

Flows of Nanofluids Past Plates and in Channels

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in
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By
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Under the guidance of
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22 December 2022

DECLARATION

I, **Neethu T S**, hereby declare that the thesis entitled, "Flows of Nanofluids Past Plates and in Channels", submitted to the University of Calicut in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy in Mathematics is a bonafide research work done by me under the supervision and guidance of Dr. Alphonsa Mathew, Assistant Professor, Department of Mathematics, St. Thomas College (Autonomous), Thrissur, Kerala.

I further declare that this thesis has not previously formed the basis of any degree, diploma or any other similar title.

Thrissur

22 December 2022



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CERTIFICATE

This is to certify that the thesis titled "**Flows of Nanofluids Past Plates and in Channels**" submitted by **Neethu T S** to University of Calicut in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy in **Mathematics** is a record of original research work carried out by her under my supervision. The content of this thesis, in full or in parts, has not been submitted by any other candidate to any other University for the award of any degree or diploma.

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CERTIFICATE

I hereby certify that, this is the revised version of the thesis entitled "**Flows of Nanofluids Past Plates and in Channels**" submitted by **Neethu T S** under my guidance, after incorporating the necessary corrections/ suggestions made by the adjudicators. I also certify that the contents in the thesis and the soft copy are one and the same.

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ABSTRACT

The present thesis is attentive to the numerical investigation of viscous incompressible nanofluid flow past plates and in channels in the presence of an applied magnetic field under different conditions. Tiwari-Das nanofluid models are employed to describe fluid motion. This thesis is arranged into six chapters in which **Chapter 1** is an introductory one that contains some basic concepts, preliminaries, and background of the work. A review of relevant literature has also been included.

Chapter 2 investigates the magnetohydrodynamic flow of water-based nanofluids between two oppositely moving vertical porous plates. The Perturbation technique is used to solve the governing equations of the flow. The consequence of various parameters on velocity, temperature, and concentration are examined via graphs utilizing MATLAB software. The physical quantities are scrutinized using statistical tools like probable error and multiple linear regression and an excellent agreement is noted. Furthermore, the simultaneous effects of parameters on drag coefficients are studied with the aid of three-dimensional surface plots.

Chapter 3 explores the three-dimensional convective hydromagnetic hybrid nanoliquid (with suspended Al_2O_3 and Fe_3O_4 nanoparticles) flow between two oppositely moving vertical porous plates utilizing the Perturbation technique. The consequence of effectual parameters on the flow profiles is analyzed with the aid of graphs using MATLAB software. Further, the rate of heat transfer is statistically scrutinized utilizing RSM (Response Surface Methodology) and sensitivity analysis. Three-dimensional surface plots are made used to illustrate the parallel effect of pertinent parameters on the drag coefficient. Moreover, this study finds applications in several engineering, geophysical, and industrial fields in heat exchangers and faulting.

Chapter 4 includes a theoretical investigation of bioconvective flow of electromagnetohydrodynamic (EMHD) hybrid nanofluid (water-based $CNT - Fe_3O_4$) over a stretching surface. The impact of viscous dissipation, chemical reaction, and stratification is also explored in it. The highly nonlinear system of partial differential equations (PDEs) is reduced to a system of ordinary differential equations

(ODEs) by effectual similarity transformations and then treated numerically using `bvp4c` (a finite difference-based built-in numerical procedure) in MATLAB. Further, the drag coefficient is statistically scrutinized for the impact of nanoparticle volume fraction of carbon nanotubes, the volume fraction of magnetite nanoparticles, the Hartmann number, and the electric field parameter by exercising the four-factor response surface methodology. This study finds applications in cancer therapy, bio-microsystems, biomedical imaging, and therapeutic drug delivery.

Chapter 5 analyses the magnetohydrodynamics of bioconvective hybrid nanofluid (water-based $TiO_2 - Ag$) flow over a permeable exponential stretching sheet. The effects of thermal radiation, heat generation, chemical reaction, porosity, and viscous dissipation have been incorporated. Similarity transformations are applied to the nonlinear system of partial differential equations that arise by the flow. The nonlinear ordinary differential system hence obtained is solved by MATLAB built-in function `bvp5c` to visualize the role of effectual parameters via tables and graphs. Physical quantities of the fluid flow are scrutinized using 3D surface plots and tables. Further, the drag coefficient and heat transfer rate are scrutinized using statistical techniques, in which multiple linear regression analysis shows a good agreement of original data and estimated data, highlighting the reliability of the study. This study can apply to manufacture the most effective and qualified products in industries

Chapter 6 presents the concluding remarks of the thesis and proposals for future work. These studies find applications in the biomedical field and industrial fields.

To My Family

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CHAPTER 1

General Introduction

Fluid- the material which exhibits continuous deformation under the action of shearing forces, frequently classified as liquids or gases. The substances having intermolecular forces, thus possess definite volume are liquids, while fluid with extremely small intermolecular forces are gases. Fluid mechanics is the stream of science that deals with the behavior of stationary and moving fluids. The analysis of fluid properties at rest is the fluid statics, whereas the study of fluid in motion, in absence of pressure force, is the fluid kinematics. And with considering pressure force, the stream of science that describes the characterization of fluid in motion is fluid dynamics.

The analysis of moving fluid is necessary as it is existing in all nature activities and our daily life. The notion of, a continuous medium developed by Aristotle, and the physical law of buoyancy initiated by Archimedes was the first step in the study of fluid flow. Sir Isaac Newton's study of fluid statistics and fluid dynamics paved the way for the analysis of fluid flow through mathematical and quantitative physics. Later Daniel Bernoulli elucidated the significance of fluid pressure and based on his notion, Euler developed elementary equations of hydrodynamics for ideal fluid.

Fluid dynamics provides a wide range of applications in transportation, industry, manufacturing, aeronautical engineering, bio-medical field, etc. It also con-

tributes in measure the evolution of planets, weather patterns, ocean tides, plate tectonics, and also blood circulation. Most of the natural motions obey conservation laws, which are treated as fundamental principles of fluid dynamics. The mathematical analysis of the fluid dynamic system is significant in engineering.

Computational fluid dynamics is a principal branch of fluid dynamics, where the numerical and analytical interpretation of fluid flow is considered. Mathematical software Matlab, Mathematica etc. are used to obtain more accurate solutions to fluid flow problems, and the accuracy of such software and numerical methods authenticates by comparing the values with previous studies. Computational fluid dynamics is important in research and engineering problems, such as aerospace and aerodynamics analysis, environment engineering, biological engineering, design and analysis of heat exchangers, and engines, and so on.

Hence the purpose of this thesis is to analyze some problems in magnetohydrodynamic nanofluid flow in channels and past plates with the impacts of radiation, viscous dissipation, heat source or sink, chemical reaction, and so on.

1.1 SOME BASIC PROPERTIES

1.1.1 Density

The density is the mass per unit volume. The density of a fluid ρ at a point P is defined as

$$\rho = \lim_{\delta V \rightarrow 0} \delta m / \delta V \quad (1.1.1)$$

Where δV is the volume element around P , and δm is the mass of the fluid within δV .

1.1.2 Pressure

Pressure of a fluid is the fraction of normal force to the area. When a fluid is contained inside a surface, it exerts a force at each point of the surface. The pressure of a fluid p at a point P is defined as

$$p = \lim_{\delta A \rightarrow 0} \delta F / \delta A \quad (1.1.2)$$

δA is the elementary area around P and δF is the normal force due to the fluid on δA .

1.1.3 Specific heat

The amount of heat needed to raise the temperature of one gram of a material by one Celcius degree. If Q is the heat energy, m -mas, and δT -the temperature difference, then specific heat capacity is

$$C = \frac{Q}{(m\delta T)} \quad (1.1.3)$$

Specific heat at constant pressure is denoted by C_p

1.1.4 Viscosity

The viscosity of a fluid is a tool for the resistance of the fluid to flow. It is also known as internal friction. Simply it is the ratio of shear stress (the tangential force per unit area when fluid is moving) to shear rate (velocity gradient).

$$\mu = \frac{\text{shear stress}}{\text{velocity gradient}} \quad (1.1.4)$$

The effect of viscosity on the motion of a fluid is determined by kinematic viscosity

$$\vartheta = \frac{\mu}{\rho} \quad (1.1.5)$$

1.2 CLASSIFICATION OF FLUIDS

1.2.1 Ideal and Real fluids

The non-viscous fluids are called ideal fluids. When a fluid is moving shear stress is zero in an ideal fluid, hence there is no resistance in the motion of the fluid. Generally, ideal fluids are not existing in nature. The viscous fluids are called real fluids. Nonzero shear stress exists always in real fluids when fluid is in motion, hence some resistance occurs in a fluid motion.

1.2.2 Compressible and incompressible fluid

The ability to change in volume of a mass of fluid is known as compressibility. Gas, vapor, steam, etc. are regarded as compressible fluids, as they have no definite volume. On the other hand, liquids are considered to possess a definite volume, hence they are incompressible fluids.

1.2.3 Newtonian and non-Newtonian fluid

Fluid flow is in such a manner that the various layers of the fluid slide uniformly over one another. Sir Isac Newton has proved that the tangential stress acting on one layer of the fluid by the other is directly proportional to the relative velocity and inversely proportional to the distance between them. If u and $u + du$ are the velocities of two adjacent layers of fluid separated by a distance dy , by Newton's law of viscosity, the tangential stress τ is

$$\tau = \mu \frac{du}{dy} \quad (1.2.1)$$

Where μ is the coefficient of viscosity. The fluids which obey this relation are referred to as Newtonian fluids. Water, vegetable oils, milk, air, and all gases are Newtonian fluids in usual circumstances.

The fluids which do not obey Newton's law of viscosity are non-Newtonian fluids. They show a non-linear correlation between shear stress and shear rate. Cake batter, polymer solutions, blood, sand in water, paste, and silicone oil are some examples of non-Newtonian fluids.

1.3 TYPES OF FLOW

1.3.1 Laminar and Turbulent flow

In fluid flow, if the individual streamlines run parallelly in a well-ordered manner, it is laminar flow. Blood flow through capillaries and oil flow through a thin tube are laminar flow. In turbulent flow, the streamlines are intertwined with each other and run improperly. Most of the flows seen in nature such as lava flow, atmosphere, and ocean current, are turbulent flows.

1.3.2 Steady and unsteady flow

The flow with time-independent fluid properties is known as steady flow. That is

$$\frac{\partial P}{\partial t} = 0, \quad (1.3.1)$$

where $P(x, y, z)$ is any property such as density, velocity, temperature, and so on. The flow-through pipes, pumps, nozzles, etc. are examples of steady flow. The flow in which the fluid properties depend on time is the unsteady flow.

1.4 HEAT TRANSFER

Heat transfer is the physical process that concerns the swapping of thermal energy between material bodies due to temperature differences. By the second law of thermodynamics, heat transfer occurs from a higher temperature region to a lower temperature region, until two regions attain the same temperature. The heat may be transferred in three modes.

1.4.1 Conduction

Heat conduction is a process in which heat transfer takes place from particles to particles by their collision. These particles do not necessarily have displacement from their position, but they vibrate against each other. These vibrating molecules interact with the neighboring molecules and transfer heat energy.

Thermal conductivity

It correlates the heat flow to the temperature gradient. If T and $T + dT$ are the temperatures of two adjacent layers of the fluid separated by a distance dy , by Fourier's law of heat conduction, the quantity of heat q transferred through a unit area in unit time is

$$q = -k \frac{dT}{dy} \quad (1.4.1)$$

k - the coefficient of thermal conductivity

The effect of conductivity on the temperature field is determined by thermal diffusivity,

$$\alpha = \frac{k}{\rho C_p} \quad (1.4.2)$$

1.4.2 Convection

Convection is a heat transfer phenomenon by the movement of the fluid between temperature differed areas of the fluid. Convection problems are classified as:

Free and Forced convection

Free convection is a mechanism, in which the fluid motion is generated only by gravity effect on heated fluids of variable density, not by any external source, also termed natural convection. The lighter (less dense) components rise, while heavier (denser) components will fall leading to a bulk fluid movement that creates natural convection.

If the motion is due to an external force (such as a fan, pump), it is forced convection. Based on forced convection, devices are developed for heat transfer in all types of heat exchangers. The convection in which both free and forced convection mechanism occurs is called mixed convection.

Bioconvection

Bioconvection occurs due to the upward-directed swimming of motile microorganisms, that are a little denser than water. The upper surface of the fluid becomes dense due to the upward movement of microorganisms, thus a hydromagnetic instability arises under some conditions that create bioconvective flow.

1.4.3 Radiation

The heat energy transfer from an object, having a higher temperature, in the form of electromagnetic waves is thermal radiation. For example, radiation from hot gas in outer space, infrared radiation emitted by an electric heater, etc. Radiation does not require a material medium for energy transfer. Thermal radiation is formed when the heat energy, by the movement of charges in material transformed into electromagnetic waves. According to Stefan- Boltzman total energy radiated per second per unit area is directly proportional to the fourth power of the absolute temperature of the surface.

$$E_t = \sigma_t^* T^4 \quad (1.4.3)$$

where T is the absolute temperature and σ_t^* is the Stefan- Boltzman constant.

1.5 MASS TRANSFER

Mass transfer is the process of movement of the mass because of the concentration difference of species in a mixture. The molecular motions generate mass transfer to reduce the concentration difference at distinct points within the mixture of two or more components. Mass transfer is mainly classified as mass diffusion and convective mass transfer.

1.5.1 Mass Diffusion

Mass diffusion is a mass transfer process in a medium at rest, in which the driving force is the concentration difference between neighboring regions in the medium.

1.5.2 Convective Mass Transfer

If the mass transfer occurs between a boundary surface and a moving fluid or between two relative moving fluids, it is known as a convective mass transfer.

Chemical Reaction

A chemical reaction is a process in which the transmission of one or more substances or reactants to one or more different substances takes place. The transmitter substances are either chemical elements or compounds. Encouraging or defeating convection has a significant role in heat transfer problems. Sustaining a non-uniform temperature gradient is an effective mechanism to delay or advance the beginning of convection. Chemical reactors have a vital role to establish a non-uniform temperature gradient. Adding a chemical reactant promotes the transformation of substances into another. If the density of the reactant is different from the density of the product formed, an isothermal reaction can cause free convection.

1.6 MAGNETOHYDRODYNAMICS

Magnetohydrodynamics (MHD); also called magneto-fluid dynamics or hydromagnetics, is the physical and mathematical framework that deals with the dynamics of electrically conducting fluids such as saltwater, ionized gases or plasmas (solar atmosphere), electrolytes, and liquid metals (molten iron, mercury, gallium) under the influence of a magnetic field.

The important observations from the historical evolution of magnetohydrodynamics are: Euler (1701-1780) generalized Newton's second law of dynamics to a continuous medium with no internal shear stress and Navier (1785-1836) modified Euler's equation corresponding to uniform viscous fluid. Stokes (1819-1903) presented the concept of internal shear stress, its mathematical frame led to the well-known Navier-Stokes equation. On the other hand in electromagnetism, Volta (1745-1827) innovated the Voltaic pile or battery. Ampere (1775-1836) found a connection between magnetism and electricity. Later Ohm (1789-1854) established the electrical conduction law and Faraday (1791-1867) discovered that the difference in magnetic flux across a loop generates variation in electric potential. Then Maxwell (1831-189) developed four equations by extending Faraday's work.

Gauss Law :

$$\text{div}D = \rho_e \quad (1.6.1)$$

Ampere's Law :

$$\text{curl}H = J \quad (1.6.2)$$

Faraday's law :

$$\text{curl}E = -\frac{\partial B}{\partial t} \quad (1.6.3)$$

Solenoidal property :

$$\text{div}B = 0 \quad (1.6.4)$$

Generalized Ohm's Law :

$$J = \sigma (E + v \times B) \quad (1.6.5)$$

where $B = \mu H$ and $D = \varepsilon_d E$

Here v denotes the velocity vector, H is the magnetic field, B is the magnetic induction, μ is the magnetic permeability, E is the electric field, J is current density, ρ_e is free electric charge density, D is the dielectric field, ε_d is the dielectric constant and σ is the electrical conductivity of the fluid.

Hartmann in 1937 analyzed the steady magnetic flux on moving liquid metal and realized thin boundary layers. Later in 1942, Alfven explained the dynamics of electrically conducting liquids indicated as magnetohydrodynamics or MHD. Hence the exploration of MHD was raised by Hennes Alfven, and Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism are together used to describe magnetohydrodynamics.

The basic theory beyond MHD is that, in a moving conductive fluid, subject to an external magnetic field, an electric current and an induced magnetic flux formed. The interaction of the original magnetic field and induced magnetic flux creates body force. More precisely, consider the flow of electrically conducting fluid with velocity v across the magnetic field of magnetic flux B . Thus a current density

$$J = \sigma (v \times B) \quad (1.6.6)$$

induced in this field and the interaction of this current with existing magnetic flux B produces electromagnetic body force,

$$F = J \times B \quad (1.6.7)$$

known as Lorentz force, where σ is the electrical conductivity.

The area that describes the dynamics of electrically conducting fluids in the presence of electric and magnetic fields is termed electromagnetohydrodynamics (EMHD). By generalized Ohm's law, the induced current density due to electric field E and magnetic field B is

$$J = \sigma (E + v \times B) \quad (1.6.8)$$

1.7 POROUS MEDIUM

The material consisting of a solid matrix with frequently distributed pores (void spaces) in it is a porous medium or a porous material. The void spaces allow the flow of fluid through these materials. The distribution of pores is irregular in a natural porous medium. How the voids are placed in the medium, their interrelation, shape, and location distinguish the porous medium. The porous medium is characterized by permeability and porosity.

1.7.1 Porosity

Porosity is the measure of pores in a medium. It is the ratio of the volume of voids to the total volume. That is porosity ϵ of a porous medium is

$$\epsilon = \frac{V_v}{V_T} \quad (1.7.1)$$

where V_v is the volume of fluid occupied in void space and V_T is the total volume of the medium (including the empty space in the medium).

1.7.2 Permeability

Permeability is the measure of the ability of a medium to transmit the fluid through it. Obviously, it is related to the size of void spaces distributed in the medium. The fluid motion in a porous medium is evaluated in terms of volume or average movement of the fluid elements over regions of space.

According to Darcy law, the flow through a porous medium is proportional to

the applied pressure gradient.

$$\delta p = -\frac{\mu}{K}u \quad (1.7.2)$$

where u is the Darcy velocity and K is the permeability of the material.

1.8 BASIC EQUATIONS

The fundamental principles of fluid dynamics are mathematically interpreted based on mass, momentum, and energy conservation laws; accordingly, the total mass, linear momentum, and energy of a closed system remain constant. The fluid flow is ruled by continuity, motion, and energy equations which are derived from the above-mentioned conservation laws.

1.8.1 Equation of continuity

The equation of continuity is reached from the law of mass conservation. Considering the fluid of density ρ flow with velocity $\vec{V}$, the continuity equation is given by

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{V}) = 0 \quad (1.8.1)$$

1.8.2 Navier –Stokes equation

The equation of motion also known as Navier-Stokes' equation or momentum equation obtained from the law of momentum conservation. Consider the fluid of density ρ flow with velocity $\vec{V}$, let $\vec{F}$ be the body force per unit mass, p is the pressure and μ is the coefficient of viscosity, the Navier-Stokes' equation is given by

$$\rho \left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right) = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{V} \quad (1.8.2)$$

1.8.3 Equation of energy

The energy equation is obtained from the law of energy conservation. Consider the fluid with density ρ , temperature T , then the equation of energy is given by,

$$\rho C_p \left(\frac{\partial T}{\partial t} + (\vec{V} \cdot \nabla) T \right) = k \frac{\partial}{\partial x_i} \left(\frac{\partial T}{\partial x_i} \right) \quad (1.8.3)$$

Where k -coefficient of thermal conductivity, C_p - specific heat

1.9 BOUNDARY LAYER

The elementary equations of hydrodynamics for inviscid fluid, developed by Euler were insufficient to explain flow separation, the pressure loss in a channel flow, and the drag force that occurs on a system moving in a fluid. Later Navier and Stokes modified Euler's equation for viscous fluids, which are known as Navier-Stokes equations, but it was difficult to make these equations analytically solved form. The boundary layer theory proposed by Ludwig Prandtl overcame the inability of Navier-Stokes equations and gives a physically strong explanation of the significance of viscosity in the evaluation of flow separation and frictional drag.

A thin layer neighboring the bounding surface, where velocity, temperature, and concentration gradients normal to the surface are important within this layer and negligible above this layer, is the boundary layer of a fluid flow. The significant classifications of boundary layers are:

1.9.1 Velocity Boundary Layer

Consider the fluid flow over a flat plate. The fluid particles have zero velocity when it attaches to the surface. These particles resist the movement of particles in the neighboring layer and the same process continues for the adjacent layers. This sequel becomes slight at a distance $y = \delta$ from the surface. This detain of fluid movement is due to the shear stress τ acting in planes that are collateral to the fluid velocity. As y values from the surface increase, the x - component of velocity u enhances and approaches the free stream velocity u_∞ .

Here δ is the value of y such that $u = 0.99 u_\infty$ and is called boundary layer

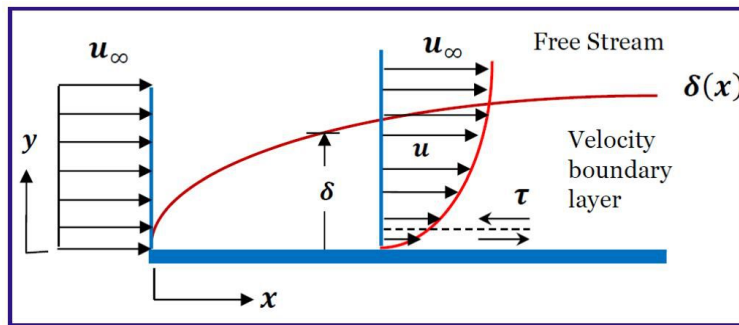


Figure 1.1: *Velocity boundary layer development on a flat plate*

thickness. The velocity boundary layer profile mentions that u differs with y through the boundary layer. The velocity gradient and shear stresses are greater within this

layer and negligible outside this layer. As reaching away from the leading edge, the influence of viscosity creeps into the free stream and the boundary layer raises.

1.9.2 Thermal Boundary Layer

The thermal boundary layer develops due to the temperature differences in the fluid-free stream and surface. Consider the fluid flow past an isothermal flat surface with temperature T_s . Let T_∞ be the uniform ambient temperature. The fluid particles lying close to the surface receive the temperature from the surface and interchange the thermal energy with the fluid particles in the adjacent layers and generate temperature gradients in the fluid. The region in which these temperature gradients exist is the thermal boundary layer. The thermal boundary layer thickness δ_t is the value of y such that $\frac{(T_s - T)}{(T_s - T_\infty)} = 0.99$. As reaching away from the leading edge, the influence of heat transfer creeps into the free stream and the thermal boundary layer grows.

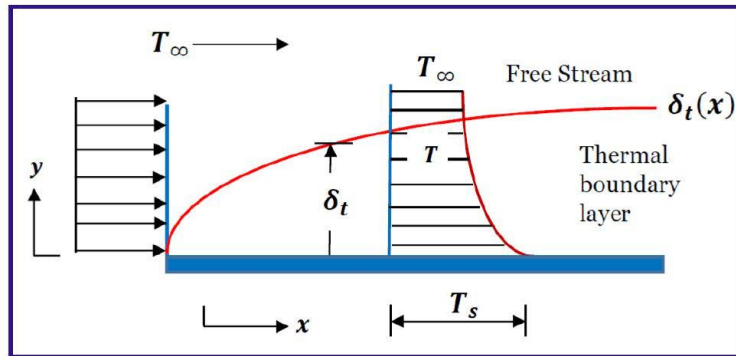


Figure 1.2: Thermal boundary layer development on a flat plate

1.9.3 Concentration Boundary Layer

The concentration boundary layer defines convection mass transfer. Consider the flow of fluid, which is the mixture of chemical species A and B, over a flat plate. Let $C_{(A,S)}$ and $C_{(A,\infty)}$ denote the concentration of species A at the surface and in the fluid-free stream respectively. If $C_{(A,S)}$ is differ from $C_{(A,\infty)}$, a concentration boundary layer generates. The region in which these concentration gradients exist is the concentration boundary layer. And the concentration boundary layer thickness δ_c is the value of y such that $\frac{(C_{(A,S)} - C_A)}{(C_{(A,S)} - C_{(A,\infty)})} = 0.99$.

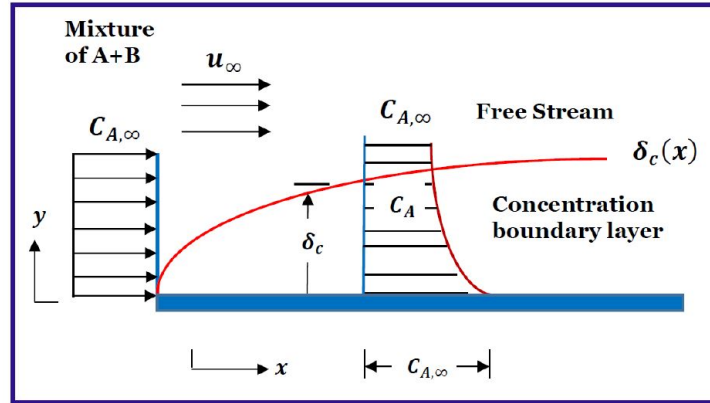


Figure 1.3: *Concentration boundary layer development on a flat plate*

1.10 PHYSICAL QUANTITIES

Physical quantities are used to evaluate friction and heat and mass transfer behavior between the boundary and the fluid. They are:

1.10.1 Drag Coefficient (C_f)

The drag coefficient is the proportion of skin shear stress at the wall to the dynamic pressure of a free stream. It measures the frictional force at the boundary between fluids and walls.

$$C_f = \frac{\tau_w}{\rho U^2} \quad (1.10.1)$$

where τ_w is the wall shear stress.

1.10.2 Nusselt Number (Nu)

It gives an idea about the movement of heat from a solid surface to a fluid. Nusselt number describes the correlation between thermal energy and thermal conductivity by the fluid movement and is given by,

$$Nu = \frac{q_w x}{k (T_w - T_\infty)} \quad (1.10.2)$$

where q_w -surface heat flux,

k -thermal conductivity,

x - the distance from the leading edge.

1.10.3 Sherwood Number (Sh)

It conveys the correlation between mass energy and mass diffusivity.

$$Sh = \frac{q_m x}{D_B (C_w - C_\infty)} \quad (1.10.3)$$

where q_m - Mass flux,

D_B -Mass diffusion coefficient

1.10.4 Microorganism Density Number (Nn)

It is the fraction of convective microbial transfer to the microorganism diffusion coefficient.

$$Nn = \frac{q_n x}{D_m (N_w - N_\infty)} \quad (1.10.4)$$

where q_n -Motile microorganism flux,

D_m -Microorganism diffusion coefficient

1.11 NON-DIMENSIONAL QUANTITIES

A research work attains its value when it is put into related implementations, and a successful solicitation is obtained after several experiments. Using geometrically similar small models instead of original full-scale objects reduces the cost of experiments. The dynamical similarity is significant here. Two fluid flows are dynamically similar if the values of the dimensionless parameters in both flows remain the same. The methods for obtaining the dimensionless parameters in a flow problem are inspectional analysis and dimensional analysis.

In the inspectional analysis method, the non-dimensional parameters are obtained by reducing the governing equations of the flow to a dimensionless form. It is possible, only if we have an entire set of equations that describe the flow.

An alternative method, with which the non-dimensional parameters may be formed from the physical quantities occurring in a flow problem is known as dimensional analysis. They are derived based on the dimensions in which each of the quantities involved in a phenomenon is expressed, and hence, must not depend on the units chosen for the calculations. The advantage of dimensional analysis is that the equations are not necessary and one is rewarded according to insight and cleverness.

The important non-dimensional parameters are:

1.11.1 Prandtl number (Pr)

The proportion of momentum diffusivity to the thermal diffusivity of the fluid is referred to Prandtl number.

$$Pr = \frac{\mu C_p}{k} = \frac{\vartheta}{\alpha} \quad (1.11.1)$$

It measures the significance of thermal conduction and viscosity of the fluid. Higher values of Pr quantify the greater viscosity and lower thermal conductivity of the fluid. Ethylene glycol is a fluid with higher Pr .

1.11.2 Schmidt number (Sc)

It is the ratio of kinematic viscosity (ϑ) to the mass diffusivity D_B . That is

$$Sc = \frac{\vartheta}{D_B} \quad (1.11.2)$$

Sc correlates the relative hydromagnetic and mass transfer boundary layer thickness. And greater Sc values constitute lower chemical molecular diffusivity.

1.11.3 Eckert number (Ec)

It is significant to regulate the correlation between kinetic energy and boundary layer enthalpy difference in the fluid system. Also, it narrates the heat dissipation. It is defined as

$$Ec = \frac{U^2}{C_p (T_w - T_\infty)} \quad (1.11.3)$$

1.11.4 Reynolds number (Re)

It is the most significant non-dimensional parameter in fluid dynamics which is supportive to identify a laminar or turbulent flow. A Higher Reynolds number specifies the domination of turbulent flow over the laminar flow. Reynolds number is defined as

$$Re = \frac{\text{Inertia force}}{\text{Viscous force}} = \frac{UL}{\vartheta} \quad (1.11.4)$$

1.11.5 Grashof number (Gr)

It measures the correlative significance of buoyancy force to the viscous force and specifies free convection. Grashof number is defined by,

$$Gr = \frac{gL^3(T_w - T_\infty)}{\nu^2 T_\infty} \quad (1.11.5)$$

1.11.6 Modified Grashof number (Gm)

Similar to Gr , it occurs in the free convection problem, when the mass transfer is considered. And is defined as

$$Gm = \frac{gL^3(C_w - C_\infty)}{\nu^2 C_\infty} \quad (1.11.6)$$

1.11.7 Magnetic field parameter(H)

It represents the correlation between electromagnetic force and viscous force. It is defined as

$$H = \sqrt{\frac{\sigma B_0^2 L^2}{\mu}} \quad (1.11.7)$$

Hartmann introduced this term, which is also known as the Hartmann number. The Lorentz force is generated at higher H values.

1.11.8 Lewis number (Le)

The Lewis number is defined as the ratio of thermal diffusivity and mass (molecular) diffusivity and it plays an important role in the simultaneous effect of heat and mass transfer processes. The Lewis Number is also the ratio of the Schmidt Number and Prandtl Number and it may also be expressed as

$$Le = \frac{\alpha}{D_B} = \frac{\nu/D_B}{\nu/\alpha} = \frac{Sc}{Pr} \quad (1.11.8)$$

1.12 NANOFUIDS

The colloidal suspension of ultrafine particles (1-100 nm) with a base fluid is termed nanofluid. The nanoparticles used in nanofluids are typically made of metals (like Ag, Ti, Cu, Fe , etc), oxides (like Fe_3O_4, Al_2O_3, TiO_2 , etc), carbides (like CaC_2, SiC , etc), or carbon nanotubes, while the base fluids include water, oil, and

ethylene glycol.

Nanoparticles possess distinct physical properties based on shape, size, dispersion state, crystallinity, and surface properties, and hence possess diverse applications in various fields. For example, potassium, silicon, iron, silver, zinc oxide, etc. are the nanoparticles employed in agricultural fields. Also, zinc oxide, silver, titanium oxide, gold, etc. are employed in cosmetics.

However, the study of nanofluids overcomes the limited thermal conductivity of regular fluids, hence the researchers focus their attention on hybrid nanofluids, the modern class of nanofluids. The colloidal suspension of two or more nanomaterials with a base fluid is termed as hybrid nanofluid. The collaborative effect of nanoparticles can improve the thermal properties of the fluid. Water-based $Al_2O_3 - Fe_3O_4$, $CNT - Fe_3O_4$, $TiO_2 - Ag$ hybrid nanofluids analyzed in this thesis.

1.12.1 Thermal properties of nanofluid

Thermophysical characteristic is an important factor to be considered in choosing nanofluids for heat transfer implementations. The effective thermophysical properties of the nanofluids are mixed properties of the base fluid and solid particles. Nanofluids are not a simple mixture of nanoparticles to the base fluid. Its thermophysical properties depend on, the methods used to prepare nanofluids and under what conditions they are prepared. Tiwari and Das developed a simpler nanofluid model considering the effective thermophysical properties of a nanofluid.

Effective Dynamic Viscosity:

$$\frac{\mu_{nf}}{\mu_f} = \frac{1}{(1 - \phi)^{2.5}} \quad (1.12.1)$$

Effective Density:

$$\frac{\rho_{nf}}{\rho_f} = (1 - \phi) + \phi \left(\frac{\rho_s}{\rho_f} \right) \quad (1.12.2)$$

Effective Electrical Conductivity:

$$\frac{\sigma_{nf}}{\sigma_f} = 1 + \frac{3 \left(\frac{\sigma_s}{\sigma_f} - 1 \right) \phi}{\left(\frac{\sigma_s}{\sigma_f} + 2 \right) - \left(\frac{\sigma_s}{\sigma_f} - 1 \right) \phi} \quad (1.12.3)$$

Effective Coefficient Of Thermal Expansion:

$$\frac{\beta_{nf}}{\beta_f} = (1 - \phi) + \phi \left(\frac{\beta_s}{\beta_f} \right) \quad (1.12.4)$$

Effective Specific Heat:

$$\frac{(\rho C_p)_{nf}}{(\rho C_p)_f} = (1 - \phi) + \phi \left(\frac{(\rho C_p)_s}{(\rho C_p)_f} \right) \quad (1.12.5)$$

Effective Thermal Conductivity:

$$\frac{K_{nf}}{K_f} = \frac{K_s + 2K_f - 2\phi(K_f - K_s)}{K_s + 2K_f + 2\phi(K_f - K_s)} \quad (1.12.6)$$

If P is a property, P_f, P_s , and P_{nf} correspond to, the property of the base fluid, solid particles, and nanofluid.

$$\phi = \frac{V_{np}}{V_f} \quad (1.12.7)$$

is the volume fraction of nanoparticles, where V_{np} and V_f denote the nanoparticle volume and base fluid volume respectively.

1.13 METHODOLOGY

The mathematical tools utilized to find the numerical solutions for the non-linear partial differential equations representing the governing flow are:

- Perturbation Technique
- Matlab bvp4c and bvp5c solver of the finite difference scheme

1.13.1 Perturbation Technique

The non-steady boundary layer problems essentially entail a steady flow, but with a minute non-steady disturbance. As this disturbance (perturbation) is supposed to be small compared with the steady flow, for the steady perturbation, the equations can be split into non-linear equations.

The necessary suggestion is that the formulation has a small parameter ε , in either the governing equation or in the boundary conditions. In a flow having a high Reynolds number $\varepsilon = \frac{1}{Re}$, in a creeping flow $\varepsilon = Re$, and in the flow around an airfoil ε is the ratio of thickness to chord length. In the perturbation technique, the solutions are assumed to be a series of ε , in which the higher order terms act as a perturbation on the lower order terms.

$$f(x, y, t) = f_0(x, y) + \sum_{i=1}^{\infty} \varepsilon^i f_i(x, y, t) \quad (1.13.1)$$

When this assumed solution is substituted in the governing flow problem, and the resulting terms are ordered by the powers of ε , a set of differential equations are obtained. All the sets of differential equations except zeroth order will be linear and can be solved.

1.13.2 **bvp4c and bvp5c**

bvp4c and bvp5c are computing programs advanced by MATLAB software to solve boundary value problems. It can be used to solve a large class of two-point boundary value problems of first-order ordinary differential equations, which is in general form:

$$y'(x) = f(x, y(x), p) \quad (1.13.2)$$

subject to the boundary conditions,

$$g(y(a), y(b), p) = 0 \quad (1.13.3)$$

where f is a continuous function in y , p is a vector of unknown parameters, and $a \leq x \leq b$. bvp4c and bvp5c are executed in the same way, but with different error tolerances. These codes have only three arguments: a function odes for evaluating ordinary differential equations, a function bcs for evaluating the residual in the boundary conditions, and a structure solinit that delivers a guess on a mesh and the solution on this mesh.

As the boundary conditions are coded $g(y(a), y(b)) = 0$, instantly $y(a)$ is approximated as ya , and $y(b)$ is approximated as yb by the function bcs, which assesses and gives back the residual $g(ya, yb)$. When solving a boundary value problem, a hypothesis of solution should be given to identify and compute the solution. The bvp4c or bvp5c solver assesses the hypothesis as a structure generated by the auxiliary function bvpint. The initial assertion of the function bvpint is a hypothesis for a mesh that discloses the nature of the solution. The next assertion is a hypothesis for the solution on the defined mesh, in which the solution contains a pair of elements $y(x)$ and $y'(x)$. The hypothesis is given in two ways, the hypothesis for $y(x)$ has an accurate shape and meets the boundary conditions or considers all the approximated solution elements to be constants and provides the vector of these constants immediately. The latter is the convenient way and often works.

The base stone of the bvp4c or bvp5c is the notion of residual, which influence the error control and mesh selection. The fundamental method to find a solution is centered on polynomial collocation with four Lobatto points for bvp4c and five Lobatto points in the case of bvp5c. The amount to be approximated and managed is residual for bvp4c and residual and error for bvp5c.

1.14 STATISTICAL ANALYSIS

The further scrutinization of numerical results obtained by the above-mentioned methods utilizing statistical techniques has a significant role in the research world due to its efficiency in producing accurate quantitative results. The statistical techniques measure the interconnection of the parameters which cannot be determined using the conventional numerical procedure. Some important statistical techniques used in this thesis are:

1.14.1 Correlation

Correlation explains the dimension and direction of interconnection with two or more variables. The correlation coefficient delineates the nature of the relationship between an independent and a dependent variable, ranging between -1 and 1. The sign of the correlation coefficient represents the nature of dependency of the independent variable on the dependent variable. If the independent variable isn't related to the dependent variable by any means then the correlation coefficient returns the value 0. Let $x_1, x_2, \dots, x_n$ are independent variables and $y_1, y_2, \dots, y_n$ are corresponding dependent variables, then the correlation coefficient is

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}} \quad (1.14.1)$$

where $\bar{x}$ and $\bar{y}$ are means of $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ respectively.

1.14.2 Probable error

Probable error (PE) is a strong statistical measure to check the reliability of the calculated coefficient. It is measured as

$$PE = \left(\frac{1 - r^2}{\sqrt{n}} \right) 0.6745 \quad (1.14.2)$$

where n denotes the number of observations. The correlation is said to be significant if $\left| \frac{r}{PE} \right| > 6$.

1.14.3 Regression analysis

Regression analysis helps to recognize the parameters, which are influencing a quantity of interest. It determines which parameters are most effective, which parameters can be neglected, and also how these variables are affected altogether. In regression analysis, a model is developed that is expressed by a function and determines a connection between a dependent variable and an independent variable is the regression model. Out of different regression models, the linear model is considered to be the best model, in which linear predictor functions are used to model the relationships.

1.14.4 Response Surface Methodology (RSM)

RSM is a statistical technique based on an experimental design that measures the mutual impact of effectual parameters (independent variables) on the response variable (dependent variable). Experimental trials are costly and time-consuming whereas RSM reduces the number of trials and optimizes the response variable. RSM estimates a model for the quantity of interest using the Central Composite Design (CCD) experimental data. RSM using CCD is a suitable sequential experimentation method which is incorporated in the present study with 20 ($2^j + 2j + o$, $j = 3, o = 6$) runs, here o denotes the number of faces and j denotes the number of factors. The relation involves 2^3 factorial 6 center and 6 axial points. A full quadratic 3-level factorial design is adopted in the study as follows

$$\begin{aligned} \text{Response } (Y) = & L_1X_1 + L_2X_2 + L_3X_3 + L_4X_1X_2 + L_5X_2X_3 + L_6X_1X_3 \\ & + L_7X_1^2 + L_8X_2^2 + L_9X_3^2 + L_{10}. \end{aligned} \quad (1.14.3)$$

Here $L_i (i = 1, 2, \dots, 10)$ denote the regression coefficients. The RSM technique becomes advantageous in finding the levels of the parameters that optimize the response. ANOVA table evaluates the quality of the fitted quadratic model in a reliable way. The significance of linear, quadratic, and interactive regression terms are computed statistically using F and p values. The regression terms in the fitted model are significant for high value of F and if $p < 0.05$.

1.14.5 Sensitivity analysis

Sensitivity analysis estimates the range and nature of dependency indicated by the parameters on the targeted variable. In other words, sensitivity analysis accounts for the variation induced by the augmenting parameter on the remaining influential parameter. The sign of sensitivity indicates the nature of the relationship between the targeted variable and the effectual parameters, and the magnitude of sensitivity indicates the intensity of the effect on the quantity of interest.

1.15 OBJECTIVES

The main objective of this research work are:

- To characterize the flow properties of nanofluids in various situations.
- To study the impact of some external effects on flow properties and physical quantities of MHD nanofluid flow.
- To Ensure the reliability of the theoretical investigations regarding the physical quantities of choice with statistical techniques.
- To explore the applications of nanofluids in bio-medical and industrial field.

1.16 REVIEW OF RELEVANT LITERATURE

1.16.1 NANOFLUID

The dynamic with regular heat transfer fluids faces a lack of thermal conductivity, which is a major drawback in engineering applications. In industries such as manufacturing, microelectronics, transport, and defense, the cooling of machinery and equipment is a major issue. The conventional thermal transport fluids such as water, oil, etc. are not sufficient to meet these cooling needs, due to their low ability to conduct heat. Even though many mechanisms have been suggested to enhance the heat transfer ability of regular fluids, the technique of adding ultrafine solid particles in a conventional fluid, introduced by Choi and Eastman is a good solution to the existing trouble. Nanofluids have a significant role to upgrade and stabilize thermophysical properties such as thermal diffusivity, thermal conductivity, convective heat transfer coefficients, and viscosity of the fluid which boosted its

acceptance. Gradually the researchers explored the influence of nanofluids in renewable energy (solar thermoelectric devices, solar collector, biomass, and geothermal), biomedical engineering (chromatography, neuro electronic links, Vivo therapy, photodynamic therapy, and drug delivery), commercial production (plastics, ceramics, metals, paints), and food processing.

Nanofluids have been a hot topic for a while now. Using nanofluids instead of the normal base fluids has yielded many awarding results. It is seen that nanofluids have better convective heat transfer capabilities relevant in industrial, engineering and medical fields. Food processing, nuclear power plants, heating or cooling, nanomagnet switches, separate oil and gas in the reservoir, solar collectors, and electronic devices are some of the industrial and engineering applications. Blood diagnostics, antibacterial, photodynamic imaging and therapy, drug loading and delivery, biosensors and cancer therapy are some significant implementations of nanofluid in biotechnology.

(Choi & Eastman, 1995) first introduced an innovative class of fluids, the nanofluid, developed by suspending metallic nanoparticles to conventional fluids. (Eastman, 1999) studied the thermal properties of nanofluids. (Lee, Choi, Li, & Eastman, 1999) generated oxide nanofluids and investigated their thermal characteristics. They noticed that nanofluids with a minute amount of nanoparticles possess comparatively larger thermal conductivity. (Xuan & Li, 2000) initiated the theoretical evaluation of the thermal properties of nanofluid. They also scrutinized the convective heat transfer and properties of nanofluid flow in a tube (Xuan & Li, 2003). (Gosselin & da Silva, 2004) highlights the significance of, optimizing the thermal properties of the free and forced convective nanofluid flow. (Akbarinia, 2008) explored the drag factor and heat transfer rate due to nanofluid flow along a pipe. (Hamad & Bashir, 2009) introduced the boundary layer flow of a non-Newtonian fluid past a stretching surface. (Abu-Nada & Eiyad, 2010) reviewed the thermal conductivity and viscosity effects of nanofluid flow under natural convection.

Numerous scientists focus their attention on nanofluids which revolutionized the field of fluid dynamics and the studies are still going on. (Kulkarni & Das, 2012) analyzed properties of nanofluids theoretically and experimentally. (Murshed & De Castro, 2014) explain the methodology for developing nanofluids, their characteristics, and principles to employ nanofluids in important areas. (Kiyani, Hayat, Ahmad,

Waqas, & Alsaedi, 2021) semi-analytically investigated the bidirectional Williamson nanofluid flow in porous space and observed that space and temperature-dependent heat sources have a positive effect on temperature. (Kumar, Bhattacharyya, Seth, & Chamkha, 2021) studied magnetite water-based nanofluid flow over a rotating disk in the presence of an external magnetic field and Arrhenius activation energy and noted that the thermophoresis parameter hurts heat transfer rate. (Hazarika, Ahmed, & Chamkha, 2021) used the fourth-order RK-shooting technique to numerically investigate the MHD flow of a chemically reacting water-based nanofluid over a permeable stretching sheet involving chemical reaction, thermophoresis, heat source, and viscous dissipation. (Seth, Bhattacharyya, Kumar, & Chamkha, 2018) examined the unsteady hydromagnetic boundary layer flow of a thermally radiating nanofluid considering Navier's velocity slip and external magnetic field past a non-linear stretching sheet.

The thermophysical properties of a nanofluid have a significant role in analyzing nanofluid flow and improving its potential utilities. (Tiwari & Das, 2007) developed a nanofluid model to characterize the behavior of nanofluids by considering the solid volume fraction of nanoparticles. (Roşca & Pop, 2017) used Tiwari Das nanofluid model to study stagnation point flow over a stretching or shrinking surface. (Eid & Nafe, 2022) employed Tiwari and Das nanofluid model to analyze the MHD nanofluid flow through an exponentially stretching surface.

1.16.2 HYBRID NANOFLUID

The latest nanotechnology research works focus on finding practices that help in boosting the efficiency and transfer properties of the considered nanoliquid. This is where hybrid nanoliquid (colloidal suspension of two or more nanomaterials) comes into the scene. The collaborative effect alters the nanoliquid's heat transfer rate proving to be beneficial in many engineering and industrial fields (like solar energy systems, car radiators, nuclear system cooling, micromanufacturing processes, etc.).

(Jana, Salehi-Khojin, & Zhong, 2007) were the first to experimentally analyze hybrid nanofluid in 2007. (Sundar, Singh, & Sousa, 2014) investigated hybrid nanofluid's friction factor and convective heat transfer coefficient. (Suresh, Venkataraj, Selvakumar, & Chandrasekar, 2011) investigated the thermophysical properties of water-based $Al_2O_3 - Cu$ hybrid composite and observed that viscosity

and thermal conductivity of the prepared composite can be enhanced by adding nanoparticles. Heat transfer aspects of regular and hybrid nanofluid flow over a stretching sheet numerically analyzed by (Devi & Devi, 2016). (Waini, Ishak, & Pop, 2019b) characterized hybrid nanofluid flow past a stretching or shrinking sheet. (Manjunatha, Kuttan, Jayanthi, Chamkha, & Gireesha, 2019) explored the impact of magnetic field and variable viscosity in improving the heat transfer ability of hybrid nanofluid flow.

(Junoh, Ali, Arifin, Bachok, & Pop, 2020) numerically explored the consequence of induced magnetic field (IMF) on the heat transfer and hydromagnetic stagnation point flow over a lengthening and shortening sheet and revealed that the heat transfer rate was higher for the hybrid nanoliquid. (Acharya & Mabood, 2021) employed the fourth-order Runge-Kutta method to numerically inspect the hybrid nanoliquid flow over a slippery permeable bended structure. They perceived that the hybrid nanoliquid exhibits lower drag coefficient and higher Nusselt number. (Waini, Ishak, & Pop, 2019a) explored the two-dimensional steady flow of water-based $Al_2O_3 - Cu$ hybrid nanoliquid over a nonuniformly shrinking/stretching permeable surface. (Ali, Asjad, & Akgül, 2021) remarked that the water-based hybrid nanoliquid showcased improved nanofluid temperature and velocity when compared with engine oil-based nanofluid.

(Khashi'ie, Arifin, Pop, & Nazar, 2021) examined the influence of gyrotactic microorganisms in hybrid nanofluids and distinguished higher heat and mass transfer rates and motile density in hybrid nanofluid compared with respective regular nanofluids. (Shahsavar, Saghafian, M.R.Salimpour, & Shafii, 2016) studied the thermal conductivity and viscosity of the hybrid nanofluid and pointed out that hybrid nanofluids act as shear thinning fluid. They concluded among nanoparticles, Carbon nanotubes provide excellent thermal, chemical, mechanical, and electrical properties. Owing to the capability of moving through various body tissues and probe into cells *CNTs* are auspicious nanostructures and used in therapeutic drug delivery system is its most desirable (Masotti & Caporali, 2013)(Bhirde et al., 2009). Because of the immense thermal conductivity of *CNT* and the strong magnetic medium of Fe_3O_4 , their combination assured an amazing future in nanoscience. (Kaiser, Buerki-Thurnherr, & Wick, 2013) noticed that without being affected by all provisions single wall carbon nanotubes can be in contact with organisms and

perchance absorbed by them and also observed that the reduced cell adhesion of cells exposed to *SWCNT*, have no major side effects on cell functions. (Raza et al., 2020) analyzed the refinement of heat transfer in fluid flow with different *CNTs*. For cooling purposes in industries combination of *CNT* and Fe_3O_4 nanoparticles are very useful (Mohebbi, Izadi, Delouei, & Sajjadi, 2019). Compared with *CNT*– H_2O nanofluid, *CNT*– Fe_3O_4 hybrid nanofluid with H_2O as base fluid possess higher thermal conductivity and more drag coefficient. Moreover, *CNT*– Fe_3O_4 hybrid nanofluid is a promising platform for magnetically engaged anti-cancer drug delivery (Fan, Jiao, Gao, Jin, & Li, 2013). However the instability of nanofluid, a limitation of applications in bio-science overcome by adding microorganisms.

Titanium dioxide (TiO_2) is an effective photocatalyst due to its strong oxidizing power, non-toxicity, and long-term chemical and physical stability. TiO_2 nanofluids provide various applications in energy systems. With the advancement of heat and mass transfer processes, the sector of application of TiO_2 nanofluids expands to the fields of heat pipes, solar collectors, energy storage refrigeration, and other energy applications. The thermal conductivity of the TiO_2 nanofluids can be affected by the ingredients of the base fluids. (Chen, Witharana, Jin, Kim, & Ding, 2009) found that the effective thermal conductivity of TiO_2 nanoparticles with ethylene glycol as base fluids are higher than that with water as a base fluid. (Duangthongsuk & Wongwises, 2009) investigated the heat transfer coefficient and friction factor of water-based TiO_2 nanofluid within a horizontal double-tube counter-current heat exchanger under turbulent flow conditions. They found that the nanofluid heat transfer coefficient depends on the nanofluid, the nanofluid temperature, and the mass flow rate of the hot water. (Reddy & Chamkha, 2016) made a comparison study of MHD convective flow of TiO_2 -water and Al_2O_3 -water nanofluids. (Acharya, Das, & Kumar Kundu, 2016) studied the effect of variable thickness on the steady two-dimensional boundary layer flows of a TiO_2 -water and *Ag*-water nanofluid past a slendering stretching sheet.

Silver nanoparticles are stable, long-lasting, and subject to controlled release. Silver-doped materials are chemically durable and release silver ions for a long time period. (Berger, Spadaro, Chapin, & Becker, 1976) observed that silver solutions and silver metal act as powerful antimicrobial agents for centuries, owing to a broad spectrum of antibacterial activity as well as low toxicity towards cells. Several stud-

ies (Dai & Bruening, 2002), (Yuranova et al., 2006) have been reported to explain the indirect effect of silver on bacteria. Silver nanoparticles perform as an excellent antibacterial coating in the food industry, water disinfection, and other disinfection-related fields. (Zhang et al., 2003) discussed the significance of combining silver and TiO_2 . it is observed that 2-4 nano meter-sized silver nanoclusters are strongly anchored to the TiO_2 nanoparticles with high dispersion, and also, suspending silver nanoparticles improves the photocatalytic and bactericidal activities of TiO_2 .

1.16.3 MAGNETOHYDRODYNAMICS

Magnetohydrodynamics (MHD), the area that deals with the study of dynamics of electrically conducting fluids under the influence of magnetic field has raised quite an interest over the years due to its versatile number of application in various fields; like in geophysics, engineering, biomedical engineering, magnetic drug targeting and many others. Magnetic drug targeting in cancer therapy, which involves establishing more explicit ways to transport cancer-killing drugs to affected areas without harm to surrounding tissues, is one of the significant implementations of magnetohydrodynamics.

(Williams, 1930) described the electromotive forces developed due to a magnetic field in a moving fluid. (Greenspan & Carrier, 1959) studied the MHD steady flow of a viscous incompressible electrically conducting fluid. (Van Blerkom, 1960) mathematically analyzed the MHD viscous fluid flow over a sphere. (Soundalgekar & Murty, 1980) numerically analyzed the thermal properties of magnetohydrodynamic flow with injection, suction, and pressure gradient. (Tezer-Sezgin & Köksal, 1989) employed the finite element method to explore the velocity and induced magnetic field of MHD steady flow through a rectangular pipe and observed that wall conductivity has an inverse proportion with flux. (Ishak, Nazar, & Pop, 2008) conducted a numerical study on the thermal transfer of magnetohydrodynamic flow due to a stretching cylinder. (Hayat, Ahmed, Sajid, & Asghar, 2007) employed homotopy analysis to study second grade fluid flow through a porous channel in presence of the magnetic field.

MHD is commonly paired with convective flows which can either be natural, forced or mixed. Natural or free convection is the type of flow where the motion is not generated by an external source. Natural Convection has called in a lot of

attention from researchers with its applications ranging from engineering to nature. (Jha & Aina, 2016) theoretically explored influence of IMF on free convective flow in a vertical microchannel and reported that augmenting magnetic Prandtl number and Hartmann number causes a decrease in volume flow rate. (Dash & Ojha, 2018) discussed about the MHD viscoelastic fluid flow betwixt two infinite horizontal permeable plates involving sinusoidal pressure gradient and noted a decline in velocity profile on amplifying Hartmann number.

1.16.4 ELECTROMAGNETOHYDRODYNAMICS (EMHD)

The realm that deals with the dynamics of electrically conducting fluids under the sway of the electric and magnetic fields are termed electromagnetohydrodynamics (EMHD). Numerous learners have prospected the significance of EMHD due to its relevancy in geophysics, micropumps, magnetic drug targeting, microscale devices, and biomedical engineering. (Liu, Jian, & Tan, 2018) conducted an entropy generation analysis of EMHD flow in a curved rectangular microchannel. They perceived that the electric field parameter had a positive impact on the entropy generation rate. An investigation on the EMHD behavior of a third-grade fluid modeled using the Darcy-Brinkman-Forchheimer model was conducted by (Zhang, Bhatti, & Michaelides, 2020). (Zainal, Nazar, Naganthran, & Pop, 2021) reviewed the unsteady EMHD flow of water-based $Al_2O_3 - Cu$ hybrid nanofluid. They detected an exaggeration in the heat transfer characteristics due to the electric and magnetic parameters. (Shah, Bonyah, Islam, & Gul, 2019) investigated the EMHD rotational flow of kerosene oil-based CNTs nanofluid over a stretching sheet and observed an ascending nature of velocity with electric field parameters. (Daniel, Aziz, Ismail, Bahar, & Salah, 2019) numerically examined the EMHD convective nanofluid flow past a permeable stretching sheet imposed with stratification. They concluded that electric and magnetic field parameters exhibit an opposite flow behavior to velocity and temperature. (Abbas, Hayat, Ayub, Bhatti, & Alsaedi, 2019) analyzed the impact of the electrical and magnetic field in nanofluid flow suspended with gyrotactic microorganisms past a porous Riga plate.

1.16.5 POROUS MEDIUM

A porous plate corresponds with a plate having frequently distributed void spaces in it. They are found to be beneficial in chemical engineering (for filtration and clar-

ification), agricultural engineering, (in the study of underwater resources), and the petroleum industry (to study the movement of natural gas, oil and water). (Muskat, 1937) was among the first to learn about the fluid flow past a porous media mathematically. (Lapwood, 1948) observed the occurrence of convective fluid flow through a porous medium. (Fulks, Guenther, & Roetman, 1971) discussed and derived conservation laws for the fluid flow through a porous medium. (Vafai & Thiyagaraja, 1987) studied the heat transfer and flow properties of fluid flow through the regions connecting two distinct porous media, a porous medium and fluid area, and a porous medium and an impermeable medium. (Patil & Kulkarni, 2008) investigated the impact of chemical reaction and heat source on a free convective flow past a porous medium.

(Nayak, Akbar, Tripathi, & Pandey, 2017) numerically analyzed the three dimensional hydromagnetic nanoliquid flow through an exponentially lengthening porous sheet with the aid of the fourth order Runge-Kutta method. They noticed a rise in velocity with augmenting porosity parameter values. (Das, Tarafdar, & Jana, 2018) inspected the impact of the transverse magnetic field, slip condition and Hall current on an unsteady hydromagnetic rotating flow over a periodically accelerated horizontal porous plate. They noted that increasing Hall current has a positive effect on the velocity profile close to the plate and a negative effect on the velocity profile away from the plate. (Biswal, Chakraverty, Ojha, & Hussein, 2021) elucidated the influence of the transverse magnetic effect on the flow of water based fluid with added silver and copper nanoparticles in a semi-porous channel utilizing the least square method.

1.16.6 BIOCONVECTION

The upward-directed swimming of microorganisms in a suspension due to its lower density than water forms an unbalanced dense surface layer that initiates the bioconvective flow. The presence of microorganisms enhances the stability of the fluid that plays a significant role in biotechnology, bio-microsystems, and bio-nano coolant systems. (Kuznetsov & Avramenko, 2004) were among the first few to investigate the stability of a suspension containing microorganisms and small particles. (Uddin, Khan, Qureshi, & Anwar Bég, 2017) employed the Runge-Kutta-Fehlberg method to numerically survey the impact of thermal and hydrodynamic slip constraints on

the water-based bio-nanomaterial containing microorganisms. They observed that the augmenting bioconvection Peclet number tends to improve the microorganism density number. The consequence of multiple slips on bioconvective flow was studied by (Alshomrani, Ullah, & Baleanu, 2020). (Bhatti, Shahid, Abbas, Alamri, & Ellahi, 2020) studied the role of activation energy on a suspension of nanoparticles and microorganisms in a magnetized fluid with the aid of successive local linearization methods.

(Pal & Mondal, 2018a) elucidated the bioconvective MHD nanofluid flow past an exponentially stretching surface and observed a decline in the motile density due to augmenting values of microorganism concentration difference parameter and bioconvection Peclet number. The relevance of three-dimensional bioconvective flow due to an exponentially stretching has been explored by (Alqarni, Waqas, Alghamdi, & Muhammad, 2022). They noticed that the microbial concentration ascends with microorganism Biot number but descends with Peclet number. (M. Khan, Salahuddin, Malik, Alqarni, & Alqahtani, 2020) and (Shafiq, Sindhu, & Khalique, 2020) numerically analyzed the bioconvective tangent hyperbolic nanofluid flow past an exponentially stretching sheets.

1.16.7 VISCOUS DISSIPATION

Viscosity is the resistance experienced by a fluid to a change in shape, or movement of neighboring portions relative to one another. It plays an important role in chemical engineering, polymer industry and lubrication processes. The viscous dissipation effect serves a major role in geophysical flows, industrial applications, and aerodynamic heating. (Lund, Omar, Raza, & Khan, 2021) illustrated the dual solutions for dissipative MHD hybrid nanofluid flow past a stretching/shrinking surface and noticed an escalation in the temperature with augmenting values of the Eckert number. (Azhar, Iqbal, & Maraj, 2019) implemented a fractional approach to investigate the consequence of viscous dissipation on the stagnation-point Jeffery fluid flow. The characteristic of viscous dissipation on micropolar fluid flow over a nonlinear stretched sheet has been elucidated by (Patel & Singh, 2019). It is observed that the Eckert number accelerates fluid flow and heat transfer process. (Hazarika et al., 2021) discussed the impact of viscous dissipation on water-based nanofluids over stretching surfaces in the presence of chemical reaction.

1.16.8 STRATIFICATION

The formation of layers due to the variations in mass, heat, and motile density profiles or the presence of discrete fluids is called stratification. (Yang, Novotny, & Cheng, 1972) studied the natural convection flow from a vertical flat plate, submerged in a temperature stratified environment. The consequence of stratification effects on hydromagnetic mixed convective nanoliquid flow was investigated by (Alsaedi, Khan, Farooq, Gull, & Hayat, 2017) with the aid of the homotopy analysis method (HAM). They observed a drop in the nanoliquid temperature with a hike in the thermal stratification parameter. (Ahmad, Nadeem, Muhammad, & Issakhov, 2020) elucidated the effect of double stratification on carbon nanotube (*CNT*) nanofluid flow and detected a decrease in the nanofluid temperature and volume fraction concerning thermal and solutal stratification parameters, respectively. (Ramzan, Riasat, Shah, Kumam, & Thounthong, 2020) examined the impact of thermal stratification parameter on unsteady ethylene glycol-based nanoliquid flow. The stratification effects on bioconvective stretched Maxwell nanofluid flow analyzed by (Ijaz Khan, Waqas, Hayat, Imran Khan, & Alsaedi, 2017) and observed a reduction in concentration and temperature profiles by the respective stratification parameters. (Naz, Tariq, & Alsulami, 2020) studied Walter's B nanofluid stratified flow consisting of swimming microorganisms.

1.16.9 THERMAL RADIATION

The phenomenon of thermal radiation has crucial importance in space technology and intense heat processes such as nuclear power plants, gas turbines, gas-cooled nuclear reactors, and space vehicles. (Viskanta & Grosh, 1962) are concerned about the impact of thermal radiation on a boundary layer flow. (Chamkha, 2000) studied the radiative hydromagnetic flow over a permeable surface. (Gireesha, Umeshaiah, Prasannakumara, Shashikumar, & Archana, 2020) executed the behavior of radiation in MHD three-dimensional Jeffrey fluid flow through a stretching sheet and observed an increment in fluid temperature and nanoparticle concentration with the changes in the radiation parameter. (Nasir et al., 2018) implemented the homotopy analysis method (HAM) to address thermal radiation and thermophoresis effects on the three-dimensional rotating nanofluid flow containing single-wall carbon nanotubes (SWCNT). They noticed that the nanofluid temperature is an expanding

function of radiation parameter. The significant influence of radiation parameters on heat and mass transfer rate is considered by (Besthapu, Haq, Bandari, & Al-Mdallal, 2019). (Gireesha, Archana, Prasannakumara, Gorla, & Makinde, 2017) and (Ramesh, Prasannakumara, Gireesha, Shehzad, & Abbasi, 2017) analyzed the radiative Casson nanofluid and Maxwell fluid flows due to a stretching surface, respectively. Further, the micropolar fluid flow due to a stretching surface considering radiation effect has been studied by (Anantha Kumar, Sugunamma, & Sandeep, 2019).

1.16.10 HEAT SOURCE

Convection induced by internal heat sources has been broadly studied because of its wide range of applications in astrophysics, geophysics, fire and combustion modeling, thermal ignition, miniaturization of electronic components, etc. In such flows, the buoyancy force is incremented due to the heat source resulting in modification of heat/mass transfer characteristics. Heat sources explore the hydro-thermal integrity of the flow. (Noghrehabadi, Saffarian, Pourrajab, & Ghalambaz, 2013) conducted an entropy analysis for nanofluid flow past a stretching sheet in the presence of heat generation/absorption and partial slip. it is analyzed that the increase of the heat generation parameter reduces the entropy generation number in the vicinity of the sheet. (2017) theoretically explored the MHD Oldroyd-B nanofluid flow with the heat source or sink induced by a stretching sheet. (Saba et al., 2018) probed the heat transfer analysis of nanofluid flow over a curved stretching surface under the influence of a heat source and a reduction in the magnitude of the local heat flux rate for increasing values of heat generation parameter is observed. (Sandeep & Sulochana, 2018) proposed a new mathematical model for analyzing the momentum and heat transfer behavior of Maxwell, Jeffrey, and Oldroyd-B nanofluids past a stretching surface in the presence of heat source/sink and observed an enhancement in thermal boundary layer thickness with a heat source or sink parameters. (Kotha, Kolipaula, Venkata Subba Rao, Penki, & Chamkha, 2020) studied the impact of internal heat generation on bioconvective MHD nanofluid flow and detected a reduction in heat transfer rate with the heat generation parameter increments.

1.16.11 FLUID FLOW THROUGH CHANNELS

The flow between two parallel plates is a common topic. (Singh, 1999) theoretically analyzed the viscous fluid flow between two parallel porous plates. (Ganesh & Krishnambal, 2006) considered the steady laminar flow of a viscous fluid between two parallel porous plates. (Sweet, Vajravelu, Van Gorder, & Pop, 2011) explore two-dimensional MHD viscous fluid flow between two moving parallel plates.

With the introduction of nanofluids, the major drawbacks against improving thermal properties by using conventional fluids were overcome. The widespread implementation of nanofluids accelerates all related areas. Many more research works on the nanofluid flow between two parallel plates have been carried out within this small period. (Sheikholeslami & Ganji, 2013) numerically analyzed the thermal properties of nanoliquid flow between parallel plates. The nanoliquid flow in presence of a magnetic field between parallel plates is reviewed in (Sheikholeslami, Hatami, & Domairry, 2015). (Mohyud-Din, Zaidi, Khan, & Ahmed, 2015) studied mass and heat transfer of nanoliquid flow rotating parallel plates. MHD nanoliquid flow in a semi-porous channel was explored by (Sheikholeslami, Hatami, & Ganji, 2013). A lot of studies have been done on hybrid nanoliquid flow between parallel plates, in a very short period. (Ramesh, Madhukesh, Prasannakumara, & Roopa, 2022) elucidated heat transfer aspects of hybrid nanofluid flow with heat source/sink and chemical reaction. The proficiency of hybrid nanoliquid flow between two parallel plates is examined by (Khashi'ie, Waini, Arifin, & Pop, 2021).

Many researchers have analyzed the hydromagnetic flow between two vertical porous plates due to free convection and varying geometrical shapes. However, only a countable number of works involving MHD-free convective flow between two vertical porous plates moving in different directions have been published. Previous studies were based on conventional (base) fluids. (Singh & Mathew, 2009b) studied three-dimensional MHD fluctuating free convective flow between two vertical porous plates moving in opposite directions. (V. Gupta, Jain, & Jha, 2016) theoretically analyzed the convective effects and heat transfer aspects of MHD flow of viscous fluids, between two vertical plates moving in opposite directions, where the channel was partially filled with a porous medium. An ascending temperature distribution and a descending main flow velocity are observed with the enhancement of the magnetic field parameter. (U. Gupta, Jha, Chaudhary, et al., 2011) discussed laminar

fully developed free-convective flow between two contrary directed moving plates and perceived that increasing Darcy number and dissipation results in a very small increment in skin friction. (Singh & Mathew, 2009a) considered the heat transfer characteristics in three-dimensional MHD flow between two parallel porous plates moving in opposite directions together with transpiration cooling.

1.16.12 FLUID FLOW PAST A STRETCHING SURFACE

The analysis of stretching surfaces still remains very important in the research field. (Sakiadis, 1961) introduced the boundary layer behavior on continuous surfaces, with a constant flow velocity. (Crane, 645–647) extended this, by considering the velocity which is proportional to the distance from the slit. (Grubka & Bobba, 1985) investigated the heat transfer properties of stretching surfaces. The flow and heat transfer properties of electrically conducting fluid past a stretching surface are considered by (Vajravelu & Hadjinicolaou, 1997). The two-dimensional boundary layer flow problem past a stretching surface was developed for the first time by (Crane, 645–647). (Magyari & Keller, 1999b) analytically and numerically described the flow and thermal properties on the exponentially stretching surface. (S. K. Khan & Sanjayanand, 2005) studied the heat transfer characteristics of viscoelastic fluid past an exponentially stretching surface. Naramgari and Sulochana (Naramgari & Sulochana, 2016) performed a study on MHD nanofluid radiative flow with heat generation through an exponentially stretching surface. (Waini, Ishak, & Pop, 2020) explored the hybrid nanofluid flow induced by exponentially stretching and shrinking sheets subject to MHD and radiation effects.

1.16.13 STATISTICAL ANALYSIS

The use of statistical tools for analyzing the effects of numerous physical parameters has intrigued a lot of researchers. A countable number of works, where the ideas of correlation, the slope of linear regression, probable error, regression analysis, sensitivity, and RSM are used to compare and analyze the outcome of various physical quantities. Analysis of physical quantities utilizing statistical techniques is convenient due to its efficiency in producing accurate quantitative results. RSM (Response Surface Methodology) analyses the conjoint impact of effectual parameters (independent variables) on the physical quantity of interest (response or dependent variable). Sensitivity analysis, on the other hand, measures the extent and na-

ture of dependency exhibited by the effectual parameters on the physical quantity of interest. (Fisher, 1950) explained the significance of the correlation coefficient. (Mahanthesh, Shashikumar, Gireesha, & Animasaun, 2019) used the slope of linear regression to characterize the drag coefficient and Nusselt number in his study. (Mackolil & Mahanthesh, 2019a) interpreted the skin friction and heat transfer rate in one of his studies with regression analysis and observed that the heat transfer rate descended with nanoparticle volume fraction and the Dufour number whereas ascending with the radiative heat parameter. (Mackolil & Mahanthesh, 2019b) analyzed heat transfer rate and drag coefficient in a study of Casson nanofluid flow incorporated with radiative heat transfer utilizing RSM. it is observed that the Nusselt number has a positive sensitivity towards thermal radiation, and it is negatively sensitive towards nanoparticle volume fraction and Dufour number.

CHAPTER 2

MHD Nanofluid Flow Between Two Vertical Porous Plates Moving In Opposite Direction *

2.1 INTRODUCTION

The theoretical analysis of MHD flow through porous channels (flow between two parallel plates) has gained great attention due to its increased advantages in the industrial, mechanical engineering, and biomedical fields. Flow in the capillaries, blood dialysis, the design of filters, and flow in blood oxygenators are some examples. Only a countable number of studies are accounted for the flow between parallel plates moving in the opposite direction.

This chapter investigates the MHD flow of water-based nanofluid between two vertical porous plates moving in the opposite direction. The perturbation technique is used to solve governing equations and the consequences of various parameters on velocity, temperature, and concentration are examined via graphs using MATLAB software. The physical quantities Nusselt number and Sherwood number are scrutinized using statistical techniques correlation, probable error, and regression

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analysis.

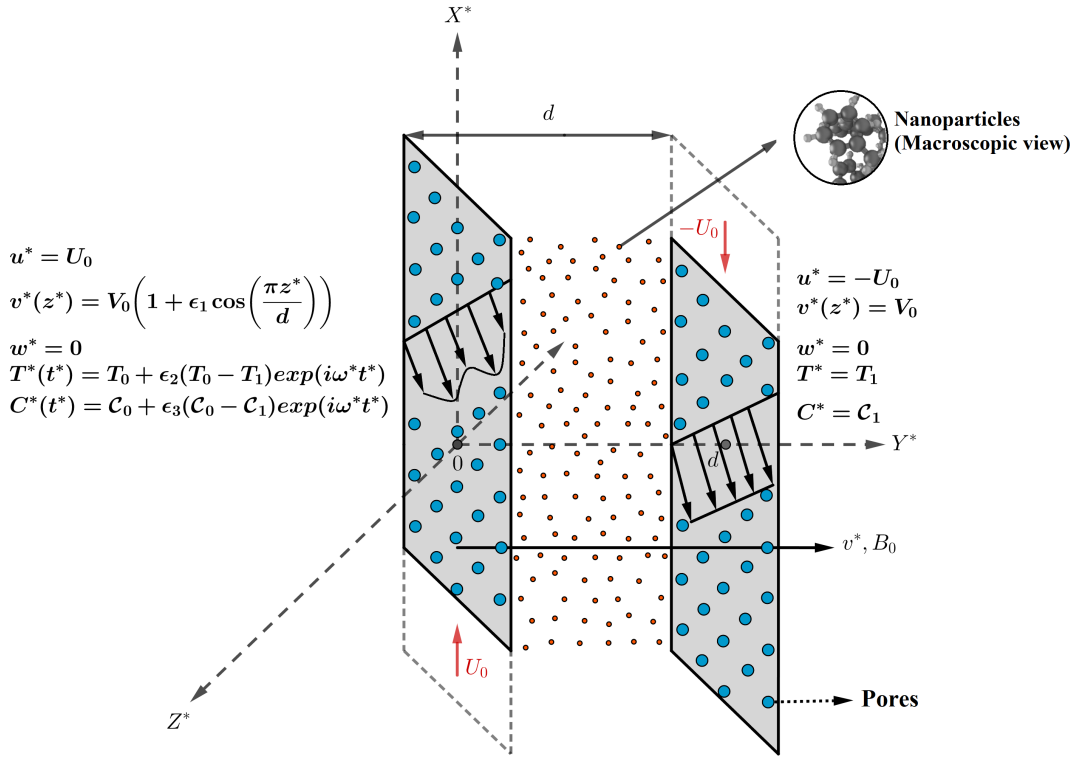


Figure 2.1: *Physical configuration*

2.2 MATHEMATICAL FORMULATION

An unsteady hydromagnetic fluctuating flow of a water based nanofluid between two insulated vertical porous plates is considered (Figure. 2.1). The problem is studied under the following assumptions:

1. Plates are moving in opposite direction with uniform velocity.
2. The upward moving plate is subjected to a transverse sinusoidal injection velocity whereas the downward moving plate is subjected to a constant suction velocity.
3. A magnetic field of uniform strength is applied perpendicular to the plane of the plate.
4. The induced magnetic field has been neglected due to the assumption of small magnetic Reynolds number.

5. The injection velocity distribution is of the form

$$v^*(z^*) = V_0(1 + \varepsilon_1 \cos(\pi z^*/d))$$

6. Without loss of generality, the distance d between the plates is taken equal to the wave length of the injection velocity.

7. The temperature of the upward moving plate fluctuating with time is given as $T^*(t^*) = T_0 + \varepsilon_2(T_0 - T_1)e^{i\omega^*t^*}$

8. The concentration of the upward moving plate fluctuating with time is given as $C^*(t^*) = C_0 + \varepsilon_3(C_0 - C_1)e^{i\omega^*t^*}$

9. The temperature and concentration of the downward moving plate is at constant temperature T_1 and constant concentration C_1 respectively

Using the above assumptions and Boussinesq's approximation, the governing equations (Singh & Mathew, 2009b) takes the form:

$$\frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0 \quad (2.2.1)$$

$$\begin{aligned} \frac{\partial u^*}{\partial t^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} = & -\frac{1}{\rho_{nf}} \left[\frac{\partial p^*}{\partial x^*} - \mu_{nf} \left(\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}} \right) + \sigma_{nf} B_0^2 u^* \right] + \\ & g\beta_{nf} (T^* - T_1) + g\beta_{nf} (C^* - C_1) \end{aligned} \quad (2.2.2)$$

$$\frac{\partial v^*}{\partial t^*} + v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z^*} = -\frac{1}{\rho_{nf}} \left[\frac{\partial p^*}{\partial y^*} - \mu_{nf} \left(\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}} \right) \right] \quad (2.2.3)$$

$$\frac{\partial w^*}{\partial t^*} + v^* \frac{\partial w^*}{\partial y^*} + w^* \frac{\partial w^*}{\partial z^*} = -\frac{1}{\rho_{nf}} \left[\frac{\partial p^*}{\partial z^*} - \mu_{nf} \left(\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}} \right) + \sigma_{nf} B_0^2 w^* \right] \quad (2.2.4)$$

$$\frac{\partial T^*}{\partial t^*} + v^* \frac{\partial T^*}{\partial y^*} + w^* \frac{\partial T^*}{\partial z^*} = \frac{K_{nf}}{(\rho C_p)_{nf}} \left[\frac{\partial^2 T^*}{\partial y^{*2}} + \frac{\partial^2 T^*}{\partial z^{*2}} \right] \quad (2.2.5)$$

$$\frac{\partial C^*}{\partial t^*} + v^* \frac{\partial C^*}{\partial y^*} + w^* \frac{\partial C^*}{\partial z^*} = D_B \left[\frac{\partial^2 C^*}{\partial y^{*2}} + \frac{\partial^2 C^*}{\partial z^{*2}} \right] - K_l (C^* - C_1) \quad (2.2.6)$$

The boundary conditions for the problem are:

$$\left. \begin{aligned} y^* = 0, u^* = U_0, v^*(z^*) = V_0 \left(1 + \varepsilon_1 \cos \frac{\pi z^*}{d} \right), w^* = 0, \\ T^*(t^* = T_0 + \varepsilon_2 (T_0 - T_1) e^{i\omega^* t^*}, C^*(t^*) = C_0 + \varepsilon_3 (C_0 - C_1) e^{i\omega^* t^*}, \\ y^* = d, u^* = -U_0, v^*(z^*) = V_0, w^* = 0, T^* = T_1, C^* = C_1 \end{aligned} \right\} \quad (2.2.7)$$

The nanofluid model used is

Effective Dynamic Viscosity:

$$\frac{1}{A_1} = \frac{\mu_{nf}}{\mu_f} = \frac{1}{(1 - \phi)^{2.5}}$$

Effective Density:

$$A_2 = \frac{\rho_{nf}}{\rho_f} = (1 - \phi) + \phi \left(\frac{\rho_s}{\rho_f} \right)$$

Effective Electrical Conductivity:

$$A_3 = \frac{\sigma_{nf}}{\sigma_f} = 1 + \frac{3 \left(\frac{\sigma_s}{\sigma_f} - 1 \right) \phi}{\left(\frac{\sigma_s}{\sigma_f} + 2 \right) - \left(\frac{\sigma_s}{\sigma_f} - 1 \right) \phi}$$

Effective Coefficient Of Thermal Expansion:

$$A_4 = \frac{\beta_{nf}}{\beta_f} = (1 - \phi) + \phi \left(\frac{\beta_s}{\beta_f} \right)$$

Effective Specific Heat:

$$A_5 = \frac{(\rho C_p)_{nf}}{(\rho C_p)_f} = (1 - \phi) + \phi \left(\frac{(\rho C_p)_s}{(\rho C_p)_f} \right)$$

Effective Thermal Conductivity:

$$A_6 = \frac{K_{nf}}{K_f} = \frac{K_s + 2K_f - 2\phi(K_f - K_s)}{K_s + 2K_f + 2\phi(K_f - K_s)}$$

Introducing the following non-dimensional quantities

$$y = \frac{y^*}{d}, z = \frac{z^*}{d}, t = t^* \omega^*, u = \frac{u^*}{U_0}, v = \frac{v^*}{V_0}, w = \frac{w^*}{V_0},$$

$$p = \frac{p^*}{\rho_{nf} V_0^2}, \quad \omega = \frac{\omega^* d^2}{\vartheta}, \quad \theta = \frac{T^* - T_1}{T_0 - T_1}, \quad C = \frac{C^* - C_1}{C_0 - C_1}$$

into equations (2.2.1) to (2.2.7), excluding (2.2.2)

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (2.2.8)$$

$$\frac{\omega}{Re} \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{A_1 A_2 Re} \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad (2.2.9)$$

$$\frac{\omega}{Re} \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{A_1 A_2 Re} \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{A_3}{A_2 Re} H^2 w \quad (2.2.10)$$

$$\frac{\omega}{Re} \frac{\partial \theta}{\partial t} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = \frac{A_6}{A_5 Pr Re} \left(\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right) \quad (2.2.11)$$

$$\frac{\omega}{Re} \frac{\partial C}{\partial t} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = \frac{1}{Re Sc} \left(\frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right) - \frac{K_c}{Re} C \quad (2.2.12)$$

Equation (2.2.2) reduces to the following partial differential equations based on the two cases:

Case 1: Magnetic Field is applied on the upward moving plate (at $y=0$)

$$\begin{aligned} \frac{\omega}{Re} \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{1}{A_1 A_2 Re} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{A_3}{A_2 Re} H^2 (u - 1) + \\ A_4 Gr Re \theta + A_4 Gm Re C \end{aligned} \quad (2.2.13)$$

Case 2: Magnetic Field is applied on the downward moving plate (at $y=1$)

$$\begin{aligned} \frac{\omega}{Re} \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{1}{A_1 A_2 Re} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{A_3}{A_2 Re} H^2 (u + 1) + \\ A_4 Gr Re \theta + A_4 Gm Re C \end{aligned} \quad (2.2.14)$$

The boundary conditions in the non-dimensional form are given by

$$\left. \begin{aligned} y = 0, u = 1, v(z) = 1 + \varepsilon_1 \cos \pi z, w = 0, \theta = 1 + \varepsilon_2 e^{it}, C = 1 + \varepsilon_3 e^{it} \\ y = 1, u = -1, v = 1, w = 0, \theta = 0, C = 0 \end{aligned} \right\} \quad (2.2.15)$$

2.3 METHOD OF SOLUTION

Since $\varepsilon = \min \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is very small, assume that the solution of the problem is of the form

$$f(y, z, t) = f_0(y) + \varepsilon f_1(y, z, t) + O(\varepsilon^2) \quad (2.3.1)$$

2.3.1 Steady Flow Solution

When $\varepsilon = 0$, the current problem contracts to a steady two dimensional hydro magnetic flow between two vertical porous plates moving in opposite direction with uniform injection/suction which is governed by the following equations:

Case 1: Magnetic Field is applied on the upward moving plate (at $y=0$)

$$\begin{aligned} u_0'' - A_1 A_2 Re u_0' - A_1 A_3 H^2 (u_0 - 1) + A_1 A_2 A_4 Re^2 Gr \theta_0 + \\ A_1 A_2 A_4 Re^2 Gm C_0 = 0 \end{aligned} \quad (2.3.2)$$

Case 2: Magnetic Field is applied on the downward moving plate (at $y=1$)

$$\begin{aligned} u_0'' - A_1 A_2 Re u_0' - A_1 A_3 H^2 (u_0 + 1) + A_1 A_2 A_4 Re^2 Gr \theta_0 + \\ A_1 A_2 A_4 Re^2 Gm C_0 = 0 \end{aligned} \quad (2.3.3)$$

with $v_0 = 1, w_0 = 0, p_0 = \text{a constant}$ and

$$\theta_0'' - \frac{A_5 Pr Re}{A_6} \theta_0' = 0 \quad (2.3.4)$$

$$C_0'' - Sc Re C_0' - K_c Sc C_0 = 0 \quad (2.3.5)$$

where primes denote the derivative with respect to y .

The corresponding boundary condition for the above two cases reduces to:

$$\left. \begin{aligned} y = 0, u_0 = 1, \theta_0 = 1, C_0 = 1 \\ y = 1, u_0 = -1, \theta_0 = 0, C_0 = 0 \end{aligned} \right\} \quad (2.3.6)$$

The solutions of equations (2.3.2) to (2.3.5) subject to boundary conditions (2.3.6) are

$$\theta_0 = \frac{1}{e^a - 1}(e^a - e^{ay}) \quad (2.3.7)$$

$$C_0 = \frac{1}{e^{m_2} - e^{m_1}}(e^{m_2}e^{m_1y} - e^{m_1}e^{m_2y}) \quad (2.3.8)$$

Case 1: Magnetic Field is applied on the upward moving plate (at y=0)

$$u_0 = \frac{1}{e^{m_4} - e^{m_3}} [(\alpha_1 e^{m_4} - \beta_1) e^{m_3y} + (\beta_1 - \alpha_1 e^{m_3}) e^{m_4y}] + B_1 e^{ay} + B_2 e^{m_1y} + B_3 e^{m_2y} + B_4 + 1 \quad (2.3.9)$$

Case 2: Magnetic Field is applied on the downward moving plate (at y=1)

$$u_0 = \frac{1}{e^{m_4} - e^{m_3}} [((\alpha_1 + 2) e^{m_4} - (\beta_1 + 2)) e^{m_3y} + ((\beta_1 + 2) - (\alpha_1 + 2) e^{m_3}) e^{m_4y}] + B_1 e^{ay} + B_2 e^{m_1y} + B_3 e^{m_2y} + B_4 - 1 \quad (2.3.10)$$

2.3.2 Cross Flow Solution

When $\varepsilon \neq 0$ substituting equation (2.3.1) in equations (2.2.8) to (2.2.10) and comparing the coefficients of ε and neglecting the terms with $O(\varepsilon^2)$, obtain the following first order equations:

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \quad (2.3.11)$$

$$\frac{\omega}{Re} \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + \frac{1}{A_1 A_2 Re} \left[\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right] \quad (2.3.12)$$

$$\frac{\omega}{Re} \frac{\partial w_1}{\partial t} + \frac{\partial w_1}{\partial y} = -\frac{\partial p_1}{\partial z} + \frac{1}{A_1 A_2 Re} \left[\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2} \right] - \frac{A_3}{A_2 Re} H^2 w_1 \quad (2.3.13)$$

Corresponding boundary conditions are

$$\left. \begin{aligned} y = 0, v_1 = \cos \pi z, w_1 = 0 \\ y = 1, v_1 = 0, w_1 = 0 \end{aligned} \right\} \quad (2.3.14)$$

These are the linear partial differential equations describing the three dimensional cross flow which are independent of the main flow component, u_1 , temperature field, θ_1 and concentration field, C_1 .

Assume that the solutions for v_1, w_1, p_1 are of the following form

$$v_1(y, z, t) = v_{11}(y)e^{it} + v_{12}(y) \cos \pi z \quad (2.3.15)$$

$$w_1(y, z, t) = -(zv_{11}'(y)e^{it} + \frac{1}{\pi}v_{12}'(y) \sin \pi z) \quad (2.3.16)$$

$$p_1(y, z, t) = p_{11}(y)e^{it} + p_{12}(y) \cos \pi z \quad (2.3.17)$$

where prime denotes the derivative with respect to y . Expressions (2.3.15) and (2.3.16) have been chosen in such a way that the continuity equation (2.3.11) is precisely satisfied. Substituting these equations into equations (2.3.12) and (2.3.13) and applying the corresponding boundary conditions, the solutions of v_1, w_1, p_1 obtained as

$$v_1 = \frac{1}{D} \sum_{i=1}^4 D_i e^{r_i y} \cos \pi z \quad (2.3.18)$$

$$w_1 = -\frac{1}{\pi D} \sum_{i=1}^4 r_i D_i e^{r_i y} \sin \pi z \quad (2.3.19)$$

$$p_1 = -\frac{1}{A_1 A_2 Re \pi^2 D} \sum_{i=1}^4 D_i (r_i^3 - A_1 A_2 Re r_i^2 - (A_1 A_3 H^2 + \pi^2) r_i) e^{r_i y} \cos \pi z \quad (2.3.20)$$

2.3.3 Temperature and Concentration Field

Similarly using equation (2.3.1) when $\varepsilon \neq 0$ the first order equation for temperature and concentration fields are

$$\frac{\omega}{Re} \frac{\partial \theta_1}{\partial t} + \frac{\partial \theta_1}{\partial y} = \frac{A_6}{A_5 Pr Re} \left(\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} \right) \quad (2.3.21)$$

$$\frac{\omega}{Re} \frac{\partial C_1}{\partial t} + \frac{\partial C_1}{\partial y} = \frac{1}{Sc Re} \left(\frac{\partial^2 C_1}{\partial y^2} + \frac{\partial^2 C_1}{\partial z^2} \right) - \frac{K_c}{Re} C_1 \quad (2.3.22)$$

with corresponding boundary conditions

$$\left. \begin{aligned} y = 0, \theta_1 = e^{it}, C_1 = e^{it} \\ y = 1, \theta_1 = 0, C_1 = 0 \end{aligned} \right\} \quad (2.3.23)$$

Equations (2.3.21)-(2.3.23) are solved with an assumption that the solutions are of the form:

$$\theta_1(y, z, t) = \theta_{11} e^{it} + \theta_{12} \cos \pi z \quad (2.3.24)$$

$$C_1(y, z, t) = C_{11} e^{it} + C_{12} \cos \pi z \quad (2.3.25)$$

Substituting equation (2.3.24) into (2.3.21) and equation (2.3.25) into (2.3.22), one can obtain the followig equations:

$$\theta_{11}'' - \frac{A_5 Pr Re}{A_6} \theta_{11}' - \frac{A_5 Pr \omega i}{A_6} \theta_{11} = 0 \quad (2.3.26)$$

$$\theta_{12}'' - \frac{A_5 Pr Re}{A_6} \theta_{12}' - \pi^2 \theta_{12} = 0 \quad (2.3.27)$$

$$C_{11}'' - Sc Re C_{11}' - (Sc \omega i + K_c Sc) C_{11} = 0 \quad (2.3.28)$$

$$C_{12}'' - Sc Re C_{12}' - (Sc K_c + \pi^2) C_{12} = 0 \quad (2.3.29)$$

with the corresponding transformed boundary conditions

$$\left. \begin{aligned} y = 0, \theta_{11} = 1, \theta_{12} = 0, C_{11} = 1, C_{12} = 0 \\ y = 1, \theta_{11} = 0, \theta_{12} = 0, C_{11} = 0, C_{12} = 0 \end{aligned} \right\} \quad (2.3.30)$$

Solving equations (2.3.26) – (2.3.29) under the boundary condition (2.3.30), the solutions are obtained as

$$\theta_1(y, z, t) = \frac{1}{e^{n_2} - e^{n_1}} (e^{n_2} e^{n_1 y} - e^{n_1} e^{n_2 y}) e^{it} \quad (2.3.31)$$

$$C_1(y, z, t) = \frac{1}{e^{m_6} - e^{m_5}} (e^{m_6} e^{m_5 y} - e^{m_5} e^{m_6 y}) e^{it} \quad (2.3.32)$$

2.3.4 Main Flow Solution

When $\varepsilon \neq 0$, with the aid of equation (2.3.1) and comparing coefficients of ε , can deduce the first order equation for the main flow component, u_1 , as

$$\frac{\omega}{Re} \frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial y} + v_1 u_0' = \frac{1}{A_1 A_2 Re} \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} \right) - \frac{A_3}{A_2 Re} H^2 u_1 + A_4 Gr Re \theta_1 + A_4 Gm Re C_1 \quad (2.3.33)$$

The corresponding boundary conditions are

$$\left. \begin{aligned} y = 0, u_1 = 0 \\ y = 1, u_1 = 0 \end{aligned} \right\} \quad (2.3.34)$$

To solve the above differential equation for the main flow velocity component, $u_1(y, z, t)$, assume that the solution is of the form:

$$u_1(y, z, t) = u_{11}(y) e^{it} + u_{12}(y) \cos \pi z \quad (2.3.35)$$

Substituting equation (2.3.35) in (2.3.33) and neglecting terms of $O(\varepsilon^2)$ one can obtain the following differential equations

$$u_{11}'' - A_1 A_2 Re u_{11}' - (A_1 A_2 \omega i + A_1 A_3 H^2) u_{11} = -A_1 A_2 A_4 Re^2 Gr \theta_{11} -$$

$$A_1 A_2 A_4 Re^2 Gm C_{11} \quad (2.3.36)$$

$$u_{12}'' - A_1 A_2 Re u_{12}' - (\pi^2 + A_1 A_3 H^2) u_{12} = -A_1 A_2 Re v_{12} u_0' \quad (2.3.37)$$

where prime denotes the derivative with respect to y . The corresponding boundary conditions are given by:

$$\left. \begin{aligned} y = 0, u_{11} = 0, u_{12} = 0 \\ y = 1, u_{11} = 0, u_{12} = 0 \end{aligned} \right\} \quad (2.3.38)$$

Solving equations (2.3.36) and (2.3.37) considering (2.3.38) and using (2.3.35) one can derive the solutions as:

Case 1: Magnetic Field is applied on the upward moving plate (at $y=0$)

$$u_1(y, z, t) = \left\{ \begin{aligned} &\frac{1}{e^{r_6} - e^{r_5}} [(\alpha_2 e^{r_6} - \beta_2) e^{r_5 y} + (\beta_2 - \alpha_2 e^{r_5}) e^{r_6 y}] + \\ &K_1 e^{n_1 y} + K_2 e^{n_2 y} + K_3 e^{m_5 y} + K_4 e^{m_6 y} \end{aligned} \right\} e^{it} + \left\{ \begin{aligned} &\frac{1}{e^{n_4} - e^{n_3}} [(\alpha_3 e^{n_4} - \beta_3) e^{n_3 y} + (\beta_3 - \alpha_3 e^{n_3}) e^{n_4 y}] + \\ &\sum_{i=1}^4 K_{i1} e^{(r_i+m_3)y} + K_{i2} e^{(r_i+m_4)y} + K_{i3} e^{(r_i+a)y} + \\ &K_{i4} e^{(r_i+m_1)y} + K_{i5} e^{(r_i+m_2)y} \end{aligned} \right\} \cos \pi z \quad (2.3.39)$$

Case 2: Magnetic Field is applied on the downward moving plate (at $y=1$)

$$u_1(y, z, t) = \left\{ \begin{aligned} &\frac{1}{e^{r_6} - e^{r_5}} [(\alpha_2 e^{r_6} - \beta_2) e^{r_5 y} + (\beta_2 - \alpha_2 e^{r_5}) e^{r_6 y}] + \\ &K_1 e^{n_1 y} + K_2 e^{n_2 y} + K_3 e^{m_5 y} + K_4 e^{m_6 y} \end{aligned} \right\} e^{it} + \left\{ \begin{aligned} &\frac{1}{e^{n_4} - e^{n_3}} [(\alpha_4 e^{n_4} - \beta_4) e^{n_3 y} + (\beta_4 - \alpha_4 e^{n_3}) e^{n_4 y}] + \\ &\sum_{i=1}^4 M_{i1} e^{(r_i+m_3)y} + M_{i2} e^{(r_i+m_4)y} + \\ &M_{i3} e^{(r_i+a)y} + M_{i4} e^{(r_i+m_1)y} + M_{i5} e^{(r_i+m_2)y} \end{aligned} \right\} \cos \pi z \quad (2.3.40)$$

2.4 PHYSICAL QUANTITIES

For practical analysis of the problem, scientists and engineers are always interested in understanding the physical quantities like drag coefficients, Nusselt number and Sherwood number as they measures the surface drag, rate of heat transfer and rate of mass transfer, respectively. The modified form of these physical quantities are

given by:

$$Sh = \frac{d}{D_B(C_0 - C_1)} \left| D_B \left(\frac{dC^*}{dy^*} \right)_{y^*=d} \right| = \left| \left(\frac{dC_0}{dy} \right)_{y=1} + \varepsilon \left(\frac{dC_1}{dy} \right)_{y=1} \right| \quad (2.4.1)$$

$$Nu = \frac{d}{K_f(T_0 - T_1)} \left| K_{nf} \left(\frac{dT^*}{dy^*} \right)_{y^*=d} \right| = A_6 \left| \left(\frac{d\theta_0}{dy} \right)_{y=1} + \varepsilon \left(\frac{d\theta_1}{dy} \right)_{y=1} \right| \quad (2.4.2)$$

Case 1: Magnetic Field is applied on the upward moving plate (at $y=0$)

$$Cf = \frac{d}{\mu_f U_0} \left| \mu_{nf} \left(\frac{du^*}{dy^*} \right)_{y^*=d} \right| = \frac{1}{A_1} \left| \left(\frac{du_0}{dy} \right)_{y=1} + \varepsilon \left(\frac{du_1}{dy} \right)_{y=1} \right| \quad (2.4.3)$$

Case 2: Magnetic Field is applied on the downward moving plate (at $y=1$)

$$Cf = \frac{d}{\mu_f U_0} \left| \mu_{nf} \left(\frac{du^*}{dy^*} \right)_{y^*=d} \right| = \frac{1}{A_1} \left| \left(\frac{du_0}{dy} \right)_{y=1} + \varepsilon \left(\frac{du_1}{dy} \right)_{y=1} \right| \quad (2.4.4)$$

2.5 RESULTS AND DISCUSSION

The impact of various nanofluids, volume fraction of nanoparticle (ϕ), Grashof number (Gr), chemical reaction parameter (K_c), injection/suction parameter (Re), Schmidt number (Sc), Hartmann number (H) and modified Grashof number (Gm) on concentration (C), temperature (θ) and velocity (u) profiles are analysed through Figures 2.2 - 2.12. The analysis has been carried out for $Cu - H_2O$ nanofluid at $z = 0, t = \frac{\pi}{2}$ and $Pr = 7$. The thermophysical properties of the base fluid and nanoparticles are given in Table 2.1.

The consequence of Gr , the ratio of buoyancy force to viscous force, on u is presented in Figure 2.2. An increase in Gr causes augmentation in u for both cases (magnetic field fixed with the upward and downward moving plate). Physically, an increase in Gr paves to the increase in buoyancy force which in turn increases the velocity. Figure 2.3 confirms that u increases in both cases when Gm is increased; the physical reason being the increase in buoyancy force due to a concentration difference. The influence of H , the ratio of electromagnetic force to the viscous force, on u is graphed in Figures 2.4 and 2.5. It is evident that a rise in H drastically

reduces the velocity in the case of a downward-moving plate. Physically this is due to the fact that the presence of a magnetic field induces a Lorentz force against the fluid flow which retards the velocity profile. However, in the case of upward moving plate, H expresses a mixed behavior on u . In the beginning, an increase in H reduces the velocity but after a certain distance, the behavior is reversed. Figures 2.6 and 2.7 illustrates the exponential increase in u both cases corresponding to the augmentation in Re . Figure 2.8 depicts the positive impact of ϕ on u . The impact of various nanofluids (water-based $Cu/Fe_3O_4/TiO_2$ nanofluid) on u is depicted in Figure 2.9.

The effect of ϕ on θ is elucidated in Figure 2.10. ϕ has a negative impact on the temperature profile, meaning an increase in ϕ fuels a decrease in θ . Figure 2.11 depicts the consequence of Re on temperature. It is observed that Re induces an increase in θ . The influence of Sc on C is illustrated in Figure 2.12. Concentration profile (C) experiences a slight increase when Sc values are improved.

The consequence of heat and mass transfer rate on fluctuating parameter along with the slope of linear regression via data points and the enhancement/decrement rate are assessed in table 2.2 and 2.3, respectively. Negative sign in slope and enhancement/decrement rate symbolises that the corresponding parameter has a negative impact on heat and mass transfer meaning augmentation in parameter reduces the heat and mass transfer rate while the magnitude of slope represents the quantity of change.

It is observed that t and Re have positive impact on Nu and ϕ and ω have negative impact on Nu . The respective rates (slope) are shown in Table 2.2. For Sh , when all other parameters are kept constant, Sh increases with an increase in ω at the rate 0.000322, Re at 0.104051 and Sc at the rate 0.355898. Also, when K_c and t increases Sh decreases at the rate of 0.03518 and 0.00751, respectively. Figures 2.13, 2.14, 2.15, and 2.16 describe the parallel effect of $\phi - Re$ and $H - Gr$ on Cf . From 2.13 and 2.15, it can be deduced that ϕ, Re, H and Gr promotes Cf on the upward moving plate. From 2.14 and 2.16, it can be summarised that ϕ, Re and Gr promotes Cf while H demotes Cf on the downward moving plate.

2.6 STATISTICAL ANALYSIS

2.6.1 Correlation and Probable Error

Correlation, a statistical technique, which enables the user in finding the degree of relationship between two or more variables, has gained a lot of interest in the recent times. The effects of different parameters on Nu and Sh are comprehensively carried out and established using tables. A more detailed study is carried out with the help of correlation coefficient (r) and Probable Error (PE). The nature of relationship and the intensity of the relationship is communicated through the sign and magnitude of r , respectively. Probable Error (PE) of r helps in determining the accuracy and reliability of the calculated correlation coefficient. According to (Fisher, 1950), correlation is said to be significant if $\left| \frac{r}{PE} \right| > 6$, where Probable Error, $PE = \left(\frac{1-r^2}{\sqrt{n}} \right) 0.6745$ and n is the number of observations

From Table 2.4, it can infer that Nu is highly negatively correlated with ϕ and ω and highly positively correlated with Re and t . From $\left| \frac{r}{PE} \right|$ values, it can conclude that all parameters are significant. Table 2.5 proposes that Sh is highly positively correlated with ω , Re and Sc and highly negatively correlated with K_c and t . As earlier, it is observed that all parameters are significant.

2.6.2 Regression Analysis

All correlations are perceived to be significant and hence further analysis can be carried out using regression. Nu and Sh are estimated using multiple linear regression models. The estimated models are of the form:

$$Nu_{est} = b_t t + b_\phi \phi + b_\omega \omega + b_{Re} Re + c$$

$$Sh_{est} = b_t t + b_\omega \omega + b_{Re} Re + b_{Sc} Sc + b_{K_c} K_c + c$$

where b_t , b_ϕ , b_ω , b_{Re} , b_{Sc} , and b_{K_c} , are the estimated regression coefficients.

Nu and Sh is estimated from 30 sets of values chosen in the range $[0, 0.04]$ for ϕ , $[5, 15]$ for ω , $[1, 1.2]$ for Re , $[0.2, 1]$ for Sc , $[0.5, 2]$ for K_c and $[0.5, \frac{\pi}{2}]$ for t and the regression coefficients are found using Microsoft Excel. As the p -values for all the physical parameters are less than 0.05, the regression coefficients are significant. The estimated regression models are given by:

$$Nu_{est} = 0.014996t - 0.807981\phi - 0.006215\omega + 6.953733Re + 0.065176$$

$$Sh_{est} = -0.007645t + 0.000344\omega + 0.103518Re + 0.359017Sc - 0.034689K_c + 0.938922$$

Negative sign of the estimated regression coefficient implies that the corresponding parameter reduces the corresponding physical quantity. The estimated regression equation corresponds with the results achieved in Tables 2.2 and 2.3. Figures 2.17 and 2.18 illustrate the accuracy of the regression model for the chosen sample.

2.7 CONCLUSION

The main conclusions drawn from the current study are listed below:

- The main flow velocity profile is directly proportional to Grashof number (Gr), volume fraction of nanoparticle (ϕ) and modified Grashof number (Gm).
- The main flow velocity profile is greater when the magnetic field is applied on the upward moving plate as compared to the main flow velocity when magnetic field is applied on the downward moving plate.
- The injection parameter (Re) has a constructive effect on the main flow velocity.
- Injection parameter enhances the temperature profile whereas nanoparticle volume fraction diminishes the temperature profile.
- On the upward moving plate, main flow velocity profile exhibits a mixed behaviour when Hartmann number (H) is increased. Initially the velocity reduces and after travelling a certain length the velocity increases.
- On the downward moving plate, velocity profile is inversely proportional to the Hartmann number.
- Schmidt number has a constructive effect on the concentration profile.
- Nusselt number is highly positively correlated with the injection parameter and highly negatively correlated with nanoparticle volume fraction.

- The regression models are found to be faultless for the chosen range of values of the parameters.

TABLES AND GRAPHS

Table 2.1: *Thermophysical properties of base fluid and nanoparticles*

Physical Properties	H_2O	Cu	TiO_2	Fe_3O_4
ρ	997.1	8933	4250	5180
C_p	4179	385	686.2	670
β	$21 * 10^5$	$1.67 * 10^5$	$0.9 * 10^5$	$1.3 * 10^5$
σ	$5 * 10^{-2}$	$5.96 * 10^7$	$2.38 * 10^6$	25000
k	0.613	400	8.9538	9.7

Table 2.2: *Variation in Nu for differing parameter values at $y = 1$ when $\phi = 0.02, \omega = 5, Re = 1, t = \pi/2, z = 0$ and $Pr = 7$*

t	ϕ	ω	Re	Nu	Enhancement/Decrement Rate
0.5				6.96773	
1				6.98036	0.18%
1.5				6.99453	0.20%
Slope				0.02681	
	0.01			7.00444	
	0.02			6.99646	-0.11%
	0.03			6.98918	-0.10%
Slope				-0.76297	
		5		6.99646	
		6		6.9855	-0.16%
		7		6.97932	-0.09%
Slope				-0.0086	
			1	6.99646	
			1.1	7.6937	9.97%
			1.2	8.39399	9.10%
Slope				6.98762	

Table 2.3: Variation in Sh for differing parameter values at $y = 1$ when $\omega = 5, Re = 1, Sc = 0.2, Kr = 1, t = \pi/2, z = 0$ and $Pr = 7$

t	ω	Re	Sc	K_c	Sh	Enhancement/Decrement Rate
0.5					1.07745	
1					1.07457	-0.27%
1.5					1.06994	-0.43%
Slope					-0.0075	
	5				1.0692	
	6				1.06954	0.03%
	7				1.06987	0.03%
Slope					0.00033	
		1			1.0692	
		1.1			1.07958	0.97%
		1.2			1.09001	0.97%
Slope					0.10405	
			0.2		1.0692	
			0.4		1.1398	6.60%
			0.6		1.21156	6.30%
Slope					0.3559	
				0.5	1.08699	
				1	1.0692	-1.64%
				1.5	1.05182	-1.63%
Slope					-0.0352	

Table 2.4: Correlation coefficient (r), Probable error (PE) and $\left|\frac{r}{PE}\right|$ values of Nu at $y = 1$ with respect to the parameters ϕ, ω, Re and t

Parameter	r	PE	$\left \frac{r}{PE}\right $
ϕ	-0.998525	0.000889	1122.9909
ω	-0.961356	0.025561	37.6096
Re	0.99999	$5.02 * 10^{-7}$	199156.61
t	0.999451	0.000428	2337.49

Table 2.5: Correlation coefficient (r), Probable error (PE) and $\left|\frac{r}{PE}\right|$ values of Sh at $y = 1$ with respect to the parameters ω, Re, Sc, Kr and t

Parameter	r	PE	$\left \frac{r}{PE}\right $
ω	0.999963802	$2.44 * 10^{-5}$	40956.47543
Re	0.999998301	$1.02 * 10^{-6}$	975880.9088
Sc	0.999968876	$1.88 * 10^{-5}$	53256.12316
K_c	-0.99994774	$3.52 * 10^{-5}$	28369.64913
t	-0.99111797	0.006887008	143.9112518

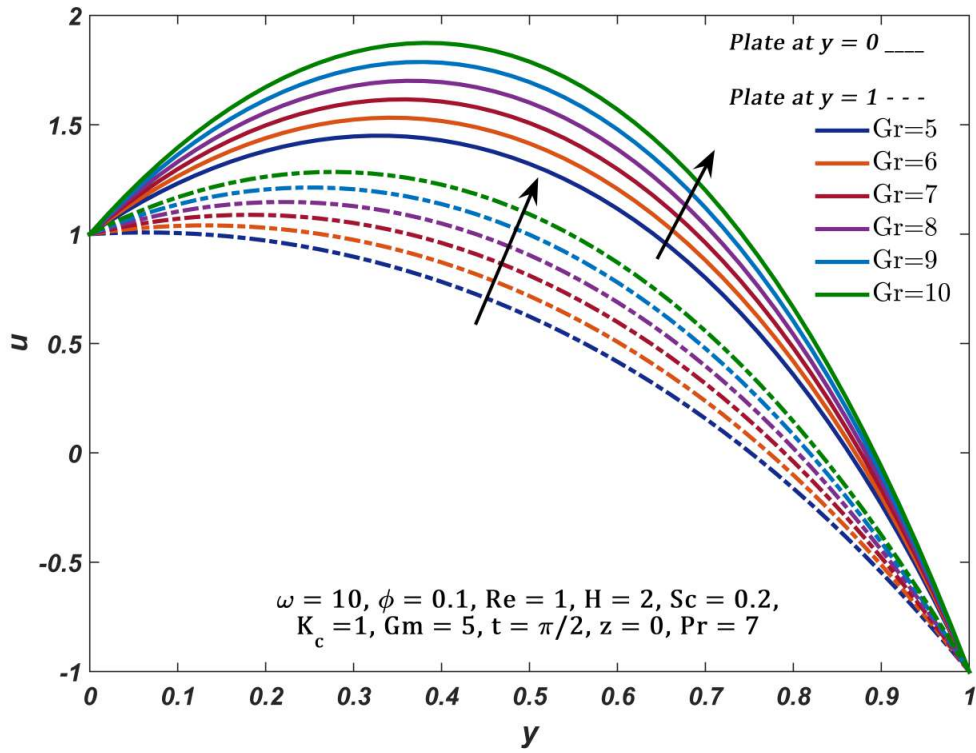


Figure 2.2: Variation in u with Gr

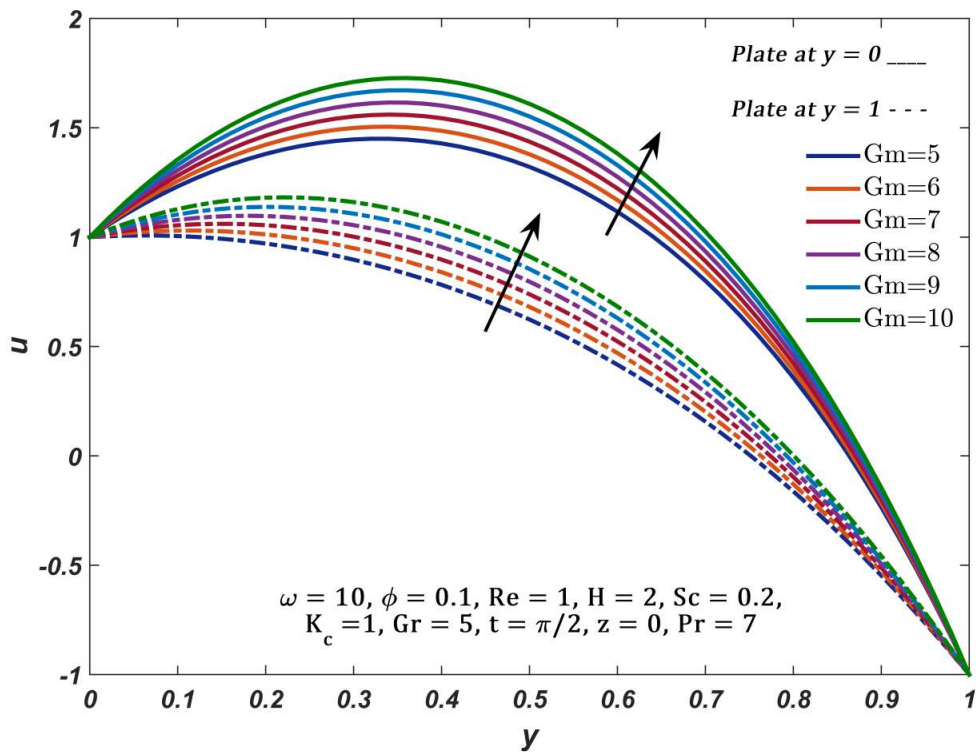


Figure 2.3: Variation in u with Gm

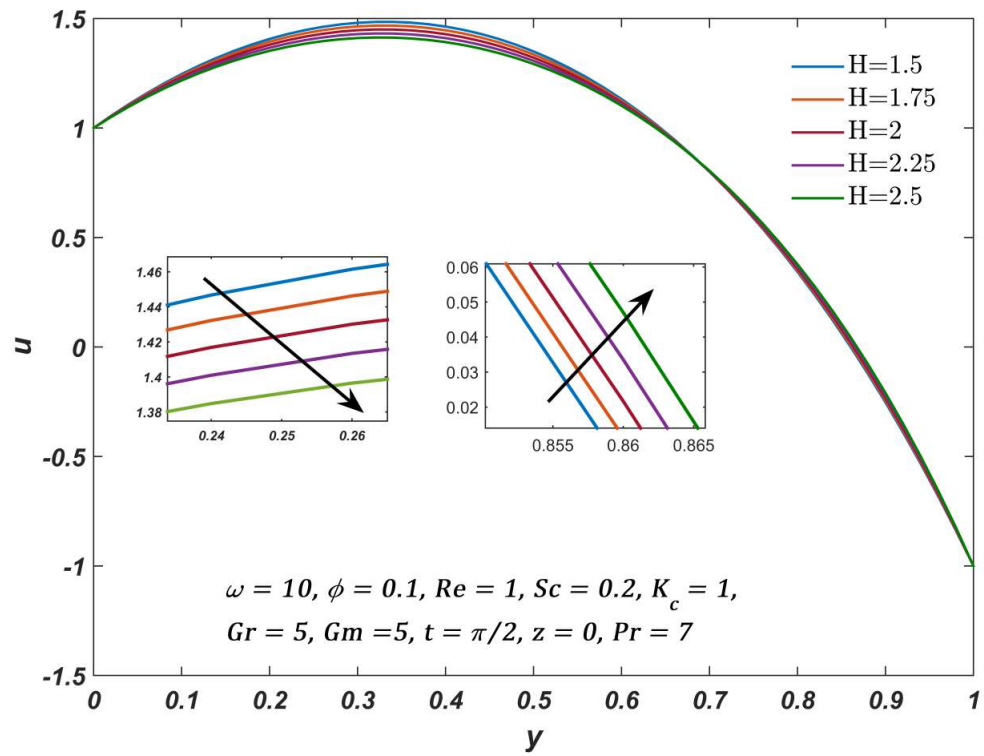


Figure 2.4: Variation in u with H at $y = 0$

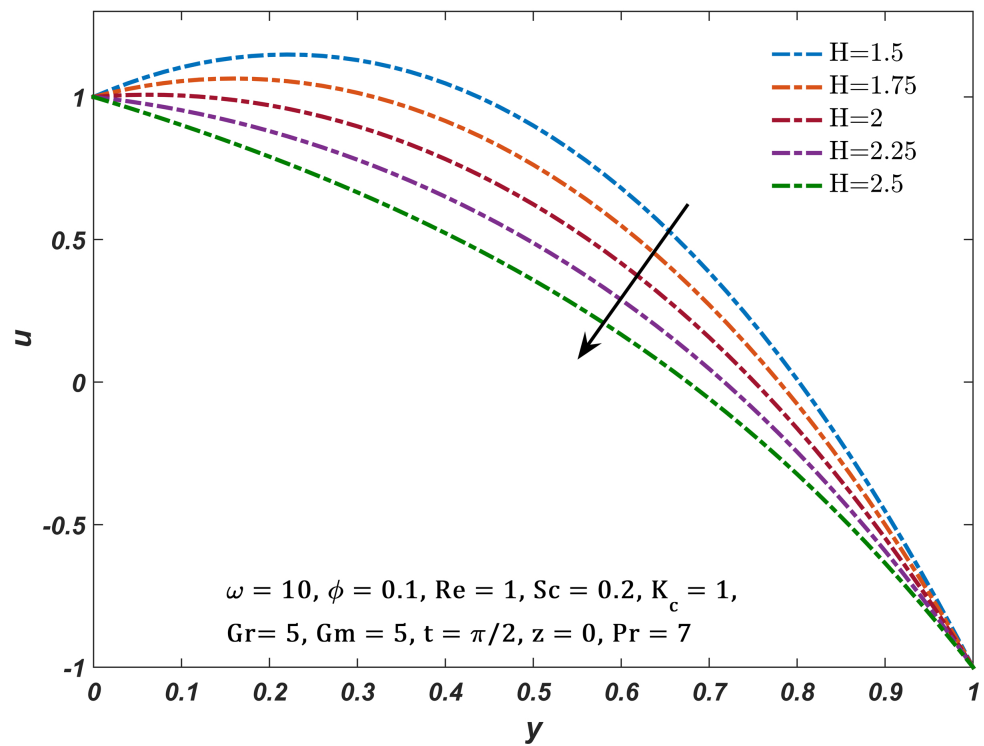


Figure 2.5: Variation in u with H at $y = 1$

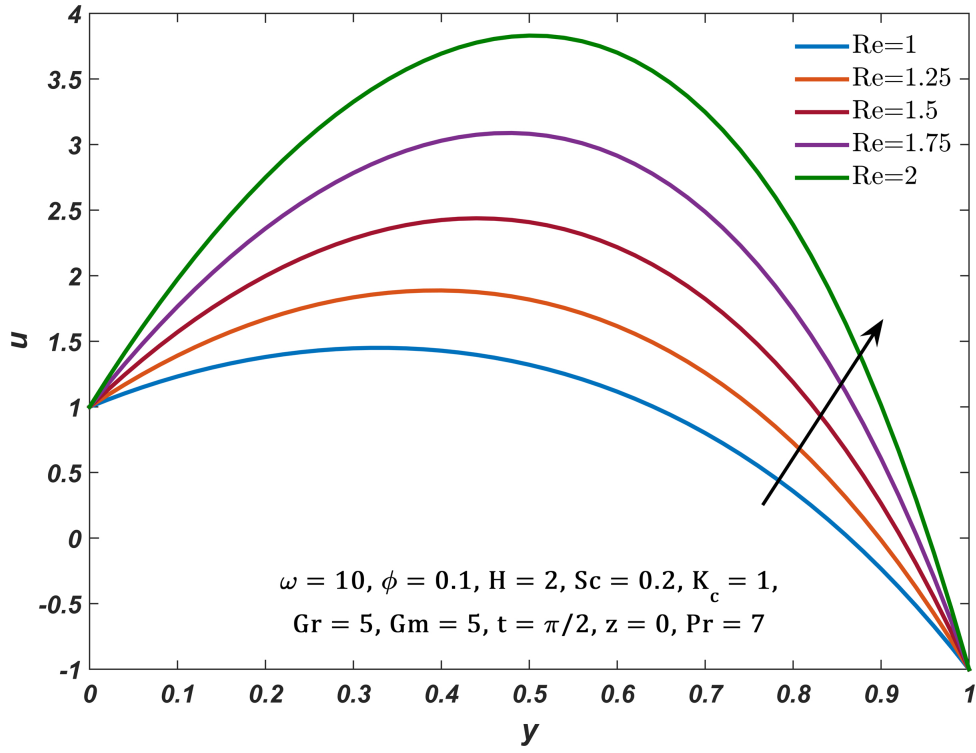


Figure 2.6: Variation in u with Re at $y = 0$

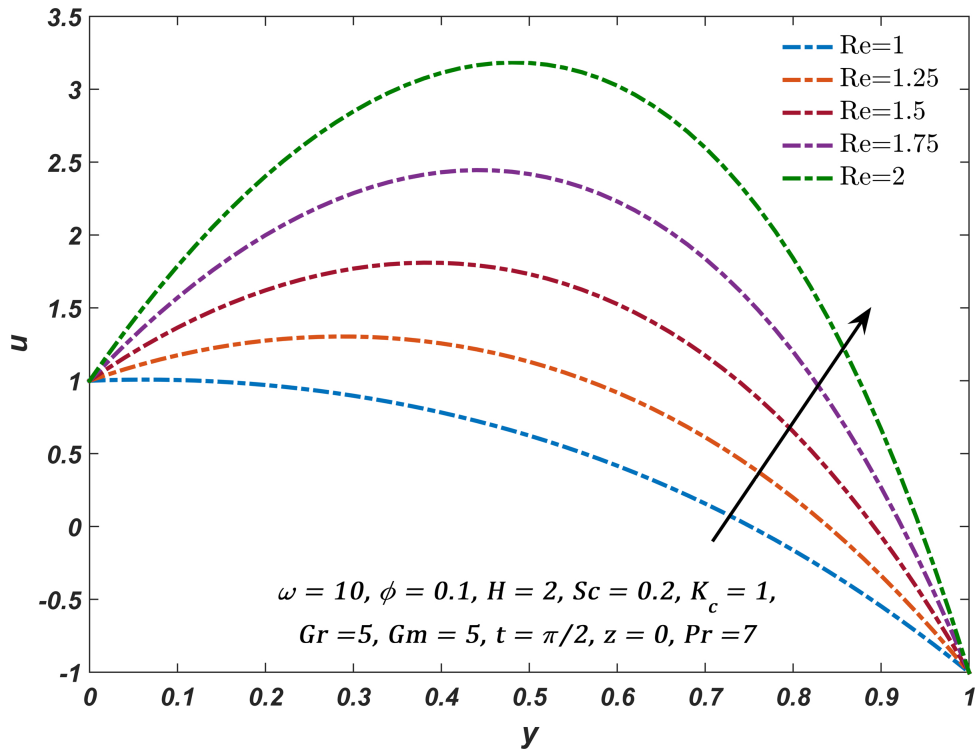
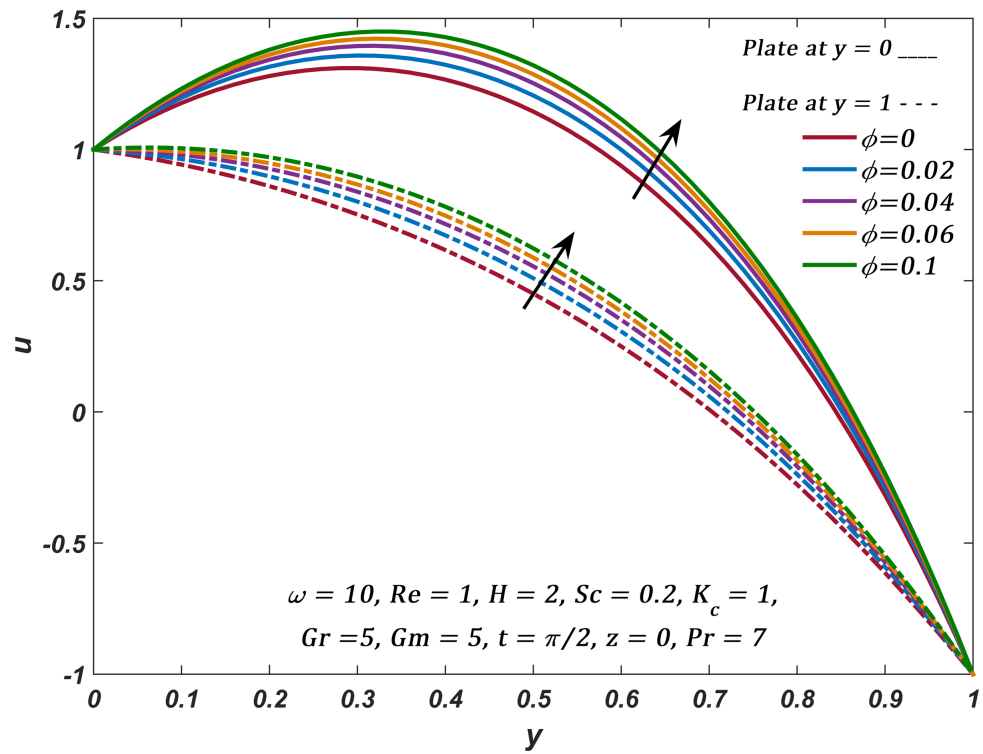
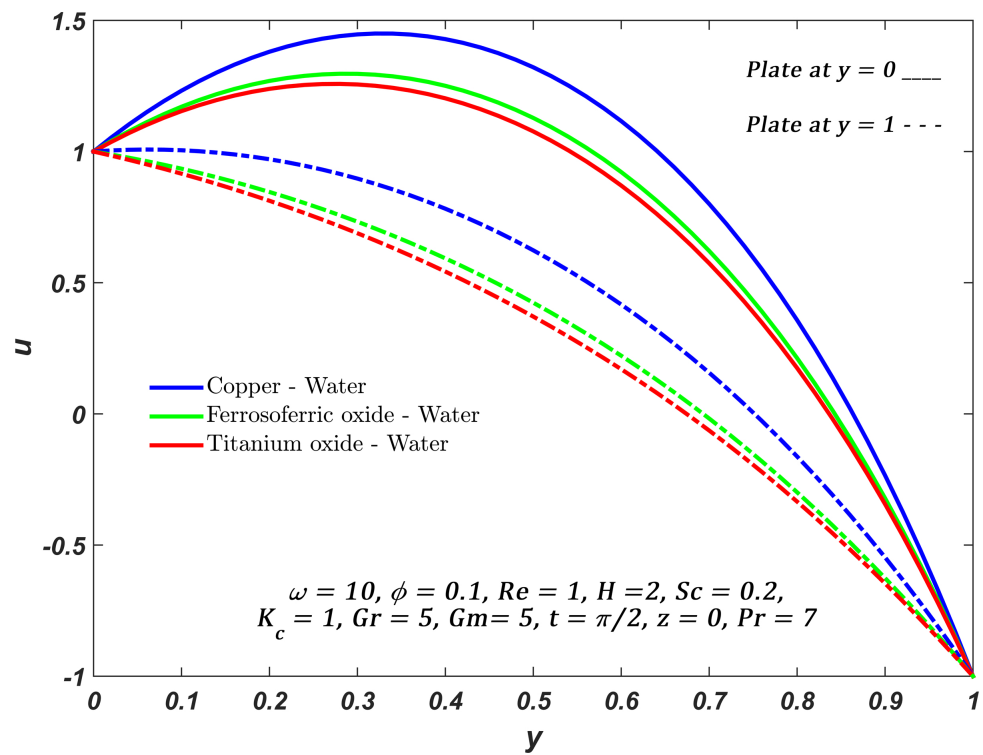


Figure 2.7: Variation in u with Re at $y = 1$


 Figure 2.8: Variation in u with ϕ

 Figure 2.9: Variation in u with different nanofluids

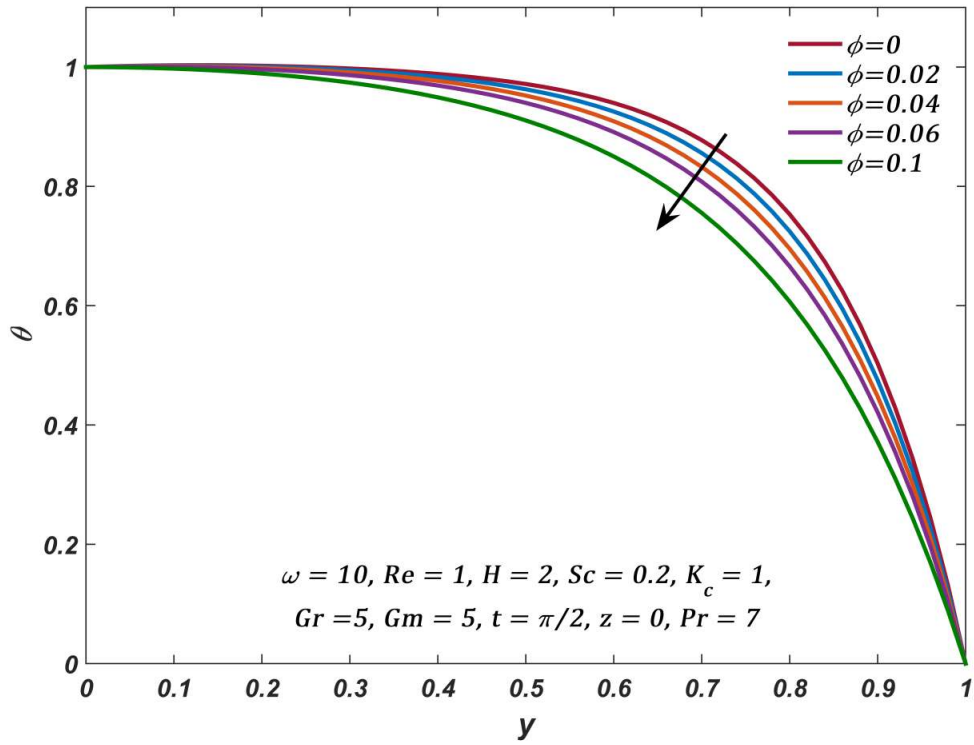


Figure 2.10: Variation in θ with ϕ

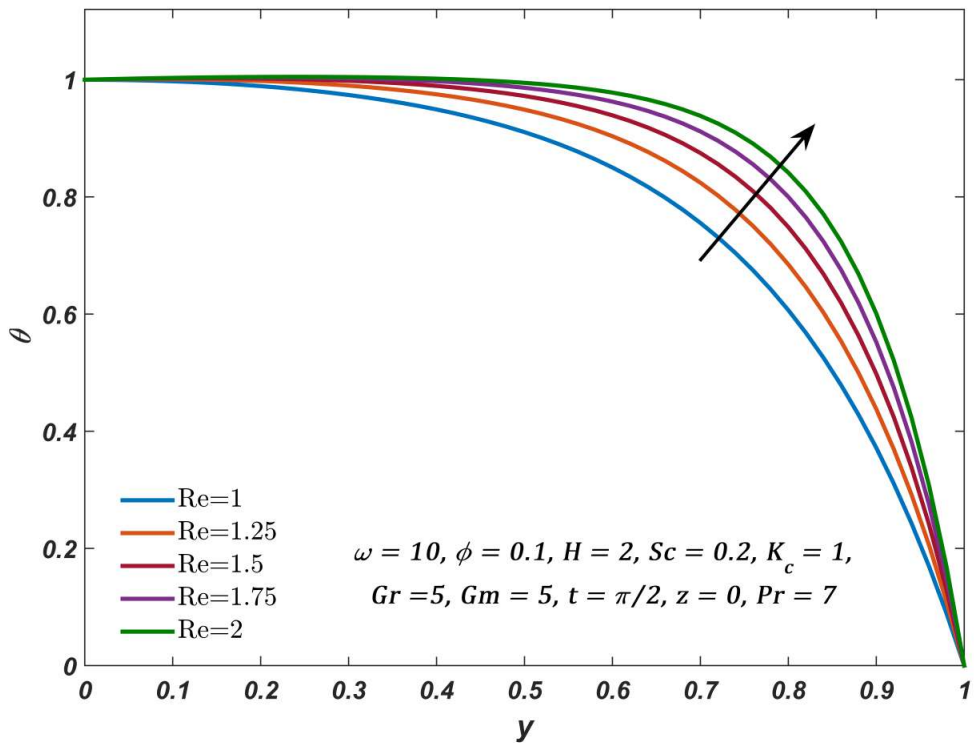


Figure 2.11: Variation in θ with Re

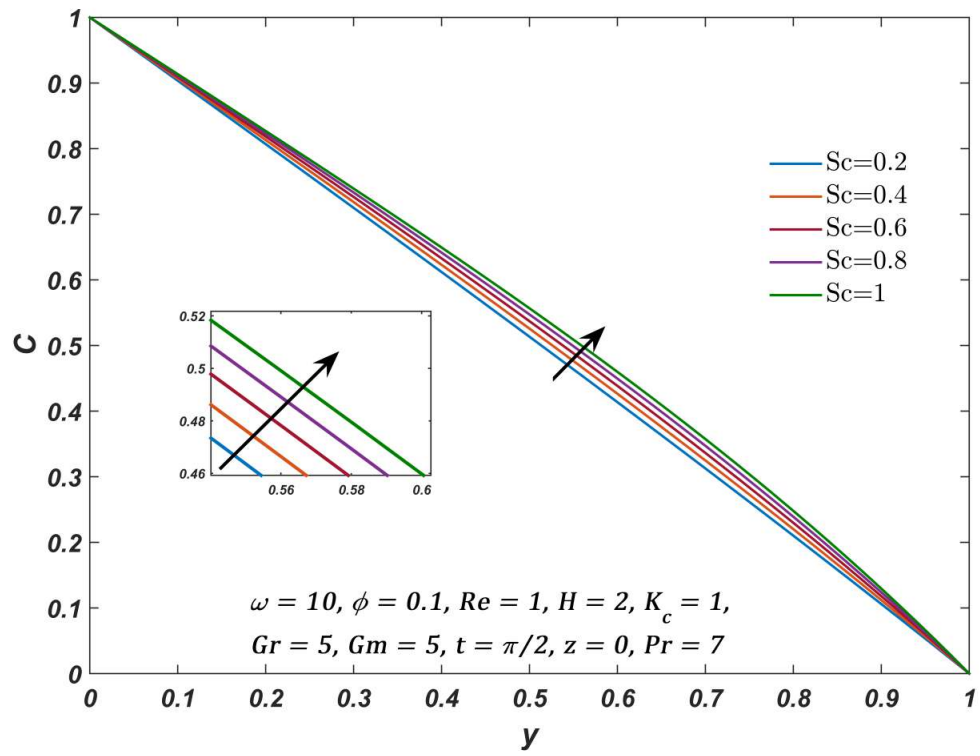


Figure 2.12: Variation in C with Sc

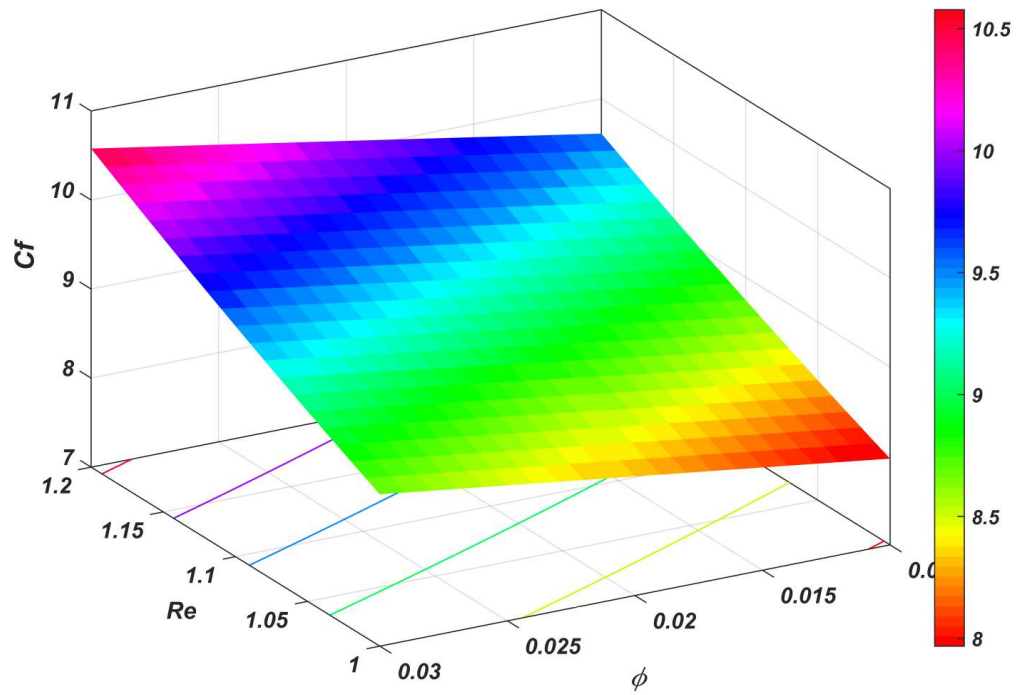


Figure 2.13: Variation in Cf with Re and ϕ at $y = 0$

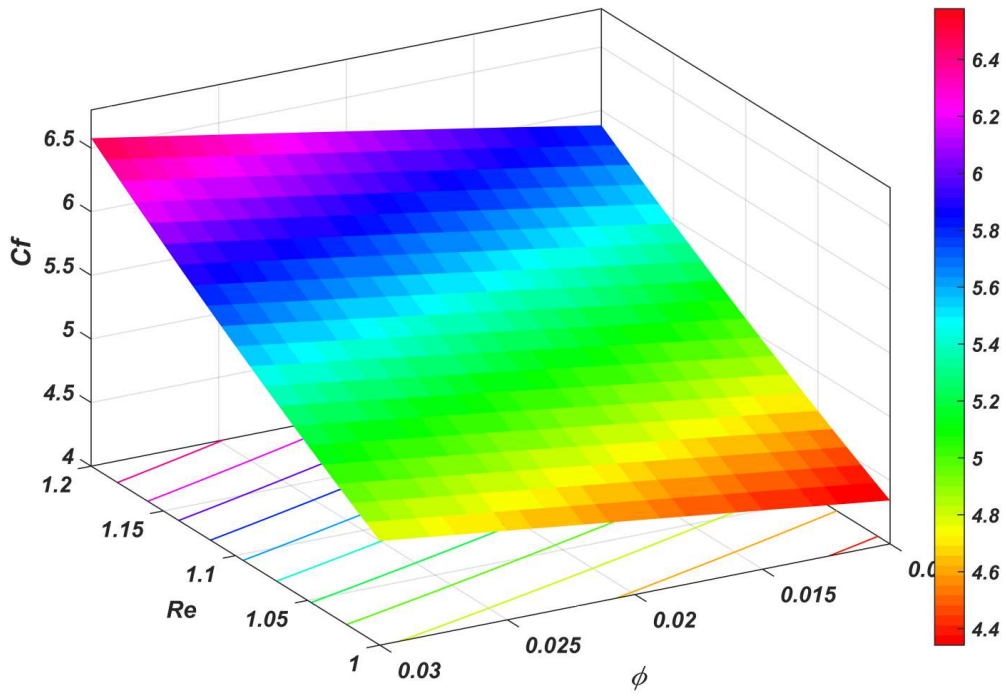


Figure 2.14: Variation in C_f with Re and ϕ at $y = 1$

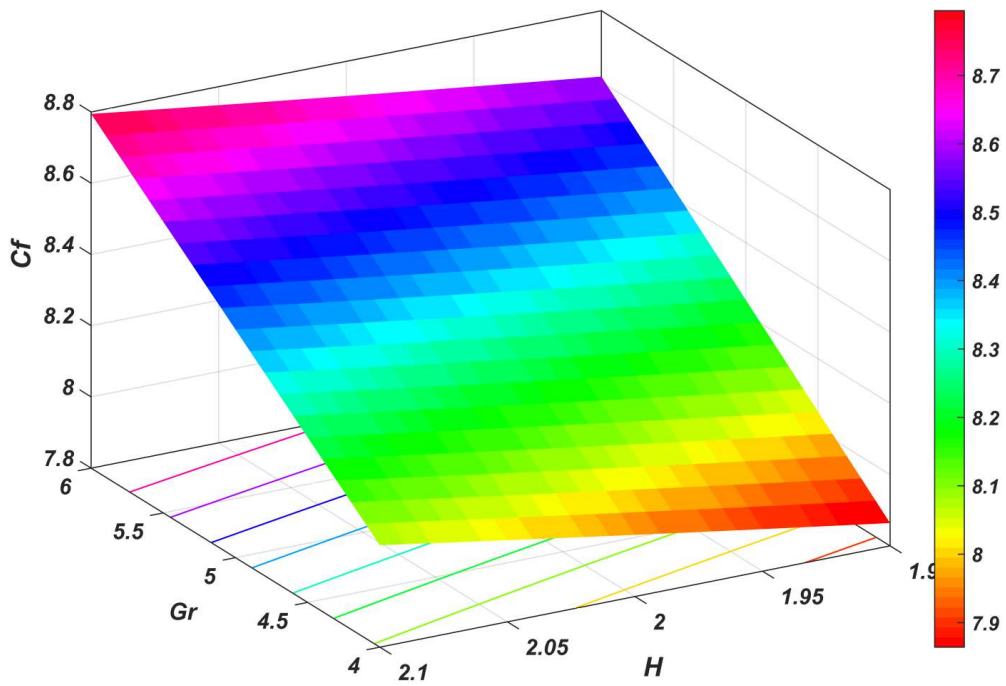


Figure 2.15: Variation in C_f with Gr and H at $y = 0$

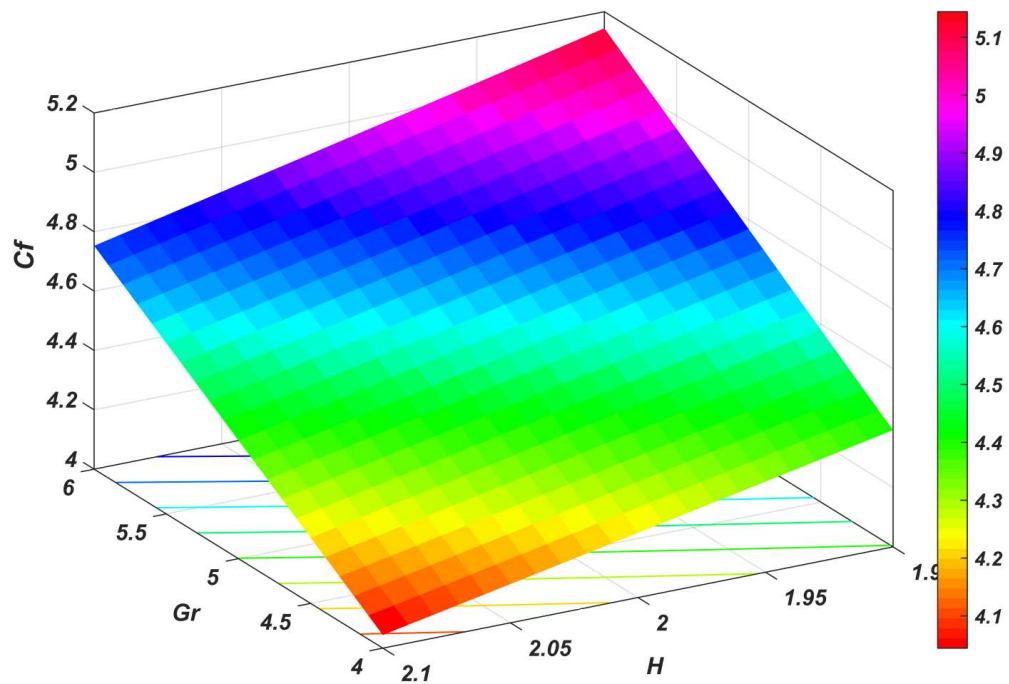


Figure 2.16: Variation in C_f with Gr and H at $y = 1$

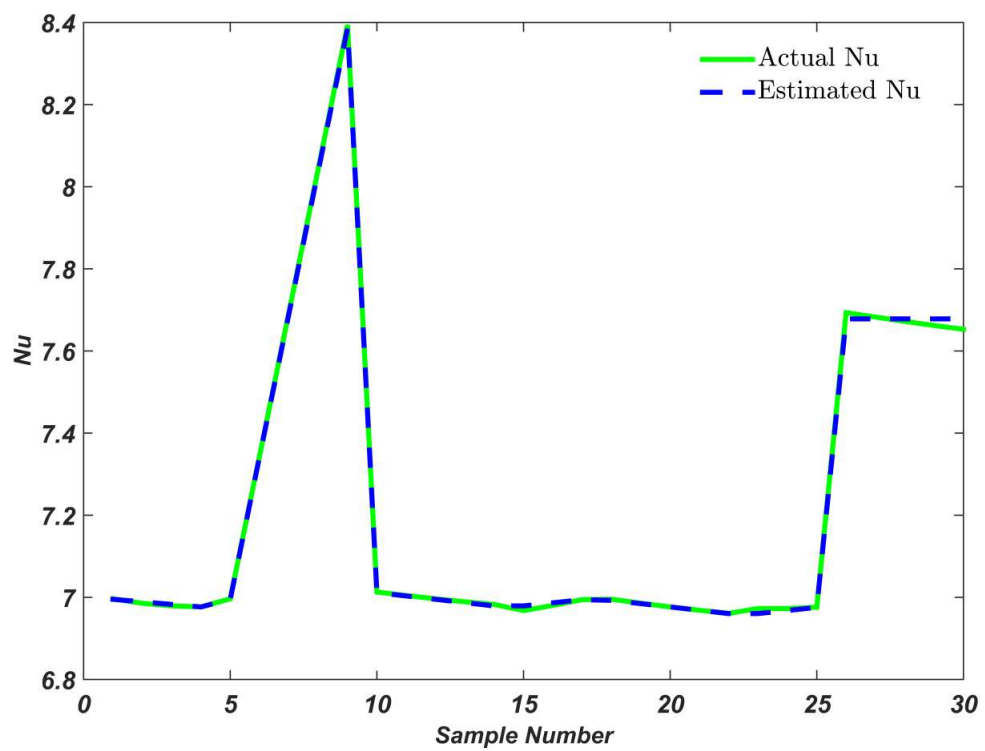


Figure 2.17: Actual Nu versus estimated Nu

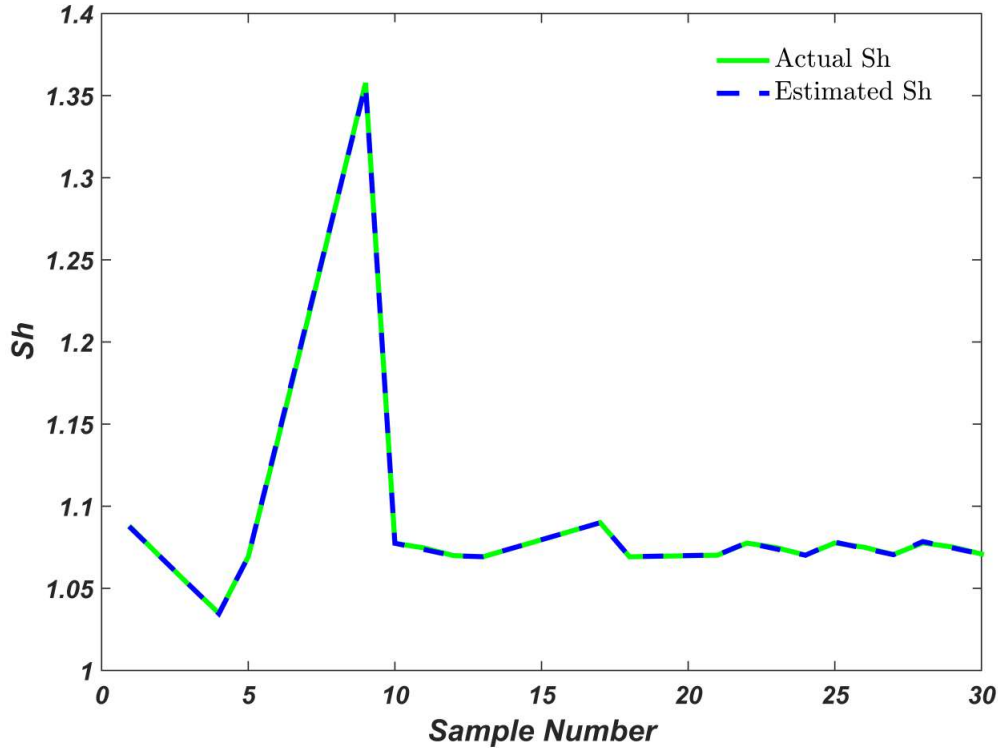


Figure 2.18: *Actual Sh versus estimated Sh*

Appendix I: Non-Dimensional Quantities

$$Pr = \frac{(\mu c_p)_f}{\kappa_f} \quad \text{Prandtl Number}$$

$$Re = \frac{V_0 d}{\vartheta_f} \quad \text{Injection/ Suction Parameter}$$

$$K_c = \frac{K_l d^2}{\vartheta_f} \quad \text{Chemical reaction Parameter}$$

$$Sc = \frac{\vartheta_f}{D_B} \quad \text{Schmidth Number}$$

$$H = B_0 d \sqrt{\frac{\sigma_f}{\rho_f \vartheta_f}} \quad \text{Hartmann Number}$$

$$Gr = \frac{g \beta_f \vartheta_f (T_0 - T_1)}{U_0 V_0^2} \quad \text{Grashof Number}$$

$$Gm = \frac{g \beta_f \vartheta_f (C_0 - C_1)}{U_0 V_0^2} \quad \text{Modified Grashof Number}$$

Appendix II: Nomenclature

u^*, v^*, w^*	Velocity components	T^*	Fluid temperature
C^*	Fluid concentration	g	Acceleration due to gravity
t^*	Time	D_B	Chemical molecular diffusivity
C_0	Nanoparticle concentration near the upward moving plate	C_1	Nanoparticle concentration near the downward moving plate
T_0	Temperature of the fluid near the upward moving plate	T_1	Temperature of the fluid near the downward moving plate
B_0	Strength of magnetic field	U_0	Velocity of the moving plates
V_0	Injection velocity	K_l	Chemical reaction coefficient
σ	Electrical conductivity	p^*	Pressure
μ	Dynamic viscosity	ρ	Density
ν	Kinematic viscosity	ω^*	Angular velocity
K	Thermal conductivity	s	Nanoparticle
C_p	Specific heat at constant pressure	f	Base fluid
ϕ	Volume fraction of nanoparticles	nf	Nanofluid
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	Very small reference constants	d	Distance between the plates

Appendix III: Short keys

$$a = \frac{A_5 Pr Re}{A_6}$$

$$m_1 = \frac{Sc Re + \sqrt{Sc^2 Re^2 + 4 Sc K_c}}{2}$$

$$m_2 = \frac{Sc Re - \sqrt{Sc^2 Re^2 + 4 Sc K_c}}{2}$$

$$m_3 = \frac{A_1 A_2 Re + \sqrt{A_1^2 A_2^2 Re^2 + 4 A_1 A_3 H^2}}{2}$$

$$m_4 = \frac{A_1 A_2 Re - \sqrt{A_1^2 A_2^2 Re^2 + 4 A_1 A_3 H^2}}{2}$$

$$B_1 = \frac{A_1 A_2 A_4 Re^2 Gr}{(e^a - 1)(a^2 - A_1 A_2 Re a - A_1 A_3 H^2)}$$

$$B_2 = \frac{A_1 A_2 A_4 Re^2 Gme^{m_2}}{(e^{m_1} - e^{m_2})(m_1^2 - A_1 A_2 Rem_1 - A_1 A_3 H^2)}$$

$$B_3 = \frac{A_1 A_2 A_4 Re^2 Gme^{m_1}}{(e^{m_2} - e^{m_1})(m_2^2 - A_1 A_2 Rem_2 - A_1 A_3 H^2)}$$

$$B_4 = \frac{A_1 A_2 A_4 Re^2 Gre^a}{(e^a - 1)(A_1 A_3 H^2)}$$

$$\alpha_1 = -(B_1 + B_2 + B_3 + B_4)$$

$$\beta_1 = -2 - (B_1 e^a + B_2 e^{m_1} + B_3 e^{m_2} + B_4)$$

$$r_1 = \frac{m_3 + \sqrt{m_3^2 + 4\pi^2}}{2}$$

$$r_2 = \frac{m_3 - \sqrt{m_3^2 + 4\pi^2}}{2}$$

$$r_3 = \frac{m_4 + \sqrt{m_4^2 + 4\pi^2}}{2}$$

$$r_4 = \frac{m_4 - \sqrt{m_4^2 + 4\pi^2}}{2}$$

$$r_5 = \frac{A_1 A_2 Re + \sqrt{A_1^2 A_2^2 Re^2 + 4(i\omega A_1 A_2 + A_1 A_3 H^2)}}{2}$$

$$r_6 = \frac{A_1 A_2 Re - \sqrt{A_1^2 A_2^2 Re^2 + 4(i\omega A_1 A_2 + A_1 A_3 H^2)}}{2}$$

$$D_1 = r_3 r_4 (e^{r_2 + r_4} - e^{r_2 + r_3}) + r_2 r_4 (e^{r_2 + r_3} - e^{r_3 + r_4}) + r_2 r_3 (e^{r_3 + r_4} - e^{r_2 + r_4})$$

$$D_2 = r_3 r_4 (e^{r_1 + r_3} - e^{r_1 + r_4}) + r_1 r_4 (e^{r_3 + r_4} - e^{r_1 + r_3}) + r_1 r_3 (e^{r_1 + r_4} - e^{r_3 + r_4})$$

$$D_3 = r_2 r_4 (e^{r_1 + r_4} - e^{r_1 + r_2}) + r_1 r_4 (e^{r_1 + r_2} - e^{r_2 + r_4}) + r_1 r_2 (e^{r_2 + r_4} - e^{r_1 + r_4})$$

$$D_4 = r_2 r_3 (e^{r_1 + r_2} - e^{r_1 + r_3}) + r_1 r_3 (e^{r_2 + r_3} - e^{r_1 + r_2}) + r_1 r_2 (e^{r_1 + r_3} - e^{r_2 + r_3})$$

$$D = D_1 + D_2 + D_3 + D_4$$

$$n_1 = \frac{\frac{A_5 Pr Re}{A_6} + \sqrt{\frac{A_5^2 Pr^2 Re^2}{A_6^2} + \frac{4 A_5 Pr \omega i}{A_6}}}{2}$$

$$n_2 = \frac{\frac{A_5 Pr Re}{A_6} - \sqrt{\frac{A_5^2 Pr^2 Re^2}{A_6^2} + \frac{4 A_5 Pr \omega i}{A_6}}}{2}$$

$$m_5 = \frac{Sc Re + \sqrt{Sc^2 Re^2 + 4 Sc(\omega i + K_c)}}{2}$$

$$m_6 = \frac{Sc Re - \sqrt{Sc^2 Re^2 + 4 Sc(\omega i + K_c)}}{2}$$

$$K_1 = \frac{A_1 A_2 A_4 Re^2 Gre^{n_2}}{(e^{n_1} - e^{n_2})(n_1^2 - A_1 A_2 Ren_1 - A_1 A_2 i\omega - A_1 A_3 H^2)}$$

$$K_2 = \frac{A_1 A_2 A_4 Re^2 Gre^{n_1}}{(e^{n_2} - e^{n_1})(n_2^2 - A_1 A_2 Ren_2 - A_1 A_2 i\omega - A_1 A_3 H^2)}$$

$$K_3 = \frac{A_1 A_2 A_4 Re^2 Gme^{m_6}}{(e^{m_5} - e^{m_6})(m_5^2 - A_1 A_2 Rem_5 - A_1 A_2 i\omega - A_1 A_3 H^2)}$$

$$K_4 = \frac{A_1 A_2 A_4 Re^2 Gme^{m_5}}{(e^{m_6} - e^{m_5})(m_6^2 - A_1 A_2 Rem_6 - A_1 A_2 i\omega - A_1 A_3 H^2)}$$

$$\alpha_2 = -(K_1 + K_2 + K_3 + K_4)$$

$$\beta_2 = -(K_1 e^{n_1} + K_2 e^{n_2} + K_3 e^{m_5} + K_4 e^{m_6})$$

$$n_3 = \frac{A_1 A_2 Re + \sqrt{A_1^2 A_2^2 Re^2 + 4(A_1 A_3 H^2 + \pi^2)}}{2}$$

$$n_4 = \frac{A_1 A_2 Re - \sqrt{A_1^2 A_2^2 Re^2 + 4(A_1 A_3 H^2 + \pi^2)}}{2}$$

$$K_{i1} = \frac{m_3(\alpha_1 e^{m_4} - \beta_1) A_1 A_2 \operatorname{Re} D_i}{(e^{m_4} - e^{m_3}) D[(r_i + m_3)^2 - A_1 A_2 \operatorname{Re}(r_i + m_3) - (A_1 A_3 H^2 + \pi^2)]}, i = 1, 2, 3, 4$$

$$K_{i2} = \frac{m_4(\beta_1 - \alpha_1 e^{m_3}) A_1 A_2 \operatorname{Re} D_i}{(e^{m_4} - e^{m_3}) D[(r_i + m_4)^2 - A_1 A_2 \operatorname{Re}(r_i + m_4) - (A_1 A_3 H^2 + \pi^2)]}, i = 1, 2, 3, 4$$

$$M_{i1} = \frac{m_3((\alpha_1 + 2)e^{m_4} - (\beta_1 + 2)) A_1 A_2 \operatorname{Re} D_i}{(e^{m_4} - e^{m_3}) D[(r_i + m_3)^2 - A_1 A_2 \operatorname{Re}(r_i + m_3) - (A_1 A_3 H^2 + \pi^2)]}, i = 1, 2, 3, 4$$

$$M_{i2} = \frac{m_4((\beta_1 + 2) - (\alpha_1 + 2)e^{m_3}) A_1 A_2 \operatorname{Re} D_i}{(e^{m_4} - e^{m_3}) D[(r_i + m_4)^2 - A_1 A_2 \operatorname{Re}(r_i + m_4) - (A_1 A_3 H^2 + \pi^2)]}, i = 1, 2, 3, 4$$

$$K_{i3} = M_{i3} = \frac{a B_1 A_1 A_2 \operatorname{Re} D_i}{D[(r_i + a)^2 - A_1 A_2 \operatorname{Re}(r_i + a) - (A_1 A_3 H^2 + \pi^2)]}, i = 1, 2, 3, 4$$

$$K_{i4} = M_{i4} = \frac{m_1 B_2 A_1 A_2 \operatorname{Re} D_i}{D[(r_i + m_1)^2 - A_1 A_2 \operatorname{Re}(r_i + m_1) - (A_1 A_3 H^2 + \pi^2)]}, i = 1, 2, 3, 4$$

$$K_{i5} = M_{i5} = \frac{m_2 B_3 A_1 A_2 \operatorname{Re} D_i}{D[(r_i + m_2)^2 - A_1 A_2 \operatorname{Re}(r_i + m_2) - (A_1 A_3 H^2 + \pi^2)]}, i = 1, 2, 3, 4$$

$$\alpha_3 = - \sum_{i=1}^4 (K_{i1} + K_{i2} + K_{i3} + K_{i4} + K_{i5})$$

$$\beta_3 = - \sum_{i=1}^4 (K_{i1} e^{r_i + m_3} + K_{i2} e^{r_i + m_4} + K_{i3} e^{r_i + a} + K_{i4} e^{r_i + m_1} + K_{i5} e^{r_i + m_2})$$

$$\alpha_4 = - \sum_{i=1}^4 (M_{i1} + M_{i2} + M_{i3} + M_{i4} + M_{i5})$$

$$\beta_4 = - \sum_{i=1}^4 (M_{i1} e^{r_i + m_3} + M_{i2} e^{r_i + m_4} + M_{i3} e^{r_i + a} + M_{i4} e^{r_i + m_1} + M_{i5} e^{r_i + m_2})$$

CHAPTER 3

Three-dimensional MHD Hybrid Nanofluid Flow Between Two Vertical Porous Plates Moving In Opposite Directions *

3.1 INTRODUCTION

The theoretical and experimental survey on nanofluids, the efficient energy transport fluids describes their enormous interpretations and implications including heat exchangers and electronic devices. The strengthening in thermophysical properties of nanofluids adjudged, hybrid nanofluid attained much attention. The synergic effects improve thermal qualities and potential utilities of nanoliquid, beneficial in the industrial area such as nuclear system cooling, thermal energy generating system, hot rolling, solar energy systems, etc.

The present chapter aims to analytically explore the three-dimensional convective hydromagnetic hybrid nanoliquid (with suspended Al_2O_3 and Fe_3O_4 nanoparticles) flow between two oppositely moving vertical porous plates. Governing equations are solved using the perturbation technique and the consequence of effectual

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parameters on velocity and temperature profiles are analyzed with aid of graphs using MATLAB software. The rate of heat transfer is statistically scrutinized using response surface methodology and sensitivity analysis.

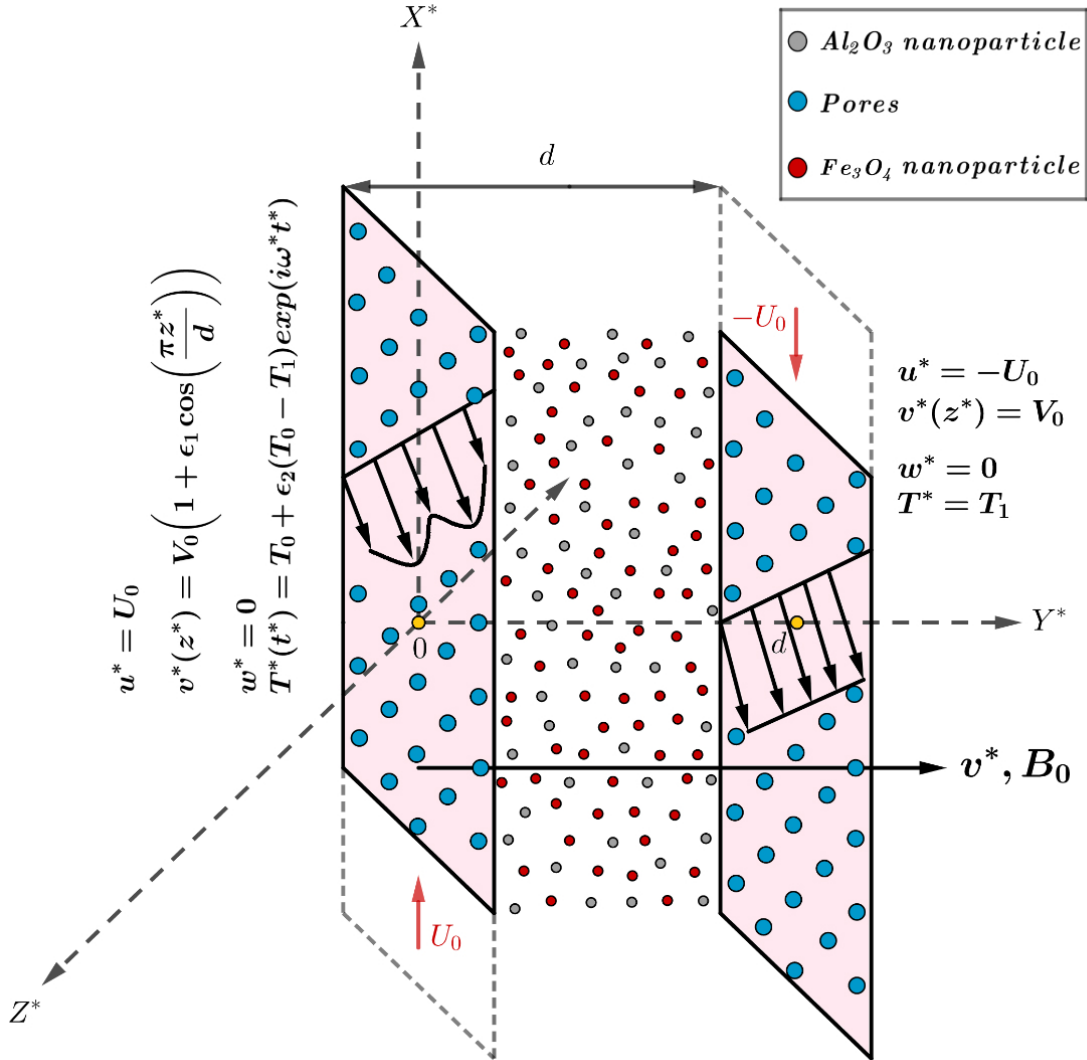


Figure 3.1: Physical configuration

3.2 MATHEMATICAL FORMULATION

An unsteady convective hybrid nanoliquid flow between two vertical porous plates involving a magnetic field (of uniform strength, B_0 applied normally to the plane of the plate) is considered (see Figure 3.1). The problem is developed utilizing the following conditions:

1. Parallel plates are traveling in different directions with uniform velocities.

2. The upward and downward moving plates are subjected to transverse sinusoidal injection velocity and constant suction velocity, respectively.
3. Induced magnetic field has been neglected due to the assumption of a small magnetic Reynolds number.
4. The injection velocity distribution is of the form:

$$v^*(z^*) = V_0(1 + \varepsilon_1 \cos(\pi z^*/d))$$

5. Without loss of generality, the distance d between the plates is taken equal to the wavelength of the injection velocity.
6. The temperature of the downward moving plate is at constant temperature T_1 and that of the upward moving plate fluctuating with time is given as:

$$T^*(t^*) = T_0 + \varepsilon_2(T_0 - T_1)e^{i\omega^*t^*}$$

Utilizing Boussinesq's approximation and the above assumptions, governing equations (referred from (Singh & Mathew, 2009b)) are given by:

$$\frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0 \quad (3.2.1)$$

$$\begin{aligned} \frac{\partial u^*}{\partial t^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} = & -\frac{1}{\rho_{hnf}} \left[\frac{\partial p^*}{\partial x^*} - \mu_{hnf} \left(\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}} \right) + \sigma_{hnf} B_0^2 u^* \right] + \\ & g\beta_{hnf}(T^* - T_1) \end{aligned} \quad (3.2.2)$$

$$\frac{\partial v^*}{\partial t^*} + v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z^*} = -\frac{1}{\rho_{hnf}} \left[\frac{\partial p^*}{\partial y^*} - \mu_{hnf} \left(\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}} \right) \right] \quad (3.2.3)$$

$$\frac{\partial w^*}{\partial t^*} + v^* \frac{\partial w^*}{\partial y^*} + w^* \frac{\partial w^*}{\partial z^*} = -\frac{1}{\rho_{hnf}} \left[\frac{\partial p^*}{\partial z^*} - \mu_{hnf} \left(\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}} \right) + \sigma_{hnf} B_0^2 w^* \right] \quad (3.2.4)$$

$$\frac{\partial T^*}{\partial t^*} + v^* \frac{\partial T^*}{\partial y^*} + w^* \frac{\partial T^*}{\partial z^*} = \frac{K_{hnf}}{(\rho C_p)_{hnf}} \left[\frac{\partial^2 T^*}{\partial y^{*2}} + \frac{\partial^2 T^*}{\partial z^{*2}} \right] \quad (3.2.5)$$

subject to the boundary conditions:

$$\left. \begin{aligned} y^* = 0, u^* = U_0, v^*(z^*) = V_0(1 + \varepsilon_1 \cos \frac{\pi z^*}{d}), w^* = 0, \\ T^*(t^*) = T_0 + \varepsilon_2(T_0 - T_1)e^{i\omega^* t^*}, \\ y^* = d, u^* = -U_0, v^*(z^*) = V_0, w^* = 0, T^* = T_1 \end{aligned} \right\} \quad (3.2.6)$$

Hybrid nanofluid model used is

Effective Dynamic Viscosity:

$$C_1 = \frac{\mu_f}{\mu_{hnf}} = (1 - \phi_1)^{2.5}(1 - \phi_2)^{2.5}$$

Effective Density:

$$C_2 = \frac{\rho_{hnf}}{\rho_f} = (1 - \phi_2) \left[1 - \phi_1 + \phi_1 \left(\frac{\rho_{s1}}{\rho_f} \right) \right] + \phi_2 \left(\frac{\rho_{s2}}{\rho_f} \right)$$

Effective Electrical Conductivity:

$$C_3 = \frac{\sigma_{hnf}}{\sigma_f} = 1 + \frac{3 \left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{\sigma_f} - (\phi_1 + \phi_2) \right)}{2 + \left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{(\phi_1 + \phi_2) \sigma_f} \right) - \left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{\sigma_f} - (\phi_1 + \phi_2) \right)}$$

Effective Coefficient Of Thermal Expansion:

$$C_4 = \frac{\beta_{hnf}}{\beta_f} = (1 - \phi_2) \left[1 - \phi_1 + \phi_1 \left(\frac{\beta_{s1}}{\beta_f} \right) \right] + \phi_2 \left(\frac{\beta_{s2}}{\beta_f} \right)$$

Effective Specific Heat:

$$C_5 = \frac{(\rho C_p)_{hnf}}{(\rho C_p)_f} = (1 - \phi_2) \left[1 - \phi_1 + \phi_1 \left(\frac{(\rho C_p)_{s1}}{(\rho C_p)_f} \right) \right] + \phi_2 \left(\frac{(\rho C_p)_{s2}}{(\rho C_p)_f} \right)$$

Effective Thermal Conductivity:

$$C_6 = \frac{K_{hnf}}{K_f}$$

where

$$\frac{K_{hnf}}{K_{nf}} = \frac{K_{s2} + 2K_{nf} - 2\phi_2(K_{nf} - K_{s2})}{K_{s2} + 2K_{nf} + 2\phi_2(K_{nf} - K_{s2})}$$

and

$$\frac{K_{nf}}{K_f} = \frac{K_{s1} + 2K_f - 2\phi_1(K_f - K_{s1})}{K_{s1} + 2K_f + 2\phi_1(K_f - K_{s1})}$$

The following dimensionless quantities are introduced into equations (3.2.1) to (3.2.6) (except (3.2.2)),

$$y = \frac{y^*}{d}, z = \frac{z^*}{d}, t = t^* \omega^*, u = \frac{u^*}{U_0}, v = \frac{v^*}{V_0}, w = \frac{w^*}{V_0},$$

$$p = \frac{p^*}{\rho_{hnf} V_0^2}, \omega = \frac{\omega^* d^2}{\vartheta}, \theta = \frac{T^* - T_1}{T_0 - T_1},$$

The reduced equations take the form:

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (3.2.7)$$

$$\frac{\omega}{Re} \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{C_1 C_2 Re} \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad (3.2.8)$$

$$\frac{\omega}{Re} \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{C_1 C_2 Re} \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{C_3}{C_2 Re} H^2 w \quad (3.2.9)$$

$$\frac{\omega}{Re} \frac{\partial \theta}{\partial t} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = \frac{C_6}{C_5 Pr Re} \left(\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right) \quad (3.2.10)$$

Equation (3.3.2) permutes to the following cases:

Case 1: Magnetic Field is applied on the upward moving plate (at $y=0$)

$$\frac{\omega}{Re} \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{1}{C_1 C_2 Re} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{C_3}{C_2 Re} H^2 (u - 1) + C_4 Gr Re \theta \quad (3.2.11)$$

Case 2: Magnetic Field is applied on the downward moving plate (at $y=1$)

$$\frac{\omega}{Re} \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{1}{C_1 C_2 Re} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{C_3}{C_2 Re} H^2 (u + 1) + C_4 Gr Re \theta \quad (3.2.12)$$

The reduced boundary conditions take the form:

$$\left. \begin{aligned} y = 0, u = 1, v(z) = 1 + \varepsilon_1 \cos \pi z, w = 0, \theta = 1 + \varepsilon_2 e^{it} \\ y = 1, u = -1, v = 1, w = 0, \theta = 0 \end{aligned} \right\} \quad (3.2.13)$$

3.3 METHOD OF SOLUTION

The reduced forms of the governing equations are resolved using the perturbation method. For this, let $\varepsilon = \min \{\varepsilon_1, \varepsilon_2\}$ be very small and suppose that solution is of the format

$$f(y, z, t) = f_0(y) + \varepsilon f_1(y, z, t) + O(\varepsilon^2) \quad (3.3.1)$$

3.3.1 Steady Flow Solution

Letting $\varepsilon = 0$, the current problem narrows to a steady two dimensional flow which is governed by the ensuing equations:

Case 1: Magnetic Field is applied on the upward moving plate (at $y=0$)

$$u_0'' - C_1 C_2 Re u_0' - C_1 C_3 H^2 (u_0 - 1) + C_1 C_2 C_4 Re^2 Gr \theta_0 = 0 \quad (3.3.2)$$

Case 2: Magnetic Field is applied on the downward moving plate (at $y=1$)

$$u_0'' - C_1 C_2 Re u_0' - C_1 C_3 H^2 (u_0 + 1) + C_1 C_2 C_4 Re^2 Gr \theta_0 = 0 \quad (3.3.3)$$

with $v_0 = 1, w_0 = 0, p_0 = \text{a constant}$ and

$$\theta_0'' - \frac{C_5 Pr Re}{C_6} \theta_0' = 0 \quad (3.3.4)$$

where prime notates the derivative with respect to y .

The analogous boundary conditions take the form:

$$\left. \begin{aligned} y = 0, u_0 = 1, \theta_0 = 1 \\ y = 1, u_0 = -1, \theta_0 = 0 \end{aligned} \right\} \quad (3.3.5)$$

Solving equations (3.3.2) to (3.3.4) with respect to (3.3.5) yields:

$$\theta_0 = \frac{1}{e^{a_1} - 1}(e^{a_1} - e^{a_1 y}) \quad (3.3.6)$$

Case 1: Magnetic Field is applied on the upward moving plate (at $y=0$)

$$u_0 = \frac{1}{e^{b_2} - e^{b_1}}[(\alpha_1 e^{b_2} - \beta_1)e^{b_1 y} + (\beta_1 - \alpha_1 e^{b_1})e^{b_2 y}] + A_1 e^{a_1 y} + A_2 + 1 \quad (3.3.7)$$

Case 2: Magnetic Field is applied on the downward moving plate (at $y=1$)

$$u_0 = \frac{1}{e^{b_2} - e^{b_1}}[((\alpha_1 + 2)e^{b_2} - (\beta_1 + 2))e^{b_1 y} + ((\beta_1 + 2) - (\alpha_1 + 2)e^{b_1})e^{b_2 y}] + A_1 e^{a_1 y} + A_2 - 1 \quad (3.3.8)$$

3.3.2 Cross Flow Solution

Letting $\varepsilon \neq 0$ applying equation (3.3.1) into equations (3.2.7) - (3.2.9) and equating like powers of ε and ignoring the higher powers of ε^2 , the following equations are derived:

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \quad (3.3.9)$$

$$\frac{\omega}{Re} \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + \frac{1}{C_1 C_2 Re} \left[\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right] \quad (3.3.10)$$

$$\frac{\omega}{Re} \frac{\partial w_1}{\partial t} + \frac{\partial w_1}{\partial y} = -\frac{\partial p_1}{\partial z} + \frac{1}{C_1 C_2 Re} \left[\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2} \right] - \frac{C_3}{C_2 Re} H^2 w_1 \quad (3.3.11)$$

Corresponding boundary conditions are

$$\left. \begin{aligned} y = 0, v_1 = \cos \pi z, w_1 = 0 \\ y = 1, v_1 = 0, w_1 = 0 \end{aligned} \right\} \quad (3.3.12)$$

These are the linear partial differential equations reporting the three-dimensional cross flow which are independent of the temperature field and the main flow com-

ponent. The solutions for v_1, w_1, p_1 are assumed to be of the form:

$$v_1(y, z, t) = v_{11}(y)e^{it} + v_{12}(y) \cos \pi z \quad (3.3.13)$$

$$w_1(y, z, t) = -(zv_{11}'(y)e^{it} + \frac{1}{\pi}v_{12}'(y) \sin \pi z) \quad (3.3.14)$$

$$p_1(y, z, t) = p_{11}(y)e^{it} + p_{12}(y) \cos \pi z \quad (3.3.15)$$

where prime notates the derivative with respect to y . Expressions (3.3.13) and (3.3.14) have been chosen so that the equation of continuity (3.3.9) is trivially satisfied. Applying these into equations (3.3.10) and (3.3.11) and employing the boundary condition (3.3.12), the solutions for v_1, w_1, p_1 are obtained as:

$$v_1 = \frac{1}{D} \sum_{i=1}^4 D_i e^{r_i y} \cos \pi z \quad (3.3.16)$$

$$w_1 = -\frac{1}{\pi D} \sum_{i=1}^4 r_i D_i e^{r_i y} \sin \pi z \quad (3.3.17)$$

$$p_1 = -\frac{1}{C_1 C_2 Re \pi^2 D} \sum_{i=1}^4 D_i (r_i^3 - C_1 C_2 Re r_i^2 - (C_1 C_3 H^2 + \pi^2) r_i) e^{r_i y} \cos \pi z \quad (3.3.18)$$

3.3.3 Temperature Field

Comparably letting $\varepsilon \neq 0$, applying equation (3.3.1) to equation (3.2.10) and comparing like powers of ε , the equation for temperature field is given by:

$$\frac{\omega}{Re} \frac{\partial \theta_1}{\partial t} + \frac{\partial \theta_1}{\partial y} = \frac{C_6}{C_5 Pr Re} \left(\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} \right) \quad (3.3.19)$$

with

$$\left. \begin{aligned} y = 0, \theta_1 &= e^{it} \\ y = 1, \theta_1 &= 0, \end{aligned} \right\} \quad (3.3.20)$$

Equation (3.3.19) along with equation (3.3.20) is solved with a supposition that the solution is of the format:

$$\theta_1(y, z, t) = \theta_{11}e^{it} + \theta_{12} \cos \pi z \quad (3.3.21)$$

Substituting equation (3.3.21) into (3.3.19), one can obtain

$$\theta_{11}'' - \frac{C_5 Pr Re}{C_6} \theta_{11}' - \frac{C_5 Pr \omega i}{C_6} \theta_{11} = 0 \quad (3.3.22)$$

$$\theta_{12}'' - \frac{C_5 Pr Re}{C_6} \theta_{12}' - \pi^2 \theta_{12} = 0 \quad (3.3.23)$$

with

$$\left. \begin{aligned} y = 0, \theta_{11} = 1, \theta_{12} = 0 \\ y = 1, \theta_{11} = 0, \theta_{12} = 0 \end{aligned} \right\} \quad (3.3.24)$$

Resolving equations (3.3.22) and (3.3.23) utilizing equation (3.3.24), the solution is given by:

$$\theta_1(y, z, t) = \frac{1}{e^{a_3} - e^{a_2}} (e^{a_3} e^{a_2 y} - e^{a_2} e^{a_3 y}) e^{it} \quad (3.3.25)$$

3.3.4 Main Flow Solution

Letting $\varepsilon \neq 0$, the first order equation for the main flow deduced with the help of equation (3.3.1) and equating like powers of ε is given by:

$$\frac{\omega}{Re} \frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial y} + v_1 u_0' = \frac{1}{C_1 C_2 Re} \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} \right) - \frac{C_3}{C_2 Re} H^2 u_1 + C_4 Gr Re \theta_1 \quad (3.3.26)$$

with

$$\left. \begin{aligned} y = 0, \quad u_1 = 0 \\ y = 1, \quad u_1 = 0 \end{aligned} \right\} \quad (3.3.27)$$

Suppose the solution is of the format:

$$u_1(y, z, t) = u_{11}e^{it} + u_{12} \cos \pi z \quad (3.3.28)$$

Applying equation (3.3.28) in equation (3.3.26) and comparing like powers of ε :

$$u_{11}'' - C_1 C_2 Re u_{11}' - (C_1 C_2 \omega i + C_1 C_3 H^2) u_{11} = -C_1 C_2 C_4 Re^2 Gr \theta_{11} \quad (3.3.29)$$

$$u_{12}'' - C_1 C_2 Re u_{12}' - (\pi^2 + C_1 C_3 H^2) u_{12} = C_1 C_2 Re v_{12} u_0' \quad (3.3.30)$$

with

$$\left. \begin{aligned} y = 0, \quad u_{11} = 0, \quad u_{12} = 0 \\ y = 1, \quad u_{11} = 0, \quad u_{12} = 0 \end{aligned} \right\} \quad (3.3.31)$$

Resolving equations (3.3.29) and (3.3.30) utilizing equation (3.3.31), the solution is derived as:

Case 1: Magnetic Field is applied on the upward moving plate (at $y=0$)

$$u_1(y, z, t) = \left\{ \frac{1}{e^{b_4} - e^{b_3}} [(\alpha_2 e^{b_4} - \beta_2) e^{b_3 y} + (\beta_2 - \alpha_2 e^{b_3}) e^{b_4 y}] + \frac{A_3 e^{a_2 y} + A_4 e^{a_3 y}}{\cos \pi z} \right\} e^{it} + \left\{ \frac{1}{e^{b_6} - e^{b_5}} [(\alpha_3 e^{b_6} - \beta_3) e^{b_5 y} + (\beta_3 - \alpha_3 e^{b_5}) e^{b_6 y}] + \frac{\sum_{i=1}^4 A_{i1} e^{(r_i+b_1)y} + A_{i2} e^{(r_i+b_2)y} + A_{i3} e^{(r_i+a_1)y}}{\cos \pi z} \right\} \cos \pi z \quad (3.3.32)$$

Case 2: Magnetic Field is applied on the downward moving plate (at $y=1$)

$$u_1(y, z, t) = \left\{ \frac{1}{e^{b_4} - e^{b_3}} [(\alpha_2 e^{b_4} - \beta_2) e^{b_3 y} + (\beta_2 - \alpha_2 e^{b_3}) e^{b_4 y}] + \frac{A_3 e^{a_2 y} + A_4 e^{a_3 y}}{\cos \pi z} \right\} e^{it} + \left\{ \frac{1}{e^{b_6} - e^{b_5}} [(\alpha_4 e^{b_6} - \beta_4) e^{b_5 y} + (\beta_4 - \alpha_4 e^{b_5}) e^{b_6 y}] + \frac{\sum_{i=1}^4 B_{i1} e^{(r_i+b_1)y} + B_{i2} e^{(r_i+b_2)y} + B_{i3} e^{(r_i+a_1)y}}{\cos \pi z} \right\} \cos \pi z \quad (3.3.33)$$

3.4 PHYSICAL QUANTITIES

Physical quantities like drag coefficient (Cf) and Nusselt number (Nu) measuring surface drag and rate of heat transfer, respectively are given by:

Case 1: Magnetic Field is applied on the upward moving plate (at $y=0$)

$$Cf = \frac{d}{\mu_f U_0} \left| \mu_{hnf} \left(\frac{du^*}{dy^*} \right)_{y^*=d} \right| = \frac{1}{C_1} \left| \left(\frac{du_0}{dy} \right)_{y=1} + \varepsilon \left(\frac{du_1}{dy} \right)_{y=1} \right| \quad (3.4.1)$$

Case 2: Magnetic Field is applied on the downward moving plate (at $y=1$)

$$Cf = \frac{d}{\mu_f U_0} \left| \mu_{hnf} \left(\frac{du^*}{dy^*} \right)_{y^*=d} \right| = \frac{1}{C_1} \left| \left(\frac{du_0}{dy} \right)_{y=1} + \varepsilon \left(\frac{du_1}{dy} \right)_{y=1} \right| \quad (3.4.2)$$

$$Nu = \frac{d}{K_f(T_0 - T_1)} \left| K_{hnf} \left(\frac{dT^*}{dy^*} \right)_{y^*=d} \right| = C_6 \left| \left(\frac{d\theta_0}{dy} \right)_{y=0} + \varepsilon \left(\frac{d\theta_1}{dy} \right)_{y=0} \right| \quad (3.4.3)$$

3.5 RESULTS AND DISCUSSION

The significance of effectual parameters on velocity (u), surface drag (Cf) and temperature (θ) profiles are depicted through Figures 3.2 - 3.13. The physical properties of nanoparticles (Al_2O_3 and Fe_3O_4) and the conventional fluid (water) are identified in Table 3.1. $\phi_1 = 0.1, \phi_2 = 0.1, \omega = 10, Re = 1, H = 2, Pr = 7, Gr = 7$ and $t = \pi/2$ are the base values of parameters employed (unless specified) throughout the analysis. Further, the validation of the obtained results is achieved through a comparative study with the previous work (Neethu, Areekara, & Mathew, 2021) (see Table 3.2) and a good agreement is noted.

Figure 3.2 explains the positive effect of Grashof number (Gr) on u meaning that an augmentation in Gr will increase the velocity. Physically, on magnifying Gr the buoyancy forces becomes prominent which results in ascending the velocity profile. Figure 3.3 manifests the consequence of Hartmann number (H) on u . The introduction of a magnetic field produces a drag force (Lorentz force) which sets up an opposite reaction on upward and downward moving plates. On varying H , it is noted that u ascends on upward moving plate whereas u descends on the downward-moving plate. The negative influence of the volume fraction of nanoparticles (ϕ_1 and ϕ_2) on u is elucidated in Figures 3.4 and 3.5, respectively. This decrease in velocity can be physically attributed to the fact that increasing the volume fraction of nanoparticles swells the viscosity of hybrid nanoliquid which causes a drop in velocity. The influence of injection/suction parameter (Re) on u is depicted in Figure 3.6. Velocity profile experiences an exponential augmentation when Re values are improved.

Figure 3.7 and 3.8 reveals that intensification in the volume fraction of nanoparticles brings about a reduction in θ . The impact of Re on θ is displayed using Figure 3.9 and it is noted that Re causes an improvement in θ . Physically, this increase in temperature can be associated with the fact that with increasing Re values the heated nanoparticles enter the opposite moving plates and the cold nanoparticles exit the opposite moving plates.

The parallel effect of effectual parameters on drag coefficient (Cf) on the up-

ward and downward-moving plates is illustrated in Figures 3.10 - 3.13 with the aid of three-dimensional surface plots. It is seen that surface drag ascends with ϕ_1 , Re , H and Gr on the upward moving plate. Further, surface drag ascends with ϕ_1 , Re , and Gr and descends with H on the downward moving plate.

3.6 STATISTICAL ANALYSIS

3.6.1 Response Surface Methodology (RSM)

RSM is a statistical approach employed in analyzing the conjoint impact of effectual parameters (independent variables) on the physical quantity of interest (response or dependent variable). In this problem, Nu is chosen as the response variable and nanoparticle volume fraction of Al_2O_3 ($0.02 \leq \phi_1 \leq 0.08$), nanoparticle volume fraction of Fe_3O_4 ($0.02 \leq \phi_2 \leq 0.08$) and injection/suction parameter ($1 \leq Re \leq 2$) are chosen as the influential parameters. Table 3.3 bespeaks the effective parameters and their levels. The general model (adopting central composite design) for response variables involving linear, interactive and quadratic terms is expressed by:

$$\text{Response} = \lambda_0 + \lambda_1 A + \lambda_2 B + \lambda_3 C + \lambda_4 AB + \lambda_5 BC + \lambda_6 AC + \lambda_7 A^2 + \lambda_8 B^2 + \lambda_9 C^2 \quad (3.6.1)$$

where $\lambda_i (i = 0, 1, \dots, 9)$ represents the regression coefficients. The experimental design and the response for the 20 runs (according to CCD) are given in Table 3.4.

The analysis of variable (ANOVA) table (Table 3.5) illustrates the efficiency of the estimated model. A parameter is claimed as significant if the corresponding p-value is less than 0.05. It is observed that the quadratic terms in ϕ_1 and ϕ_2 are not significant. Hence, they removed these terms from the model. The coefficient of determination (R^2) for the model is found to be 100% which boosts the model's accuracy.

The fitted quadratic model for Nu is given by:

$$Nu = 10.2645 - 0.07064\phi_1 - 0.0394\phi_2 + 3.403Re + 0.01807Re^2 + 0.0053\phi_1\phi_2 - 0.03258\phi_1Re - 0.0209\phi_2Re \quad (3.6.2)$$

The reliability of the estimated model for Nu is further clarified using residual plots (see Figure 3.14). All points in the normal probability plot are situated

beside a straight line with an insignificant deflection and the residual histogram is approximately bell-shaped confirming the normal nature of residuals. Furthermore, a maximum error of 0.005 can be observed from the fitted versus residual plot which also contributes to the accuracy of the model.

From equation (3.6.2), it can be inferred that ϕ_1 and ϕ_2 have a negative impact on Nu and Re has a positive effect on Nu . The parallel interaction of two parameters on Nu is graphed using surface and contour plots (see Figure 3.15) by fixing the third parameter at the medium level. From Figures 3.15 (a) - (c), it is perceived that Nu is highest for smaller values of ϕ_1 and ϕ_2 and larger values of Re .

3.6.2 Sensitivity Analysis

Sensitivity analysis is a statistical technique that measures the extent and nature of dependency exhibited by the effectual parameters on the physical quantity of interest. In other words, sensitivity analysis accounts for the variation induced by the augmenting parameter on the remaining effectual parameter. The sign of sensitivity (positive or negative) signifies the nature of the correlation between Nu and the influential parameters. Further, the magnitude of sensitivity indicates the intensity of the effect on Nu .

The quadratic model (in coded form) after neglecting the insignificant terms is given by:

$$Nu = 10.2645 - 0.07064 A - 0.0394 B + 3.403 C + 0.01807 C^2 + 0.0053 AB - 0.03258 AC - 0.0209 BC \quad (3.6.3)$$

Then the sensitivity functions are:

$$\frac{\partial Nu}{\partial A} = -0.07064 + 0.0053 B - 0.03258 C \quad (3.6.4)$$

$$\frac{\partial Nu}{\partial B} = -0.0394 + 0.0053 A - 0.0209 C \quad (3.6.5)$$

$$\frac{\partial Nu}{\partial C} = 3.403 + 0.03614 C - 0.03258 A - 0.0209 B \quad (3.6.6)$$

The sensitivity for Nu is tabulated in Table 3.6 keeping ϕ_1 in the medium level. It is noted that ϕ_1 and ϕ_2 exhibit negative sensitivity and Re exhibits a positive

sensitivity towards Nu . The sensitivity of Nu is also visualized using bar charts (Figure 3.16). It is seen that the results of sensitivity analysis are in perfect harmony with the results inferred using RSM. It is also noticed that Nu is most sensitive with Re .

3.7 CONCLUSION

The key observations are:

- Grashof number has a constructive effect on main flow velocity.
- Hartman number positively contributes towards the velocity profile on the upward moving plate and negatively contributes towards the velocity profile on the downward moving plate.
- Main flow velocity profile is higher when the magnetic field is applied on the upward-moving plate.
- Drag coefficient is directly proportional to the volume fraction of nanoparticles.
- Rate of heat transfer is the most sensitive parameter with injection/suction parameter.
- Augmentation of volume fraction of Al_2O_3 nanoparticles has more influence on the flow profiles.
- Surface drag coefficient ascends with augmenting Hartmann number on the upward moving plate and descends on the downward moving plate.
- Volume fraction of nanoparticles exhibits a destructive effect on heat transfer rate.

TABLES AND GRAPHS

Table 3.1: *Physical properties of nanoparticles and base fluid*

Physical Proper- ties	H_2O	Al_2O_3	Fe_3O_4
ρ	997.1	3970	5180
C_p	4179	765	670
β	$21 * 10^{-5}$	$0.85 * 10^{-5}$	$1.3 * 10^{-5}$
σ	$5 * 10^{-2}$	$35 * 10^6$	25000
k	0.613	40	9.7

Table 3.2: *Comparison of Nu with augmenting Re values when $\phi_1 = 0, \phi_2 = 0, \Omega = 10, t = \pi/2, Pr = 7, Gr = 5$ and $H = 2$*

Re	Nu	
	(Neethu et al., 2021)	Present study
0.5	3.60681	3.60681
1	6.9976	6.9976
1.5	10.4771	10.4771
2	13.9665	13.9665
2.5	17.4835	17.4835

Table 3.3: *Effective parameter levels*

Parameter	Symbol	Levels		
		-1(low)	0(medium)	1(high)
ϕ_1	A	0.02	0.05	0.08
ϕ_2	B	0.02	0.05	0.08
Re	C	1	1.5	2

Table 3.4: *Experimental design with response*

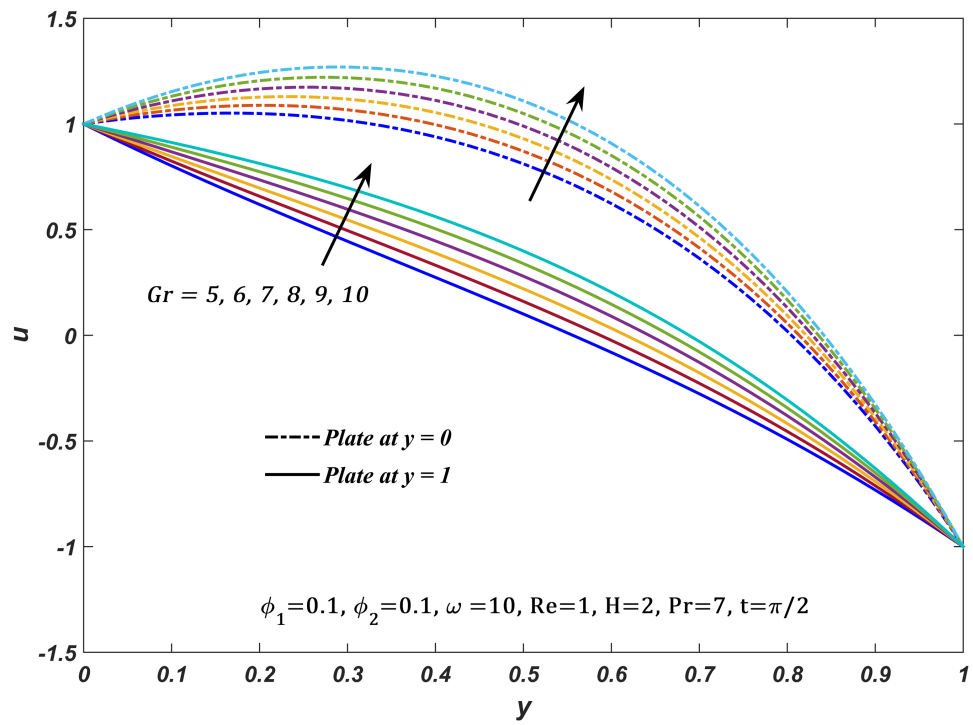
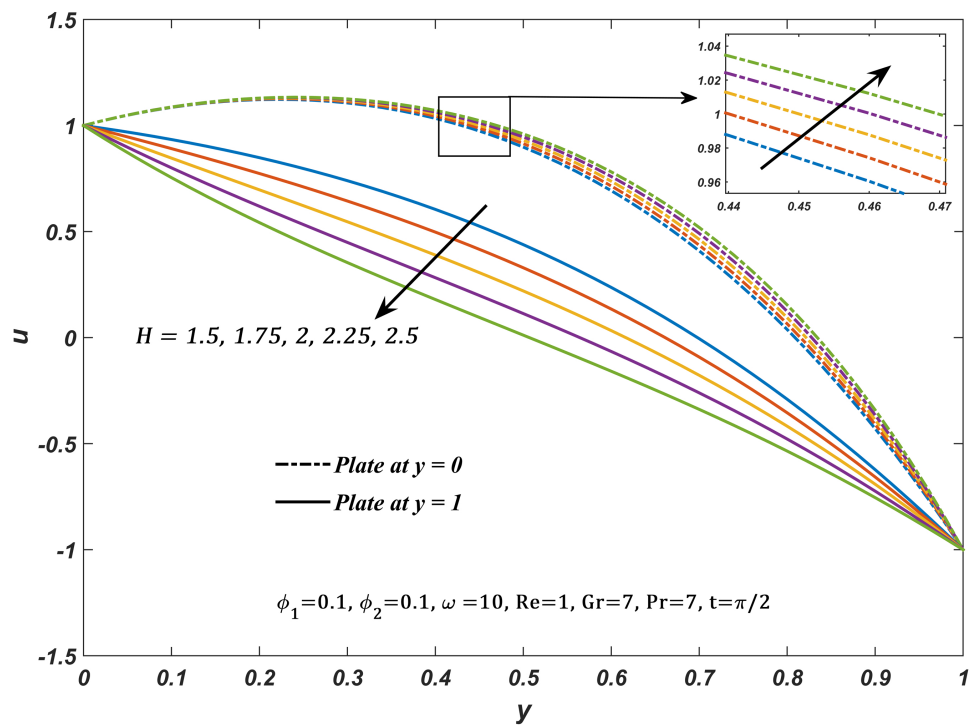
Run	Coded values			Actual values			Response
	A	B	C	ϕ_1	ϕ_2	Re	Nu
1	-1	-1	-1	0.02	0.02	1	6.9437
2	1	-1	-1	0.08	0.02	1	6.8562
3	-1	1	-1	0.02	0.08	1	6.895
4	1	1	-1	0.08	0.08	1	6.8352
5	-1	-1	1	0.02	0.02	2	13.8524
6	1	-1	1	0.08	0.02	2	13.6411
7	-1	1	1	0.02	0.08	2	13.7266
8	1	1	1	0.08	0.08	2	13.53
9	-1	0	0	0.02	0.05	1.5	10.3412
10	1	0	0	0.08	0.05	1.5	10.19
11	0	-1	0	0.05	0.02	1.5	10.309
12	0	1	0	0.05	0.08	1.5	10.2216
13	0	0	-1	0.05	0.05	1	6.8773
14	0	0	1	0.05	0.05	2	13.6873
15	0	0	0	0.05	0.05	1.5	10.2646
16	0	0	0	0.05	0.05	1.5	10.2646
17	0	0	0	0.05	0.05	1.5	10.2646
18	0	0	0	0.05	0.05	1.5	10.2646
19	0	0	0	0.05	0.05	1.5	10.2646
20	0	0	0	0.05	0.05	1.5	10.2646

Table 3.5: ANOVA table

	Deg. Of freedom	Adj. sum of squares	Adj. mean squares	Regression Coeffi- cient	F-Value	p-Value
Model	9	115.884	12.876		924250	0
<i>Linear</i>	3	115.87	38.623		2772413	0
ϕ_1	1	0.05	0.05	-0.0706	3581.88	0
ϕ_2	1	0.016	0.016	-0.0394	1114.3	0
<i>Re</i>	1	115.804	115.804	3.403	8312544	0
<i>Square</i>	3	0.002	0.001		46	0
$\phi_1 * \phi_1$	1	0	0	0.00137	0.37	0.556
$\phi_2 * \phi_2$	1	0	0	0.00107	0.23	0.644
$Re * Re$	1	0.001	0.001	0.01807	64.47	0
<i>2-Way Interac- tion</i>	3	0.012	0.004		292.11	0
$\phi_1 * \phi_2$	1	0	0	0.0053	16.13	0.002
$\phi_1 * Re$	1	0.008	0.008	-0.0326	609.35	0
$\phi_2 * Re$	1	0.003	0.003	-0.0209	250.84	0
<i>Constant</i>				10.2645		
<i>Error</i>	10	0	0			
<i>Lack-of- Fit</i>	5	0	0		*	*
<i>Pure Er- ror</i>	5	0	0			
<i>Total</i>	19	115.884				
$R^2 = 100\%$			Adjusted $R^2 = 100\%$			

Table 3.6: *Sensitivity of response Nu when $A = 0$*

B	C	Sensitivity		
		$\frac{\partial Nu}{\partial A}$	$\frac{\partial Nu}{\partial B}$	$\frac{\partial Nu}{\partial C}$
	-1	-0.0434	-0.0185	3.3878
-1	0	-0.0759	-0.0394	3.4239
	1	-0.1085	-0.0603	3.46
	-1	-0.0381	-0.0185	3.3669
0	0	-0.0706	-0.0394	3.403
	1	-0.1032	-0.0603	3.4391
	-1	-0.0328	-0.0185	3.346
1	0	-0.0653	-0.0394	3.3821
	1	-0.0979	-0.0603	3.4182


 Figure 3.2: Variation in u with Gr

 Figure 3.3: Variation in u with H

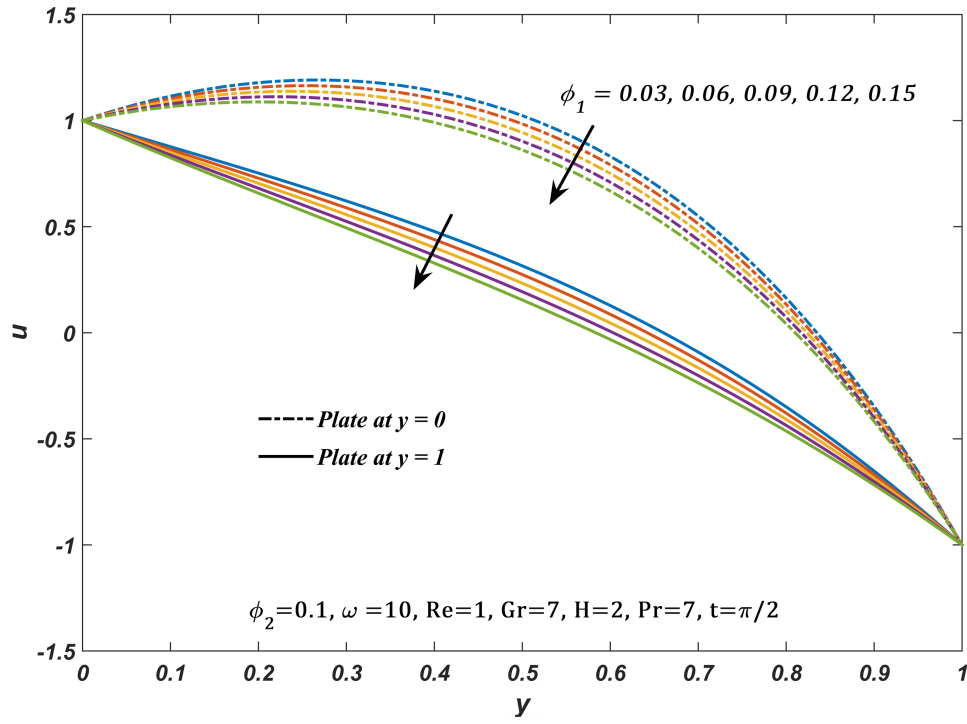


Figure 3.4: Variation in u with ϕ_1

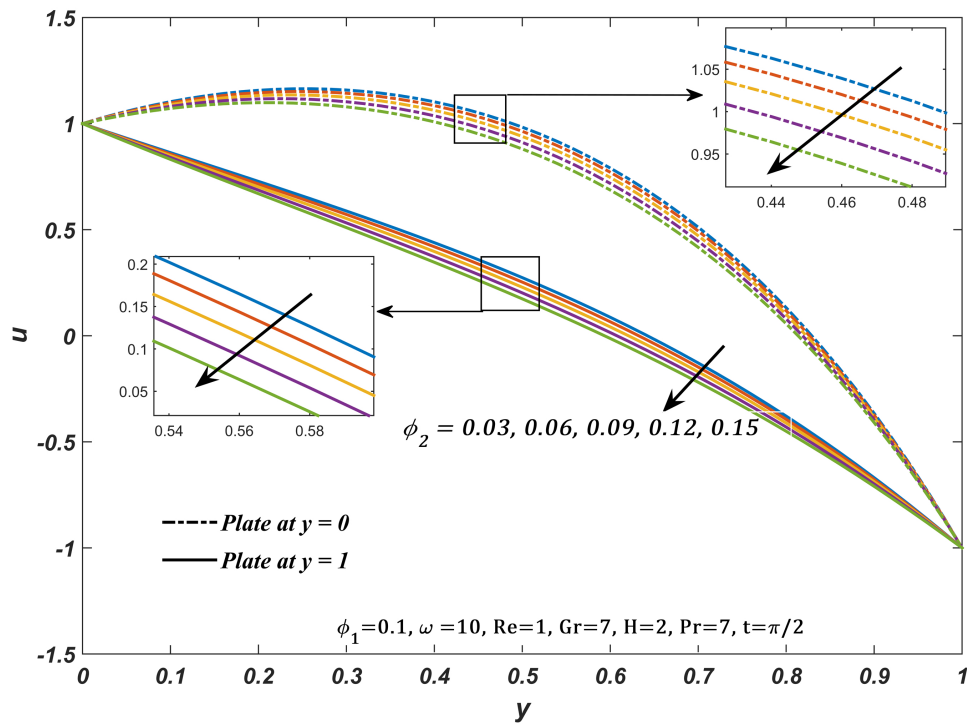
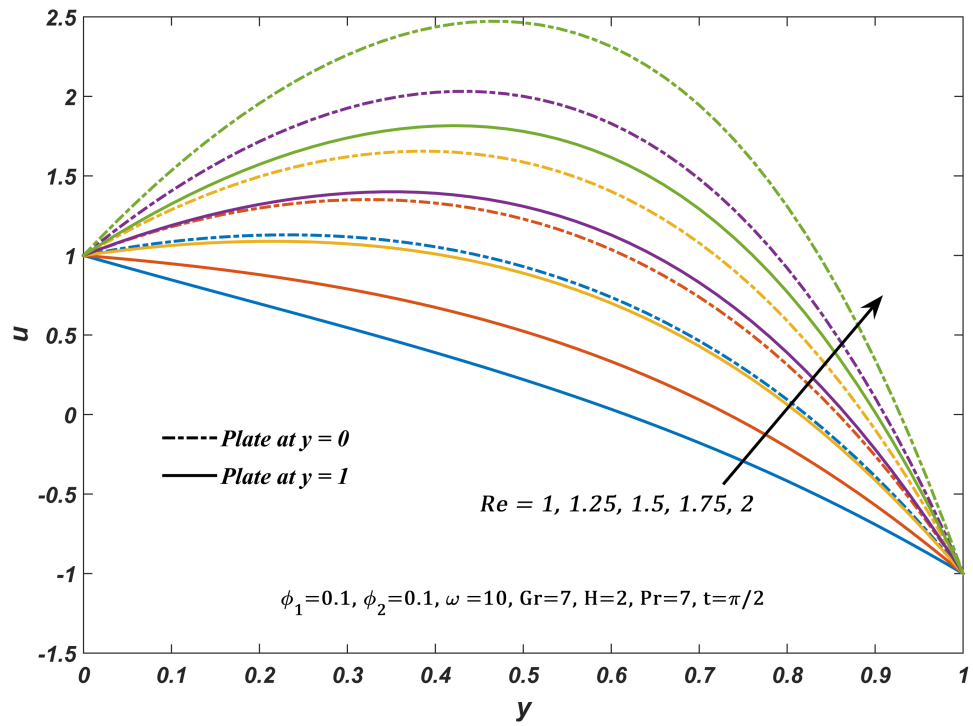
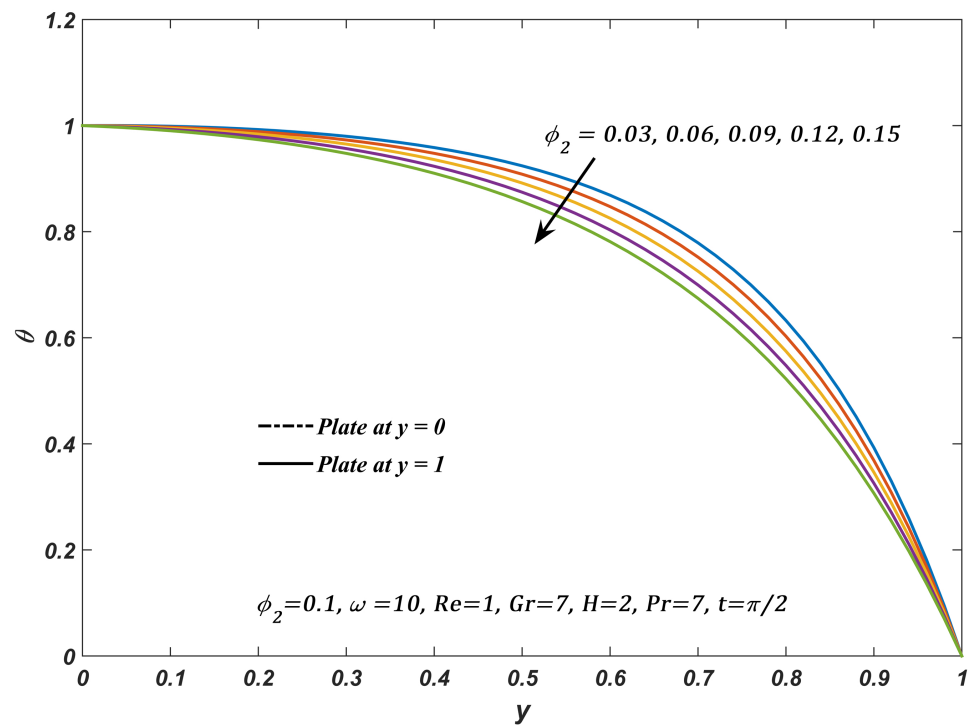


Figure 3.5: Variation in u with ϕ_2


 Figure 3.6: Variation in u with Re

 Figure 3.7: Variation in θ with ϕ_1

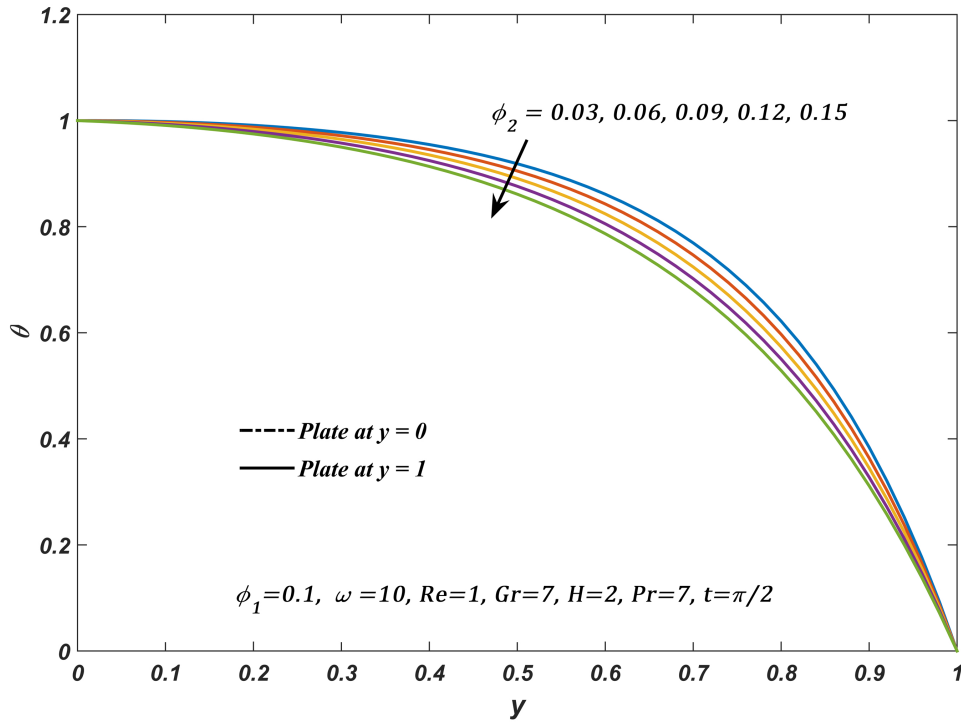


Figure 3.8: Variation in θ with ϕ_2

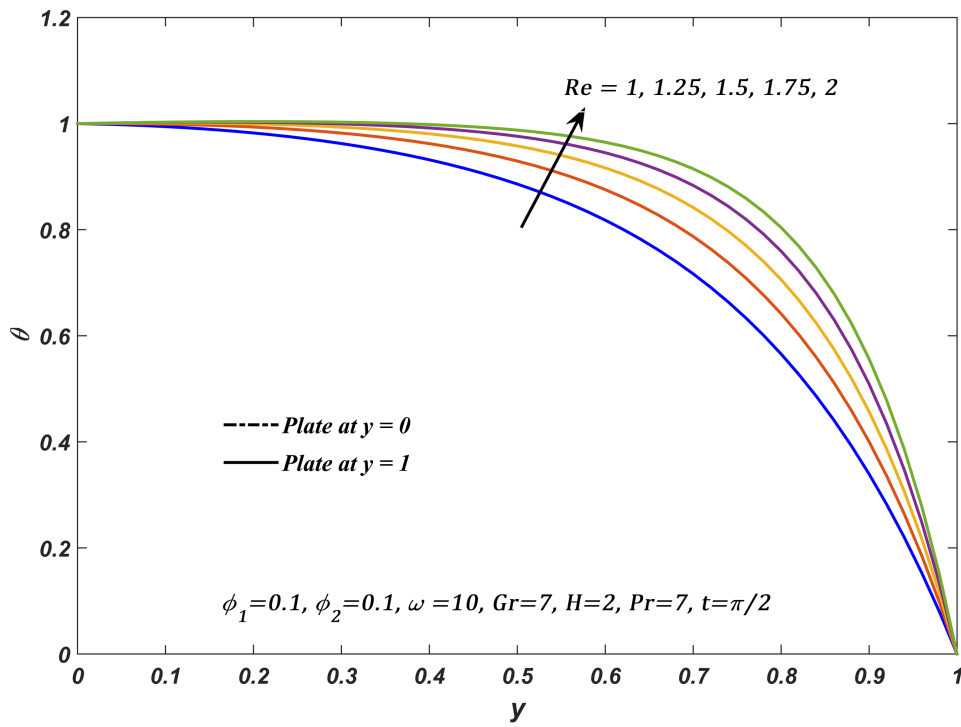


Figure 3.9: Variation in θ with Re

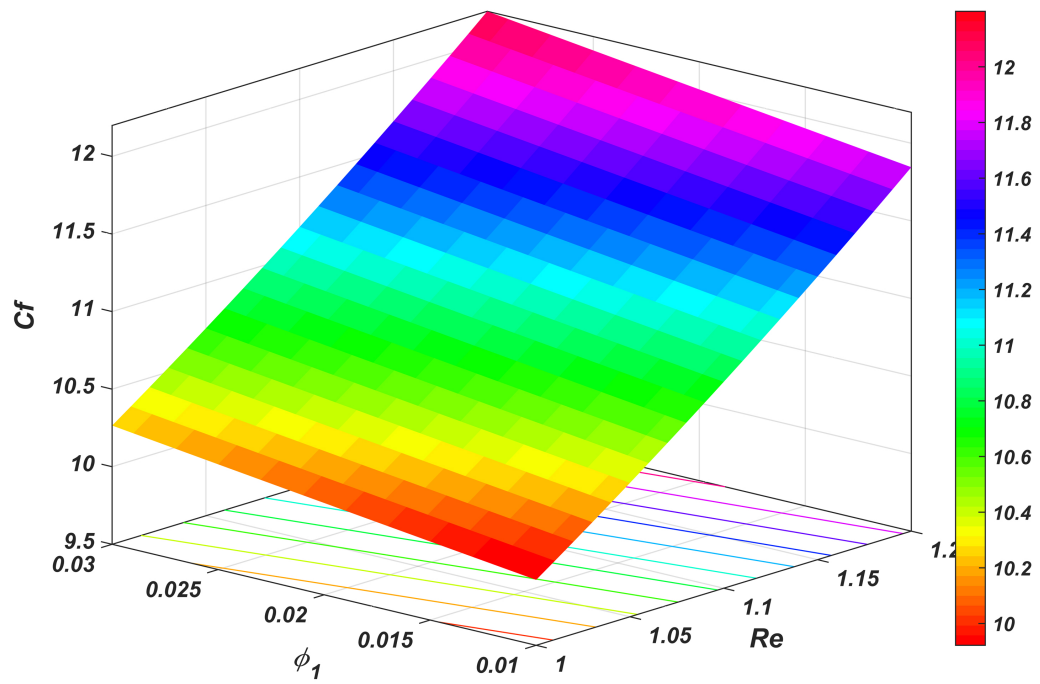


Figure 3.10: Variation in C_f with ϕ_1 and Re at $y = 0$

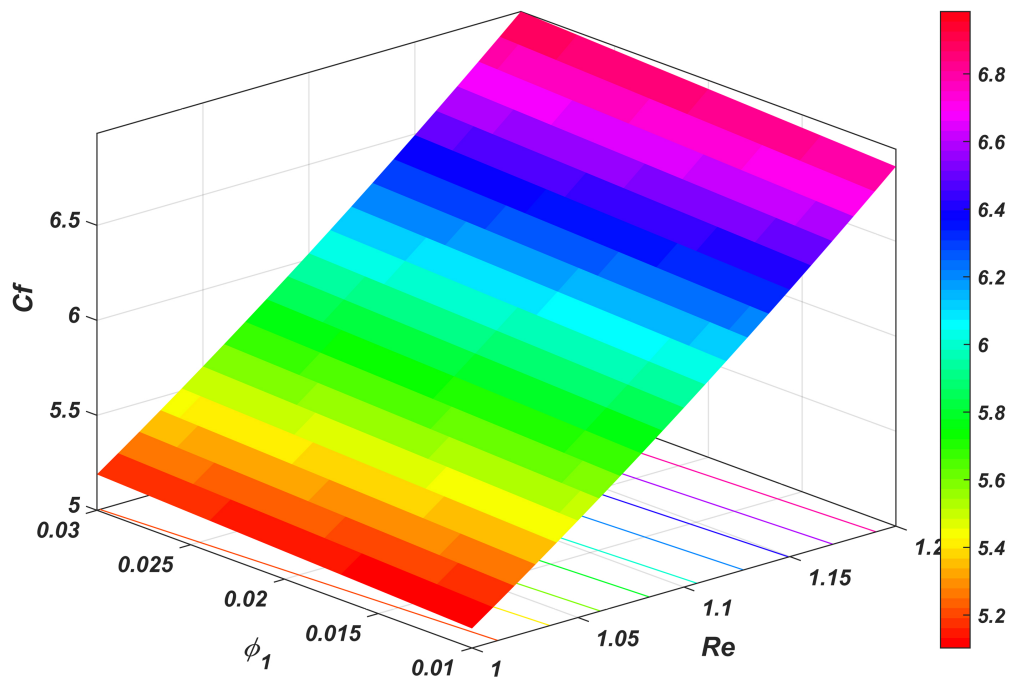


Figure 3.11: Variation in C_f with ϕ_1 and Re at $y = 1$

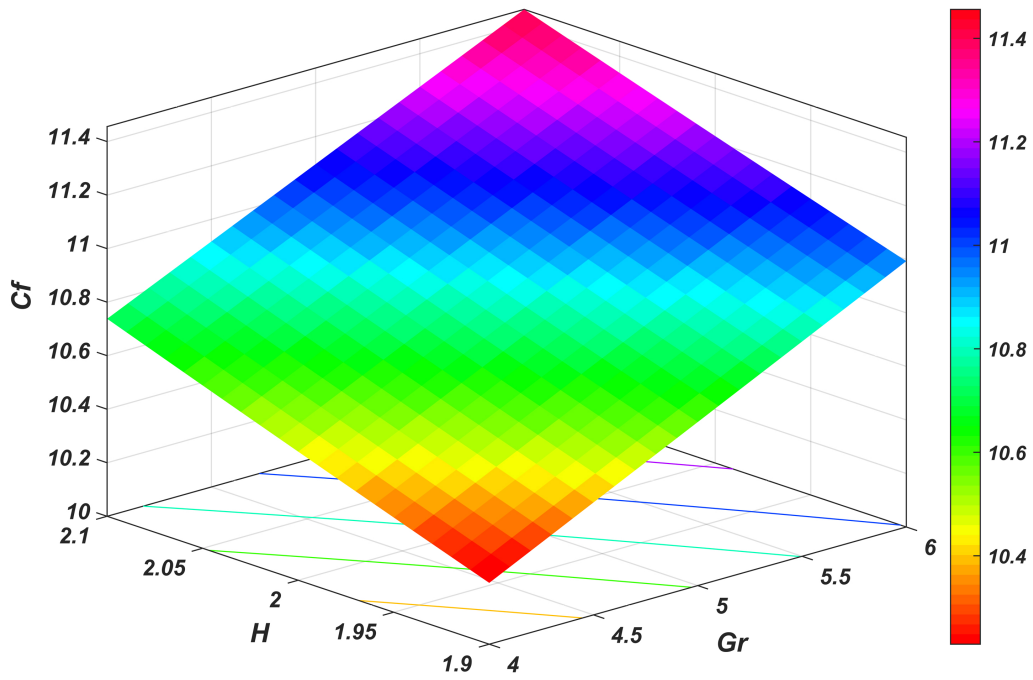


Figure 3.12: Variation in C_f with Gr and H at $y = 0$

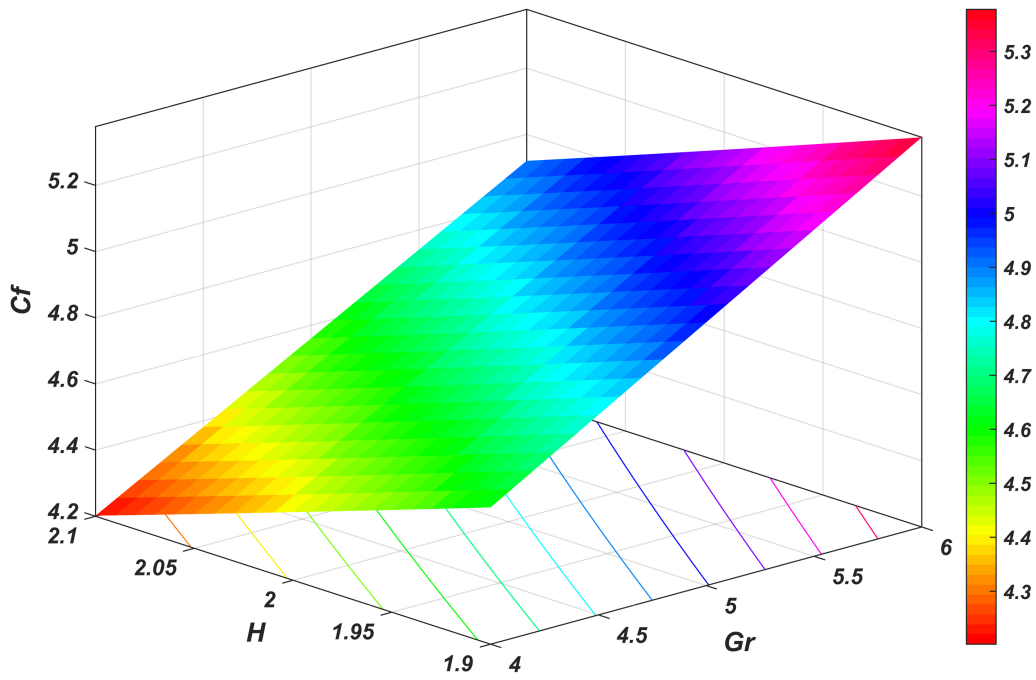


Figure 3.13: Variation in C_f with Gr and H at $y = 1$

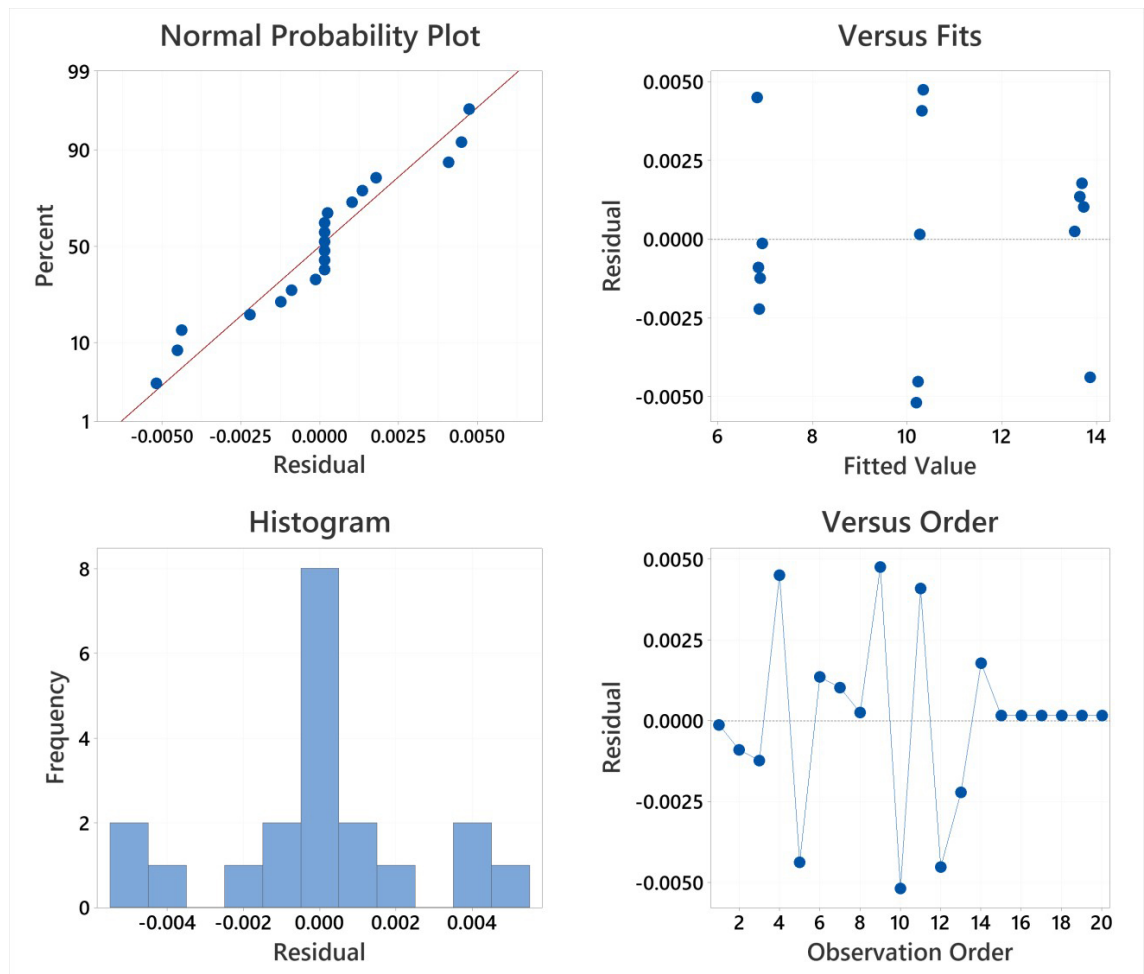


Figure 3.14: *Residual plots*

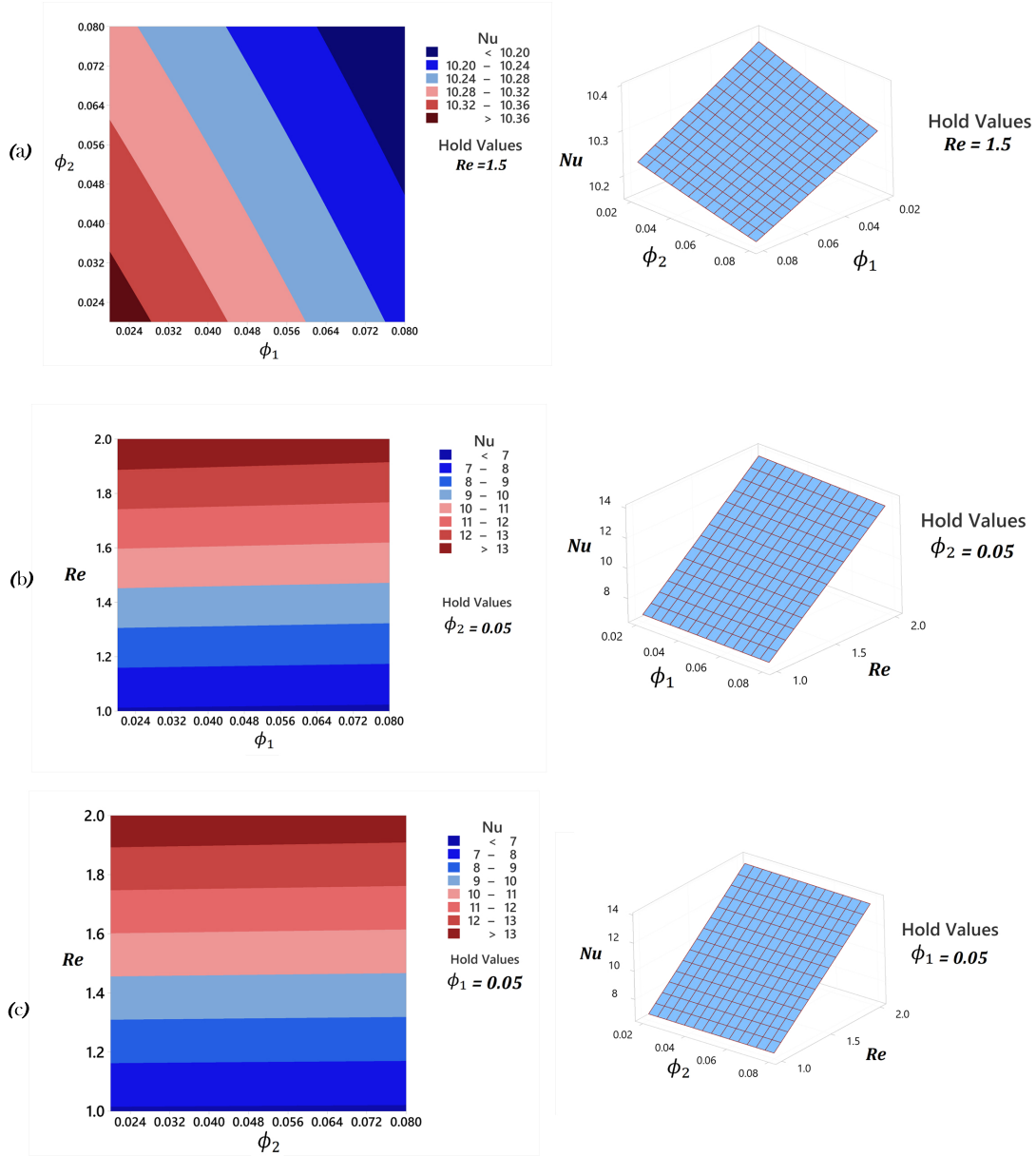


Figure 3.15: Contour and surface plots for Nu

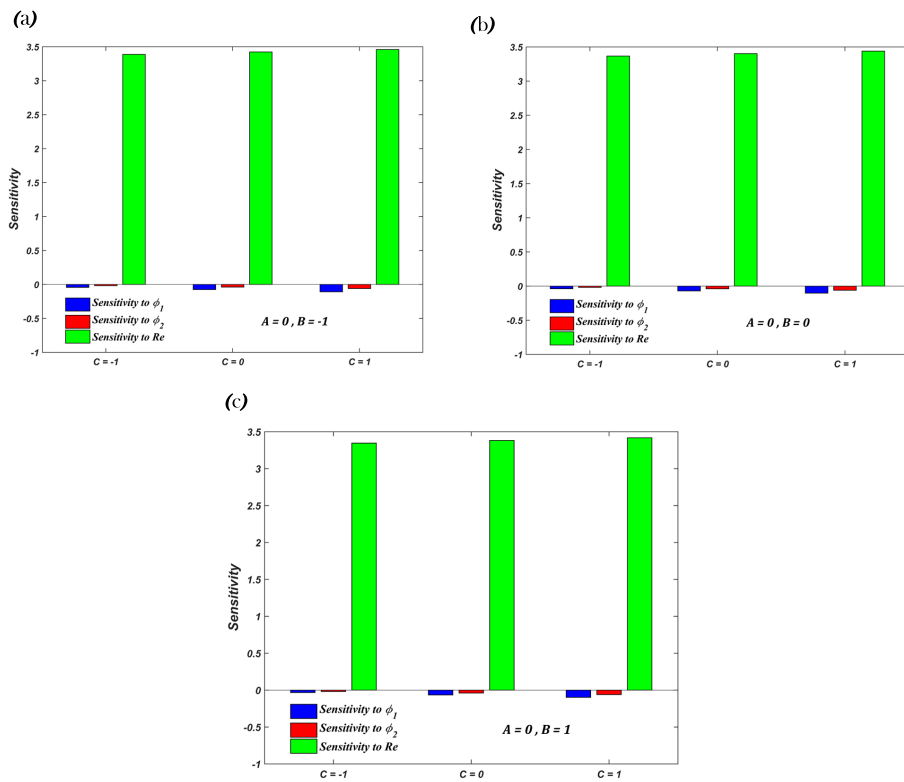


Figure 3.16: Bar charts depicting the sensitivity of Nu

Appendix I: Non-Dimensional Quantities

$$Pr = \frac{(\mu c_p)_f}{k_f} \quad \text{Prandtl Number}$$

$$Re = \frac{U_0 d}{\vartheta_f} \quad \text{Injection/ Suction Parameter}$$

$$H = B_0 d \sqrt{\frac{\sigma_f}{\rho_f \vartheta_f}} \quad \text{Hartmann Number}$$

$$Gr = \frac{g \beta_f \vartheta_f (T_0 - T_1)}{U_0^3} \quad \text{Grashof Number}$$

Appendix II: Nomenclature

u^*, v^*, w^*	Velocity components	T^*	Fluid temperature
V_0	Injection velocity	g	Acceleration due to gravity
t^*	Time	T_0, T_1	Reference temperature
p^*	Pressure	C_p	Specific heat at constant pressure
B_0	strength of magnetic field	U_0	velocity of the moving plates
σ	Electrical conductivity	ϕ_1	Volume fraction of Al_2O_3
d	Distance between the plates	ϕ_2	Volume fraction of Fe_3O_4
μ	Dynamic viscosity	ρ	Density
ν	Kinematic viscosity	ω^*	Angular velocity
K	Thermal conductivity	f	Base fluid
nf	Nanofluid	s_1	Al_2O_3 Nanoparticle
hnf	Hybrid nanofluid	s_2	Fe_3O_4 Nanoparticle
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	Very small reference constants		

Appendix III: Short keys

$$a_1 = \frac{C_5 Pr Re}{C_6}$$

$$a_2 = \frac{\frac{C_5 Pr Re}{C_6} + \sqrt{\frac{C_5^2 Pr^2 Re^2}{C_6^2} + \frac{4C_5 Pr \omega i}{C_6}}}{2}$$

$$a_3 = \frac{\frac{C_5 Pr Re}{C_6} - \sqrt{\frac{C_5^2 Pr^2 Re^2}{C_6^2} + \frac{4C_5 Pr \omega i}{C_6}}}{2}$$

$$b_1 = \frac{C_1 C_2 Re + \sqrt{C_1^2 C_2^2 Re^2 + 4C_1 C_3 H^2}}{2}$$

$$b_2 = \frac{C_1 C_2 Re - \sqrt{C_1^2 C_2^2 Re^2 + 4C_1 C_3 H^2}}{2}$$

$$b_3 = \frac{C_1 C_2 Re + \sqrt{C_1^2 C_2^2 Re^2 + 4(i\omega C_1 C_2 + C_1 C_3 H^2)}}{2}$$

$$b_4 = \frac{C_1 C_2 Re - \sqrt{C_1^2 C_2^2 Re^2 + 4(i\omega C_1 C_2 + C_1 C_3 H^2)}}{2}$$

$$b_5 = \frac{C_1 C_2 Re + \sqrt{C_1^2 C_2^2 Re^2 + 4(C_1 C_3 H^2 + \pi^2)}}{2}$$

$$b_6 = \frac{C_1 C_2 Re - \sqrt{C_1^2 C_2^2 Re^2 + 4(C_1 C_3 H^2 + \pi^2)}}{2}$$

$$A_1 = \frac{C_1 C_2 C_4 Re^2 Gr}{(e^{a_1} - 1)(a_1^2 - C_1 C_2 Re \ a_1 - C_1 C_3 H^2)}$$

$$A_2 = \frac{C_1 C_2 C_4 Re^2 Gr \ e^{a_1}}{(e^{a_1} - 1)C_1 C_3 H^2)}$$

$$A_3 = \frac{C_1 C_2 C_4 Re^2 Gr e^{a_3}}{(e^{a_2} - e^{a_3})(a_2^2 - C_1 C_2 Re \ a_2 - C_1 C_2 i\omega - C_1 C_3 H^2)}$$

$$A_4 = \frac{C_1 C_2 C_4 Re^2 Gr e^{a_2}}{(e^{a_3} - e^{a_2})(a_3^2 - C_1 C_2 Re \ a_3 - C_1 C_2 i\omega - C_1 C_3 H^2)}$$

$$\alpha_1 = -(A_1 + A_2)$$

$$\beta_1 = -(2 + A_1 e^{a_1} + A_2)$$

$$\alpha_2 = -(A_3 + A_4)$$

$$\beta_2 = -(A_3 e^{a_2} + A_4 e^{a_3})$$

$$r_1 = \frac{b_1 + \sqrt{b_1^2 + 4\pi^2}}{2}$$

$$r_2 = \frac{b_1 - \sqrt{b_1^2 + 4\pi^2}}{2}$$

$$r_3 = \frac{b_2 + \sqrt{b_2^2 + 4\pi^2}}{2}$$

$$r_4 = \frac{b_2 - \sqrt{b_2^2 + 4\pi^2}}{2}$$

$$\alpha_3 = -\sum_{i=1}^4 (A_{i1} + A_{i2} + A_{i3})$$

$$\alpha_4 = -\sum_{i=1}^4 (B_{i1} + B_{i2} + B_{i3})$$

$$\beta_3 = -\sum_{i=1}^4 (A_{i1} e^{(r_i + b_1)} + A_{i2} e^{(r_i + b_2)} + A_{i3} e^{(r_i + a_1)})$$

$$\beta_4 = -\sum_{i=1}^4 (B_{i1} e^{(r_i + b_1)} + B_{i2} e^{(r_i + b_2)} + B_{i3} e^{(r_i + a_1)})$$

$$D_1 = r_3 r_4 (e^{r_2+r_4} - e^{r_2+r_3}) + r_2 r_4 (e^{r_3+r_2} - e^{r_3+r_4}) + r_2 r_3 (e^{r_3+r_4} - e^{r_2+r_4})$$

$$D_2 = r_3 r_4 (e^{r_1+r_3} - e^{r_1+r_4}) + r_1 r_4 (e^{r_3+r_4} - e^{r_3+r_1}) + r_1 r_3 (e^{r_1+r_4} - e^{r_3+r_4})$$

$$D_3 = r_2 r_4 (e^{r_1+r_4} - e^{r_1+r_2}) + r_1 r_4 (e^{r_2+r_1} - e^{r_2+r_4}) + r_1 r_2 (e^{r_2+r_4} - e^{r_1+r_4})$$

$$D_4 = r_2 r_3 (e^{r_1+r_2} - e^{r_1+r_3}) + r_1 r_3 (e^{r_3+r_2} - e^{r_2+r_1}) + r_1 r_2 (e^{r_1+r_3} - e^{r_2+r_3})$$

$$D = D_1 + D_2 + D_3 + D_4$$

$$A_{i1} = \frac{b_1(\alpha_1 e^{b_2} - \beta_1) C_1 C_2 \operatorname{Re} D i}{(e^{b_2} - e^{b_1}) D [(r_i + b_1)^2 - C_1 C_2 \operatorname{Re}(r_i + b_1) - (C_1 C_3 H^2 + \pi^2)]}, \quad i = 1, 2, 3, 4$$

$$A_{i2} = \frac{b_2(\beta_1 - \alpha_1 e^{b_1}) C_1 C_2 \operatorname{Re} D i}{(e^{b_2} - e^{b_1}) D [(r_i + b_2)^2 - C_1 C_2 \operatorname{Re}(r_i + b_2) - (C_1 C_3 H^2 + \pi^2)]}, \quad i = 1, 2, 3, 4$$

$$B_{i1} = \frac{b_1((\alpha_1 + 2)e^{b_2} - (\beta_1 + 2)) C_1 C_2 \operatorname{Re} D i}{(e^{b_2} - e^{b_1}) D [(r_i + b_1)^2 - C_1 C_2 \operatorname{Re}(r_i + b_1) - (C_1 C_3 H^2 + \pi^2)]}, \quad i = 1, 2, 3, 4$$

$$B_{i2} = \frac{b_2((\beta_1 + 2) - (\alpha_1 + 2)e^{b_1}) C_1 C_2 \operatorname{Re} D i}{(e^{b_2} - e^{b_1}) D [(r_i + b_2)^2 - C_1 C_2 \operatorname{Re}(r_i + b_2) - (C_1 C_3 H^2 + \pi^2)]}, \quad i = 1, 2, 3, 4$$

$$B_{i3} = A_{i3} = \frac{a_1 A_1 C_1 C_2 \operatorname{Re} D i}{D [(r_i + a_1)^2 - C_1 C_2 \operatorname{Re}(r_i + a_1) - (C_1 C_3 H^2 + \pi^2)]}, \quad i = 1, 2, 3, 4$$

CHAPTER 4

Bioconvective EMHD Hybrid Nanofluid Flow Past a Stretching Sheet *

4.1 INTRODUCTION

Heat and mass transfer due to stretching sheets plays a significant role in many aspects of engineering and industrial field such as metal spinning and drawing plastic films, glass blowing, crystal growing, cooling of filaments, and textiles, etc.

This chapter includes a theoretical investigation of bioconvective electromagneticohydrodynamic (EMHD) hybrid nanofluid (water-based $CNT - Fe_3O_4$) over a stretching surface. The impact of viscous dissipation, chemical reaction, and stratification is also explored in it. Governing nonlinear partial differential equations are transformed into ordinary differential equations with similarity transformation and treated numerically using bvp4c in MATLAB software. The drag coefficient has been statistically scrutinized utilizing the four-factor response surface methodology.

4.2 MATHEMATICAL FORMULATION

Consider a steady two-dimensional bioconvective flow past a linearly stretching sheet (with velocity $u_w(x) = cx$) placed along the x -axis (see Figure 4.1). The

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water-based carbon nanotube nanofluid embodied with microorganisms occupies the region $y > 0$. An external electric field (of intensity E_0) and an external magnetic field (of magnetic strength B_0) are applied perpendicular to the fluid flow. The induced magnetic field was ignored due to the supposition of a small magnetic Reynolds number. Chemical reaction, viscous dissipation, and stratification effects are also incorporated. Using the Tiwari-Das nanoliquid model (see (Tiwari & Das, 2007)), the governing equations take the form (see (Alsaedi et al., 2017), (Bilal, 2020), (Daniel, Aziz, Ismail, & Bahar, 2020)):

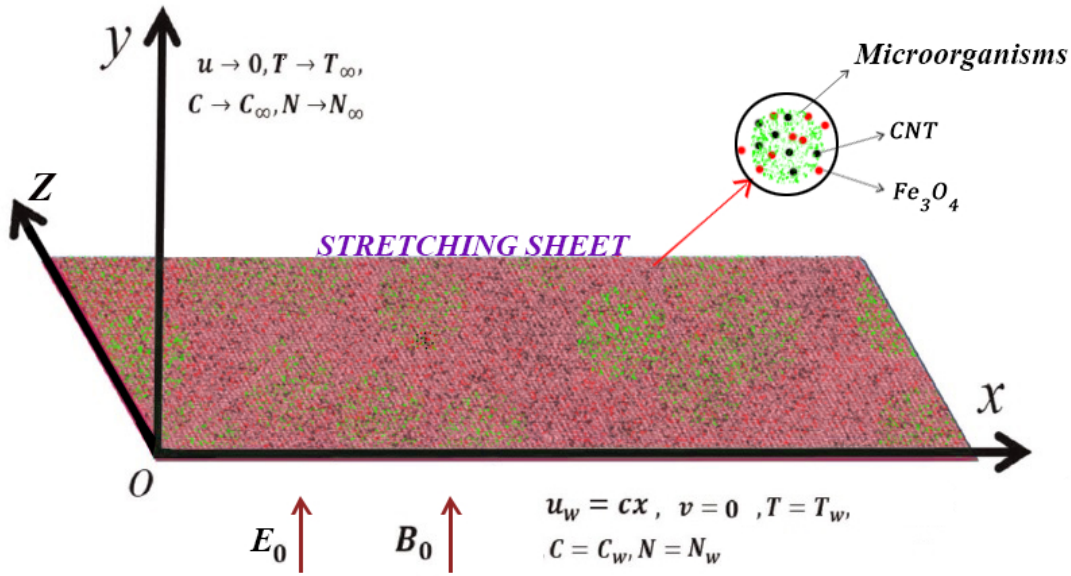


Figure 4.1: *Physical configuration*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.2.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left(\frac{\mu_{hnf}}{\rho_{hnf}} \right) \frac{\partial^2 u}{\partial y^2} + \frac{\sigma_{hnf}}{\rho_{hnf}} (E_0 B_0 - B_0^2 u) \quad (4.2.2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{hnf} \frac{\partial^2 T}{\partial y^2} + \frac{\mu_{hnf}}{(\rho C_p)_{hnf}} \left(\frac{\partial u}{\partial y} \right)^2 \quad (4.2.3)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} - K_0 (C - C_\infty) \quad (4.2.4)$$

$$u \frac{\partial N}{\partial x} + v \frac{\partial N}{\partial y} + \frac{bW_c}{C_w - C_0} \left(\frac{\partial}{\partial y} \left(N \frac{\partial C}{\partial y} \right) \right) = D_m \frac{\partial^2 N}{\partial y^2} \quad (4.2.5)$$

subject to the boundary constraints

$$\left. \begin{aligned} u = u_w(x) = cx, \quad v = 0, \quad T = T_w = T_0 + \delta_1 x, \\ C = C_w = C_0 + \varepsilon_1 x, \quad N = N_w = N_0 + \xi_1 x \end{aligned} \right\} \text{at } y = 0 \quad (4.2.6)$$

$$\left. \begin{aligned} u \rightarrow 0, \quad T \rightarrow T_\infty = T_0 + \delta_2 x, \quad C \rightarrow C_\infty = C_0 + \varepsilon_2 x, \\ N \rightarrow N_\infty = N_0 + \xi_2 x \end{aligned} \right\} \text{as } y \rightarrow \infty \quad (4.2.7)$$

where δ_i, ε_i and ξ_i ($i = 1, 2$) are dimensional constants.

The hybrid nanofluid models utilized in the current problem are (see (Sreedevi & Reddy, 2019), (Tulu & Ibrahim, 2020), (Saba, Ahmed, Khan, & Mohyud-Din, 2019)):

Effective Dynamic Viscosity:

$$\frac{\mu_{hnf}}{\mu_f} = \frac{1}{(1 - \phi_1)^{2.5} (1 - \phi_2)^{2.5}} = \frac{1}{A_1}$$

Effective Density:

$$\frac{\rho_{hnf}}{\rho_f} = (1 - \phi_2) \left[1 - \phi_1 + \phi_1 \left(\frac{\rho_{s1}}{\rho_f} \right) \right] + \phi_2 \left(\frac{\rho_{s2}}{\rho_f} \right) = A_2$$

Effective Specific Heat:

$$\frac{(\rho C_p)_{hnf}}{(\rho C_p)_f} = (1 - \phi_2) \left[1 - \phi_1 + \phi_1 \left(\frac{(\rho C_p)_{s1}}{(\rho C_p)_f} \right) \right] + \phi_2 \left(\frac{(\rho C_p)_{s2}}{(\rho C_p)_f} \right) = A_3$$

Effective Thermal Conductivity:

$$\frac{K_{hnf}}{K_f} = A_4, \text{ where}$$

$$\frac{K_{hnf}}{K_{nf}} = \frac{1 - \phi_2 + 2\phi_2 \left(\frac{K_{s2}}{K_{s2} - K_{nf}} \right) \ln \left(\frac{K_{s2} + K_{nf}}{2K_{nf}} \right)}{1 - \phi_2 + 2\phi_2 \left(\frac{K_{nf}}{K_{s2} - K_{nf}} \right) \ln \left(\frac{K_{s2} + K_{nf}}{2K_{nf}} \right)}$$

and

$$\frac{K_{nf}}{K_f} = \frac{K_{s1} + 2K_f - 2\phi_1 (K_f - K_{s1})}{K_{s1} + 2K_f + \phi_1 (K_f - K_{s1})}$$

Effective Electrical Conductivity:

$$\frac{\sigma_{hnf}}{\sigma_f} = 1 + \frac{3 \left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{\sigma_f} - (\phi_1 + \phi_2) \right)}{2 + \left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{(\phi_1 + \phi_2) \sigma_f} \right) - \left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{\sigma_f} - (\phi_1 + \phi_2) \right)} = A_5$$

Consider the following similarity transformations (referred from (Alsaedi et al., 2017) and (Bilal, 2020):

$$\left. \begin{aligned} u &= c x f'(\eta), \quad v = -\sqrt{c \vartheta_f} f(\eta), \quad \eta = y \sqrt{\frac{c}{\vartheta_f}}, \\ \theta(\eta) &= \frac{T - T_\infty}{T_w - T_0}, \quad \psi(\eta) = \frac{C - C_\infty}{C_w - C_0}, \quad \chi(\eta) = \frac{N - N_\infty}{N_w - N_0} \end{aligned} \right\} \quad (4.2.8)$$

Introducing these similarity variables and hybrid nanofluid constants the governing equations along with the boundary conditions are transmuted to:

$$f''' + A_1 A_2 f f'' - (A_1 A_5 H + A_1 A_2 f') f' + A_1 A_5 H E = 0 \quad (4.2.9)$$

$$A_4 \theta'' + A_3 Pr f \theta' + \frac{Pr Ec}{A_1} (f'')^2 = 0 \quad (4.2.10)$$

$$\psi'' + Le f \psi' - Le K_c \psi = 0 \quad (4.2.11)$$

$$\chi'' + (f Lb - Pe \psi') \chi' - Pe \psi'' \chi - Pe \Omega \psi'' = 0 \quad (4.2.12)$$

with

$$f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1 - S_1, \quad \psi(0) = 1 - S_2, \quad \chi(0) = 1 - S_3 \quad (4.2.13)$$

$$f'(\infty) \rightarrow 0, \quad \theta(\infty) = 0, \quad \psi(\infty) \rightarrow 0, \quad \chi(\infty) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty \quad (4.2.14)$$

4.3 PHYSICAL QUANTITIES

The physical quantities in non-dimensional form are given by:

Drag coefficient:

$$C f_x = \frac{\mu_{hnf} \left(\frac{\partial u}{\partial y} \right)_{y=0}}{\rho_f (u_w)^2} \Rightarrow Re_x^{1/2} C f_x = \frac{f''(0)}{A_1} \quad (4.3.1)$$

Local Nusselt number:

$$Nu_x = -\frac{x K_{hnf} \left(\frac{\partial T}{\partial y} \right)_{y=0}}{K_f (T_w - T_0)} \Rightarrow Re_x^{-1/2} Nu_x = -A_4 \theta'(0) \quad (4.3.2)$$

Local Sherwood number:

$$Sh_x = -\frac{x D_B \left(\frac{\partial C}{\partial y} \right)_{y=0}}{D_B (C_w - C_0)} \Rightarrow Re_x^{-1/2} Sh_x = -\psi'(0) \quad (4.3.3)$$

Local microorganism density number:

$$Nn_x = -\frac{x D_m \left(\frac{\partial N}{\partial y} \right)_{y=0}}{D_m (N_w - N_0)} \Rightarrow Re_x^{-1/2} Nn_x = -\chi'(0) \quad (4.3.4)$$

where $Re_x = \frac{u_w x}{\nu_f}$ is the local Reynold's number.

4.4 NUMERICAL PROCEDURE AND VALIDATION

Since equations (4.2.9) to (4.2.12) are highly nonlinear, it is very hard to find a closed-form or an exact solution. Hence an approximate solution is computed numerically by employing the bvp4c algorithm (a finite difference-based built-in numerical procedure) in MATLAB. For this purpose, let

$$\begin{aligned} f &= \Upsilon_1, \quad f' = \Upsilon_2, \quad f'' = \Upsilon_3, \quad f''' = \Upsilon_3', \quad \theta = \Upsilon_4, \quad \theta' = \Upsilon_5, \quad \theta'' = \Upsilon_5', \\ \psi &= \Upsilon_6, \quad \psi' = \Upsilon_7, \quad \psi'' = \Upsilon_7', \quad \chi = \Upsilon_8, \quad \chi' = \Upsilon_9, \quad \chi'' = \Upsilon_9'. \end{aligned}$$

The transmuted system of first-order ODEs are given as:

$$\Upsilon_1' = \Upsilon_2 \quad (4.4.1)$$

$$\Upsilon_2' = \Upsilon_3 \quad (4.4.2)$$

$$\Upsilon_3' = (A_1 A_5 H + A_1 A_2 \Upsilon_2) \Upsilon_2 - A_1 A_2 \Upsilon_1 \Upsilon_3 - A_1 A_5 H E \quad (4.4.3)$$

$$\Upsilon_4' = \Upsilon_5 \quad (4.4.4)$$

$$\Upsilon_5' = -\frac{(A_1 A_3 Pr \Upsilon_1 \Upsilon_5 + Pr Ec \Upsilon_3^2)}{A_1 A_4} \quad (4.4.5)$$

$$\Upsilon_6' = \Upsilon_7 \quad (4.4.6)$$

$$\Upsilon_7' = Le K_c \Upsilon_6 - Le \Upsilon_1 \Upsilon_7 \quad (4.4.7)$$

$$\Upsilon_8' = \Upsilon_9 \quad (4.4.8)$$

$$\Upsilon_9' = (\Upsilon_8 + \Omega) Pe \Upsilon_7' - (Lb \Upsilon_1 - Pe \Upsilon_7) \Upsilon_9 \quad (4.4.9)$$

with

$$\Upsilon_1(0) = 0, \quad \Upsilon_2(0) = 1, \quad \Upsilon_4(0) = 1 - S_1, \quad \Upsilon_6(0) = 1 - S_2, \quad \Upsilon_8(0) = 1 - S_3 \quad (4.4.10)$$

$$\Upsilon_2(\infty) = 0, \quad \Upsilon_4(\infty) = 0, \quad \Upsilon_6(\infty) = 0, \quad \Upsilon_8(\infty) = 0 \quad (4.4.11)$$

The bvp4c algorithm uses a three-stage Lobatto IIIa formula built on an association polynomial that returns a C^1 -continuous solution which is uniformly accurate to the fourth-order. The accuracy of the code and the validation of the current problem have been determined through a restrictive comparison with (W. Khan & Pop, 2010) (see Table 4.1).

4.5 RESULTS AND DISCUSSION

The upshot of effectual parameters on the physical quantities, microbial concentration profile ($\chi(\eta)$), nanofluid temperature profile ($\theta(\eta)$), velocity profile ($f'(\eta)$), and nanofluid concentration profile ($\psi(\eta)$) for $SWCNT - Fe_3O_4$ and $MWCNT - Fe_3O_4$ hybrid nanofluids has been depicted through Figures 4.2 - 4.20. The Prandtl number, Lewis number, bioconvection Lewis number, and infinity are fixed at 6.2, 2, 1.2, and 5, respectively. In addition, the thermophysical properties of the base fluid and the nanoparticles are identified in Table 4.2.

The Lorentz force produced in the presence of a magnetic field acts as a resistive force that clashes with the fluid motion and hence reduces the velocity. This decline in the velocity profile with improving Hartmann number (H), in the absence of an electric field, is exhibited in Figure 4.2. However, in the presence of an electric field, the velocity profile speeds up with increasing H values (depicted in Figure 4.3). Physically, this is due to the domination exerted by the electric field on the magnetic field. Figure 4.4 delineates the positive impact of electric field parameter E on the velocity profile. Physically, an increase in E contributes more electric field strength that in turn accelerates the fluid flow (see (Nayak, Mabood, Dogonchi, & Khan, 2021)). Further, it is noted that the velocity profile is minimum for the water-based $SWCNT - Fe_3O_4$ hybrid nanofluid.

The negative impact of thermal stratification parameter (S_1) on $\theta(\eta)$ is expressed in Figure 4.5. This is because the temperature variation between the ambient and fluid surface diminishes with augmenting S_1 values. Figure 4.6 deliberates

the consequence of Eckert number (Ec) on the nanofluid temperature. The elevation in the nanofluid temperature is physically associated with the production of frictional energy caused by the collision of fluid particles. Figures 4.7 and 4.8 elucidate the improvement in $\theta(\eta)$ with augmenting volume fraction of CNT nanoparticle (ϕ_1) and volume fraction of Fe_3O_4 nanoparticle (ϕ_2) values, respectively. Physically, it is associated with the improvement in the thermal conductivity of the nanofluid occurring due to the increased nanoparticle occupancy. Moreover, a higher nanofluid temperature is observed for water-based $SWCNT - Fe_3O_4$ hybrid nanofluid.

The effect of chemical reaction parameter (K_c) and the solutal stratification parameter (S_2) on the concentration profile has been depicted in Figures 4.9 and 4.10, respectively. A decline in $\psi(\eta)$ is noted for increasing K_c and S_2 values. Physically, higher values of K_c expedite the nanoparticle consumption causing a drop in the concentration profile. Further, an augmentation in S_2 descends the volumetric fraction between the surface and reference nanoparticles, hence reducing the concentration profile.

Figure 4.11 illustrates the decreasing nature of $\chi(\eta)$ with augmenting microorganism density stratification parameter (S_3) values. This decline in the volumetric fraction can be physically attributed to the fact that increasing S_3 values descend the concentration difference of microorganisms between the surface and away from the surface.

Figures 4.12 - 4.15 explore the concurrent influence of effective parameters on surface drag and heat transfer rate. From Figures 4.12 and 4.13, it is noticed that the drag coefficient ascends with E and descends with ϕ_1, ϕ_2 , and H . Also, it is observed that the heat transfer rate descends with augmenting ϕ_2, Ec and S_1 values (see Figures 4.14 and 4.15). Further, it is noted that the water-based $SWCNT - Fe_3O_4$ hybrid nanofluid exhibits the least surface drag and the highest heat transfer rate.

Tables 4.3 and 4.4 elucidate the impact of pertinent parameters on the mass transfer rate and the microorganism density number. Further, the slope of linear regression is calculated to statistically investigate the trend of variation over the physical quantities. The magnitude of slope represents the rate of change of considered physical quantity per unit value of the corresponding parameter and the sign of slope symbolizes the nature of this impact. The key observations drawn from

Tables 4.3 and 4.4 are:

- The mass transfer rate is a decreasing function of H and an increasing function of E and K_c .
- The water-based $MWCNT - Fe_3O_4$ hybrid nanofluid exhibits the highest mass transfer rate in comparison with water-based $SWCNT - Fe_3O_4$ hybrid nanofluid.
- A decrement rate of 166.70% (for $SWCNT - Fe_3O_4$ case) and 166.80% (for $MWCNT - Fe_3O_4$ case) are observed on the mass transfer rate when the value of solutal stratification parameter is augmented.
- The microorganism density number is a decreasing function of H and an increasing function of E and K_c .
- The microorganism density number for water-based $MWCNT - Fe_3O_4$ hybrid nanofluid is higher than the water-based $SWCNT - Fe_3O_4$ hybrid nanofluid.
- The microorganism density number descends at a rate of 124.60% (for $SWCNT - Fe_3O_4$ case) and 124.90% (for $MWCNT - Fe_3O_4$ case) when the value of microorganism density stratification parameter is increased.

4.6 STATISTICAL ANALYSIS

4.6.1 Response Surface Methodology (RSM)

The response surface methodology (RSM) is a statistical mechanism to measure the level of interaction exerted by the pertinent parameters (independent variable) chosen on the response (dependent) variable and to optimize it. In this problem, the impact of nanoparticle volume fraction of carbon nanotubes ($0.01 \leq \phi_1 \leq 0.09$), the volume fraction of magnetite nanoparticles ($0.01 \leq \phi_2 \leq 0.09$), Hartmann number ($0.4 \leq M \leq 2$), and electric field parameter ($0.1 \leq E \leq 0.5$) on the drag coefficient ($Cf_x Re_x^{1/2}$) for both $SWCNT - Fe_3O_4$ and $MWCNT - Fe_3O_4$ hybrid nanofluids (named as Cf_{SWCNT} and Cf_{MWCNT} , respectively) are scrutinized statistically using a five-level four-factor response surface optimized model. The effectual parameters and their levels are charted in Table 4.5.

The four-factor response surface optimized model (espousing CCD) for response variables involving linear, quadratic, and interactive terms is couched by:

$$\begin{aligned} Response = & b_0 + b_1 A + b_2 B + b_3 C + b_4 D + b_5 A^2 + b_6 B^2 + b_7 C^2 + \\ & b_8 D^2 + b_9 AB + b_{10} AC + b_{11} AD + b_{12} BC + b_{13} BD + b_{14} CD \end{aligned} \quad (4.6.1)$$

where $b_i (i = 0, 1...14)$ represent the regression coefficients. The experimental design and response for the 30 runs (in accordance with the four-factor CCD) are showcased in Table 4.6.

Tables 4.7 and 4.8 (ANOVA table) explain the efficiency of the approximated model. A parameter is asserted to be significant if the corresponding p-value is less than 0.05 and the corresponding F-value is greater than 1. It is noticed that the quadratic term in M is not significant for Cf_{SWCNT} and hence the term is deleted from the model. The coefficient of determination (R^2) for the model is estimated to be 99.98% which also boosts the model accuracy. The reliability and accuracy of the estimated models (Cf_{SWCNT} and Cf_{MWCNT}) and are further elucidated using residual versus fitted plots (see Figures 4.16 and 4.17). Furthermore, a maximum error of 0.006 can be noted from the fitted versus residual plot which also accounts for the correctness of the model.

The fitted quadratic models are given by:

$$\begin{aligned} Cf_{SWCNT} = & -1.01558 - 2.70667 \phi_1 - 3.78729 \phi_2 - 0.388995 H + 0.107104 E - \\ & 4.49479 \phi_1^2 - 4.58854 \phi_2^2 - 0.116042 E^2 - 6.90625 \phi_1 \phi_2 - 0.831250 \phi_1 H + \\ & 3.31875 \phi_1 E - 0.642187 \phi_2 H + 2.98750 \phi_2 E + 0.654063 H E \end{aligned} \quad (4.6.2)$$

$$\begin{aligned} Cf_{MWCNT} = & -1.01346 - 2.28813 \phi_1 - 3.79583 \phi_2 - 0.391698 H + 0.104667 E - \\ & 4.34375 \phi_1^2 - 4.56250 \phi_2^2 + 0.007734 H^2 - 0.115000 E^2 - 7.31250 \phi_1 \phi_2 - \\ & 0.906250 \phi_1 H + 3.45625 \phi_1 E - 0.637500 \phi_2 H + 2.98125 \phi_2 E + 0.655313 H E \end{aligned} \quad (4.6.3)$$

From equations (4.6.2) and (4.6.3), it can be inferred that ϕ_1, ϕ_2 and H reduce the surface drag (both cases) and E increase the surface drag (both cases). The simul-

taneous interaction of two parameters on the drag coefficient is delineated using surface plots (see Figures 4.18 and 4.19) by fixing the remaining two parameters at their corresponding medium level. From Figures 4.18 and 4.19, it is perceived that the surface drag (both cases) is highest for smaller values of ϕ_1, ϕ_2 and H and larger values of E .

4.7 CONCLUSION

The bioconvective electromagnetohydrodynamic hybrid nanoliquid flow due to a linearly elongating sheet has been theoretically investigated. The nanoliquid flow has been modeled using the Tiwari-Das model. The mathematically modeled equations are transmuted into a system of first-order ODEs using apposite similarity transformations and then resolved numerically with the aid of bvp4c (a finite difference-based built-in numerical procedure) in MATLAB. A comparative analysis of the hybrid nanoliquid flow using $SWCNT - Fe_3O_4$ and $MWCNT - Fe_3O_4$ nanoparticles has been carried out. Further, the influence of nanoparticle volume fraction of carbon nanotubes ($0.01 \leq \phi_1 \leq 0.09$), the volume fraction of magnetite nanoparticles ($0.01 \leq \phi_2 \leq 0.09$), Hartmann number ($0.4 \leq H \leq 2$), and electric field parameter ($0.1 \leq E \leq 0.5$) on the surface drag has been scrutinized utilizing the four-factor response surface optimized model. The major observations of the study are:

- The velocity profile ascends with augmenting electric field parameter values.
- The velocity profile is minimum for water-based $SWCNT - Fe_3O_4$ hybrid nanofluid.
- The surface drag coefficient is highest for smaller values of Hartmann number and larger values of electric field parameter.
- The volume fraction of nanoparticles and Eckert number intensifies the nanofluid temperature.
- A higher nanofluid temperature is observed for water-based $SWCNT - Fe_3O_4$ hybrid nanofluid.
- The solutal and microorganism density stratification parameters are negatively correlated with volume fraction and microbial concentration, respec-

tively.

- A decrement rate of 166.70% (for $SWCNT - Fe_3O_4$ case) and 166.80% (for $MWCNT - Fe_3O_4$ case) are observed on the mass transfer rate when the value of solutal stratification parameter is augmented.
- The microorganism density number descends at a rate of 124.60% (for $SWCNT - Fe_3O_4$ case) and 124.90% (for $MWCNT - Fe_3O_4$ case) when the value of microorganism density stratification parameter is increased.
- The fitted quadratic model for the drag coefficient of water-based $SWCNT - Fe_3O_4$ hybrid nanofluid is given by:

$$Cf_{SWCNT} = -1.01558 - 2.70667 \phi_1 - 3.78729 \phi_2 - 0.388995 H + 0.107104 E - 4.49479 \phi_1^2 - 4.58854 \phi_2^2 - 0.116042 E^2 - 6.90625 \phi_1 \phi_2 - 0.831250 \phi_1 H + 3.31875 \phi_1 E - 0.642187 \phi_2 H + 2.98750 \phi_2 E + 0.654063 H E$$

- The fitted quadratic model for the drag coefficient of water-based $MWCNT - Fe_3O_4$ hybrid nanofluid is given by:

$$Cf_{MWCNT} = -1.01346 - 2.28813 \phi_1 - 3.79583 \phi_2 - 0.391698 H + 0.104667 E - 4.34375 \phi_1^2 - 4.56250 \phi_2^2 + 0.007734 H^2 - 0.115000 E^2 - 7.31250 \phi_1 \phi_2 - 0.906250 \phi_1 H + 3.45625 \phi_1 E - 0.637500 \phi_2 H + 2.98125 \phi_2 E + 0.655313 H E$$

TABLES AND GRAPHS

Table 4.1: Comparison of $Nu_x Re_x^{-1/2}$ for differing Pr values when $H = E = K_c = Ec = Le = Lb = Pe = \Omega = S_1 = S_2 = S_3 = 0$

Pr	(W. Khan & Pop, 2010)	Present study
0.7	0.4539	0.454
2	0.9113	0.9114
7	1.8954	1.8954
20	3.3539	3.3539
70	6.4621	6.4622

Table 4.2: *Thermophysical properties*

Physical Properties	ρ	C_p	k	σ
Pure water	997.1	4179	0.613	0.05
SWCNT	2600	425	6600	10^6
MWCNT	1600	796	3000	10^7
Fe_3O_4	5180	670	9.7	25000

Table 4.3: *Comparison of $Sh_x Re_x^{-1/2}$ for differing H, E, K_c , and S_2 values when $Le = 2, Lb = 1.2, Ec = 0.3, \Omega = Pe = 0.5, S_1 = S_3 = \phi_1 = \phi_2 = 0.1$ and $Pr = 6.2$*

H	E	K_c	S_2	$Sh_x Re_x^{-1/2}$	
				SWCNT – Fe_3O_4	MWCNT – Fe_3O_4
0.5				1.5105	1.5118
1				1.5003	1.5014
1.5				1.492	1.4929
Slope				-0.0185	-0.0189
	0.01			1.4989	1.5
	0.03			1.5003	1.5014
	0.05			1.5017	1.5028
Slope				0.07	0.07
		0.5		1.1913	1.1929
		1		1.5003	1.5014
		1.5		1.7538	1.7546
Slope				0.5625	0.5617
			0.05	1.5837	1.5848
			0.1	1.5003	1.5014
			0.15	1.417	1.418
Slope				-1.667	-1.668

Table 4.4: Comparison of $Nn_x Re_x^{-1/2}$ for differing H, E, K_c , and S_3 values when $Le = 2, Lb = 1.2, Ec = 0.3, \Omega = Pe = 0.5, S_1 = S_2 = \phi_1 = \phi_2 = 0.1$ and $Pr = 6.2$

H	E	K_c	S_3	$Nn_x Re_x^{-1/2}$	
				$SWCNT - Fe_3O_4$	$MWCNT - Fe_3O_4$
0.5				1.4584	1.4615
1				1.4346	1.4371
1.5				1.415	1.417
Slope				-0.0434	-0.0445
	0.01			1.4302	1.4327
	0.03			1.4346	1.4371
	0.05			1.4388	1.4413
Slope				0.215	0.215
		0.5		1.2325	1.2354
		1		1.4346	1.4371
		1.5		1.6021	1.6043
Slope				0.3696	0.3689
			0.05	1.4969	1.4995
			0.1	1.4346	1.4371
			0.15	1.3723	1.3746
Slope				-1.246	-1.249

Table 4.5: Effectual parameter levels

Parameter	Symbol	Levels				
		$-2(\alpha)$	-1 (low)	0 (medium)	1 (high)	$2(\alpha)$
ϕ_1	A	0.01	0.03	0.05	0.07	0.09
ϕ_2	B	0.01	0.03	0.05	0.07	0.09
H	C	0.4	0.8	1.2	1.6	2
E	D	0.1	0.2	0.3	0.4	0.5

Table 4.6: *Four-factor CCD experimental design and the corresponding responses*

Run	ϕ_1	ϕ_2	H	E	Response	
					Cf_{SWCNT}	Cf_{MWCNT}
1	0.03 (-1)	0.03 (-1)	0.8 (-1)	0.2 (-1)	-1.4092	-1.3978
2	0.07 (1)	0.03 (-1)	0.8 (-1)	0.2 (-1)	-1.5409	-1.5138
3	0.03 (-1)	0.07 (1)	0.8 (-1)	0.2 (-1)	-1.5813	-1.5705
4	0.07 (1)	0.07 (1)	0.8 (-1)	0.2 (-1)	-1.7237	-1.6979
5	0.03 (-1)	0.03 (-1)	1.6 (1)	0.2 (-1)	-1.6344	-1.6247
6	0.07 (1)	0.03 (-1)	1.6 (1)	0.2 (-1)	-1.7968	-1.774
7	0.03 (-1)	0.07 (1)	1.6 (1)	0.2 (-1)	-1.8309	-1.8218
8	0.07 (1)	0.07 (1)	1.6 (1)	0.2 (-1)	-2.0075	-1.9858
9	0.03 (-1)	0.03 (-1)	0.8 (-1)	0.4 (1)	-1.2542	-1.2421
10	0.07 (1)	0.03 (-1)	0.8 (-1)	0.4 (1)	-1.3663	-1.3375
11	0.03 (-1)	0.07 (1)	0.8 (-1)	0.4 (1)	-1.4091	-1.3977
12	0.07 (1)	0.07 (1)	0.8 (-1)	0.4 (1)	-1.5297	-1.5023
13	0.03 (-1)	0.03 (-1)	1.6 (1)	0.4 (1)	-1.3859	-1.3755
14	0.07 (1)	0.03 (-1)	1.6 (1)	0.4 (1)	-1.5176	-1.4928
15	0.03 (-1)	0.07 (1)	1.6 (1)	0.4 (1)	-1.5546	-1.5447
16	0.07 (1)	0.07 (1)	1.6 (1)	0.4 (1)	-1.6971	-1.6735
17	0.01 (-2)	0.05 (0)	1.2 (0)	0.3 (0)	-1.441	-1.4375
18	0.09 (2)	0.05 (0)	1.2 (0)	0.3 (0)	-1.7218	-1.6894
19	0.05 (0)	0.01 (-2)	1.2 (0)	0.3 (0)	-1.4028	-1.3842
20	0.05 (0)	0.09 (2)	1.2 (0)	0.3 (0)	-1.7603	-1.7434
21	0.05 (0)	0.05 (0)	0.4 (-2)	0.3 (0)	-1.3722	-1.3509
22	0.05 (0)	0.05 (0)	2 (2)	0.3 (0)	-1.7675	-1.7522
23	0.05 (0)	0.05 (0)	1.2 (0)	0.1 (-2)	-1.8089	-1.7924
24	0.05 (0)	0.05 (0)	1.2 (0)	0.5 (2)	-1.3488	-1.3298
25	0.05 (0)	0.05 (0)	1.2 (0)	0.3 (0)	-1.5739	-1.5562
26	0.05 (0)	0.05 (0)	1.2 (0)	0.3 (0)	-1.5739	-1.5562
27	0.05 (0)	0.05 (0)	1.2 (0)	0.3 (0)	-1.5739	-1.5562
28	0.05 (0)	0.05 (0)	1.2 (0)	0.3 (0)	-1.5739	-1.5562
29	0.05 (0)	0.05 (0)	1.2 (0)	0.3 (0)	-1.5739	-1.5562
30	0.05 (0)	0.05 (0)	1.2 (0)	0.3 (0)	-1.5739	-1.5562

Table 4.7: ANOVA table (Cf_{SWCNT}).

Source	Deg. Of freedom	Adj. sum of squares	Adj. mean squares	F-value	p-value
Model	14	0.8738	0.0624	8677.55	≤ 0.0001
ϕ_1	1	0.1178	0.1178	16380.5	≤ 0.0001
ϕ_2	1	0.1915	0.1915	26617.6	≤ 0.0001
H	1	0.2402	0.2402	33393.8	≤ 0.0001
E	1	0.3106	0.3106	43185.2	≤ 0.0001
$\phi_1 * \phi_1$	1	0.0001	0.0001	12.33	0.0032
$\phi_2 * \phi_2$	1	0.0001	0.0001	12.85	0.0027
$H * H$	1	0	0	4.53	0.0504
$E * E$	1	0	0	5.13	0.0387
$\phi_1 * \phi_2$	1	0.0001	0.0001	16.98	0.0009
$\phi_1 * H$	1	0.0007	0.0007	98.37	≤ 0.0001
$\phi_1 * E$	1	0.0007	0.0007	98	≤ 0.0001
$\phi_2 * H$	1	0.0004	0.0004	58.71	≤ 0.0001
$\phi_2 * E$	1	0.0006	0.0006	79.41	≤ 0.0001
$H * E$	1	0.011	0.011	1522.55	≤ 0.0001
Error	15	0.0001	$7.193 * 10^{-6}$		
Lack-of-Fit	10	0.0001	0	*	*
Pure Error	5	0	0		
Total	29	0.8739			
$R^2 = 0.9999$			Adjusted $R^2 = 0.9998$		

Table 4.8: *ANOVA table (Cf_{MWCNT})*

Source	Deg. Of freedom	Adj. sum of squares	Adj. mean squares	F-value	p-value
Model	14	0.8632	0.0617	8557.46	≤ 0.0001
ϕ_1	1	0.0946	0.0946	13126.3	≤ 0.0001
ϕ_2	1	0.1934	0.1934	26840.9	≤ 0.0001
H	1	0.2472	0.2472	34310.6	≤ 0.0001
E	1	0.3141	0.3141	43586.9	≤ 0.0001
$\phi_1 * \phi_1$	1	0.0001	0.0001	11.49	0.004
$\phi_2 * \phi_2$	1	0.0001	0.0001	12.68	0.0028
$H * H$	1	0	0	5.83	0.029
$E * E$	1	0	0	5.03	0.0404
$\phi_1 * \phi_2$	1	0.0001	0.0001	19	0.0006
$\phi_1 * H$	1	0.0008	0.0008	116.72	≤ 0.0001
$\phi_1 * E$	1	0.0008	0.0008	106.11	≤ 0.0001
$\phi_2 * H$	1	0.0004	0.0004	57.76	≤ 0.0001
$\phi_2 * E$	1	0.0006	0.0006	78.95	≤ 0.0001
$H * E$	1	0.011	0.011	1525.78	≤ 0.0001
Error	15	0.0001	$7.205 * 10^{-6}$		
Lack-of-Fit	10	0.0001	0	*	*
Pure Error	5	0	0		
Total	29	0.8633			
$R^2 = 0.9999$			Adjusted $R^2 = 0.9998$		

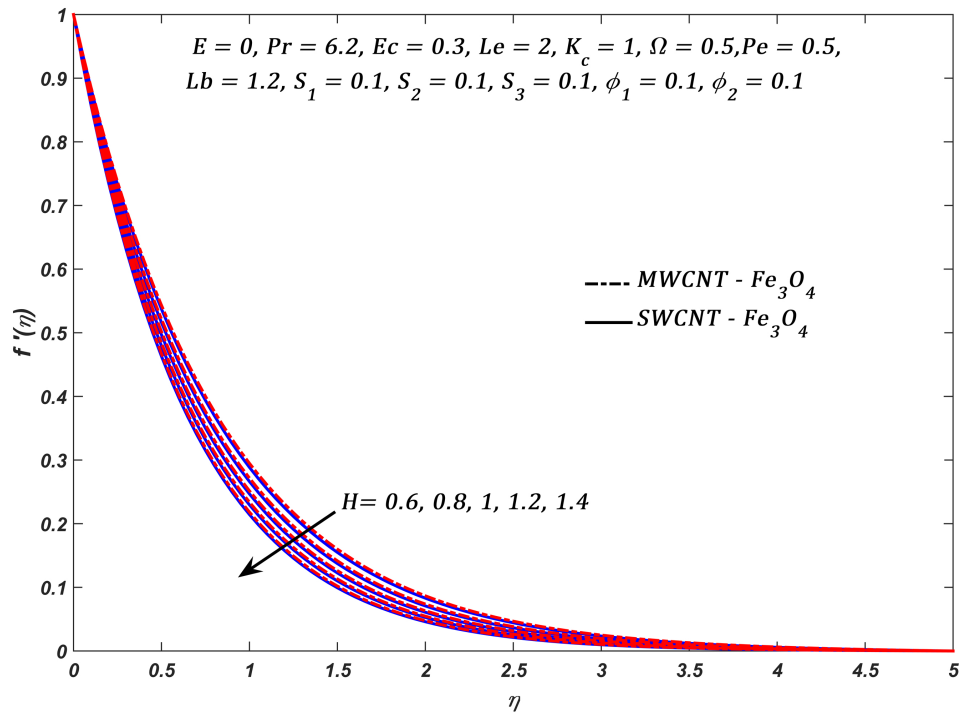


Figure 4.2: Variation in $f'(\eta)$ with H when $E = 0$

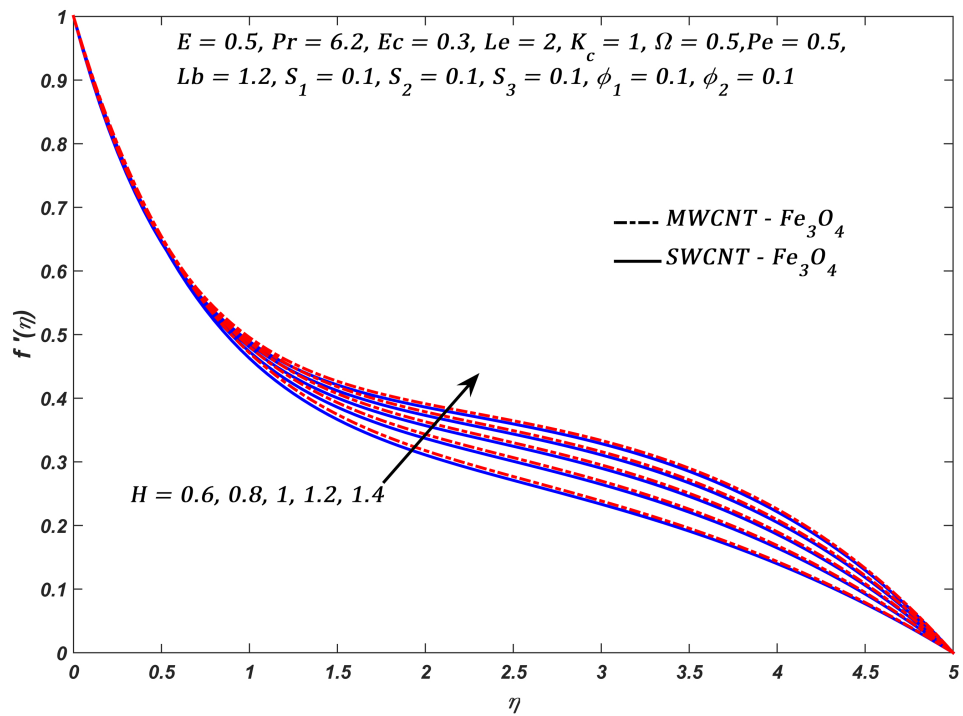


Figure 4.3: Variation in $f'(\eta)$ with H when $E \neq 0$

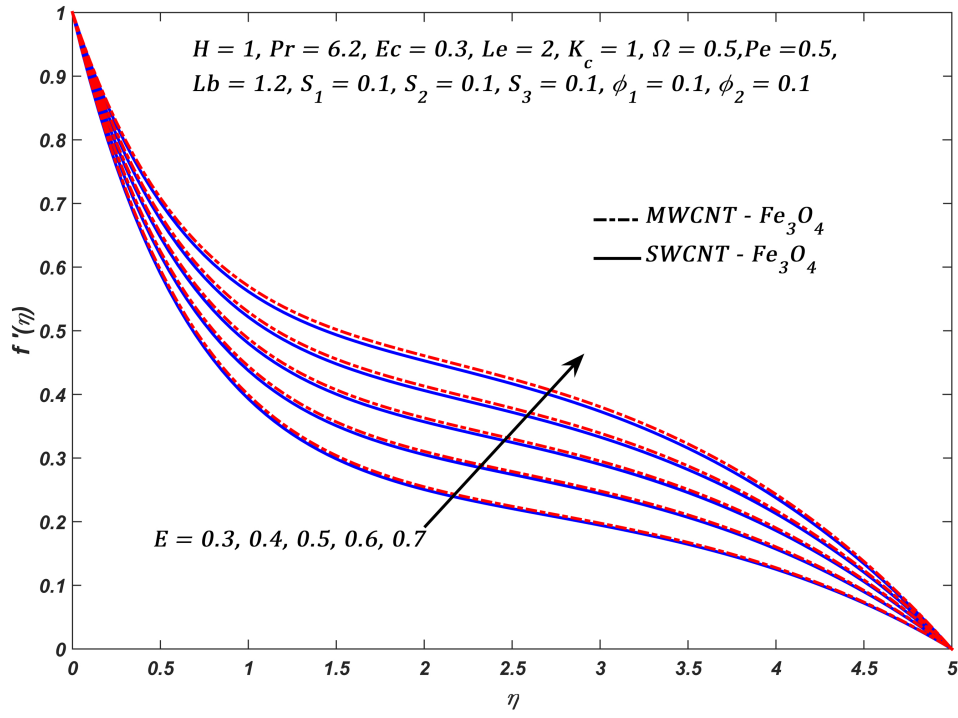


Figure 4.4: Variation in $f'(\eta)$ with E

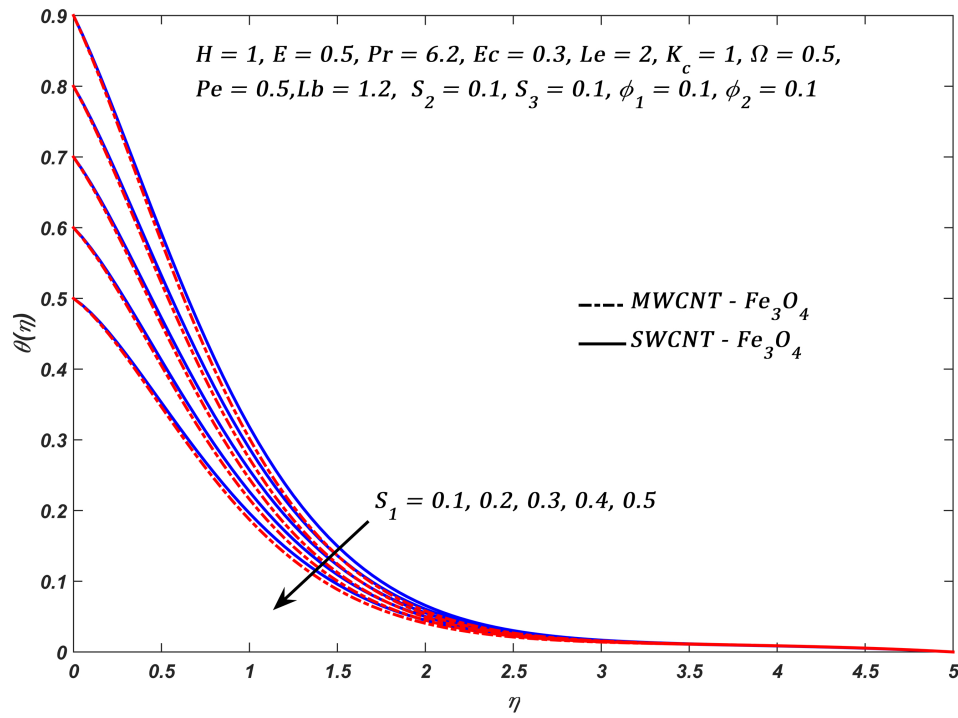
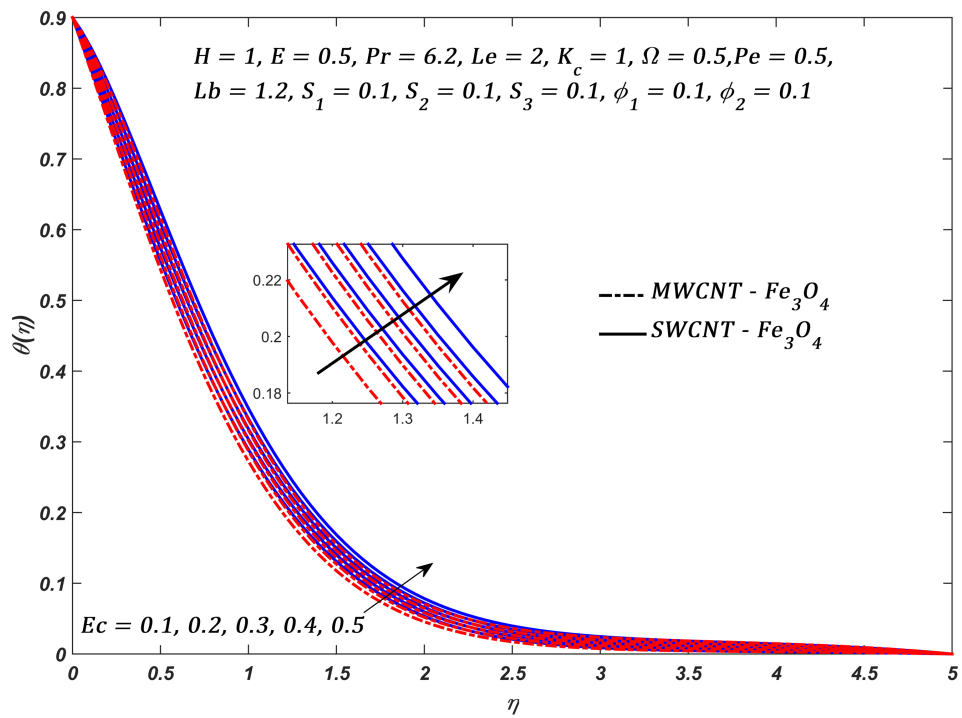
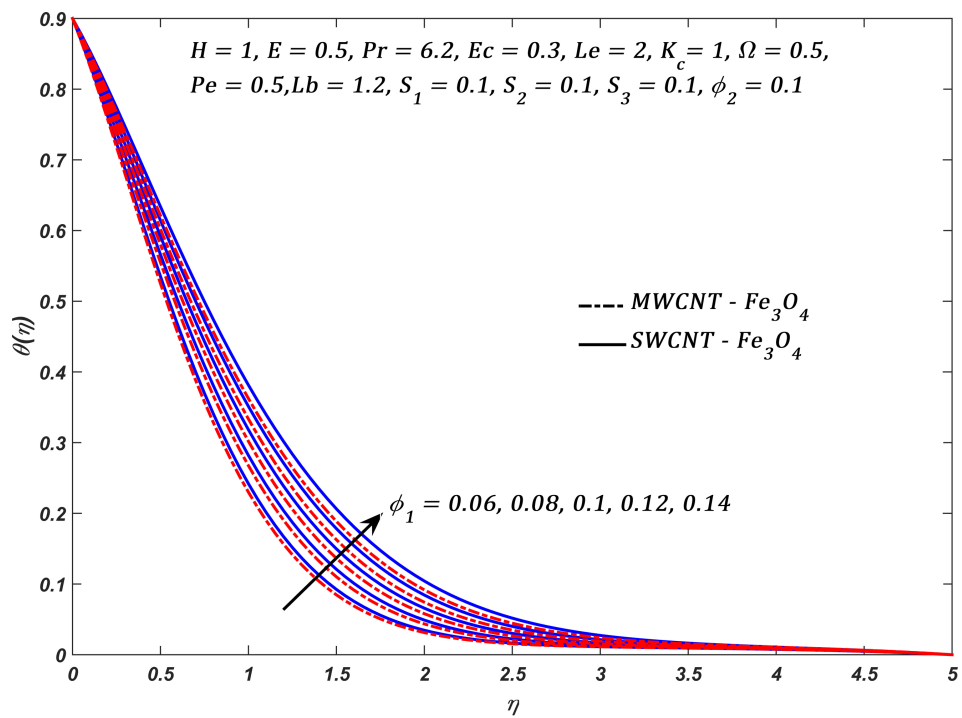


Figure 4.5: Variation in $\theta(\eta)$ with S_1


 Figure 4.6: Variation in $\theta(\eta)$ with Ec

 Figure 4.7: Variation in $\theta(\eta)$ with ϕ_1

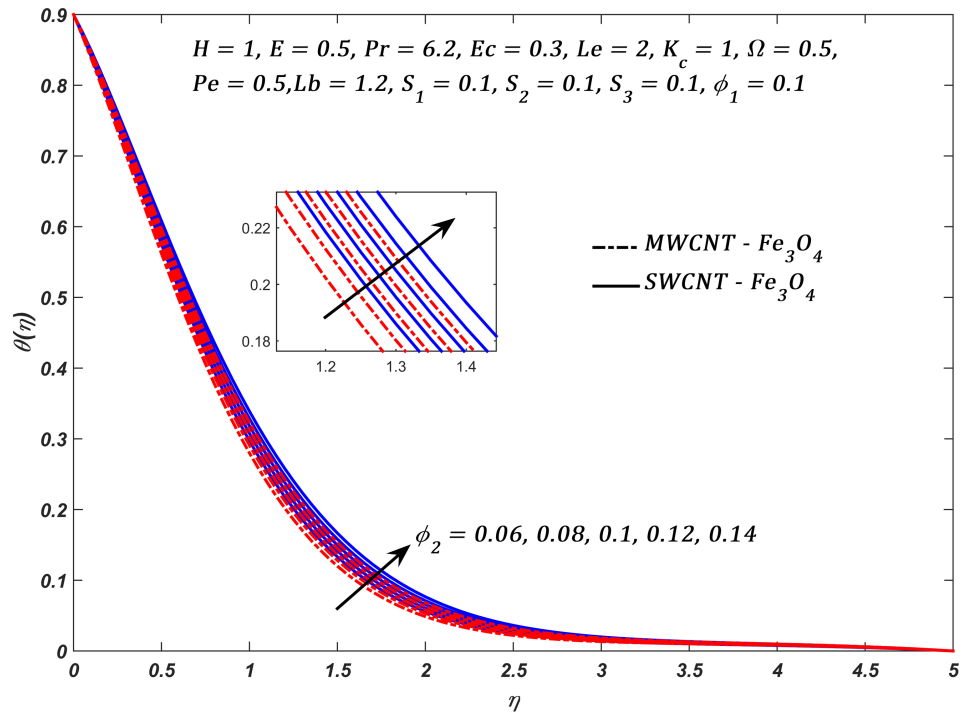


Figure 4.8: Variation in $\theta(\eta)$ with ϕ_2

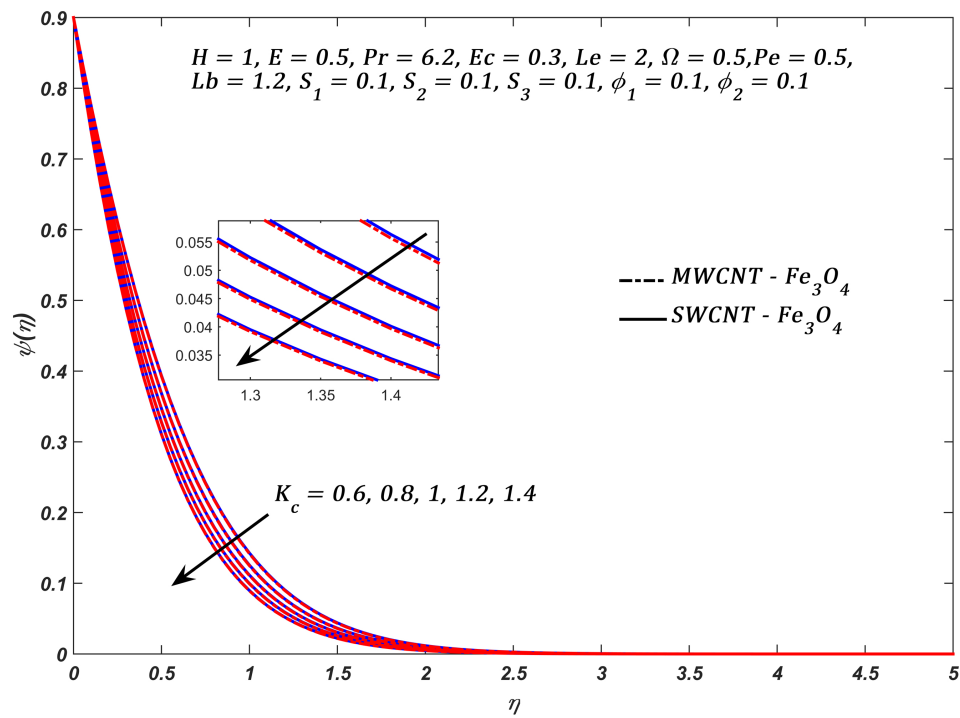
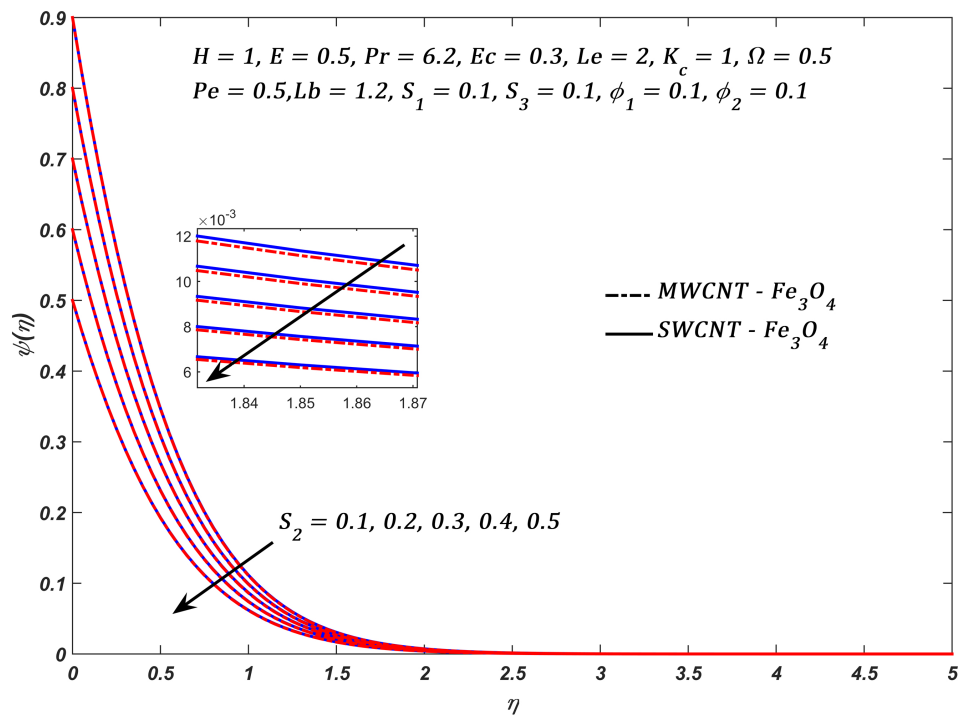
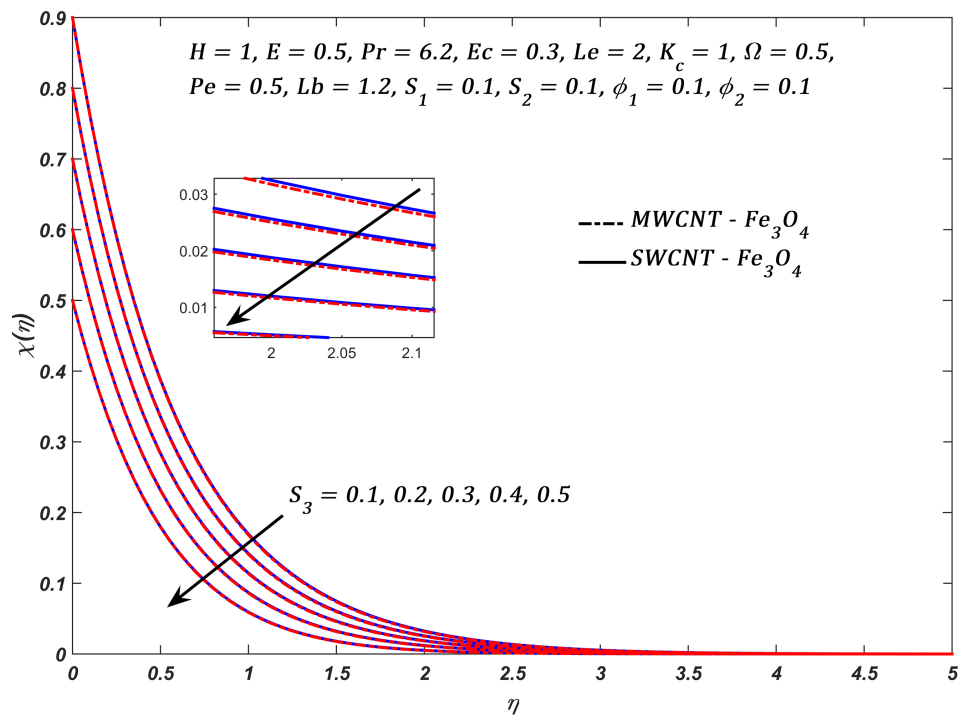


Figure 4.9: Variation in $\psi(\eta)$ with K_c


 Figure 4.10: Variation in $\psi(\eta)$ with S_2

 Figure 4.11: Variation in $\chi(\eta)$ with S_3

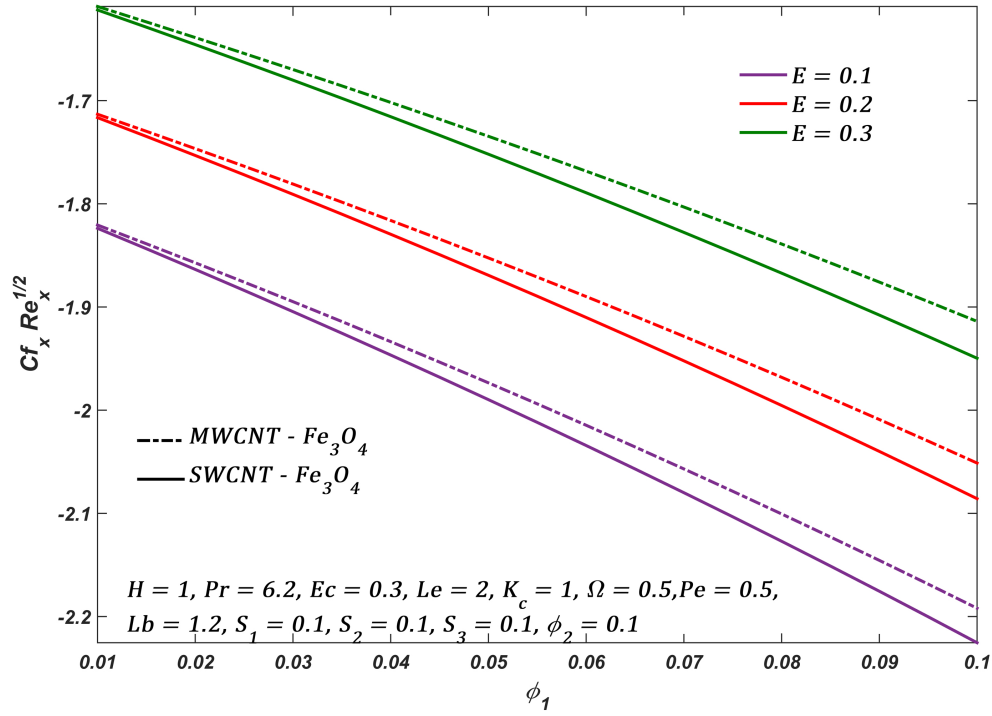


Figure 4.12: Variation in $Cf_x Re_x^{1/2}$ with ϕ_1 and E

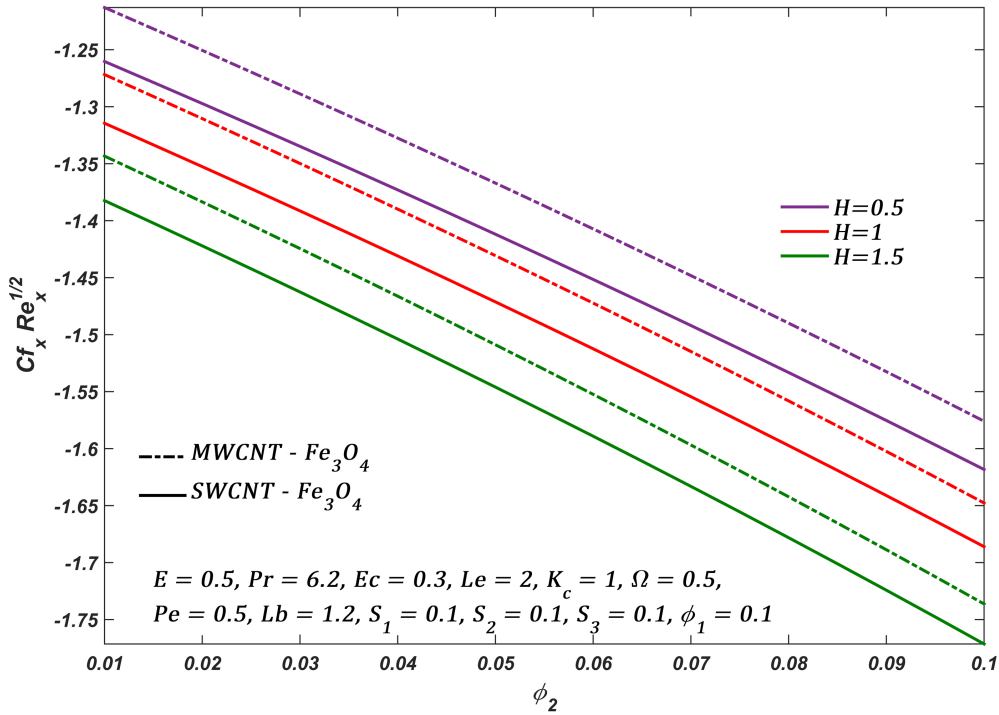


Figure 4.13: Variation in $Cf_x Re_x^{1/2}$ with ϕ_2 and M

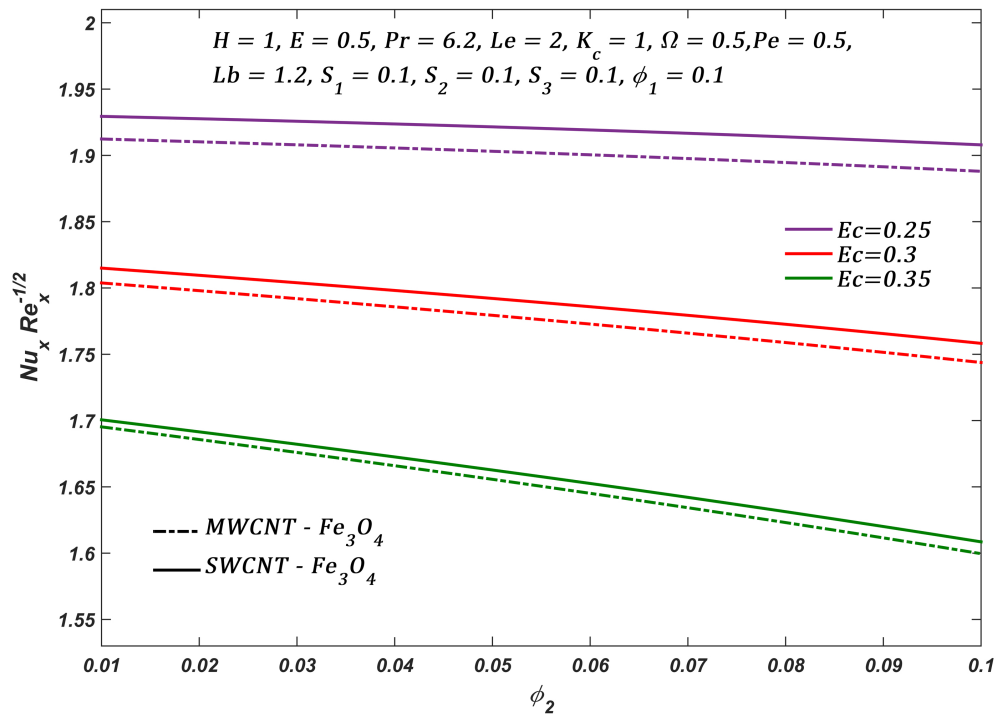


Figure 4.14: Variation in $Nu_x Re_x^{-1/2}$ with ϕ_2 and Ec

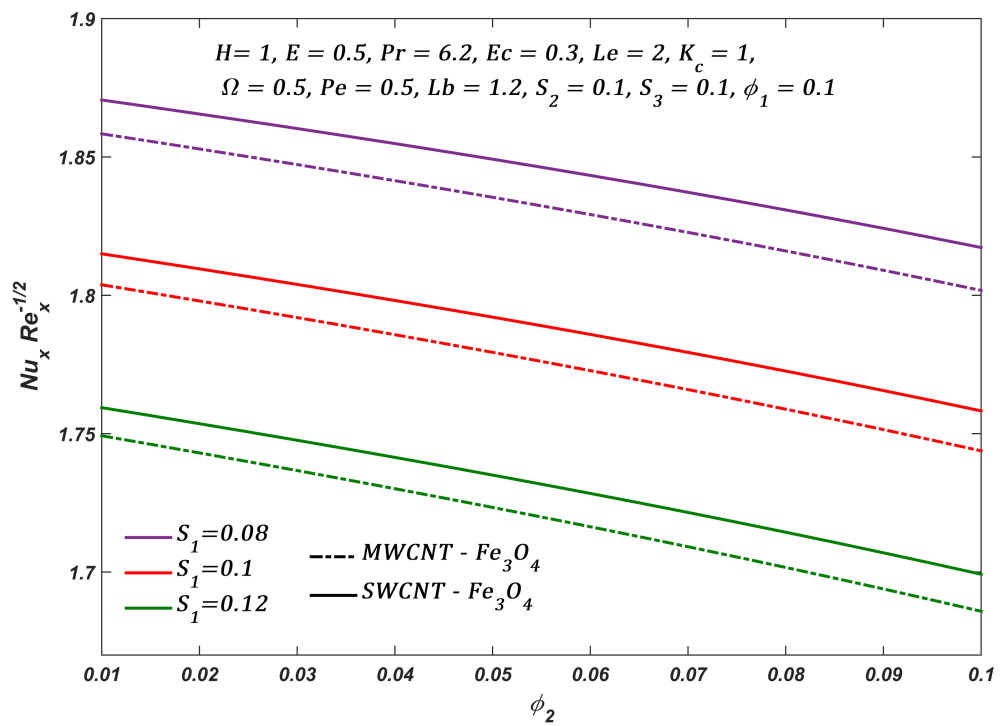


Figure 4.15: Variation in $Nu_x Re_x^{-1/2}$ with ϕ_2 and S_1

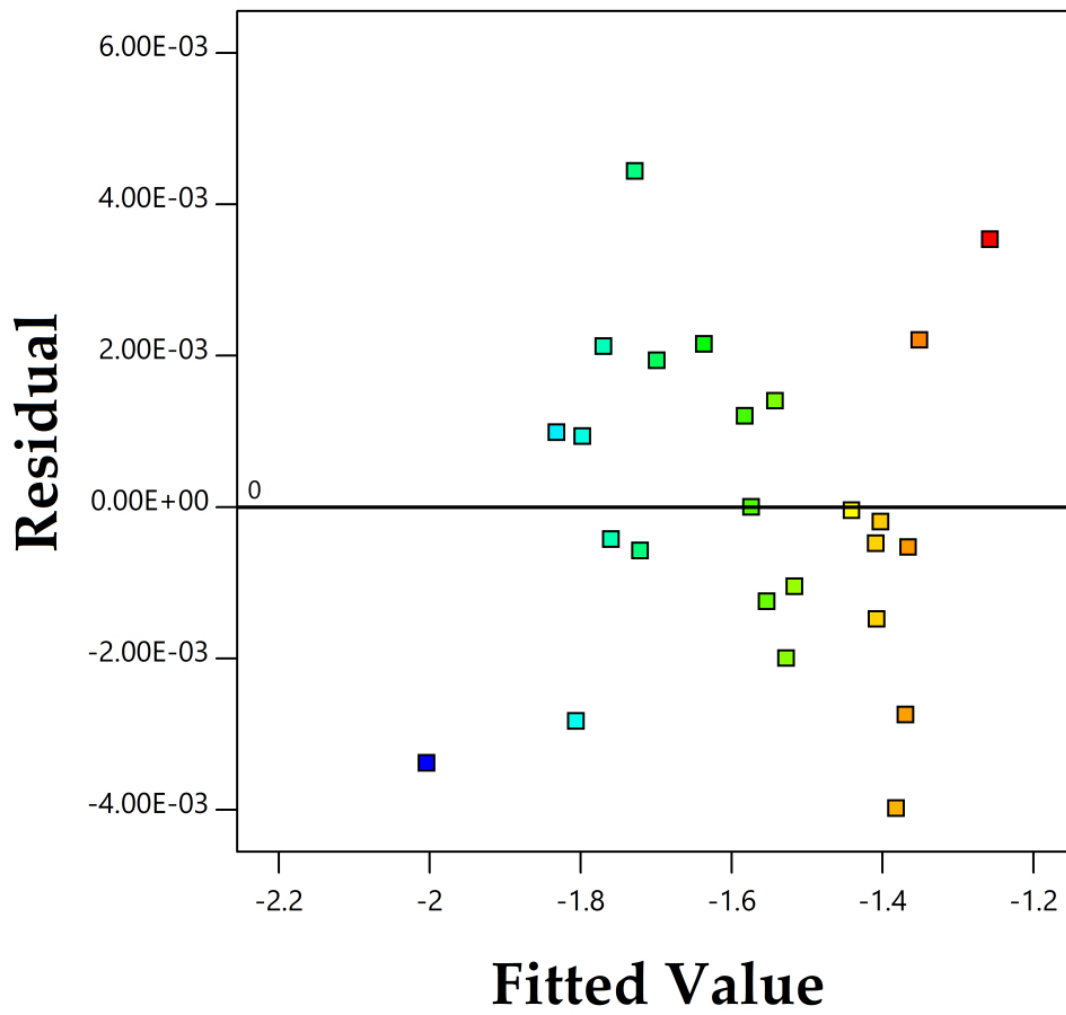


Figure 4.16: *Residual versus fitted value of $C f_{SWCNT}$*

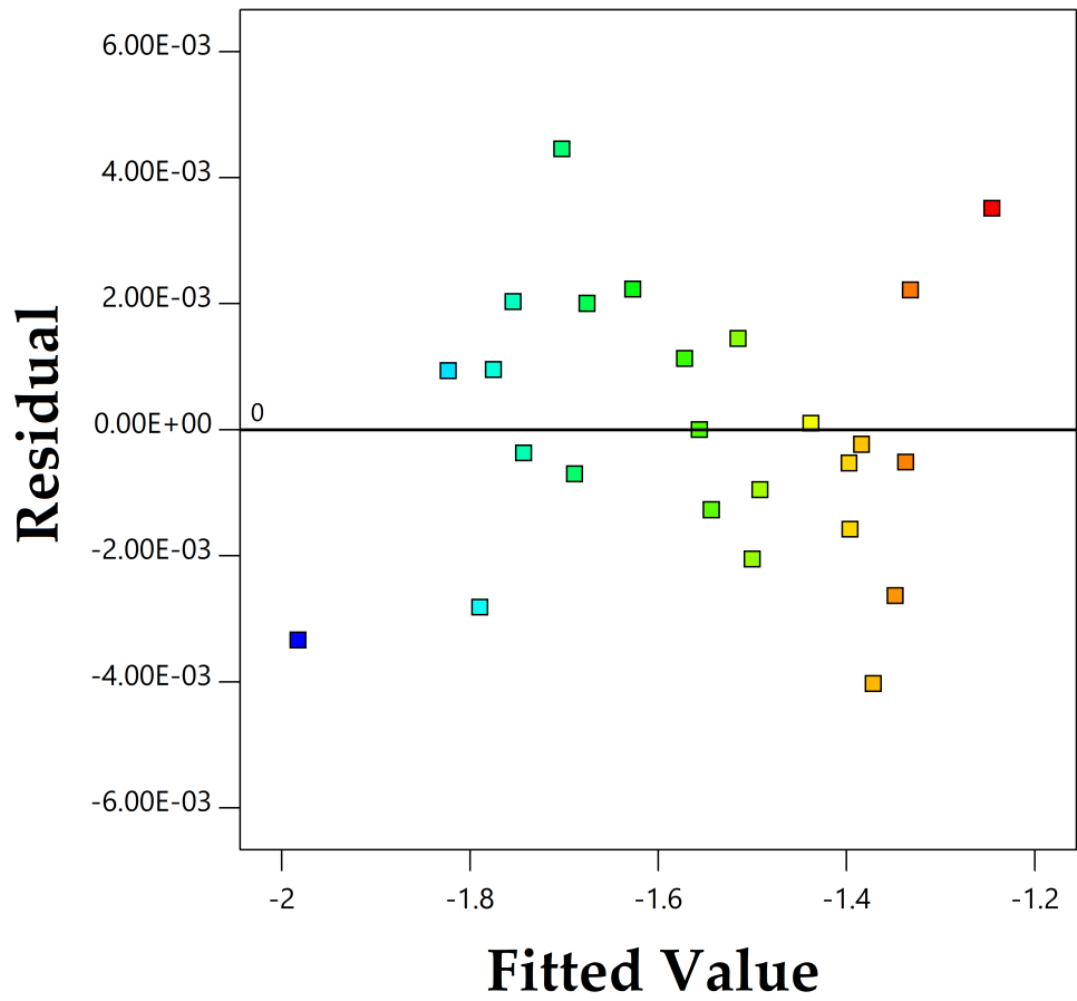


Figure 4.17: *Residual versus fitted value of Cf_{MWCNT}*

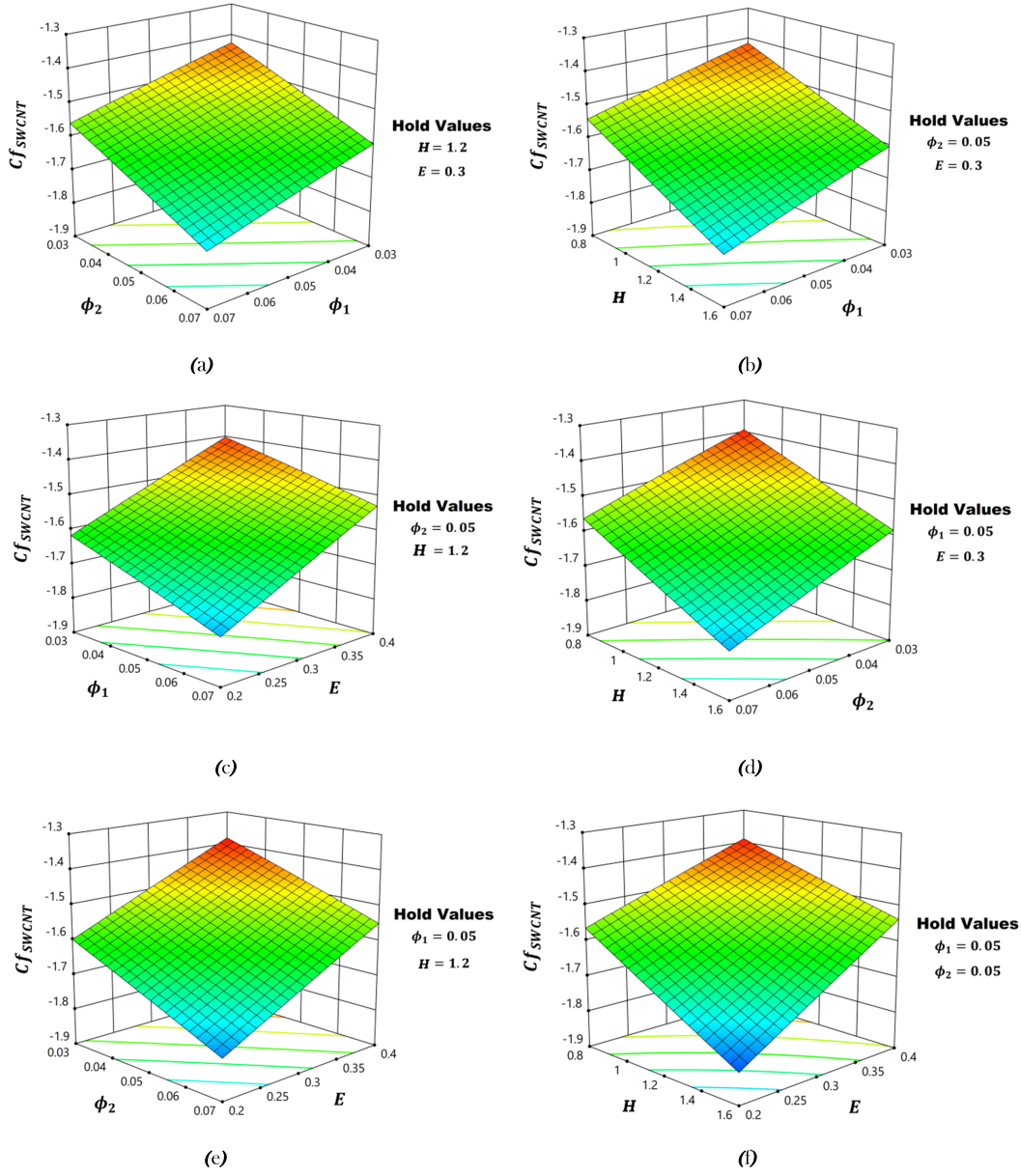
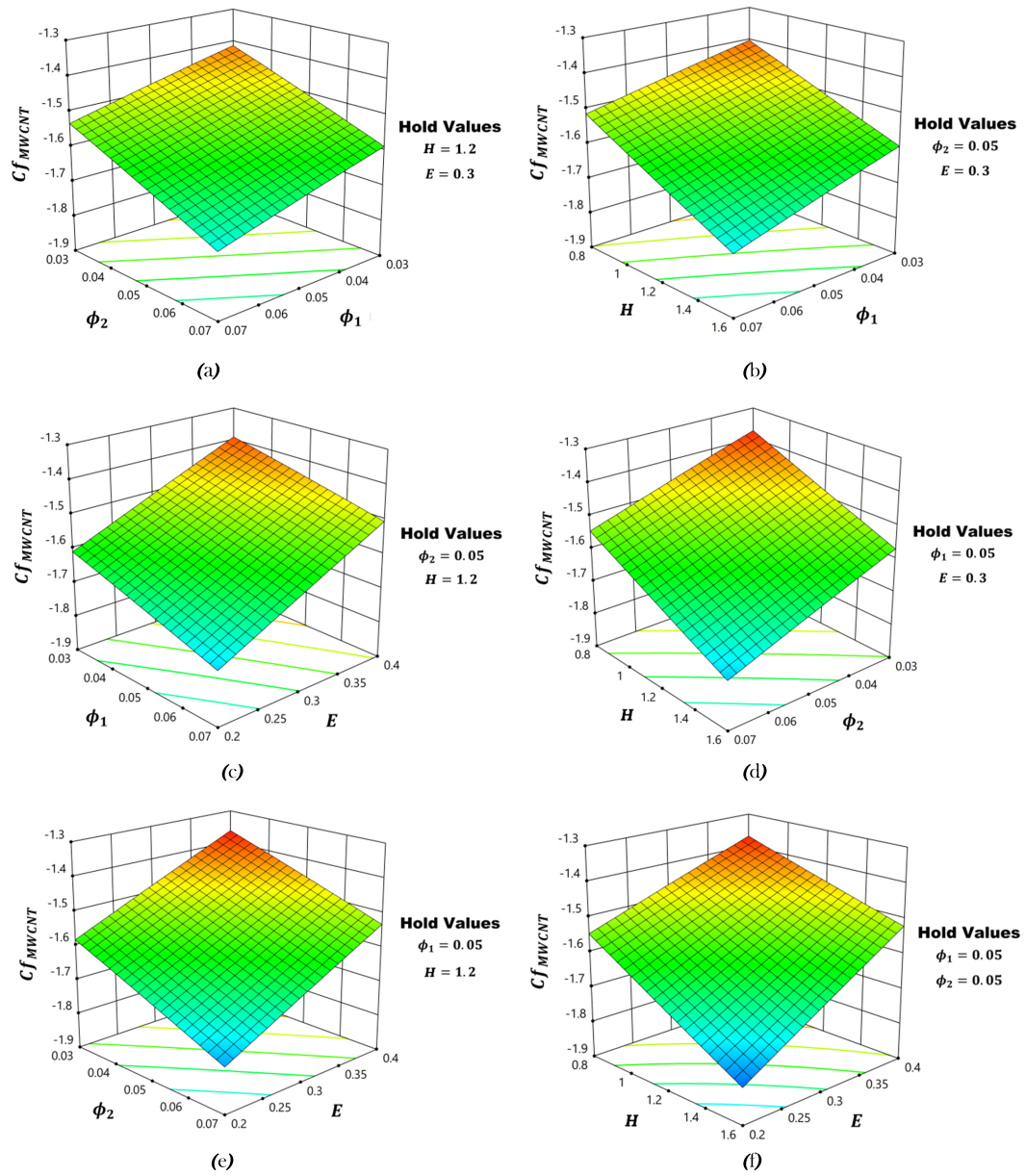


Figure 4.18: Surface plots Cf_{SWCNT}


 Figure 4.19: Surface plots Cf_{MWCNT}

APPENDIX I: Non-Dimensional Quantities

$$H = (\sigma_f B_0^2) / (C_p)_f \quad \text{Hartmann Number}$$

$$E = \frac{E_0}{B_0 u_w} \quad \text{Electric Field Parameter}$$

$$Pr = \frac{\vartheta_f}{\alpha_f} \quad \text{Prandtl Number}$$

$$K_c = \frac{K_0}{c} \quad \text{Chemical Reaction Parameter}$$

$$Le = \frac{\vartheta_f}{D_B} \quad \text{Lewis Number}$$

$$Ec = \frac{(cx)^2}{((C_p)_f (T_w - T_0))} \quad \text{Eckert Number}$$

$$Lb = \frac{\vartheta_f}{D_m} \quad \text{Bioconvection Lewis Number}$$

$$Pe = \frac{bW_c}{D_m} \quad \text{Bioconvection Peclet Number}$$

$$\Omega = \frac{N_\infty}{(N_w - N_0)} \quad \text{Microorganism Concentration Difference Parameter}$$

$$S_1 = \frac{\delta_2}{\delta_1} \quad \text{Thermal Stratification Parameter}$$

$$S_2 = \frac{\varepsilon_2}{\varepsilon_1} \quad \text{Mass Stratification Parameter}$$

$$S_3 = \frac{\xi_2}{\xi_1} \quad \text{Microorganism Density Stratification Parameter}$$

Appendix II : Nomenclature

u, v	Velocity components	T	Fluid temperature
C	Concentration of nanoparticles	μ	dynamic viscosity
T_∞	Ambient temperature	u_w	Stretching velocity
C_∞	Ambient concentration of nanoparticles	C_w	Surface concentration of nanoparticles
N_∞	Ambient concentration of Microorganism	N_w	Surface concentration of Microorganism
C_0	Reference concentration of nanoparticles	N_0	Reference concentration of Microorganism
S_3	Microorganism density stratification parameter	D_B	Brownian diffusion coefficient
ψ	Dimensionless concentration of nanoparticles	χ	Dimensionless concentration of Microorganism
θ	Dimensionless temperature	ρ	density
T_0	Reference temperature	B_0	Magnetic field strength
W_c	Maximum cell swimming speed	T_w	Surface temperature
c	Stretching rate	α	Thermal diffusivity
C_p	Specific heat	hnf	Hybrid nanofluid
D_m	Microorganisms diffusion coefficient	σ	electrical conductivity
E_0	Applied electric field	η	Dimensionless variable
N	Concentration of Microorganism	ν	kinematic viscosity
S_1	Thermal stratification parameter	s_1	CNT nanoparticle
S_2	Mass stratification parameter	s_2	Fe_3O_4 nanoparticle
ϕ_2	Volume fraction of Fe_3O_4	f	Base fluid
ϕ_1	Volume fraction of CNT	nf	Nanofluid
K_0	Chemical reaction coefficient	$\delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \xi_1, \xi_2$	constants

Bioconvective MHD Hybrid Nanofluid Flow past an Exponential Stretching Sheet *

5.1 INTRODUCTION

The stretching rate of sheets strongly affects the final product quality and, the exponential velocity and temperature distributions influence the admirable grade products attained in the annealing and thinning of copper wires.

This chapter analyses the magnetohydrodynamics of bioconvective hybrid nanofluid (TiO_2 and Ag in water) flow over a permeable exponential stretching sheet. The effects of thermal radiation, heat generation, chemical reaction, porosity, and viscous dissipation have been incorporated. Apposite similarity variables are applied in transforming the modeled PDE into an ODE system, and transmuted equations are solved numerically with the aid of the finite-difference-based `bvp5c` algorithm using MATLAB software. Multiple linear regression has been utilized to statistically scrutinize the effect of relevant variables on drag coefficient and heat transfer rate.

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5.2 MATHEMATICAL FORMULATION

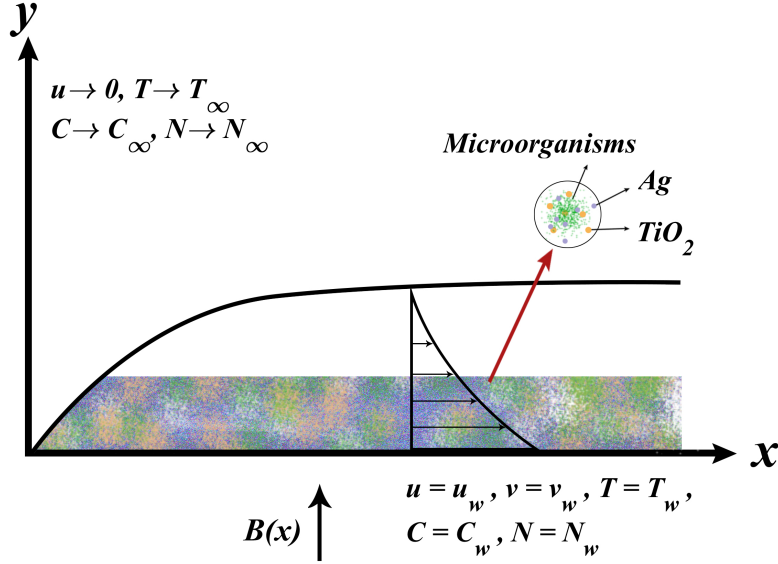


Figure 5.1: *Physical configuration*

Here a two-dimensional steady dissipative flow of hybrid nanofluid suspended with gyrotactic microorganisms through an exponentially stretching permeable sheet is contemplated. The x -axis is taken along the stretching surface and the y -axis is normal to the surface (illustrated in Figure 5.1). The sheet is stretched in x -direction ($x \geq 0$) with exponential velocity $u_w(x) = ce^{x/L}$. Velocity in y -direction is considered to be $v_w = v_0e^{x/2L}$, since it is a permeable sheet. Here c and v_0 are positive constants and L is the reference length. The sheet is maintained at temperature $T_w = T_\infty + T_0e^{x/2L}$, nanoparticle concentration $C_w = C_\infty + C_0e^{x/2L}$, and microbial concentration $N_w = N_\infty + N_0e^{x/2L}$ where T_0, C_0 and N_0 are constants, T_∞, C_∞ , and N_∞ are corresponding, the ambient fluid temperature, nanoparticle concentration, and microbial concentration. The variable magnetic field $B(x) = B_0e^{x/2L}$, where B_0 is uniform magnetic strength inflicted normally to the sheet. The consequence of the variable heat source $Q(x) = Q_0e^{x/L}$, chemical reaction $K_l = K_0e^{x/L}$, and poros-

ity $K = K_r e^{-x/L}$ effects are also taken into account. Hence the governing equations see (Waini et al., 2020), (Pal & Mondal, (2018b), (Raju, Sandeep, Sugunamma, Babu, & Reddy, 2016) are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (5.2.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left(\frac{\mu_{hnf}}{\rho_{hnf}} \right) \frac{\partial^2 u}{\partial y^2} - \frac{\sigma_{hnf}}{\rho_{hnf}} B^2(x) u - \frac{\mu_{hnf}}{\rho_{hnf}} \frac{u}{K} \quad (5.2.2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{hnf} \frac{\partial^2 T}{\partial y^2} - \frac{1}{(\rho C_p)_{hnf}} \frac{\partial q_r}{\partial y} + \frac{Q}{(\rho C_p)_{hnf}} (T - T_\infty) + \frac{\mu_{hnf}}{(\rho C_p)_{hnf}} \left(\frac{\partial u}{\partial y} \right)^2 \quad (5.2.3)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} - K_l (C - C_\infty) \quad (5.2.4)$$

$$u \frac{\partial N}{\partial x} + v \frac{\partial N}{\partial y} + \frac{bW_c}{C_w - C_\infty} \left(\frac{\partial}{\partial y} \left(N \frac{\partial C}{\partial y} \right) \right) = D_m \frac{\partial^2 N}{\partial y^2} \quad (5.2.5)$$

subject to the boundary conditions:

$$\left. \begin{aligned} u = u_w = ce^{x/L} \quad v = v_w = v_0 e^{x/2L} \quad T = T_w = T_\infty + T_0 e^{x/2L} \\ C = C_w = C_\infty + C_0 e^{x/2L}, \quad N = N_w = N_\infty + N_0 e^{x/2L} \end{aligned} \right\} \text{ at } y = 0 \quad (5.2.6)$$

$$u \rightarrow 0, \quad T \rightarrow T_\infty, \quad C \rightarrow C_\infty, \quad N \rightarrow N_\infty \text{ as } y \rightarrow \infty \quad (5.2.7)$$

The radiative heat flux q_r (in equation (5.2.3)) can be expressed as $q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y}$, where k^* and σ^* denote the mean absorption coefficient and Stefan-Boltzman constant, respectively. Using Taylor's series T^4 can be expressed as $T^4 \cong 4T_\infty^3 T - 3T_\infty^4$. Consider the following similarity variables (see (Waini et al., 2020) and (Shafiq et al., 2020)):

$$\begin{aligned} \eta = ye^{x/2L} \sqrt{c/2\vartheta_f L}, \quad u = ce^{x/L} f'(\eta), \quad v = -e^{\frac{x}{2L}} \sqrt{\frac{\vartheta_f c}{2L}} (f(\eta) + \eta f'(\eta)), \\ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \psi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}, \quad \chi(\eta) = \frac{N - N_\infty}{N_w - N_\infty} \end{aligned}$$

The effective thermophysical models of the hybrid nanofluid are given by:

Effective Dynamic Viscosity:

$$\frac{\mu_{hnf}}{\mu_f} = \frac{1}{(1 - \phi_1)^{2.5} (1 - \phi_2)^{2.5}} = \frac{1}{a_1}$$

Effective Density:

$$\frac{\rho_{hnf}}{\rho_f} = (1 - \phi_2) \left[1 - \phi_1 + \phi_1 \left(\frac{\rho_{s1}}{\rho_f} \right) \right] + \phi_2 \left(\frac{\rho_{s2}}{\rho_f} \right) = a_2$$

Effective Electrical Conductivity:

$$\frac{\sigma_{hnf}}{\sigma_f} = 1 + \frac{3 \left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{\sigma_f} - (\phi_1 + \phi_2) \right)}{2 + \left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{(\phi_1 + \phi_2) \sigma_f} \right) - \left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{\sigma_f} - (\phi_1 + \phi_2) \right)} = a_3$$

Effective Specific Heat:

$$\frac{(\rho C_p)_{hnf}}{(\rho C_p)_f} = (1 - \phi_2) \left[1 - \phi_1 + \phi_1 \left(\frac{(\rho C_p)_{s1}}{(\rho C_p)_f} \right) \right] + \phi_2 \left(\frac{(\rho C_p)_{s2}}{(\rho C_p)_f} \right) = a_4$$

Effective Thermal Conductivity:

$$\frac{K_{hnf}}{K_f} = a_5$$

where

$$\frac{K_{hnf}}{K_{nf}} = \frac{K_{s2} + 2K_{nf} - 2\phi_2 (K_{nf} - K_{s2})}{K_{s2} + 2K_{nf} + 2\phi_2 (K_{nf} - K_{s2})}$$

and

$$\frac{K_{nf}}{K_f} = \frac{K_{s1} + 2K_f - 2\phi_1 (K_f - K_{s1})}{K_{s1} + 2K_f + \phi_1 (K_f - K_{s1})}$$

In view of the above-mentioned similarity variables and effective thermophysical model, one can get the following from equations (5.2.1) – (5.2.7):

$$f''' + a_1 a_2 f f'' - f' (a_1 a_3 H + K_p + 2a_1 a_2 f') = 0 \quad (5.2.8)$$

$$a_1 \left(a_5 + \frac{4}{3} R \right) \theta'' + a_1 a_4 Pr f \theta' + a_1 Pr (\beta - a_4 f') \theta + Pr Ec f''^2 = 0 \quad (5.2.9)$$

$$\psi'' + Le f \psi' - Le (f' + K_c) \psi = 0 \quad (5.2.10)$$

$$\chi'' - Lb (f' \chi - f \chi') - Pe \{ (\chi + \Omega) \psi'' + \psi' \chi' \} = 0 \quad (5.2.11)$$

with

$$f(0) = S, \quad f'(0) = 1, \quad \theta(0) = 1, \quad \psi(0) = 1, \quad \chi(0) = 1 \quad (5.2.12)$$

$$f'(\infty) \rightarrow 0, \quad \theta(\infty) \rightarrow 0, \quad \psi(\infty) \rightarrow 0, \quad \chi(\infty) \rightarrow 0 \quad (5.2.13)$$

5.3 PHYSICAL QUANTITIES

The drag coefficients, local Nusselt number, local Sherwood number, and local microorganisms density number are given by (see (Waini et al., 2020) and (Pal & Mondal, (2018b)):

$$Cf_x = \frac{\tau_w}{\rho_f(u_w)^2} = \frac{\mu_{hnf} \left(\frac{\partial u}{\partial y} \right)_{y=0}}{\rho_f(u_w)^2} \Rightarrow Re_x^{1/2} Cf_x = \frac{f''(0)}{a_1}. \quad (5.3.1)$$

$$Nu_x = \frac{2Lq_w}{K_f(T_w - T_\infty)} = \frac{2L \left(-K_{hnf} \left(\frac{\partial T}{\partial y} \right)_{y=0} + (q_r)_{y=0} \right)}{K_f(T_w - T_\infty)} \Rightarrow Re_x^{-1/2} Nu_x = - \left(a_5 + \frac{4}{3}R \right) \theta'(0) \quad (5.3.2)$$

$$Sh_x = \frac{2Lq_m}{D_B(C_w - C_\infty)} = - \frac{2LD_B \left(\frac{\partial C}{\partial y} \right)_{y=0}}{D_B(C_w - C_\infty)} \Rightarrow Re_x^{-1/2} Sh_x = -\psi'(0) \quad (5.3.3)$$

$$Nn_x = \frac{2Lq_n}{D_m(N_w - N_\infty)} = - \frac{2LD_m \left(\frac{\partial N}{\partial y} \right)_{y=0}}{D_m(N_w - N_\infty)} \Rightarrow Re_x^{-1/2} Nn_x = -\chi'(0) \quad (5.3.4)$$

where $Re_x = \frac{2Lu_w}{\vartheta_f}$ is the local Reynold's number.

5.4 NUMERICAL PROCEDURE

The transmuted Eqs. (5.2.8) – (5.2.13) are reduced into a system of single-order ordinary differential equations (ODEs) by setting:

$$f = y_1, \quad f' = y_2, \quad f'' = y_3, \quad \theta = y_4, \quad \theta' = y_5,$$

$$\psi = y_6, \quad \psi' = y_7, \quad \chi = y_8, \quad \chi' = y_9$$

The subsequent system of single-order ODEs is given by:

$$y_1' = y_2, \quad y_2' = y_3, \quad y_3' = y_2(a_1 a_3 H + K_p + 2a_1 a_2 y_2) - a_1 a_2 y_1 y_3,$$

$$y_4' = y_5, \quad y_5' = - \frac{(a_1 a_4 Pr y_1 y_5 + a_1 Pr(\beta - a_4 y_2) y_4 + Pr Ec y_3^2)}{a_1(a_5 + \frac{4}{3}R)},$$

$$y_6' = y_7, \quad y_7' = Le((y_2 + K_c) y_6 - y_1 y_7),$$

$$y_8' = y_9, \quad y_9' = Lb(y_2 y_8 - y_1 y_9) + Pe \{ (y_8 + \Omega) y_7' + y_7 y_9 \}$$

with

$$y_1(0) = S, y_2(0) = 1, y_4(0) = 1, y_6(0) = 1, y_8(0) = 1,$$

$$y_2(\infty) = 0, y_4(\infty) = 0, y_6(\infty) = 0, y_8(\infty) = 0.$$

The above system has been solved numerically using the finite-difference based bvp5c algorithm with infinity fixed at 5. The reliability of the adopted numerical method has been validated through a restrictive correspondence of the surface temperature gradient ($-\theta'(0)$) with the previously published works (Waini et al., 2020), (Magyari & Keller, 1999a) and (Abd El-Aziz, 2009) and a commendable agreement is observed (see Table 5.1).

5.5 RESULTS AND DISCUSSION

The impact of pertinent parameters on velocity ($f'(\eta)$), temperature ($\theta(\eta)$), nanoparticle concentration ($\psi(\eta)$), and microbial concentration ($\chi(\eta)$) profiles are illustrated through Figures 5.2 – 5.21. The thermophysical values of base fluid (water) and nanoparticles (TiO_2 and Ag) are displayed in Table 5.2. The Prandtl number (Pr) is set to 6.2 and $0.5 \leq H, R, K_p \leq 3, 0.1 \leq S, Ec, Pe \leq 0.6, 0 \leq \beta \leq 0.25, 0.3 \leq K_c, \Omega \leq 0.8, 1.2 \leq Le \leq 2.2, 0.6 \leq Lb \leq 1.6, 0 \leq \phi_1, \phi_2 \leq 0.1$ represent the considered range of the effectual parameters.

The Lorentz force generated due to an enhancement in the magnetic field parameter (H) resists the fluid flow that reduces the fluid velocity (displayed in Figure 5.2). Figure 5.3 illustrates the negative impact of the porosity parameter (K_p) on the velocity profile. This is physically attributed to the fact that an increase in K_p drops the magnitude of Darcian body force and hence slows the fluid. Figure 5.4 describes the changes in the suction parameter (S) is inversely proportional to $f'(\eta)$. This is in accordance with the physical fact with increasing values of S , the momentum boundary layer tends to stick with the stretching sheet that disturbs the flow momentum.

Figure 5.5 indicates the enhancement in $\theta(\eta)$ with H . This is because the generated Lorentz force (due to the changes in H) increases the friction between the fluid layers that enhance the temperature profile. The upshot of radiation parameter (R) on $\theta(\eta)$ is exhibited in Figure 5.6. Greater values of R produce greater surface heat flux that improves the fluid temperature. Figure 5.7 depicts

the positive impact of Eckert number (Ec) on $\theta(\eta)$. The elevation in the temperature profile is physically associated with the production of frictional energy caused by the collision of fluid particles. The heat source (β) and porosity (K_p) parameters generate an internal heat (see (Mandal & Mukhopadhyay, 2013) and (Naramgari & Sulochana, 2016)) that elevates the temperature profile (see Figures 5.8 and 5.9, respectively). Figure 5.10 elucidate the consequence of S on $\theta(\eta)$. Augmentation in S increases the fluid's absorption rate (towards the surface) and hence reduces the fluid temperature. The positive influence of volume fraction of TiO_2 and Ag nanoparticles on $\theta(\eta)$ is depicted in Figures 5.11 and 5.12, respectively. Physically, this is due to the increased particle collision and improved thermal conductivity of the fluid caused by the addition of TiO_2 and Ag nanoparticles.

The Lorentz force generated with the magnification of H induces a disturbance in the fluid motion that enlarges the solutal boundary layer thickness (see Figure 5.13). An increase in the chemical reaction parameter (K_c) expedites the nanoparticle consumption causing a drop in $\psi(\eta)$ (shown in Figure 5.14). The negative response of S on $\psi(\eta)$ has been graphed in Figure 5.15. This decrease in $\psi(\eta)$ is physically attributed to the fact that an improvement in S brings the fluid closer to the surface that reduces the solutal boundary layer thickness. An increase in Lewis number (Le) reduces the mass diffusivity thereby causing a decline in the concentration profile (see Figure 5.16).

Figures 5.17 and 5.18 elucidate the positive impact of H and K_p on $\chi(\eta)$, respectively. The disturbance in the fluid motion caused by the changes in H and K_p generates heat that boosts the microorganism boundary layer thickness. Figure 5.19 graphs the decreasing nature of $\chi(\eta)$ with respect to Peclet number (Pe). Physically, augmentation in Pe intensifies the microbial movement that reduces the microorganism density near the surface. The diffusivity of the microorganisms descends with mounting values of bioconvection Lewis number (Lb) that reduces the microorganism density of the fluid (shown in Figure 5.20). Figure 5.21 bespeaks the deviations in $\chi(\eta)$ with respect to microbial concentration difference parameter (Ω). Physically, this decrease in $\chi(\eta)$ is due to the fact that higher values of Ω implies larger density difference between the gyrotactic microorganisms and base fluid that makes the surface of the fluid unstable forcing the microorganisms to swim back to the fluid's bottom layer.

The simultaneous influence of influential parameters on the mass transfer rate has been displayed in Figures 5.22 and 5.23 with the aid of three-dimensional surface plots. It is observed that the mass transfer rate is a decreasing function of K_p and an increasing function of S . Higher values of K_c consume more nanoparticles thereby lowering the concentration of the fluid that enhances the mass transfer rate (see Figure 5.22). Furthermore, enhanced kinetic energy due to the addition of TiO_2 nanoparticles increase the mass transfer rate (See Figure 5.23).

The consequence of prominent parameters on drag coefficient, Nusselt number, and microorganism density number have been carried out in Tables 5.3 – 5.5. The slope of linear regression has also been calculated. The magnitude of slope represents the rate of change of the considered physical quantity per unit value of the corresponding parameter and the sign of slope symbolizes the nature of this impact. The drag coefficient declines with changes in H , K_p , S , ϕ_1 , and ϕ_2 values (see Table 5.3). The restricted flow due to these variables reduces the drag coefficient. The rate of heat transfer ascends with R (since R improves surface heat flux) but descends with an increase in H , Ec , β , ϕ_1 , and ϕ_2 values (see Table 5.4). The internal heat generated due to these variables reduces the temperature difference between the surface and the fluid which lowers the heat transfer rate. The microorganism density number descends with K_p and ascends with K_c , Pe , Ω , Le , and Lb (see Table 5.5). Augmentation in K_p values reduce the fluid motion that demotes the microorganism density number. A rise in the chemical reaction parameter consumes more nanoparticles and decreases the concentration of the chemical species. This causes a decrease in the fluid's density that promotes the movement of more microorganisms to the vacant area and thereby increases the microorganism density number.

5.6 STATISTICAL ANALYSIS

The use of statistical techniques like probable error, correlation, and multiple linear regression to scrutinize the numerical and estimated results have gained a lot of interest from the research community. In this study, the effect of pertinent parameters on drag coefficient and heat transfer rate have been statistically scrutinized.

5.6.1 Correlation and probable error

Correlation measures the degree of association between two or more variables. The magnitude of the correlation coefficient (r), where $r \in [-1, 1]$, indicates the strength

of the relationship and the sign of r represents the nature of this relationship. The integrity of r is further clarified using probable error, $PE = (\frac{1-r^2}{\sqrt{n}})0.6745$ where n is the number of observations. If $|\frac{r}{PE}| > 6$, then the correlation is said to be significant.

From Table 5.6, it is noted that $Cf_x Re_x^{1/2}$ is negatively correlated with H, K_p, S, ϕ_1 and ϕ_2 . $Nu_x Re_x^{-1/2}$ is positively correlated with R and negatively correlated with H, Ec, β, ϕ_1 , and ϕ_2 (see Table 5.7). Furthermore, it is observed that these findings coincide with the results obtained in Tables 5.3 and 5.4.

5.6.2 Multiple linear regression

Regression analysis judges the relationship between responses (dependent variable) and one or more predictors (independent variables). All correlations are found to be significant and hence $Cf_x Re_x^{1/2}$ and $Nu_x Re_x^{-1/2}$ are estimated using multiple regression analysis. The general forms of the estimated models are given by:

$$Cf_{est} = b_H H + b_{K_p} K_p + b_S S + b_{\phi_1} \phi_1 + b_{\phi_2} \phi_2 + b_0$$

$$Nu_{est} = b_H H + b_{Ec} Ec + b_R R + b_{\beta} \beta + b_{\phi_1} \phi_1 + b_{\phi_2} \phi_2 + b_0$$

where $b_H, b_{K_p}, b_S, b_{Ec}, b_R, b_{\beta}, b_{\phi_1}, b_{\phi_2}, b_0$ are the estimated regression coefficients.

The drag coefficient is estimated from 30 sets of values chosen in the range [0.5,3] for H and K_p , [0.01,0.1] for ϕ_1 and ϕ_2 , and [0.1,0.6] for S . Further, the heat transfer rate is evaluated from 36 sets of values chosen in the range [0.5,3] for H and R , [0.01,0.1] for ϕ_1 and ϕ_2 , [0.1,0.6] for Ec , and [0.05,0.3] for β . The regression coefficients (for both cases) are found using Microsoft Excel. The estimated regression models are given by:

$$Cf_{est} = -0.39712 H - 0.38537 K_p - 1.21825 S - 7.4983 \phi_1 - 10.3397 \phi_2 - 0.89767$$

$$Nu_{est} = -0.61526 H - 9.26793 Ec + 0.189206 R - 9.16537 \beta - 5.50554 \phi_1 \\ - 9.71937 \phi_2 + 6.557836$$

The positive sign of the estimated regression coefficient denotes that the corresponding parameter ascends the drag coefficient or the heat transfer rate and a negative

sign of the estimated regression coefficient denotes that the corresponding parameter descends the drag coefficient or the heat transfer rate. From the estimated models, it can be observed that the drag coefficient declines with increasing values of H, K_p, S, ϕ_1 , and ϕ_2 . Furthermore, it is also noted that the heat transfer rate enhances with augmentation in R and reduces with increasing values of H, Ec, β, ϕ_1 , and ϕ_2 . These results are in perfect agreement with the findings in Tables 5.3, 5.4, 5.6 and 5.7. The accuracy of the estimated regression model (for the chosen sample) has been adjudged through Figures. 5.24 and 5.25. A commendable agreement is noted between the actual and estimated values.

5.7 CONCLUSION

For its application in metal spinning, drawing of plastic films, glass blowing, crystal growing, and cooling of filaments; the dynamics of bioconvective MHD hybrid nanofluid (TiO_2 and Ag in water) flow over a permeable exponential stretching surface in the presence of thermal radiation, heat generation, chemical reaction, porosity, and dissipative effects has been investigated. The consequence of effectual variables on the flow profiles has been numerically solved with the aid of the finite-difference-based bvp5c algorithm. Further, multiple linear regression has been utilized to statistically scrutinize the effect of pertinent parameters on drag coefficient and heat transfer rate. The major conclusions drawn from the study are:

- The magnetic field has a destructive effect on the velocity profile and a constructive effect on the temperature, nanoparticle concentration, and microbial concentration profiles.
- An increase in the porosity parameter descends the velocity profile and ascends the temperature and microbial concentration profiles.
- The temperature profile is proportional to augmentations in radiation parameter, magnetic field parameter, Eckert number, and volume fraction of TiO_2 and Ag nanoparticles.
- Nanoparticle concentration is a decreasing function of Lewis number and chemical reaction parameter.
- The mass transfer rate is inversely proportional to the changes in the porosity

parameter and directly proportional to the changes in the chemical reaction and suction parameters.

- The local microbial density number declines with growing values of porosity parameter and improves with growing values of the chemical reaction parameter and Peclet number.
- The estimated regression model for the drag coefficient is given by:

$$Cf_{est} = -0.39712 H - 0.38537 K_p - 1.21825 S - 7.4983 \phi_1 - 10.3397 \phi_2 - 0.89767$$

- The estimated regression model for the heat transfer rate is given by:

$$Nu_{est} = -0.61526 H - 9.26793 Ec + 0.189206 R - 9.16537 \beta - 5.50554 \phi_1 - 9.71937 \phi_2 + 6.557836$$

- The drag coefficient is negatively correlated with the magnetic parameter, suction parameter, porosity parameter, and volume fraction of TiO_2 and Ag nanoparticles.
- The heat transfer rate is positively correlated with the radiation parameter and negatively correlated with the magnetic parameter, heat source parameter, viscous dissipation parameter, and volume fraction of TiO_2 and Ag nanoparticles.

TABLES AND GRAPHS

Table 5.1: Comparison of $-\theta'(0)$ for different values of Pr when $\phi_1 = \phi_2 = M = R = Ec = Le = Lb = K_c = K_p = \beta = \Omega = Pe = S = 0$

Pr	(Magyari & Keller, 1999a)	(Abd El-Aziz, 2009)	(Waini et al., 2020)	Present Study
0.5	0.5943	0.5945	0.5943	0.5945
1	0.9548	0.9548	0.9548	0.9548
3	1.8691	1.8691	1.8691	1.8691
5	2.5001	2.5001	2.5001	2.5001
10	3.6604	3.6604	3.6604	3.6604

Table 5.2: *Thermophysical properties*

Properties	H_2O	TiO_2	Ag
ρ	997.1	4250	10500
C_p	4179	686.2	235
k	0.613	8.9538	429
σ	$5 * 10^{-2}$	$2.38 * 10^6$	$3.5 * 10^6$

Table 5.3: *Variation in $Cf_x Re_x^{1/2}$ when $Pr = 6.2, R = 1, Ec = 0.3, Le = 2, K_c = \Omega = Pe = 0.5, \beta = 0.1$, and $Lb = 1.2$*

H	K_p	S	ϕ_1	ϕ_2	$Cf_x Re_x^{1/2}$
0.5	1	0.1	0.1	0.1	-3.3727
1	1	0.1	0.1	0.1	-3.5937
1.5	1	0.1	0.1	0.1	-3.8012
2	1	0.1	0.1	0.1	-3.9975
Slope					-0.4166
1	0.5	0.1	0.1	0.1	-3.38
1	1	0.1	0.1	0.1	-3.5937
1	1.5	0.1	0.1	0.1	-3.7947
1	2	0.1	0.1	0.1	-3.9852
Slope					-0.4034
1	1	0.1	0.1	0.1	-3.5937
1	1	0.2	0.1	0.1	-3.7076
1	1	0.3	0.1	0.1	-3.8253
1	1	0.4	0.1	0.1	-3.9466
Slope					-1.1765
1	1	0.1	0.01	0.1	-2.924
1	1	0.1	0.02	0.1	-2.9904
1	1	0.1	0.03	0.1	-3.0586
1	1	0.1	0.04	0.1	-3.1287
Slope					-6.8242
1	1	0.1	0.1	0.01	-2.6673
1	1	0.1	0.1	0.02	-2.7629
1	1	0.1	0.1	0.03	-2.86
1	1	0.1	0.1	0.04	-2.9588
Slope					-9.7153

Table 5.4: Variation in $Cf_x Re_x^{1/2}$ when $Pr = 6.2$, $R = 1$, $Ec = 0.3$, $Le = 2$, $K_c = \Omega = Pe = 0.5$, $\beta = 0.1$, and $Lb = 1.2$

H	K_p	S	ϕ_1	ϕ_2	$Cf_x Re_x^{1/2}$
0.5	1	0.1	0.1	0.1	-3.3727
1	1	0.1	0.1	0.1	-3.5937
1.5	1	0.1	0.1	0.1	-3.8012
2	1	0.1	0.1	0.1	-3.9975
Slope					-0.4166
1	0.5	0.1	0.1	0.1	-3.38
1	1	0.1	0.1	0.1	-3.5937
1	1.5	0.1	0.1	0.1	-3.7947
1	2	0.1	0.1	0.1	-3.9852
Slope					-0.4034
1	1	0.1	0.1	0.1	-3.5937
1	1	0.2	0.1	0.1	-3.7076
1	1	0.3	0.1	0.1	-3.8253
1	1	0.4	0.1	0.1	-3.9466
Slope					-1.1765
1	1	0.1	0.01	0.1	-2.924
1	1	0.1	0.02	0.1	-2.9904
1	1	0.1	0.03	0.1	-3.0586
1	1	0.1	0.04	0.1	-3.1287
Slope					-6.8242
1	1	0.1	0.1	0.01	-2.6673
1	1	0.1	0.1	0.02	-2.7629
1	1	0.1	0.1	0.03	-2.86
1	1	0.1	0.1	0.04	-2.9588
Slope					-9.7153

Table 5.5: Variation in $Nu_x Re_x^{-1/2}$ when $Pr = 6.2, Le = 2, K_c = \Omega = Pe = 0.5, K_p = 1, Lb = 1.2$ and $S = 0.1$

H	Ec	R	β	ϕ_1	ϕ_2	$Nu_x Re_x^{-1/2}$
0.5	0.3	1	0.1	0.1	0.1	1.25895
1	0.3	1	0.1	0.1	0.1	0.90999
1.5	0.3	1	0.1	0.1	0.1	0.58361
Slope						-0.6753
1	0.1	1	0.1	0.1	0.1	2.76283
1	0.2	1	0.1	0.1	0.1	1.83641
1	0.3	1	0.1	0.1	0.1	0.90999
Slope						-9.2642
1	0.3	0.5	0.1	0.1	0.1	0.78785
1	0.3	1	0.1	0.1	0.1	0.90999
1	0.3	1.5	0.1	0.1	0.1	1.00861
Slope						0.22075
1	0.3	1	0.15	0.1	0.1	0.61246
1	0.3	1	0.2	0.1	0.1	0.23675
1	0.3	1	0.25	0.1	0.1	-0.2966
Slope						-9.0907
1	0.3	1	0.1	0.01	0.1	1.39817
1	0.3	1	0.1	0.02	0.1	1.34932
1	0.3	1	0.1	0.03	0.1	1.29927
Slope						-4.945
1	0.3	1	0.1	0.1	0.01	1.78003
1	0.3	1	0.1	0.1	0.02	1.68727
1	0.3	1	0.1	0.1	0.03	1.59375
Slope						-9.3141

Table 5.6: Variation in $Nn_x Re_x^{-1/2}$ when $Pr = 6.2, H = R = 1, S = \beta = \phi_1 = \phi_2 = 0.1$, and $Ec = 0.3$

K_p	K_c	Pe	Ω	Le	Lb	$Nn_x Re_x^{-1/2}$
0.5	0.5	0.5	0.5	2	1.2	2.13076
1	0.5	0.5	0.5	2	1.2	2.1024
1.5	0.5	0.5	0.5	2	1.2	2.07662
Slope						-0.0542
1	0.4	0.5	0.5	2	1.2	2.0549
1	0.5	0.5	0.5	2	1.2	2.1024
1	0.6	0.5	0.5	2	1.2	2.14748
Slope						0.46289
1	0.5	0.4	0.5	2	1.2	1.87087
1	0.5	0.5	0.5	2	1.2	2.1024
1	0.5	0.6	0.5	2	1.2	2.33564
Slope						2.32388
1	0.5	0.5	0.4	2	1.2	2.02874
1	0.5	0.5	0.5	2	1.2	2.1024
1	0.5	0.5	0.6	2	1.2	2.17607
Slope						0.73665
1	0.5	0.5	0.5	1.8	1.2	2.02589
1	0.5	0.5	0.5	2	1.2	2.1024
1	0.5	0.5	0.5	2.2	1.2	2.17615
Slope						0.37565
1	0.5	0.5	0.5	2	1	2.00053
1	0.5	0.5	0.5	2	1.2	2.1024
1	0.5	0.5	0.5	2	1.4	2.20067
Slope						0.50036

Table 5.7: Correlation coefficient (r), probable error (PE), and $\left|\frac{r}{PE}\right|$ values of $Cf_x Re_x^{1/2}$

Parameter	r	PE	$\left \frac{r}{PE}\right $
H	-0.9992	0.00042	2357.25
K_p	-0.9993	0.0004	2521.01
S	-0.9998	0.00014	7262.22
ϕ_1	-0.9998	0.00011	9078
ϕ_2	-0.9999	4.41E-05	22696.4

Table 5.8: Correlation coefficient (r), probable error (PE), and $\left| \frac{r}{PE} \right|$ values of $Nu_x Re_x^{-1/2}$

Parameter	r	PE	$\left \frac{r}{PE} \right $
H	-0.9991	0.00051	1972.77
Ec	-1	0	#DIV/0!
R	0.99864	0.00075	1332.26
β	-0.9657	0.01858	51.9837
ϕ_1	-0.9998	9.36E-05	10680.2
ϕ_2	-1	1.10E-05	90788.2

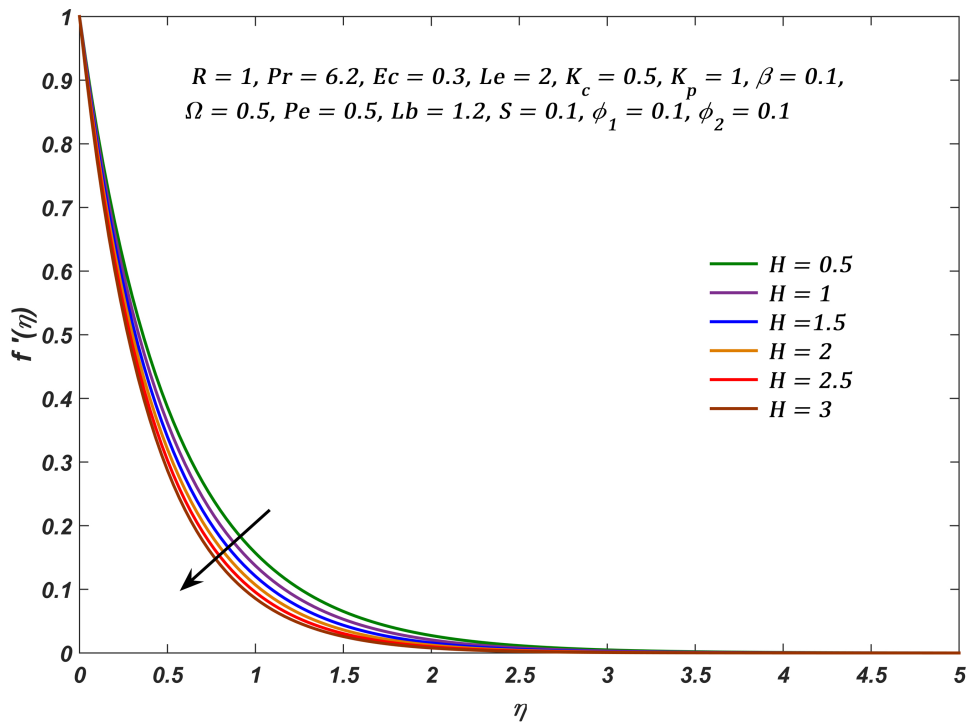


Figure 5.2: Variation in $f'(\eta)$ with M

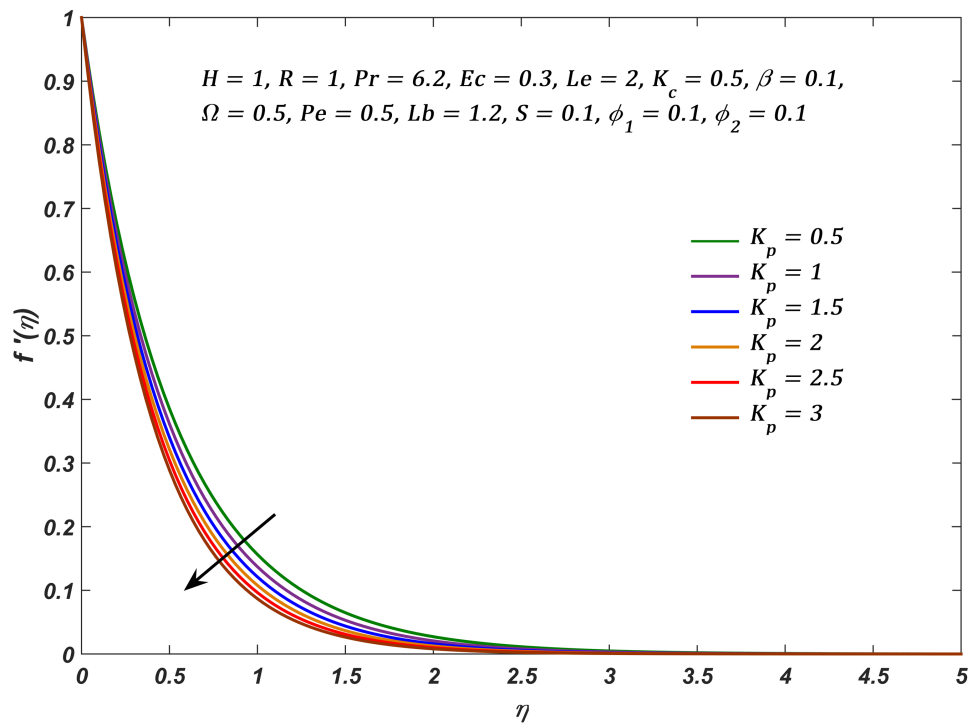


Figure 5.3: Variation in $f'(\eta)$ with K_p

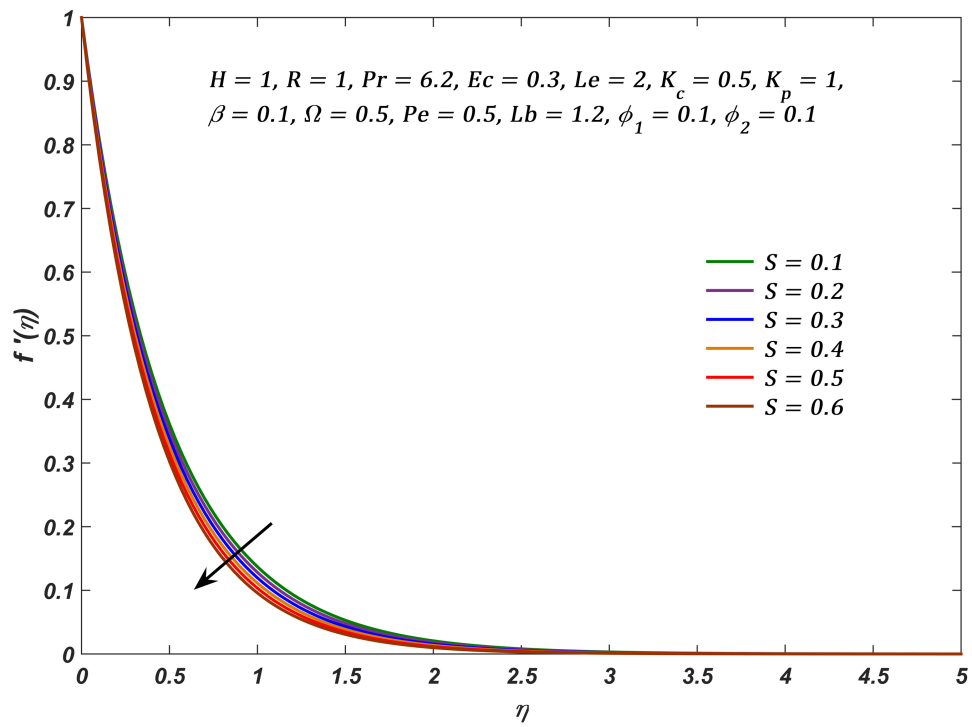


Figure 5.4: Variation in $f'(\eta)$ with S

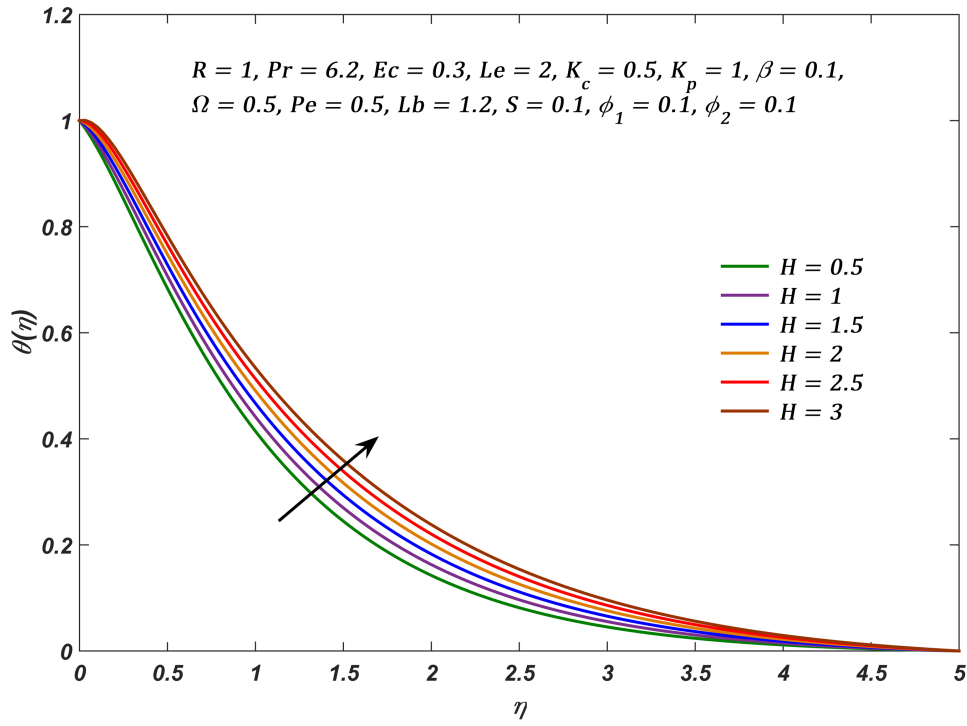


Figure 5.5: Variation in $\theta(\eta)$ with M

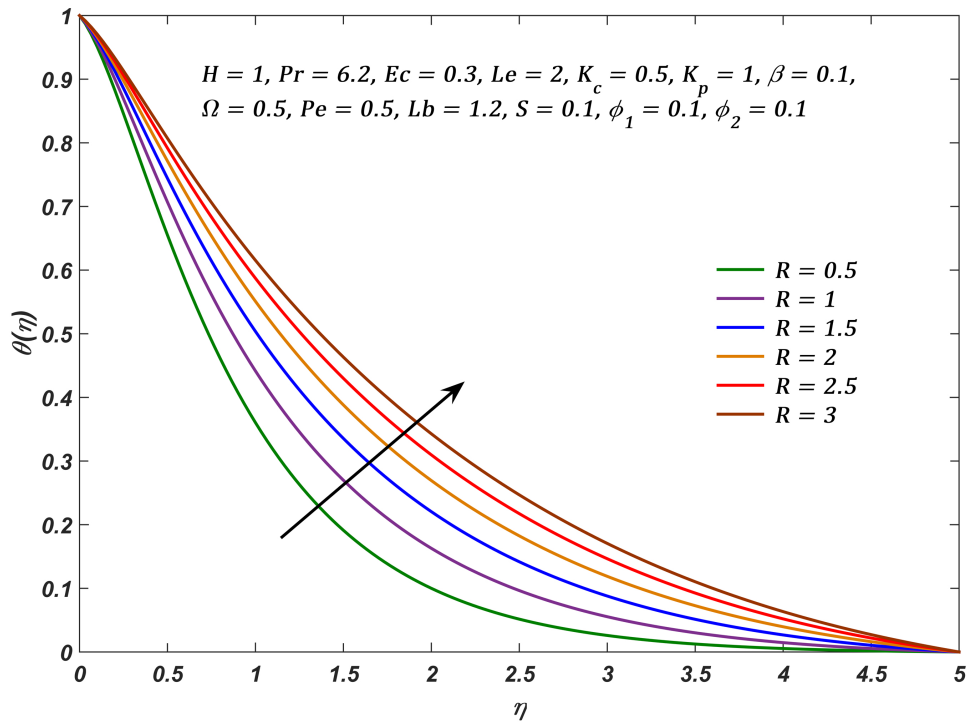


Figure 5.6: Variation in $\theta(\eta)$ with R

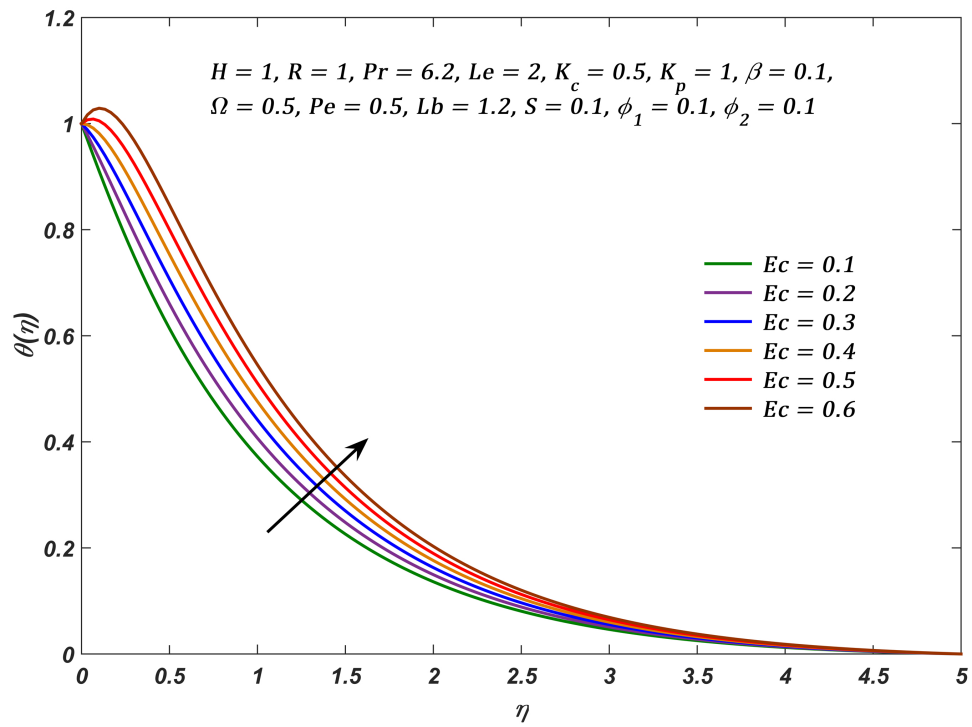


Figure 5.7: Variation in $\theta(\eta)$ with Ec

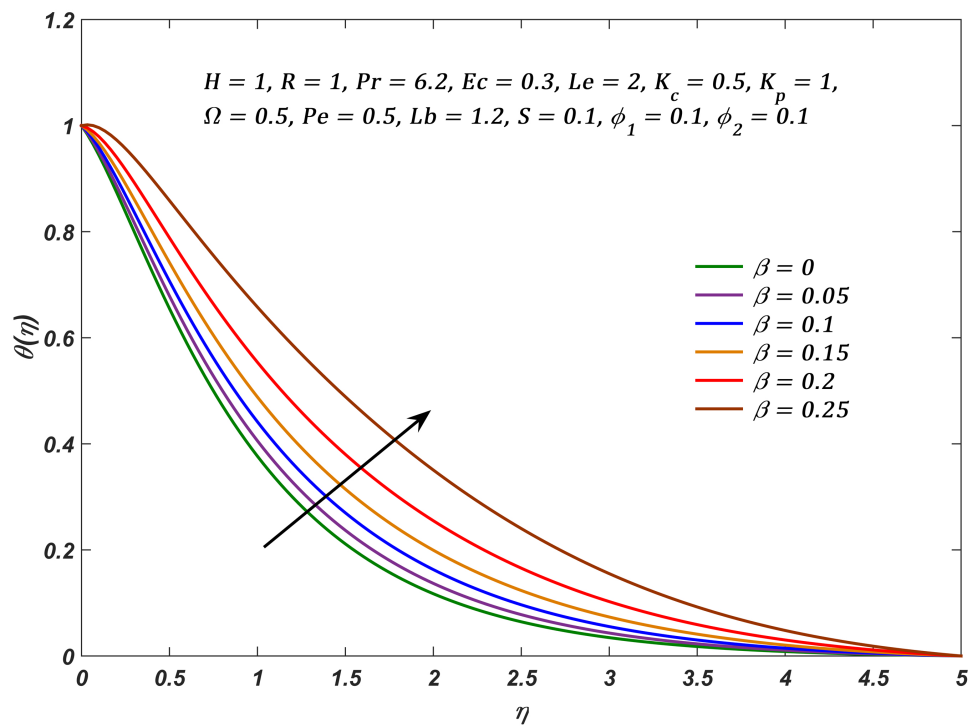


Figure 5.8: Variation in $\theta(\eta)$ with β

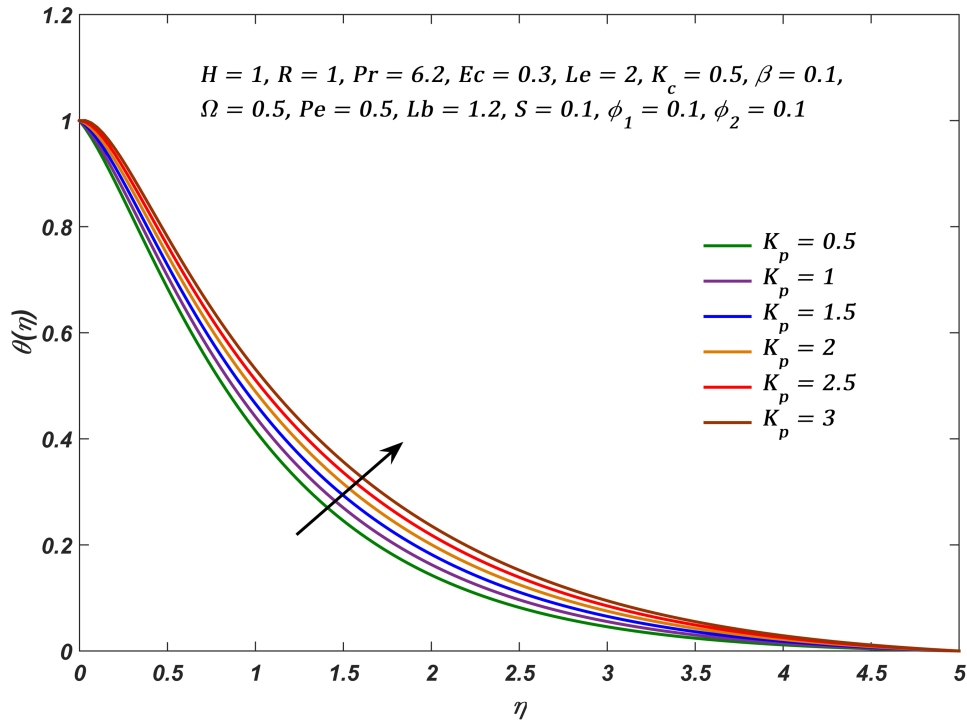


Figure 5.9: Variation in $\theta(\eta)$ with K_p

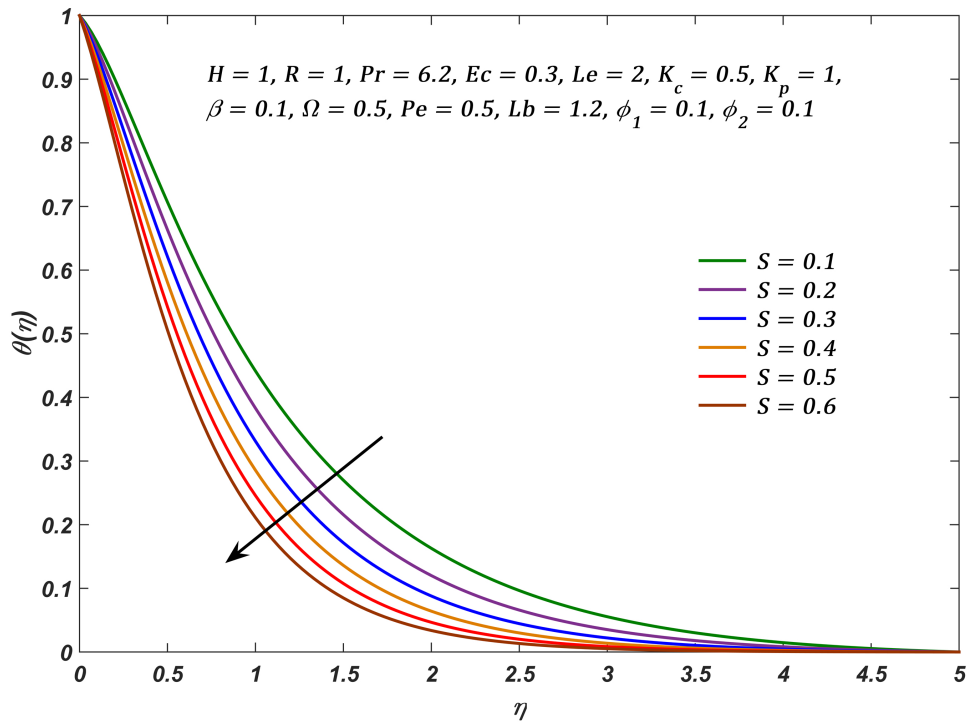
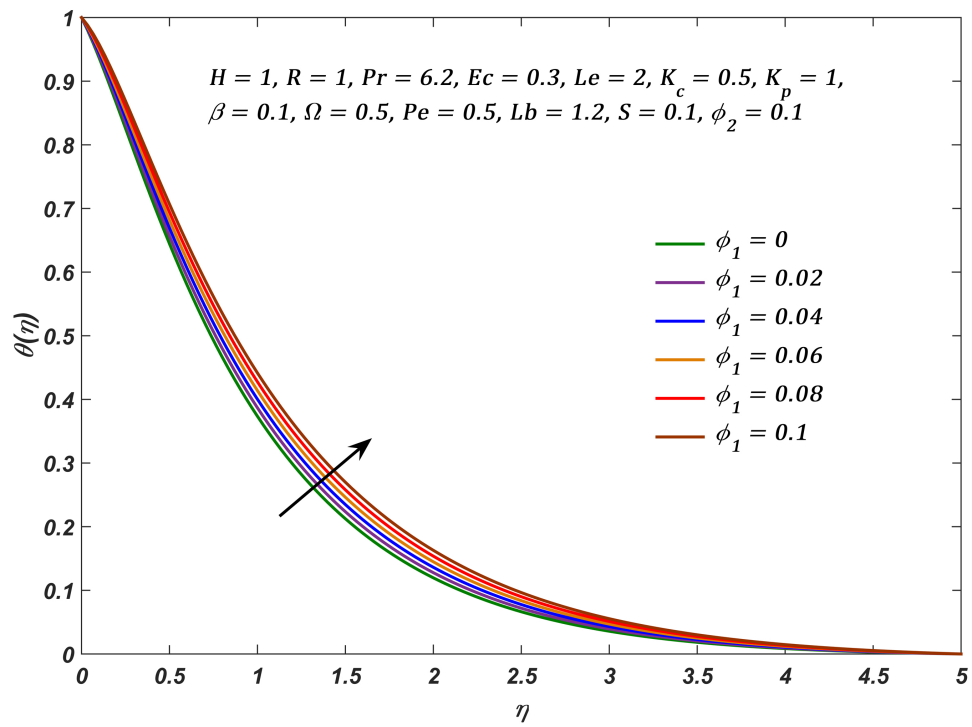
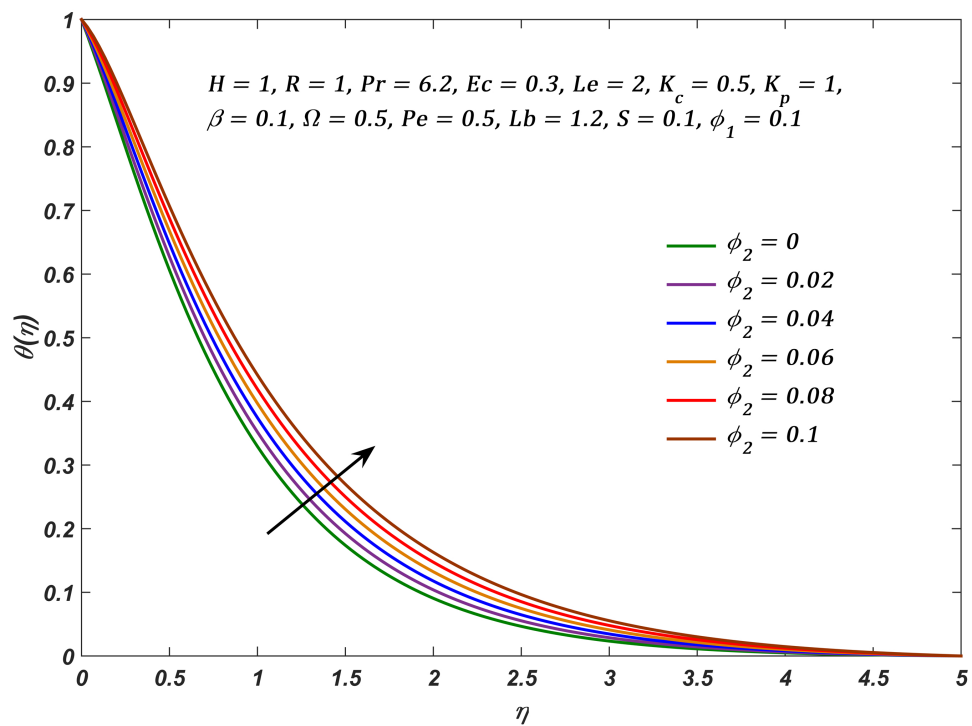


Figure 5.10: Variation in $\theta(\eta)$ with S


 Figure 5.11: Variation in $\theta(\eta)$ with ϕ_1

 Figure 5.12: Variation in $\theta(\eta)$ with ϕ_2

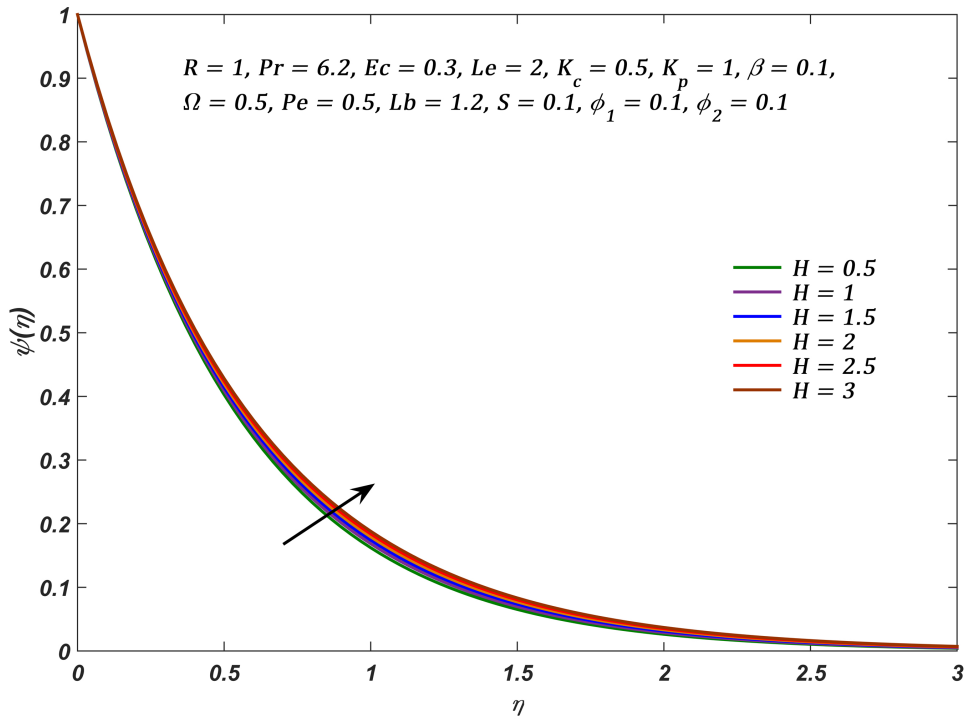


Figure 5.13: Variation in $\psi(\eta)$ with M

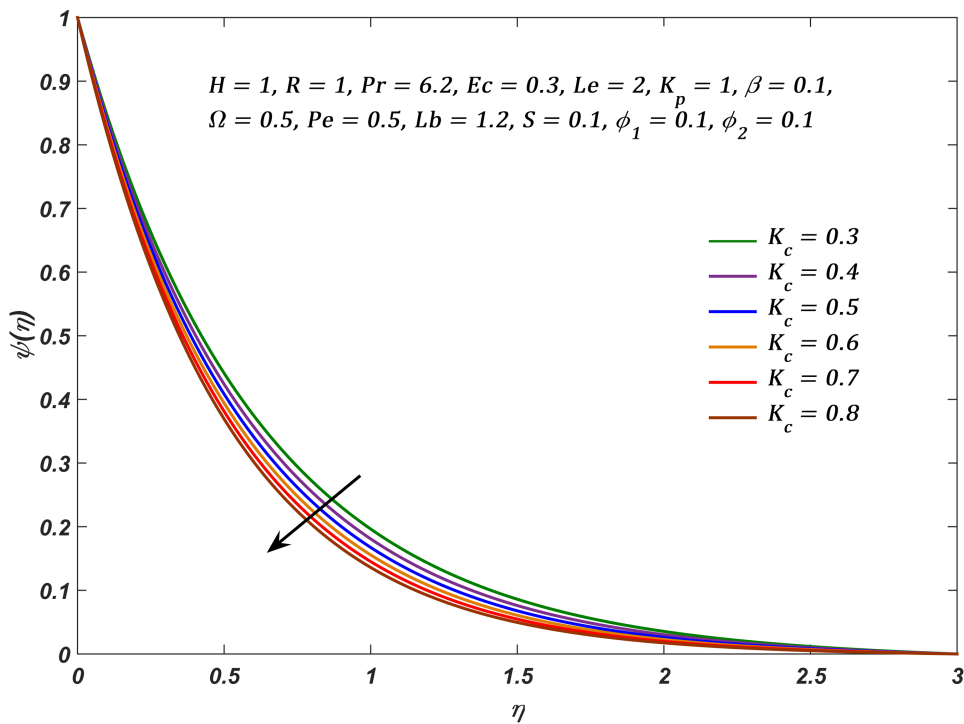


Figure 5.14: Variation in $\psi(\eta)$ with K_c

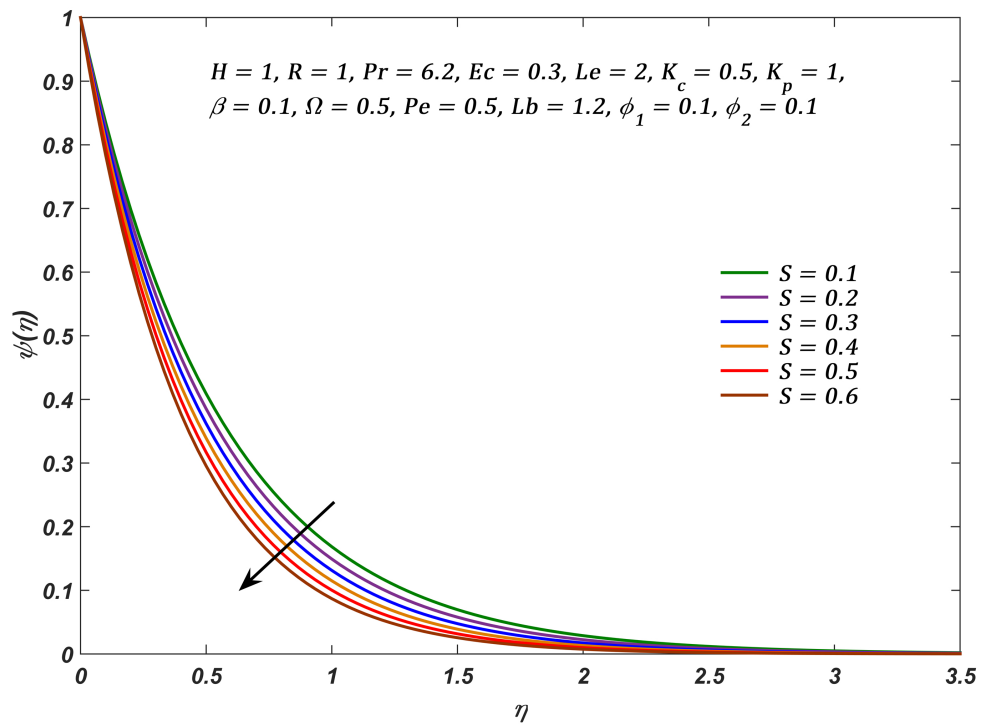


Figure 5.15: Variation in $\psi(\eta)$ with S

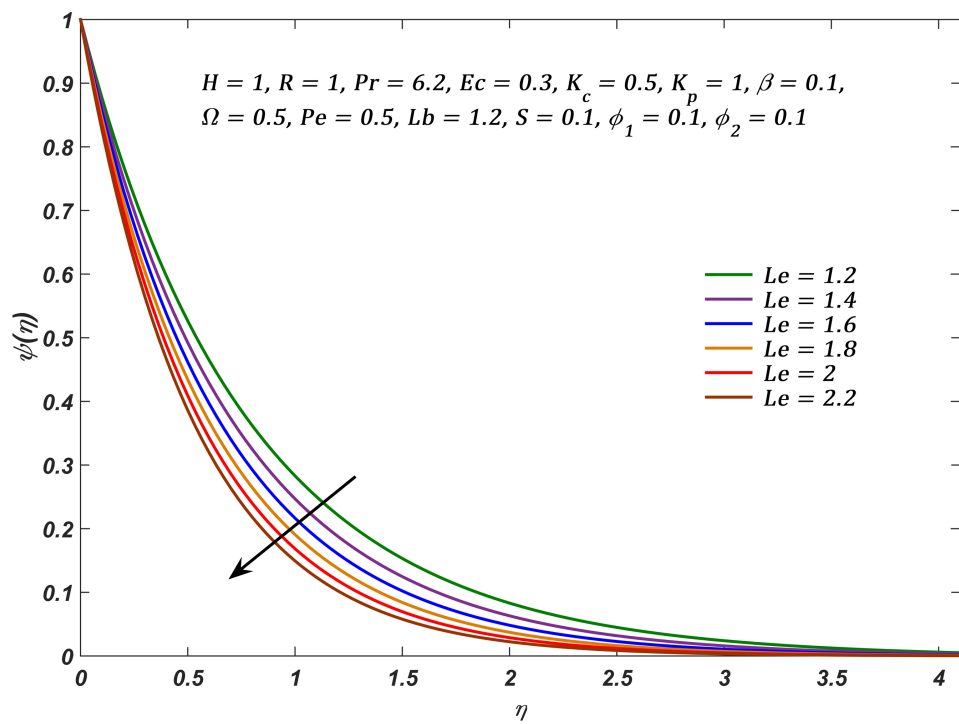


Figure 5.16: Variation in $\psi(\eta)$ with Le

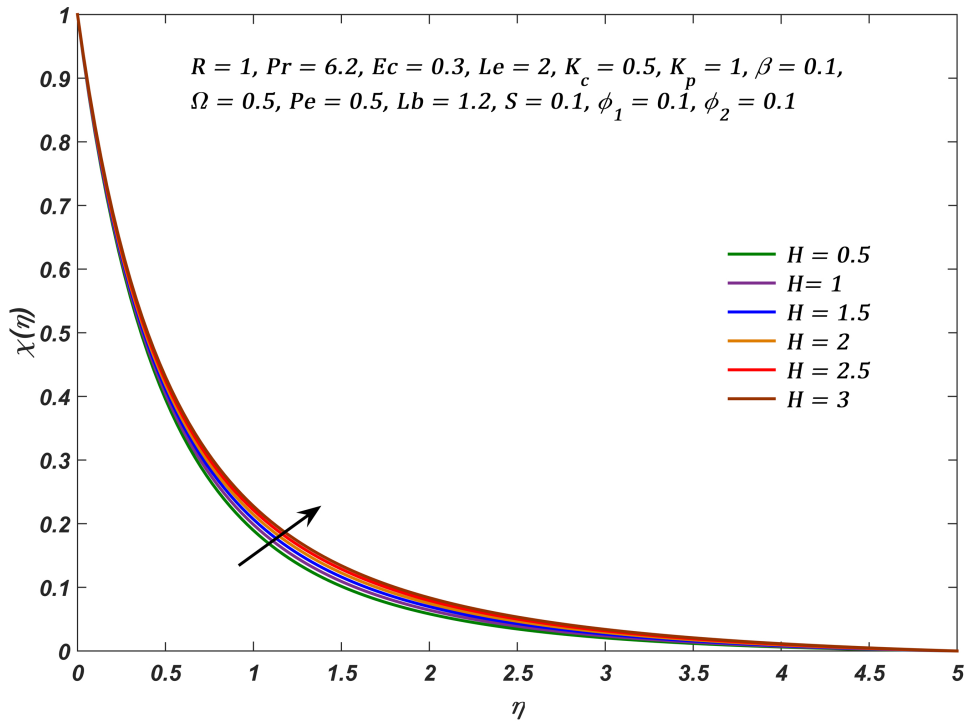


Figure 5.17: Variation in $\chi(\eta)$ with M

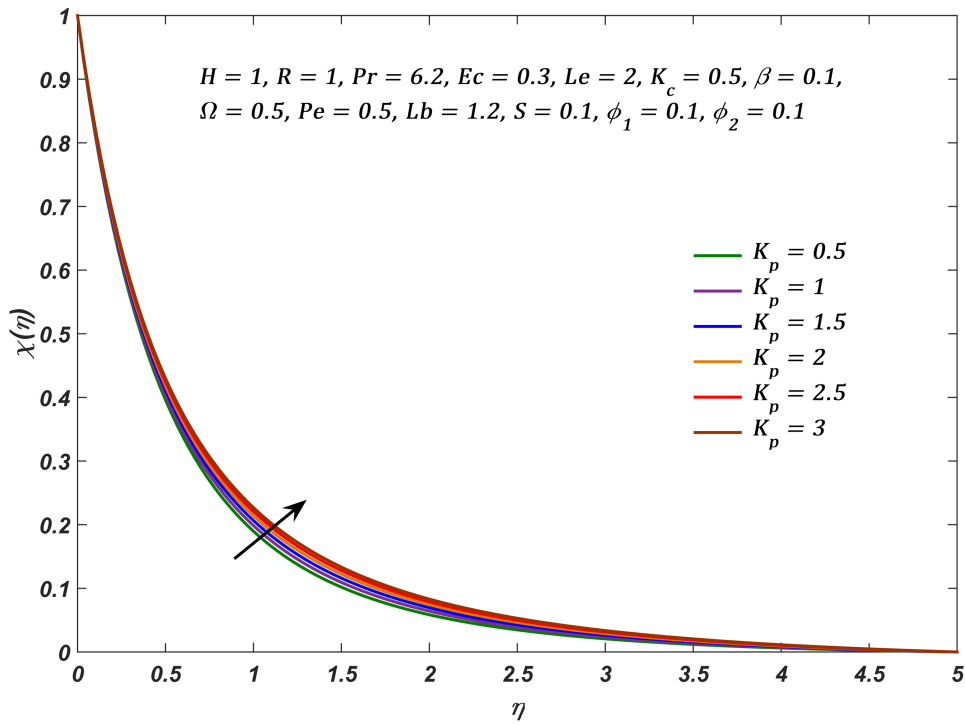
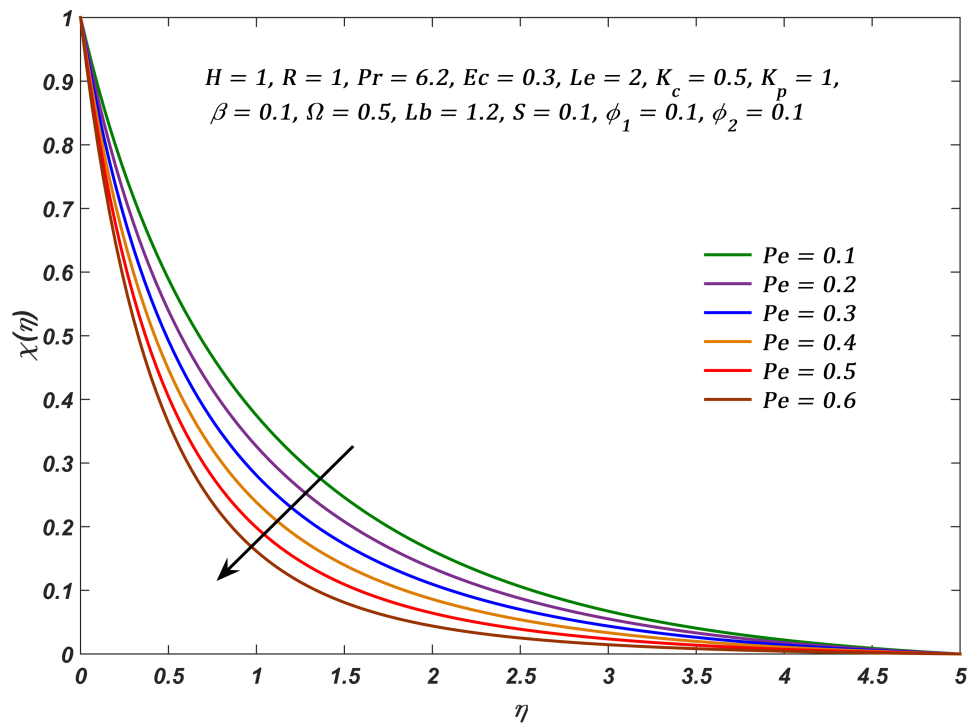
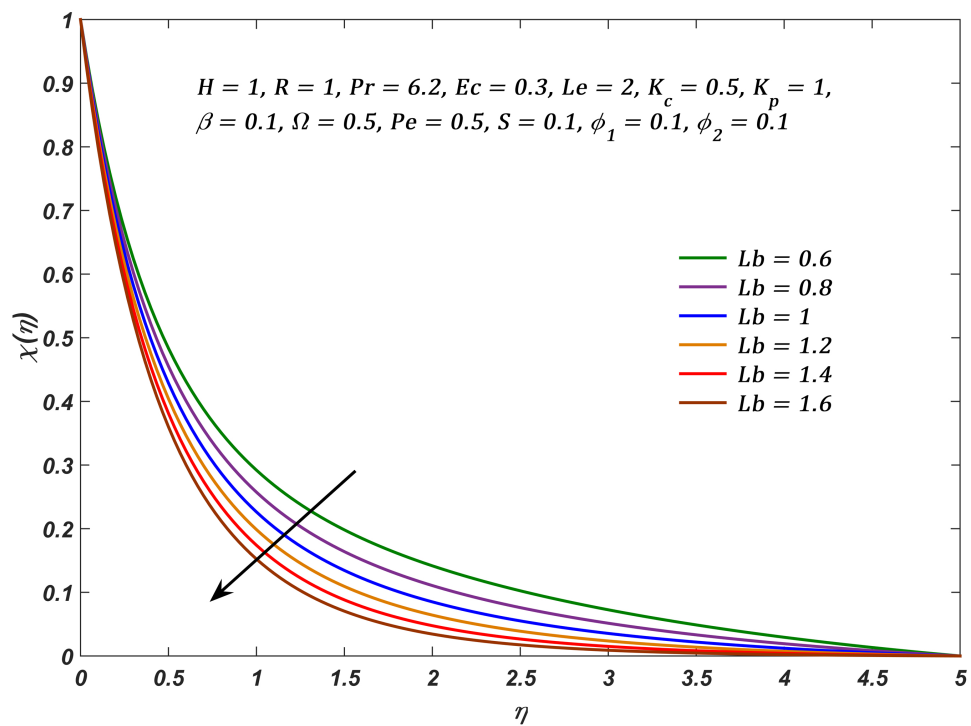


Figure 5.18: Variation in $\chi(\eta)$ with K_p


 Figure 5.19: Variation in $\chi(\eta)$ with Pe

 Figure 5.20: Variation in $\chi(\eta)$ with Lb

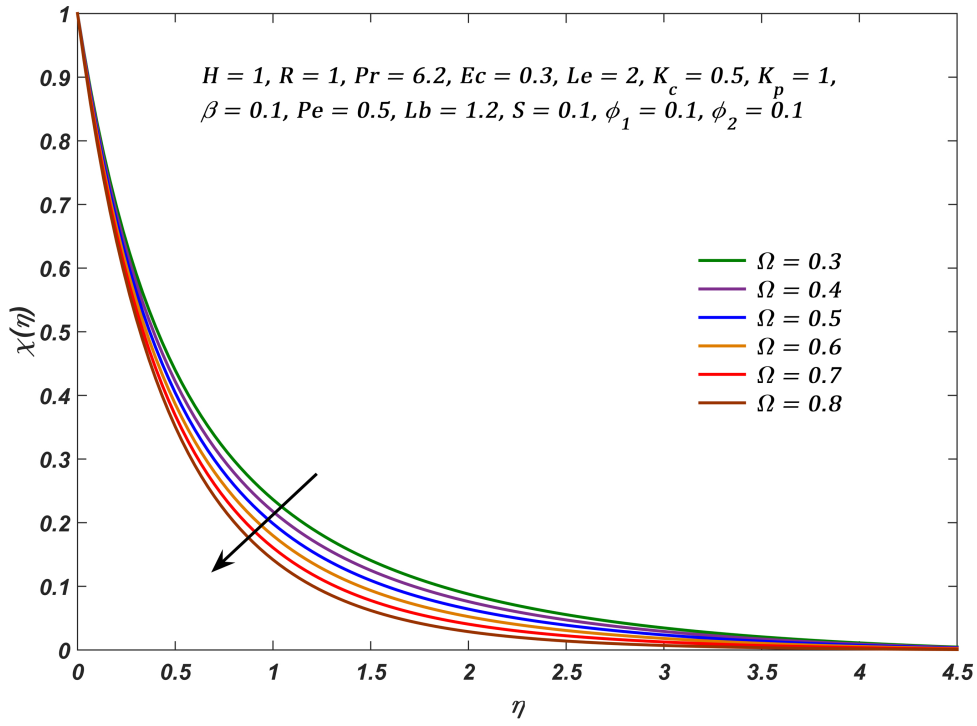


Figure 5.21: Variation in $\chi(\eta)$ with Ω

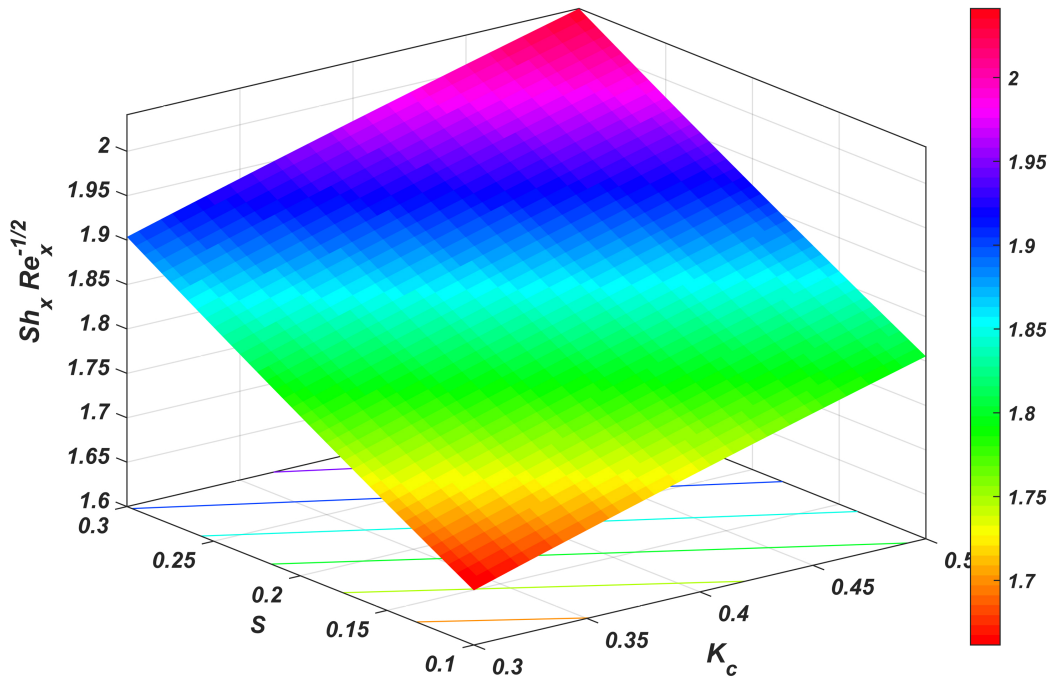


Figure 5.22: Variation in $Sh_x Re_x^{-1/2}$ with K_c and S

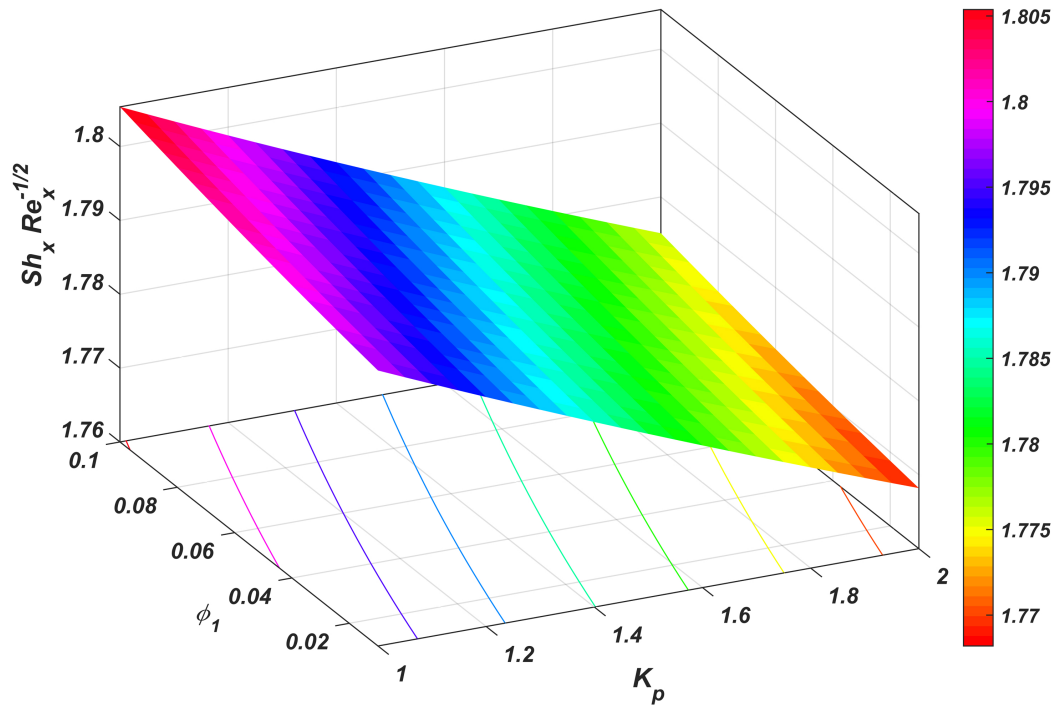


Figure 5.23: Variation in $Sh_x Re_x^{-1/2}$ with K_p and ϕ_1

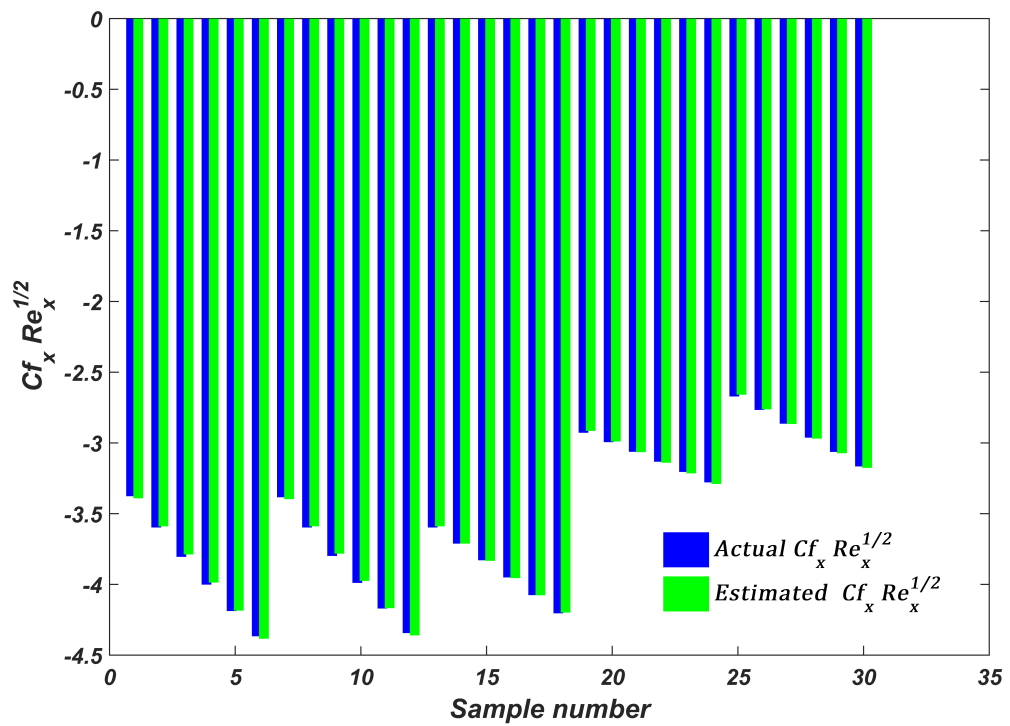


Figure 5.24: Actual versus estimated $Cf_x Re_x^{1/2}$

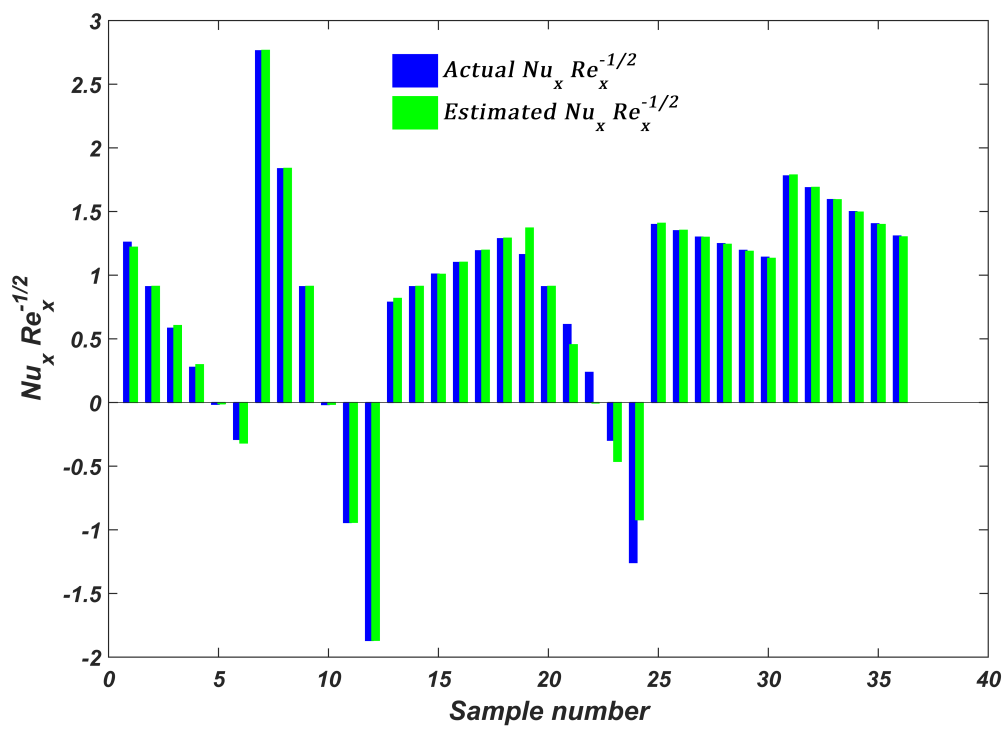


Figure 5.25: Actual versus estimated $Nu_x Re_x^{-1/2}$

APPENDIX I: Non-Dimensional Quantities

$$H = \frac{2LB_0^2\sigma_f}{c\rho_f} \quad \text{Hartmann Number}$$

$$Pr = \frac{\mu_f(C_p)_f}{k_f} \quad \text{Prandtl Number}$$

$$Ec = \frac{u_w^2}{(C_p)_f(T_w - T_\infty)} \quad \text{Eckert Number}$$

$$R = \frac{4\sigma^*T_\infty^3}{k^*k_f} \quad \text{Radiation Parameter}$$

$$Le = \frac{\vartheta_f}{D_B} \quad \text{LewisNumber}$$

$$\Omega = \frac{N_\infty}{N_w - N_\infty} \quad \text{Microorganism Concentration Difference Parameter}$$

$$Pe = \frac{bW_c}{D_m} \quad \text{Bioconvection Peclet Number}$$

$$Lb = \frac{\vartheta_f}{D_m} \quad \text{Bioconvection Lewis Number}$$

$$K_c = \frac{2LK_0}{c} \quad \text{Chemical Reaction Parameter}$$

$$S = -v_0\sqrt{\frac{2L}{\vartheta_fc}} \quad \text{Suction Parameter}$$

$$\beta = \frac{2LQ_0}{c(\rho C_p)_f} \quad \text{Heat Source Parameter}$$

$$K_p = \frac{2L\vartheta_f}{cK_r} \quad \text{Porosity Parameter}$$

Appendix II : Nomenclature

u, v	Velocity components	T	Fluid temperature
C	Concentration of nanoparticles	N	Microbial concentration
C_w	Surface concentration of nanoparticles	T_w	Surface temperature
N_w	Surface concentration of Microorganism	b, c	Constants
C_∞	Ambient concentration of nanoparticles	T_∞	Ambient temperature
N_∞	Ambient concentration of Microorganism	T_0	Reference temperature
C_0	Reference concentration of nanoparticles	Re_x	Reynolds number
N_0	Reference concentration of Microorganism	ν	Kinematic viscosity
θ	Dimensionless temperature	L	Reference length
ψ	Dimensionless concentration of nanoparticles	σ	Electrical conductivity
χ	Dimensionless concentration of Microorganism	ρ	Density
K_r	Permeability of the porous medium	μ	Dynamic viscosity
K_0	Chemical reaction coefficient	α	Thermal diffusivity
D_m	Microorganisms diffusion coefficient	q_r	Radiative heat flux
W_c	Maximum cell swimming speed	C_p	Specific heat
v_0	Initial strength of suction	f'	Dimensionless velocity
ϕ_1	Volume fraction of TiO_2	f	Base fluid
ϕ_2	Volume fraction of Ag	nf	Nanofluid
B_0	Magnetic field strength	hnf	Hybrid nanofluid
D_B	Brownian diffusion coefficient	s_1	TiO_2 nanoparticle
Q_0	Heat generation coefficient	s_2	Ag nanoparticle

CHAPTER 6

Conclusion

The prime objective of the present thesis is the theoretical and numerical analysis of magnetohydrodynamic nanofluid flow past plates and in channels. MHD flow is significant in engineering, biomedical engineering, aeronautics, MHD pumps, magnetic drug targeting, and so on. Transportation of cancer-killing drugs to affected areas without harm to surrounding tissues is one of the significant implementations of MHD flow. The flow properties such as velocity, temperature, and concentration, and physical quantities such as drag coefficients and, heat and mass transfer rates are discussed. In bioconvective flow, the microbial concentration, and the microorganism density number are also considered. The external magnetic field is applied normally to the fluid flow and due to the assumption of a small magnetic Reynolds number, the induced magnetic field has been neglected. Tiwari-Das nanofluid models are employed to describe fluid motion. The thesis involved the discussion of water-based nanofluids with nanoparticles Cu , Fe_3O_4 , TiO_2 , $Al_2O_3 - Fe_3O_4$, $CNT - Fe_3O_4$, and $TiO_2 - Ag$. The impact of external constraints such as thermal radiation, chemical reaction, viscous dissipation, stratification, and heat generation, to amplify or delay the beginning of convection as required in realistic implementations, is analyzed by observing the results of augmenting associated parameters.

- The Lorentz force generated due to the presence of a magnetic field, resists the fluid flow, and hence reduces the fluid velocity. This resistance enhances the friction between fluid layers which implies, the generation of heat and extension of solutal and microorganisms boundary layer thickness.
- The velocity field lines are grown due to the strengthening of the electric field.
- The electric field accelerates fluid motion and it dominates the resistive force obtained due to the magnetic field. Hence in the presence of an electric field, the fluid velocity enhances even the magnetic field intensity increases.
- Volume fraction of nanoparticles enhances the temperature field.
- Among water-based nanofluids with Cu , Fe_3O_4 and TiO_2 nanoparticles, Cu has a significant role to enhance fluid velocity.
- As the positive values of the Reynolds number describes the injection effects, the heated nanoparticles move to the fluids from surfaces that augment fluid velocity and fluid temperature. The inverse of this process occurs in the case of suction. Due to suction, the fluid particles tend to stick with surfaces, and heat is absorbed by surfaces, which reduces momentum, temperature, and solutal boundary layer thickness.
- Greater chemical reaction parameter values accelerate nanoparticle consumption, reducing the concentration of chemical species and increasing mass transfer rates. The fluid's density reduces by this; hence, the microorganisms tend to move to the vacant area in a bioconvective flow.
- The presence of a porous medium disturbs the fluid flow which causes the fluid particles to collide and generate internal heat energy. This thermal energy boosts the microorganism boundary layer thickness.
- Dissipative flow enhances the collision of fluid particles, thus generating frictional energy and hence internal heat.
- Fluid temperature is significantly increased by improving heat source, porosity, and radiation.

- The improvement of fluid temperature reduces the heat transfer rates (or Nusselt number), as it measures the rate of heat transferred from the surface to the fluid. But, when greater radiation is applied to the fluid flow, an immense surface heat flux is produced, which promotes higher heat transfer rates, even the fluid temperature enhances.

Magnetic fields and electric fields have a crucial role to control flow with the desired velocity. It can be implemented to improve the magnetic drug delivery system, by establishing more explicit ways to transport cancer-killing drugs to affected areas. The thermal conductivity of the fluid is influenced by improving temperature and heat transfer aspects that are implemented to prevent exhaustion of energy in engineering and industrial field. High temperature is used to destroy diseased cells in the biomedical field. Chemical reactions are taken into account as they find applications in the biochemical system, catalysis, batteries, corrosion phenomenon, combustion, and electrolytic cells. The flow modeled with the porous channel can be implemented in heat exchangers, transpiration cooling of gas turbine blades, porous walled flow reactors, solar energy collectors, and so on, by improving the thermophysical properties of the fluid under various situations. The stretching rate of sheets strongly affects the final product quality and, the exponential velocity and temperature distributions influence the admirable grade products attained in the annealing and thinning of copper wires.

FUTURE SCOPE OF THE STUDY

The scope of this research work lies in analyzing and interpreting the flow properties and physical quantities of the nanofluid flow in the presence of a magnetic field. Using ternary hybrid nanofluids instead of regular nanofluids or hybrid nanofluids, improves the stability and thermal properties at an advanced level, and considering proper ternary hybrid composite assures its potential utilities. However, a lot of research work is still required in the fields of characterization and application of ternary hybrid nanofluids to overcome challenges. The transportation of fluids in some biological organisms can be modeled with specific fluids, by modifying the flow models in chapter 2 and chapter 3. Not much has been learned about exponential stretching surfaces. It's a new way to make a lot of inventions that can be applied on an industrial scale. Studies on bioconvective flow through exponential stretch-

ing surfaces have a long way to go. Each chapter can be extended by considering different flow mediums and flow models. Also can be extended by considering various non-Newtonian fluids and various advantageous effects like nonlinear thermal radiation and space-dependent heat source.

Publications in Journals and Presentations

Publications:

1. Thyvalappil Sugathan Neethu and Sujesh Areekara and Alphonsa Mathew, “A Statistical Approach On Three Dimensional Free Convective Flow of Water Based Nanofluids Between Two Vertical Porous Plates Moving In Opposite Direction,” Heat Transfer Wiley , vol. 50, no. 5, pp. 5170-5197, July. 2021, <https://doi.org/10.1002/htj.22120>
2. Alphonsa Mathew and T.S. Neethu and Sujesh Areekara, “Three-dimensional hydromagnetic hybrid nanoliquid flow and heat transfer between two vertical porous plates moving in opposite directions: Sensitivity analysis,” Heat Transfer Wiley, vol. 50, no. 7, pp. 6548-6571, May. 2021, <https://doi.org/10.1002/htj.22192>
3. T.S.Neethu and Sujesh Areekara and A.S. Sabu and Alphonsa Mathew and K.K. Anakha, ”Bioconvective electromagnetohydrodynamic hybrid nanoliquid flow over a stretching sheet with stratification effects: A four-factor response surface optimized model,” Waves in Random and Complex Media, April. 2022, <https://doi.org/10.1080/17455030.2022.2066218>
4. T.S. Neethu and A.S. Sabu Alphonsa Mathew and A. Wakif and Sujesh

Areekara, “Multiple Linear Regression on Bioconvective MHD Hybrid Nanofluid Flow past an Exponential Stretching Sheet with Radiation and Dissipation effects,” *International Communications in Heat and Mass Transfer*, vol. 135, pp. 106115, June. 2022, <https://doi.org/10.1016/j.icheatmasstransfer.2022.106115>

Presentations:

1. T.S. Neethu, Heat transfer in free convective MHD flow past a hot vertical plate, International Conference on Pure and Applied Mathematics, SCSVMV, Enathur, Kanchipuram, Tamilnadu, held on 17 - 19 December 2018.
2. T.S. Neethu, An oscillatory free convective MHD flow in a rotating vertical channel, International Conference on Mathematics 2018, St. Thomas College (Autonomous), Thrissur, Kerala, held on 29 - 30 June 2018.

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