta+s}e^{-\beta y} + \frac{k_2}{\beta+s}ye^{-\beta y} + \frac{k_2}{(\beta+s)^2}e^{-\beta y}} \quad (5.14)$$

The inverse Laplace transform of (5.14), and simplifying

$$g(u, y) = \left\{ \frac{(\beta - r_1)^2}{(r_2 - r_1)} + \frac{(\beta - r_1)^2(\beta - r_2)y}{(r_1 - r_2)} \right\} e^{-r_1 u - \beta y} + \left\{ \frac{(\beta - r_2)^2}{(r_1 - r_2)} + \frac{(\beta - r_1)(\beta - r_2)^2 y}{(r_2 - r_1)} \right\} e^{-r_2 u - \beta y}$$

where $r_1 (= R)$ and r_2 are the positive roots of the equation (1.11).

Table: 2
Severity of Ruin $G(u, y)$
 $\beta = 3, \lambda = 2, c = 1.1$

$u \setminus y$	1	2	3	4	5
0	0.663798	0.740657	0.746665	0.747073	0.747099
1	0.402647	0.440183	0.442922	0.443102	0.443113
2	0.232498	0.254034	0.255602	0.255705	0.255711
3	0.134071	0.146488	0.147392	0.147451	0.147455
4	0.077310	0.084469	0.084991	0.085025	0.085027
5	0.044579	0.048708	0.049008	0.049028	0.049029

5.3.3 Mixed Exponential distribution

Assuming that a single claim amount has a Mixed Exponential distribution with

$$f(x) = p_1 \beta_1 e^{-\beta_1 x} + p_2 \beta_2 e^{-\beta_2 x}, \quad x > 0, \quad 0 < \beta_1 < \beta_2, \quad p_1 + p_2 = 1.$$

Therefore,

$$\hat{g}(s) = \frac{(s + \beta_1)(s + \beta_2)(2m\lambda^2 - 3c\lambda + 3c\lambda\varphi(0)) - 2\lambda^2\varphi(0)(\beta_1 p_2 + \beta_2 p_1)}{(s + \beta_1)(s + \beta_2)(2m\lambda^2 - 3c\lambda + c^2\beta\varphi(0))} \quad (5.15)$$

Hence the inverse Laplace transform of (5.15)

$$g(0, y) = k_1 e^{-\beta_1 y} + k_2 e^{-\beta_2 y} \quad (5.16)$$

$$\text{where } k_1 = \frac{2p_1 \lambda^2 \varphi(0)}{3c\lambda - 2m\lambda^2 + c^2 \beta_1 \varphi(0)}$$

$$\text{and } k_2 = \frac{2p_2 \lambda^2 \varphi(0)}{3c\lambda - 2m\lambda^2 + c^2 \beta_2 \varphi(0)}.$$

Then:

$$\hat{g}(s) = \frac{k_1}{s + \beta_1} + \frac{k_2}{s + \beta_2}$$

$$g(0, u + y) = k_1 e^{-\beta_1(u+y)} + k_2 e^{-\beta_2(u+y)}$$

and:

$$\hat{\zeta}(s, y) = \frac{k_1}{s + \beta_1} e^{-\beta_1 y} + \frac{k_2}{s + \beta_2} e^{-\beta_2 y}.$$

Hence, we have:

$$\hat{g}(s, y) = \frac{\frac{k_1}{s + \beta_1} e^{-\beta_1 y} + \frac{k_2}{s + \beta_2} e^{-\beta_2 y}}{1 - \frac{k_1}{s + \beta_1} - \frac{k_2}{s + \beta_2}} \quad (5.17)$$

Taking the inverse Laplace transform of (5.17) and after simplification, we get

$$\begin{aligned} g(u, y) = & \frac{(\beta_1 - r_1)(\beta_1 - r_2)(\beta_2 - r_1)}{(\beta_1 - \beta_2)(r_2 - r_1)} e^{-r_1 u - \beta_1 y} + \\ & \frac{(\beta_1 - r_1)(\beta_1 - r_2)(\beta_2 - r_2)}{(\beta_1 - \beta_2)(r_1 - r_2)} e^{-r_2 u - \beta_1 y} + \\ & \frac{(\beta_2 - r_1)(\beta_1 - r_1)(\beta_2 - r_2)}{(\beta_1 - \beta_2)(r_1 - r_2)} e^{-r_1 u - \beta_2 y} + \\ & \frac{(\beta_2 - r_1)(\beta_2 - r_2)(\beta_1 - r_2)}{(\beta_1 - \beta_2)(r_2 - r_1)} e^{-r_2 u - \beta_2 y} \end{aligned}$$

where $r_1 (= R)$ and r_2 are the positive roots of the equation (1.11).

Table: 3

Severity of Ruin $G(u, y)$
 $\beta_1 = 0.5, \beta_2 = 2, \lambda = 1, p_1 = 1/3, c = 1.1$

$u \setminus y$	1	2	3	4	5	10
0	0.268064	0.378087	0.437706	0.472904	0.494122	0.524098
1	0.162899	0.247954	0.297682	0.327592	0.345699	0.371307
2	0.117621	0.182917	0.221702	0.245116	0.259302	0.279370
3	0.088902	0.138952	0.168784	0.186808	0.197730	0.213181
4	0.067909	0.106260	0.129136	0.142959	0.151337	0.163188
5	0.051995	0.081379	0.098910	0.109504	0.115924	0.125006

In the following figure (5.2), $G(u, y)$ is plotted for $u = 0, 1, 2, \dots$. It can be seen

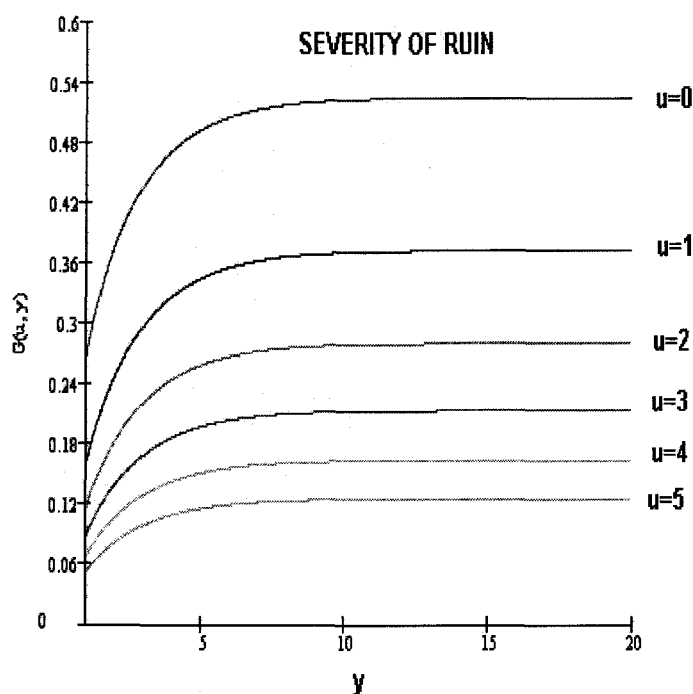


Figure 5.2: Severity of Ruin

that for a given value of u , the distribution function $G(u, y)$ is increasing in y and

tend to a limiting value after a certain limit of y . Hence as y is tending to infinity, the distribution $G(u, y)$ tends towards the ruin probability $\psi(u)$, as shown in the table given below :

u	0	1	2	3	4	5
$\lim_{y \rightarrow \infty} G(u, y)$	0.526778	0.373597	0.281164	0.214562	0.164227	0.125818

5.3.4 Generalized Exponential distribution

If the individual claim amount has a GE distribution with $GE(2, \beta)$ then

$$\hat{g}(s) = \frac{(s + \beta)(s + 2\beta)(2m\lambda^2 - 3c\lambda + 3c\lambda\varphi(0)) - 2\lambda^2\varphi(0)(s + 3\beta)}{(s + \beta)(s + 2\beta)(2m\lambda^2 - 3c\lambda + c^2\beta\varphi(0))} \quad (5.18)$$

Taking the inverse Laplace transform of (5.18), we have:

$$g(0, y) = k_1 e^{-\beta y} - k_2 e^{-2\beta y}$$

$$\text{where } k_1 = \frac{4\lambda^2\varphi(0)}{3c\lambda - 2m\lambda^2 + c^2\beta\varphi(0)}$$

$$\text{and } k_2 = \frac{2\lambda^2\varphi(0)}{3c\lambda - 2m\lambda^2 + 2c^2\beta\varphi(0)}$$

Then:

$$\hat{g}(s) = \frac{k_1}{s + \beta} - \frac{k_2}{s + 2\beta}$$

$$g(0, u + y) = k_1 e^{-\beta(u+y)} - k_2 e^{-2\beta(u+y)}$$

$$\hat{\zeta}(s, y) = \frac{k_1}{s + \beta} e^{-\beta y} - \frac{k_2}{s + 2\beta} e^{-2\beta y}$$

Therefore,

$$\hat{g}(s, y) = \frac{\frac{k_1}{s+\beta}e^{-\beta y} - \frac{k_2}{s+2\beta}e^{-2\beta y}}{1 - \frac{k_1}{s+\beta} + \frac{k_2}{s+2\beta}}$$

Taking the inverse Laplace transform of the above expression, and after simplification, we get:

$$\begin{aligned} g(u, y) = & \frac{(\beta - r_1)(\beta - r_2)(2\beta - r_1)}{\beta(r_1 - r_2)} e^{-r_1 u - \beta y} + \\ & \frac{(\beta - r_1)(\beta - r_2)(2\beta - r_2)}{\beta(r_2 - r_1)} e^{-r_2 u - \beta y} + \\ & \frac{(\beta - r_1)(2\beta - r_1)(2\beta - r_2)}{\beta(r_2 - r_1)} e^{-r_1 u - 2\beta y} + \\ & \frac{(2\beta - r_1)(2\beta - r_2)(\beta - r_2)}{\beta(r_1 - r_2)} e^{-r_2 u - 2\beta y} \end{aligned}$$

where $r_1 (= R)$ and r_2 are the positive roots of the equation (1.11).

Table: 4

Severity of Ruin $G(u, y)$

$\beta = 3, \lambda = 2, c = 1.1$

$u \setminus y$	1	2	3	4	5
0	0.736325	0.859406	0.876492	0.878812	0.879127
1	0.607585	0.697941	0.710318	0.711996	0.712223
2	0.488700	0.561246	0.571182	0.572529	0.572711
3	0.392931	0.451258	0.459247	0.460329	0.460476
4	0.315927	0.362824	0.369247	0.370117	0.370235
5	0.254014	0.291721	0.296885	0.297585	0.297679

The following graph (5.3) shows a realization of $G(u, y)$ for $u=0, 1, 2, \dots$

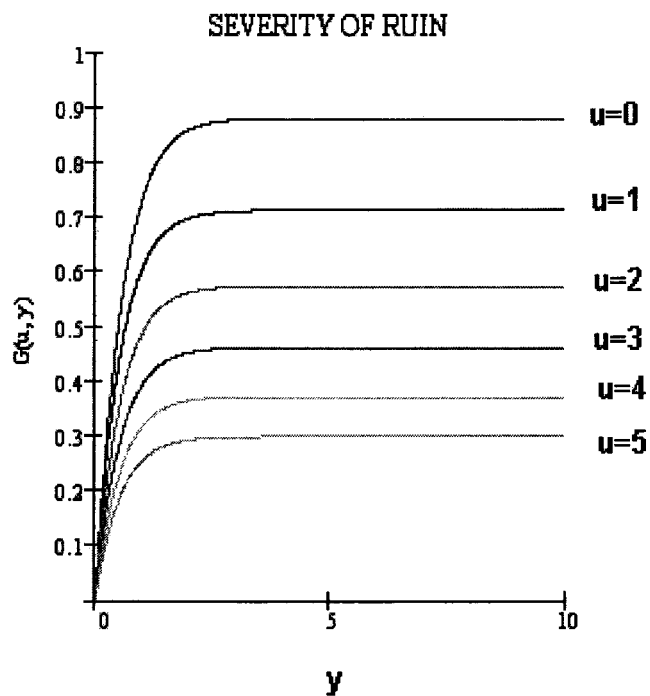


Figure 5.3: Severity of Ruin

The limiting case of $G(u, y)$, when the claim amount distribution $GE(2, \beta)$, is shown in the table given below.

u	0	1	2	3	4	5
$\lim_{y \rightarrow \infty} G(u, y)$	0.879178	0.712263	0.572745	0.460505	0.370260	0.297700

Barrier Probabilities and Maximum Severity of Ruin

K.K.Thampi “Probabilistic analysis of some Sparre Andersen models” Thesis.
Department of Statistics , University of Calicut, 2007

Chapter 6

Barrier Probabilities and Maximum Severity of Ruin*

6.1 Introduction

One of the important probabilities associated with ruin is the probability that the surplus process attains the level b from initial surplus u without first falling below zero. In classical risk model, this barrier probability, $B(u, b)$, has been discussed by Dickson and Gray[19]. The upper barrier, b , plays an important role in the study of risk theory as dividends can be paid continuously whenever the level is not below b and the overflow with respect to the level b is paid as dividends.

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The renewal risk process with inter-occurrence times of Erlangian type is discussed by Dickson[24]. He analyzed $B(u, b)$ in which inter-occurrence times between claims are distributed as $\Gamma(2, 2)$ and the claim amount is also $\Gamma(2, 2)$ distribution. In this chapter, the Poisson number process of the classical risk model is replaced with a more general renewal risk process in which claim inter-arrival time follows the $GE(2, \lambda)$.

6.2 Integro - Differential Equation for $B(u, b)$

By conditioning on the time and amount of first claim, we have the integral equation

$$B(u, b) = \int_0^{t_b} p(t) \int_0^{u+ct} f(x)B(u+ct-x, b)dxdt + \int_{t_b}^{\infty} p(t)dt \quad (6.1)$$

where $u+ct_b = b$, so that the surplus process will reach b at time t_b on the assumption no claims occur by time t_b .

Let $s = u + ct$, we get

$$cB(u, b) = \int_u^b p\left(\frac{s-u}{c}\right) \int_0^s f(x)B(s-x, b)dxds + \int_b^{\infty} p\left(\frac{s-u}{c}\right)ds \quad (6.2)$$

Differentiating (6.2) w.r.t. u twice, then we have

$$c^2B^{(2)}(u, b) - 3\lambda cB^{(1)}(u, b) + 2\lambda^2B(u, b) = 2\lambda^2 \int_0^u f(u-x)B(x, b)dx \quad (6.3)$$

Note that if survival occurs from initial surplus u , then the surplus process must be at level $b > u$ at some point in time, as the condition (1.10) necessarily mean

that $U(t) \rightarrow \infty$ as $t \rightarrow \infty$. In the case of classical risk model, the distribution of the time to the next claim from the time the surplus attains b is exponential, the probabilistic behavior of the surplus when it attains b is independent of its prior probabilistic behavior because of the lack of memory property. Therefore, we can write $B(u, b) = \frac{1-\psi(u)}{1-\psi(b)}$ (Dickson and Gray[19]). This result does not hold for a renewal model, but an approximated value for $B(u, b)$ is given in the case of Erlang(2, 2) claim occurrence (Dickson[24]).

6.3 Different Claim Amount Distributions

6.3.1 Exponential Distribution

Assume that claim amount distribution is exponential with mean $1/\beta$.

Setting

$$k(u) = 2\lambda^2 \int_0^\infty p(u-x)B(x, b)dx \quad (6.4)$$

Differentiating (6.4) w.r.t u , we get

$$k^{(1)}(u) + \beta k(u) = 2\beta\lambda^2 B(u, b) \quad (6.5)$$

We can eliminate the integral term from (6.3) by differentiating it again w.r.t. u .

$$c^2 B^{(3)}(u, b) + (c^2\beta - 3c\lambda)B^{(2)}(u, b) + (2\lambda^2 - 3c\beta\lambda)B^{(1)}(u, b) = 0 \quad (6.6)$$

We can write the solution of the differential equation (6.6) as

$$B(u, b) = \sum_{i=0}^2 \alpha_i e^{r_i u} \quad (6.7)$$

where $\{r_i\}_{i=0}^2$ are the roots of the characteristic equation (6.6) and $\{\alpha_i\}_{i=0}^2$ are the values depending on b . Now we search the values of $\{\alpha_i\}_{i=0}^2$.

Using boundary condition, $B(b, b) = 1$, we have $\sum \alpha_i e^{r_i b} = 1$.

Substituting (6.7), the LHS of (6.3) becomes:

$$2\lambda^2 \alpha_0 + \sum_{i=1}^2 \alpha_i (cr_i - 2\lambda)(cr_i - \lambda) e^{r_i u} \quad (6.8)$$

and when substituted in the RHS of (6.3), we get:

$$2\lambda^2 \left\{ \alpha_0 + \beta \sum_{i=1}^2 \frac{\alpha_i e^{r_i u}}{\beta + r_i} - \alpha_0 e^{-\beta u} - \beta \sum_{i=1}^2 \frac{\alpha_i e^{-\beta u}}{\beta + r_i} \right\} \quad (6.9)$$

Equating (6.8) and (6.9) and comparing the coefficient of $e^{-\beta u}$, then:

$$\alpha_0 + \beta \sum_{i=1}^2 \frac{\alpha_i}{\beta + r_i} = 0$$

Differentiating (6.3) w.r.t. u , and rearranging the terms, we have:

$$cB^{(1)}(u, b) - \lambda B(u, b) = -\frac{2\lambda^2}{c} \left\{ \int_u^b e^{-2\lambda(\frac{s-u}{c})} \int_0^s p(x) B(s-x, b) dx ds + \int_b^\infty e^{-2\lambda(\frac{s-u}{c})} ds \right\}$$

Substituting $B(u, b)$ in the above equation and solving the integral in the RHS,

$$\sum_{i=0}^2 \alpha_i e^{r_i u} (cr_i - \lambda) = -\lambda \alpha_0 (1 - e^{-2\lambda t_b}) \sum_{i=1}^2 \alpha_i (cr_i - \lambda) (e^{r_i u} - e^{r_i b} e^{-2\lambda t_b}) - \lambda e^{-2\lambda t_b}$$

After simplification of the above equation, we have:

$$\alpha_0 + \frac{1}{\lambda} \sum_{i=0}^2 \alpha_i (\lambda - cr_i) e^{r_i b} = 1.$$

Now we have 3 equations to solve for $\{\alpha_i\}_{i=0}^2$

$$\begin{aligned}\alpha_0 + \sum_{i=0}^2 \alpha_i e^{r_i b} &= 1 \\ \alpha_0 + \beta \sum_{i=0}^2 \frac{\alpha_i}{\beta + r_i} &= 0 \\ \alpha_0 + \frac{1}{\lambda} \sum_{i=0}^2 \alpha_i (\lambda - cr_i) e^{r_i b} &= 1.\end{aligned}$$

If we assume that $B(u, b) = \sum_{i=0}^3 \alpha_i e^{-R_i u}$ is the solution of the differential equation (6.6), then $\{R_i\}_{i=1}^3$ are the roots of the equation(1.11) and without loss of generality we can take $R_1 = R$, the Lundberg exponent. $\{\alpha_i\}_{i=0}^3$ can be obtained from the above set of equations replacing r_i with $-R_i$.

6.3.2 Mixed Exponential Distribution

We assume that the individual claim amount has Mixed exponential distribution with p.d.f.

$$f(x) = p_1 \beta_1 e^{-\beta_1 x} + p_2 \beta_2 e^{-\beta_2 x}, \quad x > 0, \quad 0 < \beta_1 < \beta_2, \quad p_1 + p_2 = 1.$$

Let $k(u) = 2\lambda^2 \int_0^u f(u-x)B(x, b)dx$

Therefore,

$$k(u) = 2\lambda^2 \left\{ p_1 \beta_1 e^{-\beta_1 u} \int_0^u e^{\beta_1 x} B(x, b) dx + p_2 \beta_2 e^{-\beta_2 u} \int_0^u e^{\beta_2 x} B(x, b) dx \right\} \quad (6.10)$$

Differentiating (6.10) with respect to u twice, and simplifying

$$k^{(1)}(u) = 2\lambda^2 \left\{ p_1\beta_1(-\beta_1) \int_0^u e^{\beta_1 x} B(x, b) dx + p_2\beta_2(-\beta_2) \int_0^u e^{\beta_2 x} B(x, b) dx + B(u, b)(p_1\beta_1 + p_2\beta_2) \right\}$$

$$k^{(2)}(u) = 2\lambda^2 \left\{ \beta_1^2 p_1 \beta_1 e^{-\beta_1 u} \int_0^u e^{\beta_1 x} B(x, b) dx + \beta_2^2 p_2 \beta_2 e^{-\beta_2 u} \int_0^u e^{\beta_2 x} B(x, b) dx - p_1 \beta_1^2 B(u, b) - p_2 \beta_2^2 B(u, b) + p_1 \beta_1 B^{(1)}(u, b) + p_2 \beta_2 B^{(1)}(u, b) \right\} \quad (6.11)$$

$$\begin{aligned} k^{(2)}(u) + (\beta_1 + \beta_2)k^{(1)}(u) + \beta_1\beta_2 k(u) - 2\lambda^2 B^{(1)}(u, b)(p_1\beta_1 + p_2\beta_2) \\ = 2\lambda^2 \beta_1\beta_2 B(u, b) \end{aligned} \quad (6.12)$$

We eliminate the integral term from (6.3) by differentiating it twice, we get:

$$\begin{aligned} c^2 B^{(4)}(u, b) + \{c^2(\beta_1 + \beta_2) - 3c\lambda\} B^{(3)}(u, b) + \\ \{2\lambda^2 - 3c\lambda(\beta_1 + \beta_2) + c^2\beta_1\beta_2\} B^{(2)}(u, b) + \\ \{2\lambda^2(\beta_1 + \beta_2) - 2\lambda^2(p_1\beta_1 + p_2\beta_2) - 3c\lambda\beta_1\beta_2\} B^{(1)}(u, b) = 0 \end{aligned} \quad (6.13)$$

We can write the solution of the differential equation (6.13) as

$$B(u, b) = \sum_{i=0}^3 \alpha_i e^{r_i u} \quad (6.14)$$

where $\{r_i\}_{i=0}^3$ are roots of the characteristic equation of (6.13) and $\{\alpha_i\}_{i=0}^3$ are values depending on b . We need four equations to obtain the values of $\{\alpha_i\}_{i=0}^3$.

Using the boundary condition, $B(b, b) = 1$ we get $\sum_{i=0}^3 \alpha_i e^{r_i b} = 1$.

Substituting (6.14), the LHS of (6.3) becomes:

$$\sum_{i=0}^3 \alpha_i e^{r_i u} (c r_i - 2\lambda)(c r_i - \lambda) \quad (6.15)$$

and when substituted in the RHS of (6.3) and rearranging, we get:

$$\begin{aligned}
2\lambda^2 \int_0^u (p_1\beta_1 e^{-\beta_1 x} + p_2\beta_2 e^{-\beta_2 x}) \sum_{i=0}^3 \alpha_i e^{r_i(u-x)} dx \\
= 2\lambda^2 \left\{ \sum_0^3 \alpha_i e^{r_i u} \frac{\beta_1\beta_2 + r_i(p_1\beta_1 + p_2\beta_2)}{(\beta_1 + r_i)(\beta_2 + r_i)} \right. \\
\left. - p_1\beta_1 \sum_0^3 \frac{\alpha_i}{\beta_1 + r_i} e^{-\beta_1 u} - p_2\beta_2 \sum_0^3 \frac{\alpha_i}{\beta_2 + r_i} e^{-\beta_2 u} \right\} \quad (6.16)
\end{aligned}$$

Again, equating (6.15) and (6.16), comparing the coefficients of $p_1 e^{-\beta_1 u}$ and $p_2 e^{-\beta_2 u}$, we obtain

$$\begin{aligned}
\beta_1 \sum_{i=0}^3 \frac{\alpha_i}{\beta_1 + r_i} &= 0 \\
\beta_2 \sum_{i=0}^3 \frac{\alpha_i}{\beta_2 + r_i} &= 0.
\end{aligned}$$

Now differentiating (6.2) with respect to u , and rearranging the terms, we have:

$$cB^{(1)}(u, b) - \lambda B(u, b) = -\frac{2\lambda^2}{c} \left\{ \int_u^b e^{-2\lambda(\frac{s-u}{c})} \int_0^s f(x)B(s-x, b) dx ds + \int_b^\infty e^{-2\lambda(\frac{s-u}{c})} ds \right\}$$

Substituting $B(u, b)$ in the above equation and solving the integral in the RHS, we obtain:

$$\sum_{i=0}^3 \alpha_i e^{r_i u} (cr_i - \lambda) = \sum_{i=0}^3 \alpha_i (cr_i - \lambda) (-e^{r_i b} e^{-2\lambda b} + e^{r_i u}) - \lambda e^{-2\lambda b}.$$

Therefore,

$$\frac{1}{\lambda} \sum_{i=0}^3 \alpha_i (\lambda - cr_i) e^{r_i b} = 1.$$

Now we have four equations to solve for $\{\alpha_i\}_{i=0}^3$, viz,

$$\begin{aligned} \sum_{i=0}^3 \alpha_i e^{r_i b} &= 1 \\ \sum_{i=0}^3 \frac{\alpha_i}{\beta_1 + r_i} &= 0 \\ \sum_{i=0}^3 \frac{\alpha_i}{\beta_2 + r_i} &= 0 \\ \frac{1}{\lambda} \sum_{i=0}^3 \alpha_i (\lambda - cr_i) e^{r_i b} &= 1. \end{aligned}$$

In the following table, $B(u, b)$ is calculated for different values of u and b .

Table: 1

$\beta_1 = 0.5, \beta_2 = 2, \lambda = 1, p_1 = 1/3$, and $c = 1.1$

$b \setminus u$	0	1	2	3	4	5
1	0.811931	1.000000
2	0.690460	0.909491	1.000000
3	0.622541	0.823747	0.942265	1.000000
4	0.579602	0.767195	0.880181	0.959567	1.000000	..
5	0.550631	0.728867	0.836403	0.913735	0.970690	1.000000

6.3.3 Gamma Distribution

Let us assume that the claim amount distribution has a $\Gamma(2, \beta)$ distribution. We have:

$$k(u) = 2\lambda^2 \int_0^u f(u-x)B(x, b)dx \quad (6.17)$$

Inserting $f(x)$ and differentiating (6.17) with respect to u twice, we obtain

$$k^{(2)}(u) + 2\beta k^{(1)}(u) + \beta^2 k(u) = 2\lambda^2 \beta^2 B(u, b).$$

Using the same argument as described above, we have the differential equation

$$\begin{aligned} c^2 B^{(4)}(u, b) + (2\beta c^2 - 3\lambda c) B^{(3)}(u, b) + (2\lambda^2 - 6\beta\lambda c + \beta^2 c^2) B^{(2)}(u, b) + \\ (4\beta\lambda^2 - 3\beta\lambda c) B^{(1)}(u, b) = 0 \end{aligned} \quad (6.18)$$

with solutions

$$B(u, b) = \sum_{i=0}^3 \alpha_i e^{r_i u}. \quad (6.19)$$

where $\{r_i\}_{i=0}^3$ are the roots of the characteristic equation (6.18) and $\{\alpha_i\}_{i=0}^3$ are obtained by solving the set of equations.

$$\begin{aligned} \sum_{i=0}^3 \alpha_i e^{r_i b} &= 1 \\ \sum_{i=0}^3 \frac{\alpha_i}{\beta + r_i} &= 0 \\ \sum_{i=0}^3 \frac{\alpha_i}{(\beta + r_i)^2} &= 0 \\ \frac{1}{\lambda} \sum_{i=0}^3 \alpha_i (\lambda - cr_i) e^{r_i b} &= 1. \end{aligned}$$

Therefore, we can find the probability of reaching an upper barrier b before ruin occurs.

Table: 2 $\beta = 3, \lambda = 2, \text{ and } c = 1.1$

b \ u	0	1	2	3	4	5
1	0.505977	1.000000
2	0.356045	0.783272	1.000000
3	0.303625	0.668578	0.8932	1.000000
4	0.279860	0.616254	0.823632	0.943233	1.000000	..
5	0.267775	0.589641	0.788066	0.902689	0.96868	1.000000

6.4 Maximum Severity of Ruin

The method used for finding the probability, $B(u,b)$, that the surplus process to attain the level b before ruin occurs, can also be used to find the maximum severity of ruin. Let us define $G(u, y) = P\{T_u < \infty \text{ and } U(T_u) \geq -y\}$ to be the probability of ruin with initial surplus u and deficit immediately after causing ruin is at most y . Note that $\psi(u) = \lim_{y \rightarrow \infty} G(u, y)$ so that $\frac{G(u,y)}{\psi(u)} = P\{|U(T_u)| \leq y/T_u < \infty\}$ is a proper distribution function. Hence $G(u, y)$ is a defective distribution with density function $g(u, y) = \frac{\partial}{\partial y} G(u, y)$.

Suppose the surplus process continues even after ruin, we consider the distribution of maximum level of the severity until the surplus process crosses upward through level zero. The assumption (1.10), definitely leads the surplus process to attain this level.

Let T'_u be the time of the first up crossing the surplus process through the level zero, hence:

$$T'_u = \inf\{t : t > T_u \text{ and } U(t) > 0\}.$$

Define the random variable Z_u such that

$$Z_u = \sup\{|U(t)|, T_u \leq t \leq T'_u\}.$$

Let

$$H_u(z) = P\{Z_u \leq z/T_u < \infty\} \quad (6.20)$$

be the distribution function of the random variable Z_u given ruin occurs. Note that the maximum severity of ruin will be less than z if ruin occurs with severity $y \leq z$ and if the surplus doesn't fall below $-z$ from the level $-y$, where $z > y$. The probability of this event is that of reaching the level zero from $-y$ without falling below $-z$ and it is nothing but, $B(z-y, z)$, the probability of reaching the level z from the level $z - y$ before falling to zero. Let $\xi(u, y)$ be the conditional distribution of severity of ruin, given ruin occurs. Then,

$$H_u(z) = \int_0^z \xi(u, y) B(z - y, z) dy \quad (6.21)$$

Letting $\xi(u, y) = \frac{g(u, y)}{\psi(u)}$, we have:

$$H_u(z) = \frac{1}{\psi(u)} \int_0^z g(u, y) B(z - y, z) dy \quad (6.22)$$

Now we shall consider two specific claim amount distributions in which the maximum deficit at the time of ruin can be obtained.

Let us assume that the claim amount distribution is *Exponential* with mean $1/\beta$, then

$$g(u, y) = (\beta - R)e^{-Ru - \beta y} \quad (6.23)$$

where R is the adjustment coefficient.

$$\begin{aligned} H_u(z) &= \frac{1}{\psi(u)} \int_0^z (\beta - R)e^{-Ru - \beta y} B(z - y, z) dy \\ &= \beta \left\{ \alpha_0 \int_0^z e^{-\beta y} dy + \sum_{i=1}^2 \alpha_i e^{-R_i z} \int_0^z e^{-(\beta - R_i)y} dy \right\} \\ &= \alpha_0 (1 - e^{-\beta z}) + \beta \sum_{i=1}^2 \frac{\alpha_i (e^{-R_i z} - e^{-\beta z})}{\beta - R_i} \\ &= \alpha_0 + \beta \sum_{i=1}^2 \frac{\alpha_i e^{-R_i z}}{\beta - R_i} - e^{-\beta z} \left\{ \alpha_0 + \beta \sum_{i=1}^2 \frac{\alpha_i}{\beta - R_i} \right\} \end{aligned} \quad (6.24)$$

where $R_1 (= R)$ and R_2 are the roots of the equation (1.11).

In this model, it can be seen that $H_u(z)$ is *independent* of the *initial reserve* u .

Now, if the *claim amount* has distribution is $\Gamma(2, \beta)$, then

$$\begin{aligned} g(u, y) &= \left\{ \frac{(\beta - R_1)^2}{(R_2 - R_1)} + \frac{(\beta - R_1)^2 (\beta - R_2) y}{(R_1 - R_2)} \right\} e^{-R_1 u - \beta y} + \\ &\quad \left\{ \frac{(\beta - R_2)^2}{(R_1 - R_2)} + \frac{(\beta - R_1) (\beta - R_2)^2 y}{(R_2 - R_1)} \right\} e^{-R_2 u - \beta y} \end{aligned} \quad (6.25)$$

where $R_1 (= R)$ and R_2 are the positive roots of the equation (1.11). Therefore,

$$\begin{aligned} H_u(z) &= \frac{1}{\psi(u)} \left\{ \int_0^z \left[\left\{ \frac{(\beta - R_1)^2}{(R_2 - R_1)} e^{-R_1 u} + \frac{(\beta - R_2)^2}{(R_1 - R_2)} e^{-R_1 u} \right\} e^{-\beta y} + \right. \right. \\ &\quad \left. \left\{ \frac{(\beta - R_1)^2 (\beta - R_2)}{(R_1 - R_2)} e^{-R_1 u} + \frac{(\beta - R_1) (\beta - R_2)^2}{(R_2 - R_1)} e^{-R_2 u} \right\} y e^{-\beta y} \right] \\ &\quad \left. \left[\alpha_0 + \sum_{i=1}^3 \alpha_i e^{-R_i(z-y)} \right] dy \right\}. \end{aligned}$$

After simplification, we get

$$\begin{aligned}
 H_u(z) = \frac{1}{\psi(u)} & \left\{ \left[\frac{(\beta - R_1)^2}{R_2 - R_1} e^{-R_1 u} + \frac{(\beta - R_2)^2}{R_1 - R_2} e^{-R_2 u} \right] \right. \\
 & \left[\left(\frac{\alpha_0}{\beta} + \sum_{i=1}^3 \frac{\alpha_i e^{-R_i z}}{\beta - R_i} \right) - e^{-\beta z} \left(\frac{\alpha_0}{\beta} + \sum_{i=1}^3 \frac{\alpha_i}{\beta - R_i} \right) \right] + \\
 & \left[\frac{(\beta - R_1)^2 (\beta - R_2)}{(R_1 - R_2)} e^{-R_1 u} + \frac{(\beta - R_1) (\beta - R_2)^2}{(R_2 - R_1)} e^{-R_2 u} \right] \\
 & \left[\left(\frac{\alpha_0}{\beta^2} + \sum_{i=1}^3 \frac{\alpha_i e^{-R_i z}}{(\beta - R_i)^2} \right) - e^{-\beta z} \left(\frac{\alpha_0}{\beta^2} + \sum_{i=1}^3 \frac{\alpha_i}{(\beta - R_i)^2} \right) - \right. \\
 & \left. \left. z e^{-\beta z} \sum_{i=1}^3 \frac{\alpha_i}{(\beta - R_i)} \right] \right\}
 \end{aligned}$$

where R_3 is the negative root of the equation (1.11).

The following table gives the probability of maximum severity of ruin for various values of u and z .

Table: 3
 Probability of Maximum severity of ruin $H_u(z)$
 $\beta = 3, \lambda = 2,$ and $c = 1.1$

$z \setminus u$	0	1	2	3	4	5
1	0.779172	0.807461	0.808224	0.808245	0.808246	0.808246
2	0.910711	0.922513	0.922832	0.922840	0.922841	0.922841
3	0.956096	0.961904	0.962061	0.962065	0.962065	0.962065
4	0.976665	0.979752	0.979835	0.979838	0.979838	0.979838
5	0.987125	0.988829	0.988875	0.988876	0.988876	0.988876
10	0.999222	0.999325	0.999328	0.999328	0.999328	0.999328
20	0.999997	0.999997	0.999997	0.999997	0.999997	0.999997

On The Time of Ruin with Penalty

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Chapter 7

On The Time of Ruin with Penalty*

7.1 Introduction

In recent years, many researchers in risk theory have shown considerable interest in the study of the distribution of time to ruin, surplus prior to ruin, and deficit at the time of ruin. But in most cases, the identification of the joint distribution of these factors is difficult since explicit expressions do not exist. In general, closed form solutions for the moments of these distributions do not exist. Different approaches have been used to study the distributions. Gerber, Goevaerts and Kass[37] considered the probability that ruin occurs with initial surplus u and that the deficit at

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the time of ruin is less than y , $y \geq 0$. Later, Dufresne and Gerber[29] obtained the distribution of the surplus immediately prior to ruin in the classical Poisson risk model. Dickson and Waters[20] considered recursive calculation of the distribution of the surplus prior to ruin, and the distribution of the deficit at the time of ruin when the individual claim size is discrete. Similar results were obtained in the paper Dickson[21] using a different approach to study the distribution function. Gerber and Shiu[39] obtained the joint distribution of the time to ruin, surplus before ruin and deficit at ruin in the classical risk model. To calculate an expected discounted penalty, which is a function of the initial reserve u , they show that it satisfies a defective renewal equation. This was a major break through in the study of these distributions. Another approach was suggested by Lin and Willmot[52] to solve the defective renewal equation. They obtained a closed form expression for the expected discounted penalty function in the compound Poisson model. Dickson and Hipp[25] obtained an explicit expression for the moments of time to ruin in the Erlang(2) risk model by assuming the penalty function $\omega(x_1, x_2) = 1$. Cheng and Tang[12] complemented the work of Dickson and Hipp[25] by deriving explicit expression for the moments of the surplus before ruin and deficit at ruin in the Erlang(2) risk process. Recently Gerber and Shiu[40] focussed on the analysis of the function $\phi_\delta(u)$, the expected discounted penalty function, when the inter claim time is that of a sum of n independent exponential (different parameters) random variables. They obtained a closed form expression for the joint probability density of the deficit at ruin and surplus prior to ruin, when the initial surplus is zero.

7.2 Preliminaries

In the classical continuous time risk model, the premiums are received continuously at the rate c , and the aggregate claims form a compound Poisson Process with individual claim sizes X_1, X_2, \dots . These X_i 's are assumed to be i.i.d random variables and let $f(x)$ be the probability density function with the Laplace-Stieltjes transform $\hat{f}(s) = \int_0^\infty e^{-sx} dF(x)$

Let $\{T_i\}_{i=1}^\infty$ be a sequence of independent and identically distributed random variables, where T_1 denotes the time until the first claim and, for $i > 1$, T_i denotes the time elapsed between the $(i-1)^{th}$ and i^{th} claim with exponential distribution of mean $1/\lambda$.

The renewal risk model, introduced by Sparre Andersen[1], is more general in that the common distribution of the independent and identically distributed random variables T_i is an arbitrary general distribution. In this chapter, we assume that each of the inter claim time random variables T_i is a Generalized Exponential (5.3) with $\alpha = 2$.

We also consider two non-negative random variables in connection with the time of ruin, the surplus immediately before ruin $U(T^-)$, where T^- is the left limit of T and deficit at the time of ruin $|U(T)|$. Let $\omega(x_1, x_2)$, $0 \leq x_1, x_2 < \infty$, be a non-negative function. For $\delta > 0$, define

$$\phi_\delta(u) = E\{e^{-\delta T} \omega(U(T^-), |U(T)|) I(T < \infty) | U(0) = u\}$$

where

$$I(T < \infty) = \begin{cases} 1 & \text{if } T < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\omega(U(T^-), |U(T)|)$ can be interpreted as the penalty at the time of ruin. $\phi_\delta(u)$ is the expected discounted penalty where δ is interpreted as the force of interest.

Note that if $\omega(x_1, x_2) = 1$ and $\delta = 0$, then

$$\phi_\delta(u) = \psi(u).$$

Let U_k denote the surplus after the payment of the k^{th} claim. Thus $U_0 = u$ at $T_0 = 0$. For $k = 1, 2, 3, \dots$

$$U_k = u + \sum_{i=1}^k (cT_i - X_i).$$

A process of the form

$$\left\{ e^{-\delta \sum_{i=1}^k T_i + s U_k}, k = 0, 1, 2, \dots \right\}, \text{ for } s > 0$$

is a martingale if and only if

$$E\left(e^{-\delta \sum_{i=1}^k T_i + s U_k} \mid U_0 = u\right) = e^{su}.$$

Therefore we have the martingale condition:

$$\hat{p}(\delta - cs)\hat{f}(s) = 1 \tag{7.1}$$

The equation (7.1) is the general form of the Lundberg's fundamental equation and has a negative solution $-R$ (when $\delta = 0$, R is the Lundberg's exponent).

7.3 An Integro-Differential Equation

By conditioning the time of the first claim T_1 , and the amount of the first claim X_1 , we can write:

$$\begin{aligned} \phi_\delta(u) = & \int_0^\infty e^{-\delta t} p(t) \int_0^{u+ct} \phi_\delta(u+ct-x) f(x) dx dt + \\ & \int_0^\infty e^{-\delta t} p(t) \int_{u+ct}^\infty \omega(u+ct, x-(u+ct)) f(x) dx dt \end{aligned}$$

Substituting $s = u+ct$ and differentiating w.r.t. u

$$\begin{aligned} c\phi'_\delta(u) = & \frac{\delta}{c} \left\{ \int_u^\infty \int_0^s e^{-\delta(\frac{s-u}{c})} \phi_\delta(s-x) f(x) p\left(\frac{s-u}{c}\right) dx ds + \right. \\ & \left. \int_u^\infty \int_s^\infty e^{-\delta(\frac{s-u}{c})} \omega(s, x-s) f(x) p\left(\frac{s-u}{c}\right) dx ds \right\} - \\ & \frac{1}{c} \left\{ \int_u^\infty \int_0^s e^{-\delta(\frac{s-u}{c})} \phi_\delta(s-x) f(x) p'\left(\frac{s-u}{c}\right) dx ds + \right. \\ & \left. \int_u^\infty \int_s^\infty e^{-\delta(\frac{s-u}{c})} \omega(s, x-s) f(x) p'\left(\frac{s-u}{c}\right) dx ds \right\} - \\ & p(0) \left\{ \int_0^u \phi_\delta(u-x) f(x) dx - \int_u^\infty \omega(u, x-u) f(x) dx \right\} \end{aligned} \quad (7.2)$$

$$\begin{aligned} c\phi''_\delta(u) = & \frac{\delta^2}{c^2} \left\{ \int_u^\infty \int_0^s e^{-\delta(\frac{s-u}{c})} \phi_\delta(s-x) f(x) p\left(\frac{s-u}{c}\right) dx ds + \right. \\ & \left. \int_u^\infty \int_s^\infty e^{-\delta(\frac{s-u}{c})} \omega(s, x-s) f(x) p\left(\frac{s-u}{c}\right) dx ds \right\} - \\ & \frac{2\delta}{c^2} \left\{ \int_u^\infty \int_0^s e^{-\delta(\frac{s-u}{c})} \phi_\delta(s-x) f(x) p'\left(\frac{s-u}{c}\right) dx ds + \right. \\ & \left. \int_u^\infty \int_s^\infty e^{-\delta(\frac{s-u}{c})} \omega(s, x-s) f(x) p'\left(\frac{s-u}{c}\right) dx ds \right\} + \\ & \frac{1}{c^2} \left\{ \int_u^\infty \int_0^s e^{-\delta(\frac{s-u}{c})} \phi_\delta(s-x) f(x) p''\left(\frac{s-u}{c}\right) dx ds + \right. \\ & \left. \int_u^\infty \int_s^\infty e^{-\delta(\frac{s-u}{c})} \omega(s, x-s) f(x) p''\left(\frac{s-u}{c}\right) dx ds \right\} - \end{aligned}$$

$$\begin{aligned} & \frac{\delta}{c} p(0) \left\{ \int_0^u \phi_\delta(u-x) f(x) dx - \int_u^\infty \omega(u, x-u) f(x) dx \right\} - \\ & \frac{1}{c} p'(0) \left\{ \int_0^u \phi_\delta(u-x) f(x) dx - \int_u^\infty \omega(u, x-u) f(x) dx \right\} \end{aligned} \quad (7.3)$$

But $p(0) = 0$, $p'(0) = 2\lambda^2$, and $p''(\frac{s-u}{c}) = -2\lambda^2 p(\frac{s-u}{c}) - 3\lambda p'(\frac{s-u}{c})$, then

$$\begin{aligned} c^2 \phi_\delta''(u) = & (\delta^2 - 2\lambda^2) \phi_\delta(u) - (2\delta + 3\lambda) \delta \phi_\delta(u) + c(2\delta + 3\lambda) \phi_\delta'(u) + \\ & 2\lambda^2 \int_0^u \phi(u-x) f(x) dx + \int_u^\infty \omega(u, x-u) f(x) dx \end{aligned} \quad (7.4)$$

7.4 Lundberg's fundamental Equation

For the model considered, the Lundberg's equation is

$$c^2 s^2 - c(2\delta + 3\lambda)s + (2\lambda + \delta)(\lambda + \delta) = 2\lambda^2 \hat{f}(s) \quad (7.5)$$

Assume that $\delta > 0$ and $h(s) = c^2 s^2 - c(2\delta + 3\lambda)s + (2\lambda + \delta)(\lambda + \delta)$ then there exist at least two positive roots $r_1 < \frac{3\lambda + 2\delta}{2c} < r_2$ such that $h(r_i) = 2\lambda^2 \hat{f}(s)$, $i = 1, 2$.

We have $h(0) = (2\lambda + \delta)(\lambda + \delta) > 2\lambda^2 = 2\lambda^2 \hat{p}(0)$.

Also $h'(s) = 2c^2 s - c(2\delta + 3\lambda)$ and $h''(s) = 2c^2 > 0$. Hence $h(s)$ has its minimum value at $s = \frac{3\lambda + 2\delta}{2c}$.

It can be seen that

$$h\left(\frac{3\lambda + 2\delta}{2c}\right) = -\frac{\lambda^2}{4} \quad \text{and} \quad \lim_{s \rightarrow \infty} h(s) = \infty.$$

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Also, note that $2\lambda^2 \frac{d}{ds} \hat{f}(s) = -2\lambda^2 \int_0^\infty x e^{-sx} f(x) dx$.

Hence $2\lambda^2 \hat{f}(s)$ is a decreasing function of s . Therefore, for $s > 0$, $h(s)$ intersects $2\lambda^2 \hat{f}(s)$ at two distinct points r_1 and r_2 such that $r_1 < \frac{3\lambda + 2\delta}{2c} < r_2$.

7.5 Laplace Transform of $\phi(u)$

The operator $T_r f$, we call it *Translated Laplace Transform*, used in recent literature, is defined as

$$T_r f(x) = \int_x^\infty e^{-r(u-x)} f(u) du.$$

Then

$$T_{r_1} T_{r_2} f(x) = T_{r_2} T_{r_1} f(x) = \frac{T_{r_1} f(x) - T_{r_2} f(x)}{r_2 - r_1}.$$

Proof

$$\begin{aligned} T_{r_1} T_{r_2} f(x) &= \int_x^\infty e^{-r_1(y-x)} \left(\int_y^\infty e^{-r_2(u-y)} f(u) du \right) dy \\ &= e^{r_1 x} \int_x^\infty e^{-y(r_1-r_2)} \left(\int_y^\infty e^{-r_2 u} f(u) du \right) dy \\ &= e^{r_1 x} \left(\int_x^\infty e^{-r_2 u} f(u) du \frac{e^{-x(r_1-r_2)}}{r_1-r_2} - \int_x^\infty e^{-r_2 y} f(y) \frac{e^{-y(r_1-r_2)}}{r_1-r_2} dy \right) \\ &= \frac{1}{r_1-r_2} \int_x^\infty e^{-r_2(u-x)} f(u) du - \frac{1}{r_1-r_2} \int_x^\infty e^{-r_2(y-x)} f(y) dy \end{aligned}$$

Hence the result.

Now let us denote

$$\begin{aligned}\omega(u) &= \int_u^\infty \omega(u, x-u)f(x)dx, \\ \gamma(u) &= T_{r_2}T_0\omega(u) = \frac{T_0\omega(u) - T_{r_2}\omega(u)}{r_2}, \\ \zeta(u) &= T_{r_2}T_0f(u) = \frac{T_0f(u) - T_{r_2}f(u)}{r_2}.\end{aligned}$$

Theorem

The Laplace transform of $\phi_\delta(u)$ is

$$\hat{\phi}_\delta(s) = \frac{2\lambda^2\hat{\zeta}(s)}{c^2 - 2\lambda^2\hat{\gamma}(s)}. \quad (7.6)$$

Proof

Taking the Laplace Transform of (7.4), we obtain

$$\begin{aligned}c^2(s^2\hat{\phi}_\delta(s) - s\phi_\delta(0) - \phi'_\delta(0)) &= (\delta^2 - 2\lambda^2)\hat{\phi}_\delta(s) - (2\delta + 3\lambda)\delta\hat{\phi}_\delta(s) + \\ &+ c(2\delta + 3\lambda)(s\hat{\phi}_\delta(s) - \phi_\delta(0)) + 2\lambda^2\hat{\phi}_\delta(s)\hat{f}(s) + 2\lambda^2\hat{\omega}(s)\end{aligned} \quad (7.7)$$

Therefore

$$\hat{\phi}_\delta(s) = \frac{c^2s\phi_\delta(0) - c(2\delta + 3\lambda)\phi_\delta(0) + c^2\phi'_\delta(0) + 2\lambda^2\hat{\omega}(s)}{h(s) - 2\lambda^2\hat{f}(s)} \quad (7.8)$$

If r_1 is a root of the equation (7.5), it is also the root of the numerator of (7.8).

Therefore,

$$c^2r_1\phi_\delta(0) + 2\lambda^2\hat{\omega}(r_1) = c(2\delta + 3\lambda)\phi_\delta(0) - c^2\phi'_\delta(0).$$

Then:

$$\hat{\phi}_\delta(s) = \frac{c^2(s - r_1)\phi_\delta(0) + 2\lambda^2(\hat{\omega}(s) - \hat{\omega}(r_1))}{h(s) - 2\lambda^2\hat{f}(s)}.$$

But:

$$L\{T_{r_1}\omega(u)\} = \frac{\hat{\omega}(s) - \hat{\omega}(r_1)}{r_1 - s} = T_{r_1}\hat{\omega}(s)$$

where L denotes the Laplace transform.

Hence, we get:

$$\hat{\phi}_\delta(s) = \frac{c^2(s - r_1)\phi_\delta(0) + 2\lambda^2(r_1 - s)T_{r_1}\hat{\omega}(s)}{h(s) - 2\lambda^2\hat{f}(s)}$$

Since $r_1 (< r_2)$ is a zero of the Lundberg equation for our model, and using (7.8),

we get:

$$2\lambda^2\hat{\omega}(r_1) = \phi_\delta(0)\{c(2\delta + 3\lambda) - c^2r_1\} - c^2\phi'_\delta(0)$$

Similarly,

$$2\lambda^2\hat{\omega}(r_2) = \phi_\delta(0)\{c(2\delta + 3\lambda) - c^2r_2\} - c^2\phi'_\delta(0)$$

Solving the above set of equations, obtain

$$\phi_\delta(0) = \frac{2\lambda^2(\hat{\omega}(r_1) - \hat{\omega}(r_2))}{c^2(r_2 - r_1)} \quad (7.9)$$

Also

$$2\lambda^2\hat{\omega}(r_2) = \frac{2\lambda^2(\hat{\omega}(r_1) - \hat{\omega}(r_2))}{c^2(r_2 - r_1)}(c(2\delta + 3\lambda) - c^2r_2) - c^2\phi'_\delta(0)$$

and, therefore

$$\phi'_\delta(0) = \frac{2\lambda^2\left((2\delta + 3\lambda)(\hat{\omega}(r_1) - \hat{\omega}(r_2)) + c(r_1\hat{\omega}(r_2) - r_2\hat{\omega}(r_1))\right)}{c^3(r_2 - r_1)} \quad (7.10)$$

Substituting $\phi_\delta(0)$ and $\phi'_\delta(0)$ in (7.8) and simplifying, we get

$$\hat{\phi}_\delta(s)(h(s) - 2\lambda^2\hat{f}(s)) = 2\lambda^2\left(\hat{\omega}(s) + \frac{s - r_2}{r_2 - r_1}\hat{\omega}(r_1) - \frac{s - r_1}{r_2 - r_1}\hat{\omega}(r_2)\right) \quad (7.11)$$

But r_1 is a zero of (7.5), therefore:

$$\begin{aligned} h(s) - 2\lambda^2 \hat{f}(s) &= h(s) - 2\lambda^2 \hat{f}(s) - h(r_1) + 2\lambda^2 \hat{f}(r_1) \\ &= h(s) - h(r_1) - 2\lambda^2 (\hat{f}(s) - \hat{f}(r_1)) \end{aligned}$$

and:

$$h(r_1) = (cr_1 - (\lambda + \delta))(cr_1 - (2\lambda + \delta)).$$

Therefore,

$$\begin{aligned} h(s) - 2\lambda^2 \hat{f}(s) &= c^2(s^2 - r_1^2) - c(2\delta + 3\lambda)(s - r_1) - 2\lambda^2(\hat{f}(s) - \hat{f}(r_1)) \\ &= (s - r_1) \left\{ c^2(s + r_1) - c(2\delta + 3\lambda) - \frac{2\lambda^2(\hat{f}(s) - \hat{f}(r_1))}{s - r_1} \right\} \end{aligned}$$

Since r_2 is also a zero of (7.5), we can write

$$c^2(r_2 + r_1) - c(2\delta + 3\lambda) - \frac{2\lambda^2(\hat{f}(r_2) - \hat{f}(r_1))}{r_2 - r_1} = 0$$

Therefore

$$c(2\delta + 3\lambda) = c^2(r_2 + r_1) - \frac{2\lambda^2(\hat{f}(r_2) - \hat{f}(r_1))}{r_2 - r_1} \quad (7.12)$$

Therefore, equation (7.11) becomes

$$\begin{aligned} \hat{\phi}_\delta(s) &\left\{ c^2(s - r_1)(s - r_2) + 2\lambda^2(s - r_2) \frac{\hat{f}(s) - \hat{f}(r_1)}{r_2 - r_1} - 2\lambda^2(s - r_1) \frac{\hat{f}(s) - \hat{f}(r_2)}{r_2 - r_1} \right\} \\ &= 2\lambda^2 \left\{ \hat{\omega}(s) + \frac{s - r_2}{r_2 - r_1} \hat{\omega}(r_1) - \frac{s - r_1}{r_2 - r_1} \hat{\omega}(r_2) \right\} \quad (7.13) \end{aligned}$$

Let us consider the R.H.S of (7.13) separately

$$\frac{2\lambda^2 \left(\hat{\omega}(s)(r_2 - r_1) + (s - r_2)\hat{\omega}(r_1) - (s - r_1)\hat{\omega}(r_2) \right)}{(s - r_1)(s - r_2)(r_2 - r_1)}$$

Adding and subtracting $s\hat{\omega}(s)$ and simplifying, we obtain

$$2\lambda^2 \left(\frac{1}{r_2 - r_1} \frac{\hat{\omega}(r_1) - \hat{\omega}(s)}{s - r_1} - \frac{1}{r_2 - r_1} \frac{\hat{\omega}(r_2) - \hat{\omega}(s)}{s - r_2} \right)$$

$$\begin{aligned}
&= 2\lambda^2 \frac{(T_{r_1}\hat{\omega}(s) - T_{r_2}\hat{\omega}(s))}{r_2 - r_1} \\
&= 2\lambda^2 \hat{\zeta}(s)
\end{aligned} \tag{7.14}$$

Similarly, after simplification, the L.H.S of (7.13) becomes

$$\begin{aligned}
&\hat{\phi}_\delta(s) \left(c^2 - 2\lambda^2 \frac{1}{r_2 - r_1} \frac{\hat{f}(s) - \hat{f}(r_1)}{r_1 - s} + 2\lambda^2 \frac{1}{r_2 - r_1} \frac{\hat{f}(s) - \hat{f}(r_2)}{r_2 - s} \right) \\
&= \hat{\phi}_\delta(s) \left(c^2 - 2\lambda^2 \frac{T_{r_1}\hat{f}(s) - T_{r_2}\hat{f}(s)}{r_2 - r_1} \right) \\
&= \hat{\phi}_\delta(s) (c^2 - 2\lambda^2 \hat{\gamma}(s))
\end{aligned} \tag{7.15}$$

(7.14) and (7.15) prove the theorem.

7.6 Moments of Time to Ruin

Now we consider special cases when claim amount distribution is either Exponential or Gamma, to illustrate how we can find the moments of time to ruin in this renewal risk model. We obtain the mean and variance in each case.

7.6.1 Exponential individual claims

Suppose that the penalty at ruin is independent of the surplus immediately before ruin, $\omega(x, y) = \omega(y)$ under the assumption that the claim amount distribution is

exponential with mean $1/\beta$. Then

$$\begin{aligned}\omega(u) &= \int_u^\infty \omega(u, x-u)f(x)dx \\ &= \int_0^\infty \omega(u, y)f(u+y)dy \\ &= \int_0^\infty \omega(y)f(u+y)dy \\ &= \beta e^{-\beta u} \hat{\omega}(\beta).\end{aligned}$$

Let $\omega(y) = e^{-r_2 y}$, then

$$\begin{aligned}\omega(u) &= \frac{\beta}{\beta+r_2} e^{-\beta u} \\ T_{r_1} &= \frac{\beta}{(\beta+r_1)(\beta+r_2)} e^{-\beta x} \\ T_{r_2} &= \frac{\beta}{(\beta+r_2)^2} e^{-\beta x} \\ T_{r_1} T_{r_2} \omega(x) &= \frac{\beta}{(\beta+r_1)(\beta+r_2)^2} e^{-\beta x} \\ \hat{\zeta}(s) &= \frac{\beta}{(\beta+r_1)(\beta+r_2)^2} \frac{\beta}{\beta+s}\end{aligned}\tag{7.16}$$

Similarly,

$$\hat{\gamma}(s) = \frac{1}{(\beta+r_1)(\beta+r_2)} \frac{\beta}{\beta+s}\tag{7.17}$$

Hence,

$$\hat{\phi}_\delta(s) = \frac{2\lambda^2 \beta}{(\beta+r_2)(c^2(\beta+r_1)(\beta+r_2)(\beta+s) - 2\lambda^2 \beta)}\tag{7.18}$$

Taking the inverse Laplace transform of (7.18), we obtain

$$\begin{aligned}\phi_\delta(u) &= \frac{2\beta\lambda^2}{c^2(\beta+r_1)(\beta+r_2)^2} e^{-\beta u \left(1 - \frac{2\lambda^2}{c^2(\beta+r_1)(\beta+r_2)}\right)} \\ &= \frac{\beta - R}{\beta+r_2} e^{-Ru}\end{aligned}\tag{7.19}$$

where $R = \beta(1 - \frac{2\lambda^2}{c^2(\beta+r_1)(\beta+r_2)})$.

Now the k^{th} moment at the time of ruin, given that ruin has occurred, is given by

$$E\{T^k | T < \infty\} = \frac{\sigma_k(u)}{\psi(u)}$$

where $\sigma_k(u) = (-1)^k \frac{d^k}{d\delta^k} \phi_\delta(u) |_{\delta=0}$.

$$\frac{d}{d\delta} \phi_\delta(u) = - \left\{ \frac{e^{-R(\delta)u} R'(\delta)}{\beta + r_2(\delta)} + \frac{e^{-R(\delta)u} R'(\delta)(\beta - R(\delta))u}{\beta + r_2(\delta)} + \frac{e^{-R(\delta)u} r_2'(\delta)(\beta - R(\delta))}{(\beta + r_2(\delta))^2} \right\}.$$

Since $-R$ is the negative root of the Lundberg's fundamental equation (7.5), therefore:

$$c^2 R(\delta)^2 + c(2\delta + 3\lambda)R(\delta) + (2\lambda + \delta)(\lambda + \delta) = \frac{2\lambda^2\beta}{\beta - R(\delta)} \quad (7.20)$$

Differentiating (7.20) w.r.t. δ and setting $\delta = 0$, obtain

$$R'(0) = \frac{2cR(0) + 3\lambda}{2\lambda^2\beta(\beta - R(0))^{-2} - 2c^2R(0) - 3\lambda c}$$

Similarly r_2 is a positive root of the Lundberg's equation, then

$$r_2'(0) = \frac{2cr_2(0) - 3\lambda}{2c^2r_2(0) - 3\lambda c + 2\lambda^2\beta(\beta + r_2(0))^{-2}}$$

Therefore,

$$\begin{aligned} \sigma_1(u) = & \frac{e^{-R(0)u}}{(\beta + r_2(0))^2} \left(\frac{(2cr_2(0) - 3\lambda)(\beta - R(0))}{2\beta\lambda^2(\beta + r_2(0))^{-2} + 2c^2r_2(0) - 3\lambda c} + \right. \\ & \left. \frac{(\beta + r_2(0))(2cR(0) + 3\lambda)}{2\beta\lambda^2(\beta + R(0))^{-2} - 2c^2R(0) - 3\lambda c} (1 + u(\beta + r_2(0))) \right) \quad (7.21) \end{aligned}$$

Since $\psi(u) = \frac{(\beta - R(0))}{\beta} e^{-R(0)u}$, we have

$$E(T|T < \infty) = \frac{\beta}{(\beta + r_2(0))^2 (\beta - R(0))} \left\{ \frac{(2cr_2(0) - 3\lambda)(\beta - R(0))}{2\beta\lambda^2(\beta + r_2(0))^{-2} + 2c^2r_2(0) - 3c\lambda} + \frac{(\beta + r_2(0))(2cR(0) + 3\lambda)}{2\beta\lambda^2(\beta + R(0))^{-2} - 2c^2R(0) - 3c\lambda} (1 + u(\beta - R(0))) \right\}.$$

To find the variance at the time of ruin,

$$\frac{\sigma_2(u)}{\psi(u)} = \frac{\beta}{(\beta + r_2(0))} \left(\frac{2uR'(0)^2}{\beta - R(0)} + u^2R'(0)^2 + \frac{2r_2'(0)^2}{(\beta + r_2(0))^2} + \frac{2R'(0)r_2'(0)}{(\beta - R(0))(\beta + r_2(0))} + \frac{2uR'(0)r_2'(0)}{\beta + r_2(0)} - \frac{R''(0)}{\beta - R(0)} - uR''(0) - \frac{r_2''(0)}{\beta + r_2(0)} \right).$$

With

$$R''(0) = \frac{4\beta\lambda^2R'(0)^2(\beta - R(0))^{-3} - 2c^2R'(0)^2 - 4cR'(0) - 2}{2c^2R(0) + 3c\lambda - 2\beta\lambda^2(\beta - R(0))^{-2}} \quad \text{and}$$

$$r_2''(0) = \frac{4\beta\lambda^2r_2'(0)^2(\beta + r_2(0))^{-3} - 2c^2r_2'(0)^2 + 4cr_2'(0) - 2}{2c^2r_2(0) - 3c\lambda + 2\beta\lambda^2(\beta + r_2(0))^{-2}}.$$

Therefore, we obtain $E(T|T < \infty)$ and $V(T|T < \infty)$.

For $c = 1.1$, $\lambda = 2$, $\beta = 2$

$$E(T|T < \infty) = 0.43962 + 0.41094u \quad \text{and}$$

$$V(T|T < \infty) = 1.84269 + 2.7782u + 0.37368u^2$$

7.6.2 Gamma individual claims

If the individual claim amount has Gamma distribution with scale parameter β and shape parameter two, then

$$\omega(u) = \frac{\beta^2 e^{-\beta u}}{(\beta + r_2)^2} (1 + \beta u + r_2 u)$$

where $\omega(x, y)$ is same as in the previous section.

$$T_{r_1} T_{r_2} \omega(x) = \frac{\beta^2}{(\beta + r_1)^2 (\beta + r_2)^3} [3\beta e^{-\beta x} + \beta^2 x e^{-\beta x} + r_2 e^{-\beta x} + \beta r_2 x e^{-\beta x} + 2r_1 e^{-\beta x} + \beta r_1 x e^{-\beta x} + r_1 r_2 x e^{-\beta x}].$$

Therefore,

$$\hat{\zeta}(s) = \frac{\beta^2}{(\beta + r_1)^2 (\beta + r_2)^3} \left\{ \frac{3\beta}{\beta + s} + \frac{\beta^2}{(\beta + s)^2} + \frac{r_2}{\beta + s} + \frac{\beta r_2}{(\beta + s)^2} + \frac{2r_1}{\beta + s} + \frac{\beta r_1}{(\beta + s)^2} + \frac{r_1 r_2}{(\beta + s)^2} \right\}$$

Similarly,

$$\hat{\gamma}(s) = \frac{\beta^2}{(\beta + r_1)^2 (\beta + r_2)^2} \left(\frac{2\beta}{\beta + s} + \frac{\beta^2}{(\beta + s)^2} + \frac{r_1}{\beta + s} + \frac{\beta r_1}{(\beta + s)^2} + \frac{r_2}{\beta + s} + \frac{r_1 r_2}{(\beta + s)^2} \right)$$

Using (7.6), we have

$$\hat{\phi}_\delta(s) = \frac{A}{B}$$

where $A = 2\beta^2 \lambda^2 \{4\beta^2 + 3\beta r_1 + 2\beta r_2 + r_1 r_2 + s(3\beta + 2r_1 + r_2)\}$ and

$$B = (\beta + r_2) \left\{ c^2 (\beta + r_1)^2 (\beta + r_2)^2 (\beta + s)^2 - 2\beta^2 \lambda^2 (2\beta + r_1 + r_2) s - 2\beta^2 \lambda^2 (2\beta r_1 + r_1 r_2 + 3\beta^2 + 2\beta r_2) \right\}.$$

Setting

$$\begin{aligned}
 k_1 &= 2\beta^2\lambda^2(4\beta^2 + 3\beta r_1 + 2\beta r_2 + r_1 r_2) \\
 k_2 &= 2\beta^2\lambda^2(3\beta + 2r_1 + r_2) \\
 k_3 &= -2\beta^2\lambda^2(2\beta r_1 + r_1 r_2 + 3\beta^2 + 2\beta r_2)(\beta + r_2) \\
 k_4 &= -2\beta^2\lambda^2(2\beta + r_1 + r_2)(\beta + r_2) \\
 k_5 &= (\beta + r_2)(c^2(\beta + r_1)^2(\beta + r_2)^2).
 \end{aligned}$$

$$\hat{\phi}_\delta(s) = \frac{k_1 + k_2 s}{k_3 + k_4 s + k_5 (s + \beta)^2} \quad (7.22)$$

Taking inverse Laplace transform of (7.22) and simplifying

$$\phi_\delta(u) = \frac{(\beta - r)^2(R + r_2)}{(R - r)(\beta + r_2)^2} e^{-ru} - \frac{(\beta - R)^2(r + r_2)}{(R - r)(\beta + r_2)^2} e^{-Ru} \quad (7.23)$$

where

$$\begin{aligned}
 R &= \frac{k_4 + 2\beta k_5 - \sqrt{k_4^2 - 4k_3 k_5 + 4\beta k_4 k_5}}{2k_5} \\
 r &= \frac{k_4 + 2\beta k_5 + \sqrt{k_4^2 - 4k_3 k_5 + 4\beta k_4 k_5}}{2k_5}
 \end{aligned}$$

$-R$ and $-r$ are the negative roots of the Lundberg's fundamental equation (7.5).

We can solve for the moments of the time of ruin following the method described in the previous section.

$$\begin{aligned}
 \frac{d}{d\delta} \phi_\delta(u) &= \frac{(\beta - R(\delta))^2(r(\delta) + r_2(\delta))}{(R(\delta) - r(\delta))(\beta + r_2(\delta))^2} e^{-R(\delta)u} \left\{ \frac{2R'(\delta)}{\beta - R(\delta)} - uR'(\delta) + \frac{R'(\delta) - r'(\delta)}{R(\delta) - r(\delta)} + \right. \\
 &\quad \left. \frac{2r_2'(\delta)}{\beta + r_2(\delta)} + \frac{r'(\delta) + r_2'(\delta)}{r(\delta) - r_2(\delta)} \right\} - \frac{(\beta - r(\delta))^2(R(\delta) + r_2(\delta))}{(R(\delta) - r(\delta))(\beta + r_2(\delta))^2} e^{-r(\delta)u} \left\{ \frac{2r'(\delta)}{\beta - r(\delta)} + \right. \\
 &\quad \left. ur'(\delta) + \frac{R'(\delta) - r'(\delta)}{R(\delta) - r(\delta)} + \frac{2r_2'(\delta)}{\beta + r_2(\delta)} - \frac{R'(\delta) + r_2'(\delta)}{R(\delta) + r_2(\delta)} \right\}
 \end{aligned}$$

Differentiating the Lundberg's equation with respect to δ , and setting $\delta = 0$, we obtain

$$R'(0) = \frac{3\lambda + 2cR(0)}{4\beta^2\lambda^2(\beta - R(0))^{-3} - 3c\lambda - 2c^2R(0)} \quad (7.24)$$

$$r_2'(0) = \frac{3\lambda - 2cr_2(0)}{3c\lambda - 4\beta^2\lambda^2(\beta - r_2(0))^{-3} - 2c^2r_2(0)} \quad (7.25)$$

$r'(0)$ is obtained by replacing R by r in (7.24). Therefore,

$$E\{T|T < \infty\} = \frac{\sigma_1}{\psi(u)}$$

where

$$\psi(u) = \frac{R(0)(\beta - r(0))^2}{\beta^2(R(0) - r(0))} e^{-r(0)u} - \frac{r(0)(\beta - R(0))^2}{\beta^2(R(0) - r(0))} e^{-R(0)u}.$$

Similarly we can find $\frac{d^2}{d\delta^2}\phi_\delta(u)\Big|_{\delta=0}$ and then $E\{T^2|T < \infty\}$, where

$$R''(0) = \frac{12\beta^2\lambda^2R'(0)^2(\beta - R(0))^{-4} - 2c^2R'(0)^2 - 4cR'(0) - 2}{2c^2R(0) + 3c\lambda - 4\beta^2\lambda^2(\beta - R(0))^{-3}} \quad (7.26)$$

$$r_2''(0) = \frac{12\beta^2\lambda^2r_2'(0)^2(\beta + r(0))^{-4} - 2c^2r_2'(0)^2 + 4cr_2'(0) - 2}{2c^2r_2(0) + 3c\lambda - 4\beta^2\lambda^2(\beta - r_2(0))^{-3}} \quad (7.27)$$

For $\beta = 3, \lambda = 2, c = 1.1$

$$E\{T|T < \infty\} = \frac{0.59407 + 1.050174u + (0.0546142 + 0.00432u)e^{-3.58368u}}{0.76906 - 0.02195e^{-3.58368u}}.$$

$$V\{T|T < \infty\} = \frac{7.33586 - 16.1786u + 2.01814u^2 + e^{-3.58368u}(0.47039 - 0.62754u - 0.09858u^2)}{(0.76906 - 0.02195e^{-3.58368u})^2}.$$

Conclusion

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Conclusion

This is a detailed and in-depth study on some Sparre-Andersen models in Risk theory. We have introduced the renewal risk model in which the inter-claims times are Generalized exponential and obtained explicit expressions for finite/infinite time ruin probabilities. We have considered the severity of ruin, barrier probabilities and moments of time to ruin in these models and provided analytic as well as numerical solutions.

A first goal is to describe the risk theory from the basic level and then to introduce renewal risk model. We have addressed several aspects of risk and present a contribution in the context of Sparre Andersen model. In Chapters 1 and 2 we have provided definitions of various terms involved in ruin theory and a detailed description of Cramer-Lundberg models and Sparre Andersen models. In Chapter 3, we discussed a classical risk model where batch arrival of claims occur and obtained finite time ruin probabilities.

A Sparre Andersen model in which the inter-arrival of claims is Generalized

exponential distribution is considered in Chapter 4. We obtain the bounds for the ultimate ruin probabilities for different claim amount distributions. Our interest is not only in the computation of the probability of ruin, but also in the amount of insurer's deficit at the time of ruin. Chapter 5 is a work towards this direction and obtained the severity of ruin for a renewal risk model.

In Chapter 6, we obtained the barrier probability, ie, the probability that the surplus process attains the barrier b , without first falling below zero. Again, we have gone beyond this and there by obtained the probability of maximum severity of ruin for this Sparre-Andersen model.

The moments of time to ruin was considered in Chapter 7. The study was focused on the function $\phi_\delta(u)$, the expected discounted penalty, which is due at ruin, may depend on the deficit at the time of ruin. We have obtained closed form expressions for the moments of time of ruin, given that ruin occurs.

Scope for Future Research

Chapters 3 to 7 can be viewed as individual contribution in their own. The study may be continued further considering renewal risk models with Generalized exponential distribution having arbitrary parameters. Very little work has been done on renewal risk models with heavy tailed distribution as either inter-claim or claim size distributions. Such models are of great interest to many researchers in

risk theory and some study in this direction may be taken up. We can also think of bridging the theory of risk models and pricing of options in stock markets, which is a potential field for research.

We hope to have established grounds for future research in these directions that will lead to further insight in the stochastic behavior of renewal risk models.

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