Modelling and Analysis of Lifetime Data Using Various Failure Rate Distributions

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DOCTOR OF PHILOSOPHY
in
STATISTICS

By

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Under the guidance of

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CERTIFICATE

This is to certify that the adjudicators of the PhD thesis of Ms. Anakha K K, titled, ***"Modelling and Analysis of Lifetime Data Using Various Failure Rate Distributions"*** have not given any directions for corrections or suggestions for change in their reports. The content of the thesis in both hardcopy and soft copy are one and same.

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DECLARATION

I, **Anakha K K**, hereby declare that the work presented in the thesis entitled, ***"Modelling and Analysis of Lifetime Data Using Various Failure Rate Distributions"***, is based on the original work done by me under the guidance of Dr. Chacko V M, Professer, Department of Statistics, St. Thomas College (Autonomous), Thrissur, Kerala and has not been included in any other thesis submitted previously for the award of any agree. The contents of the thesis are undergone plagiarism check using iThenticate software at C.H.M.K. Library, University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.

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CERTIFICATE

This is to certify that the thesis titled ***"Modelling and Analysis of Lifetime Data Using Various Failure Rate Distributions"*** submitted by **Anakha K K** to the **University of Calicut** in partial fulfilment of the requirements for the award of the Degree of ***Doctor of Philosophy in Statistics*** is a record of original research work carried out by her under my supervision. The content of this thesis, in full or in parts, has not been submitted by any other candidate to any other University for the award of any degree or diploma.

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To My Family

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ABSTRACT

Lifetime data analysis plays a vital role in fields such as reliability engineering, biostatistics, and quality control by focusing on the time to occurrence of specific events like system failure, death, or recovery. The key objective in modelling such data is to uncover insights into the causes of failure or survival, enabling risk assessment and informed decision-making. A central element of this analysis is selecting appropriate failure rate distributions that capture the diverse patterns observed in real-world systems-ranging from monotonic to non-monotonic failure rates.

This thesis, titled "Modelling and Analysis of Lifetime Data Using Various Failure Rate Distributions", is organized into ten chapters. Chapter 1 introduces essential concepts of lifetime data and failure rate modelling. Chapter 2 develops a new flexible distribution using the DUS transformation and the Kumaraswamy distribution, effectively capturing a wide range of failure rate behaviors. The applicability of the model is demonstrated using both simulation and real data studies.

Chapter 3 introduces another novel distribution derived via the DUS transformation with the inverse Kumaraswamy distribution, offering strong modelling capabilities for non-monotonic hazard rates. Its utility is verified through characterizations, reliability analysis, and real data applications. Chapter 4 presents a bivariate model based on the Farlie-Gumbel-Morgenstern copula with inverse Weibull marginals, designed to capture weakly dependent lifetime data, and demonstrates its applicability through a real-life medical dataset.

Chapters 5 to 7 focus on censoring mechanisms in life-testing experiments. Chapter 5 analyzes progressive Type-II censoring, progressive Type-II hybrid censoring, and adaptive progressive Type-II censoring schemes using the exponential Power distribution, comparing parameter estimation via maximum likelihood and Bayesian methods. Chapter 6 applies the joint adaptive progressive Type-II censoring scheme to the generalized Lindley distribution, emphasizing efficient estimation using Markov Chain Monte Carlo and bootstrap techniques. Chapter 7 proposes a novel T1-T2 mixture censoring scheme based on the Weibull distribution, offering advantages in maximizing the number of failures within a given supplementary time. All the above models are illustrated using real-world data to ensure practical

relevance and applicability. Chapter 8 addresses long-term survivors through cure fraction modelling using the exponentiated Weibull distribution under mixture and non-mixture frameworks. Both frequentist and Bayesian approaches are employed, and the models are validated using cancer survival data from Kerala, India.

Finally, Chapter 9 summarizes the key contributions of the thesis, highlighting the development of new lifetime distributions, innovative censoring schemes, and their practical applications. These contributions provide valuable tools for real-world lifetime data analysis and lay the foundation for future research in this field.

Key words: Lifetime data analysis; failure rate distributions; copula models; censoring schemes; cure fraction modelling

പഠനസംഗ്രഹം

റിലയബിലിറ്റി എഞ്ചിനീയറിംഗ്, ബയോസ്റ്റാറ്റിസ്റ്റിക്സ്, ക്വാളിറ്റി കണ്ട്രോൾ തുടങ്ങിയ മേഖലകളിൽ ലൈഫ് ടൈം ഡാറ്റ വിശകലനം നിർണായകമാണ്, കാരണം സിസ്റ്റം പരാജയം, മരണം അല്ലെങ്കിൽ അതിജീവനം പോലുള്ള നിർദ്ദിഷ്ട സംഭവങ്ങളുടെ സമയം മനസ്സിലാക്കുന്നതിൽ ഇത് ശ്രദ്ധ കേന്ദ്രീകരിക്കുന്നു. അത്തരം ഡാറ്റ മോഡലിംഗ് ചെയ്യുന്നതിലെ പ്രാഥമിക ലക്ഷ്യം പരാജയത്തിന്റെയോ അതിജീവനത്തിന്റെയോ അടിസ്ഥാന കാരണങ്ങളെക്കുറിച്ച് അർത്ഥവത്തായ ഉൾക്കാഴ്ചകൾ നേടുക എന്നതാണ്. ഇത് ഗവേഷകരെ സിസ്റ്റം പ്രവർത്തനരീതി പ്രവചിക്കാനും, അപകടസാധ്യതകൾ വിലയിരുത്താനും, കാര്യക്ഷമമായ പരിഹാരങ്ങൾ രൂപപ്പെടുത്താനും സഹായിക്കുന്നു. ഈ വിശകലനത്തിന്റെ ഒരു പ്രധാന വശം ഉചിതമായ ഒരു പരാജയ നിരക്ക് വിതരണം തിരഞ്ഞെടുക്കുക എന്നതാണ്. വ്യത്യസ്ത സിസ്റ്റങ്ങൾ പരാജയത്തിന്റെ സവിശേഷമായ പാറ്റേണുകൾ പ്രദർശിപ്പിക്കുന്നു, ഇത് വിശാലമായ സ്റ്റാറ്റിസ്റ്റിക്കൽ ഡിസ്ട്രിബ്യൂഷൻസ് പ്രയോഗിച്ചുകൊണ്ട് വിവരിക്കാം. ഓരോ ഫംഗ്ഷന്റെയും പരാജയ നിരക്ക് അതിന്റെ സ്വഭാവ സവിശേഷതയാണ്. വർദ്ധിച്ചുവരുന്നതോ, കുറയുന്നതോ, സ്ഥിരമായതോ, അല്ലെങ്കിൽ ഏകതാനമല്ലാത്തതോ ആയ ഈ ഫംഗ്ഷനുകൾ, സിസ്റ്റങ്ങളുടെ സങ്കീർണ്ണതയെയും കാലക്രമേണ അവയുടെ വ്യത്യസ്ത അപകടസാധ്യത നിലകളെയും പ്രതിഫലിപ്പിക്കുന്നു.

"മോഡലിങ് ആൻഡ് അനാലിസിസ് ഓഫ് ലൈഫ്ടൈം ഡാറ്റാ യൂസിങ് വേരിയസ് ഫെയ്‌ലിയർ റേറ്റ് ഡിസ്ട്രിബ്യൂഷൻസ്" എന്ന തലക്കെട്ടിലുള്ള ഈ പ്രബന്ധം പത്ത് അധ്യായങ്ങളായി ക്രമീകരിച്ചിരിക്കുന്നു. ഈ പ്രബന്ധത്തിന്റെ അടിത്തറ രൂപപ്പെടുത്തുന്ന അടിസ്ഥാന ആശയങ്ങളും നിർവചനങ്ങളും അധ്യായം 1 പരിചയപ്പെടുത്തുന്നു. മാത്രമല്ല, ഇത് ഒരു ആഴത്തിലുള്ള സാഹിത്യ അവലോകനം അവതരിപ്പിക്കുന്നു.

DUS പരിവർത്തനം ഉപയോഗിച്ച് വികസിപ്പിച്ചെടുത്ത പുതിയ മോഡലുകൾ 2ഉം 3ഉം അധ്യായങ്ങൾ പരിചയപ്പെടുത്തുന്നു. ഗവേഷണ പഠനം മോഡലുകളുടെ പ്രകടനങ്ങളെയും സ്വാഭാവ സവിശേഷതകളെയും വിലയിരുത്തുകയും കൂടാതെ നിലവിലുള്ള മോഡലുകളുമായി താരതമ്യപ്പെടുത്തുമ്പോൾ മികച്ച പ്രവർത്തനം ഉറപ്പിക്കുകയും ചെയ്യുന്നു.

പല ഉപയോഗങ്ങളിലും ഡാറ്റ സങ്കീർണ്ണമാണെന്ന് മാത്രമല്ല, ഡാറ്റകൾ നിരീക്ഷണങ്ങൾക്കിടയിലുള്ള ആശ്രിതത്വവും പ്രകടമാക്കാറുണ്ട്. ഓരോ വേരിയബിളിന്റെയും മാർജിനൽ സ്വഭാവം നിലനിർത്തിക്കൊണ്ട് അത്തരം ആശ്രിതത്വങ്ങളെ മോഡൽ ചെയ്യുന്നതിന് കോപ്പുല സിദ്ധാന്തം ശക്തമായ അടിത്തറ നൽകുന്നു. അദ്ധ്യായം 4-ൽ, വിശ്വാസ്യത വിശകലനത്തിനും ആശ്രിതത്വ അളവുകൾക്കുമായി ഫാർലി-ഗംബെൽ-മോർഗെൻസ്റ്റേൺ (FGM) കോപ്പുലയെ അടിസ്ഥാനമാക്കി രൂപകൽപ്പന ചെയ്ത ഒരു ദ്വിവേരിയേറ്റ് ഡിസ്ട്രിബ്യൂഷൻ ചർച്ചചെയ്യുന്നു. വൃക്ക രോഗിയുടെ ആവർത്തന സമയങ്ങളെക്കുറിച്ചുള്ള ഒരു യഥാർത്ഥ ഡാറ്റാസെറ്റ് മോഡലിന്റെ പ്രായോഗിക ഉപയോഗം പ്രകടമാക്കുന്നു.

ലൈഫ് ടൈം പരീക്ഷണങ്ങളിലെ ഒരു പ്രധാന വെല്ലുവിളി ഡാറ്റയുടെ പൂർണ്ണ വിവരങ്ങൾ ലഭ്യമല്ലാത്തപ്പോൾ ഉണ്ടാകുന്ന സെൻസറിംഗ് ആണ്. 5-ആം അധ്യായത്തിൽ ലൈഫ്-ടെസ്റ്റിംഗ് പരീക്ഷണങ്ങളിൽ ഉപയോഗിക്കുന്ന പ്രോഗ്രസീവ് സെൻസറിംഗ് രീതികളായ പ്രോഗ്രസീവ് ടൈപ്പ്-II (PT2), പ്രോഗ്രസീവ് ടൈപ്പ്-II ഹൈബ്രിഡ് (PT2H), അഡാപ്റ്റീവ് പ്രോഗ്രസീവ് ടൈപ്പ്-II (APT2) സെൻസറിംങ്ങുകളെ താരതമ്യം ചെയ്യുന്നു.

ലൈഫ് ടൈം താരതമ്യം ചെയ്യുന്നതിനായി നിർമിതപെടുത്തിയിട്ടിട്ടുള്ള ജോയിന്റ് അഡാപ്റ്റീവ് പ്രോഗ്രസീവ് ടൈപ്പ്-II (JAPT) സെൻസറിംങ്ങിൽ രണ്ട് ഡാറ്റകൾ

ഉപയോഗിച്ചുള്ള ഒരു ലൈഫ്-ടെസ്റ്റിംഗ് പരീക്ഷണത്തിന്റെ സ്റ്റാറ്റിസ്റ്റിക്കൽ അനാലിസിസിൽ അധ്യായം 6 ശ്രദ്ധ കേന്ദ്രീകരിക്കുന്നു. പരീക്ഷണ സമയവും ചെലവും കുറയ്ക്കുന്നതിനാണ് JAPT രീതി ഉപയോഗിക്കുന്നത്.

7-ആം അധ്യായം ഒരു പുതിയ T1-T2 മിക്സ്ചർ സെൻസറിംഗ് സ്കീം നിർദ്ദേശിക്കുകയും വെയ്ബുൾ ഡിസ്ട്രിബ്യൂഷനെ അടിസ്ഥാനമാക്കി അതിന്റെ സ്റ്റാറ്റിസ്റ്റിക്കൽ അനുമാനം പര്യവേക്ഷണം ചെയ്യുകയും ചെയ്യുന്നു. ഒരു നിശ്ചിത കൂട്ടിച്ചേർത്ത സമയത്തിനുള്ളിൽ പരമാവധി പരാജയങ്ങൾ ഉറപ്പാക്കുന്ന പുതിയ രീതിയുടെ ഗുണങ്ങൾ അധ്യായം എടുത്തുകാണിക്കുന്നു. ഒരു യഥാർത്ഥ ഡാറ്റയിൽ T1-T2 മിക്സ്ചർ സെൻസറിങ്ങിന്റെ പ്രവർത്തനം അവതരിപ്പിച്ചിരിക്കുന്നു.

അതിജീവന വിശകലനത്തിലെ മറ്റൊരു പ്രധാന വെല്ലുവിളി ദീർഘകാലം അതിജീവിച്ചവരുടെയോ, അല്ലെങ്കിൽ സുഖം പ്രാപിച്ചതായി കണക്കാക്കപ്പെടുന്നവരും ഇനി ഈ സംഭവത്തിന്റെ അപകടസാധ്യതയില്ലാത്തവരുമായ വ്യക്തികളുടെയോ സാന്നിധ്യമാണ്. ക്യൂർ ഫ്രാക്ഷൻ മോഡലിംഗ് ഈ പ്രശ്നത്തെ പ്രത്യേകമായി മോഡലിൽ ഉൾപ്പെടുത്തിക്കൊണ്ട് അഭിസംബോധന ചെയ്യുന്നു. അധ്യായം 8-ൽ മലബാർ കാൻസർ സെന്ററിൽ (തലശ്ശേരി, കണ്ണൂർ, കേരള) നിന്നുള്ള കൊളോറെക്ടൽ കാൻസർ അതിജീവന ഡാറ്റ ഉപയോഗിച്ച് ക്യൂർ ഫ്രാക്ഷൻ മോഡലിന്റെ പ്രായോഗികത പ്രകടമാക്കിയിരിക്കുന്നു.

പുതിയ മോഡലുകളുടെ വികസനം, നൂതന സെൻസറിംഗ് രീതികൾ, മറ്റ് സുപ്രധാന ഗവേഷണ സംഭാവനകൾ എന്നിവ എടുത്തുകാണിച്ചുകൊണ്ട്, പ്രബന്ധത്തിന്റെ പ്രധാന കണ്ടെത്തലുകൾ 9-ആം അധ്യായത്തിൽ സംഗ്രഹിക്കുന്നു. നിർദ്ദിഷ്ട മോഡലുകളും രീതികളും യഥാർത്ഥ ലോക സാഹചര്യങ്ങളിൽ പ്രയോഗിക്കുന്നതിനുള്ള പ്രായോഗിക ശുപാർശകളും ഈ പ്രബന്ധം നൽകുന്നു.

പ്രധാന വാക്കുകൾ: ലൈഫ്ടൈം ഡാറ്റ അനാലിസിസ്; ഫെയ്ലിയർ റേറ്റ് ഡിസ്ട്രിബ്യൂഷൻസ്; കോപുല മോഡൽസ്; സെൻസറിംഗ് സ്കീംസ്; ക്യൂർ ഫ്രാക്ഷൻ മോഡലിങ്.

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CHAPTER 1

Introduction and Review of Literature

1.1 Introduction

Lifetime data, also known as survival time or failure time data, mainly observed in the fields of medicine, engineering, and social sciences, refers to the time until an event of interest occurs. It has received considerable attention from many statisticians and researchers for the computation of reliability or survival probabilistic measures. The term ‘lifetime’ indicates the duration measured from a specific starting point to the time of failure. The terms ‘death’ or ‘failure’ are often metaphorically used to describe the occurrence of events in different contexts.

Statistical modelling and analysis of lifetime data have an important role in many scientific and technological areas, especially to predict the lifetime of mechanical parts or electronic devices, to assess the efficacy of medical treatments in clinical trials, and to evaluate customer retention in marketing. Analyzing lifetime data needs specialized statistical models to portray and understand the statistical behaviors. These investigations aim to introduce and describe distributions, compare them to pre-existing distributions, and explore the results over a wide range of topics.

In the parametric modelling the data follows a specific probabilistic distribution and has an essential role in analyzing lifetime data. These models allow accurate inferences about the statistical characteristics such as mean lifetime, failure rates, and survival probability. Statistical distributions are widely applied in different fields for modelling lifetime data. For example, they are applied in reliability engineering to determine the life span of manufactured mechanical or electrical components. In demography, they help study the duration of marital life until divorce. In medical sciences, they are applied to analyze the survival time of patients with specific diseases from the date of diagnosis to death. In computer science, they are used to model the failure rates of software systems, while insurance companies use them to estimate policy durations without claims.

Failure rate distributions are crucial for survival and reliability studies as they show the risk of an event occurring over time. The prominent fundamental failure rate distributions, including exponential, gamma, Weibull, Pareto, etc., exhibit significant limitations in their properties and lack substantial flexibility in modelling lifetime data. Many researchers have created different classes of distributions, and generalized distributions have been extensively studied in the literature. Generalized distributions extend basic models like the Weibull and exponential distributions, which provide greater flexibility for analyzing complex patterns. Developing new distributions is essentials to address real-life data challenges.

In many applications, the data is not only complex but also exhibits dependence among observations, as in the case of bivariate or multivariate survival times. For instance, in medical research, the joint survival times of patients undergoing different treatments may be dependent, as treatment outcomes can affect overall survival rates. The Copula theory provides a strong basis to model such dependencies while maintaining the marginal behavior of each variable. This approach is widely used in finance, insurance, and medical research to study joint survival probabilities and dependence structures.

A significant challenge in the lifetime experiments is the occurrence of censoring, a common phenomenon in many fields such as survival analysis, reliability engineering, and clinical trials, which arises when complete information about the event of interest is not available for some observations. Censoring schemes are often used to minimize total test time and costs. An effective censoring scheme should balance the experiment's duration, the number of units tested, and the efficiency of the statistical analysis based on the collected data. Various censoring schemes, such as right, left, or interval censoring, are employed to handle such incomplete data effectively.

Another significant challenge in survival analysis is the presence of long-term survivors, or individuals who are considered cured, meaning they are no longer at risk of the event of interest. Cure fraction modelling addresses this issue by specifically including a cured proportion into the model. This is especially important in medical studies, where advances in treatments have led to substantial improvements in patient outcomes.

The thesis focuses on studying lifetime data by addressing the key challenges to develop new methods. This research provides an insight into the failure rate distributions and introduces flexible lifetime data models, also considering dependencies in different contexts. This study also examines various censoring schemes and proposes a new method to handle incomplete data effectively. Advanced modelling techniques of Bayesian analysis are also applied to tackle specific challenges in survival analysis, with real-life applications including data from Malabar Cancer Centre, Thalassery, Kerala, India. These contributions aim to improve the understanding and interpretation of lifetime data and to develop practical solutions for scientific and real-world problems.

1.2 Fundamental Ideas

Lifetime distributions offer a clear perception of reliability theory and survival analysis as the characteristic terms, such as failure rate function and mean residual life function, gives comprehensive accounts through real-time data modelling (Barlow and Proschan (1975)). Over the preceding decades, a considerable thorough study has been done on lifetime distributions. The failure rate has several applications in diverse areas and it is known by distinctive names. In actuaries, failure rates are used to compute mortality tables, so-called 'force of mortality' (Steffensen (1930)). In statistics, the reciprocal of failure rate for the normal distribution is known 'Mills ratio'. While in extreme value theory, failure rate plays an important role in the so-called 'intensity function' (Gumbel (1958)), which is also called the 'hazard rate' in reliability theory.

The concept of the failure rate has been beneficial in modelling scenarios involving imperfect repair when proper maintenance is performed. For a given lifetime random variable T, the probability density function (PDF) and cumulative distribution function (CDF) are denoted by $f(t)$ and $F(t) = P(T \leq t)$, respectively. The survival function, which represents the probability of the system or component surviving beyond time t, is denoted by $S(t) = P(T > t) = 1 - F(t)$ and is also often notated as $\bar{F}(t)$.

The failure rate function, also known as the hazard rate, is defined as:

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t | T > t)}{\Delta t} = \frac{f(t)}{S(t)},$$

where $h(t)$ represents the instantaneous rate of failure at time t, given that the component has survived up to time t.

The cumulative hazard function $H(t)$ measures the accumulated risk of failure up to a given time t. It is defined as the integral of the hazard function $h(t)$ over time from 0 to t. The $H(t)$ is given by the expression:

$$H(t) = \int_0^t h(u)du,$$

where $h(u)$ is the hazard rate function at time u. In survival analysis, $H(t)$ reflects the total 'hazard' accumulated by time t and is related to the survival function $S(t)$ by the relationship $S(t) = e^{-H(t)}$.

In lifetime experiments, the mean residual life (MRL) function describes the average additional lifetime of components that have survived up to a given time t. The MRL is characterized by

$$M(t) = E[T - t | T > t] = \int_t^\infty \frac{1 - F(u)}{1 - F(t)} du,\ t \geq 0.$$

Since the MRL is closely connected to the failure rate function, some studies examined the monotonicity of the MRL concerning the monotonicity of the failure rate function. For a detailed account of MRL, it is recommended to read Guess & Proschan (1988).

In many cases, it is reasonable to assume that a system's lifetime decreases after a certain age, indicating that its survival probability declines as the system ages. Distributions exhibiting this behavior are referred to as positive ageing distributions. To identify which real-world distributions are suitable for lifetime data analysis, the concept of ageing must be considered. Ageing is commonly characterized through the failure rate function, which provides a convenient way to describe how the likelihood of failure evolves over time. Understanding this relationship is necessary for choosing appropriate lifetime distributions in reliability modelling.

1.2.1 Concept of Ageing

The concept of 'ageing' has a significant role in reliability since lifetime distributions are often characterized by the ageing properties. Barlow & Proschan (1975) and Lai & Xie (2006) define 'no ageing' as the circumstance where the age of a component does not affect the distribution of its residual life. In a probabilistic context, 'positive ageing' (also known as 'adverse ageing') refers to the situation in which the residual life decreases as the age of the component increases. Conversely, 'negative ageing' (or 'beneficial ageing') describes the opposite effect, where the residual life increases as the age of the component grows.

Ageing behaviors including increasing failure rate (IFR), decreasing failure rate (DFR), bathtub-shaped failure rate (BFR), upside-down bathtub shaped failure rate (UBFR), and others have been discussed using these distinctive terms. 'increasing failure rate on average' (IFRA), 'decreasing failure rate on average' (DFRA), 'new better than used' (NBU), 'new better than used in expectation' (NBUE), 'new worse than used' (NWU), 'new worse than used in expectation' (NWUE), 'decreasing mean residual life' (DMRL), 'increasing mean residual life' (IMRL), 'harmonic new better than used in expectation' (HNBUE), 'harmonic new worse than used in expectation' (HNWUE), and so on are examples of additional behaviors. See Barlow & Proschan (1975) and Hollander & Proschan (1984) for a full description.

A lifetime random variable T is NBU if:

$$P(T > x + t | T > t) \leq P(T > x) \, \text{for all} \, x, t \geq 0.$$

This means that the conditional probability of the system surviving an additional x time units, given that it has already survived t time units, is less than or equal to the probability that a brand-new system would survive x time units. The concept of NBUE is a refinement of the NBU property in reliability theory. A system or component is classified as NBUE if the expected remaining lifetime of a new item is at least as great as that of an item that has already been operating for a period of time.

A lifetime distribution is classified as DFR if $-\log \bar{F}(t)$ (where $\bar{F}(t)$ is the survival function) is a concave function of t. This concavity indicates that the failure rate decreases over time. Conversely, the distribution is considered IFR if $-\log \bar{F}(t)$ is a convex function of t, meaning the failure rate increases as time progresses.

Consider a random variable T, representing the lifetime of a unit that does not 'age' in a probabilistic sense. This means that the probability distribution of the remaining lifetime at any age t, given by $P(T > t + x \mid T > t)$, is independent of t. Mathematically, this implies:

$$P(T > t + x \mid T > t) = P(T > x) \quad \text{for all } t, x > 0,$$

or equivalently,

$$\bar{F}(t + x) = \bar{F}(t)\bar{F}(x),$$

where $\bar{F}(x) = P(T > x)$ is the survival function of T. It is well-known that, among continuous distributions, only the exponential distribution, with distribution function $F(x) = 1 - e^{-\delta x}$ (for $x > 0$ and $\delta > 0$), satisfies this condition. This unique property, called the lack-of-memory (no-ageing) property, characterizes the exponential distribution in reliability theory and reflects the idea that the probability of failure does not depend on how long the unit has already operated.

Definition 1.2.1. *Assume the failure rate function $h(t) : \mathbb{R}^+ \to \mathbb{R}^+$. The CDF $F(t)$ is classified based on the behavior of the failure rate function $h(t)$ as follows:*

1. *The distribution is said to have IFR if $h'(t) > 0$ for all t;*

2. *DFR if $h'(t) < 0$ for all t;*

3. *BFR if $h'(t) < 0$ for $t \in (0, t_0)$, $h'(t_0) = 0$ and $h'(t) > 0$ for $t > t_0$, for a comprehensive account of bathtub failure rate distributions, see Rajarshi & Rajarshi (1988);*

4. *UBFR if $h'(t) > 0$ for $t \in (0, t_0)$, $h'(t_0) = 0$ and $h'(t) < 0$ for $t > t_0$;*

5. *Modified bathtub shaped failure rate (MBFR) if $h(t)$ is first increasing and then bathtub shaped;*

6. *Roller-coaster shaped failure rate if there exist n consecutive change points $0 < t_1 < t_2 < \ldots < t_n < \infty$ such that in each interval $[t_{j-1}, t_j]$, $1 \leq j \leq$*

$n+1$, where $t_0 = 0$, $t_{n+1} = \infty$, $h(t)$ is strictly monotone and it has opposite monotonicity in any two adjacent such intervals.

Definition 1.2.2. *(Lariviere & Porteus (2001))*
Let X be a non-negative random variable, then the generalized failure rate of X is $g(x) = xh(x)$.

If $g(x)$ is increasing weakly in x for $F(x) < 1$, then X is said to have an increasing generalized failure rate (IGFR) or equivalently, $F(x)$ is an IGFR distribution. Similarly, a decreasing generalized failure rate (DGFR) can be defined when $g(x)$ is weakly decreasing in x. IGFR distributions are mainly considered in pricing and supply chain management.

Definition 1.2.3. *(Lariviere (2006)) The function $f(x, y)$ is considered totally positive of order 2 (TP$_2$) if, for $x_1 < x_2$ and $y_1 < y_2$, the following inequality holds: $f(x_1, y_1)f(x_2, y_2) - f(x_1, y_2)f(x_2, y_1) \geq 0$, as stated by Barlow and Proschan (1996). The following statements are equivalent:*

1. *X is IGFR.*
2. *$\log(X)$ is IFR.*
3. *$X \leq_{hr} \alpha X$ for $\alpha \geq 1$.*
4. *$f(x;\theta) = \bar{F}(x/\theta)$ is TP$_2$.*

Here, $X_1 \leq_{hr} X_2$ denotes that X_1 is smaller than X_2 in the hazard rate order. Specifically, the random variable X_1 with hazard rate $h_1(x)$ is said to be smaller than the random variable X_2 with hazard rate $h_2(x)$ in the hazard rate order if $h_1(x) \geq h_2(x)$ for all $x \geq 0$ (Ross (1995)).

Definition 1.2.4. *Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ represent the ordered failure times of n items from a random variable X under test. Suppose $X_{(r)} < t \leq X_{(r+1)}$ for some r. The total time on test (TTT) statistic up to time t is defined as:*

$$TTT(t) = nX_{(1)} + (n-1)(X_{(2)} - X_{(1)}) + \cdots + (n-r)(t - X_{(r)}), \tag{1.2.1}$$

which accounts for the time observed between failure times, weighted by the number of items still under test. The total time up to the r-th failure is given by:

$$TTT(X_{(r)}) = nX_{(1)} + (n-1)(X_{(2)} - X_{(1)}) + \cdots + (n-r+1)(X_{(r)} - X_{(r-1)}), \tag{1.2.2}$$

Alternatively, Eq. (1.2.2) can be expressed as:

$$TTT(t) = X_{(1)} + X_{(2)} + \cdots + X_{(r)} + (n-r)t. \tag{1.2.3}$$

The scaled TTT statistic ϕ_r, which represents the proportion of the total time up to the r-th failure relative to the total time for all failures, is defined as:

$$\phi(r) = \frac{TTT(X_{(r)})}{TTT(X_{(n)})} = \frac{\sum_{j=1}^{r}(n-j+1)(X_{(j)} - X_{(j-1)})}{\sum_{j=1}^{n}(n-j+1)(X_{(j)} - X_{(j-1)})}, \tag{1.2.4}$$

with $X_{(0)} = 0$.

The scaled TTT transform of a lifetime distribution F can also be expressed as: $\phi(p) = \frac{H_F^{-1}(p)}{H_F^{-1}(1)}$, where $H_F^{-1}(p) = \int_0^{F^{-1}(p)}(1-F(x))dx,\ p \in [0,1]$, and $F^{-1}(p) = \inf\{x : F(x) \geq p\}$. A detailed discussion of the TTT transform can be found in Barlow and Campo (1975), while a quantile-based expression is provided in Nair et al. (2013).

Properties of Ageing Classes

Some properties of ageing distributions have been discussed in the works of Barlow and Campo (1975), Lee & Thompson (1976), Bergman (1977), Langberg et al. (1980), Klefsjö (1980), Aarset (1987), and Xie (1989). A few of them are listed below.

Suppose $\phi(p)$ be the scaled TTT transform of a continuous lifetime distribution F, then

1. F is IFR (DFR) if and only if $\phi(p)$ is concave (convex) in $p \in [0,1]$.

2. F is IFRA (DFRA) if and only if $\phi(p)/p$ is decreasing (increasing) in $p \in [0,1]$.

3. F is NBUE (NWUE) if and only if $\phi(p) \geq p\,(\phi(p) \leq p)$ for $p \in [0,1]$.

4. F is DMRL (IMRL) if and only if $(1-\phi(p))/(1-p)$ is decreasing (increasing) in $p \in [0,1]$.

5. F is HNBUE (HNWUE) if and only if $\phi(p) \leq 1 - exp\{-F^{-1}(p)/\mu\}$ $\phi(p) \geq 1 - exp\{-F^{-1}(p)/\mu\}$ for $p \in [0,1]$, where μ is the mean of the distribution.

6. F is BFR (UBFR) if $\phi(p)$ has only reflection point p_0 such that $0 < p_0 < 1$ and it is convex (concave) on $[0, p_0]$ and concave (convex) on $[p_0, 1]$.

Distributions with monotone failure rates have been considered a significant factor because of their practical interest in reliability. Such distributions comprise a massive class. In the class of failure rate distributions, IFR distributions have their relevance. (see, for example, Marshall & Proschan (1965), Kvam (2002) and Koutras (2011)). The Weibull and Gamma are two primer distributions that are effectively using in many industries for optimizing system reliability, especially for redundancy allocation problems (RAP) (see, Fyffe et al. (1968)).

If F is IFR, it is equivalent to say that F is Polya frequency function of order 2 (PF2). Schoenberg (1951) introduced PF2 is defined by, a PF2 function is a non-negative measurable function $g(x)$ defined for all real x such that

$$\begin{vmatrix} g(x_1 - y_1) & g(x_1 - y_2) \\ g(x_2 - y_1) & g(x_2 - y_2) \end{vmatrix} \geq 0,$$

whenever $x_1 < x_2$ and $y_1 < y_2$; in addition, $g(x) \neq 0$ for at least two distinct values of x. If the density function f has PF2, then it would be easy to verify that the distribution has IFR, but the converse is need not to be true. PF2 has many useful properties including unimodality, closure under convolution, variation diminishing and certain moment properties (see Schoenberg (1951) and Karlin et al. (1961)).

Theorem 1.2.5. *(Barlow et al. (1963))*

If F and G are IFR, then their convolution H, given by $H(t) = \int_{-\infty}^{\infty} F(t - x)dG(x)$, is also IFR.

Bakouch et al. (2014) proposed a model (Binomial-exponential 2) using zero-truncated binomial random variable having IFR, $h(x) = \lambda(1 - \frac{\theta}{2-\theta+\lambda\theta x})$. It is increasing in x for $0 \leq \theta \leq 1$ and $\lambda > 0$. Guilani et al. (2016) investigated RAP of a series-parallel system with components having IFR based on Weibull distribution. Bugatekin (2017) developed a two-parameter mixed distribution named RL (Rayleigh-Logarithmic) distribution with IFR. Elbarmi et al. (2021) provide IFRA model with an estimator of F that is uniformly strongly consistent and they derive convergence of the estimator at the point where F is IFRA using the arg max theorem.

It is reasonable to assume that the failure rate of the life distribution is increasing (IFR). The wear-outphase easily interprets this fact, implying that the age-old units have a higher chance of failure. However, it appears to be more difficult to explain why a unit's life expectancy is increasing while its failure rate is decreasing. DFR distributions may appear in different ways including when the strength of some metals increases with usage or when an instrument exhibit an 'infant mortality' rate during their lifetime. Normally DFR represents the apparatus having betterment with age so that age-old units have less chance of failure.

Theorem 1.2.6. *(Barlow, Marshall, Proschan (1963))*

- *F is DFR if and only if the support of F is $[0, \infty]$, and $1 - F(x + y)$ is TP_2 for $x + y \geq 0$.*

- *DFR property is not preserved under the reliability conditions of convolution and coherent systems. However, mixtures of DFR distributions are DFR.*

- *If $F(t, \phi)$ is a DFR distribution in t for each ϕ in Φ, then $G(t) = \int_{\Phi} F(t, \phi) d\mu(\phi)$ is DFR where μ is a probability measure in Φ.*

Adamidis & Loukas (1998) presented a DFR distribution named Exponential-Geometric distribution obtained as a mixture of the exponential and geometric distribution. Its failure rate $h(x) = \beta(1 - pe^{-\beta x})^{-1}$, $p \in (0, 1)$, $\beta > 0$, $x > 0$ is decreasing in x. Finkelstein & Esaulova (2001) observed a mixture of several continuous IFR distributions and analyzed the limiting behaviour of the mixture failure rate function. They found the mixture can be DFR, and for that, the limiting behaviour of conditional mean and the conditional variance of the mixing parameters are essential. Chahkandi & Ganjali (2009) introduced a DFR distribution which is a mixture of power-series distribution and exponential distribution. Generally, mixtures of exponential distributions are DFR.

Pham (2006) provides some properties of IFR and DFR distributions:

- Suppose X_1 and X_2 are IFR, then $X_1 + X_2$ are also IFR; but DFR property may not be preserved. In the case of mixtures, the opposite happens.

- IFR distributions are preserved under coherent systems.

- Order statistics of an IFR distribution is again IFR; but not necessarily true for DFR distributions.

- Spacings of DFR distributions give DFR; but not true in the case of IFRdistributions.
- The PDF of IFR distributions may not be unimodal but a decreasing function for DFR distributions.

In most real-life situations, failure rates may not exhibit monotonic nature; rather they start with a peak value, decrease, may then persist constant and in the end increases rapidly (see, Rajarshi & Rajarshi (1988)). These indicate the different aspects of a component or system, i.e., early life (infant mortality phase), useful life and wear-out phase. They are also known, respectively, as burn-in, random and wear-out phases in the reliability context. This occurs mainly due to diverse amplitudes of the life of a device when they are concerned by a collection of deformities. In modelling real-life datasets, bathtub failure rates are beneficial because the lifetimes of mechanical products, electromechanical and electronics are frequently modelled with this aspect. Mitra and Basu (1996) provided various properties of BFR distributions. Preservations of lifetime distributions under the reliability operations like convolution, mixtures and coherent systems are viewed.

- Suppose F is BFR and G is exponential with mean $\{h(t_0)\}^{-1}$, then $\bar{F}(t) \leq \bar{G}(t)$, where t_0 is the change point for $h(t)$ is minimum.
- $E(X^k) \leq \frac{\Gamma(k+1)}{r(t_0)^k}$, $k > 0$.
- If a BFR distribution has a k^{th} raw moment, $E(X^k) = \frac{\Gamma(k+1)}{r(t_0)^k}$, then it is necessarily exponential.
- If the BFR distribution has a finite turning point t_0, then $\{\frac{\bar{F}(x)}{\bar{F}(t_0)}\}^{1/(x-t_0)}$ is decreasing in x, $x \in (t_0, \infty)$.
- The moment sequence uniquely determines the distribution if it is BFR with a finite turning point.
- Convolution of BFR distributions is not necessarily BFR.
- A mixture of BFR distributions may not be BFR.
- A parallel system consisting of two independent components from BFR distributions may not be BFR.

Definition 1.2.7. *(Glaser (1980)) Let F be a lifetime distribution with a differentiable CDF, a density function f, and a failure rate λ. The distribution F is classified as a BFR distribution if there exists a point t_0 within $(0, \infty)$ such that $\lambda(t)$ decreases for $t \leq t_0$ and increases for $t \geq t_0$.*

Definition 1.2.8. *(Deshpande and Suresh (1990)) A lifetime distribution F is defined as a BFR distribution if there exists a point $t_0 > 0$ such that F is concave on $[0, t_0]$ and convex on $[t_0, \infty)$.*

Definition 1.2.9. *(Deshpande and Suresh (1990)) A lifetime distribution is classified as an 'increasing initially, then decreasing mean residual life' (IDMRL) distribution if there exists a point $t_0 > 0$ such that:*

- $\mu(s) \leq \mu(t)$ *for* $0 \leq s < t < t_0$, *and*
- $\mu(u) \geq \mu(\nu)$ *for* $t_0 \leq u \leq \nu < \infty$.

These distributions are also referred to as upside-down bathtub-shaped mean residual life distributions (see Rajarshi & Rajarshi (1988)).

Definition 1.2.10. *(Mi (1995))*
A distribution function F is classified as a BFR (UBFR) type if there exist two points t_1 and t_2 such that $0 \leq t_1 \leq t_2 < \infty$, and the failure rate function $h(t)$ satisfies the following conditions:

- *it is strictly decreasing (increasing) for* $0 \leq t \leq t_1$;
- *it remains constant for* $t_1 \leq t \leq t_2$; *and*
- *it is strictly increasing (decreasing) for* $t_2 \leq t$.

The points t_1 and t_2 are referred to as the change points of $h(t)$. The BFR becomes strictly increasing (IFR) when $t_1 = t_2 = 0$. Similarly, if $t_1 = t_2 \to \infty$, $h(t)$ becomes strictly decreasing (DFR). In general, when $t_1 = t_2$, the interval where $h(t)$ is constant reduces to a single point. This definition treats monotonic failure rates IFR and DFR as a special case of BFR.

Definition 1.2.11. *(Mitra & Basu (1996)) An absolutely continuous distribution with support $[0, \infty)$ is referred to as a BFR distribution if its failure rate function $h(t)$ is non-increasing over $[0, t_0)$ and non-decreasing over $[t_0, \infty)$ for some $t_0 \geq 0$.*

By definition, the distribution allows for at most two change points. According to Mi (1995), points within the interval (t_1, t_2) are not considered true change points, although they were treated as such in the work of Mitra & Basu (1996).

A generalization of Definition 1.2.7 is given by,

Definition 1.2.12. *A life distribution F is said to be BFR, if there is a point t_0, $0 < t_0 < \infty$, and $\bar{F}(x|t) = \bar{F}(t+x)/\bar{F}(t)$, for which:*

1. *$\bar{F}(x|t)$ is increasing w.r.t. t, for $0 \leq t \leq t_0$, and $0 \leq x \leq t_0 - t$; and*

2. *$\bar{F}(x|t)$ is decreasing w.r.t. t, for $t_0 \leq t \leq t_\infty$ and $x \geq 0$.*

Various techniques are used to construct BFR type distributions. A few of them are given.

- **Glaser's technique**

  Glaser (1980) adopted a function $\eta(t)$ that accomplishes the succeeding norms:

  1. $\eta(t) = -\frac{f'(t)}{f(t)}$ and $f(t)$ is a density function;
  2. there exist a $t_0 > 0$ such that $\eta'(t) < 0$ for all $t \in (0, t_0)$, $\eta'(t_0) = 0$ and $\eta'(t) > 0$ for all $t > 0$;
  3. there exist a $y_0 > 0$ such that $\int_{y_0}^{\infty} [f(y)/f(y_0)]\eta(y_0)dy - 1 = 0$.

- **Convex function**

  From the description of BFR distribution, we can construct a BFR by selecting a positive convex function $h(t)$ that satisfies $\int_0^\infty h(t)dt = \infty$ (Rajarshi and Rajarshi (1988)).

- **Function of random variables**

  This technique due to Griffith (1982), is for differentiable failure rate functions.

- **Reliability and stochastic Mechanisms**

  A series system with two independent components, where one has an IFR distribution and the other a DFR distribution, may result in a BFR distribution, as demonstrated by Murthy et al. (1973)

- **Mixtures**

  Mixtures of BFR distributions frequently produce BFR distributions.

- **Truncating DFR models**

  BFR models can be constructed by truncating DFR type distributions.

Often, some distributions arise with more than one turning point for the failure rates. Roller-coaster failure rates appear in such situations (see Wong (1988), Wong & Lindstrom (1988) and Wong (1991)). The monotonicity of failure rates with one turning point can be easily determined using Glaser's technique. To study the monotonicity of failure rates with more than one turning point, Gupta & Warren (2001) introduced generalized Glaser's technique and examined several examples. Moreover, they found out a relation between turning points of failure rates and Glaser's eta function and put forward the number of turning points of failure rates that do not surpass the number of turning points of Glaser's eta function.

Gupta & Viles (2011) have taken on the monotonicity of Glaser's eta function, failure rate and mean residual life function, and the results are used to investigate the results in extended generalized inverse Gaussian distribution.

1.2.2 Lifetime Distributions

In disciplines like survival analysis and reliability theory, lifetime distributions serve as essential tools for modelling the time until a specific event occur such as the survival time of patients or the malfunctioning of a machines. These distributions are used to estimate survival probabilities, failure rates, and cumulative risks over time. Prominent lifetime distributions include the exponential, Weibull, and gamma distributions, each useful for different types of failure patterns, such as constant or varying failure rates. By offering ways to predict the likelihood and timing of events, lifetime distributions are essential for analyzing the durability of products, planning maintenance, and determining the treatment effectiveness in healthcare. Some commonly used lifetime distributions are described below.

Exponential Distribution

The exponential distribution is often used in reliability and survival analysis because of its simplicity and important features. It assumes a constant failure rate, meaning the chance of an event, like a failure, occurring in the next moment is always the same, no matter how much time has already passed. This "memoryless" property means that past events don't affect the likelihood of future ones. The distribution has just one parameter, δ, which represents both the failure rate and the average lifetime of a system. Although this model is helpful in forecasting the lifespan of many objects, it's important to be careful, as real-world data may not always follow this consistent rate, and statistical techniques may be sensitive to these differences. The PDF of exponential distribution is given by

$$f(x;\delta) = \delta e^{-\delta x},\ x \geq 0,\ \delta > 0.$$

Its corresponding CDF and failure rate function are given by, respectively

$$F(x;\delta) = 1 - e^{-\delta x},\ x \geq 0,\ \delta > 0,$$

and

$$h(x;\delta) = \delta,\ \delta > 0.$$

The exponential distribution models the intervals between events in a Poisson process, where the events occur independently and at a steady rate. The mean (expected value) of an exponential distribution is $\frac{1}{\delta}$, and its variance is $\frac{1}{\delta^2}$. This property links the rate of events (δ) directly to the average lifetime and the variability around it.

Weibull Distribution

The Weibull distribution is one of the most commonly used distributions for modelling time-to-event data. By adjusting its shape parameter, it can represent various failure rate behaviors, such as increasing, decreasing, or constant failure rates. Due to its flexibility, the Weibull distribution can effectively model a wide range of lifetime data, including biological survival times and mechanical failure rates. The PDF and CDF of the Weibull distribution are given, respectively, by:

$$f(x;\gamma,\delta) = \gamma\delta x^{\gamma-1} e^{-\delta x^{\gamma}};\ x,\delta,\gamma > 0,$$

and

$$F(x;\gamma,\delta) = 1 - e^{-\delta x^{\gamma}};\ x,\delta,\gamma > 0.$$

Here $\delta > 0$ is the scale parameter and $\gamma > 0$ is the shape parameter.

The failure rate of the Weibull distribution can be expressed as:

$$h(x;\gamma,\delta)=\gamma\delta x^{\gamma-1};\quad x,\delta,\gamma>0,$$

where the behavior of the failure rate depends on the value of the shape parameter γ. For $\gamma=1$, the Weibull distribution simplifies to the exponential distribution, which has a constant failure rate. When $\gamma>1$, the failure rate increases over time, which is typical for ageing processes where the likelihood of failure grows as the item or individual gets older. When $0<\gamma<1$, the failure rate decreases over time, making it suitable for scenarios where early-life failures are more likely, and the probability of failure decreases as time progresses. The mean and variance of the Weibull distribution are given by $\frac{\Gamma(\frac{1}{\gamma}+1)}{\delta^{1/\gamma}}$ and $\frac{1}{\delta^{2/\gamma}}\left[\Gamma(\frac{2}{\gamma}+1)-(\Gamma(\frac{1}{\gamma}+1))^2\right]$, respectively.

Inverse Weibull distribution

A random variable X is said to follow an inverse Weibull distribution if its PDF and CDF, for $x>0$, are given by:

$$f(x;\gamma,\delta)=\gamma\delta x^{-(\gamma+1)}e^{-\delta x^{-\gamma}},$$

and

$$F(x;\gamma,\delta)=e^{-\delta x^{-\gamma}},$$

respectively. Here, $\gamma>0$ and $\delta>0$ denote the shape and scale parameters, respectively. The failure rate function can be expressed as:

$$h(x;\gamma,\delta)=\frac{\gamma\delta x^{-(\gamma+1)}e^{-\delta x^{-\gamma}}}{1-e^{-\delta x^{-\gamma}}}.$$

The failure rate function of the inverse Weibull distribution exhibits a unimodal shape, which contrasts with the Weibull distribution, whose failure rate function is limited to being either increasing, decreasing, or constant.

The Exponentiated Weibull Distribution

The exponentiated Weibull (EW) distribution is a three-parameter extension of the Weibull distribution, introduced by Mudholkar and Srivastava (1993). This distribution adds an extra shape parameter, which increases the flexibility of the Weibull model. As a result, it can effectively represent various failure rate behaviors, including increasing, decreasing, unimodal, bathtub-shaped, and constant patterns. The PDF of the EW distribution for a random variable X is given by:

$$f(x;\Theta) = \frac{\alpha\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}\left(1-e^{-(x/\lambda)^{\theta}}\right)^{\alpha-1}e^{-(x/\lambda)^{\theta}}, \quad x>0,$$

where $\Theta = (\alpha,\lambda,\theta)$ represents the parameter set, with $\alpha > 0$ as the additional shape parameter, $\lambda > 0$ as the scale parameter, and $\theta > 0$ determining the shape of the Weibull component.

The CDF is given by:

$$F(x;\Theta) = \left(1-e^{-(x/\lambda)^{\theta}}\right)^{\alpha}, \quad x>0.$$

The failure rate function, which describes the instantaneous failure rate, is expressed as:

$$h(x;\Theta) = \frac{\alpha\theta x^{\theta-1}\left(1-e^{-(x/\lambda)^{\theta}}\right)^{\alpha-1}e^{-(x/\lambda)^{\theta}}}{\lambda^{\theta}\left(1-\left(1-e^{-(x/\lambda)^{\theta}}\right)^{\alpha}\right)}.$$

The EW distribution proves beneficial for its capacity to represent a variety of failure rate behaviors, making it relevant in fields like reliability analysis, survival studies, population modelling, and extreme value analysis. Also, it serves as a generalization of several other distributions, including the Weibull, exponentiated Exponential, Rayleigh, and exponential distributions, which can be seen as special cases.

Gamma Distribution

The gamma distribution is most helpful for modelling life lengths and waiting periods. It is extensively utilized in domains such as queuing theory, insurance, and hydrology, where systems involve events that occur at varying rates or cumulative waiting times.When α, the shape parameter, is an integer, the gamma distribution and the Erlang distribution are equal. It reduces to the exponential distribution for $\alpha = 1$ as well. The PDF and CDF of the gamma distribution are, respectively,

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}; x>0,\ \alpha,\beta>0,$$

and

$$F(x;\alpha,\beta) = \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)};\ x > 0, \alpha, \beta > 0,$$

where $\gamma(\alpha,\beta x)$ is the incomplete gamma function. The gamma distribution models right-skewed data and has an IFR property for $\alpha > 1$, a DFR property for $\alpha < 1$, and a constant failure rate for $\alpha = 1$. Its skewness depends only on its shape parameter α and is given by $\frac{2}{\sqrt{\alpha}}$.

Pareto Distribution

The Pareto distribution is important in disciplines like economics, finance, and survival analysis and is frequently used to represent heavy-tailed data. It is especially useful for analyzing income inequality and functions as a lifetime model in actuarial science and engineering. A random variable follows a Pareto distribution if its PDF is defined as:

$$f(x;a,b) = \frac{ab^a}{x^{a+1}};\ x > b,\ a > 0, b > 0,$$

where a is the shape parameter and b is the scale parameter. The CDF of the Pareto distribution is expressed as:

$$F(x;a,b) = 1 - \left(\frac{b}{x}\right)^a, \quad x > b,\ a > 0,\ b > 0.$$

The failure function $h(x;a,b)$ of the Pareto distribution is given by:

$$h(x;a,b) = \frac{a}{x}, \quad x > b,\ a > 0,\ b > 0.$$

The failure rate function is monotonically decreasing with x.

Lomax Distribution

Being a special case of the generalized Pareto distribution, Lomax distribution is often referd to as the Pareto Type-II distribution. This distribution is commonly used for modelling heavy-tailed datasets and beneficial in financial risk, insurance, and failure-time data modelling. The PDF of the Lomax distribution is defined as:

$$f(x;a,b) = ab(1+bx)^{-(a+1)}, \quad x > 0,\ a > 0,\ b > 0,$$

where a and b denotes the shape and scale parameters, respectively. The CDF is given by:

$$F(x; a, b) = 1 - (1 + bx)^{-a}, \quad x > 0,\, a > 0,\, b > 0.$$

The failure rate function $h(x)$ of the Lomax distribution is expressed as:

$$h(x; a, b) = \frac{ab}{1 + bx}, \quad x > 0,\, a > 0,\, b > 0.$$

The failure rate function is monotonically decreasing in x. In scenarios where the likelihood of event occurrence decreases over time, such as in situations involving wear-out failure or age-based risk reduction, this distribution would be appropriate.

Exponential Power Distribution

For a random variable X, the CDF of the exponential Power (EP) distribution is defined as:

$$F(x) = 1 - e^{1-e^{(\lambda x)^{\alpha}}};\ x > 0,\ \alpha,\ \lambda > 0. \tag{1.2.5}$$

The corresponding PDF of X is expressed as:

$$f(x) = \alpha\lambda^{\alpha} x^{\alpha-1} e^{1+(\lambda x)^{\alpha} - e^{(\lambda x)^{\alpha}}};\ x > 0,\ \alpha,\ \lambda > 0. \tag{1.2.6}$$

This distribution is versatile, as it can represent both increasing and bathtub-shaped failure rates, making it ideal for modelling the reliability and risk of electronic and mechanical systems.

The failure rate function of the EP distribution expressed as $h(x) = \alpha\lambda^{\alpha} x^{\alpha-1} e^{(\lambda x)^{\alpha}}$, displays significant characteristics.

1. If $\alpha \geq 1$, $h(x)$ is an increasing function for $x > 0$.
2. If $\alpha < 1$, $h(x)$ has a bathtub shape and approaches its minimum at $x = \left(\frac{1-\alpha}{\alpha\lambda^{\alpha}}\right)^{\frac{1}{\alpha}}$.

This demonstrates the EP distribution's versatility in modelling different failure rate patterns.

Kumaraswamy Distribution

The Kumaraswamy (K) distribution have similar characteristics with the Beta distribution, particularly different density shapes and boundary behavior. The density can demonstrate unimodal, anti-unimodal, increasing decreasing, or constant patterns, depending on the values of its parameters. The K distribution has tractability advantages over the Beta distribution, such as closed-form formulations for its quantile function and CDF.

The two-parameter Kumaraswamy distribution has the following properties:

The PDF is given by:

$$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1},\ 0 < x < 1,\ \alpha > 0, \beta > 0.$$

The CDF is given by:

$$F(x;\alpha,\beta) = 1-(1-x^{\alpha})^{\beta}.$$

The failure rate function is expressed as:

$$h(x;\alpha,\beta) = \frac{\alpha\beta x^{\alpha-1}}{1-x^{\alpha}}.$$

With the shape parameters α and β, this distribution is quite versatile for modelling data that is bounded within $(0,1)$. Its failure rate function demonstrates a variety of characteristics, such as monotonically increasing for $\alpha \geq 1$ and exhibits a bathtub shape for $\alpha < 1$.

Inverse Kumaraswamy Distribution

Inverse Kumaraswamy (IK) distribution is obtained by transforming the K distribution as $X = (1-T)/T$, where the random variable T follows K distribution. Thereafter, the random variable X is said to have the IK(a, b) distribution. The PDF and CDF of the IK distribution are respectively given by:

$$f(x;a,b) = ab(1+x)^{-(a+1)}(1-(1+x)^{-a})^{b-1},\ x > 0,\ a > 0,\ b > 0 \tag{1.2.7}$$

and

$$F(x;a,b) = (1-(1+x)^{-a})^{b},\ x > 0,\ a > 0,\ b > 0, \tag{1.2.8}$$

where a and b are shape parameters. The versatility of the IK distribution makes it popular in areas including biology, engineering, and economics. In contrast to other distributions, this distribution may be used to predict long-term reliability as it provides comparatively optimistic predictions about uncommon events located in the upper tail of the distribution.

The failure rate function of IK distribution is shown as

$$h(x; a, b) = \frac{ab(1 + x)^{-(a+1)}(1 - (1 + x)^{-a})^{b-1}}{1 - (1 - (1 + x)^{-a})^b}; \; x > 0, \; a, b > 0.$$

The IK distribution has both monotonic and non-monotonic failure rates. For $b \leq 1$, $h(x)$ is decreasing, and for $b > 1$, $h(x)$ has upside-down bathtub shape.

Lindley Distribution

Lindley (1958) proposed the Lindley distribution, as a mixture of an exponential distribution and a gamma distribution with a shape parameter of 2. The PDF of the proposed model is defined by:

$$f(x; \lambda) = \frac{\lambda^2(1 + x)}{1 + \lambda} e^{-\lambda x}; \quad x > 0, \; \lambda > 0,$$

where λ is the scale parameter. The CDF and failure rate functions are respectively given by:

$$F(x) = 1 - \frac{1 + \lambda + \lambda}{1 + \lambda} e^{-\lambda x}; x > 0, \lambda > 0,$$

and

$$h(x) = \frac{\lambda^2(1 + x)}{1 + \lambda + \lambda x}.$$

With respect to both x and λ, $h(x)$ increases and satisfies the inequality $\lambda^2/(\lambda + 1) < h(x) < \lambda$. Since it can be a better fit in certain circumstances, the Lindley distribution is often preferred over other distributions like the exponential, when the data exhibits more complexity.

The mode of the distribution is defined as:

$$\text{Mode}(X) = \begin{cases} \frac{1-\lambda}{\lambda}, & \text{if } 0 < \lambda < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Ghitany et al. (2008) highlighted the gaining popularity of Lindley distribution for modelling lifetime data. Its applications spread to areas like survival analysis, actuarial studies, and queue modelling.

Generalized Lindley Distribution

Nadarajah et al. (2011) proposed the generalized Lindley (GL) Distribution as a generalization of the Lindley distribution. It has been introduced to increase its flexibility in modelling numerous data types. The PDF of the GL distribution is expressed as:

$$f(x) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right]^{\alpha-1} \exp(-\lambda x); x > 0,\ \alpha, \lambda > 0,$$

where λ is the scale parameter and α is the shape parameter.
Suppose that the independent random variables $X_1, X_2, ..., X_\alpha$, each following a Lindley distribution, represents the failure times of the components in a parallel system. Assuming the components functions independently, the probability that the parallel system fails before a given time x follows a GL Distribution. The CDF of GL distribution is given by:

$$F(x;\alpha,\lambda) = \left(1 - \frac{1+\lambda+\lambda x}{(1+\lambda)}e^{-\lambda x}\right)^{\alpha}; x > 0,\ \alpha, \lambda > 0.$$

The presence of the shape parameter α improves the flexibility of the distribution, which enables it to reflect a range of failure rate characteristics, such as increasing, decreasing, and bathtub-shaped failure rates. The GL distribution provides more flexibility to simulate complicated data structures while maintaining the simplicity of the Lindley distribution. Its applications extend to fields such as queuing theory, reliability engineering, and environmental studies, where versatile and robust models are required for real-world data.

1.2.3 Copula

Copula is a function that connects multivariate distribution functions into uniform [0,1] marginals. A copula function can be effectively used to describe a multivariate distribution with a dependence structure. For $x_1, x_2, ..., x_n$, the n-dimensional copula can be represented as follows: if F is the continuous joint CDF, then there is a unique copula C such that $F(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n))$. Here we are interested in the bivariate form. Sklar (1973) presents that, for two random variables X and Y with CDFs $F_1(x)$ and $F_2(y)$ and PDFs $f_1(x)$ and $f_2(y)$, respectively, the copula CDF and copula PDF are as follows:

$$
\begin{aligned}
F(x,y) &= C(F_1(x), F_2(y)) \\
f(x,y) &= f_1(x) f_2(y) c(F_1(x), F_2(y)).
\end{aligned}
\tag{1.2.9}
$$

When the margins are continuous, C is uniquely determined. In cases where they are not continuous, C is uniquely defined only over $\mathcal{R}(F_1) \times \cdots \times \mathcal{R}(F_n)$, with $\mathcal{R}(F_i)$ representing the range of F_i. Copulas are very useful in advancing the study of two random variables that exhibit a dependence within bivariate distributions. Notably, there exists a variety of well-known copula models, including the Gaussian, Clayton, Gumbel, Frank, Farlie-Gumbel-Morgenstern, plackett, and Archimedean copulas.

Farlie-Gumbel-Morgenstern

The Farlie-Gumbel-Morgenstern (FGM) copula is a type of copula used in statistics and probability theory to model dependence between random variables. The FGM copula is particularly suitable for identifying weak or moderate dependence structures. It has a simple mathematical form and is defined as:

$$
C(u_1, u_2) = u_1 u_2 + \omega u_1 u_2 (1 - u_1)(1 - u_2),
$$

where u_1 and u_2 represents the marginal distributions of the variables and ω is the dependence parameter. The parameter ω ranges from -1 to 1, with $\omega = 0$ representing independence, $\omega > 0$ indicating positive dependence, and $\omega < 0$ indicating negative dependence. The FGM copula is most appropriate for situations when the dependencies are relatively weak as it is able to determine a limited degree of dependence. Despite its drawbacks, the FGM copula is commonly used in fields like finance, engineering, and actuarial science, where it provides an efficient way to model interactions between variables with weak or moderate dependencies.

1.2.4 Censoring

In life-testing experiments, it is not always possible to obtain all of the information. Occasionally, experimental units or items may be removed from the experiment for a variety of reasons. Censoring takes place in these situations when only partial information about the data is available. Censored data can take several forms, with one of the primary reasons for censoring being to reduce the total test time and associated costs. An ideal censoring scheme should strike a balance between (i) the total time spent on the experiment, (ii) the number of units involved, and (iii) the efficiency of statistical inference derived from the experiment results. Depending on how the data is rendered incomplete, censoring is classified into three types:

- Left censoring
- Interval censoring
- Right censoring

Left censoring happens when the event of interest has occurred before the observation period begins, though the exact time of occurrence is not known. Examples include the development of undetected hypertension or the onset of type 2 diabetes before diagnosis.

Interval censoring occurs when the exact time of an event is unknown, but a time interval in which the event occurred is available. If the interval is very short, such as one day or one hour, it is a common practice to disregard the interval and assign a consistent endpoint to simplify analysis. Examples of interval censoring include the onset of symptoms of an illness between routine doctor visits or a power outage occurring between two maintenance checks.

Right censoring occurs when we only know that an individual's lifetime exceeds a certain observed value, without knowing the exact time of the event. This type of censoring can arise for various reasons, either by design, such as when a life test is terminated before all items fail, or unexpectedly, as when a participant in a longitudinal study is "lost to follow-up" after moving out of the study area. For example, a lung cancer patient might be recruited for a clinical trial to evaluate a drug's effect on survival, but if the patient dies in a car accident after T years, this would result in right-censored data for the study.
Two widely used right-censoring methods are Type-I and Type-II censoring.

Type-I Censoring

Type-I censoring typically occurs when a study runs over a fixed time period. In this scheme, the duration of the experiment is predetermined, but the number of observed failures is random. In this scheme, each individual has a specified censoring time $C_i > 0$, which serves as the cutoff for observation. If the event time T_i occurs before or at C_i, it is recorded; otherwise, we only know that T_i is greater than C_i. The likelihood for Type-I censoring is the product of two components: for uncensored observations, it includes the PDF $f(t_i)$, while for censored observations, it includes the survival function $S(C_i)$, which represents the probability that the event time exceeds the censoring time. The likelihood function is expressed as:

$$L(\theta) = \prod_{i=1}^{n} \left[f(t_i;\theta)^{\delta_i} \times S(C_i;\theta)^{1-\delta_i} \right].$$

Here, $\delta_i = 1$ indicates an uncensored observation, and $\delta_i = 0$ indicates a censored observation. This formulation allows for the estimation of parameters θ based on both observed and censored data.

Type-II Censoring

In Type-II censoring, the experiment ends when a pre-specified number of r (where $1 \leq r \leq n$) failures are observed. Here, the number of failures is fixed, but the experimental duration is random. This scheme arises when n individuals start the study together, and the study terminates once r failures are recorded. Although used in some life tests, Type-II censoring has the drawback that the total duration $t_{(r)}$ is unknown at the start since it depends on when the failures occur.

In Type-II censoring, the value of r is determined before data collection begins, and the data consists of the r smallest lifetimes from a random sample $T_1, T_2, ..., T_n$. For continuous distributions, ties are disregarded, and the r smallest lifetimes are denoted as $T_{(1)} \leq T_{(2)} \leq ... \leq T_{(r)}$. If the T_i have a PDF $f(t)$ and a survival function $S(t)$, then, according to general results in order statistics, the joint PDF of $T_1, T_2, ..., T_r$ is:

$$L(\theta) = \frac{n!}{(n-r)!} \left\{ \prod_{i=1}^{r} f(t_{(i)};\theta) \right\} S(t_{(r)};\theta)^{n-r}.$$

A combination of Type-I and Type-II censoring schemes, known as a hybrid censoring scheme (HCS), has gained immense prevalence in reliability and life testing experiments. Some common types of hybrid censoring schemes include:

1. **Type-I HCS**
   Consider a life testing experiment where n units are tested, with the lifetimes of the units assumed to be independent and identically distributed random variables. Let the ordered lifetimes of these units be denoted accordingly. The test ends when either a pre-determined number, r, of units have failed, or when a fixed time, T, has passed. In other words, the life test concludes

at a random time $T^* = min\{X_{r:n}, T\}$, depending on which condition is met first. It is typically assumed that the failed units are not replaced during the experiment. This type of censoring scheme is known as the Type-I hybrid censoring scheme.

2. **Type-II HCS**

   An alternative hybrid censoring scheme is the Type-II hybrid censoring, where the experiment ends at a random time $T^* = \max\{X_{r:n}, T\}$, with r and T being predetermined. This scheme ensures that at least r failures are observed by the end of the experiment. If r failures occur before time T, the experiment continues until time T, potentially resulting in more than r failures in the data. Conversely, if the r-th failure doesn't occur before time T, the experiment continues until the r-th failure is observed, ensuring exactly r failures. Although the termination time in this scheme is a random variable, which can be a disadvantage from the experimenter's perspective, it guarantees that at least r failures are observed, potentially more, leading to more efficient inferential procedures.

3. **Generalized Type-I HCS**

   Suppose we fix integers r and k, where $k, r \in \{1, 2, 3, ..., n\}$, $T \in (0, \infty)$, and $k < r < n$. Under this setup, the experiment stops at $\min\{X_{r:n}, T\}$ if the k-th failure occurs before time T. If the k-th failure takes place after T, the experiment ends at $X_{k:n}$ instead. This censoring scheme builds on Type-I hybrid censoring by allowing the study to extend beyond time T if only a limited number of failures are observed by then. Here, the goal is for the experimenter to observe r failures, but a minimum of k failures is acceptable.

4. **Generalized Type-II HCS**

   At the outset of the experiment, fix $r \in 1, 2, ..., n$ and two time points $T_1, T_2 \in (0, \infty)$ with $T_1 < T_2$. If the r-th failure occurs before T_1, the experiment ends at T_1. If the r-th failure happens between T_1 and T_2, the experiment concludes at $X_{r:n}$. Otherwise, it terminates at T_2. This hybrid censoring scheme adjusts the Type-I scheme by ensuring that the experiment will not extend beyond T_2, which sets an absolute maximum duration for the study.

Progressive Censoring Schemes

Removing units during an experiment can be valuable, especially in studies of wear and ageing, where disassembling items at various stages helps in understanding the ageing process. Intermediate removal is also useful when balancing reduced experiment time with ongoing observations or when some surviving units, particularly those that are rare or costly, can be used in other tests. In some cases, unit loss before the experiment's end is unavoidable, such as with accidental breakage or loss of contact with study participants. These practical needs and challenges lead reliability researchers to explore progressive censoring methods. The main types of progressive censoring schemes include:

1. **Progressive Type-I Censoring (PT1)**
   In the PT1 censoring approach, units are removed from a life test at specified censoring times $T_1 < T_2 < \cdots < T_m$, with the test ending at T_m. Initially, n units are placed on the test, and at each inspection time T_i, a set number of surviving units R_i are randomly removed from the study, for $i = 1, 2, \ldots, m$. The values $R_1, R_2, \ldots, R_m$ can be set as proportions of the units surviving at each time T_i, calculated as $R_i = \lfloor p_i \times \text{number of remaining units at } T_i \rfloor$, with $p_m = 1$ to ensure all remaining units are removed at T_m. Here, $\lfloor a \rfloor$ represents the largest integer less than or equal to a, since the number of remaining units varies at each time point. In a different method, $R_1, R_2, \ldots, R_m$ can be set as fixed integers, provided there are at least R_i units to remove at each T_i. In this case, the actual number of units removed at T_i is $R_i^{\circ} = \min(R_i, \text{remaining units at } T_i)$, with all remaining units removed at T_m. The observed data, $X_1, X_2, \ldots, X_m$, represents the number of units that fail in each interval $[0, T_1), [T_1, T_2), \ldots, [T_{m-1}, T_m)$. For simplicity, we denote the observed censoring scheme $(R_1^{\circ}, R_2^{\circ}, \ldots, R_m^{\circ})$ as $(R_1, R_2, \ldots, R_m)$, with the initial sample size n calculated as $n = \sum_{i=1}^{m}(X_i + R_i)$.

2. **Progressive Type-II Censoring (PT2)**
   A life-testing experiment is set up with n units according to the PT2 censoring. When the first failure in the experiment occurs, R_1 of the remaining $(n-1)$ units are randomly removed. At the time of second failure, R_2 of the remaining $(n - R_2 - 2)$ units are removed randomly. This procedure is repeated till

the predetermined m^{th} failure occurs, at which all remaining units ($R_m = n - R_1 - R_2 - \cdots R_{m-1} - m$) are removed. Note that, $\sum_{i=1}^{m} R_i + m = n$. For $R = (R_1 = R_2 = \cdots = R_m = 0)$, the PT2 censoring method becomes a complete sampling and for $R = (R_1 = R_2 = \cdots R_{m-1} = 0, R_m = n - m)$, it reduces to conventional Type-II censoring method. Here, $R = (R_1, R_2, ..., R_m)$ is predetermined, which is an essential aspect in the construction of the progressively censored experiment.

The conventional order statistics are essentially a subset of the progressively Type-II right-censored order statistics. As a result, any findings derived from the progressively Type-II right-censored order statistics extend to and generalize the respective findings for the conventional order statistics.

1.2.5 Cure Fraction Models

Censoring schemes are crucial for managing incomplete data in survival analysis. However, they do not always address a unique phenomenon often seen in long-term studies: the presence of individuals who are essentially "cured." These individuals, after a certain point, are no longer at risk of experiencing the event being studied, such as a recurrence of a disease or failure of an individual. The concept of a cure fraction is particularly important in fields like cancer research, where some patients achieve long-lasting remission. Cure fraction models expand on traditional survival analysis by taking this cured group into account, allowing for a better understanding of overall survival trends and treatment outcomes.

Let T denote the time to the event of interest, with X as the cure indicator. Here, $X = 1$ signifies that the subject is not cured, while $X = 0$ indicates otherwise. It is assumed that $P(T < \eta \mid X = 1) = 1$, where η represents the maximum survival time for uncured individuals. Typically, η is set to infinity for theoretical simplicity, making $X = 0$ unobservable and ensuring that cured individuals ($X = 0$) are always censored. However, in practical applications, η can be assigned a finite value, allowing $X = 0$ to be observed. For example, in studies on pregnancy and time to abortion, η is often set to 20 weeks since a woman is no longer considered at risk of abortion after reaching 20 weeks of gestation.

Consider $S_c(t)$ and $S_{\bar{c}}(t)$ as the survival functions for cured and uncured individuals in a population, respectively. Specifically, $S_{\bar{c}}(t) = P(T > t \mid X = 1)$ represents the survival function of uncured individuals, while $S_c(t) = P(T > t \mid X = 0)$ corresponds to the survival function of cured individuals, valid for $t < \eta$. By definition, cured individuals have $S_c(t) = P(T > t \mid X = 0) \equiv 1$, making $S_c(t)$ a degenerate survival function. The mixture cure model combines these two components into an overall survival function for T when $t < \eta$, given by:

$$P(T > t) = S(t) = (1-\delta)S_{\bar{c}}(t) + \delta S_c(t) = \delta + (1-\delta)S_{\bar{c}}(t),$$

where $\delta = P(X = 0)$ denotes the probability of being cured.

The mixture cure model consists of two components: δ, known as the incidence part, which describes the probability of being cured or uncured, and $S_{\bar{c}}(t)$, referred to as the latency part, which represents the survival time distribution of uncured individuals. This model permits covariates to influence the cure probability and the survival time of uncured individuals independently. Thereby, it offers clear interpretation of covariate effects and effortless extension of the model to more complex scenarios.

Another technique to model cure fraction is the non-mixture cure model. The non-mixing model, in contrast to the mixture model, facilitates the assumption that the cure fraction is expressed by the survival curve plateauing at a non-zero number. It achieves this by directly embedding the cure fraction into the cumulative hazard or survival function rather than assuming the population is divided into 'cured' and 'uncured' subgroups.

The survival function in the non-mixture cure model is defined as:

$$S(t) = \exp\left[\log(\delta)F_{\bar{c}}(t)\right],$$

where δ is the cure fraction with $0 < \delta < 1$, $F_{\bar{c}}(t) = 1 - S_{\bar{c}}(t)$ is the CDF for the uncured population, and $S_{\bar{c}}(t)$ is the survival function for the uncured population. As $t \to \infty$, $F_{\bar{c}}(t) \to 1$, and consequently, $S(t) \to \delta$, indicating that the survival curve approaches a plateau at the cure fraction δ.

The failure rate function in the non-mixture model is given by:

$$h(t) = -\log(\delta)f_{\bar{c}}(t).$$

The hazard function is directly proportional to the baseline hazard of the uncured population, scaled by $-\log(\delta)$, which reflects the influence of the cure fraction on the overall hazard.

Unlike the mixture cure model, the non-mixture model integrates the cure fraction directly into the survival function via the cumulative hazard, without dividing the population into cured and uncured subgroups. This difference allows the non-mixture model to retain a simpler structure while taking long-term survival behavior.

1.2.6 Model Performance Measures

In this section, we examine different model performance measures used to figure out the goodness-of-fit and choose the best-fitting model for the data. These metrics contains statistical tests and standards that measures the difference between the observed data and the theoretical model. The Kolmogorov-Smirnov test, Cramér-von Mises test, and Anderson-Darling test are frequently used to test the fit of a distribution to the data. In addition, model selection criteria such as the Akaike Information Criterion, Bayesian Information Criterion, and Corrected Akaike's Information Criterion, are taken into account for comparing competing models. Also, the Nikulin-Rao-Robson statistic is included for the validation of models, making sure that the model we chose performs well and correctly represents the underlying data distribution.

Kolmogorov-Smirnov Test

The Kolmogorov-Smirnov (KS) test, introduced by Kolmogorov in 1933 and later refined by Smirnov in 1948, is a non-parametric method for determining whether a random sample $x_1, x_2, \ldots, x_n$ is drawn from a population with a particular CDF. The maximum distance between the empirical distribution, $\hat{G}(x)$, and the specified CDF, $G(x)$ is measured through this test. The empirical distribution function is expressed as:

$$\hat{G}(x) = \frac{1}{n}\sum_{i=1}^{n} I(x_i \leq x),$$

where I is the indicator function that equals 1 if $x_i \leq x$, and 0 otherwise.
The test statistic is given by:

$$D = \sup_x |\hat{G}(x) - G(x)|,$$

where D denotes the maximum absolute difference between the empirical distribution function and the specified CDF.
The null hypothesis and the alternative hypotheses for the KS test are as follows:
H_0: The sample follows the specified distribution, i.e., $G(x)$ is the true CDF.
H_1: The sample does not follow the specified distribution, i.e., $G(x)$ is not the true CDF.

The critical value of D is compared to a standard derived from the sample size n and the significance level α. H_0 is rejected in favor of H_1, if D exceeds the standard. This technique is popular for testing goodness-of-fit and is particularly suitable in comparing distributions in a range of applications.

Cramér-von Mises Test

The objective of Cramér-von Mises (CVM) test is to evaluate whether a sample $x_1, x_2, ..., x_n$ obtained from a population with a specified CDF, $G(x)$. In comparison to the KS test, which concentrates on the maximum distance between the empirical distribution function and the specified CDF, the CVM test takes into account the squared differences between them across all values of x, thereby giving an estimate of the overall discrepancy.
The test statistic is expressed as:

$$W^2 = n \int_{-\infty}^{\infty} [\hat{G}(x) - G(x)]^2 dG(x).$$

For a given sample, the test statistic can be computed as:

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2i-1}{2n} - G(x_i) \right)^2,$$

where $x_1, x_2, ..., x_n$ represents ordered sample values.
The critical values derived from the null distribution of the test statistic is compared to the value of W^2. H_0 is rejected in favor of H_1, if W^2 surpasses the critical value

at a given significance level α. The CVM test is especially sensitive to variations within the distribution, making it a reliable method for evaluating goodness-of-fit in various applications.

Anderson-Darling Test

The Anderson-Darling (AD) test is a modification of the CVM test, offering a goodness-of-fit measure by placing more weight on the tails of the distribution. Because of this characteristic, the AD test is especially sensitive to tail discrepancies, which can be important in areas involving extreme values like risk analysis and reliability.

The test is depends on the squared differences between the empirical distribution function, $\hat{G}(x)$, and the specified CDF, $G(x)$, weighted by the variance of $G(x)$. The test statistic is expressed as:

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log G(x_i) + \log(1 - G(x_{n+1-i}))], \tag{1.2.10}$$

where $x_1, x_2, ..., x_n$ represents the ordered sample values.
Compared the AD statistic A^2 to the critical values obtained from the null distribution of the test statistic. The null hypothesis H_0 is rejected in favor of the alternative H_1, if A^2 exceeds the critical value at a chosen significance level α. The AD test outperforms upon the CVM test by adjusting for the variability of $F(x)$, particularly at the extremes. This modification makes it a powerful tool for identifying deviations in the tails of a distribution, enhancing its applicability in areas like survival analysis, reliability engineering, and financial risk modelling.

Akaike Information Criterion

The Akaike Information Criterion (AIC) is a popular method for selecting the best statistical model among a set of candidates. AIC was introduced by Hirotugu Akaike in 1973, uses the idea of information entropy and quantifies this trade-off mathematically.
The AIC is given by

$$\text{AIC} = -2\ln(\mathcal{L}) + 2r, \tag{1.2.11}$$

where $\mathcal{L}$ and r are represents the maximum value of likelihood and the number of parameters respectively. When comparing multiple models, the one with the lowest

AIC is taken as the most appropriate.It's crucial to remember that AIC values are relative, and comparisons are valid only among models fit to the same dataset.

Bayesian Information Criterion

The Bayesian Information Criterion (BIC) also called, the Schwarz Criterion, a tool for choosing the best statistical model by considering model fit and its complex nature. This criterian does not perform well with models with higher parameters or higher sample sizes.
The BIC can be computed by

$$\text{BIC} = -2\ln(\mathcal{L}) + r\ln(n), \tag{1.2.12}$$

where $\mathcal{L}$ is the maximum likelihood value, r is the number of model parameters, and n is the sample size.
The first part in Eq. 1.2.12 quantifies how well the model fits the data and the second part reduces the complexity. A better model is indicated by a lower BIC value.

Corrected Akaike's Information Criterion

The Corrected Akaike Information Criterion (AICC) is introduced as an extension of the AIC that accounts for small sample sizes. While AIC is used in selecting models effectively, it may lead to biased results when the sample size (n) is small relative to the number of parameters (r).
The AICC introduces a correction term to handle this bias and is computed by

$$\text{AICC} = \text{AIC} + \frac{2r(r+1)}{n-r-1}.$$

In small datasets, the modification guarantees that models with more parameters are properly penalized. The AICC converges to the AIC as n increases. When choosing a model for small or moderate datasets, it is especially reliable as it provides a better predictive accuracy.

Validation

It is always important to evaluate how well a statistical model fits observed data in order to confirm that the model properly represents the underlying distribution.

The Nikulin-Rao-Robson (NRR) goodness-of-fit test is used for evaluating whether a given distribution describes the data. This method is an extension of Pearson's classical chi-square test by accommodating both complete and censored data with small sample sizes. Nikulin (1973) and Rao & Robson (1974) independently proposed a statistic that is now collectively known as the NRR statistic. Being a refined version of the Pearson statistic, the NRR statistic also follows chi-square distribution. This method has been applied to fit different models, such as the power-generalized Weibull by Voinov et al. (2013), the unit modified Burr-III by Haq et al. (2020), the unit generalized inverse Weibull by Dey et al. (2021), and the Xgamma-exponential by Yadav et al. (2022), among others.

Nikulin-Rao-Robson Test Statistic

Let $X_1, X_2, ..., X_n$ be a sample of size n from a parametric family with CDF $G(x)$. Suppose the null hypothesis H_0 to be tested is given by

$$H_0 : P\{X_i \leq x\} = G(x, \boldsymbol{\theta}), \quad x \in \mathbb{R}, \qquad \boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_r)^T,$$

where $\boldsymbol{\theta}$ represents the unknown model parameters. The Y^2 statistic, commonly referred to as the NRR statistic introduced by Nikulin (1973) and Rao & Robson (1974), is defined as follows.

Let the random samples $X_1, X_2, ..., X_n$ are grouped into k mutually disjoint subintervals $\boldsymbol{I}_1, \boldsymbol{I}_2, ..., \boldsymbol{I}_k$, where $\boldsymbol{I}_i =]a_{i-1}, a_i]$, where $i = \overline{1; k}$.

The boundaries, a_i, of the intervals $\boldsymbol{I}_i$ are evaluated to satisfy,

$$p_i(\boldsymbol{\theta}) = \int_{a_{i-1}}^{a_i} g(x, \boldsymbol{\theta}) dx; \quad i = 1, 2, ..., k.$$

Therefore, $a_i = G^{-1}(\frac{i}{k});\ i = 1, 2, ..., k-1$.

Let $\nu_i = (\nu_1, \nu_2, ..., \nu_k)^T$ represents the frequency vector resulting from organizing the data into the $\boldsymbol{I}_i$ intervals, thus,

$$\nu_i = \sum_{j=1}^{n} 1_{\{x_j \in \boldsymbol{I}_i\}}; \quad i = 1, 2, ..., k.$$

Now the NRR statistic is expressed as,

$$Y_n^2(\hat{\boldsymbol{\theta}}) = \chi^2(\hat{\boldsymbol{\theta}}) + \frac{1}{n}\boldsymbol{H}^T(\hat{\boldsymbol{\theta}})\left(\boldsymbol{I}(\hat{\boldsymbol{\theta}}) - \boldsymbol{L}(\hat{\boldsymbol{\theta}})\right)^{-1}\boldsymbol{H}(\hat{\boldsymbol{\theta}}),$$

where $\chi^2(\hat{\boldsymbol{\theta}}) = \left(\frac{\nu_1 - np_1(\boldsymbol{\theta})}{\sqrt{np_1(\boldsymbol{\theta})}}, \frac{\nu_2 - np_2(\boldsymbol{\theta})}{\sqrt{np_2(\boldsymbol{\theta})}}, ..., \frac{\nu_k - np_k(\boldsymbol{\theta})}{\sqrt{np_k(\boldsymbol{\theta})}}\right)^T$, and $L(\boldsymbol{\theta})$ refers the information matrix for the data that has been grouped as described by,

$$\boldsymbol{L}(\boldsymbol{\theta}) = D(\boldsymbol{\theta})^T D(\boldsymbol{\theta}),$$

with $D(\boldsymbol{\theta}) = \left[\frac{1}{\sqrt{p_j}}\frac{\partial p_j(\boldsymbol{\theta})}{\partial \mu}\right]_{k\times r}$; $j = 1, 2, ..., k$ and $s = 1, 2, ..., r$, then $H(\boldsymbol{\theta}) = (H_1(\boldsymbol{\theta}), H_2(\boldsymbol{\theta}), ..., H_r(\boldsymbol{\theta}))^T$ with $H_s(\boldsymbol{\theta}) = \sum_{j=1}^{k}\frac{\nu_j}{p_j}\frac{\partial p_j(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_s}$.

In this context, $\hat{\boldsymbol{\theta}}$ refers to the ML estimates of the parameter vector, while $\boldsymbol{I}(\hat{\boldsymbol{\theta}})$ represents the corresponding estimated Fisher information matrix. Moreover, the Y^2 statistic is distributed according to a chi-square distribution χ^2_{k-1} with $(k-1)$ degrees of freedom.

1.3 Review of Literature

This section provides an overview of existing research significant to establish the foundation for the current study. The main objective of this research review is to identify the research gaps by analysing prior work on lifetime distributions, censoring and statistical inference methods.

1.3.1 DUS Transformation

Different transformation methods have been introduced in statistical literature for better modelling purposes. These transformations also increases the applicability of existing distributions by improving its flexibility. One such transformation is the DUS transformation introduced by Kumar et al. (2015) which gives parsimonious distribution in terms of computation and interpretation. Since it doesn't add a new parameter, it is clearly a transformation and not a generalization. The DUS transformation has gained wide appreciation in different areas, including censoring and neutrosophic statistics, due to its adaptability and effectiveness in various contexts. According to Kumar et al. (2015), if $f(x)$ and $F(x)$ be the PDF and CDF of some baseline distribution, then the PDF and CDF of DUS transformed distribution are $g(x) = \frac{1}{e-1}f(x)e^{F(x)}$ and $G(x) = \frac{1}{e-1}[e^{F(x)} - 1]$, respectively. They considered exponential distribution as the baseline distribution and the resultant gives an IFR distribution even the exponential distribution has only a constant failure rate. The failure rate of the DUS-exponential distribution is, $h(x) = \theta e^{-\theta x}\left[e^{e^{-\theta x}} - 1\right]^{-1}$, $x > 0$, $\theta > 0$. Maurya et al. (2017) considered Lindley distribution as the baseline distribution and named it exponential transformed Lindley distribution. Its failure rate is

$$h(x) = \frac{\theta^2(1+x)e^{-\theta x}}{(\theta+1)\left(e^{\left[e^{\theta x}\frac{1+\theta+\theta x}{1+\theta}\right]} - 1\right)},\ x > 0,\ \theta > 0.$$

The distribution has IFR for all the values of θ.

The DUS transformation of the Lomax distribution, referred to as the DUS-Lomax distribution, was introduced by Deepthi & Chacko (2020). Its failure rate is given by

$$h(x) = \frac{\alpha\beta(1+\beta x)^{-(\alpha+1)}}{e^{(1+\beta x)^{-\alpha}} - 1},\ x > 0,\ \alpha, \beta > 0.$$

The DUS-Lomax distribution exhibits both DFR and UBFR properties, despite the original Lomax distribution belonging to the DFR family.

A generalized lifetime model has been proposed by Kavya & Manoharan (2020) using generalized DUS transformation with Weibull distribution as the baseline distribution (GDUS-Weibull distribution). The Weibull distribution has only monotone failure rates, but GDUS-Weibull distribution has IFR, DFR and UBFR shape properties.

The versatility of the DUS transformation is evident in various studies. Tripathi et al. (2021) explored Bayesian parameter estimation of the DUS-exponential distribution in the context of upper record values. Anakha & Chacko (2021) demonstrated that the DUS transformation enhances the flexibility of the K distribution when modelling real-world data. Similarly, Gauthami & Chacko (2021) introduced the DUS-inverse Weibull distribution, which also displays DFR and UBFR properties.

The Power Generalized DUS (PGDUS) transformation was introduced by Thomas & Chacko (2021) in order to extend the idea for modelling parallel systems in which the components follow DUS-transformed lifetimes. Thereafter, by make use of the PGDUS transformation, Thomas & Chacko (2023) modeled parallel systems using Weibull and Lomax as the baseline distributions. Similarly, Amrutha & Chacko (2024) introduced the PGDUS inverse Kumaraswamy distribution, resulting in a novel lifetime model with wider applicability.

1.3.2 Copula

According to the Sklar's theorem, provided by Sklar (1959), any multivariate joint distribution can be described by its marginal distributions and a copula function that represents their dependency structure. The Sklar's theorem considers as the foundational result in copula theory. By popularizing the idea of the copula concept, Nelson (2006) made significant contributions and provided an elaborate framework to the copula theory.

Farlie (1960), Gumbel (1960), and Morgenstern (1956) proposed the FGM copula, which represents a prominent parametric family in copula theory. It is specifically known for modelling weak dependencies in bivariate distributions. The simple and closed-form representation of the FGM copula make it suitable for both theoretical studies and real-world applications. Balakrishnan and Lai (2009) provided an immense study on copula's properties, generalizations, and various applications. Also, they highlight the adaptability of the model in fields such as survival analysis, reliability, and risk management, where variable dependencies are often moderate.

Initial significant contributions include those by Huang & Kotz (1984, 1999) and Sarmanov (1966), who proposed modifications to expand the copula's application over more extensive dependence structures. The FGM copula known for its beneficiary in depicting a wide range of dependency agreements, ranging from complete independence to moderate dependence. As Joe (1997) points out, its simple mathematical form makes it computationally efficient and easy to implement in real-world applications.

Copula techniques provide a flexible structure for creating a range of bivariate lifetime distributions, making them appropriate for handling different data types and analyzing two related lifetimes in an individual observation. This is especially important when examining the time between repeated hospitalizations for a specific disease or when studying paired organs, such as kidneys or eyes (see Rinne (2008); Bhattacharjee & Misra (2016)).

Achcar et al. (2015) introduced a bivariate model of generalized exponential distribution using FGM copula. Vaidyanathan et al. (2016) proposed a bivariate Lindley distribution through the application of the Morgenstern method and examined some of its properties. Abd Elaal & Jarwan (2017) explored bivariate generalized exponential distributions developed from FGM and Plackett copulas. They studied and highlighted the practical usage of these distributions by applying them to real-world data, highlighting their effectiveness in modelling dependence structures. The statistical properties of a bivariate Dagum distribution obtained using copulas were studied by Popović et al. (2018). Samanthi & Sepanski (2019) introduced a new bivariate extension of beta-generated distributions by making use of Archimedean copulas. They highlighted how this model can appropriately capture dependencies in financial data and enhance risk assessment.

A likelihood-based estimation method for the common mean vector in bivariate meta-analysis under the FGM copula was proposed by Shih et al. (2019), who also provided confidence interval approaches and theoretical results. The study emphasizes the benefits of the FGM model in comparison to the bivariate normal model, underscoring its capacity for use in sensitivity analyses and future survival data meta-analysis. Many researchers created a bivariate model using the FGM copula following which they examined dependence in stress-strength models. See also, Domma et al. (2013), Almetwally et al. (2020), James & Chandra (2022), James et al. (2022), El-Sherpieny et al. (2022), Abulebda et al. (2022), Abulebda et al. (2023). Recently, Chesneau (2024) introduced a novel FGM-type power copula with a unique one-parameter formulation, enabling flexible modelling of negative dependence. The study explored its mathematical properties, dependence measures, data generation, and derived a new bivariate normal distribution, advancing copula theory and its applications.

1.3.3 Censoring Schemes

The conventional censoring schemes, Type-I and Type-II, exhibit distinct features. The Type-I censoring scheme involves running the experiment until a fixed time point T is reached, while the Type-II censoring scheme allows the experiment to continue until the occurrence of the m-th failure. These schemes lack complete event time information for censored observations. This loss of information can limit the precision and accuracy of statistical analysis, potentially leading to biased estimates or reduced statistical power.

To address the drawback, Epstein(1954) introduced a mixture of Type-I and Type-II censoring scheme, known as hybrid censoring scheme. This scheme also referred to as Type-I hybrid censoring scheme. Childs et al.(2003) introduced a new censoring scheme known as Type-II hybrid censoring scheme. However, in recent years, hybrid censoring scheme has grown increasingly prevalent in reliability and life-testing investigations, these schemes lack the flexibility to allow units to be removed from the experiment before it reaches its termination.

Later, Cohen (1963) introduced the progressive censoring scheme. Progressive censoring allows for the removal of experimental objects at various stages to maintain the total cost and time associated with the experiment. The concept of PT2 censoring was first introduced by Herd (1956) in his doctoral thesis and has been discussed by various authors (see Balakrishnan & Aggarwala (2000) and Balakrishnan & Cramer (2014)). Subsequently, Gajjar & Khatri (1969) considered PT1 right-censoring, where the population parameters changed with each removal. Despite the efficiency benefits, Burkschat (2008) points out that progressive censoring has a longer test time than the conventional Type-II censoring scheme. To address this drawback, Kundu & Joarder (2006) introduced the Type-II progressively hybrid (PT2H) censoring scheme with a fixed experimental time.

Thereafter Ng et al. (2009) introduced a censoring scheme called adaptive Type-II progressive (APT2) censoring scheme (also referred to as Ng-Kundu-Chan model). Cramer & Iliopoulos (2010) presented a highly general description of APT2 censoring using a conditional distribution design, enabling a general examination of such models. Moreover, Cramer & Iliopoulos (2015) broadened the idea of progressive censoring in their paper, creating a flexible model that works with various inspection times and unit removals. This model includes adaptive progressive Type-I and Type-II censoring schemes with either random or fixed experimental times. A modified APT2 censoring scheme, a Type-I censored variant of the Ng-Kundu-Chan model proposed by Yan et al. (2021), concluded life tests at a second threshold $T^* > T$. Balakrishnan et al. (2023) describe this model as a hybrid censored version, which improves flexibility in modelling the data dynamics. An algorithm and simulation results were presented by Schmiedt & Cramer (2024) in the (generalized) Ng-Kundu-Chan model, providing new insights that extend to the original Ng-Kundu-Chan model. These findings have important relevance to the broader literature.

The objective of comparative studies is frequently to assess and compare reliability or performance of various populations or product lines. Joint censoring techniques are commonly utilized, enabling simulataneous analysis across multiple groups. Recently, there has been a lot of interest in using these schemes to evaluate the durability of products from different production sources, especially when using jointly censored testing schemes. Bhattacharyya & Mehrotra (1981) were the first to describe the challenge of evaluating the relative merits of products from different product lines under such schemes. A comprehensive review of these advancements, covering both parametric and nonparametric approaches within the framework of the joint Type-II censoring scheme, was posited by Bhattacharyya (1995). Subsequently, under joint Type-II censoring, Balakrishnan & Rasouli (2008) developed exact inferential methods for comparing two exponential populations. A combined progressive Type-II censoring strategy was presented by Rasouli & Balakrishnan (2010) and implemented on two exponential samples.

Mondal & Kundu (2018) presented the Balanced Joint Type-II Progressive Censoring (BJPT) scheme and suggested likelihood-based inference for two exponential populations. As it allows clearer articulation of estimator properties and offers certain advantages over the Joint Progressive Censoring (JPC) scheme, this scheme is systematically easier to handle than the JPC scheme. In order to apply it to two Weibull populations Mondal & Kundu (2020a) expanded their findings. Based on these developments, Sultana et al. (2021) proposed a new joint adaptive Type-II progressive censoring (JAPT) scheme for independent samples from two populations, assuming exponential lifetimes. They evaluated Bayesian inference under a Beta-Gamma prior, calculated estimated confidence intervals, and developed the maximum likelihood estimators (MLEs) and their exact distributions. Additionally, Goel & Krishna (2022) constructed statistical inference methods for the BJPT scheme and implemented to two Lindley populations.

Sultana et al. (2023) further examined the JAPT censoring scheme for two weibull populations to reduce experimental time and cost. They applied both frequentist and Bayesian inference methods, including confidence and credible intervals, and validated the technique through simulations and real-world data, relating it to BJPT censoring scheme by Sultana et al. (2021). A statistical inference for comparative generalized inverted exponential populations within a set computing time was proposed by Almuhayfith (2024). A joint adaptive Type-II Hybrid Progressive Censoring Scheme (JAPTH) was employed to balance the total test time with the number of observed failures.

1.3.4 Cure Fraction Modelling

Cure rate models are survival models that comprises two groups within a population: those who have been 'cured' and those who are still at risk. These models have gained lots of popularity in the analysis of lifetime data for cancer research in clinical trials. Boag (1949) was the first to introduce the concept of estimation of cure fraction and later it was advanced by Berkson & Gage (1952). This method, called mixture cure model, is also commonly called standard cure rate model and it is a popular method in survival analysis for modelling populations with a cured fraction. This approach have certain drawbacks, though, especially when covariates are taken into account. Yakovlev et al. (1993) proposed an alternative model called the bounded cumulative hazard (BCH) model in order to address these drawbacks. This BCH model allows better estimation of survival probabilities. An elaborated study on the BCH model, its theoretical foundations, and benefits can be found in Chapter 5 of the book *Bayesian Survival Analysis* by Ibrahim et al. (2001). By applying the mixture cure rate model to grouped survival data Yu et al. (2004) observed that the cure fraction estimate is dependent on the latency distribution and follow-up time. They discovered the generalized gamma distribution provide a stable estimate. They pointed out that short follow-up times might make it difficult to estimate the survival functions and suggested longer follow-ups to improve accuracy. In the analysis of electric motor insulation, Nelson (2005) applied the cure rate model in reliability studies. He observed that motors operated at low temperatures lasted indefinitely, while those at higher temperatures failed quickly, and used the mixture model to capture the immune components in this context.

The Cox proportional hazards model, introduced by Cox in (1972), is one of the most widely used methods in survival analysis. This semi-parametric regression model assesses the relationship between survival time and covariates, assuming that every individual in the study is at risk of the event. However, this assumption is often unsuitable for modern cancer trials, where many patients are cured after sufficient follow-up. Lambert et al. (2007) pointed out that standard Cox models often overlook long-term survivors in such studies. They recommend using parametric models with a cure fraction, as these models provide better estimates of survival proportions and mean survival time, offering more clinically meaningful interpretations.

Kannan et al. (2010) studied cure rate models with covariates using the generalized exponential distribution, demonstrating an excellent data fit and enabling the testing of covariate effects on immune probability. Martinez et al. (2013) analyzed both mixture and non-mixture cure fraction models using the generalized modified Weibull distribution for gastric cancer data. They concluded that cure fraction models, particularly when long-term survivors are involved, are better suited for lifetime data. In addition, they proposed a non-mixture cure model as an alternative to the standard Cox model for such data. Similarly, Yusuf & Baker (2016) compared mixture and non-mixture cure models, emphasizing the importance of cure fraction models in survival analysis, particularly with the Weibull distribution, and highlighted the advantages of Bayesian methodologies for parameter estimation. Further advancing this research, Looha et al. (2018) examined prognostic factors in the long-term survival of colorectal cancer patients, using a Weibull-based non-mixture cure model. Their work demonstrated that this model provides optimal survival data estimation for both male and female patients. Similarly, by classifyiing the population into immune and susceptible groups, Gharaaghaji et al. (2021) used mixture cure models to cervical cancer survival data and observed their effectiveness in analyzing both short-term and long-term survival. Recently, Martinez et al. (2022) extended the exploration of cure fraction models for survival data by employing both frequentist and Bayesian methods to estimate parameters of the exponentiated Weibull distribution. Their study confirmed the importance of the model in survival analysis by highlighting its versatility and ability to handle different hazard function shapes.

Wang et al.(2024) emphasized the challenge of defining treatment effects in cure models due to censoring, making it more difficult to observe cure status. To address this, they proposed two causal estimands-the timewise risk difference and mean survival time difference-based on the always-uncured group. These estimands help study treatment effects on failure times in this group, alongside cure rates. A detailed overview of the mixture cure model in survival analysis is provided by Maller et al. (2024), highlighting both significant recent advancements and earlier findings.

They discuss key aspects such as testing for long-term survivors in the absence of cures, estimating the probability of an individual being cured, and analyzing the importance of sufficient follow-up. The study emphasizes the role of extreme value methods and identifies several challenging open problems that require further exploration in this field.

1.4 Objectives

The primary objectives of this study are outlined as follows:

- To study the various failure rate distributions and its applications for modelling lifetime data.
- To propose new failure rate models and compare existing various failure rate models.
- To enhance the applications of various failure rate distributions in system engineering, biostatistics and other scientific areas.
- To investigate existing censoring schemes and propose better scheme than the existing schemes.
- Apply cure fraction modelling to address the presence of long-term survivors in medical studies.

1.5 Outline of the Thesis

The thesis is structured into ten chapters, outlined as follows: *Chapter 1* introduces the fundamental concepts and definitions that form the foundation of this thesis. Moreover, it presents an in-depth literature review. The chapter includes a thorough examination of ageing concepts, failure rate distributions, copula function, censoring methods, and cure fraction modelling, which were integral to achieving the objectives of this research.

Chapter 2 presents a new distribution developed using the DUS transformation with the Kumaraswamy distribution as the baseline. This distribution is flexible, capturing both monotonic and non-monotonic failure rate behaviors, making it useful for various reliability and lifetime modelling applications. Parameters are estimated using the ML method, and these estimates are applied in reliability analysis using the stress-strength model to evaluate system resilience under stress. A simulation study examines the performance of the ML method under different sample sizes and parameter values. Finally, a real data application shows that the DUS-K distribution provides a better fit than existing models.

Chapter 3 introduces a novel non-monotonic failure rate distribution developed using the DUS transformation with the inverse Kumaraswamy distribution as its base. The statistical properties of the distribution are elaborated. Characterization properties of the proposed model based on truncated moments, hazard rate, reversed hazard rate, and conditional expectation are examined. Using this distribution, the reliability of the single-component stress-strength model is computed. Parameter estimation is coducted using the ML method and its efficiency examined in terms of mean squared errors and biases. In order to demonstrate the distribution's ability to fit real-world data two real datasets are used. The NRR statistic provides additional support for the validation of real data applications and emphasizing the usefulness of distribution in different scenarios.

In *Chapter 4*, a new distribution named Bivariate Farlie-Gumbel-Morgenstern Inverse Weibull model designed for reliability analysis and dependency measures is introduced. By combining the FGM copula with inverse Weibull marginal distributions, the model can be used to evaluate the stress-strength reliability and also, making it appropriate for modelling data with weak correlations. The chapter examines the statistical properties of the model, including conditional distributions, the moment generating function, positive quadrant dependence, and reliability measures. To estimate the dependence parameter and reliability, parameter estimation is conducted via both ML and inference function margin methods. Lastly, to demonstrate the practical application, the model is applied to a real dataset on kidney patient recurrence times.

In *Chapter 5*, different censoring methods used in life-testing experiments, focusing on PT2 censoring and its modifications: PT2H and APT2 censoring are examined. The PT2H and APT2 methods address the concerns of long experiment durations in PT2 censoring. As a case study, the chapter make use of the EP distribution, which is suitable for modelling data with increasing and bathtub-shaped failure rate behaviours. In comparison to the popularly used Weibull distribution, the EP distribution appears to be a useful alternative in certain reliability-related decision-making and cost analysis. Through a simulation study, the effectiveness of parameter estimation using ML and Bayesian methods is evaluated. Conclusively, the chapter compares the performance of PT2, PT2H, and APT2 censoring methods using a real-world data.

A statistical inference for two independent samples modelled by the generalized Lindley distribution in a life-testing experiment using the JAPT censoring scheme for comparing lifetimes is discussed in *Chapter 6*. To minimize the experimental time and costs, the JAPT scheme has been introduced. Point estimation is done using ML and Bayesian methods, while interval estimation is performed using asymptotic confidence intervals, Highest posterior density credible intervals, and bootstrap confidence intervals. Bayes estimates are obtained using the Markov Chain Monte Carlo (MCMC) method, along with the importance sampling algorithm, and the corresponding Bayes credible intervals are derived. A real dataset is analyzed to show how the methods work in practice, and the performance of the estimators is evaluated through a MCMC simulation study.

Chapter 7 proposes a novel T1-T2 mixture censoring scheme and explores its statistical inference based on the Weibull distribution. The chapter highlights the advantages of the new scheme, which ensures a maximum number of failures within a given supplementary time. It provides computational formulas for the expected number of failures and the expected failure time. The Fisher information for the T1-T2 mixture censoring scheme is derived to support parameter estimation. The effectiveness of ML and Bayesian methods, using squared error and LINEX loss functions, is demonstrated through simulations. A real dataset is used to illustrate the practical application of the T1-T2 mixture censoring scheme, showcasing its functionality in a real-world context.

Chapter 8 explores cure fraction models in survival data analysis, focusing on cases where some individuals are not at risk of the event. It uses frequentist and Bayesian methods to estimate the parameters of the EW distribution, an extension of the Weibull distribution with an extra shape parameter. Three models are considered: the mixture cure fraction, non-mixture cure fraction, and cure rate proportional odds models. Bayesian inference is performed using MCMC methods, and the performance of ML estimators is evaluated through simulations. The model's practicality is demonstrated using colorectal cancer survival data from the Malabar Cancer Centre, Thalassery. The EW distribution's flexibility in modelling different hazard shapes makes it a strong choice for survival data with a cure fraction.

Chapter 9 summarizes the key findings of the thesis, highlighting the development of new models, innovative censoring schemes, and other significant research contributions. *Chapter 10* offers practical recommendations for implementing the proposed models and methods in real-world applications.

CHAPTER 2

DUS Transformation of Kumaraswamy Distribution

2.1 Introduction

The study of lifetime distributions plays an important role in many areas of science and technology. Choosing the right distribution is key to understanding data and drawing meaningful conclusions. This makes it essential to explore and develop models that fit the data more accurately.

The motivation to create more flexible and reliable distributions comes from the need to better understand the complexities of real-world data. Improving how we model and study data helps us gain clearer insights and make better decisions. This is especially important in fields like healthcare, survival analysis, and reliability engineering, where the results can greatly affect people's lives. Transformations are a powerful tool in creating new probability distributions since they allow us to adjust existing models to better fit complex data with or without the need for adding extra parameters. By applying transformations, we can modify the shape, skewness, or tails of a distribution to better represent real-world data, whether it's data with extreme values, uneven patterns, or other complexities. This approach helps overcome the limitations of standard distributions, making models more flexible and accurate.

There are many distributions in the reliability literature to model lifetime data. Among the various lifetime distributions, the Beta distribution received considerable attention and has been used extensively for over one hundred years. Beta distribution is resilience to model data sets which are confined to bounded intervals.

Kumaraswamy (1980) introduced Kumaraswamy (K) distribution, having some similar properties to Beta distribution namely; the boundary behavior, density behaviors, etc. The density has unimodal, uni-antimodal, increasing, decreasing or constant behaviors with respect to its parameter values. K distribution has several benefits as compared to Beta distribution in the matter of tractability and has a closed form for its CDF and quantile function. Recently K distribution has observed one of the prominent distributions for its comprehensive applicability including modelling data of biomedical, reliability engineering, finance, and hydrology.

In the field of reliability analysis, Nadarajah (2008) pointed out that K distribution is considered as a well substitute to the Beta distribution hence it received great attention especially in the hydrological literature. Lemonte (2011) considered a bias correction procedure developed on the parametric bootstrap for the K distribution. A statistical study based on record values by K distribution is provided in Nadar et al. (2013). Mitnik & Baek (2013) proposed a new model which is based on the re-parameterized K distribution. Lemonte et al. (2013) introduced generalization of K distribution and named it as exponentiated Kumaraswamy distribution. Mitnik (2013) proved that the variables of K distribution show closeness under exponentiation and linear transformation, and derived the structural properties of the general form of the distribution. Gosh & Hamedani (2017) introduced a new generalization, Gamma-Kumaraswamy distribution. Exponentiated generalized Kumaraswamy distribution is proposed by Elgarhy et al. (2018). Kizilaslan & Nadar (2018) and Kohansal (2019) have worked on the estimation of stress-strength reliability of K distribution.

In this chapter, a new distribution based on DUS transformation method with Kumaraswamy as the baseline distribution is studied. DUS transformation gives a parsimonious distribution concerning computation and interpretation as it involves no new parameters other than parameters involved in the baseline distribution (Kumar (2015)). The DUS transformation of K distribution which provides a better fit and gives more accurate estimates when modelling bathtub shaped failure rate data. The remaining sections arranged as follows. In Section 2.2, the PDF, CDF, and failure rate function of the DUS transformation of the K distribution are provided. Shapes of PDF and failure rate function are discussed in Section 2.3. Statistical properties including moments, moment generating function, characteristic function, cumulant generating function, quantile function and entropy of DUS-K distribution are discussed in Section 2.4. Section 2.5 discussed the distribution of extreme order statistics. Estimation of the parameters using ML method and the asymptotic Normality of MLEs are discussed in Section 2.6. Single component and multicomponent stress-strength reliability estimations are discussed in Section 2.7. In Section 2.8, a simulation study is performed to validate the MLEs. In section 2.9, a real data set is used to demonstrate the adequacy of the proposed distribution and final conclusions are given in Section 2.10.

2.2 DUS-Kumaraswamy Distribution

Suppose X be a random variable with a baseline distribution, then the PDF, $g(x)$ and CDF, $G(x)$ of DUS transformation of the distribution proposed by Kumar (2015) are,

$$g(x) = \frac{1}{e-1} f(x) e^{F(x)} \tag{2.2.1}$$

and

$$G(x) = \frac{1}{e-1} [e^{F(x)} - 1], \tag{2.2.2}$$

respectively, where $f(x)$ and $F(x)$ are PDF and CDF of the baseline distribution. The failure rate function $h(x)$ of DUS transformation of the distribution is given by,

$$h(x) = \frac{1}{e - e^{F(x)}} f(x) e^{F(x)}. \tag{2.2.3}$$

It is clearly a transformation not a generalization. For $0 < x < 1$, consider the DUS transformation of two parameter K distribution with PDF

$$f(x) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, \tag{2.2.4}$$

CDF

$$F(x) = 1 - (1-x^{\alpha})^{\beta}, \tag{2.2.5}$$

and failure rate function

$$h(x) = \frac{\alpha\beta x^{\alpha-1}}{1-x^{\alpha}}, \tag{2.2.6}$$

respectively, where $\alpha > 0$ and $\beta > 0$. Using (2.2.4), (2.2.5) & (2.2.6) in (2.2.1), (2.2.2) & (2.2.3), the PDF of DUS-Kumaraswamy distribution (DUS-K(α, β)) is obtained as

$$g(x) = \frac{1}{e-1}\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}e^{1-(1-x^{\alpha})^{\beta}};\ 0 < x < 1,\ \alpha > 0,\ \beta > 0. \tag{2.2.7}$$

The CDF and failure rate of DUS-K(α, β) distribution are

$$G(x) = \frac{1}{e-1}[e^{1-(1-x^{\alpha})^{\beta}} - 1];\ 0 < x < 1, \tag{2.2.8}$$

and

$$h(x) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}[e^{(1-x^{\alpha})^{\beta}} - 1]^{-1};\ 0 < x < 1, \tag{2.2.9}$$

respectively, where $0 < x < 1$, $\alpha > 0$, $\beta > 0$.

2.3 Shapes of PDF and failure rate function

It can see that the density function of DUS-K(α, β) has the shape properties, specifically;

$\alpha < 1, \beta \geq 1 \Rightarrow g(x)$ is decreasing ; $\alpha = 1, \beta > 1 \Rightarrow g(x)$ is decreasing ;
$\alpha < 1, \beta < 1 \Rightarrow g(x)$ is uniantimodal; $\alpha \geq 1, \beta \leq 1 \Rightarrow g(x)$ is increasing;
$\alpha > 1, \beta > 1 \Rightarrow g(x)$ is unimodal.

Mode of the distribution can be found as a solution of the equation, $\frac{d}{dx}\log g(x) = 0$. By substituting the PDF in the given equation, we get

$$\begin{aligned}\frac{d}{dx}\log g(x) = \frac{d}{dx}\log\left\{\frac{1}{e-1}\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}e^{1-(1-x^{\alpha})^{\beta}}\right\} = 0\\ \Rightarrow \frac{\alpha-1}{x} - \frac{\alpha(\beta-1)x^{\alpha-1}}{1-x^{\alpha}} + \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1} = 0.\end{aligned}$$

The equation mentioned above is challenging to solve analytically; however, it can be solved numerically by using fixed-point method or bisection method. The PDF of the DUS-K(α, β) distribution for various parameter values is illustrated in Figures

2.1 and 2.2, respectively. Additionally, Figure 2.3 depicts the failure rate function of the distribution.

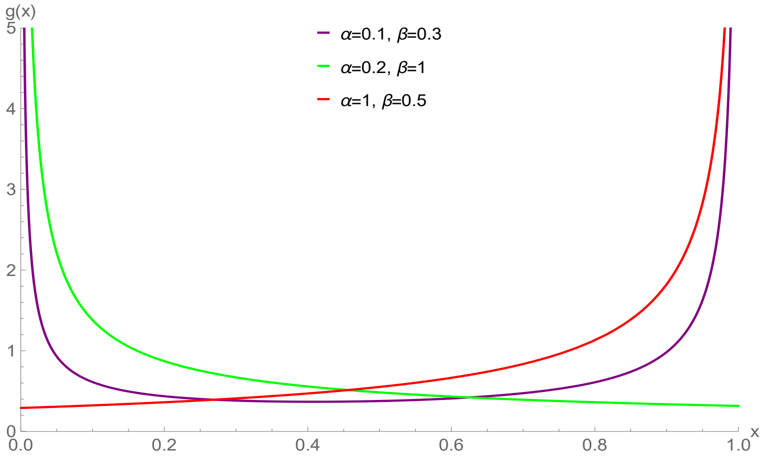


Figure 2.1: PDF of DUS-K(α, β) for (0.1, 0.3), (0.2, 1), and (1, 0.5).

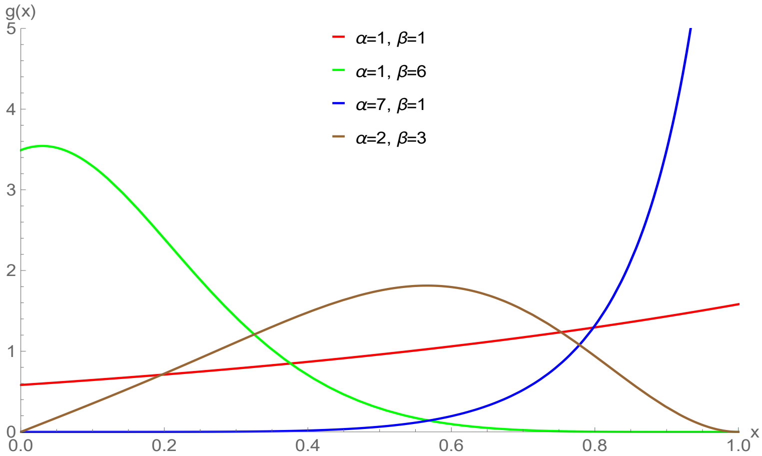


Figure 2.2: PDF of DUS-K(α, β) for (1, 1), (1, 6), (7, 1), and (2, 3).

In Figure 2.3, the graph gives failure rate function of DUS-K(α,β) distribution for the parameter values (0.4, 0.4), (1, 0.5), (0.7, 1) and (5, 4). Failure rate function $h(x)$ has both monotonic and nonmonotonic behaviors. It is monotonically increasing for $\alpha \geq 1$ and bathtub shape for $\alpha < 1$.

2.4 Statistical Properties

In this section we present some statistical properties of the two parameter DUS-K distribution.

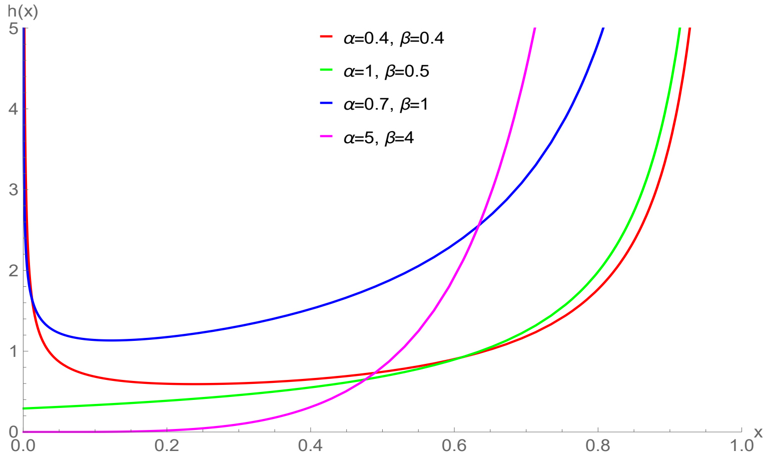


Figure 2.3: Failure rate function of DUS-K(α, β)

2.4.1 Moments

For $r \in N$, the r^{th} raw moment μ_r' is,

$$\begin{aligned}
\mu_r' &= E(X^r)\\
&= \int_0^1 x^r \frac{1}{e-1}\alpha\beta x^{\alpha-1}(1-x^\alpha)^{\beta-1}e^{1-(1-x^\alpha)^\beta}dx\\
&= \frac{e\alpha\beta}{e-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_0^1 x^{r+\alpha-1}(1-x^\alpha)^{\beta+\beta m-1}dx\\
&= \frac{e\beta}{e-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}B\big(\frac{r}{\alpha}+1,\beta+\beta m\big).
\end{aligned}$$

where B(.,.) denotes the Beta function.

From the r^{th} raw moment we have the mean and variance say, $E(X)$ and $Var(X)$ as

$$E(X) = \frac{e\beta}{e-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}B\left(\frac{1}{\alpha}+1,\beta+\beta m\right) \tag{2.4.1}$$

and

$$Var(X) = \frac{e\beta}{e-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}B\left(\frac{2}{\alpha}+1,\beta+\beta m\right)-\left(\frac{e\beta}{e-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}B\left(\frac{1}{\alpha}+1,\beta+\beta m\right)\right)^2. \tag{2.4.2}$$

2.4.2 Moment Generating Function

If $X \sim$DUS-K(α, β) then the moment generating function $M_X(t)$ is,

$$M_X(t) = E(e^{tX}) = \int_0^1 e^{tx}\frac{1}{e-1}\alpha\beta x^{\alpha-1}(1-x^\alpha)^{\beta-1}e^{1-(1-x^\alpha)^\beta}dx$$

$$= \frac{e\alpha\beta}{e-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n t^m}{m!n!}\int_0^1 x^{\alpha+m-1}(1-x^{\alpha})^{\beta n+\beta-1}dx$$
$$= \frac{e\beta}{e-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n t^m}{m!n!}B\bigg(\frac{m}{\alpha}+1, \beta n+\beta\bigg). \tag{2.4.3}$$

2.4.3 Characteristic Function

Characteristic function, $\phi_X(t)$ of DUS-K(α, β) is,

$$\phi_X(t) = E(e^{itX}) = \frac{e\beta}{e-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (it)^m}{m!n!}B\bigg(\frac{m}{\alpha}+1, \beta n+\beta\bigg), i=\sqrt{-1}. \tag{2.4.4}$$

2.4.4 Cumulant Generating Function

Cumulant generating function, $K_X(t) = \log\phi_X(t)$ of DUS-K(α, β) is,

$$K_X(t) = \log\frac{e\beta}{e-1} + \log\Bigg\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (it)^m}{m!n!}B\bigg(\frac{m}{\alpha}+1, \beta n+\beta\bigg)\Bigg\},\ i=\sqrt{-1}. \tag{2.4.5}$$

2.4.5 Quantiles

The p^{th} quantile function, denoted by $Q(p)$ of DUS-K(α, β) distribution will obtain as a solution of the equation $F(Q(p)) = p$, where $0 < p < 1$. Implies,

$$\frac{1}{e-1}\Big[e^{1-(1-Q(p)^{\alpha})^{\beta}} - 1\Big] = p.$$

i.e,

$$Q(p) = \Big[1-(1-\log(1+p(e-1)))^{\frac{1}{\beta}}\Big]^{\frac{1}{\alpha}}. \tag{2.4.6}$$

Median of DUS-K(α, β) distribution is obtained by substituting $p = 1/2$ in Eq. (2.4.6),

$$i.e., \text{ Median} = \bigg[1-(1-\log(1+\frac{1}{2}(e-1)))^{\frac{1}{\beta}}\bigg]^{\frac{1}{\alpha}}.$$

2.4.6 Entropy

Entropy can be describing as the measure of variability or a measure of information. Renyi entropy of a random variable is defined by, $\tau_R(\gamma) = \frac{1}{1-\gamma}\log\{\int g^{\gamma}(x)dx\}$, where $\gamma > 0$ and $\gamma \neq 1$.

If X has DUS-K(α, β), then

$$\begin{aligned}\int g^{\gamma}(x) &= \int_0^1 \left(\frac{1}{e-1}\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}e^{1-(1-x^{\alpha})^{\beta}}\right)^{\gamma} dx \\ &= \left(\frac{e}{e-1}\right)^{\gamma} \alpha^{\gamma-1}\beta^{\gamma} \sum_{m=0}^{\infty} \frac{(-1)^m \gamma^m}{m!} B\left(\gamma + \frac{1-\gamma}{\alpha}, \beta\gamma + \beta m - \gamma + 1\right).\end{aligned}$$

Thus the Renyi entropy for X is obtained as,

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\left(\frac{e}{e-1}\right)^{\gamma} \alpha^{\gamma-1}\beta^{\gamma} \sum_{m=0}^{\infty} \frac{(-1)^m \gamma^m}{m!} B\left(\gamma + \frac{1-\gamma}{\alpha}, \gamma\beta + \beta m - \gamma + 1\right)\right\}. \tag{2.4.7}$$

The Shannon entropy is defined by $E(-\log g(X))$. Then Shannon entropy of DUS-K$(\alpha,\ \beta)$ is obtained as,

$$\begin{aligned}E(-\log g(X)) = &-\log\left(\frac{\alpha\beta}{e-1}\right) - (\alpha-1)E(\log X) + \\ &(\beta-1)\sum_{k=1}^{\infty}\frac{E(X^{\alpha k})}{k} + \sum_{r=0}^{\infty}(-1)^r\binom{\beta}{r}E(X^{\alpha r}) - 1.\end{aligned} \tag{2.4.8}$$

The q-entropy is defined as $H_q(x) = \frac{1}{1-q}\log\left(1 - \int_{-\infty}^{\infty} g^q(x)dx\right)$, $q > 0$ and $q \neq 1$. Then q-entropy of DUS-K(α, β) is

$$H_q(x) = \frac{1}{1-q}\log\left(1 - \left(\frac{e}{e-1}\right)^{q} \alpha^{q-1}\beta^{q} \sum_{m=0}^{\infty} \frac{(-1)^m q^m}{m!} B\left(q + \frac{1-q}{\alpha}, \beta q + \beta m - q + 1\right)\right). \tag{2.4.9}$$

2.5 Distribution of the Extreme Values

Distribution of the extreme order statistics plays a major role in reliability analysis of series and parallel systems (Barlow & Proschan (1975)). Consecutive order statistics $X_{(r)}$, $X_{(r+1)}$ are frequently found to be useful in ranking and selection problems (Garg (2008)).

Let X_1, $X_2, ..., X_n$ are independent and identically distributed (*i.i.d.*) random variables of size n with PDF $g(x)$ and CDF $G(x)$ and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be their corresponding order statistics. Then PDF of the r^{th} order statistic is

$$g_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} g(x) G^{r-1}(x)\{1 - G(x)\}^{n-r}. \tag{2.5.1}$$

The PDF of the r^{th} order statistic of DUS-K(α, β) is obtained by using (2.2.7) and (2.2.8) is

$$g_{X_{(r)}}(x;\alpha,\beta) = \frac{n!}{(r-1)!(n-r)!}\frac{1}{(e-1)^n}\alpha\beta x^{\alpha-1}(1-x^\alpha)^{\beta-1}e^{1-(1-x^\alpha)^\beta}$$
$$(e^{1-(1-x^\alpha)^\beta}-1)^{r-1}\Big(e-e^{1-(1-x^\alpha)^\beta}\Big)^{n-r},\ 0<x<1,\ \alpha>0,\beta>0.$$

For, $\alpha > 0$, $\beta > 0$ and $0 < x < 1$, the PDF of $X_{(1)} = min(X_1, X_2, ..., X_n) =$ and $X_{(n)} = max(X_1, X_2, ...X_n)$ are respectively

$$g_{X_{(1)}}(x;\alpha,\beta) = \frac{n}{(e-1)^n}\alpha\beta x^{\alpha-1}(1-x^\alpha)^{\beta-1}e^{1-(1-x^\alpha)^\beta}(e-e^{1-(1-x^\alpha)^\beta})^{n-1},$$

and

$$g_{X_{(n)}}(x;\alpha,\beta) = \frac{n}{(e-1)^n}\alpha\beta x^{\alpha-1}(1-x^\alpha)^{\beta-1}e^{1-(1-x^\alpha)^\beta}(e^{1-(1-x^\alpha)^\beta}-1)^{n-1}.$$

For, $\alpha > 0$, $\beta > 0$ and $0 < x < 1$, the CDF of $X_{(1)}$ and $X_{(n)}$ are respectively

$$G_{X_{(1)}}(x;\alpha,\beta) = 1-\left[\frac{e-e^{1-(1-x^\alpha)^\beta}}{e-1}\right]^n$$

and

$$G_{X_{(n)}}(x;\alpha,\beta) = \left[\frac{e^{1-(1-x^\alpha)^\beta}-1}{e-1}\right]^n.$$

2.6 Estimation

In this section, first we discuss the parameter estimation using method of moments followed by the ML method for the DUS-K(α, β) distribution.

Let consider a random sample of size n from DUS-K(α, β) distribution and if $m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ be the first and second sample moments. Equating the sample moments m_1 and m_2, with the corresponding population moments $E(X)$ and $E(X^2)$, we get

$$m_1 = \frac{e\beta}{e-1}\sum_{i=o}^{\infty}\frac{(-1)^i}{i!}\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma\left(i\beta+\beta\right)}{\Gamma\left(i\beta+\beta+\frac{1}{\alpha}+1\right)}$$

$$m_2 = \frac{e\beta}{e-1}\sum_{i=o}^{\infty}\frac{(-1)^i}{i!}\frac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma\left(i\beta+\beta\right)}{\Gamma\left(i\beta+\beta+\frac{2}{\alpha}+1\right)}.$$

The method of moments provides a straightforward approach to parameter estimation by equating sample moments to their corresponding population moments. The resulting equations for m_1 and m_2 can be solved numerically to estimate the parameters α and β. These estimates serve as initial values for more refined estimation techniques, such as the ML method, which is discussed below.

MLEs of the parameters can be obtained as follows. Let $\log L(\alpha,\beta)$ be the log-likelihood function for unknown parameters α and β. Given n observations, $x_1,\ x_2, ...,\ x_n$, $\log L(\alpha,\beta)$ is

$$\log L(\alpha,\beta) = n\log\alpha + n\log\beta - n\log(e-1) + \sum_{i=1}^{n}\left(1-(1-x_i^{\alpha})^{\beta}\right) + (\alpha-1)\sum_{i=1}^{n}\log x_i + (\beta-1)\sum_{i=1}^{n}\log(1-x_i^{\alpha}), \tag{2.6.1}$$

$\log L(\alpha,\beta)$ is differentiable with respect to the parameters α and β. We will attain the MLEs $\hat{\theta} = (\hat{\alpha},\hat{\beta})$ by maximizing the log-likelihood with the concerning parameters.

$$\frac{\partial \log L(\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} + \beta\sum_{i=1}^{n}x_i^{\alpha}(1-x_i^{\alpha})^{\beta-1}\log x_i + \sum_{i=1}^{n}\log x_i - (\beta-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{1-x_i^{\alpha}}$$

and

$$\frac{\partial \log L(\alpha,\beta)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}(1-x_i^{\alpha})^{\beta}\log(1-x_i^{\alpha}) + \sum_{i=1}^{n}\log(1-x_i^{\alpha}).$$

The preceding equations can be solved numerically by using statistical software with arbitrary initial values.

To obtain the confidence intervals of α and β, one requires the observed information matrix:

$$I = \begin{bmatrix} E\left(-\frac{\partial^2 \log L}{\partial\alpha^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial\alpha\partial\beta}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial\beta\partial\alpha}\right) & E\left(-\frac{\partial^2 \log L}{\partial\beta^2}\right) \end{bmatrix}.$$

The second order partial derivatives of $\log L(\alpha,\beta)$ are given by,

$$\frac{\partial^2 \log L}{\partial\alpha^2} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n}\beta x_i^{\alpha}(1-x_i^{\alpha})^{\beta-2}(\log x_i)^2\left(1-\beta x_i^{\alpha}\right) - (\beta-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}(\log x_i)^2}{(1-x_i^{\alpha})^2},$$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\beta} = \frac{\partial^2 \log L}{\partial\beta\partial\alpha} = \sum_{i=1}^{n} \left[1 + \beta \log(1 - x_i^{\alpha})\right] x_i^{\alpha} (1 - x_i^{\alpha})^{\beta-1} \log x_i - \sum_{i=1}^{n} \frac{x_i^{\alpha} \log x_i}{1 - x_i^{\alpha}},$$

and

$$\frac{\partial^2 \log L}{\partial\beta^2} = -\frac{n}{\beta^2} - \sum_{i=1}^{n} (1 - x_i^{\alpha})^{\beta} (\log(1 - x_i^{\alpha}))^2.$$

Since the MLEs are not in the explicit forms, we check out the asymptotic distribution of the estimates. As $n \to \infty$, $\hat{\theta}$ converges normally with mean zero and variance $I^{-1}(\theta)$, where $I^{-1}(\theta)$ is the variance covariance matrix of θ, the inverse of Fisher information matrix.

The asymptotic distribution of the MLE, $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ of $\theta = (\alpha, \beta)$ is, $\sqrt{n}(\hat{\theta} - \theta) \to N(0, I^{-1}(\theta))$, where $I^{-1}(\theta)$ is the variance covariance matrix of θ, the inverse of Fisher information matrix. Thus, the asymptotic $100(1 - \delta)\%$ confidence intervals of α and β are $\hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\alpha})}$ and $\hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\beta})}$ respectively, where $Z_{\frac{\delta}{2}}$ is the upper $100(\frac{\delta}{2})^{th}$ percentile of standard Normal distribution.

2.7 Stress-Strength Reliability

Stress-Strength model has a significant role in reliability engineering. The reliability in stress-strength context defines the probability that the random strength greater than the random stress of a component or system. The concept of stress-strength reliability estimation was first mentioned by Birnbaum (1956). Inference on stress-strength model has been considered substantially by various authors. Recently, Pakdaman & Ahmadi (2018) derived an expression for stress-strength reliability estimation for a system having independent components. Xavier & Jose (2021), Chaturvedi & Malhotra (2020) and Biswas et al. (2021) have studied the stress-strength reliability under generalization of power transformed half-logistic distribution, proportional reversed hazard family with lower record values and log-Lindley distribution, respectively.

In this section reliability estimation of single component stress-strength model (SSS) and multicomponent stress-strength model (MSS) are considered.

2.7.1 Single Component Stress-Strength Reliability

This section deals with the reliability of SSS based on independent stress and strength variables. Let X and Y represents the 'strength' and 'stress' having PDF DUS-K(α,

β_1) and DUS-K(α, β_2) respectively. Then the system reliability $R = P(Y < X)$ is

$$\begin{aligned}
R = P[Y < X] = &\int_0^\infty g_X(x)G_Y(x)dx \\
=&\frac{1}{(e-1)^2}\alpha\beta_1 \int_0^1 x^{\alpha-1}(1-x^\alpha)^{\beta_1-1}e^{1-(1-x^\alpha)^{\beta_1}}\left[e^{1-(1-x^\alpha)^{\beta_2}}-1\right]dx \\
=&\left(\frac{e}{e-1}\right)^2\alpha\beta_1\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(-1)^{m+n}}{m!n!}\int_0^1 x^{\alpha-1}(1-x^\alpha)^{\beta_1 m+\beta_2 n+\beta_1-1}dx- \\
&\left(\frac{1}{e-1}\right)^2\alpha\beta_1\int_0^1 x^{\alpha-1}(1-x^\alpha)^{\beta_1-1}e^{1-(1-x^\alpha)^{\beta_1}}dx.
\end{aligned}$$

$$i.e.,\ R = \left(\frac{e}{e-1}\right)^2\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(-1)^{m+n}}{m!n!}\frac{\beta_1}{\beta_1 m+\beta_2 n+\beta_1}-\frac{1}{e-1}. \tag{2.7.1}$$

2.7.2 Multicomponent Stress-Strength Reliability

The stress-strength model can be broaden into a MSS system consists of k independent and identical strength components subjected to a common random stress. The system functions only if s out of k ($1 \le s \le k$) components concurrently outlive. Let $G(\cdot)$ and $F(\cdot)$ represents the continuous distribution function of independent strength and stress components, then the reliability in a multicomponent stress-strength model developed by Bhattacharyya & Johnson (1974) is

$$\begin{aligned}
R_{s,k} =&P[at\ least\ s\ of\ the\ (X_1, X_2, ...X_k)\ exceed\ Y] \\
=&\sum_{i=s}^k\binom{k}{i}\int_{-\infty}^\infty[1-G(x)]^i[G(x)]^{k-i}dF(x).
\end{aligned} \tag{2.7.2}$$

Reliability in a multicomponent has been discussed by several authors on different lifetime distributions for stress and strength random variables. Recently, Sharma & Dey (2019), Kayal et al. (2020), Kohansal & Nadarajah (2019) and Hassan et al. (2020) have studied reliability estimation in multicomponent stress-strength model.

Let $X_1, X_2, ..., X_k$ denote the random strength of k components with DUS-K(α, β_1) and Y be the common stress with DUS-K(α, β_2) of a MSS system. The reliability in MSS using (2.7.2) is given by,

$$R_{s,k} = \sum_{i=s}^k\binom{k}{i}\int_0^1\left[1-\left(\frac{1}{e-1}\left(e^{1-(1-x^\alpha)^{\beta_1}}-1\right)\right)\right]^i\left[\frac{1}{e-1}\left(e^{1-(1-x^\alpha)^{\beta_1}}-1\right)\right]^{k-i}$$

$$
\begin{aligned}
&\frac{1}{e-1}\alpha\beta_2 x^{\alpha-1}(1-x^{\alpha})^{\beta_2-1}e^{1-(1-x^{\alpha})^{\beta_2}}dx \\
=&\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}(-1)^{l_1+l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\frac{\alpha\beta_2 e^{i+1}}{(e-1)^{k+1}}\int_0^1 e^{-l_1(1-x^{\alpha})^{\beta_1}} \\
&e^{(k-i-l_2)[1-(1-x^{\alpha})^{\beta_1}]}x^{\alpha-1}(1-x^{\alpha})^{\beta_2-1}e^{-(1-x^{\alpha})^{\beta_2}}dx.
\end{aligned}
$$

After simplification we get,

$$
R_{s,k}=\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{\infty}\sum_{l_4=0}^{\infty}\frac{(-1)^{l_1+l_2+l_3+l_4}}{l_3!\;l_4!}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\frac{\beta_2 e^{k-l_2-1}}{(e-1)^{k+1}}\frac{(k-i-l_2+l_1)^{l_3}}{\beta_1 l_3+\beta_2 l_4+\beta_2}. \tag{2.7.3}
$$

2.7.3 Estimation of Reliability

Here we discuss the reliability estimation in SSS and MSS. Suppose (X_1, X_2, ..., X_n) and (Y_1, Y_2, ..., Y_m) are independent random samples from DUS-K(α,β_1) and DUS-K(α,β_2) respectively. To obtain the MLE of R, consider their log-likelihood of the observed joint sample, x_1, x_2, ..., x_n, y_1, y_2, ..., y_m,

$$
\begin{aligned}
l(\alpha,\beta_1,\beta_2)=&(m+n)\log\left(\frac{e}{e-1}\right)+(m+n)\log\alpha+n\log\beta_1+m\log\beta_2+ \\
&(\alpha-1)\sum_{i=1}^{n}\log x_i+(\beta_1-1)\sum_{i=1}^{n}\log(1-x_i^{\alpha})-\sum_{i=1}^{n}(1-x_i^{\alpha})^{\beta_1}+ \\
&(\alpha-1)\sum_{j=1}^{m}\log y_j+(\beta_2-1)\sum_{j=1}^{m}\log\left(1-y_j^{\alpha}\right)-\sum_{j=1}^{m}\left(1-y_j^{\alpha}\right)^{\beta_2}.
\end{aligned} \tag{2.7.4}
$$

MLE of the parameters are the solutions of the following non-linear equations:

$$
\begin{aligned}
\frac{\partial\log l}{\partial\alpha}=&\frac{m+n}{\alpha}+\sum_{i=1}^{n}\log x_i-(\beta_1-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{1-x_i^{\alpha}}+\beta_1\sum_{i=1}^{n}x_i^{\alpha}(1-x_i^{\alpha})^{\beta_1-1}\log x_i+ \\
&\sum_{j=1}^{m}\log y_j-(\beta_2-1)\sum_{j=1}^{m}\frac{y_j^{\alpha}\log y_j}{1-y_j^{\alpha}}+\beta_2\sum_{j=1}^{m}y_j^{\alpha}(1-y_j^{\alpha})^{\beta_2-1}\log y_j, \\
\frac{\partial\log l}{\partial\beta_1}=&\frac{n}{\beta_1}+\sum_{i=1}^{n}\log(1-x_i^{\alpha})-\sum_{i=1}^{n}(1-x_i^{\alpha})^{\beta_1}\log(1-x_i^{\alpha}),
\end{aligned}
$$

and

$$
\frac{\partial\log l}{\partial\beta_2}=\frac{m}{\beta_2}+\sum_{j=1}^{m}\log(1-y_j^{\alpha})-\sum_{j=1}^{m}(1-y_j^{\alpha})^{\beta_2}\log(1-y_j^{\alpha}).
$$

Let $\hat{\alpha}^R$, $\hat{\beta_1}^R$, and $\hat{\beta_2}^R$ be the estimates of the parameters α, β_1, and β_2 respectively. Using (2.7.1), by the invariance property, MLE of SSS reliability of R is obtained as,

$$\hat{R} = \left(\frac{e}{e-1}\right)^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{m!n!} \frac{\hat{\beta_1}^R}{\hat{\beta_1}^R m + \hat{\beta_2}^R n + \hat{\beta_1}^R} - \frac{1}{e-1}. \tag{2.7.5}$$

Using (2.7.3), MLE of MSS reliability, $\hat{R}_{s,k},\ s = 1, 2, ...k$ is obtained as,

$$\hat{R}_{s,k} = \sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{\infty}\sum_{l_4=0}^{\infty} \frac{(-1)^{l_1+l_2+l_3+l_4}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}}{l_3!\ l_4!} \frac{\hat{\beta_2}^R e^{k-l_2-1}}{(e-1)^{k+1}} \frac{(k-i-l_2+l_1)^{l_3}}{\hat{\beta_1}^R l_3 + \hat{\beta_2}^R l_4 + \hat{\beta_2}^R}. \tag{2.7.6}$$

2.7.4 Asymptotic Distribution and Confidence Interval

To find out the asymptotic variance of $\hat{R}$ and $\hat{R}_{s,k}$ we proceed as follows.
The asymptotic distribution of $\hat{R}$ is obtained as, $\sqrt{(n+m)}(\hat{R}-R)^d \to N(0, \Delta'(\theta)I^{-1}(\theta)\Delta(\theta))$, where $\Delta(\theta) = \left(\frac{\partial R}{\partial \alpha}, \frac{\partial R}{\partial \beta_1}, \frac{\partial R}{\partial \beta_2}\right) = (\delta_1, \delta_2, \delta_3)'$ and I_R is the Fisher information matrix of unknown parameter $\theta = (\alpha, \beta_1, \beta_2)$ and is given by,

$$I_R = \begin{bmatrix} E\left(-\frac{\partial^2 \log l}{\partial \alpha^2}\right) & E\left(-\frac{\partial^2 \log l}{\partial \alpha \partial \beta_1}\right) & E\left(-\frac{\partial^2 \log l}{\partial \alpha \partial \beta_2}\right) \\ E\left(-\frac{\partial^2 \log l}{\partial \beta_1 \partial \alpha}\right) & E\left(-\frac{\partial^2 \log l}{\partial \beta_1^2}\right) & E\left(-\frac{\partial^2 \log l}{\partial \beta_1 \partial \beta_2}\right) \\ E\left(-\frac{\partial^2 \log l}{\partial \beta_2 \partial \alpha}\right) & E\left(-\frac{\partial^2 \log l}{\partial \beta_2 \partial \beta_1}\right) & E\left(-\frac{\partial^2 \log l}{\partial \beta_2^2}\right) \end{bmatrix}.$$

The terms inside the Fisher information matrix will get directly. Then the asymptotic variance (AV) of $\hat{R}$ is

$$\begin{aligned} AV(\hat{R}) =& \frac{1}{n+m}\Delta'(\theta)I^{-1}(\theta)\Delta(\theta) \\ =& Var(\hat{\alpha}^R)\delta_1^2 + Var(\hat{\beta_1}^R)\delta_2^2 + Var(\hat{\beta_2}^R)\delta_3^2 + 2\delta_1\delta_2 Cov(\hat{\alpha}^R, \hat{\beta}_1^R) \\ &+ 2\delta_1\delta_3 Cov(\hat{\alpha}^R, \hat{\beta}_2^R) + 2\delta_2\delta_3 Cov(\hat{\beta}_1^R, \hat{\beta}_2^R) \end{aligned} \tag{2.7.7}$$

The asymptotic 95% confidence interval for R is obtained as $\hat{R} \pm 1.96\sqrt{AV(\hat{R})}$.

To determine the confidence interval for $\hat{R}_{s,k}$, $1 \leq s \leq k$, consider the asymptotic variance of $\hat{R}_{s,k}$,

$$AV(\hat{R}_{s,k}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial R_{s,k}}{\partial \theta_i}\frac{\partial R_{s,k}}{\partial \theta_j} I^{-1}(\theta).$$

As $m \to \infty$ and $n \to \infty$, $\frac{\hat{R}_{s,k} - R_{s,k}}{\sqrt{AV(\hat{R}_{s,k})}} \to N(0, 1)$. Therefore, the asymptotic 95% confidence interval for $\hat{R}_{s,k}$ is obtained as $\hat{R}_{s,k} \pm 1.96\sqrt{AV(\hat{R}_{s,k})}$.

2.8 Simulation

Here, a comprehensive numerical investigation is executed to evaluate the behavior of MLEs and their asymptotic results for the DUS-K model. For different choices of parameters and sample sizes, the biases and mean square error (MSE)s are calculated. Samples of DUS-K(α, β) of sizes 50, 100, 250, 500 and 1000 for the parameter combinations (1.5, 2), (0.5, 0.5) and (2, 2.5) corresponding to (α, β) are generated. Inversion method is used to generate the sample and is done via statistical software R. As shown in Tables 2.1-2.4, the biases and MSEs of the parameter estimates decrease as the sample size increases.

Table 2.1: Estimates, Biases and MSEs for DUS-K model at $\alpha = 1.5$ and $\beta = 2$

n	Parameter estimates	Bias	MSE
50	$\hat{\alpha}$=1.5443	0.0443	0.0696
	$\hat{\beta}$=2.1088	0.1088	0.2008
100	$\hat{\alpha}$=1.5140	0.0140	0.0354
	$\hat{\beta}$=2.0332	0.0332	0.0871
250	$\hat{\alpha}$=1.4886	0.0113	0.0155
	$\hat{\beta}$=1.9975	0.0187	0.0157
500	$\hat{\alpha}$=1.5113	0.0113	0.0065
	$\hat{\beta}$=2.0187	0.0124	0.0300
1000	$\hat{\alpha}$=1.5015	0.0015	0.0029
	$\hat{\beta}$=2.0086	0.0086	0.0069

2.9 Application

Here, a real data set is used to demonstrate the applicability of proposed distribution. The proposed distribution is compared with K distribution by Kumaraswamy (1980), generalized Lindley (GL) distribution by Nadarajah (2011), DUS exponential (DUS-E) distribution by Tripathi (2021), DUS-Lomax (DUS-L) distribution by Deepthi & Chacko (2020) and generalized DUS-Weibull (GDUS-W) by Kavya & Manoharan (2020).

The data given in Aarset (1987) is considered, which contains the time to failure of 50 devices put on life test at time 0, see Table 2.5.

Using R software, the MLEs of the model parameters, the KS statistic, p-value,

Table 2.2: Estimates, Biases and MSEs for DUS-K model at $\alpha = 0.5$ and $\beta = 0.5$

n	Parameter estimates	Bias	MSE
50	$\hat{\alpha}$=0.5378	0.0378	0.0274
	$\hat{\beta}$=0.5148	0.0148	0.0060
100	$\hat{\alpha}$=0.5321	0.0321	0.0114
	$\hat{\beta}$=0.5032	0.0038	0.0027
250	$\hat{\alpha}$=0.5126	0.0126	0.004
	$\hat{\beta}$=0.5038	0.0032	0.0011
500	$\hat{\alpha}$=0.5124	0.0124	0.0020
	$\hat{\beta}$=0.5043	0.0043	0.0005
1000	$\hat{\alpha}$=0.5056	0.0056	0.0008
	$\hat{\beta}$=0.5007	0.0007	0.0002

Table 2.3: Estimates, Biases and MSEs for DUS-K model at $\alpha = 2$ and $\beta = 2.5$

n	Parameter estimates	Bias	MSE
50	$\hat{\alpha}$=2.0305	0.0305	0.1227
	$\hat{\beta}$=2.6232	0.1232	0.3785
100	$\hat{\alpha}$=2.0214	0.0214	0.0553
	$\hat{\beta}$=2.6120	0.1120	0.1701
250	$\hat{\alpha}$=2.0392	0.0392	0.0247
	$\hat{\beta}$=2.5602	0.0602	0.0601
500	$\hat{\alpha}$=2.0363	0.0363	0.0127
	$\hat{\beta}$=2.5544	0.0544	0.0289
1000	$\hat{\alpha}$=2.0029	0.0029	0.0057
	$\hat{\beta}$=2.5030	0.0030	0.0135

log-likelihood (Log-L) value, AIC, and BIC are computed and presented in Table 2.6. The table indicates that the DUS-K distribution provides a better fit among the competing models, as it has the lowest KS statistic value, AIC, and BIC, along with the highest p-value and log-L value compared to other models. Figure 2.4 represents the empirical CDF of data set which give weight to the fitness of DUS-K(α, β).

Table 2.4: Estimates, Biases and MSEs for DUS-K model at $\alpha = 3$ and $\beta = 2$

n	Parameter estimates	Bias	MSE
50	$\hat{\alpha}$=3.1746	0.1746	0.3388
	$\hat{\beta}$=2.1872	0.1872	0.2391
100	$\hat{\alpha}$=3.0910	0.0910	0.1689
	$\hat{\beta}$=2.0837	0.0837	0.0945
250	$\hat{\alpha}$=2.9919	0.0801	0.0518
	$\hat{\beta}$=2.0076	0.0761	0.0278
500	$\hat{\alpha}$=3.0661	0.0661	0.0317
	$\hat{\beta}$=2.0454	0.0454	0.0173
1000	$\hat{\alpha}$=3.0430	0.0430	0.0152
	$\hat{\beta}$=2.0012	0.0012	0.0067

Table 2.5: Aarset data

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

2.10 Summary

In this chapter, we introduced a distribution developed using the DUS transformation with the K distribution as a baseline, which exhibits both monotonic and non-monotonic behaviors in its failure rate function, making it versatile for various reliability and lifetime modelling applications. We rigorously derived the statistical properties of the DUS-K distribution and also analyze the distributions of its sample maximum and minimum, which are key for understanding extremal behavior relevant to risk assessment and quality control. For parameter estimation, we employed the ML method, allowing us to obtain reliable parameter estimates based on observed data and apply these estimates in reliability analysis using the stress-strength model- a framework widely used in engineering to evaluate the probability of a system's resilience under stress. A simulation study is conducted to assess the performance of

Table 2.6: MLEs, KS statistic, p-value,log-likelihood, AIC and BIC for the fitted models

Disribution	Estimates	KS value	p-value	Log-L	AIC	BIC
DUS-K	$\hat{\alpha} = \mathbf{0.507}$ $\hat{\beta} = \mathbf{0.973}$	**0.130**	**0.361**	**6.587**	**-9.174**	**-5.350**
K	$\hat{\alpha}$=0.586 $\hat{\beta}$=0.847	0.144	0.247	5.598	-7.197	-3.373
GDUS-W	$\hat{\alpha}$=0.127 $\hat{\beta}$=1.153 $\hat{\gamma}$=4.484	0.180	0.075	2.884	0.231	5.967
DUS-L	$\hat{\alpha}$=210 $\hat{\beta}$=0.013	0.241	0.005	-10.094	24.189	28.014
DUS-E	$\hat{\theta}$=2.699	0.175	0.090	-10.034	22.068	23.980
GL	$\hat{\alpha}$=0.075 $\hat{\beta}$=2.372	0.196	0.042	-8.82	21.64	25.463

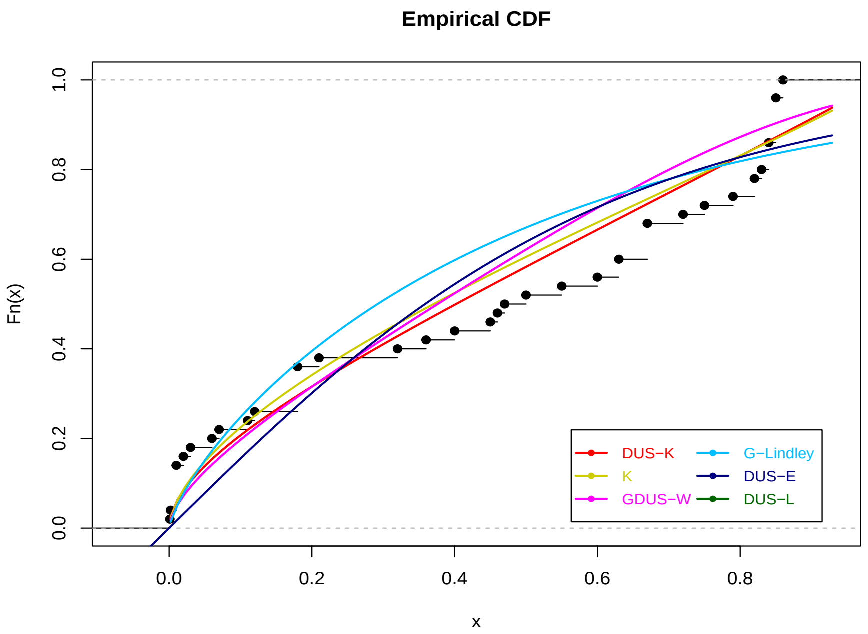


Figure 2.4: Empirical CDF

the ML method under various sample sizes and parameter values, helping to validate the theoretical results. Finally, an application to a real dataset demonstrates the DUS-K distribution's superior fit compared to existing distributions, highlighting its potential as an effective tool for modelling complex lifetime data.

CHAPTER 3

DUS-Inverse Kumaraswamy Distribution

3.1 Introduction

In modern decades proposing new probability distributions became a common enactment. Many researchers are interested to propose new probability distributions by attaching an extra parameter to the baseline distribution. Some researchers are interested in using transformations whereas others introduce new probability distributions by using generators or by combining two or more distributions. Over the last few years researchers have extensive desires for the pertinence of inverse transformations of probability distributions and their applications. Modelling the real life data is the foremost purpose of such amendment. What differentiates the new distributions with the existing probability distributions is the increase in model flexibility for modelling complex data structures. Inverse probability distributions are broadly using in different areas including biomedical, environmental studies, economics, engineering and reliability.

Al-Fattah et al. (2017) proposed inverse Kumaraswamy (IK) distribution by make use of the transformation $X = (1-T)/T$, where the random variable T follows K distribution with density function,

$$f(t; a, b) = abt^{a-1}(1-t^a)^{b-1},\ 0 < t < 1,\ a > 0,\ b > 0, \tag{3.1.1}$$

where a and b are shape parameters.

Thus, the random variable X has IK(a, b) distribution and its PDF and CDF are given respectively as,

$$f(x; a, b) = ab(1+x)^{-(a+1)}(1-(1+x)^{-a})^{b-1},\ x > 0,\ a > 0,\ b > 0, \tag{3.1.2}$$

and

$$F(x; a, b) = (1-(1+x)^{-a})^{b},\ x > 0,\ a > 0,\ b > 0, \tag{3.1.3}$$

where a and b are shape parameters. The IK distribution is widely used in fields such as biology, engineering, and economics for its versatility. This distribution has potential applications in forecasting long-term reliability, offering relatively optimistic forecasts regarding rare occurrences situated in the upper tail of the distribution compared to alternative distributions.

Based on the CDF of K distribution, Shahbaz et al. (2012) suggested a four parameter inverse Weibull distribution. Iqbal et al. (2017) developed a generalization of IK distribution. Jamal et al. (2019) proposed the “Generalized inverted Kumaraswamy-G” family of distributions, which is based on the generalized IK distribution. A comparative study in terms of Bayesian estimates and maximum likelihood estimates of IK distribution in related to PT2 censored data can see in Rashad et al. (2019). Estimation in stress-strength reliability of IK distribution has discussed by Hameed et al. (2020). Aly & Abuelamayem (2020) derived bivariate and multivariate case of IK distribution and presented bivariate IK distribution as a new bivariate Marshall–Olkin distribution as a means to implement effectively in several areas. A precise formulation for individual, cumulative, and dependent moments for the IK distribution using dual generalized order statistics are established by Khatoon et al. (2021). Nagy et al. (2022) studied ranked set sampling methodology-based estimation using IK distribution parameters and applied it within the realms of life testing and reliability analysis. Yousef et al. (2023) explores the estimation of lifetimes in step stress accelerated life tests using Type-II censored samples, with the IK distribution. Alsadat et al. (2023) introduces the Kavya–Manoharan generalized inverse Kumaraswamy distribution, which represents an enhanced variant of the generalized inverse Kumaraswamy distribution, incorporating three parameters.

In this work a new class of distribution based on DUS transformation with IK distribution as the baseline distribution is proposed. DUS transformation produces a parsimonious distribution as it doesn't contain any new parameters hence it is easy to analyze and interpret. DUS transformation primarily discussed by Kumar et al. (2015). Maurya et al. (2017) studied the transformation with Lindley as the baseline distribution and applied to a real data set.

By building upon the IK distribution, DUS transformation enhances the flexibility of the parent model, facilitating a more thorough examination of its behavior and essential properties. This novel approach also overcomes the IK distribution's intrinsic drawbacks, which may make it difficult to accurately represent the subtleties of actual data patterns.

The following reasons justify the study of this distribution:

- Its probability density function exhibits various asymmetric forms, including unimodal, inverse J-shaped, and right-skewed.
- The hazard function of the proposed model displays decreasing and up-side-down bathtub shapes.
- The distribution features a closed-form quantile function, facilitating the computation of numerous properties and the generation of random numbers.
- The distribution offers well-defined characterization properties based on truncated moments, hazard rate, reverse hazard rate, and conditional expectation of specific functions of the random variable.
- Efficient estimation of the distribution's parameters can be achieved through maximum likelihood estimation.
- Validation of the model's suitability is performed using the modified Chi-square goodness-of-fit test based on the NRR statistic, further supporting its practical relevance.
- Due to its high flexibility, the proposed distribution outperform other well-known distributions in terms of data fitting. This is demonstrated using two actual datasets, with model selection results indicating that the suggested distribution is the most suitable choice for them.

This chapter organized as follows. Section 3.1 is the introductory part. In section 3.2, a new model DUS-IK distribution has been derived. The shape properties of PDF and hazard rate function of the proposed model are considered in section 3.3.

In section 3.4, various statistical properties including limiting behaviors, moments, moment generating function, quantiles, distribution of order statistics, etc., have been derived. Section 3.5 provides a discussion on the various characterizations of the proposed model. Three renowned entropies are included in section 3.6. Section 3.7, discussed ML estimation method for the proposed distribution. The stress-strength reliability and its asymptotic properties are derived in section 3.8. Section 3.9 corroborated the efficiency of estimates through simulation process. In Section 3.10, two real data sets are used to illustrate the effectiveness of the model. Finally, Section 3.11 includes the validation using the NRR statistic, and the concluding remarks are provided in Section 3.12.

3.2 The DUS-Inverse Kumaraswamy Distribution

The proposed DUS-Inverse Kumaraswamy (DUS-IK) distribution *via* DUS transformation can be attained as follows: Let $f(x)$ and $F(x)$ be the PDF and CDF of some baseline distribution. Then the PDF and CDF of DUS transformation of the distribution are given by,

$$g(x) = \frac{1}{e-1} f(x) e^{F(x)} \tag{3.2.1}$$

and

$$G(x) = \frac{1}{e-1} \left[e^{F(x)} - 1 \right] \tag{3.2.2}$$

respectively. Thus the PDF and CDF of DUS-IK(a, b) distribution with shape parameters a and b, are determined by incorporating Eq.(3.1.2) & Eq.(3.1.3) to Eq.(3.2.1) as,

$$g(x) = \frac{1}{e-1} ab(1+x)^{-(a+1)}(1-(1+x)^{-a})^{b-1} e^{(1-(1+x)^{-a})^{b}},\ x>0,\ a>0,\ b>0, \tag{3.2.3}$$

and the corresponding CDF is obtained as

$$G(x) = \frac{1}{e-1} \left[e^{(1-(1+x)^{-a})^{b}} - 1 \right],\ x>0,\ a>0,\ b>0. \tag{3.2.4}$$

The hazard rate function ($h(x)$) and reversed hazard rate function ($r(x)$) of

DUS-IK(a, b) distribution are, respectively,

$$h(x) = \frac{g(x)}{1 - G(x)} = \frac{ab(1+x)^{-a-1}\left[1 - (1+x)^{-a}\right]^{b-1} e^{[1-(1+x)^{-a}]^b}}{e - e^{[1-(1+x)^{-a}]^b}} \tag{3.2.5}$$

and

$$r(x) = \frac{g(x)}{G(x)} = \frac{ab(1+x)^{-a-1}[1 - (1+x)^{-a}]^{b-1} e^{[1-(1+x)^{-a}]^b}}{e^{[1-(1+x)^{-a}]^b} - 1}. \tag{3.2.6}$$

3.3 Shape of DUS-Inverse Kumaraswamy Distribution

This section discloses the shape properties of DUS-IK(a, b) distribution. Figure 3.1 illustrate the shapes of PDF of DUS-IK(a, b) for different choices of parameter values. It can be deduced that the PDF of the proposed distribution is exponentially decreasing (inverse J shaped) for $b \leq 1$ and unimodal with positively skewed for $b > 1$. The unimodal probability distributions offer simplicity, interpretability, and computational advantages, making them valuable tools for statistical analysis and modelling in various fields. Moreover, they can often provide robust estimates and predictions even in the presence of moderate deviations from the assumed distributional form, making them versatile tools for analyzing diverse datasets. Their widespread use across various fields of study, including engineering, finance, biology, and social sciences, underscores their practical utility.

The mode of DUS-IK(a, b) can be attained by solving, $\frac{d}{dx}\log g(x) = 0$. $i.e$,

$$ab\left(1 - (1+x)^{-a}\right)^{b-1}(1+x)^{-a-1} + \frac{a(b-1)(1+x)^{-a-1}}{(1-(1+x)^{-a})} - \frac{a+1}{1+x} = 0. \tag{3.3.1}$$

The Eq.(3.3.1) can be solved numerically by the fixed-point method or the bisection method by choosing some initial values.

The plots depicting the Bowley skewness and Moors kurtosis of DUS-IK(a, b) across various parameter selections are presented in Figures 3.2 and 3.3, respectively. Figure 3.2 illustrates that the skewness of the proposed model is positively skewed and decreases as both parameters a and b increase. Additionally, Figure 3.3 demonstrates that kurtosis also decreases with increasing values of parameters a and b. Figure 3.4 represents the plot of hazard rate function of DUS-IK(a, b) distribution. The proposed distribution has both monotonic and non-monotonic hazard rates. For $b \leq 1$, $h(x)$ is decreasing and for $b > 1$, $h(x)$ has upside-down bathtub shape.

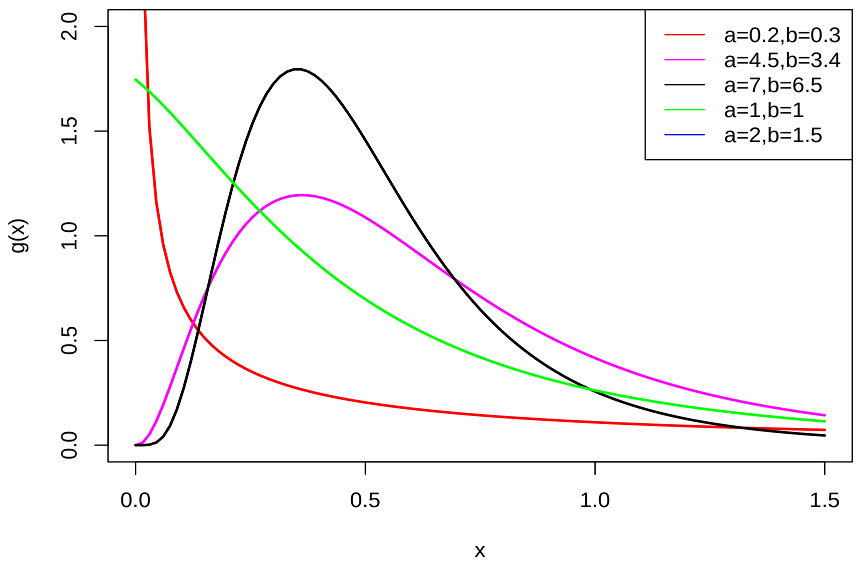


Figure 3.1: Density plot of DUS-IK(a, b) for the parameters (0.2, 0.3), (4.5, 3.4), (7, 6.5), (1, 1), and (2, 1.5) for (a, b)

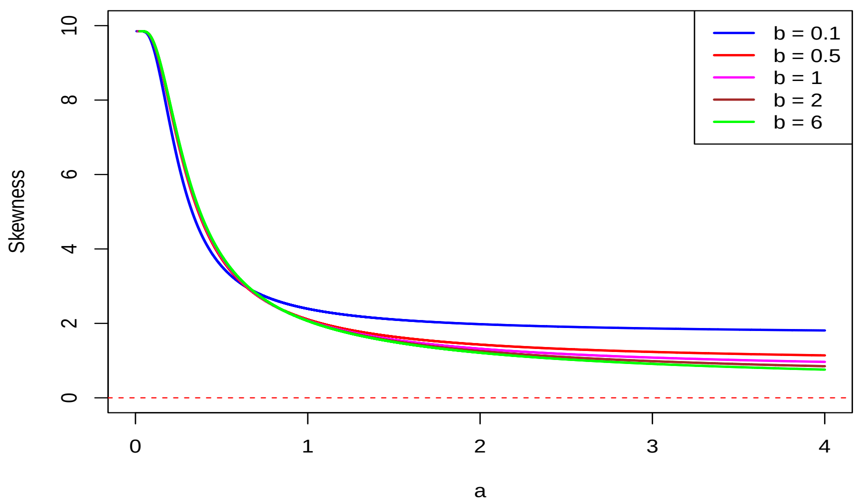


Figure 3.2: Bowley skewness plot of DUS-IK(a, b) for some parameter values

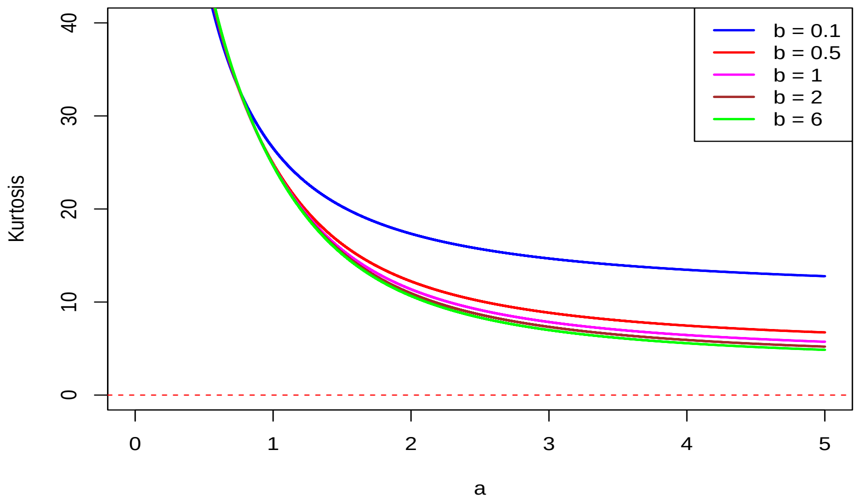


Figure 3.3: Moors kurtosis plot of DUS-IK(a, b) for some parameter values

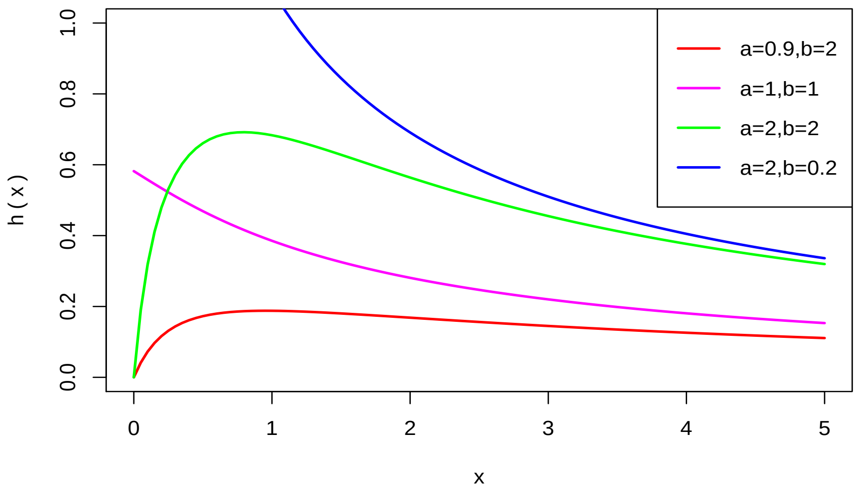


Figure 3.4: Plot of Hazard Rate Function of DUS-IK(a, b).

3.4 Structural Properties

3.4.1 Limiting Behaviors

Lemma 3.4.1. *The limiting behavior of the density function of DUS-IK(a, b) distribution when x tends to zero is as follows:*

$$\lim_{x \to 0} g(x) = \begin{cases} \infty & ; \quad b < 1 \\ \frac{a}{e-1} & ; \quad b = 1 \\ 0 & ; \quad b > 1. \end{cases}$$

Proof.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{1}{e-1}ab(1+x)^{-(a+1)}(1-(1+x)^{-a})^{b-1}e^{(1-(1+x)^{-a})^b}.$$

Clearly the terms, $\lim_{x\to 0}(1+x)^{-(a+1)} \cong 1$ and $\lim_{x\to 0} e^{(1-(1+x)^{-a})^b} \cong 1$.
Thus we have,

$$\begin{aligned}\lim_{x\to 0} g(x) &= \lim_{x\to 0}\frac{ab}{e-1}(1-(1+x)^{-a})^{b-1}\\ &= \lim_{x\to 0}\frac{ab}{e-1}\left(1-\left(1-ax+\frac{(ax)^2}{2!}-\frac{(ax)^3}{3!}+\cdots\right)\right)^{b-1}\\ &= \lim_{x\to 0}\frac{a^b b}{e-1}x^{b-1}\left(1-\frac{ax}{2!}+\frac{(ax)^2}{3!}-\cdots\right)^{b-1}.\end{aligned}$$

Now the results are directly. □

3.4.2 Moments

Moments are inevitable to describe as it is useful to study the important characteristics and attributes of a distribution. If the random variable X has DUS-IK(a, b), the r^{th} raw moment of X, is given as follows:

$$E(X^r) = \int_0^\infty x^r \frac{1}{e-1}ab(1+x)^{-(a+1)}[1-(1+x)^{-a}]^{b-1}e^{[1-(1+x)^{-a}]^b}dx. \tag{3.4.1}$$

By using the Substitution $u = [1-(1+x)^{-a}]^b$ and susequently the expansion $e^{-t} = \sum_{i=0}^{\infty}(-1)^i\frac{x^i}{i!}$, one can write,

$$\begin{aligned}E(X^r) &= \frac{1}{e-1}\sum_{j=0}^{r}\sum_{l=0}^{\infty}\frac{(-1)^{r-j}}{l!}\binom{r}{j}\int_0^1(1-u^{\frac{1}{b}})^{-\frac{j}{a}}u^l du\\ &= \frac{b}{e-1}\sum_{j=0}^{r}\sum_{l=0}^{\infty}\frac{(-1)^{r-j}}{l!}\binom{r}{j}B\left(1-\frac{j}{a}, bl+b\right),\end{aligned} \tag{3.4.2}$$

where $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$ represents the Beta function.
By substituting $r=1$ and $r=2$ in Eq. (3.4.2), the first and second raw moments of DUS-IK(a, b) are obtained as,

$$E(X) = \frac{b}{e-1}\sum_{l=0}^{\infty}\frac{1}{l!}B\left(1-\frac{1}{a}, bl+b\right) - \frac{1}{(e-1)}\sum_{l=0}^{\infty}\frac{1}{l!(l+1)} \quad \text{and} \tag{3.4.3}$$

$$E(X^2) = \frac{1}{e-1}\sum_{l=0}^{\infty}\frac{1}{l!(l+1)} - \frac{2b}{e-1}\sum_{l=0}^{\infty}\frac{1}{l!}B\left(1-\frac{1}{a}, bl+b\right) + \frac{b}{e-1}\sum_{l=0}^{\infty}\frac{1}{l!}B\left(1-\frac{2}{a}, bl+b\right), \tag{3.4.4}$$

respectively. Thus, the variance can be derived using the formula $Var(X) = E(X^2) - (E(X))^2$.

We calculate the mean, variance, coefficient of variation (CV), skewness, and kurtosis using R software and present the results in Table 3.1 for different choices of parameters a and b.

From Table 3.1, it is evident that the DUS-IK(a, b) distribution exhibits positive skewness and demonstrates variability in its kurtosis properties, encompassing platykurtic, mesokurtic, and leptokurtic tendencies.

The values of mean, variance, CV, skewness, and kurtosis decrease with an increase in parameter a. However, the mean and variance increase with an increase in parameter b. Additionally, the CV, skewness, and kurtosis decrease with an increase in b.

3.4.3 Moment Generating Function, Characteristic Function, and Cumulant Generating Function

If the random variable X has DUS-IK(a, b), then the moment generating function of X, denoted by $M_X(t)$ is derived as follows:

$$\begin{aligned}
M_X(t) =& E(e^{tX}) \\
=& \int_0^{\infty} e^{tx}\frac{1}{e-1}ab(1+x)^{-(a+1)}\left[1-(1+x)^{-a}\right]^{b-1} e^{(1-(1+x)^{-a})^b}dx \\
=& \frac{ab}{e-1}\sum_{j=0}^{\infty}\frac{t^j}{j!}\int_0^{\infty} x^j ab(1+x)^{-(a+1)}\left[1-(1+x)^{-a}\right]^{b-1} e^{(1-(1+x)^{-a})^b}dx \\
=& \frac{1}{e-1}\sum_{j=0}^{\infty}\frac{t^j}{j!}\int_0^{1}\left[\left(1-u^{\frac{1}{b}}\right)^{-\frac{1}{a}}-1\right]^j e^u du,
\end{aligned}$$

where $u = [1-(1+x)^{-a}]^b$. Using the series expansion $(x-1)^m = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}x^k$,

Table 3.1: Mean, variance, CV, skewness, and kurtosis of DUS-IK(a, b) distribution for different choice of parameters

	b	**0.1**	**0.5**	**1**	**2**	**6**
	Mean	4.9×10^4	5.5×10^5	1.6×10^6	5.1×10^6	1.2×10^{10}
	Variance	2.4×10^{17}	1.5×10^{17}	1.3×10^{14}	1.3×10^{17}	7.4×10^{21}
a=0.2	CV	9.9923	7.0507	7.0865	7.1038	7.115
	Skewness	9.7019	7.8946	7.9226	7.9366	7.9459
	Kurtosis	93.059	64.781	65.2351	65.4615	65.6122
	Mean	0.2238	2.3595	5.8846	14.2586	56.6236
	Variance	0.608	36.149	202.395	1135.25	17572.18
a=0.8	CV	3.4839	2.5481	2.4175	2.363	2.341
	Skewness	5.2808	4.9344	4.9292	4.9371	4.947
	Kurtosis	29.467	26.461	26.4349	26.5098	26.599
	Mean	0.1556	1.3785	3.085	6.5721	20.6249
	Variance	0.2564	8.4137	34.3802	138.407	1248.22
a=1	CV	3.2525	2.1041	1.9006	1.79	1.7129
	Skewness	4.9992	4.3277	4.259	4.2371	4.23
	Kurtosis	26.5972	20.984	20.4661	20.304	20.2523
	Mean	0.0608	0.1718	0.3086	0.5002	0.9228
	Variance	0.0226	0.0456	0.0848	0.1404	0.2719
a=4	CV	2.4757	1.2427	0.944	0.7491	0.565
	Skewness	3.8323	2.244	1.8407	1.6036	1.4343
	Kurtosis	15.6913	5.8507	4.0465	3.1715	2.6304
	Mean	0.0177	0.1079	0.1906	0.3025	0.535
	Variance	0.0021	0.0163	0.0284	0.0433	0.0717
a=6	CV	2.6303	1.1853	0.8852	0.6883	0.5005
	Skewness	3.9854	2.0456	1.6111	1.3483	1.1554
	Kurtosis	17.253	4.7418	2.9795	2.1448	1.643

one can calculate the integration as,

$$\begin{aligned} M_X(t) &= \frac{1}{e-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\frac{(-1)^{j-k}t^j}{j!l!}\binom{j}{k}\int_0^1\left(1-u^{\frac{1}{b}}\right)^{-\frac{k}{a}}u^l du \\ &= \frac{b}{e-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\frac{(-1)^{j-k}t^j}{j!l!}\binom{j}{k}B\left(1-\frac{k}{a}, bl+b\right). \end{aligned} \tag{3.4.5}$$

Similarly, the characteristic function, $\phi_X(t)$ can be derived as,

$$\phi_X(t) = E(e^{itX}) = \frac{b}{e-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\frac{(-1)^{j-k}(it)^j}{j!l!}\binom{j}{k}B\left(1-\frac{k}{a}, bl+b\right). \tag{3.4.6}$$

Hence the cumulant generating function can be written as,

$$\begin{aligned} K_X(t) &= \log\phi_X(t) \\ &= log\left\{\frac{b}{e-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\frac{(-1)^{j-k}(it)^j}{j!l!}\binom{j}{k}B\left(1-\frac{k}{a}, bl+b\right)\right\}. \end{aligned} \tag{3.4.7}$$

3.4.4 Quantiles

The quantile function $Q(p)$ of the random variable X is,

$$Q(p) = \left(1-\left[\log\left(p(e-1)+1\right)\right]^{\frac{1}{b}}\right)^{-\frac{1}{a}} - 1, \tag{3.4.8}$$

where $p\,\epsilon\,(0,1)$. When $p=\frac{1}{2}$, one obtains the median of DUS-IK(a, b), *i.e.,*

$$Median = \left(1-\left[\log\left(\left(\frac{e-1}{2}\right)+1\right)\right]^{\frac{1}{b}}\right)^{-\frac{1}{a}} - 1.$$

Using Eq. (3.4.8), the interquartile range (IQR) of DUS-IK(a, b) is,

$$IQR = \left(1-\left[\log\left(\left(\frac{3(e-1)}{4}\right)+1\right)\right]^{\frac{1}{b}}\right)^{-\frac{1}{a}} - \left(1-\left[\log\left(\left(\frac{e-1}{4}\right)+1\right)\right]^{\frac{1}{b}}\right)^{-\frac{1}{a}}. \tag{3.4.9}$$

3.4.5 Order Statistics

Many domains of statistics, including as reliability and life testing, have extensively used order statistics. Let $X_{(1)}, X_{(2)}, X_{(3)}, ..., X_{(n)}$ be the order statistics of *i.i.d.* random variables $X_1, X_2, X_3, ..., X_n$ of size n. Then the PDF of the r^{th} order statistic is given by,

$$g_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} g(x) G^{r-1}(x) \{1 - G(x)\}^{n-r}.$$

Accordingly, one can get the PDF of r^{th} order statistic of DUS-IK(a, b) distribution as,

$$\begin{aligned} g_{X_{(r)}}(x) =& \frac{n!}{(r-1)!(n-r)!} \frac{ab}{(e-1)^n} (1+x)^{-a-1} \left[1-(1+x)^{-a}\right]^{b-1} e^{\left[1-(1+x)^{-a}\right]^b} \times \\ & \left[e^{\left[1-(1+x)^{-a}\right]^b} - 1\right]^{r-1} \left[e - e^{\left[1-(1+x)^{-a}\right]^b}\right]^{n-r}; \; x > 0; \; a, \; b > 0. \end{aligned} \tag{3.4.10}$$

When $r = 1$, Eq. (3.4.10) gives the PDF of 1^{st} order statistic, *i.e.*,

$$\begin{aligned} g_{X_{(1)}}(x) = & \frac{nab}{(e-1)^n} (1+x)^{-a-1} \left[1-(1+x)^{-a}\right]^{b-1} e^{\left[1-(1+x)^{-a}\right]^b} \times \\ & \left[e - e^{\left[1-(1+x)^{-a}\right]^b}\right]^{n-1}; \; x > 0; \; a, b > 0. \end{aligned} \tag{3.4.11}$$

When $r = n$, Eq. (3.4.10) gives the PDF of n^{th} order statistic, *i.e.,*

$$\begin{aligned} g_{X_{(n)}}(x) = & \frac{nab}{(e-1)^n} (1+x)^{-a-1} \left[1-(1+x)^{-a}\right]^{b-1} e^{\left[1-(1+x)^{-a}\right]^b} \times \\ & \left[e^{\left[1-(1+x)^{-a}\right]^b} - 1\right]^{n-1}; \; x > 0; \; a, b > 0. \end{aligned} \tag{3.4.12}$$

For $a > 0$, $b > 0$, $x > 0$, the CDF of 1^{st} and n^{th} order statistics are,

$$G_{X_{(1)}} = 1 - \left[\frac{e - e^{\left[1-(1+x)^{-a}\right]^b}}{e-1}\right]^n \text{ and } G_{X_{(n)}} = \left[\frac{e^{\left[1-(1+x)^{-a}\right]^b} - 1}{e-1}\right]^n \text{ respectively.}$$

3.5 Characterization

In this section, the characterization of the DUS-IK(a, b) distribution are obtained based on different criteria: (i) based on two truncated moments, (ii) based on the hazard function, (iii) based on the reversed hazard function, and (iv) based on the conditional expectation of a specific function of the random variable.

3.5.1 Characterizations based on two truncated moments

Initially, we examine the characterization of the DUS-IK(a, b) distribution by considering the ratio of two truncated moments. By applying Theorem 3.5.1, as proposed by Glänzel (1987) we can derive the characterization based on truncated moments.

Theorem 3.5.1. *Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, $D = [a, b]$ is an interval with $(a < b)$ (where $a = -\infty$ and $b = \infty$ are also possible). Now, if X is a continuous random variable with CDF $G(x)$, defined from $\Omega \to D$, and $Q_1(x)$ and $Q_2(x)$ are two real functions defined on D such that*

$$\frac{E[Q_2(x)|X \geq x]}{E[Q_1(x)|X \geq x]} = \phi(x),\ x \in D, \tag{3.5.1}$$

here, $\phi(x)$ is a real function. Suppose $Q_1(x)$ and $Q_2(x)$ belong to the class $C^1(D)$, ϕ is in $C^2(D)$, and G is a twice continuously differentiable, strictly monotone function defined on the set D. Also, consider that the equation $\phi Q_1 = Q_2$ lacks a real solution within the confines of the interior of D. Under these conditions, the function G, denoted by

$$G(x) = \int_a^x \mathcal{K}\left|\frac{\phi^{'}(u)}{\phi(u)Q_1(u) - Q_2(u)}\right| e^{-w(u)} du, \tag{3.5.2}$$

is uniquely defined by the functions Q_1, Q_2 and ϕ. Here, w is the solution to the differential equation $w^{'} = \frac{\phi^{'} Q_1}{\phi Q_1 - Q_2}$, and $\mathcal{K}$ serves as the normalization constant ensuring that $\int_D dG = 1$.

Proposition 3.5.2. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $Q_1(x) = 2(1-(1+x)^{-a})^{1-b} e^{-(1-(1+x)^{-a})^b}(1+x)^{-a}$ and $Q_2(x) = 3(1-(1+x)^{-a})^{1-b} e^{-(1-(1+x)^{-a})^b}(1+x)^{-2a}$ for $x \in (0, \infty)$. Then the PDF of the random variable X conforms to Eq. (3.2.3) if and only if the function $\phi(x)$, as specified in Theorem 3.5.1, assumes the prescribed structure $\phi(x) = (1+x)^{-a}$, $x > 0$.*

Proof. Assuming the PDF of the random variable X is given by Eq. (3.2.3) then,

$$[1 - F(x)]E[Q_1(x)|X \geq x] = \frac{b}{e-1}(1+x)^{-2a},\ x > 0,$$

and

$$[1 - F(x)]E[Q_2(x)|X \geq x] = \frac{b}{e-1}(1+x)^{-3a},\ x > 0.$$

Further, $\phi(x)Q_1(x) - Q_2(x) = -(1-(1+x)^{-a})^{1-b}e^{-(1-(1+x)^{-a})^b}(1+x)^{-2a} < 0$ for $x > 0$. On the other hand, if ϕ is in the form mentioned above, then

$$w^{'}(x) = \frac{\phi^{'}(x)Q_1(x)}{\phi(x)Q_1(x) - Q_2(x)} = 2a(1+x)^{-1},\ x > 0, \tag{3.5.3}$$

and hence $w(x) = 2a\log(1+x)$, $x > 0$.

In light of Theorem 3.5.1, the random variable X possesses the PDF represented by Eq. (3.2.3). □

Corollary 3.5.3. *For a continuous random variable $X : \Omega \to (0,\infty)$ and $Q_1(x)$ defined per Proposition 3.5.2, the PDF of X is expressed by Eq. (3.2.3) if and only if there exist functions $Q_1(x)$ and $Q_2(x)$, as specified in Theorem 3.5.1, that fulfill the ensuing differential equation,*

$$\frac{\phi^{'}(x)Q_1(x)}{\phi(x)Q_1(x) - Q_2(x)} = 2a(1+x)^{-1},\ x > 0. \tag{3.5.4}$$

Corollary 3.5.4. *The general solution to the differential equation provided in Corollary 3.5.3 is*

$$\phi(x) = (1+x)^a\left[-\int 2a(1+x)^{-1}(1+x)^{-a}(Q_1(x))^{-1}Q_2(x)dx + C\right], \tag{3.5.5}$$

where C is a constant. Take note that Proposition 3.5.2 provides a set of functions that satisfy the described differential equation when C is equal to 0. However, it's crucial to emphasize that there exist other sets of triplets $(Q_1(x), Q_2(x), \phi(x))$ that also meet the conditions outlined in Theorem 3.5.1.

3.5.2 Characterization based on the hazard function

Proposition 3.5.5. *Assume a continuous random variable X, mapping from the sample space Ω to values ranging from 0 to ∞, possesses a PDF of Eq. (3.2.3) if and only if its hazard function $h(x)$ complies with the following differential equation*

$$h^{'}(x) - \frac{f^{'}(x)}{f(x)}h(x) = \frac{ea^2b^2(1+x)^{-2a-2}(1-(1+x)^{-a})^{2b-2}e^{(1-(1+x)^{-a})^b}}{(e - e^{(1-(1+x)^{-a})^b})^2} \tag{3.5.6}$$

with boundary condition $\lim_{x\to\infty} h(x) = 0$.

Proof. If X follows a PDF of Eq. (3.2.3), it is evident that the previously mentioned differential equation is satisfied. Subsequently, if the stated differential equation is valid, then

$$\frac{d}{dx}\left\{(f(x))^{-1}h(x)\right\} = \frac{d}{dx}\left\{\frac{e^{(1-(1+x)^{-a})^b}}{e - e^{(1-(1+x)^{-a})^b}}\right\}. \tag{3.5.7}$$

Hence, $h(x) = \frac{ab(1+x)^{-a-1}\left[1-(1+x)^{-a}\right]^{b-1} e^{[1-(1+x)^{-a}]^b}}{e-e^{[1-(1+x)^{-a}]^b}}$, denotes the hazard function of the DUS-IK(a, b) distribution. □

3.5.3 Characterization based on the reversed hazard function

Proposition 3.5.6. *Assume a continuous random variable X, mapping from the sample space Ω to values ranging from 0 to ∞, possesses a PDF of Eq. (3.2.3) if and only if its reversed hazard function $r(x)$ complies with the following differential equation*

$$r^{'}(x) - \frac{f^{'}(x)}{f(x)}r(x) + \frac{a^2b^2(1+x)^{-2a-2}(1-(1+x)^{-a})^{2b-2}e^{(1-(1+x)^{-a})^b}}{(e^{(1-(1+x)^{-a})^b}-1)^2} = 0 \tag{3.5.8}$$

with boundary condition $\lim_{x\to\infty} r(x) = 0$.

Proof. If X follows a PDF of Eq.(3.2.3), then it is clear that the above differential equation holds. Now, suppose that Eq. (3.5.8) satisfies, then

$$\frac{d}{dx}\left\{(f(x))^{-1}r(x)\right\} = \frac{d}{dx}\left\{\frac{e^{(1-(1+x)^{-a})^b}}{e^{(1-(1+x)^{-a})^b} - 1}\right\}. \tag{3.5.9}$$

i.e., $r(x) = \frac{ab(1+x)^{-a-1}[1-(1+x)^{-a}]^{b-1}e^{[1-(1+x)^{-a}]^b}}{e^{[1-(1+x)^{-a}]^b}-1}$, is the reverse hazard function of DUS-IK(a, b). □

3.5.4 Characterization based on the conditional expectation of specific function of the random variable

Proposition 3.5.7. *Assume a continuous random variable X, mapping from the sample Ω to values ranging from 0 to ∞, possesses a CDF of Eq. (3.2.4). Consider a differentiable function $\mathcal{K}(x)$ defined for values from 0 to ∞ as given by:*

$$\mathcal{K}(x) = \theta\left[1 - \left(\frac{1}{e-1}\left(e^{(1-(1+x)^{-a})^b} - 1\right)\right)^{\frac{1}{\theta-1}}\right]^{-1}, \tag{3.5.10}$$

where $\theta > 1$. *Then*

$$E[(\mathcal{K}(x))^{\theta}|X \leq x] = \theta(\mathcal{K}(x))^{\theta-1},\, x \in (0, \infty) \tag{3.5.11}$$

if and only if $X \sim$ *DUS-IK distribution.*

Proof. Suppose that $E[(\mathcal{K}(x))^{\theta}|X \leq x] = \theta(\mathcal{K}(x))^{\theta-1}$ holds. Then

$$\int_0^x (\mathcal{K}(z))^{\theta} g(z)dz = \theta(\mathcal{K}(x))^{\theta-1}G(x). \tag{3.5.12}$$

Differentiating each sides of the equation,

$$(\mathcal{K}(x))^{\theta}g(x) = \theta\left[(\theta-1)\mathcal{K}^{'}(x)(\mathcal{K}(x))^{\theta-2}G(x) + (\mathcal{K}(x))^{\theta-1}g(x)\right]. \tag{3.5.13}$$

Upon integrating after considering partial fractions, the integral expression

$$\int \frac{g(x)}{G(x)}dx = (\theta-1)\int\left[-\frac{\mathcal{K}^{'}(x)}{\mathcal{K}(x)} + \frac{\mathcal{K}^{'}(x)}{\mathcal{K}(x)-\theta}\right]dx \tag{3.5.14}$$

can be simplified to

$$G(x) = \left[1 - \frac{\theta}{\mathcal{K}(x)}\right]^{\theta-1}. \tag{3.5.15}$$

Since $\mathcal{K}(x) = \theta\left[1 - \left(\frac{1}{e-1}\left(e^{(1-(1+x)^{-a})^b} - 1\right)\right)^{\frac{1}{\theta-1}}\right]^{-1}$, where $\theta > 1$. Then we have $G(x) = \frac{1}{e-1}\left(e^{(1-(1+x)^{-a})^b} - 1\right)$.

Conversely, if X follows the DUS-IK distribution, then it is evident that Eq. (3.5.11) holds. $\square$

Proposition 3.5.8. *Assume a continuous random variable* X*, mapping from the sample space* Ω *to values ranging from 0 to* ∞*, possesses a CDF of Eq. (3.2.4). Consider a differentiable function* $\psi(x)$ *defined for values from 0 to* ∞ *as given by:*

$$\psi(x) = \theta\left[1 + \left(\frac{e - e^{(1-(1+x)^{-a})^b}}{e-1}\right)^{\frac{1}{\theta-1}}\right]^{-1},\, \theta > 1. \tag{3.5.16}$$

Then

$$E[(\psi(x))^{\theta}|X \geq x] = \theta(\psi(x))^{\theta-1},\, x \in (0, \infty), \tag{3.5.17}$$

if and only if $X \sim$ *DUS-IK distribution.*

Proof. Suppose that $E[(\psi(x))^{\theta}|X \geq x] = \theta(\psi(x))^{\theta-1}$ holds. Then

$$\int_x^{\infty} (\psi(z))^{\theta} g(z)dz = \theta(\psi(x))^{\theta-1}\bar{G}(x). \tag{3.5.18}$$

Differentiating each sides of the equation,

$$-(\psi(x))^{\theta} g(x) = \theta \left[(\theta-1)\psi^{'}(x)(\psi(x))^{\theta-2}\bar{G}(x) - (\psi(x))^{\theta-1}g(x)\right]. \tag{3.5.19}$$

Upon integrating after considering partial fractions, the integral expression

$$\int \frac{g(x)}{\bar{G}(x)}dx = (\theta-1)\int \left[-\frac{\psi^{'}(x)}{\psi(x)} + \frac{\psi^{'}(x)}{\theta-\psi(x)}\right] dx \tag{3.5.20}$$

can be simplified to

$$\bar{G}(x) = \left[-1 + \frac{\theta}{\psi(x)}\right]^{\theta-1}. \tag{3.5.21}$$

Since $\psi(x) = \theta \left[1 + \left(\frac{e-e^{(1-(1+x)^{-a})^b}}{e-1}\right)^{\frac{1}{\theta-1}}\right]^{-1}$, $\theta > 1$. Then we have $\bar{G}(x) = \frac{e-e^{(1-(1+x)^{-a})^b}}{e-1}$.

Conversely, if X follows the DUS-IK distribution, then it is evident that Eq. (3.5.17) holds. $\square$

3.6 Entropy

The term entropy of a random variable X represents the measure of variability and it has been extensively applied in a variety of areas. This section discusses three types of entropies namely, Reǹyi entropy, Shannon entropy and q-entropy.

The Reǹyi entropy for a random variable X with PDF $g(x)$ is defined by,

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\left(\int g^{\gamma}(x)dx\right),$$

where $\gamma > 0$ and $\gamma \neq 1$. Suppose the random variable X has DUS-IK(a, b) distribution, the term $\int g^{\gamma}(x)dx$ can be computed as,

$$\int g^{\gamma}(x)dx = \frac{a^{\gamma}b^{\gamma}}{(e-1)^{\gamma}} \int_0^{\infty} (1+x)^{-\gamma(a+1)} \left[1-(1+x)^{-a}\right]^{\gamma(b-1)} e^{\gamma[1-(1+x)^{-a}]^b} dx.$$

Using Binomial expansion, we have

$$\begin{aligned}\int g^{\gamma}(x)dx &= \frac{a^{\gamma}b^{\gamma}}{(e-1)^{\gamma}}\sum_{i=0}^{\infty}\frac{r^i}{i!}\int_0^{\infty}(1+x)^{-\gamma(a+1)}\left[1-(1+x)^{-a}\right]^{\gamma b+bi-\gamma}dx\\ &= \frac{a^{\gamma-1}b^{\gamma}}{(e-1)^{\gamma}}\sum_{i=0}^{\infty}\frac{r^i}{i!}B\left(\gamma b+bi-\gamma+1,\gamma+\frac{\gamma-1}{a}\right).\end{aligned}$$

Thus the Reǹyi entropy of X is obtained as,

$$\tau_R(\gamma)=\frac{1}{1-\gamma}\log\left(\frac{a^{\gamma-1}b^{\gamma}}{(e-1)^{\gamma}}\sum_{i=0}^{\infty}\frac{r^i}{i!}B\left(\gamma b+bi-\gamma+1,\gamma+\frac{\gamma-1}{a}\right)\right). \qquad (3.6.1)$$

The Shannon entropy of the random variable X is given by,

$$\begin{aligned}E(-\log g(X)) = -\log\frac{ab}{e-1}&+(a+1)E(\log(1+X))+(b-1)\sum_{i=1}^{\infty}\frac{E(1+X)^{-ai}}{i}\\ &-\sum_{r=0}^{\infty}(-1)^r\binom{b}{r}E(1+X)^{-ar}. \qquad (3.6.2)\end{aligned}$$

The q-entropy of the random variable X is defined as

$$H_q(x)=\frac{1}{1-q}\log\left(1-\int_{-\infty}^{\infty}g^q(x)dx\right),\ q>0\,\text{and}\,q\neq 1.$$

$$H_q(x)=\frac{1}{1-q}\log\left\{1-\frac{a^{q-1}b^q}{(e-1)^q}\sum_{i=0}^{\infty}\frac{q^i}{i!}B\left(qb+bi-q+1,q+\frac{q-1}{a}\right)\right\}. \qquad (3.6.3)$$

3.7 Maximum Likelihood Estimation

The parameters of the DUS-IK(a, b) distribution are estimated in this section. If $X_1, X_2, ..., X_n$ is a random sample from DUS-IK(a, b) with parameters a and b then the distribution's log-likelihood function is given by

$$\begin{aligned}\log L = n\log a + n\log b - n\log(e-1) - (a-1)\sum_{i=1}^{n}\log(1+x_i) +\\ (b-1)\sum_{i=1}^{n}\log(1-(1+x_i)^{-a})+\sum_{i=1}^{n}(1-(1+x_i)^{-a})^b.\end{aligned} \qquad (3.7.1)$$

Let the parameters a and b are assumed to be unknown and the normal equations for them are obtained by taking their derivatives partially with respect to its parameters, are as follows:

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} \log(1+x_i) + (b-1)\sum_{i=1}^{n} \frac{(1+x_i)^{-a}\log(1+x_i)}{(1-(1+x_i)^{-a})} + b\sum_{i=1}^{n}(1-(1+x_i)^{-a})^{b-1}(1+x_i)^{-a}\log(1+x_i) \tag{3.7.2}$$

$$\frac{\partial \log L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log(1-(1+x_i)^{-a}) + \sum_{i=1}^{n}(1-(1+x_i)^{-a})^{b}\log(1-(1+x_i)^{-a}) \tag{3.7.3}$$

The MLEs of the parameters a and b are obtained by solving the preceding system of equations. Finding the system's exact solution is tough. Iterative processes such as the Newton-Raphson approach, which are accessible in software like as R, Python, Mathematica, and others, can be used to maximize the likelihood function.

Next, we can look into the existence and uniqueness of the MLEs when the other parameter is given.

Theorem 3.7.1. *Let $g_1(a; b, x)$ represent the function on the right-hand side (RHS) of Eq.(3.7.2), where the parameter b is given. In this case, there exist at least one solution for $g_1(a; b, x) = 0$ for $b > 0$ and this solution is unique when*

$$-a(b-1)\sum_{i=1}^{n} \frac{(1+x_i)^{-a-1}\log(1+x_i)}{[1-(1+x_i)^{-a}]^2} < \frac{n}{a^2}.$$

Proof. Given,

$$g_1(a; b, x) = \frac{n}{a} - \sum_{i=1}^{n} \log(1+x_i) + (b-1)\sum_{i=1}^{n} \frac{(1+x_i)^{-a}\log(1+x_i)}{(1-(1+x_i)^{-a})} + b\sum_{i=1}^{n}(1-(1+x_i)^{-a})^{b-1}(1+x_i)^{-a}\log(1+x_i)$$

Consider, $\lim_{a\to 0} g_1(a; b, x) = \infty - \sum_{i=1}^{n} \log(1+x_i) + \infty + 0 = \infty$. Also $\lim_{a\to\infty} g_1(a; b, x) = 0 - \sum_{i=1}^{n} \log(1+x_i) + 0 + 0 < 0$. Thus there exist at least one solution $\hat{a} \in (0, \infty)$ such that $g_1(a; b, x) = 0$. Morover the solution will be unique when $\frac{\partial g_1(a;b,x)}{\partial a} < 0$, where $\frac{\partial g_1(a;b,x)}{\partial a} = -\frac{n}{a^2} - a(b-1)\sum_{i=1}^{n} \frac{(1+x_i)^{-a-1}\log(1+x_i)}{[1-(1+x_i)^{-a}]^2}$. □

Theorem 3.7.2. *Let $g_2(b; a, x)$ represent the function on the right-hand side (RHS) of Eq.(3.7.3), where the parameter a is given. In this case, there exist at least one* *solution for $g_2(b; a, x) = 0$ for $a > 0$ and this solution is unique when*

$$\sum_{i=1}^{n}(1-(1+x_i)^{-a})^b[\log(1-(1+x_i)^{-a})]^2 < \frac{n}{b^2}.$$

Proof. Given

$$g_2(b; a, x) = \frac{n}{b} + \sum_{i=1}^{n}\log(1-(1+x_i)^{-a}) + \sum_{i=1}^{n}(1-(1+x_i)^{-a})^b\log(1-(1+x_i)^{-a})$$

Consider, $\lim_{b\to 0} g_2(b; a, x) = \infty + \sum_{i=1}^{n}\log(1-(1+x_i)^{-a}) + \sum_{i=1}^{n}\log(1-(1+x_i)^{-a}) = \infty$. Also $\lim_{b\to\infty} g_2(b; a, x) = 0 + \sum_{i=1}^{n}\log(1-(1+x_i)^{-a}) - \infty < 0$. Thus there exist at least one solution $\hat{b} \in (0, \infty)$ such that $g_2(b; a, x) = 0$. Morover the solution will be unique when $\frac{\partial g_2(b;a,x)}{\partial b} < 0$, where $\frac{\partial g_2(b;a,x)}{\partial b} = -\frac{n}{b^2} + \sum_{i=1}^{n}(1-(1+x_i)^{-a})^b[\log(1-(1+x_i)^{-a})]^2$. □

To determine the confidence intervals of a and b, it is essential to consider the Fisher information matrix, I as;

$$I = \begin{bmatrix} E\left(-\frac{\partial^2 \log L}{\partial a^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial a \partial b}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial b \partial a}\right) & E\left(-\frac{\partial^2 \log L}{\partial b^2}\right) \end{bmatrix}$$

. The second order partial derivatives inside the information matrix are:

$$\begin{aligned}
\frac{\partial^2 \log L}{\partial a^2} &= -\frac{n}{a^2} - a(b-1)\sum_{i=1}^{n}\frac{(1+x_i)^{-a-1}\log(1+x_i)}{[1-(1+x_i)^{-a}]^2} \\
\frac{\partial^2 \log L}{\partial b^2} &= -\frac{n}{b^2} + \sum_{i=1}^{n}(1-(1+x_i)^{-a})^b[\log(1-(1+x_i)^{-a})]^2 \\
\frac{\partial^2 \log L}{\partial a \partial b} &= \sum_{i=1}^{n}\frac{a(1+x_i)^{-a-1}}{1-(1+x_i)^{-a}} + \sum_{i=1}^{n}\frac{a+b(1+x_i)\log(1+x_i-(1+x_i)^{-a+1})}{(1+x_i)^{a+1}[1-(1+x_i)^{-a}]^{1-b}} \\
&= \frac{\partial^2 \log L}{\partial b \partial a}
\end{aligned}$$

As for large sample property, the asymptotic distribution of MLE of $\theta = (a, b)$ could be regarded as Normally distributed with mean zero and variance-covariance matrix I^{-1}, i.e. $n(\hat{\theta} - \theta) \to N(0, I^{-1})$. Where I^{-1} is the inverse of Fisher information matrix and n is the sample size. Therefore, the asymptotic $100(1-\delta)\%$ confidence intervals of a and b are $\hat{a} \pm Z_{\frac{\delta}{2}}\sqrt{Var(\hat{a})}$ and $\hat{b} \pm Z_{\frac{\delta}{2}}\sqrt{Var(\hat{b})}$ respectively. Where

$Z_{\frac{\delta}{2}}$ denotes the upper $100(\frac{\delta}{2})^{th}$ percentile of standard normal distribution.

3.8 Stress-Strength Reliability

Stress-strength model has a significant role in reliability engineering. Let X indicate the strength of a component or system that is subjected to a random stress Y. The system's functioning is then defined by SSR. When applied random stress exceeds the system's strength, it fails. If X and Y are distributed as DUS-IK(a, b_1) and DUS-IK(a, b_2), respectively, then the system reliability $R = P[Y < X]$ is,

$$\begin{aligned}
R =& P[Y < X] = \int_0^\infty g_X(x) G_Y(x) dx \\
=& \int_0^\infty \frac{1}{(e-1)^2} ab_1 (1+x)^{-a-1} \left[1-(1+x)^{-a}\right]^{b_1-1} e^{[1-(1+x)^{-a}]^{b_1}} \left[e^{[1-(1+x)^{-a}]^{b_2}} - 1\right] dx \\
=& \frac{ab_1}{(e-1)^2} \int_0^\infty (1+x)^{-a-1} \left[1-(1+x)^{-a}\right]^{b_1-1} e^{[1-(1+x)^{-a}]^{b_1}} e^{[1-(1+x)^{-a}]^{b_2}} dx - \frac{1}{e-1} \\
=& \frac{ab_1}{(e-1)^2} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{1}{m!n!} \int_0^\infty (1+x)^{-a-1} \left[1-(1+x)^{-a}\right]^{b_1 m + b_2 n + b_1 - 1} dx - \frac{1}{e-1}.
\end{aligned}$$

By setting $u = 1 - (1+x)^{-a}$, the integral yields,

$$R = \frac{b_1}{(e-1)^2} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{1}{m!n!} \int_0^1 u^{b_1 m + b_2 n + b_1 - 1} du - \frac{1}{e-1}.$$

$$i.e.,\ R = \frac{b_1}{(e-1)^2} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{1}{m!n!} \frac{1}{b_1 m + b_2 n + b_1} - \frac{1}{e-1}. \tag{3.8.1}$$

3.8.1 Estimation of Reliability

This section deals with the estimation of SSR. To obtain the estimate of SSR, consider the joint likelihood of the stress-strength sample X_1, X_2, X_3,..., X_n, Y_1, Y_2, Y_3,..., Y_m,

$$\begin{aligned}
L(x, y, a, b_1, b_2) = & \prod_{i=1}^n \frac{1}{e-1} ab_1 (1+x_i)^{-a-1} \left[1-(1+x_i)^{-a}\right]^{b_1-1} e^{[1-(1+x_i)^{-a}]^{b_1}} \times \\
& \prod_{j=1}^m \frac{1}{e-1} ab_2 (1+y_j)^{-a-1} \left[1-(1+y_j)^{-a}\right]^{b_2-1} e^{[1-(1+y_j)^{-a}]^{b_2}}.
\end{aligned} \tag{3.8.2}$$

The log-likelihood of the joint sample is obtained as,

$$\log L(x, y, a, b_1, b_2) = -(m+n)\log(e-1) + (m+n)\log a + n \log b_1 + m \log b_2$$

$$- (a+1)\sum_{i=1}^{n}\log(1+x_i) + (b_1-1)\sum_{i=1}^{n}\log[1-(1+x_i)^{-a}] + \sum_{i=1}^{n}\Big[1-(1+x_i)^{-a}\Big]^{b_1}$$
$$- (a+1)\sum_{j=1}^{m}\log(1+y_j) + (b_2-1)\sum_{j=1}^{m}\log\Big[1-(1+y_j)^{-a}\Big] + \sum_{j=1}^{m}[1-(1+y_j)^{-a}]^{b_2}. \tag{3.8.3}$$

The MLEs of the parameters can be obtained as a solution of the following non-linear equations:

$$\frac{\partial \log L}{\partial a} = \frac{m+n}{a} - \sum_{i=1}^{n}\log(1+x_i) - \sum_{j=1}^{m}\log(1+y_j) + (b_1-1)\sum_{i=1}^{n}\frac{(1+xi)^{-a}\log(1+x_i)}{[1-(1+x_i)^{-a}]}$$
$$+ b_1\sum_{i=1}^{n}(1+x_i)^{-a}\Big[1-(1+x_i)^{-a}\Big]^{b_1-1}\log(1+x_i) + (b_2-1)\sum_{j=1}^{m}\frac{(1+y_j)^{-a}\log(1+y_j)}{[1-(1+y_j)^{-a}]}$$
$$+ b_2\sum_{j=1}^{m}(1+y_j)^{-a}\Big[1-(1+y_j)^{-a}\Big]^{b_2-1}\log(1+y_j)$$

$$\frac{\partial \log L}{\partial b_1} = \frac{n}{b_1} + \sum_{i=1}^{n}\log[1-(1+x_i)^{-a}] + \sum_{i=1}^{n}[1-(1+x_i)^{-a}]^{b_1}\log[1-(1+x_i)^{-a}]$$
$$\frac{\partial \log L}{\partial b_2} = \frac{m}{b_2} + \sum_{j=1}^{m}\log[1-(1+y_j)^{-a}] + \sum_{j=1}^{m}[1-(1+y_j)^{-a}]^{b_2}\log[1-(1+y_j)^{-a}]$$

Suppose that $\hat{a}$, $\hat{b_1}$ and $\hat{b_2}$ represents the MLEs of a, b_1 and b_2 respectively. Using Invariance property, MLE of SSR can be written as,

$$\hat{R} = \frac{\hat{b_1}}{(e-1)^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}\frac{1}{\hat{b}_1 m + \hat{b}_2 n + \hat{b_1}} - \frac{1}{e-1}. \tag{3.8.4}$$

3.8.2 Asymptotic Confidence Interval of SSR

For large sample size, MLEs are asymptotically normal with variance-covariance matrix equal to the inverse of Fisher information matrix. Morover when the sample size approaches infinity the bias will tend to zero and their variance attains Cramer-Rao lower bound. To derive the asymptotic variance of $\hat{R}$, the section continued as follows.

The asymptotic distribution of MLE of SSR ($\hat{R}$) is given by,

$$\sqrt{(n+m)}(\hat{R}-R) \xrightarrow{d} N\Big(0, \Delta'(\theta)I^{-1}(\theta)\Delta(\theta)\Big),$$

where $\Delta(\theta) = \left(\frac{\partial R}{\partial a}, \frac{\partial R}{\partial b_1}, \frac{\partial R}{\partial b_2}\right)' = (\xi_1, \xi_2, \xi_3)'$ and $I^{-1}(\theta)$ denote the inverse of Fisher information matrix for the unknown parameters $\theta = (\hat{a}, \hat{b_1}, \hat{b_2})$ and is given by,

$$I(\theta) = \begin{bmatrix} E\left(-\frac{\partial^2 \log l}{\partial a^2}\right) & E\left(-\frac{\partial^2 \log l}{\partial a \partial b_1}\right) & E\left(-\frac{\partial^2 \log l}{\partial a \partial b_2}\right) \\ E\left(-\frac{\partial^2 \log l}{\partial b_1 \partial a}\right) & E\left(-\frac{\partial^2 \log l}{\partial b_1^2}\right) & E\left(-\frac{\partial^2 \log l}{\partial b_1 \partial b_2}\right) \\ E\left(-\frac{\partial^2 \log l}{\partial b_2 \partial a}\right) & E\left(-\frac{\partial^2 \log l}{\partial b_2 \partial b_1}\right) & E\left(-\frac{\partial^2 \log l}{\partial b_2^2}\right) \end{bmatrix}.$$

Ther terms inside the matrix can derive directly. Thus the asymptotic variance of $\hat{R}$ is,

$$\begin{aligned} AV(\hat{R}) &= \frac{1}{n+m}\Delta'(\theta)I^{-1}(\theta)\Delta(\theta) \\ &= Var(\hat{a})\xi_1^2 + Var(\hat{b_1})\xi_2^2 + Var(\hat{b_2})\xi_3^2 + 2\xi_1\xi_2 Cov(\hat{a}, \hat{b_1}) + \\ & 2\xi_1\xi_3 Cov(\hat{a}, \hat{b_2}) + 2\xi_2\xi_3 Cov(\hat{b_1}, \hat{b_2}). \end{aligned}$$

Therefore, the 95% asymptotic confidence interval of $\hat{R}$ is $\hat{R} \pm 1.96\sqrt{AV(\hat{R})}$.

3.9 Simulation

To validate the results obtained from the above sections a simulation study has been conducted for various sample sizes and parameter values. The performances of MLEs are justified based on data generated through inversion method. Here the sample sizes are taken as 25, 50, 100, 300, 500, 1000, 2000, 3000 and the parameters as $(0.5, 1)$, $(0.4, 0.2)$, $(5, 3)$ corresponding to (a, b). To measure the bias and MSE, MLEs are performed for each combination of sample sizes and parameter values, and the process is replicated 1000 times. The complete process of simulation techniques is done using statistical software R.

The analysis of Table 3.2 reveals a consistent pattern: as the sample size increases, both biases and MSEs decrease across all parameter value sets, indicating a trend towards more accurate estimates. Conversely, higher parameter values correspond to elevated biases and MSEs, emphasizing the challenge of estimation accuracy with larger parameter values. Moreover, with increasing sample sizes, estimates tend to converge towards their true values, highlighting the importance of robust sample sizes for reliable statistical inference. This underscores the necessity for careful consideration of sample size and parameter values in statistical analysis to ensure accurate and trustworthy results.

Table 3.2: Simulation Table

(a, b)	Sample size	$\hat{a}$	$\hat{b}$	Bias($\hat{a}$)	Bias($\hat{b}$)	MSE($\hat{a}$)	MSE($\hat{b}$)
(0.5,1)	25	0.5509	1.1564	0.05098	0.15646	0.02289	0.187
	50	0.5252	1.07	0.02522	0.07001	0.00914	0.06526
	100	0.5102	1.0307	0.01028	0.03075	0.00405	0.02873
	300	0.5039	1.0126	0.00394	0.01265	0.00124	0.00814
	500	0.5027	1.0099	0.00278	0.00998	0.00072	0.00463
	1000	0.5015	1.0058	0.00152	0.00585	0.00034	0.00228
	2000	0.5004	1.0024	0.00041	0.00243	0.00017	0.00117
	3000	0.5003	1.0013	0.00033	0.00135	0.00011	0.00078
(0.4,0.2)	25	0.5033	0.2193	0.10332	0.01931	0.07946	0.00367
	50	0.436	0.2092	0.03606	0.00926	0.01711	0.00157
	100	0.4198	0.2042	0.0198	0.00424	0.00745	0.00069
	300	0.4084	0.2024	0.00843	0.00244	0.00202	0.00018
	500	0.4062	0.2021	0.0062	0.00213	0.00111	0.0001
	1000	0.4037	0.2014	0.00374	0.00148	0.00044	4.66×10^{-5}
	2000	0.4018	0.2006	0.00184	0.00064	0.00016	1.64×10^{-5}
	3000	0.4012	0.2003	0.00127	0.00038	8.24×10^{-5}	8.07×10^{-6}
(5,3)	25	5.3936	3.7418	0.39369	0.74186	1.42056	3.59249
	50	5.1981	3.3236	0.19812	0.32363	0.59184	1.06374
	100	5.0813	3.1401	0.0813	0.14016	0.26821	4.27×10^{-1}
	300	5.0312	3.0541	0.03124	0.05417	0.08356	1.16×10^{-1}
	500	5.0224	3.0406	0.0224	0.04069	0.04855	6.50×10^{-2}
	1000	5.0116	3.0225	0.0116	0.02251	0.02297	3.14×10^{-2}
	2000	5.0027	3.0088	0.00271	0.0088	0.01208	1.62×10^{-2}
	3000	5.0023	3.0051	0.0023	0.00518	0.00794	1.07×10^{-2}

3.10 Application

This section provides a comparison of proposed model with some existing models. Two data sets presented to examine the relevance of the model. The proposed model DUS-IK distribution has been compared with IK distribution, K distribution, DUS Kumaraswamy (DUS-K) distribution (Anakha & Chacko (2021)), DUS-nverse Weibull (DUS-IW) distribution (Gauthami & Chacko (2021)) and generalized DUS Weibull (GDUSW) distribution (Kavya & Manoharan (2020)). To measure the goodness of fit, the KS test and CVM test are computed. Additionally, AIC and BIC are used for the comparison of proposed model with the other models.

The first data set is from Dumonceaux & Antle (1973), representing measurements of maximum flood discharges over a four-year period consisting of 20 observations (in millions of cubic feet per second). The second dataset pertains to the endurance of deep-groove ball bearings provided by Lieblein & Zelen (1956); see Table 3.3 and 3.4. Specifically, the ball bearing data consists of observations on the number of million revolutions before failure for each of 23 ball bearings. According to the data provided in Table 3.5, it is evident that both datasets exhibit positive skewness and leptokurtic properties. Consequently, we opt to employ the DUS-IK(a, b) distribution to model them.

The values of estimates, log-likelihood, KS statistic and corresponding p-value, CVM test statistic and corresponding p-value, AIC and BIC for the data sets 1 and 2, are given in Table 3.6 and Table 3.7 respectively. In each table, the proposed model has the lowest KS statistic, CVM statistic, AIC and BIC values as well as greatest log-likelihood value and p-values as compared to other models. The table values provide evidence that the proposed distribution performs better than the given existing models.

Table 3.3: Flood Data

0.265	0.392	0.297	0.3235	0.402
0.269	0.315	0.654	0.338	0.379
0.418	0.423	0.379	0.412	0.416
0.449	0.484	0.494	0.613	0.74

Table 3.4: Endurance of deep-groove ball bearings data

17.88	45.60	54.12	68.88	105.84	28.92
48.40	55.56	84.12	127.92	33.00	51.84
67.80	93.12	128.04	41.52	51.96	68.64
98.64	173.40	42.12	54.12	68.64	105.12

Table 3.5: Summary statistics of the datasets

	Min	**Median**	**Mean**	**Max**	**SD**	**Skewness**	**Kurtosis**
Dataset I	0.265	0.407	0.4231	0.74	0.1252	0.9882	0.248
Dataset II	17.88	61.68	71.47	173.4	36.852	0.9423	0.3513

3.11 Validation

Validation of statistical models is important to ensure their reliability and adequacy in representing observed data. The NRR statistic is a powerful goodness-of-fit test

Table 3.6: Estimates, Log-Likelihood, KS statistic, CVM test statistic, AIC, and BIC for the data set I

	Distributions					
	DUS-IK	**IK**	**DUS-K**	**K**	**DUS-L**	**GDUSW**
Estimates	$\hat{a}$=**17.0157** $\hat{b}$=**147.9997**	$\hat{a}$=15.5745 $\hat{b}$=127.8151	$\hat{\alpha}$=2.8604 $\hat{\beta}$=10.7337	$\hat{\alpha}$=3.3632 $\hat{\beta}$=11.7890	$\hat{\alpha}$=96 $\hat{\beta}$=0.0331	$\hat{\alpha}$=1097 $\hat{\beta}$=110 $\hat{\gamma}$=0.5397
Log-L	**30.1895**	16.2727	12.4558	12.8661	-0.2822	15.9247
KS	**0.1179**	0.1232	0.2042	0.2109	0.4401	0.1354
p-value	**0.9438**	0.9216	0.3742	0.3359	0.0008	0.8565
CVM	**0.0419**	0.0423	0.1591	0.1635	0.9433	0.0549
p-value	**0.9277**	0.9253	0.3652	0.3528	0.0028	0.8512
AIC	**-56.379**	-28.5454	-20.9116	-21.7323	4.5645	-25.8496
BIC	**-54.3875**	-26.554	-18.9201	-19.7409	6.5560	-22.8624

Table 3.7: Estimates, Log-Likelihood, KS statistic, CVM test statistic, AIC, and BIC for the data set II

	Distributions					
	DUS-IK	**IK**	**DUS-K**	**K**	**DUS-L**	**GDUSW**
Estimates	$\hat{a}$=**2.1256** $\hat{b}$=**2904.9999**	$\hat{a}$=1.9178 $\hat{b}$=1789.9999	$\hat{\alpha}$=1.8397 $\hat{\beta}$=134.9998	$\hat{\alpha}$=2.1102 $\hat{\beta}$=199.9997	$\hat{\alpha}$=2.01$\times 10^2$ $\hat{\beta}$=9.059$\times 10^{-5}$	$\hat{\alpha}$=1097 $\hat{\beta}$=110 $\hat{\gamma}$=0.2316
Log-L	**80.2209**	-120.0357	47.2876	47.5510	-124.1931	-119.5335
KS	**0.1327**	0.1348	0.1588	0.1593	0.9988	0.1461
p-value	**0.7912**	0.7757	0.5797	0.5758	2.2$\times 10^{-16}$	0.6845
CVM	**0.0752**	0.0829	0.0777	0.0756	7.9937	0.1040
p-value	**0.7241**	0.6792	0.7097	0.7221	2.2$\times 10^{-16}$	0.5689
AIC	**-156.4418**	244.0714	-90.5752	-88.7460	252.3862	245.0671
BIC	**-154.0857**	246.4275	-88.2191	-88.7460	254.7423	248.6013

designed specifically for validating parametric models. By comparing observed and expected frequencies under the assumed model, this method assesses how well the model aligns with the data. Its application offers an efficient approach for testing the suitability of lifetime distributions and ensuring accurate statistical inference.

3.11.1 NRR statistic for DUS-IK distribution

Given a sample $X = (X_1, X_2, ..., X_n)^T$, we aim to determine if the data fit the DUS-IK distribution, where $P\{X_i \leq x\} = G_{DUS-IK}(x, \boldsymbol{\theta})$ with parameters, $\boldsymbol{\theta} = (a, b)^T$. We perform a chi-square goodness-of-fit test using the NRR statistic, starting by calculating the MLEs $\hat{\boldsymbol{\theta}}$ for these parameters. Since the Y^2 statistic is independent of the parameters, we utilize the Fisher information matrix $\boldsymbol{I}(\boldsymbol{\theta})$ to facilitate the calculations. With all required components for Y^2 provided for the DUS-IK distribution, computing Y^2 is straightforward.

Simulation studies of NRR statistic

To validate the results of this work, we carry out an extensive study through numerical simulations. We aim to illustrate that the Y^2 statistic asymptotically follows a chi-squared distribution with $k-1$ degrees of freedom by performing 10,000 simulations of the $Y^2(\boldsymbol{\theta})$ statistic under the null hypothesis H_0. These simulations incorporate various parameter values for the DUS-IK(a, b) distribution and $k = 12$ intervals. We then compare the simulated distribution of Y^2 to a chi-squared distribution with 11 degrees of freedom. The resulting histograms are presented in Figure 3.5, juxtaposed with the chi-squared distribution for $k-1$ degrees of freedom.

As shown in Figure 3.5, the distribution of the Y^2 statistic across different parameter values and grouping cell counts k aligns with a chi-squared distribution with $k-1$ degrees of freedom, within the expected range of simulation variability. The same results are obtained for varying numbers of equally likely grouping intervals and parameter values, indicating that the limiting distribution of the generalized chi-squared Y^2 statistic is independent of the specific underlying distribution.

Next, we focus on testing the null hypothesis H_0 that the sample follows the DUS-IK distribution. To achieve this, we calculate the NRR statistic Y^2 for 10,000 simulated samples with sizes $n = 25$, 50, 100, 300, and 500 at different theoretical significance levels $\epsilon = 0.01$, 0.02, 0.05, 0.1. We then compute the average number of times the null hypothesis is not rejected when $Y^2 \leq \chi^2_{\epsilon}(k-1)$. The results, which compare empirical and theoretical significance levels, are presented in Table 3.8. It is evident that as the sample size increases, the empirical levels closely approach the theoretical levels. This indicates that the test is well-suited for the DUS-IK distribution.

Table 3.8: Empirical level versus theoretical level

$N = 10,000$	$\epsilon = 0.01$	$\epsilon = 0.02$	$\epsilon = 0.05$	$\epsilon = 0.1$
25	0.9882	0.9883	0.9586	0.9142
50	0.9896	0.9877	0.9543	0.9058
100	0.9891	0.9843	0.9541	0.9044
300	0.9899	0.9835	0.9518	0.9004
500	0.99	0.9818	0.9507	0.9001

Application to flood data

To test the null hypothesis H_0 that the flood data follows the DUS-IK distribution, we apply the NRR statistic derived earlier. First, we compute the MLEs of the parameters using the `nlm` function in R, obtaining $\hat{a} = 17.01996$ and $\hat{b} = 148.19883$. Using these estimates, we then calculate the NRR Y^2 statistic for $k = 8$, which resulting in $Y^2 = 6.400336$. The critical value of the chi-square distribution at a significance level of $\epsilon = 0.05$ for 7 degrees of freedom is $\chi^2_{k-1} = 14.06714$. Since the calculated Y^2 statistic is less than the critical value, we conclude that there is no significant evidence to reject the null hypothesis, indicating that the DUS-IK distribution adequately fits the flood data.

Application using endurance of deep-groove ball bearings data

To test the null hypothesis H_0 that the endurance of deep-groove ball bearings data follows the DUS-IK distribution, we apply the NRR statistic derived earlier. First, we compute the MLEs of the parameters using the `nlm` function in R, obtaining $\hat{a} = 2.12571$ and $\hat{b} = 2905.7143$. Using these estimates, we then calculate the NRR Y^2 statistic for $k = 8$, which resulting in $Y^2 = 8.021857$. The critical value of the chi-square distribution at a significance level of $\epsilon = 0.05$ for 7 degrees of freedom is $\chi^2_{k-1} = 14.06714$. Since the calculated Y^2 statistic is less than the critical value, we conclude that there is no significant evidence to reject the null hypothesis, indicating

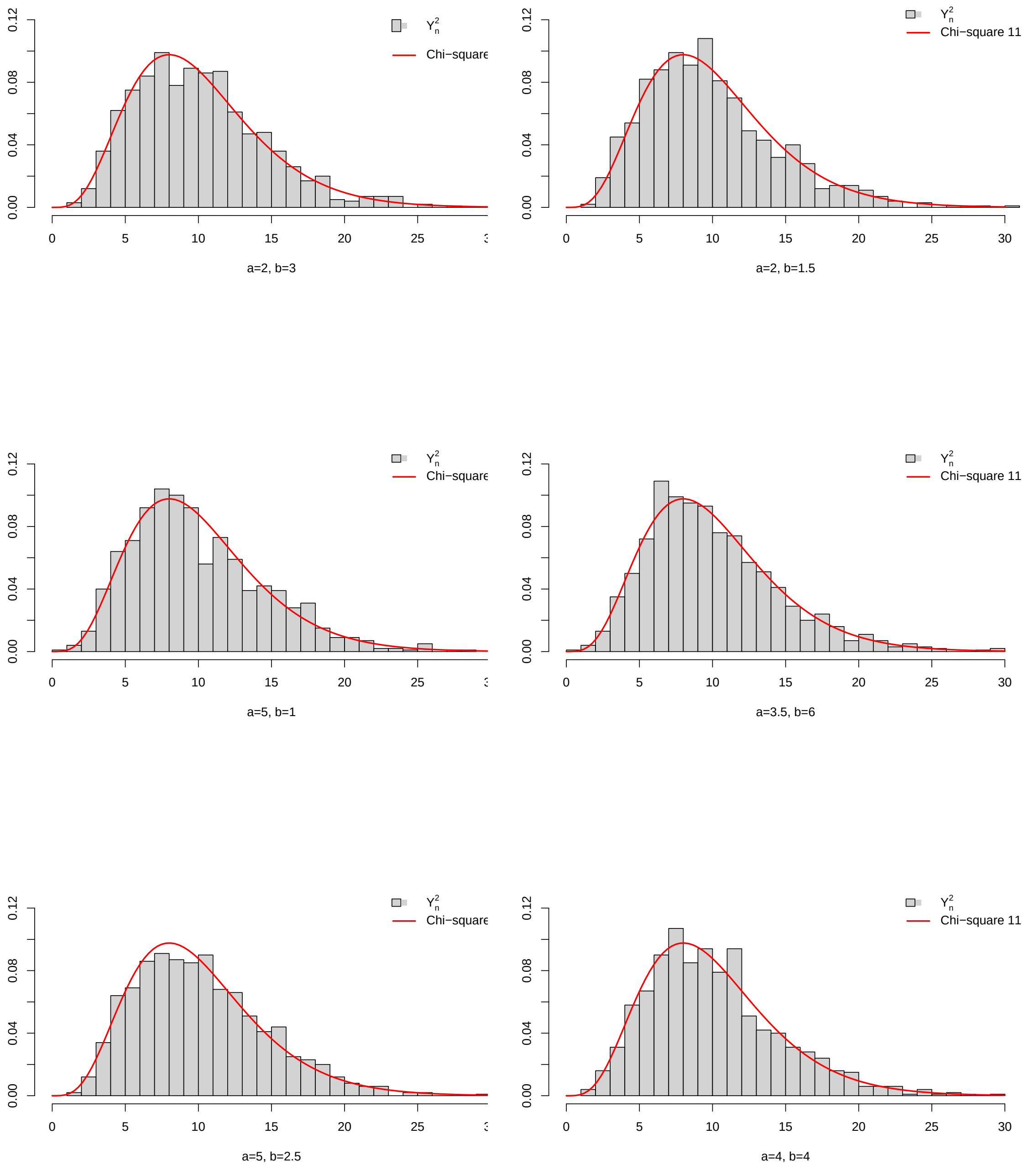


Figure 3.5: Simulated distribution of NRR statistic versus Chi-square distribution curve for 11 degrees of freedom $(n = 300, k = 12, N = 10,000)$

that the DUS-IK distribution adequately fits the endurance of deep-groove ball bearings data.

3.12 Summary

In this study, we introduced the novel DUS-IK distribution, which leverages the DUS transformation with the IK distribution as its foundation. The distribution exhibits diverse forms of asymmetry in its PDF, including unimodal, inverse J-shaped, and right-skewed distributions, making it a versatile tool for modelling various real-world phenomena. Additionally, the hazard function of the IK distribution displays distinctive decreasing and upside-down bathtub shapes, providing valuable insights into survival analysis and reliability modelling. The closed-form quantile function of the DUS-IK distribution facilitates efficient computation of properties and generation of random numbers, enhancing its practical utility in statistical analysis. Parameter estimation using ML estimation ensures robustness and accuracy in modelling, with simulation results demonstrating the convergence of estimates towards true values with increasing sample sizes. Furthermore, model selection results from actual datasets favor the proposed distribution over other renowned distributions, highlighting its superior performance in data fitting. Despite these strengths, higher parameter values correspond to elevated biases and MSEs, which serve as limitations to estimation accuracy. Future research directions could explore applications across different fields and under various types of censoring cases, thereby advancing statistical modelling and distribution theory.

CHAPTER 4

Bivariate Inverse Weibull Distribution for Dependence Reliability Measures

4.1 Introduction

In reliability and survival analysis, data are often dependent, which requires careful analysis to ensure accurate conclusions. Systems frequently incorporate parallel components, allowing continued operation even in the event of component failure. Understanding the nature of dependence among these parallel components is pivotal for assessing the overall system reliability. For instance, in medical studies, the survival times of patients diagnosed with specific types of cancers. Here, the patient's survival chances may depend on various factors such as age, treatment type, genetic mark, etc. Similarly, in finance, survival analysis can apply to study the survival times until loan default. Here, the risk associated with different borrowers depends on factors like credit score, income, loan amount, etc. To study the dependence structure between the factors, one can use the copula function, which is the architect of statistics for dependence measures and is a powerful tool for determining how variables relate.

Over the past few years, there have been many attempts to create bivariate distributions using the copula function. According to Nelson (2006), a copula is a function that connects multivariate distribution functions into uniform [0,1] marginals. A copula function can be effectively used to describe a multivariate distribution with a dependence structure. In this study, the primary emphasis is on the exploration of the FGM copula. The FGM copula has been extensively studied to learn more about its properties and uses.

The exploration of dependence structures through tools like the FGM copula is closely tied to the choice of marginal distributions. In the context of reliability and survival analysis, the Weibull distribution emerges as a foundational model due to its versatility in capturing varying hazard behaviors. However, when the data exhibit non-monotonic hazard patterns, alternative models such as the inverse Weibull (IW) distribution become indispensable for accurate analysis.

The IW distribution is well suited to serve as a representation of heavy-tailed distributions, particularly in scenarios involving extreme events unfolding over prolonged periods. This indicates the applicability of modelling data with outliers or events that happen infrequently but have a great impact. The IW distribution models are useful for understanding exactly how often systems fail, how severe those failures are, how reliable they are, and for making plans to maintain them effectively. The IW distribution has numerous applications across various fields, including medicine, reliability analysis, environmental sciences, economics, telecommunications, network analysis, and demography. By examining failures of mechanical components affected by deterioration phenomena, Keller et al. (1985) developed the IW model. Jiang et al. (2001) examined mixture models, competing risk models, and multiplicative models with the application of IW distribution. Later, Khan et al. (2008) studied IW distribution theoretically. Using the maximization process, Muhammed (2016) proposed a bivariate inverse Weibull (BIW) distribution from three independent IW distributions. Independently, Kundu & Gupta (2017) introduced a BIW distribution that can be used when the data has ties. The BIW distribution contains a singular component; hence, it cannot be used if there are no ties in the dataset. By excluding the singular component, Shuvashree & Kundu (2020b) proposed an absolutely continuous BIW model, suitable for cases when the marginals exhibit heavy-tailed distributions. This model is appropriate when there are no ties in the data and is employed for modelling dependent complementary risk data. AL-Moisheer et al. (2020) introduced another BIW distribution, whose marginal parameters have

Bernoulli distributions.

The principle objective of this chapter is to introduce a bivariate Farlie-Gumbel-Morgenstern Inverse Weibull (BFGMIW) model and subsequently analyze its statistical and reliability characteristics. The expression for dependence stress-strength reliability has been derived. Model parameters and dependence parameters have been estimated using ML estimation and inference function margins methods.

The remainder of the chapter is arranged as follows. In Section 4.2, a description of the proposed model has been given. Section 4.3 deals with the statistical properties, including conditional distributions, moment generating function, and positive quadrant dependence. Section 4.4 includes the dependent stress-strength reliability of the proposed model and other reliability measures. In Section 4.5, the ML method and the inference function margin method are discussed for parameter estimation. The asymptotic confidence intervals for the parameter estimates are presented in Section 4.6. A simulation study has been conducted in Section 4.7. In Section 4.8, a real data set is used for the application of the proposed model. Finally, the chapter is concluded in Section 4.9.

4.2 Bivariate Inverse Weibull Distribution

The FGM copula stands as one of the prominent copula families that has gained considerable popularity and was initially examined by Eyraud (1936), Morgenstern (1956), Gumbel (1960), and Farlie (1960). The CDF of the FGM family of distributions is given by,

$$F_{X,Y}(x,y) = F_X(x)G_Y(y)(1+\omega(1-F_X(x))(1-G_Y(y))), \quad -1 \leq \omega \leq 1, \quad (4.2.1)$$

where F_X and G_Y represents the marginal CDFs and ω represent the dependence parameter. The corresponding PDF is,

$$f_{X,Y}(x,y) = f_X(x)g_Y(y)(1+\omega(1-2F_X(x))(1-2G_Y(y))), \quad -1 \leq \omega \leq 1. \quad (4.2.2)$$

The ranges for the Spearman's and Kendall's correlation coefficients for FGM copula are $(-1/3, 1/3)$ and $(-2/9, 2/9)$ respectively.

Here we assume that the marginal CDFs follows IW distributions, not identically but independently. Then the marginal CDFs are given by,

$$F_X(x;\gamma_1,\delta_1) = e^{-\delta_1 x^{-\gamma_1}}, \tag{4.2.3}$$

$$G_Y(y;\gamma_2,\delta_2) = e^{-\delta_2 y^{-\gamma_2}}. \tag{4.2.4}$$

The corresponding marginal PDFs are,

$$f_X(x;\gamma_1,\delta_1) = \gamma_1\delta_1 x^{-(\gamma_1+1)} e^{-\delta_1 x^{-\gamma_1}}, \tag{4.2.5}$$

$$g_Y(y;\gamma_2,\delta_2) = \gamma_2\delta_2 y^{-(\gamma_2+1)} e^{-\delta_2 y^{-\gamma_2}}. \tag{4.2.6}$$

The joint PDF and CDF of BFGMIW distribution are obtained as,

$$\begin{aligned} F_{X,Y}(x,y) = e^{-\delta_1 x^{-\gamma_1}} e^{-\delta_2 y^{-\gamma_2}} &\left[1+\omega(1-e^{-\delta_1 x^{-\gamma_1}})(1-e^{-\delta_2 y^{-\gamma_2}})\right], \\ & x,y>0, \quad -1 \le \omega \le 1, \quad \gamma_1,\gamma_2,\delta_1,\delta_2>0, \end{aligned} \tag{4.2.7}$$

$$\begin{aligned} f_{X,Y}(x,y) = \gamma_1\gamma_2\delta_1\delta_2 & x^{-(\gamma_1+1)} y^{-(\gamma_2+1)} e^{-\delta_1 x^{-\gamma_1}} e^{-\delta_2 y^{-\gamma_2}} \\ & \left[1+\omega(1-2e^{-\delta_1 x^{-\gamma_1}})(1-2e^{-\delta_2 y^{-\gamma_2}})\right], \\ & x,y>0, \quad -1 \le \omega \le 1, \quad \gamma_1,\gamma_2,\delta_1,\delta_2>0. \end{aligned} \tag{4.2.8}$$

The bivariate hazard function of BFGMIW distribution is defined as,

$$\begin{aligned} h(x,y) &= \frac{f(x,y)}{S(x,y)} \\ &= \frac{\gamma_1\gamma_2\delta_1\delta_2 x^{-(\gamma_1+1)} y^{-(\gamma_2+1)}\left[1+\omega P(x)P(y)\right]}{(1-e^{-\delta_1 x^{-\gamma_1}})(1-e^{-\delta_2 y^{-\gamma_2}})\left[\omega + e^{\delta_1 x^{-\gamma_1}} e^{\delta_2 y^{-\gamma_2}}\right]}, \end{aligned} \tag{4.2.9}$$

where $P(x) = (1-2e^{-\delta_1 x^{-\gamma_1}})$ and $P(y) = (1-2e^{-\delta_2 y^{-\gamma_2}})$.

The plots of PDFs, CDFs, and hazard functions for various choices of parameters are given in the Figure 4.1.

4.3 Statistical Properties

This section deals with some important statistical characteristics of the BFGMIW distribution, including conditional distributions, the moment generating function, and positive quadrant dependence.

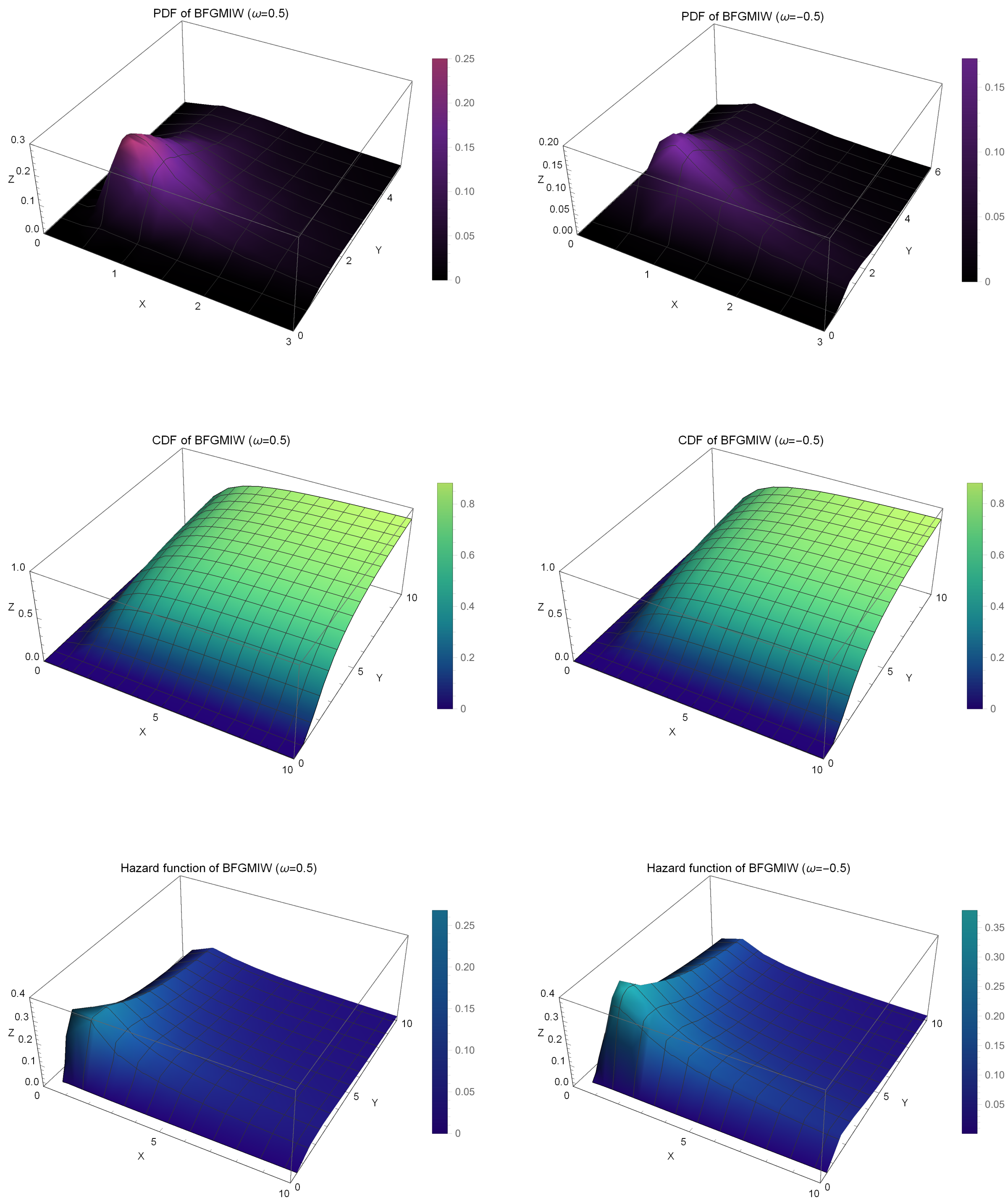


Figure 4.1: The PDF, CDF, and hazard function of the BFGMIW distribution for $\gamma_1 = 1.5$, $\delta_1 = 1$, $\gamma_2 = 1.5$, $\delta_2 = 3$, and $\omega = \{0.5, -0.5\}$.

4.3.1 Conditional distributions

The conditional CDF of X given $Y = y$ of BFGMIW distribution is given by,

$$F_{X/Y} = e^{-\delta_1 x^{-\gamma_1}}(1 + \omega(1 - e^{-\delta_1 x^{-\gamma_1}})(1 - 2e^{-\delta_2 y^{-\delta_2}})),$$

the corresponding conditional PDF is,

$$f_{X/Y} = \gamma_1\delta_1 x^{-(\gamma_1+1)} e^{-\delta_1 x^{-\gamma_1}}(1 + \omega(1 - 2e^{-\delta_1 x^{-\gamma_1}})(1 - 2e^{-\delta_2 y^{-\gamma_2}})).$$

Then, the conditional expectation of X given $Y = y$ can be derived as,

$$E(X/Y) = \delta_1^{\frac{1}{\gamma_1}}\Gamma\left(1 - \frac{1}{\gamma_1}\right)\left[1 + \omega(1 - 2e^{-\delta_2 y^{-\gamma_2}})(1 - 2^{\frac{1}{\gamma_1}})\right]; \quad \gamma_1 > 1.$$

4.3.2 Moment generating function

The moment generating function of BFGMIW distribution is given by,

$$\begin{aligned} M_{X,Y}(t_1, t_2) &= E(e^{t_1 x} e^{t_2 y}) \\ &= \int_0^\infty \int_0^\infty e^{t_1 x} e^{t_2 y} f_{X,Y}(x, y) dy dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{t_1^i t_2^j}{i! j!} \delta_1^{\frac{i}{\gamma_1}} \delta_2^{\frac{j}{\gamma_2}} \Gamma\left(1 - \frac{i}{\gamma_1}\right)\Gamma\left(1 - \frac{j}{\gamma_2}\right)\left[1 + \omega\Omega_1\Omega_2\right], \end{aligned}$$

where $\Omega_1 = \left(1 - 2^{\frac{i}{\gamma_1}}\right)$ and $\Omega_2 = \left(1 - 2^{\frac{j}{\gamma_2}}\right)$.

4.3.3 Positive Quadrant Dependence

The concepts of positive quadrant dependence (PQD) and negative quadrant dependence (NQD) in bivariate distributions indicate the dependence between random variables based on their positions in the positive and negative quadrants, respectively (see, Lehmann (1966)). Since then found extensive applications in various fields in statistics, econometrics, finance, and actuarial science. For a pair of random variables (X, Y), PQD is defined as:

$$PQD(X, Y) = P(X > x, Y > y) \geq P(X > x)P(Y > y), \text{for all}\, x, y. \tag{4.3.1}$$

The NQD is characterized by the reverse inequality of (4.3.1).

Theorem 4.3.1. *BFGMIW distribution possesses both PQD and NQD properties.*

Proof. Consider,

$$
\begin{aligned}
P(X > x, Y > y) - P(X > x)P(Y > y) \\
&= \left(1 - e^{-\delta_1 x^{-\gamma_1}}\right)\left(1 - e^{-\delta_2 y^{-\gamma_2}}\right)\left[\omega e^{-\delta_1 x^{-\gamma_1}} e^{-\delta_2 y^{-\gamma_2}}\right] \\
&= \omega\kappa(x, y),
\end{aligned}
$$

where $\kappa(x, y) = \left(1 - e^{-\delta_1 x^{-\gamma_1}}\right)\left(1 - e^{-\delta_2 y^{-\gamma_2}}\right) e^{-\delta_1 x^{-\gamma_1}} e^{-\delta_2 y^{-\gamma_2}}$, is the products of marginal CDFs and survival functions of x and y. Therefore, the subsequent set of inequalities for $\omega\kappa(x, y)$ is presented as follows:

$$
\omega\kappa(x, y) = \begin{cases} > 0 & \text{if } \omega > 0 \quad \text{for all } x, y > 0, \\ < 0 & \text{if } \omega < 0 \quad \text{for all } x, y > 0. \end{cases}
$$

Hence, the BFGMIW distribution exhibits both PQD and NQD properties. Specifically, the distribution demonstrates PQD for $\omega > 0$ and NQD for $\omega < 0$. □

4.4 Reliability and Dependence Measures

This section covers various dependence measures, including dependence stress-strength reliability, hazard gradient function, local dependence function, left (right) tail decreasing (increasing) property, and Clayton-Oakes Association Measure.

4.4.1 Dependence Stress-Strength Reliability

The relationship between the strength (X) and stress (Y) of a system can be described using mathematical notation. If we assume that X and Y are variables that conform to a BFGMIW distribution with a dependence parameter ω, we can define the system's reliability in terms of stress-strength as R. This reliability, represented by R, is determined by calculating the probability that Y is lower than X, which can be expressed as $R = P(Y < X)$.

A great deal of work has been done to estimate R using the stress-strength dependence assumption. These investigations often utilize bivariate continuous lifetime distributions, such as the bivariate Marshall-Olkin exponential distribution studied by Chandra & Pandey (2012), the bivariate gamma distribution analyzed by

Nadarajah (2005), and the bivariate model of generalized exponential and generalized inverse exponential distributions studied by Ahmed (2020), along with other related works cited in the literature.

To simplify the calculations, we have assumed that the marginal distributions have the same shape parameter γ. The stress-strength reliability R is defined as follows:

$$\begin{aligned} R = P[Y < X] &= \int_0^\infty \int_0^x f_{X,Y}(x,y)dydx \\ &= R_1 + \omega(R_1 + 2I) \end{aligned}$$

where

$$R_1 = \int_0^\infty \int_0^x f_X(x)g_Y(y)dydx \quad \text{and}$$

$$I = \int_0^\infty \int_0^x \left(F_X(x)\left(2G_Y(y) - 1\right) - G_Y(y)\right) f_X(x)g_Y(y)dydx.$$

Proposition 4.4.1. *Let (X, Y) follows BFGMIW distribution with marginal distributions $IW(\gamma, \delta_1)$ and $IW(\gamma, \delta_2)$ respectively for X and Y, then the stress-strength reliability R, is,*

$$R = \frac{\delta_1}{\delta_1 + \delta_2}\left\{1 + \omega\frac{\delta_2(\delta_1 - \delta_2)}{(2\delta_1 + \delta_2)(\delta_1 + 2\delta_2)}\right\} \tag{4.4.1}$$

In case of independence between X and Y, ω will be zero, leading to the expression for stress-strength reliability as,

$$R = R_1 = \frac{\delta_1}{\delta_1 + \delta_2}.$$

4.4.2 Hazard Gradient Function

The survival function of BFGMIW distribution is,

$$S(x,y) = (1 - F_X(x))(1 - G_Y(y))[1 + \omega F_X(x)G_Y(y)].$$

i.e.,

$$S(x,y) = (1 - e^{-\delta_1 x^{-\gamma_1}})(1 - e^{-\delta_2 y^{-\gamma_2}})[1 + \omega e^{-\delta_1 x^{-\gamma_1}} e^{-\delta_2 y^{-\gamma_2}}].$$

Johnson & Kotz (1975) defined hazard gradient function as a vector components

$(\eta_1(x,y), \eta_2(x,y))$, where

$$\eta_1(x,y) = -\frac{\partial \ln S(x,y)}{\partial x},$$
$$\eta_2(x,y) = -\frac{\partial \ln S(x,y)}{\partial y}.$$

The vector components are obtained as

$$\eta_1(x,y) = \frac{\gamma_1\delta_1 x^{-(\gamma_1+1)}e^{-\delta_1 x^{-\gamma_1}}}{1-e^{-\delta_1 x^{-\gamma_1}}}\left[1-\left(\left(1-e^{-\delta_1 x^{-\gamma_1}}\right)^{-1}\left(\left(\omega e^{-\delta_2 y^{-\gamma_2}}\right)^{-1}+1\right)-1\right)^{-1}\right], \tag{4.4.2}$$

$$\eta_2(x,y) = \frac{\gamma_2\delta_2 y^{-(\gamma_2+1)}e^{-\delta_2 y^{-\gamma_2}}}{1-e^{-\delta_2 y^{-\gamma_2}}}\left[1-\left(\left(1-e^{-\delta_2 y^{-\gamma_2}}\right)^{-1}\left(\left(\omega e^{-\delta_2 x^{-\gamma_1}}\right)^{-1}+1\right)-1\right)^{-1}\right]. \tag{4.4.3}$$

Theorem 4.4.2. *Let (X,Y) follows BFGMIW distribution, then it has increasing (decreasing) hazard rate for $\omega > 0\,(< 0)$.*

Proof. Consider (4.4.2), $\frac{\gamma_1\delta_1 x^{-(\gamma_1+1)}e^{-\delta_1 x^{-\gamma_1}}}{1-e^{-\delta_1 x^{-\gamma_1}}}$ is the hazard function h_X and an increasing function in x. Further, $\omega^{-1} > 1$ for $0 \le \omega \le 1$ and $\left(e^{-\delta_2 y^{-\gamma_2}}\right)^{-1} > 1$ implies that $\left(\left(\omega e^{-\delta_2 y^{-\gamma_2}}\right)^{-1}+1\right) > 1$. Also $\left(1-e^{-\delta_1 x^{-\gamma_1}}\right)^{-1}$ is an increasing function in x. Therefore $\left[1-\left(\left(1-e^{-\delta_1 x^{-\gamma_1}}\right)^{-1}\left(\left(\omega e^{-\delta_2 y^{-\gamma_2}}\right)^{-1}+1\right)-1\right)^{-1}\right]$ is a positive increasing function. Hence $\eta_1(x,y)$ is a positive increasing function in x for positive values of ω. Similarly, $\eta_2(x,y)$ is a positive increasing function in y for, $0 \le \omega \le 1$. In the same way, it is apparent that the evidence for the counterpart is straightforward. $\square$

4.4.3 Local Dependence Function

Holland & Wang (1987) introduced a captivating local dependence function $\rho(x,y)$, designed to meticulously investigate the dependence between random variables X and Y, defined by

$$\rho(x,y) = \frac{\partial^2}{\partial x \partial y}\ln f_{X,Y}(x,y).$$

Utilizing the local dependence function in the analysis of bivariate distributions provides a robust approach to investigating the concept of TP_2.

Definition 4.4.3. *A bivariate distribution $f_{X,Y}(x,y)$ is said to be TP_2 if*

$$f(x_1,y_1)f(x_2,y_2) \ge f(x_1,y_2)f(x_2,y_1),\ x_1 < x_2,\ y_1 < y_2.$$

The bivariate density function $f_{X,Y}(x,y)$ is said to have TP_2 property if and only if $\rho(x,y) > 0$.

Proposition 4.4.4. *Let* (X,Y) *follows BFGMIW distribution, then*

$$\rho(x,y) = \frac{4\omega\gamma_1\gamma_2\delta_1\delta_2 x^{-(\gamma_1+1)}y^{-(\gamma_2+1)}e^{-\delta_1 x^{-\gamma_1}}e^{-\delta_2 y^{-\gamma_2}}}{\left[1+\omega\left(1-2e^{-\delta_1 x^{-\gamma_1}}\right)\left(1-2e^{-\delta_2 y^{-\gamma_2}}\right)\right]^2},$$

which is > 0 *for* $\omega > 0$ *indicating the* TP_2 *property.*

4.4.4 Left tail decreasing (LTD) and Right tail increasing (RTI)

A bivariate random variable (X,Y) is said to have Left tail decreasing (LTD) or Right tail increasing (RTI) if and only if, $LTD_{Y/X} = P(Y \leq y\,|X \leq x)$, is decreasing or $RTI_{Y/X} = P(Y > y\,|X > x)$ is increasing for every x and y.

For BFGMIW distribution,

$$LTD_{Y/X} = e^{-\delta_2 y^{-\gamma_2}}\left[1+\omega\left(1-e^{-\delta_1 x^{-\gamma_1}}\right)\left(1-e^{-\delta_2 y^{-\gamma_2}}\right)\right],$$

is decreasing for $\omega \leq 0$, and

$$RTI_{Y/X} = \left(1-e^{-\delta_2 y^{-\gamma_2}}\right)\left[1+\omega e^{-\delta_1 x^{-\gamma_1}}e^{-\delta_2 y^{-\gamma_2}}\right],$$

is increasing for $\omega > 0$. Thus BFGMIW distribution has $LTD_{Y/X}$ for $\omega \leq 0$ and $RTI_{Y/X}$ for $\omega > 0$.

4.4.5 Clayton-Oakes Association Measure

The Clayton-Oakes association measure determines the dependence between two random variables in a bivariate data set. Clayton (1978) and Oakes (1989) defined the association measure using the survival function as,

$$\theta(x,y) = \frac{f(x,y)S(x,y)}{S_x(x,y)S_y(x,y)}, \tag{4.4.4}$$

where $S_x(x,y) = \frac{\partial}{\partial x}S(x,y)$ and $S_y(x,y) = \frac{\partial}{\partial y}S(x,y)$.

Proposition 4.4.5. *Suppose* (X,Y) *follows BFGMIW distribution, then*

$$\theta(x,y) = \frac{(1+\omega(1-2e^{-\delta_1 x^{-\gamma_1}})(1-2e^{-\delta_2 y^{-\gamma_2}}))(1+\omega e^{-\delta_1 x^{-\gamma_1}}e^{-\delta_2 y^{-\gamma_2}})}{(1-\omega e^{-\delta_2 y^{-\gamma_2}}(1-2e^{-\delta_1 x^{-\gamma_1}}))(1-\omega e^{-\delta_1 x^{-\gamma_1}}(1-2e^{-\delta_2 y^{-\gamma_2}}))}. \tag{4.4.5}$$

When X and Y are independent, i.e., $\omega = 0$, $\theta(x, y)$ takes the value 1.

4.5 Parameter Estimation Methods

For parameter estimation, two methods have been used: ML method and Inference Function Margin (IFM).

4.5.1 Maximum Likelihood Estimation

In this section, we apply the ML method to estimate the parameters of the BFGMIW distribution. Assuming that $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ constitute a sample of size n from the BFGMIW distribution, the log-likelihood of the distribution is expressed as follows.

$$\begin{aligned}\ln L =& n \ln \gamma_1 + n \ln \gamma_2 + n \ln \delta_1 + n \ln \delta_2 - (\gamma_1 + 1) \sum_{i=1}^{n} \ln x_i - (\gamma_2 + 1) \sum_{i=1}^{n} \ln y_i \\ & - \delta_1 \sum_{i=1}^{n} x_i^{-\gamma_1} - \delta_2 \sum_{i=1}^{n} y_i^{-\gamma_2} + \sum_{i=1}^{n} \ln \left[1 + \omega \left(1 - 2e^{-\delta_1 x_i^{-\gamma_1}} \right) \left(1 - 2e^{-\delta_2 y_i^{-\gamma_2}} \right) \right].\end{aligned}$$

The partial derivatives of the log-likelihood function with respect to its parameters are as follows:

$$\begin{aligned}\frac{\partial \ln L}{\partial \gamma_1} &= \frac{n}{\gamma_1} - \sum_{i=1}^{n} \ln x_i + \delta_1 \sum_{i=1}^{n} x_i^{-\gamma_1} \ln x_i - \sum_{i=1}^{n} \frac{2\omega \delta_1 x_i^{-\gamma_1} \ln x_i e^{-\delta_1 x_i^{-\gamma_1}} \left(1 - 2e^{-\delta_2 y_i^{-\gamma_2}} \right)}{1 + \omega \left(1 - 2e^{-\delta_1 x_i^{-\gamma_1}} \right) \left(1 - 2e^{-\delta_2 y_i^{-\gamma_2}} \right)}, \\ \frac{\partial \ln L}{\partial \gamma_2} &= \frac{n}{\gamma_2} - \sum_{i=1}^{n} \ln y_i + \delta_2 \sum_{i=1}^{n} y_i^{-\gamma_2} \ln y_i - \sum_{i=1}^{n} \frac{2\omega \delta_2 y_i^{-\gamma_2} \ln y_i e^{-\delta_2 y_i^{-\gamma_2}} \left(1 - 2e^{-\delta_1 x_i^{-\gamma_1}} \right)}{1 + \omega \left(1 - 2e^{-\delta_1 x_i^{-\gamma_1}} \right) \left(1 - 2e^{-\delta_2 y_i^{-\gamma_2}} \right)}, \\ \frac{\partial \ln L}{\partial \delta_1} &= \frac{n}{\delta_1} - \sum_{i=1}^{n} x_i^{-\gamma_1} + \sum_{i=1}^{n} \frac{2\omega x_i^{-\gamma_1} e^{-\delta_1 x_i^{-\gamma_1}} \left(1 - 2e^{-\delta_2 y_i^{-\gamma_2}} \right)}{1 + \omega \left(1 - 2e^{-\delta_1 x_i^{-\gamma_1}} \right) \left(1 - 2e^{-\delta_2 y_i^{-\gamma_2}} \right)}, \\ \frac{\partial \ln L}{\partial \delta_2} &= \frac{n}{\delta_2} - \sum_{i=1}^{n} y_i^{-\gamma_2} + \sum_{i=1}^{n} \frac{2\omega y_i^{-\gamma_2} e^{-\delta_2 y_i^{-\gamma_2}} \left(1 - 2e^{-\delta_2 x_i^{-\gamma_1}} \right)}{1 + \omega \left(1 - 2e^{-\delta_1 x_i^{-\gamma_1}} \right) \left(1 - 2e^{-\delta_2 y_i^{-\gamma_2}} \right)}, \\ \frac{\partial \ln L}{\partial \omega} &= \frac{\left(1 - 2e^{-\delta_1 x_i^{-\gamma_1}} \right) \left(1 - 2e^{-\delta_2 y_i^{-\gamma_2}} \right)}{1 + \omega \left(1 - 2e^{-\delta_1 x_i^{-\gamma_1}} \right) \left(1 - 2e^{-\delta_2 y_i^{-\gamma_2}} \right)}.\end{aligned}$$

Parameter estimates are derived by equating the partial derivatives to zero and then solving for the values. As the partial derivatives lack an explicit form, numerical methods are necessary to solve the equations.

4.5.2 Inference Function Margin

IFM is an estimation technique used in semiparametric models to estimate the parameters related to the marginal distributions while accounting for the dependence between the random variables. This method proposed by Xu (1996) and Joe (2005) consists of two-stage estimation process, where the first stage is dedicated to estimating the marginal distributions. Suppose L_1 and L_2 represents the log-likelihood function of the marginal distributions,

$$L_1 = \sum_{i=1}^{n} \ln f(x_i; \gamma_1, \delta_1) \quad ; \quad L_2 = \sum_{i=1}^{n} \ln g(y_i; \gamma_2, \delta_2),$$

can be written as,

$$L_1 = n \ln \gamma_1 + n \ln \delta_1 - (\gamma_1 + 1) \sum_{i=1}^{n} \ln x_i - \delta_1 \sum_{i=1}^{n} x_i^{-\gamma_1},$$
$$L_2 = n \ln \gamma_2 + n \ln \delta_2 - (\gamma_2 + 1) \sum_{i=1}^{n} \ln y_i - \delta_2 \sum_{i=1}^{n} y_i^{-\gamma_2}.$$

By solving $\frac{\partial L_1}{\partial \gamma_1} = 0$, $\frac{\partial L_1}{\partial \delta_1} = 0$, $\frac{\partial L_2}{\partial \gamma_2} = 0$, $\frac{\partial L_2}{\partial \delta_2} = 0$, the MLEs $(\hat{\gamma_1}, \hat{\delta_1}, \hat{\gamma_2}, \hat{\delta_2})$ can be derived. Then by invariance property, $\hat{F}(x) = e^{-\hat{\delta}_1 x^{-\hat{\gamma}_1}}$ and $\hat{G}(y) = e^{-\hat{\delta}_2 y^{-\hat{\gamma}_2}}$.

In the second stage, our objective is to optimize the joint density. This requires taking into account both the dependence parameter and the MLEs derived from the marginal distributions. Now the IFM estimate of BFGMIW distribution can be written as,

$$L_{IFM} = \sum_{i=1}^{n} \ln(1 + \omega(1 - 2\hat{F}(x))(1 - 2\hat{G}(y))). \tag{4.5.1}$$

The following expression represents the result obtained by taking the partial derivative of L_{IFM} with respect to the dependence parameter ω and setting it equal to zero.

$$\frac{\partial L_{IFM}}{\partial \omega} = \sum_{i=1}^{n} \frac{(1 - 2e^{-\delta_1 x_i^{-\gamma_1}})(1 - 2e^{-\delta_2 y_i^{-\gamma_2}})}{1 + \omega(1 - 2e^{-\delta_1 x_i^{-\gamma_1}})(1 - 2e^{-\delta_2 y_i^{-\gamma_2}})} = 0. \tag{4.5.2}$$

By solving (4.5.2) using numerical techniques, one can obtain the IFM estimate.

4.6 Asymptotic Confidence Interval

This section presents the asymptotic confidence intervals for the parameters of the BFGMIW distribution using the ML and IFM methods. To compute these confidence intervals, the Fisher information matrix is required as a preliminary step. The Fisher information matrix for the parameter space $\Theta = (\gamma_1, \delta_1, \gamma_2, \delta_2, \omega)$ is expressed as follows:

$$I(\hat{\Theta}) = \begin{bmatrix} \frac{\partial^2 l}{\partial \gamma_1^2} & \frac{\partial^2 l}{\partial \gamma_1 \partial \delta_1} & \frac{\partial^2 l}{\partial \gamma_1 \partial \gamma_2} & \frac{\partial^2 l}{\partial \gamma_1 \partial \delta_2} & \frac{\partial^2 l}{\partial \gamma_1 \partial \omega} \\ \frac{\partial^2 l}{\partial \delta_1 \partial \gamma_1} & \frac{\partial^2 l}{\partial \delta_1^2} & \frac{\partial^2 l}{\partial \delta_1 \partial \gamma_2} & \frac{\partial^2 l}{\partial \delta_1 \partial \delta_2} & \frac{\partial^2 l}{\partial \delta_1 \partial \omega} \\ \frac{\partial^2 l}{\partial \gamma_2 \partial \gamma_1} & \frac{\partial^2 l}{\partial \gamma_2 \partial \delta_1} & \frac{\partial^2 l}{\partial \gamma_2^2} & \frac{\partial^2 l}{\partial \gamma_2 \partial \delta_2} & \frac{\partial^2 l}{\partial \gamma_2 \partial \omega} \\ \frac{\partial^2 l}{\partial \delta_2 \partial \gamma_1} & \frac{\partial^2 l}{\partial \delta_2 \partial \delta_1} & \frac{\partial^2 l}{\partial \delta_2 \partial \gamma_2} & \frac{\partial^2 l}{\partial \delta_2^2} & \frac{\partial^2 l}{\partial \delta_2 \partial \omega} \\ \frac{\partial^2 l}{\partial \omega \partial \gamma_1} & \frac{\partial^2 l}{\partial \omega \partial \delta_1} & \frac{\partial^2 l}{\partial \omega \partial \gamma_2} & \frac{\partial^2 l}{\partial \omega \partial \delta_2} & \frac{\partial^2 l}{\partial \omega^2} \end{bmatrix} = -H(\hat{\Theta}),$$

where $\hat{\Theta}$ represents the parameter estimates under ML and $H(\hat{\Theta})$ is the Hessian matrix.

We are focused on constructing the confidence interval for the dependence parameter ω, so the hessian matrix for ω under IFM can be written as,

$$H = \frac{\partial^2 l}{\partial \omega^2}.$$

The $100(1-\alpha)\%$ confidence interval for ω is $\hat{\omega} \pm Z_{\frac{\alpha}{2}} \sqrt{Var(\hat{\omega})}$, where $Z_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}^{th}$ percentile of standard normal variate and $Var(\hat{\omega})$ is the diagonal element of $I^{-1}(\hat{\Theta})$.

4.7 Simulation study

We conduct a simulation study to validate the estimation techniques of ML and IFM for the BFGMIW distribution. The performance of the dependence parameter, along with other parameters and stress strength reliability R, is validated in this section. We use the `MaxLik` and `copula` packages in R software to generate samples and estimate the parameters. The following algorithm, discussed by Nelsen (2006) is used to generate the samples from the BFGMIW distribution.

- Generate u_1 and u_2 from uniform(0,1) distribution, where u_1 and u_2 are indepedent.
- Set $x = F^{-1}(u_1)$, where F^{-1} is the inverse CDF of the marginal CDF $F(x)$.
- Set $F(y/x) = u_2$, and solve for y.

- In order to obtain n data points $(x_i, y_i), i = 1, 2, ..n$, repeat the above steps n times.

Bivariate data from the BFGMIW distribution is generated for sample sizes 50, 100, 250, and 500 using the preceding steps. Three sets of parameter values, $(\gamma_1, \delta_1, \gamma_2, \delta_2) = (2, 2, 2, 1.5), (0.5, 1, 0.5, 1.5)$, and $(0.8, 0.2, 0.8, 0.6)$, are chosen for different values of the dependence parameter $\omega = -0.8, -0.5, 0.5$, and 0.8. The parameter estimates under ML and IFM techniques are calculated based on 1000 iterations. The stress-strength reliability R has been computed for each estimation method. The MSE corresponding to parameters and R has been derived and is provided in Tables 4.1-4.3. Additionally, the average length of the confidence interval (ACL) and coverage probability (CP) for the dependence parameter are given in Table 4.4. From the tables, the following conclusions can be drawn:

- It can be observed that both ML and IFM methods perform well in estimating the parameters as well as in finding the confidence intervals.
- The MSE corresponds to each parameter, and R decreases with the increase in sample size under both ML and IFM methods.
- As the sample size increases in ML and IFM, the value of ACL for the dependence parameter decreases, while the CP maintains its nominal level.
- It can be noticed that when the value of the dependence parameter increases, the variation in the R value increases.

Table 4.1: MLE and IFM estimates and corresponding MSEs for the parameter set $(\gamma_1, \delta_1, \gamma_2, \delta_2)$=(2,2,2,1.5).

		MLE						IFM					
$(\gamma_1=2, \delta_1=2,$ $\gamma_2=2, \delta_2=1.5)$	Sample size	$\hat{\gamma}_1$ MSE	$\hat{\delta}_1$ MSE	$\hat{\gamma}_2$ MSE	$\hat{\delta}_2$ MSE	$\hat{\omega}$ MSE	$\hat{R}_{MLE}$ MSE	$\hat{\gamma}_1$ MSE	$\hat{\delta}_1$ MSE	$\hat{\gamma}_2$ MSE	$\hat{\delta}_2$ MSE	$\hat{\omega}$ MSE	$\hat{R}_{IFM}$ MSE
$\omega=-0.8$	50	2.0437	2.0548	2.0451	1.4996	-0.8989	0.563	2.0795	2.062	2.0899	1.5271	-0.8605	0.561
		0.0592	0.0957	0.0538	0.0677	0.2893	0.0026	0.066	0.1097	0.0531	0.0575	0.5023	0.0018
	100	2.0295	2.0358	2.0296	1.5021	-0.795	0.5616	1.9946	2.0214	2.0269	1.5115	-0.7903	0.5598
		0.0232	0.0462	0.0264	0.0196	0.1137	0.001	0.0306	0.0433	0.018	0.0228	0.143	0.0008
	250	2.0077	2.0028	2.0069	1.493	-0.8099	0.5601	2.0088	2.0055	2.0159	1.4902	-0.8032	0.5609
		0.0075	0.0161	0.0097	0.0094	0.0283	0.0004	0.0074	0.0162	0.0126	0.0116	0.0302	0.0004
	500	1.9978	2.0154	2.0015	1.4968	-0.8058	0.5609	1.9853	2.003	1.9945	1.4925	-0.8088	0.5602
		0.0052	0.0093	0.0046	0.0051	0.013	0.0002	0.0042	0.0064	0.0048	0.0049	0.014	0.0002
$\omega=-0.5$	50	2.0176	2.0551	2.0818	1.5373	-0.5118	0.5623	2.0861	2.082	2.0304	1.5443	-0.5016	0.5654
		0.0625	0.1152	0.0648	0.0512	0.2272	0.0024	0.0498	0.1216	0.0686	0.0579	0.2008	0.0023
	100	2.0176	2.0422	2.0636	1.5282	-0.5108	0.5633	2.043	2.007	2.0311	1.5195	-0.4952	0.561
		0.0245	0.0539	0.0359	0.0238	0.0752	0.0011	0.0172	0.0453	0.0265	0.0237	0.0751	0.001
	250	2.0138	2.0256	1.994	1.5042	-0.5053	0.5656	2.0087	2.0263	2.0222	1.4887	-0.4996	0.5681
		0.0122	0.0137	0.0088	0.0074	0.0301	0.0004	0.0075	0.0166	0.0103	0.0102	0.0297	0.0004
	500	2.0131	2.0102	2.0066	1.5027	-0.4781	0.5647	2.0117	1.9998	1.9966	1.5032	-0.4768	0.5634
		0.0063	0.0077	0.006	0.0046	0.0176	0.0002	0.0041	0.0091	0.0055	0.0043	0.0175	0.0002
$\omega=0.5$	50	2.014	2.0643	2.0867	1.5536	0.4479	0.5771	2.1049	2.0854	2.0746	1.5222	0.4043	0.5846
		0.049	0.0974	0.0746	0.0692	0.2428	0.0032	0.0587	0.1078	0.0674	0.0564	0.2126	0.0028
	100	2.0252	2.0368	2.0242	1.5302	0.5248	0.5785	2.0133	1.9857	2.0176	1.544	0.5237	0.5697
		0.028	0.0418	0.0237	0.0211	0.0809	0.0011	0.0321	0.0395	0.0261	0.0261	0.0799	0.0013
	250	2.0338	2.0115	2.0293	1.5038	0.4829	0.5799	2.0159	2.0332	1.9919	1.4934	0.4868	0.5841
		0.0085	0.0133	0.0105	0.0105	0.0356	0.0005	0.0097	0.0263	0.01	0.0098	0.0354	0.0007
	500	1.9979	2.0015	1.9972	1.503	0.5096	0.5788	1.9944	2.0037	2.0055	1.5017	0.5063	0.5794
		0.0054	0.0105	0.0047	0.0044	0.021	0.0003	0.0048	0.0084	0.0049	0.0041	0.0204	0.0003
$\omega=0.8$	50	2.0726	2.0935	2.0188	1.537	0.8982	0.589	2.0824	2.086	2.0604	1.5337	0.8093	0.589
		0.0648	0.1154	0.0628	0.0475	0.3146	0.003	0.0553	0.1174	0.064	0.0481	0.2265	0.003
	100	2.0165	2.0032	2.0481	1.5147	0.8185	0.5812	2.0504	2.0542	2.0574	1.5128	0.7889	0.5877
		0.0275	0.0486	0.0326	0.0271	0.0919	0.0013	0.0371	0.0545	0.0279	0.0237	0.073	0.0016
	250	2.032	2.0393	2.0074	1.5013	0.8102	0.5892	2.0268	2.0219	2.016	1.4943	0.8076	0.5879
		0.0094	0.0186	0.0101	0.0098	0.0256	0.0006	0.0096	0.0203	0.0095	0.0109	0.0249	0.0008
	500	2.0034	2.0074	1.9926	1.4916	0.805	0.5864	2.0059	2.0235	2.0131	1.4961	0.7979	0.5877
		0.0037	0.0111	0.0048	0.0046	0.0117	0.0003	0.0049	0.0094	0.0065	0.0043	0.0124	0.0003

Table 4.2: MLE and IFM estimates and corresponding MSEs for the parameter set $(\gamma_1, \delta_1, \gamma_2, \delta_2)$=(0.5,1,0.5,1.5).

		MLE						IFM					
$(\gamma_1 = 0.5, \delta_1 = 1, \gamma_2 = 0.5, \delta_2 = 1.5)$	Sample size	$\hat{\gamma}_1$ MSE	$\hat{\delta}_1$ MSE	$\hat{\gamma}_2$ MSE	$\hat{\delta}_2$ MSE	$\hat{\omega}$ MSE	$\hat{R}_{MLE}$ MSE	$\hat{\gamma}_1$ MSE	$\hat{\delta}_1$ MSE	$\hat{\gamma}_2$ MSE	$\hat{\delta}_2$ MSE	$\hat{\omega}$ MSE	$\hat{R}_{IFM}$ MSE
$\omega = -0.8$	50	0.521	1.0289	0.5122	1.5597	-0.8473	0.4161	0.5082	0.9981	0.5168	1.5506	-0.816	0.4103
		0.0032	0.024	0.0032	0.0587	0.2364	0.0024	0.0036	0.0232	0.004	0.0548	0.2267	0.0021
	100	0.5156	0.9925	0.4988	1.5378	-0.794	0.4107	0.5028	1.0155	0.5053	1.5232	-0.7786	0.4168
		0.0019	0.0135	0.0018	0.0282	0.1027	0.0013	0.0014	0.0127	0.0014	0.0246	0.0926	0.001
	250	0.5023	1.0064	0.5025	1.5125	-0.8125	0.417	0.5005	0.9976	0.5016	1.4926	-0.806	0.4179
		0.0006	0.0042	0.0005	0.0087	0.0335	0.0004	0.0006	0.0036	0.0005	0.0097	0.0316	0.0004
	500	0.5044	1.0022	0.5028	1.5105	-0.8027	0.4162	0.5009	1.0006	0.5033	1.4975	-0.8001	0.4175
		0.0004	0.003	0.0003	0.0053	0.0167	0.0003	0.0004	0.0022	0.0003	0.0047	0.0151	0.0002
$\omega = -0.5$	50	0.5122	1.0045	0.5116	1.5035	-0.5387	0.4123	0.5088	1.0335	0.5137	1.5643	-0.5032	0.4097
		0.0038	0.0245	0.0038	0.0509	0.2112	0.0027	0.0036	0.0268	0.003	0.0765	0.2154	0.0025
	100	0.5064	1.0147	0.5153	1.5255	-0.5033	0.4107	0.5102	0.9989	0.5134	1.4984	-0.4867	0.4105
		0.0019	0.0145	0.0021	0.0279	0.0786	0.0014	0.0019	0.0109	0.0016	0.0236	0.0793	0.0009
	250	0.5024	1.0001	0.5004	1.4998	-0.4714	0.4103	0.5015	0.9995	0.5052	1.5056	-0.4636	0.4089
		0.0007	0.0044	0.0006	0.0131	0.0341	0.0005	0.0006	0.0041	0.0005	0.0075	0.0339	0.0004
	500	0.5029	1.0045	0.5013	1.496	-0.4937	0.4122	0.5026	1.0008	0.5017	1.5161	-0.4916	0.4084
		0.0004	0.0016	0.0003	0.004	0.0182	0.0002	0.0003	0.0023	0.0003	0.0046	0.0172	0.0002
$\omega = 0.5$	50	0.5191	1.0339	0.5167	1.5654	0.4655	0.3895	0.5143	0.9997	0.5123	1.5268	0.4517	0.3867
		0.0047	0.0193	0.0029	0.0618	0.1241	0.0023	0.0031	0.0231	0.0033	0.0458	0.1184	0.0023
	100	0.5085	1.0072	0.504	1.5414	0.4631	0.3853	0.5112	0.9942	0.5002	1.5236	0.4528	0.385
		0.0017	0.0117	0.0012	0.0247	0.0826	0.0013	0.0017	0.0151	0.0017	0.0278	0.087	0.0019
	250	0.5015	1.0048	0.5046	1.5106	0.5124	0.3885	0.5006	1.006	0.5035	1.4871	0.5105	0.3933
		0.0004	0.0041	0.0007	0.01	0.0285	0.0005	0.0006	0.0039	0.0006	0.0085	0.0295	0.0006
	500	0.5022	1.0003	0.5022	1.5153	0.4997	0.3869	0.5003	1.0067	0.5029	1.502	0.4957	0.3909
		0.0003	0.0023	0.0004	0.0059	0.0176	0.0003	0.0003	0.0032	0.0003	0.0041	0.0172	0.0004
$\omega = 0.8$	50	0.5178	1.0086	0.5071	1.5489	0.795	0.3772	0.5129	1.0013	0.5099	1.5579	0.7175	0.3763
		0.0029	0.0204	0.0026	0.0481	0.3387	0.0027	0.0036	0.023	0.0029	0.0511	0.1948	0.0035
	100	0.5032	1.0306	0.5051	1.5096	0.8158	0.3896	0.505	1.0191	0.5138	1.4962	0.7949	0.3899
		0.0013	0.0131	0.0016	0.0226	0.0863	0.0015	0.0016	0.0115	0.0018	0.0252	0.0841	0.0015
	250	0.5063	0.9943	0.5026	1.4961	0.8044	0.3821	0.4997	0.9966	0.5013	1.5113	0.8107	0.3821
		0.0007	0.0044	0.0007	0.0118	0.0298	0.0006	0.0007	0.0036	0.0006	0.0076	0.0342	0.0006
	500	0.4982	1.0013	0.503	1.4957	0.7956	0.3843	0.5014	1.0167	0.5002	1.499	0.7897	0.3879
		0.0003	0.0023	0.0003	0.0055	0.0131	0.0002	0.0003	0.0029	0.0003	0.0048	0.0122	0.0004

Table 4.3: MLE and IFM estimates and corresponding MSEs for the parameter set $(\gamma_1, \delta_1, \gamma_2, \delta_2)$=(0.8,0.2,0.8,0.6).

		MLE						**IFM**					
$(\gamma_1 = 0.8, \delta_1 = 0.2,$ $\gamma_2 = 0.8, \delta_2 = 0.6)$	Sample size	$\hat{\gamma}_1$ MSE	$\hat{\delta}_1$ MSE	$\hat{\gamma}_2$ MSE	$\hat{\delta}_2$ MSE	$\hat{\omega}$ MSE	$\hat{R}_{MLE}$ MSE	$\hat{\gamma}_1$ MSE	$\hat{\delta}_1$ MSE	$\hat{\gamma}_2$ MSE	$\hat{\delta}_2$ MSE	$\hat{\omega}$ MSE	$\hat{R}_{IFM}$ MSE
$\omega = -0.8$	50	0.8244	0.2009	0.8261	0.6107	-0.8329	0.2823	0.8172	0.2071	0.8291	0.6151	-0.7829	0.2844
		0.0108	0.0035	0.0115	0.0123	0.2853	0.005	0.011	0.0028	0.0095	0.0113	0.1937	0.0036
	100	0.8111	0.1993	0.825	0.5875	-0.8473	0.2897	0.8148	0.2007	0.8137	0.5976	-0.836	0.2869
		0.0036	0.001	0.0057	0.0053	0.0964	0.0018	0.0043	0.0014	0.0047	0.0066	0.1029	0.002
	250	0.8077	0.1977	0.803	0.5998	-0.803	0.2823	0.804	0.1999	0.8036	0.5997	-0.7988	0.2839
		0.0018	0.0006	0.0013	0.0021	0.0255	0.0007	0.0016	0.0006	0.0018	0.0019	0.0264	0.0007
	500	0.7995	0.201	0.8055	0.5989	-0.803	0.2856	0.8038	0.1986	0.8008	0.6035	-0.798	0.2819
		0.0008	0.0003	0.001	0.0012	0.0147	0.0005	0.0008	0.0003	0.0008	0.0013	0.0139	0.0004
$\omega = -0.5$	50	0.8186	0.2042	0.8431	0.5774	-0.5485	0.2859	0.8278	0.1966	0.8175	0.6014	-0.5324	0.2692
		0.0076	0.0025	0.0117	0.0126	0.1359	0.0041	0.0121	0.0028	0.0111	0.0138	0.1538	0.0039
	100	0.8066	0.2006	0.8085	0.6155	-0.4805	0.2668	0.8028	0.2038	0.8139	0.5902	-0.4762	0.2773
		0.0039	0.0014	0.0047	0.0064	0.1238	0.0025	0.0035	0.0012	0.0045	0.0057	0.1044	0.0021
	250	0.8015	0.1995	0.8013	0.5967	-0.5109	0.2727	0.8023	0.2016	0.7967	0.602	-0.5035	0.2724
		0.0013	0.0005	0.0013	0.0026	0.0315	0.0007	0.0016	0.0005	0.0015	0.0022	0.0314	0.0007
	500	0.8033	0.1988	0.8029	0.6037	-0.5129	0.2698	0.7969	0.2022	0.7997	0.6068	-0.5116	0.2719
		0.0007	0.0002	0.0008	0.0011	0.0142	0.0003	0.0007	0.0003	0.0009	0.0011	0.0141	0.0004
$\omega = 0.5$	50	0.8305	0.1962	0.8148	0.6134	0.554	0.2207	0.8211	0.1998	0.8225	0.6146	0.5149	0.225
		0.012	0.0029	0.0091	0.0131	0.1706	0.0042	0.0076	0.0029	0.0085	0.0099	0.1521	0.0038
	100	0.8161	0.1983	0.8093	0.6065	0.4672	0.2274	0.8071	0.2021	0.8098	0.6093	0.4634	0.2274
		0.0045	0.0013	0.0036	0.0048	0.0945	0.0018	0.0041	0.0016	0.0036	0.0045	0.0954	0.0018
	250	0.8084	0.1979	0.8076	0.6011	0.5128	0.2256	0.8042	0.2015	0.8034	0.6022	0.5146	0.2292
		0.0019	0.0007	0.0017	0.0019	0.0269	0.0006	0.0018	0.0006	0.002	0.0024	0.0279	0.0009
	500	0.802	0.1998	0.8015	0.6027	0.5206	0.2253	0.8044	0.1988	0.7978	0.6048	0.5183	0.2253
		0.0008	0.0003	0.0007	0.0012	0.0159	0.0004	0.0009	0.0002	0.0008	0.001	0.0159	0.0004
$\omega = 0.8$	50	0.815	0.1994	0.8295	0.6057	0.8834	0.2121	0.8384	0.1928	0.8266	0.603	0.795	0.2112
		0.0066	0.0027	0.0111	0.0154	0.2633	0.0026	0.0084	0.0025	0.0097	0.0102	0.1854	0.0047
	100	0.8108	0.2019	0.811	0.5987	0.7634	0.2204	0.8154	0.1944	0.821	0.5958	0.7433	0.2154
		0.0042	0.0014	0.0041	0.0047	0.0827	0.0018	0.0046	0.0013	0.0045	0.005	0.0766	0.0018
	250	0.7982	0.2049	0.7971	0.6099	0.8065	0.2177	0.809	0.1961	0.8008	0.6028	0.7939	0.2119
		0.0016	0.0006	0.0013	0.0029	0.0313	0.0008	0.0017	0.0006	0.0015	0.0023	0.0296	0.0009
	500	0.8045	0.1979	0.8008	0.5973	0.8012	0.2147	0.8021	0.1997	0.7988	0.599	0.8035	0.2159
		0.0009	0.0003	0.0009	0.0012	0.0155	0.0003	0.0009	0.0003	0.0009	0.0013	0.0181	0.0004

Table 4.4: Confidence length and coverage probabilities for the dependence parameter ω

Parameter set	**Sample size**	ACL_{MLE}	CP_{MLE}	ACL_{IFM}	CP_{IFM}
$\gamma_1 = 2, \delta_1 = 2$ $\gamma_2 = 2, \delta_2 = 1.5, \omega = -0.8$	50	1.618	0.91	1.5234	1
	100	1.056	0.93	0.9675	1
	250	0.6464	0.97	0.7125	1
	500	0.4528	0.92	0.4559	1
$\gamma_1 = 2, \delta_1 = 2$ $\gamma_2 = 2, \delta_2 = 1.5, \omega = -0.5$	50	1.5783	0.93	1.7329	1
	100	1.1252	0.95	1.2041	1
	250	0.7016	0.93	0.6279	1
	500	0.5042	0.91	0.5264	1
$\gamma_1 = 2, \delta_1 = 2$ $\gamma_2 = 2, \delta_2 = 1.5, \omega = 0.5$	50	1.6109	0.91	1.4519	1
	100	1.1263	0.94	1.1673	1
	250	0.7023	0.95	0.7335	1
	500	0.4987	0.9	0.4938	1
$\gamma_1 = 2, \delta_1 = 2$ $\gamma_2 = 2, \delta_2 = 1.5, \omega = 0.8$	50	1.5607	0.93	1.5749	1
	100	1.0376	0.88	0.8703	1
	250	0.6334	0.96	0.6139	1
	500	0.4626	0.94	0.4219	1
$\gamma_1 = 0.5, \delta_1 = 1$ $\gamma_2 = 0.5, \delta_2 = 1.5, \omega = -0.8$	50	1.5409	0.93	1.5896	1
	100	1.0641	0.96	1.3393	1
	250	0.6459	0.91	0.7204	1
	500	0.454	0.92	0.4437	1
$\gamma_1 = 0.5, \delta_1 = 1$ $\gamma_2 = 0.5, \delta_2 = 1.5, \omega = -0.5$	50	1.5675	0.93	1.6373	1
	100	1.1276	0.93	1.2499	1
	250	0.712	0.95	0.6979	1
	500	0.5065	0.95	0.4736	1
$\gamma_1 = 0.5, \delta_1 = 1$ $\gamma_2 = 0.5, \delta_2 = 1.5, \omega = 0.5$	50	1.5934	0.97	1.5071	1
	100	1.1238	0.95	1.2316	1
	250	0.7073	0.95	0.7408	1
	500	0.502	0.94	0.4978	1
$\gamma_1 = 0.5, \delta_1 = 1$ $\gamma_2 = 0.5, \delta_2 = 1.5, \omega = 0.8$	50	1.6534	0.93	0.7555	1
	100	1.0463	0.94	1.0176	1
	250	0.6366	0.92	0.7161	1
	500	0.4495	0.93	0.5197	1
$\gamma_1 = 0.8, \delta_1 = 0.2$ $\gamma_2 = 0.8, \delta_2 = 0.6, \omega = -0.8$	50	1.5761	0.92	1.4197	1
	100	1.0628	0.95	1.3342	1
	250	0.6674	0.93	0.7087	1
	500	0.6499	0.95	0.6938	1
$\gamma_1 = 0.8, \delta_1 = 0.2$ $\gamma_2 = 0.8, \delta_2 = 0.6, \omega = -0.5$	50	1.5783	0.98	2.0654	1
	100	1.1193	0.89	1.0513	1
	250	0.7048	0.92	0.7451	1
	500	0.4987	0.96	0.522	1
$\gamma_1 = 0.8, \delta_1 = 0.2$ $\gamma_2 = 0.8, \delta_2 = 0.6, \omega = 0.5$	50	1.5743	0.94	1.6385	1
	100	1.1361	0.96	1.1181	1
	250	0.7035	0.97	0.7501	1
	500	0.4993	0.94	0.5307	1
$\gamma_1 = 0.8, \delta_1 = 0.2$ $\gamma_2 = 0.8, \delta_2 = 0.6, \omega = 0.8$	50	1.5838	0.92	1.4872	1
	100	1.0598	0.94	1.0362	1
	250	0.6442	0.93	0.6831	1
	500	0.452	0.9	0.4233	1

4.8 Real Data Analysis

In this section, we make use of a real data set to verify the effectiveness of the methodologies introduced in the preceding sections. We examine the dataset provided by McGilchrist & Aisbett (1991), which pertains the reccurence time to infection at point of insertion of the catheter for kidney patients using portable dialysis equipment. The data are given in Table 4.5.

Table 4.5: First and second recurrence times of 38 kidney patients

	8	23	22	447	30	24	7	511	53	15
X	7	141	96	149	536	17	185	292	22	15
	152	402	13	39	12	113	132	34	2	130
	27	5	152	190	119	54	6	63		
	16	13	28	318	12	245	9	30	196	154
Y	333	8	38	70	25	4	117	114	159	108
	562	24	66	46	40	201	156	30	25	26
	58	43	30	5	8	16	78	8		

The Kendall's and Spearman's correlation coefficients for the dataset are weak. Therefore, it is appropriate to choose a bivariate distribution derived from the FGM copula. The correlation measures for the dataset are reported in Table 4.6. First, we are interested in checking the effectiveness of the IW distribution by fitting it separately with the individual marginal distributions of X and Y. Figure 4.2, including the empirical CDF, P-P plot, Q-Q plot, and histogram with density plot for the marginal distributions, indicate a good fit for the data sets. Furthermore, the consistency of the results was confirmed by the KS test, as shown in Table 4.7.

Table 4.6: Correlation test results for the real data

	Correlation measure	
	Kendall's	Spearman's
Correlation	0.0014347	0.0001095
p-value	0.3914	0.4191

We use Multiplier bootstrap-based goodness-of-fit test introduced by Genest (2013) to fit FGM copula by using the parametric bootstrap M=10,000 times, reported in Table 4.8. The KS goodness-of-fit test is available in the R package

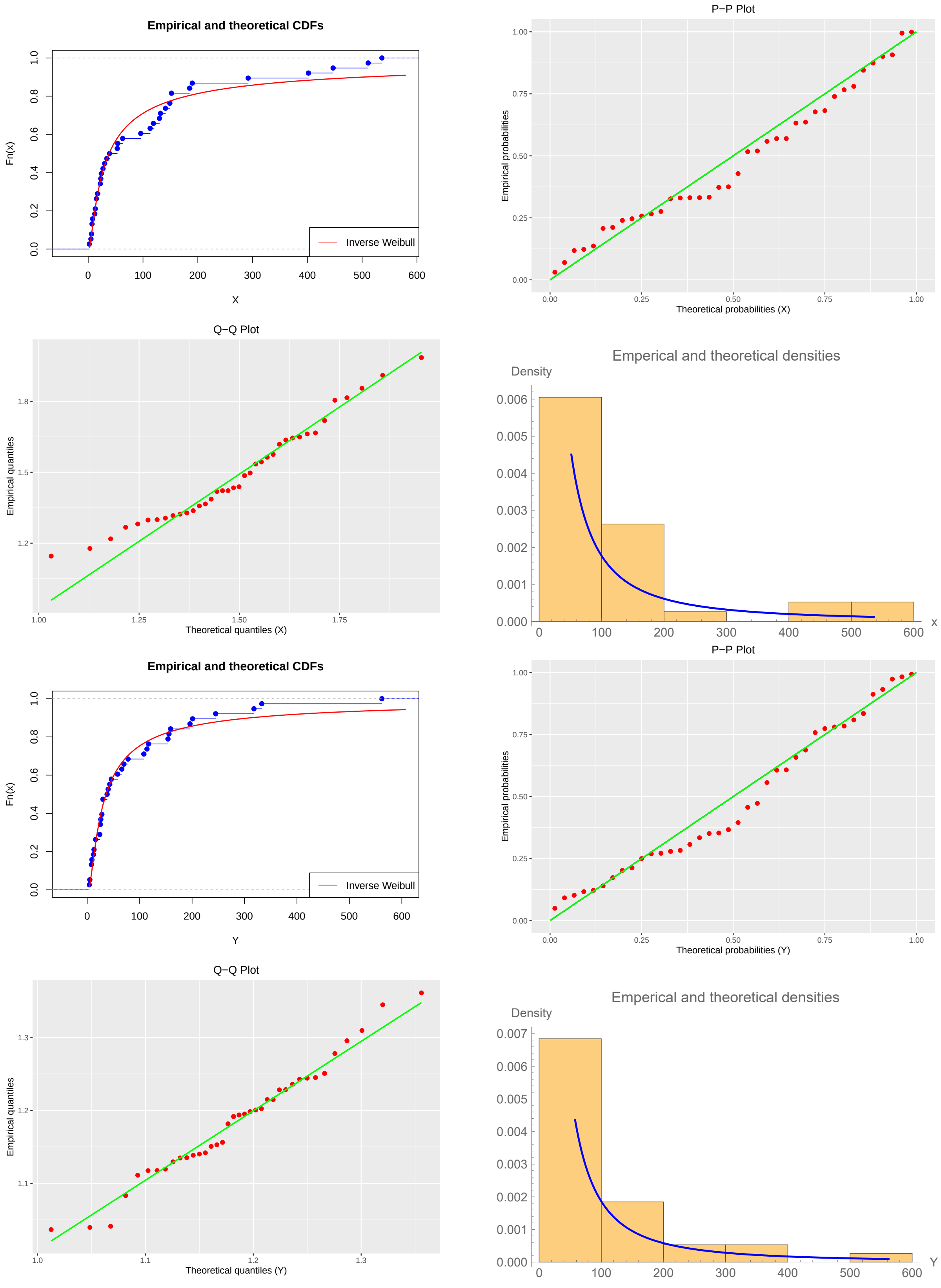


Figure 4.2: Probability plots for the real datasets

Table 4.7: Goodness-of-fit test for inverse Weibull distribution

	D statistic	p-value	Estimates
X	0.12619	0.5805	$\hat{\gamma}_1$ =0.72996 & $\hat{\delta}_1$ =9.8586
Y	0.1109	0.7383	$\hat{\gamma}_2$ =0.87173 & $\hat{\delta}_2$ =15.69919

`gofCopula`, which compares the empirical copula against a parametric estimate of the copula.

Table 4.8: The goodness-of-fit test for FGM copula

	p-value	Test statistic	$\hat{\omega}$
KS	0.9999	0.29294	0.04863

Next, the proposed BFGMIW distribution has been fitted to the real datasets and compared with the FGM models, FGMBW (FGM bivariate weibull) by El-Sherpieny (2022) and BFGMPLx (FGM bivariate power lomax) by Qura et al. (2023) distributions. For comparison purposes, the AIC, the AICC, and the BIC were used. The results shown in Table 4.9 indicate that BFGMIW has the better fit. Table 4.10 displays parameter estimates with corresponding standard errors (SE), AIC, AICC, and BIC values for ML and IFM methods. Clearly, both methods perform similarly, as they have almost similar values.

4.9 Summary

In this chapter, we consider the bivariate inverse Weibull model using the Farlie-Gumbel-Morgenstern copula. The proposed model (BFGMIW) can assess the dependent stress-strength reliability. The statistical characteristics, including conditional distributions, moment generating function, positive quadrant dependence properties, reliability measures, etc., have been derived. For parameter estimation, ML estimation and IFM methods are proposed. These methods are used to estimate the dependence parameter as well as stress-strength reliability. From the simulation study, it has been observed that when the value of the dependence parameter increases, the variation in the R value increases. The model is suitable when there is a weak correlation between the data sets. A real data set containing

Table 4.9: Estimates, AIC, AICC, and BIC values for FGM models.

FGM model	Estimates	AIC	AICC	BIC
BFGMIW	$\hat{\gamma}_1$=**0.7317** $\hat{\delta}_1$=**9.9499** $\hat{\gamma}_2$=**0.8706** $\hat{\delta}_2$=**15.6809** $\hat{\omega}$=**0.04125**	**857.1087**	**858.9837**	**865.2966**
FGMBW	$\hat{\alpha}_1$=0.0979 $\hat{\beta}_1$=0.0428 $\hat{\alpha}_2$= 0.1421 $\hat{\beta}_2$= 0.1677 $\hat{\theta}$=-0.3802	1143.273	1145.148	1151.461
BFGMPLx	$\hat{\gamma}_1$=0.01345 $\hat{\beta}_1$=0.0145 $\hat{\lambda}_1$=0.0008 $\hat{\gamma}_2$=0.0166 $\hat{\beta}_2$=0.01302 $\hat{\lambda}_2$=6.44×10^{-15} $\hat{\theta}$=0.0142	1753.568	1755.443	1761.756

Table 4.10: Parameter estimates, SE, AIC, AICC, and BIC values under MLE and IFM methods

Estimation	$\hat{\gamma}_1$	$\hat{\delta}_1$	$\hat{\gamma}_2$	$\hat{\delta}_2$	$\hat{\omega}$	**AIC**	**AICC**	**BIC**
MLE	0.73169	9.94996	0.8706	15.68089	0.04125	857.1087	858.9837	865.2966
(SE)	(0.09242)	(2.92023)	(0.08249)	(3.42849)	(0.51802)			
IFM	0.73003	9.86139	0.8719	15.7041	0.03654	857.1069	858.9819	865.2948
(SE)	(0.08613)	(2.605)	(0.1297)	(6.4356)	(0.51923)			

the recurrence times of kidney patients is used to examine the applicability of the BFGMIW distribution. We expect that the suggested model will find significant usage across diverse areas.

CHAPTER 5

Exponential Power Distribution Under Progressive Type-II Censoring Schemes

5.1 Introducion

In life-testing experiments, it is not always possible to gather all of the information. For a variety of reasons, experimental units or items may occasionally be taken out of the experiment. Censoring occurs in these situations when only partial information about the data is available (Lawless (2011)). In recent years, the reliability literature has seen the development of various censoring methods (Balakrishnan and Kundu (2013)). These advancements in censoring techniques have expanded the range of options available for dealing with incomplete data.

Censoring schemes, such as Type-I, Type-II, and their hybrid extensions, are widely used in reliability and life-testing studies. However, these traditional schemes have limitations that can make them less effective in practical experimental settings. The key flaw with the above-mentioned censoring schemes is that most inferential conclusions must be drawn under the assumption that there has been at least one observed failure. In addition, there may have been very few failures up to the pre-determined time T. The lack of a sufficient number of failures may lead to inefficiency in parameter estimation. Moreover, these schemes lack the flexibility to allow units to be removed from the experiment before it reaches its termination. Items that are lost in experimental locations besides the eventual end point may also be unavoidable, like in the situation of accidentally breaking experimental apparatus or losing contact with study subjects. These justifications and motives direct reliability researchers straight into the field of progressive censoring. The PT2 censoring scheme was introduced by Herd (1956) and studied by many authors (see, Balakrishnan & Aggarwala (2000) and Balakrishnan & Cramer (2014)). Later, Gajjar & Khatri (1969) examined a scenario of PT1 right censoring where the population parameters changed at every time of removal.

The progressive censoring scheme permits the withdrawal of experimental objects at various stages to reduce the total cost and time associated with the experiment. This approach could be useful in terms of identifying an early appropriate censoring scheme from the standpoint of experimental design. Nevertheless, while this notion is frequently made in the literature, it is not really met in real-world trials as the investigator can alter the censoring numbers during the study regardless of the circumstances. As a result, having a model that supports this type of adaptation is critical.

In exchange for the efficiency benefit, Burkschat (2008) pointed out that progressive censoring has a greater test time than the conventional Type-II censoring method. In order to overcome the disadvantage of PT2 censoring scheme, Kundu & Joarder (2006) proposed PT2H censoring method having fixed experimental time T. In this scheme, we are likely to encounter a very small number (even zero) of units because the sample size is random. Therefore, in many real-life scenarios, statistical inference cannot be employed. As a result, Ng et al. (2009) offered a novel censoring scheme called the APT2 censoring scheme. Utilizing both the prior censoring numbers and failure times, this approach enables the selection of the subsequent censoring number. Hemmati & Khorram (2017), Ateya & Mohammed (2017), Almetwally et al. (2020), and Mohan & Chacko (2021) have thoroughly reviewed this scheme.

Various lifetime distributions are commonly utilized for analyzing the censoring techniques. One of them is the two-parameter exponential Power (EP) distribution proposed by Smith & Bain (1975), which is known for its amenable nature and bathtub-shaped failure rate. The bathtub form is a useful conceptual paradigm for several technological and mechanical failures. Moreover, this distribution may be thought of as a truncated extreme-value distribution with a Weibull-type parameterization rather than the usual location-scale parameterization. The EP distribution has drawn a lot of interest in the literature and has been examined by numerous researchers. See, Leemis (1986), Rajarshi & Rajarshi (1988), and Chen (2000), who derived statistical inferences about the parameters. Akdam (2017) derived statistical inference about the stress-strength model of EP distribution based on PT2 censored samples.

Many authors have researched various progressive censoring schemes that provide the basis for statistical inferences about different lifetime distributions. However, the main focus of this chapter is on the effectiveness of the three censoring techniques PT2, PT2H, and APT2 on the EP distribution. It is worth noting that the behavior of the EP distribution has only been discussed under PT2 (Wu (2008)) and not under PT2H and APT2 in literature, despite the fact that the latter two are designed to address PT2's limitations. This study primarily estimates the EP distribution parameters using PT2, PT2H, and APT2 censored data. We assess the bias, MSE, confidence intervals, and coverage probability through a Monte Carlo simulation study in R programming. Subsequently, we compare the performance of PT2, PT2H, and APT2 when applied to the EP distribution against popular time models using real data.

This chapter is structured as follows: Section 5.2 discusses EP distribution as a lifetime model. The model description and parameter estimation for PT2, PT2H and APT2 censoring schemes are described in Sections 5.3, 5.4, and 5.5, respectively. Section 5.6 discusses the construction of asymptotic, percentile bootstrap, and Highest Posterior Density (HPD) intervals for the unknown parameters of the models. Moving on to Section 5.7, a simulation study is conducted to compare the effectiveness of the MLEs and Bayesian estimates for the unknown parameters. The results are demonstrated through a real data set in Section 5.8, and the chapter is summarized in Section 5.9.

5.2 The Exponential Power Distribution

A lifetime distribution is a mathematical effort to determine how long a system or device will last. Every day, as technology advances, more and more complicated forms of consumable things are created. So it requires credible statistical distributions to be incorporated as lifetime models to model the failure data of these innovative gadgets. The modelling of such kinds of real data sets has led to the introduction of numerous lifetime distributions. Among them, distributions with bathtub-shaped failure rate functions are very useful in practice; hence, they are studied by many authors. The EP distribution has an increasing and bathtub-shaped failure rate, which serves as a suitable conceptual metaphor for the risk posed by various electronic and mechanical items. A comparison of the EP model to the Weibull distribution reveals that the failure rate function of the EP model increases exponentially for large values of x, whereas the failure rate function of the Weibull model increases polynomially. Additionally, the failure rate of the EP distribution exhibits a bathtub shape, while the Weibull failure rate does not. Therefore, the EP model proves to be a valuable alternative to the Weibull distribution in certain cases.

The CDF of the EP distribution for a random variable X can be expressed as follows:

$$F(x) = 1 - e^{1-e^{(\lambda x)^{\alpha}}};\, x > 0,\, \alpha,\, \lambda > 0. \tag{5.2.1}$$

The corresponding PDF of X is,

$$f(x) = \alpha\lambda^{\alpha}x^{\alpha-1}e^{1+(\lambda x)^{\alpha}-e^{(\lambda x)^{\alpha}}};\, x > 0,\, \alpha,\, \lambda > 0. \tag{5.2.2}$$

Figure 5.1 demonstrates the behavior of the PDF of the EP distribution for different parameter values.

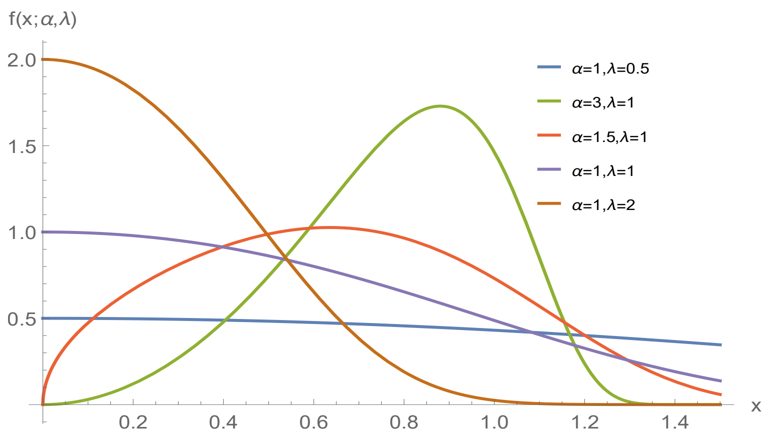


Figure 5.1: The plot of the EP density function for distinct choices of parameter values

The failure rate function of the EP distribution, expressed as $h(x) = \alpha\lambda^{\alpha}x^{\alpha-1}e^{(\lambda x)^{\alpha}}$, is noteworthy for its distinct behavior, characterized by a bathtub shape and increasing failure rate over time. For values of $\alpha \geq 1$, $h(x)$ is an increasing function for $x > 0$. Additionally, for values of $\alpha < 1$, the failure rate function has a bathtub shape and attains its minimum at $\left(\frac{1-\alpha}{\alpha\lambda^{\alpha}}\right)^{\frac{1}{\alpha}}$. Figure 5.2 illustrates the bathtub-shaped failure rate function of the EP distribution, highlighting its distinct characteristics.

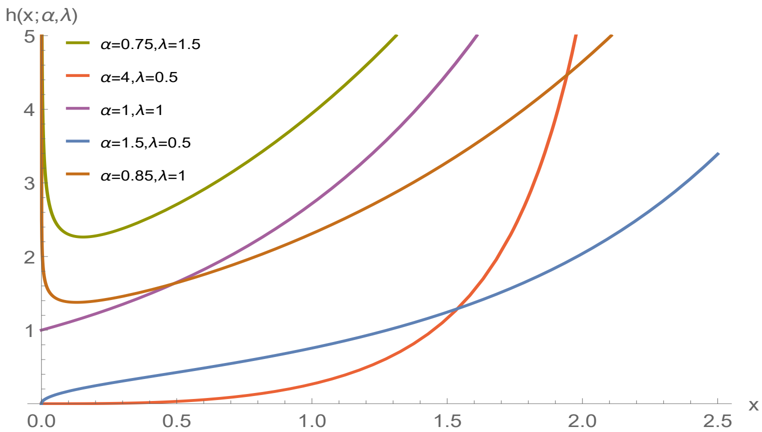


Figure 5.2: The plot of the failure rate function, $h(x)$, of the EP distribution for different parameter values

5.3 Progressive Type-II Censoring

A life-testing experiment is set up with n units according to the PT2 censoring. When the first failure in the experiment occurs, R_1 of the remaining $(n-1)$ units are randomly removed. At the time of second failure, R_2 of the remaining $(n - R_2 - 2)$ units are removed randomly. This procedure is repeated till the predetermined m^{th} failure occurs, at which all remaining units $(R_m = n - R_1 - R_2 - \cdots R_{m-1} - m)$ are removed. Note that, $\sum_{i=1}^{m} R_i + m = n$. For $R = (R_1 = R_2 = \cdots = R_m = 0)$, the PT2 censoring method becomes a complete sampling and for $R = (R_1 = R_2 = \cdots R_{m-1} = 0, R_m = n - m)$ it reduces to conventional Type-II censoring method. Here, $R = (R_1, R_2, ..., R_m)$ is predetermined, which is an essential aspect in the construction of the progressively censored experiment. This method can be intriguing from the perspective of experimental design because it identifies an initially optimal scheme (see Burkschat et al. (2006), Balakrishnan & Narayanaswamy (2007), Balakrishnan et al. (2008)).

5.3.1 Model Description

Let $x_{1:m:n} < x_{2:m:n} < \cdots < x_{i:m:n} < \cdots < x_{m:m:n}$, be the PT2 censored observations with censoring scheme R from $EP(\alpha, \lambda)$. Then the likelihood function based on PT2 censored sample is given by,

$$L_1(x_{i:m:n}; \theta) = A_1 \prod_{i=1}^{m} f(x_{i:m:n}; \theta) \left[1 - F(x_{i:m:n}; \theta)\right]^{R_i},$$

where A_1 is a constant which is independent of parameter values and given by,

$$A_1 = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - R_1 - R_2 - \cdots R_{m-1} - (m - 1)).$$

To simulate a PT2 censored sample from the EP distribution, we adopt the algorithm described by Balakrishnan & Aggarwala (2000), which entails the following steps:

- Step 1. Generate m independent random observations $(U_1, U_2, \cdots, U_m)$ from Uniform distribution.
- Step 2. Set $W_i = U_i^{1/\left(i+\sum_{j=m-i+1}^{m} R_j\right)}$, for $i = 1, 2, .., m$.
- Step 3. Set $Z_{i:m:n} = 1 - W_m W_{m-1} \cdots W_{m-i+1}$, for $i = 1, 2, ..., m$. So that $(Z_{1:m:n}, Z_{2:m:n}, ..., Z_{m:m:n})$ are the PT2 censored observations from the Uniform(0,1) distribution.
- Step 4. Set $X_{i:m:n} = F^{-1}(Z_{i:m:n})$, where $F(\cdot)$ is the CDF of EP distribution.

Now $(X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n})$ are the PT2 censored samples from EP distribution with censoring scheme R.

5.3.2 MLE

In this section the MLE of the unknown parameters of $EP(\alpha, \lambda)$ are obtained using PT2 censored data. Assume $(X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n})$ are the PT2 censored observations of an experiment with n units in it. Then the log-likelihood function according to PT2 censoring can be written as (for convenience x_i represents for $x_{i:m:n}$);

$$\log L_1(\alpha, \lambda) = \log A_1 + m \log \alpha + m\alpha \log \lambda + (\alpha - 1) \sum_{i=1}^{m} \log x_i$$

$$+\sum_{i=1}^{m}\left[1+(\lambda x_i)^{\alpha}-e^{(\lambda x_i)^{\alpha}}\right]+\sum_{i=1}^{m}R_i\left[1-e^{(\lambda x_i)^{\alpha}}\right]. \tag{5.3.1}$$

Equating the first derivatives of log-likelihood w.r.t. its unknown parameters, α and λ, to zero gives the following:

$$\frac{\partial \log L_1(\alpha,\lambda)}{\partial \alpha}=\frac{m}{\alpha}+m\log\lambda+\sum_{i=1}^{m}\log x_i+\sum_{i=1}^{m}\left[1-(1+R_i)e^{(\lambda x_i)^{\alpha}}\right](\lambda x_i)^{\alpha}\log(\lambda x_i)=0,$$
$$\frac{\partial \log L_1(\alpha,\lambda)}{\partial \lambda}=\frac{m\alpha}{\lambda}+\sum_{i=1}^{m}\left[1-(1+R_i)e^{(\lambda x_i)^{\alpha}}\right]\alpha\lambda^{\alpha-1}x_i^{\alpha}=0.$$

A solution of the above system of equations gives the MLEs of the unknown parameters. These nonlinear equations have no closed form solution. To solve the system of non-linear equations, the Newton-Raphson iteration technique can be utilized.

5.3.3 Bayesian Estimation

Here we examines the Bayesian estimate of the parameters α and λ by analyzing both symmetric and asymmetric loss functions under the three censoring methods PT2, PT2H, and APT2. Both the squared error loss (L_{SE}) function and the LINEX loss (L_{LL}) function are analyzed for the symmetric and asymmetric loss functions, respectively. The L_{SE} is described as,

$$L_{SE}(g(\theta),\hat{g}(\theta))=(g(\theta)-\hat{g}(\theta))^2, \tag{5.3.2}$$

where $\hat{g}(\theta)$ is the estimate of $g(\theta)$. The L_{LL} can be defined as,

$$L_{LL}(g(\theta),\hat{g}(\theta))=e^{d(g(\theta)-\hat{g}(\theta))}-d(g(\theta)-\hat{g}(\theta))-1,\ d\neq 0, \tag{5.3.3}$$

where d is known as the loss parameter. For the LINEX loss function, the Bayes estimate $\hat{\theta}_{LL}$ can be calculated as,

$$\hat{\theta}_{LL}=-\frac{1}{d}\log\left\{E_{\theta}\left(e^{-d\theta}\right)\right\}, \tag{5.3.4}$$

provided that the expectation exists and is finite. Here, it is assumed that the prior distributions of α and λ follows independent $Gamma(a_1,b_1)$ and $Gamma(a_2,b_2)$, respectively, where a_1, b_1, a_2, and b_2 are the hyper parameters. In cases where both α and λ are unknown, it's not possible to have a conjugate prior. As a result, gamma priors can be used instead since they're flexible and can account for non-informative priors. Hence, the independent prior distributions of α and λ are,

$$\pi_1(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha}, \tag{5.3.5}$$

$$\pi_2(\lambda) \propto \lambda^{a_2-1}e^{-b_2\lambda}. \tag{5.3.6}$$

The joint prior of α and λ can be written as

$$\pi(\alpha,\lambda) \propto \alpha^{a_1-1}\lambda^{a_2-1}e^{-b_1\alpha-b_2\lambda}. \tag{5.3.7}$$

Using Bayes Theorem, the joint posterior density function of the parameters α and λ under PT2 is,

$$\pi(\alpha,\lambda|\underline{X}) \propto L_1(\alpha,\lambda|\underline{X})\pi(\alpha,\lambda),$$

$$\pi(\alpha,\lambda|\underline{X}) \propto \alpha^{m+a_1-1}\lambda^{\alpha m+a_2-1}e^{-b_1\alpha-b_2\lambda+\sum_{i=1}^{m}(1+(\lambda x_i)^\alpha-e^{(\lambda x_i)^\alpha}+R_i(1-e^{(\lambda x_i)^\alpha}))}\prod_{i=1}^{m}x_i^{\alpha-1}. \tag{5.3.8}$$

The conditional distribution of the unknown parameters are,

$$\pi(\alpha|\lambda,\underline{X}) \propto \alpha^{m+a_1-1}\lambda^{\alpha m}e^{-b_1\alpha+\sum_{i=1}^{m}(1+(\lambda x_i)^\alpha-e^{(\lambda x_i)^\alpha}+R_i(1-e^{(\lambda x_i)^\alpha}))}\prod_{i=1}^{m}x_i^{\alpha-1}, \tag{5.3.9}$$

$$\pi(\lambda|\alpha,\underline{X}) \propto \lambda^{\alpha m+a_2-1}e^{-b_2\lambda+\sum_{i=1}^{m}(1+(\lambda x_i)^\alpha-e^{(\lambda x_i)^\alpha}+R_i(1-e^{(\lambda x_i)^\alpha}))}. \tag{5.3.10}$$

To generate Bayesian estimates of α and λ, we use the Metropolis-Hastings (M-H) algorithm proposed by Metropolis et al. (1953) and Hastings (1970). This method is particularly useful when the posterior density is operationally infeasible, and posterior samples need to be constructed using random proposal distributions. Based on the posterior samples, we can draw Bayesian inference about the unknown parameters α and λ. The following steps can be used to construct the posterior samples:

Step 1: Initialize the values (α^0, γ^0), and let $\sigma = \alpha^{i-1}$.

Step 2: Generate a proposal value α^i using a random proposal distribution.

Step 3: Calculate the posterior density for the proposed values.

Step 4: Compute the acceptance probability using the M-H acceptance rule.
Step 5: Determine whether to accept or reject the proposed values based on the acceptance probability and a random draw from a uniform distribution.
Step 6: If the proposed values are accepted, add them to the posterior sample. If they are rejected, add the current values to the posterior sample.
Step 7: Repeat steps 2-6 for a sufficient number (N) of iterations to obtain a representative posterior sample.

To ensure convergence and eliminate the impact of initial value selection, we exclude the first M estimates from the posterior sample. The remaining sample of size N-M, consisting of (α^i, λ^i), with i varies from $M + 1$ to N, constitutes an estimated posterior sample that can be used for Bayesian inference.

5.4 Progressive Type-II Hybrid Censoring

The main disadvantages of PT2 censoring approach is that the convenient sample size is determined at random, and it might be quite small. As a result, traditional statistical inference approaches will either be ineffective or incompatible. To solve these drawbacks, Kundu & Joarder (2006) devised a censoring scheme known as the PT2H censoring scheme, in which the experiment ends at a predetermined time.

Assume n identical units are tested, and $X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$ denotes the life times of corresponding items. Here, the integers $m < n$, $R_1, R_2, ..., R_m$ with $R_1 + R_2 + ..., R_m + m = n$ and T are pre-fixed. When the first failure $X_{1:m:n}$ occurs, R_1 of the rest of the items are randomly eliminated. Likewise, R_2 of the remaining observations is eliminated when the second failure $X_{2:m:n}$ occurs. The procedure will continue until either the m^{th} failure or the time point T is reached. The experiment ends at the time point $X_{m:m:n}$, if the m^{th} failure $X_{m:m:n}$ happens prior to the time point T. In the event that only J failures $(0 \leq J < m)$ occur prior to time T, all remaining $(n - J - \sum_{i=1}^{J} R_i)$ units will be removed at time T, and the experiment will end at that time. Here, the two situations are referred to as Case I and Case II, respectively, and the censoring technique is called PT2H censoring scheme.

5.4.1 Model Description

The two possible outcomes of PT2H censoring scheme can be represented as follows:

$$\text{Case I} \quad : \quad \{X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}\}; \quad \text{if} \quad X_{m:m:n} < T, \quad \text{or} \tag{5.4.1}$$

$$\text{Case II} \quad : \quad \{X_{1:m:n}, X_{2:m:n}, \ldots, X_{J:m:n}\}; \quad \text{if} \quad X_{J:m:n} < T < X_{J+1:m:n}. \tag{5.4.2}$$

It should be noted that the obersvations $X_{J+1:m:n}, X_{J+2:m:n}, ..., X_{m:m:n}$ are not observed in Case II.

The form of likelihood function corresponding to both cases respectively are,

$$\text{Case I} \quad : \quad L_2(\alpha, \lambda) \propto \prod_{i=1}^{m} \{f(x_{i:m:n})\,(1 - F(x_{i:m:n}))^{R_i}\}, \quad \text{or} \tag{5.4.3}$$

$$\text{Case II} \quad : \quad L_2(\alpha, \lambda) \propto \prod_{i=1}^{J} \{f(x_{i:m:n})\,(1 - F(x_{i:m:n}))^{R_i}\}\,(1 - F(T))^{n-J-\sum_{i=1}^{J} R_i}. \tag{5.4.4}$$

Case I and Case II can be combined as follows:

$$L_2(\alpha, \lambda) \propto \quad \prod_{i=1}^{D} \{f(x_{i:m:n})\,(1 - F(x_{i:m:n}))^{R_i}\}\,(1 - F(T))^{n-D-\sum_{i=1}^{D} R_i},$$

where $D = m$, for Case I and $D = J$ for Case II.

PT2H censored sample can be easily generated using the method described in Balakrishnan & Aggarwala (2000).

5.4.2 MLE

Assume $(X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n})$ are the PT2H censored observations of an experiment with n units in it, the log-likelihood function according to the sample can be written as,

$$\log L_2(\alpha, \lambda) \propto D \log \alpha + D\alpha \log \lambda + (\alpha - 1) \sum_{i=1}^{D} \log x_i + \sum_{i=1}^{D} \Big(1 + (\lambda x_i)^\alpha - e^{(\lambda x_i)^\alpha}\Big)$$
$$+ \sum_{i=1}^{D} R_i \Big(1 - e^{(\lambda x_i)^\alpha}\Big) + \Big(n - D - \sum_{i=1}^{D} R_i\Big)\Big(1 - e^{(\lambda T)^\alpha}\Big).$$

A solution of the system of equations given below, will give the MLE of the parameters, α and λ under the scheme PT2H.

$$\frac{\partial \log L_2}{\partial \alpha} = \frac{D}{\alpha} + D \log \lambda + \sum_{i=1}^{D} \log x_i + \sum_{i=1}^{D} (\lambda x_i)^\alpha \log(\lambda x_i) \Big(1 - e^{(\lambda x_i)^\alpha}\Big)$$
$$- \sum_{i=1}^{D} R_i (\lambda x_i)^\alpha \log(\lambda x_i) e^{(\lambda x_i)^\alpha} - (n - D - \sum_{i=1}^{D} R_i)(\lambda T)^\alpha \log(\lambda T) e^{(\lambda T)^\alpha}$$

$$\frac{\partial \log L_2}{\partial \lambda} = \frac{D\alpha}{\lambda} + \alpha\lambda^{\alpha-1}\sum_{i=1}^{D} x_i^{\alpha}\left(1 - e^{(\lambda x_i)^{\alpha}}\right) - \alpha\lambda^{\alpha-1}\sum_{i=1}^{D} R_i x_i^{\alpha} e^{(\lambda x_i)^{\alpha}}$$
$$- \alpha\lambda^{\alpha-1}T^{\alpha}\left(n - D - \sum_{i=1}^{D} R_i\right)e^{(\lambda T)^{\alpha}}$$

Similar to Section 5.3.2, this system of equations can be solved using the Newton-Raphson method.

5.4.3 Bayesian estimation

Under the PT2H framework, let us assume that the prior distributions of α and λ follows independent $Gamma(c_1, d_1)$ and $Gamma(c_2, d_2)$, respectively, where $c_1, d_1, c_2, \& \, d_2$ are the hyper parameters. The prior distributions of α and λ are,

$$\psi_1(\alpha) \propto \alpha^{c_1-1}e^{-d_1\alpha}, \tag{5.4.5}$$

$$\psi_2(\lambda) \propto \lambda^{c_2-1}e^{-d_2\lambda}. \tag{5.4.6}$$

Then the joint prior distribution takes the form,

$$\psi(\alpha, \lambda) \propto \alpha^{c_1-1}\lambda^{c_2-1}e^{-d_1\alpha-d_2\lambda}. \tag{5.4.7}$$

The joint posterior density function is given by,

$$\psi(\alpha, \lambda|\underline{X}) \propto L(\alpha, \lambda|\underline{X})\psi(\alpha, \lambda)$$
$$\psi(\alpha, \lambda|\underline{X}) \propto \alpha^{c_1+D-1}\lambda^{\alpha D+c_2-1}e^{-d_1\alpha-d_2\lambda+\sum_{i=1}^{D}(1+(\lambda x_i)^{\alpha}-e^{(\lambda x_i)^{\alpha}})}\prod_{i=1}^{D} x_i^{\alpha-1}$$
$$e^{\sum_{i=1}^{D} R_i(1-e^{(\lambda x_i)^{\alpha}})+(1-e^{(\lambda T)^{\alpha}})(n-D-\sum_{i=1}^{D} R_i)}. \tag{5.4.8}$$

Then the conditional distributions of α and λ are obtained by,

$$\psi(\alpha|\lambda, \underline{X}) \propto \alpha^{c_1+D-1}\lambda^{\alpha D}e^{-d_1\alpha+\sum_{i=1}^{D}(1+(\lambda x_i)^{\alpha}-e^{(\lambda x_i)^{\alpha}})}\prod_{i=1}^{D} x_i^{\alpha-1}$$
$$e^{\sum_{i=1}^{D} R_i(1-e^{(\lambda x_i)^{\alpha}})+(1-e^{(\lambda T)^{\alpha}})(n-D-\sum_{i=1}^{D} R_i)}, \tag{5.4.9}$$

$$\psi(\lambda|\alpha, \underline{X}) \propto \lambda^{\alpha D+c_2-1}e^{-d_2\lambda+\sum_{i=1}^{D}(1+(\lambda x_i)^{\alpha}-e^{(\lambda x_i)^{\alpha}})}$$
$$e^{\sum_{i=1}^{D} R_i(1-e^{(\lambda x_i)^{\alpha}})+(1-e^{(\lambda T)^{\alpha}})(n-D-\sum_{i=1}^{D} R_i)}. \tag{5.4.10}$$

To calculate Bayesian estimates of the unknown parameters α and λ, we need to use the M-H technique discussed in Section 5.3.3 because the conditional distribution of α and λ doesn't have an explicit form.

5.5 Adaptive Type-II Progressive Censoring

Let's say the experimenter gives us a time T that is an optimal overall test period, but we can extend the experiment beyond time T. Here, the values m, n, R, and T are prefixed. If the m^{th} failure happens prior to T, the study will terminate at the time $X_{m:m:n}(< T)$. Instead, if the experimental time reaches T but the number of observed failures does not reach m, we should stop the study as quickly as possible. Till the m^{th} failure happens, the survival units will not be eliminated. Suppose that the experimental time surpasses T immediately following the J-th failure, (i.e., $X_{J:m:n} < T < X_{J+1:m:n}$, $X_{0:m:n} \equiv 0$, $X_{m+1:m:n} \equiv \infty, J = 0, 1, 2, \cdots, m$). As a result, when the real time exceeds T, the aforementioned censoring scheme becomes, $R_{J+1} = R_{J+2} = \cdots = R_{m-2} = R_{m-1} = 0$ and $R_m = n - m - \sum_{i=1}^{J} R_i$. Whenever the $(J+1)^{th}$ failure time is larger than T for $(J+1 < m)$, this formulation compels us to end the experiment straightaway. When $T = 0$, this scheme transforms into a Type-II censoring. $T = \infty$, is the related premise if the test duration is adequate. This model evolves into a PT2 censoring model over time. For more details about APT2 see Ng et al. (2009).

This arrangement can be considered as a design in which we are guaranteed to have m failure times while keeping the overall test duration close to the optimum time T. This method exemplifies how an investigator can manage the experiment. The experimentor will take out fewer units if he wants to receive observations earlier. More units will be removed if he wants to see bigger observed failure times. Therefore, a more accommodating approach to choosing the censoring method is preferred.

5.5.1 Model Description

For $J = j$, the likelihood function is given by,

$$L_3(x_{i:m:n}; \theta | J = j) = d_j \left[\prod_{i=1}^{m} f(x_{i:m:n}; \theta)\right] \left\{\prod_{i=1}^{j} \left[1 - F(x_{i:m:n}; \theta)\right]^{R_i}\right\} \left[1 - F(x_{m:m:n}; \theta)\right]^{(n-m-\sum_{i=1}^{j} R_i)},$$

where

$$d_j = \prod_{i=1}^{m} \left[n - i + 1 - \sum_{k=1}^{\max\{i-1,j\}} R_k\right].$$

For a fixed value of m, n, T, and R, the method described by Ng et al. (2009) is used to simulate APT2 censored data.

5.5.2 MLE

Assume $(X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n})$ are the APT2 censored observations of an experiment with n units in it. For a given $(J = j)$, the log-likelihood function of $EP(\alpha, \lambda)$ according to APT2 censoring is,

$$\log L_3(x_i; \alpha, \lambda | J = j) = Constant + m \log \alpha + m\alpha \log \lambda + (\alpha - 1) \sum_{i=1}^{m} \log x_i$$
$$+ \sum_{i=1}^{m} \left(1 + (\lambda x_i)^\alpha - e^{(\lambda x_i)^\alpha}\right) + \sum_{i=1}^{j} R_i \left(1 - e^{(\lambda x_i)^\alpha}\right) + (n - m - \sum_{i=1}^{j} R_i) \left(1 - e^{(\lambda x_m)^\alpha}\right).$$

The second derivates w.r.t. α and λ are obtained as,

$$\frac{\partial \log L_3}{\partial \alpha} = \frac{m}{\alpha} + m \log \lambda + \sum_{i=1}^{m} \log x_i + \sum_{i=1}^{m} (\lambda x_i)^\alpha \log(\lambda x_i) \left(1 - e^{(\lambda x_i)^\alpha}\right) -$$
$$\sum_{i=1}^{j} R_i (\lambda x_i)^\alpha \log(\lambda x_i) e^{(\lambda x_i)^\alpha} - \left(n - m - \sum_{i=1}^{j} R_i\right) (\lambda x_m)^\alpha \log(\lambda x_m) e^{(\lambda x_m)^\alpha}$$

and

$$\frac{\partial \log L_3}{\partial \lambda} = \frac{m\alpha}{\lambda} + \alpha \lambda^{\alpha - 1} \sum_{i=1}^{m} x_i^\alpha \left(1 - e^{(\lambda x_i)^\alpha}\right) - \alpha \lambda^{\alpha - 1} \sum_{i=1}^{j} R_i x_i^\alpha e^{(\lambda x_i)^\alpha}$$
$$- (n - m - \sum_{i=1}^{j} R_i) \alpha \lambda^{\alpha - 1} x_m^\alpha e^{(\lambda x_m)^\alpha}.$$

Likewise, in the Sections 5.3.2 and 5.4.2, the estimates can be optimised using the Newton-Raphson method.

5.5.3 Bayesian Estimation

For APT2 method, we assume that the parameters α and λ follows $Gamma(e_1, f_1)$ and $Gamma(e_2, f_2)$ respectively, where e_1, f_1, e_2, and f_2 are the hyper parameters. The independent prior distributions are given by,

$$\phi_1(\alpha) \propto \alpha^{e_1 - 1} e^{-f_1 \alpha}, \tag{5.5.1}$$

$$\phi_2(\lambda) \propto \lambda^{e_2 - 1} e^{-f_2 \lambda}. \tag{5.5.2}$$

The joint prior distribution is,

$$\phi(\alpha,\lambda) \propto \alpha^{e_1-1}\lambda^{e_2-1}e^{-f_1\alpha-f_2\lambda}. \tag{5.5.3}$$

The joint posterior density function is derived as,

$$\begin{aligned}\phi(\alpha,\lambda|\underline{X}) &\propto L(\alpha,\lambda|\underline{X})\phi(\alpha,\lambda)\\ \phi(\alpha,\lambda|\underline{X}) &\propto \alpha^{m+e_1-1}\lambda^{\alpha m+e_2-1}e^{-f_1\alpha-f_2\lambda+\sum_{i=1}^{m}(1+(\lambda x_i)^{\alpha}-e^{(\lambda x_i)^{\alpha}})}\prod_{i=1}^{m}x_i^{\alpha-1}\\ &\quad e^{\sum_{i=1}^{j}R_i(1-e^{(\lambda x_i)^{\alpha}})+(1-e^{(\lambda x_m)^{\alpha}})(n-m-\sum_{i=1}^{j}R_i)}.\end{aligned} \tag{5.5.4}$$

Now the conditional distributions of α and λ are obtained by,

$$\begin{aligned}\phi(\alpha|\lambda|\underline{X}) \propto \alpha^{m+e_1-1}\lambda^{\alpha m}e^{-f_1\alpha+\sum_{i=1}^{m}(1+(\lambda x_i)^{\alpha}-e^{(\lambda x_i)^{\alpha}})}\prod_{i=1}^{m}x_i^{\alpha-1}\\ e^{\sum_{i=1}^{j}R_i(1-e^{(\lambda x_i)^{\alpha}})+(1-e^{(\lambda x_m)^{\alpha}})(n-m-\sum_{i=1}^{j}R_i)},\end{aligned} \tag{5.5.5}$$

$$\begin{aligned}\phi(\lambda|\alpha,\underline{X}) \propto \lambda^{\alpha m+e_2-1}e^{-f_2\lambda+\sum_{i=1}^{m}(1+(\lambda x_i)^{\alpha}-e^{(\lambda x_i)^{\alpha}})}\\ e^{\sum_{i=1}^{j}R_i(1-e^{(\lambda x_i)^{\alpha}})+(1-e^{(\lambda x_m)^{\alpha}})(n-m-\sum_{i=1}^{j}R_i)}.\end{aligned} \tag{5.5.6}$$

As mentioned previously, the expressions for α and λ end up in unexplicit forms, so we'll have to resort to using the M-H algorithm to produce the Bayesian estimates for both parameters.

5.6 Confidence Intervals

In this section, we'll be delving into various methods for generating confidence intervals for the unknown parameters α and λ. Initially, we'll be discussing the asymptotic confidence interval. However, since obtaining the PDF of α and λ can be a bit challenging, we'll also be creating Bootstrap confidence intervals for them. Additionally, we'll be exploring HPD credible intervals in this section as well.

5.6.1 Asymptotic Confidence Intervals (ACI)

The ACI for the parameters α and λ are derived in this section using the asymptotic variance–covariance matrix for the MLE. The inverse of observed Fisher information matrix can be used to calculate the asymptotic variance-covariance matrix.

The observed Fisher information matrix is,

$$I_0(\alpha,\lambda) = \begin{bmatrix} E\left(-\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial \alpha \partial \lambda}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial \lambda \partial \alpha}\right) & E\left(-\frac{\partial^2 \log L}{\partial \lambda^2}\right) \end{bmatrix},$$

where the second derivatives inside the matrix for each censoring scheme can be directly calculated and can be found in the Appendix A.

An inverted form of the above matrix will give the estimate of asymptotic variance-covariance matrix of MLEs of α and λ. The ACI for α and λ can be obtained as: $\hat{\alpha} \pm Z_{\frac{\delta}{2}}\sqrt{\upsilon_{11}}$ and $\hat{\lambda} \pm Z_{\frac{\delta}{2}}\sqrt{\upsilon_{22}}$, respectively, where υ_{11} and υ_{22} are main diagonal entries of $I_0(\alpha,\lambda)$ and Z_δ is the 100δ-th percentile of standard normal variate.

5.6.2 Percentile Bootstrap Confidence Interval

The algorithm proposed by Efron (1982) is used to construct the parametric percentile bootstrap (BOOT-P) confidence interval for the unknown parameters α and λ:

- STEP 1: Obtain $\hat{\alpha}$ and $\hat{\lambda}$ from the complete sample $(x_{1:m:n}, x_{2:m:n}, \cdots, x_{m;m;n})$.
- STEP 2: Generate a censored sample $(y_{1:m:n}, y_{2:m:n}, \cdots, y_{m:m:n})$ from $EP(\alpha,\lambda)$ distribution for a pre-fixed censoring scheme $(R_1, R_2, \cdots, R_m)$ and time T.
- STEP 3: Determine the estimates $\hat{\alpha}^*$ and $\hat{\lambda}^*$ based on the sample $(y_{1:m:n}, y_{2:m:n}, \cdots, y_{m:m:n})$.
- STEP 4: Repeat steps 2 and 3 NBOOT times.
- STEP 5: Suppose $F_\alpha(y) = P(\hat{\alpha}^* \leq y)$ and $F_\lambda(y) = P(\hat{\lambda}^* \leq y)$, where F is the CDF. Then compute $\hat{\alpha}_{BOOT-P}(y) = F_\alpha^{-1}(y)$ and $\hat{\lambda}_{BOOT-P}(y) = F_\lambda^{-1}(y)$ for a particular y. The approximate $100(1-\delta)\%$ BOOT-P confidence interval for α and λ are given by $(\hat{\alpha}^*_{BOOT-P}(\frac{\delta}{2}), \hat{\alpha}^*_{BOOT-P}(1-\frac{\delta}{2}))$ and $(\hat{\lambda}^*_{BOOT-P}(\frac{\delta}{2}), \hat{\lambda}^*_{BOOT-P}(1-\frac{\delta}{2}))$, respectively.

5.6.3 HPD Credible Interval

To construct the $100(1-\delta)\%$ HPD credible interval of the unknown parameters α and λ we can use the ordered samples $(\alpha_{M+1}, \alpha_{M+2}, ..., \alpha_N)$ and $(\lambda_{M+1}, \lambda_{M+2}, ..., \lambda_N)$

obtained from Bayesian estimation. The HPD intervals is determined by selecting the shortest interval among these samples. These credible intervals play a crucial role in characterizing the uncertainty associated with our parameter estimates.

5.7 Simulation Study

In this section, we compare the effectiveness of MLE, Bayes estimates, and interval estimation techniques for the three censoring methods mentioned above using a Monte Carlo simulation technique. The effectiveness of various methods can be determined by comparing their bias, MSE, CP, and ACL. For the purpose of comparison, the simulation schemes are created by taking into account the size of the samples (n), effective samples (m), and the ideal experimental time (T) for various values of R. Without losing generality, the parameter values ($\alpha = 4, \lambda = 3$) and ($\alpha = 1, \lambda = 0.5$) are selected as the initial values along with the experimental time $T = 0.3$ and $T = 1$ respectively. The censoring schemes are shortened for discussion purposes. For example, the scheme $(1_3, 0_4)$ represents $(1, 1, 1, 0, 0, 0, 0)$. The censoring schemes used in this simulation technique are given in Table 5.1.

To estimate the MLE of α and λ, we generate EP distribution samples using the Monte Carlo method under the PT2, PT2H, and APT2 censoring methods for each scheme and the specified experimental time T. By applying the `optim` command in the R programme and prefixing the initial value to be the true value, the MLEs of unknown parameters based on generated samples are calculated. The MLE values and corresponding bias and MSE values for ($\alpha = 4, \lambda = 3$) and ($\alpha = 1, \lambda = 0.5$) are presented in Tables 5.2 and 5.3 respectively. In addition, detailed results of interval estimation based on asymptotic confidence intervals and percentile bootstrap confidence intervals at the 95% confidence level for ($\alpha = 4, \lambda = 3$) and ($\alpha = 1, \lambda = 0.5$) are shown in Tables 5.4 and 5.5, respectively, through the CP and ACL values. The sampling process is carried out 1000 times, and the average of the computation results is used as the final value.

To derive the Bayes estimates for squared error and LINEX loss functions, we use the M-H algorithm. For the LINEX loss function, we arbitrarily choose values of the loss parameter d as -1 and 1. For the prior distribution, the hyper parameters are chosen in such a way that the corresponding prior means are reasonably close to the actual parameter values while minimizing the variance. The estimated bias and

Table 5.1: Different censoring schemes used in the simulation study.

Sl. No.	**n**	**m**	**R**
(1)	30	20	$(1_{10}, 0_{10})$
(2)			$(2_5, 0_{15})$
(3)	30	25	$(1_5, 0_{20})$
(4)			$(1, 0, 1, 0, 1, 0, 1, 0, 1, 0_{16})$
(5)	50	30	$(1_{20}, 0_{10})$
(6)			$(2_{10}, 0_{20})$
(7)	50	40	$(2_5, 0_{35})$
(8)			$(1_{10}, 0_{30})$
(9)	50	45	$(1_5, 0_{40})$
(10)			$(1, 0, 1, 0, 1, 0, 1, 0, 1, 0_{36})$

risk for different censoring schemes under the loss functions mentioned above are presented in Tables 5.6 and 5.7. Additionally, Table 5.8 showcases the CL and CP based on the 95% HPD credible interval (HPDC).

The following conclusions can be made based on the Tables 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8.

(1) In each censoring method, there is a general tendency for the MLE and Bayesian estimates to approach the true value, while the corresponding MSEs and risks decrease as the sample size (n) and effective failure numbers (m) increase.

(2) The Bayesian estimates are more likely to be closer to the true values of the parameters, and they tend to have less bias and less risk compared to MLE's.

(3) The bias and MSE of the scale parameter (λ) in each estimation technique are lower in comparison to the shape parameter (α).

(4) In general, the value of ACL is narrower for each censoring method as sample size n and effective failure numbers m increase.

(5) Regardless of schemes, ACL decreases as the effective failure number rises.

(6) The value of CP in PT2, PT2H, and APT2 typically maintains its nominal level in bootstrap and HPDC intervals. However, the value of CP fails to sustain the nominal level in the asymptotic approach, particularly for the scale parameter.

Table 5.2: Average biases and MSEs for MLEs of $\alpha = 4$ and $\lambda = 3$

(n,m)	R	Scheme	α MLE	α Bias	α MSE	λ MLE	λ Bias	λ MSE
(30,20)	1	PT2	4.2850	0.2850	0.6963	3.0291	0.0291	0.0130
		PT2H	4.2918	0.2918	1.2133	3.0124	0.0124	0.0378
		APT2	4.2788	0.2788	0.6826	3.0284	0.0284	0.0134
	2	PT2	4.2778	0.2778	0.6919	3.0268	0.0268	0.0124
		PT2H	4.2850	0.2850	1.2128	3.0105	0.0105	0.0378
		APT2	4.2614	0.2614	0.6703	3.0334	0.0334	0.0129
(30,25)	3	PT2	4.2573	0.2573	0.5988	3.0186	0.0186	0.0098
		PT2H	4.2437	0.2437	1.1712	2.9951	-0.0049	0.0318
		APT2	4.2388	0.2388	0.5935	3.0273	0.0273	0.0105
	4	PT2	4.2586	0.2586	0.5961	3.0192	0.0192	0.0100
		PT2H	4.2467	0.2467	1.1662	2.9964	-0.0036	0.0307
		APT2	4.2654	0.2654	0.5878	3.0221	0.0221	0.0103
(50,30)	5	PT2	4.2038	0.2038	0.3914	3.0215	0.0215	0.0093
		PT2H	4.1824	0.1824	0.6840	3.0069	0.0069	0.0245
		APT2	4.1127	0.1127	0.3546	3.0071	0.0071	0.0136
	6	PT2	4.1887	0.1887	0.3745	3.0182	0.0182	0.0085
		PT2H	4.1625	0.1625	0.6608	3.0021	0.0021	0.0255
		APT2	4.1821	0.1821	0.3794	3.0184	0.0184	0.0083
(50,40)	7	PT2	4.1559	0.1559	0.3288	3.0107	0.0107	0.0055
		PT2H	4.1600	0.1600	0.6668	2.9979	-0.0021	0.0200
		APT2	4.1633	0.1633	0.3292	3.0129	0.0129	0.0062
	8	PT2	4.1545	0.1545	0.3225	3.0110	0.0110	0.0055
		PT2H	4.1595	0.1595	0.6632	2.9980	-0.0020	0.0204
		APT2	4.1525	0.1525	0.3321	3.0121	0.0121	0.0056
(50,45)	9	PT2	4.1321	0.1321	0.2945	3.0093	0.0093	0.0050
		PT2H	4.1125	0.1125	0.5792	2.9942	-0.0058	0.0180
		APT2	4.1386	0.1386	0.3149	3.0122	0.0122	0.0049
	10	PT2	4.1317	0.1317	0.2925	3.0094	0.0094	0.0050
		PT2H	4.1132	0.1132	0.5732	2.9945	-0.0055	0.0179
		APT2	4.1362	0.1362	0.3130	3.0114	0.0114	0.0049

Table 5.3: Average biases and MSEs for MLEs of $\alpha = 1$ and $\lambda = 0.5$

(n,m)	R	Scheme	α			λ		
			MLE	BIAS	MSE	MLE	BIAS	MSE
(30,20)	1	PT2	1.0714	0.0714	0.0435	0.5238	0.0238	0.0067
		PT2H	1.1000	0.1000	0.1128	0.5327	0.0327	0.0265
		APT2	1.0548	0.0548	0.0331	0.5119	0.0119	0.0086
	2	PT2	1.0694	0.0694	0.0432	0.5221	0.0221	0.0063
		PT2H	1.0930	0.0930	0.1167	0.5258	0.0258	0.0276
		APT2	1.0665	0.0665	0.0432	0.5220	0.0220	0.0068
(30,25)	3	PT2	1.0643	0.0643	0.0374	0.5158	0.0158	0.0049
		PT2H	1.0866	0.0866	0.1135	0.5172	0.0172	0.0231
		APT2	1.0667	0.0667	0.0382	0.5161	0.0161	0.0054
	4	PT2	1.0647	0.0647	0.0373	0.5162	0.0162	0.0049
		PT2H	1.0897	0.0897	0.1163	0.5188	0.0188	0.0228
		APT2	1.0566	0.0566	0.0343	0.5183	0.0183	0.0059
(50,30)	5	PT2	1.0510	0.0510	0.0245	0.5175	0.0175	0.0046
		PT2H	1.0585	0.0585	0.0560	0.5201	0.0201	0.0166
		APT2	0.9463	-0.0537	0.0183	0.4574	-0.0426	0.0109
	6	PT2	1.0472	0.0472	0.0234	0.5150	0.0150	0.0042
		PT2H	1.0558	0.0558	0.0607	0.5171	0.0171	0.0195
		APT2	1.0466	0.0466	0.0263	0.5184	0.0184	0.0039
(50,40)	7	PT2	1.0390	0.0390	0.0206	0.5090	0.0090	0.0026
		PT2H	1.0453	0.0453	0.0532	0.5066	0.0066	0.0145
		APT2	1.0358	0.0358	0.0212	0.5111	0.0111	0.0030
	8	PT2	1.0386	0.0386	0.0202	0.5092	0.0092	0.0026
		PT2H	1.0448	0.0448	0.0526	0.5065	0.0065	0.0144
		APT2	1.0441	0.0441	0.0206	0.5118	0.0118	0.0029
(50,45)	9	PT2	1.0330	0.0330	0.0184	0.5079	0.0079	0.0024
		PT2H	1.0431	0.0431	0.0513	0.5087	0.0087	0.0123
		APT2	1.0326	0.0326	0.0181	0.5107	0.0107	0.0023
	10	PT2	1.0329	0.0329	0.0183	0.5079	0.0079	0.0024
		PT2H	1.0423	0.0423	0.0508	0.5082	0.0082	0.0125
		APT2	1.0346	0.0346	0.0178	0.5081	0.0081	0.0022

Table 5.4: CP and ACL of confidence intervals for $\alpha = 4$ and $\lambda = 3$

			BOOT-P				Asymptotic CI			
			α		λ		α		λ	
(n,m)	R	Censoring Scheme	CP	ACL	CP	ACL	CP	ACL	CP	ACL
(30,20)	1	PT2	1	2.8209	1	0.4468	0.954	2.8918	0.916	0.4038
		PT2H	1	4.0596	1	0.7550	0.964	4.0872	0.916	0.7669
		APT2	1	2.8917	1	0.4372	0.956	2.8919	0.92	0.4046
	2	PT2	1	2.8256	1	0.4362	0.954	2.9148	0.917	0.3980
		PT2H	1	4.0960	1	0.7695	0.957	4.1084	0.916	0.7663
		APT2	1	2.9882	1	0.4069	0.948	2.9045	0.911	0.4003
(30,25)	3	PT2	0.99	2.3953	0.99	0.4261	0.959	2.7325	0.92	0.3538
		PT2H	0.98	3.9344	0.98	0.6730	0.955	3.9358	0.93	0.6920
		APT2	1	2.7104	1	0.3719	0.96	2.7242	0.916	0.3563
	4	PT2	1	2.3894	0.99	0.4276	0.959	2.7196	0.917	0.3547
		PT2H	0.99	3.9433	0.99	0.6705	0.959	3.9283	0.926	0.6921
		APT2	1	2.7439	1	0.3699	0.967	2.7241	0.917	0.3541
(50,30)	5	PT2	1	2.0997	1	0.3874	0.946	2.2645	0.925	0.3411
		PT2H	0.98	3.1051	0.98	0.6015	0.961	3.1457	0.931	0.6202
		APT2	1	2.1693	1	0.4294	0.959	2.2227	0.847	0.3484
	6	PT2	1	2.0681	1	0.3706	0.948	2.2463	0.93	0.3290
		PT2H	0.98	3.1119	0.98	0.6177	0.964	3.1547	0.937	0.6350
		APT2	1	2.1986	1	0.3346	0.959	2.2421	0.923	0.3295
(50,40)	7	PT2	1	1.9038	1	0.3149	0.952	2.0909	0.951	0.2822
		PT2H	1	2.9845	1	0.5306	0.945	3.0240	0.939	0.5396
		APT2	1	2.0512	1	0.2856	0.959	2.0969	0.917	0.2817
	8	PT2	1	1.8835	1	0.3160	0.951	2.0685	0.952	0.2828
		PT2H	1	2.9623	1	0.5326	0.949	2.9961	0.943	0.5427
		APT2	1	2.0209	1	0.2818	0.948	2.0698	0.939	0.2832
(50,45)	9	PT2	1	1.6663	1	0.2726	0.951	2.0059	0.94	0.2671
		PT2H	1	2.8454	1	0.4890	0.955	2.9190	0.946	0.5126
		APT2	1	1.9727	1	0.2650	0.937	2.0028	0.939	0.2673
	10	PT2	1	1.6620	1	0.3357	0.951	1.9979	0.94	0.2673
		PT2H	1	2.8398	1	0.4901	0.956	2.9094	0.95	0.5133
		APT2	1	1.9893	1	0.2674	0.932	1.9928	0.929	0.2674

Table 5.5: CP and ACL of confidence intervals for $\alpha = 1$ and $\lambda = 0.5$

			BOOT-P CI				Asymptotic CI			
(n,m)	R	Censoring Scheme	α		λ		α		λ	
			CP	ACL	CP	ACL	CP	ACL	CP	ACL
(30,20)	1	PT2	1	0.6609	1	0.3599	0.954	0.7230	0.922	0.2793
		PT2H	0.99	1.1865	0.99	0.5841	0.954	1.1893	0.923	0.6390
		APT2	1	0.6576	1	0.3473	0.964	0.7193	0.865	0.2769
	2	PT2	1	0.6207	1	0.4033	0.954	0.7287	0.927	0.2749
		PT2H	0.99	1.1825	0.99	0.5885	0.95	1.1972	0.916	0.6478
		APT2	0.99	0.7334	0.99	0.2924	0.947	0.7273	0.93	0.2754
(30,25)	3	PT2	1	0.6207	1	0.4033	0.959	0.6831	0.922	0.2419
		PT2H	1	1.1685	1	0.5570	0.949	1.1526	0.921	0.5845
		APT2	1	0.6828	1	0.2503	0.954	0.6842	0.913	0.2419
	4	PT2	1	0.7074	1	0.2307	0.959	0.6799	0.919	0.2426
		PT2H	0.99	1.1626	0.99	0.5515	0.948	1.1517	0.918	0.5829
		APT2	1	0.6725	1	0.2651	0.958	0.6769	0.9	0.2458
(50,30)	5	PT2	1	0.5892	1	0.2322	0.946	0.5661	0.933	0.2335
		PT2H	1	0.8909	1	0.4908	0.968	0.8936	0.933	0.5118
		APT2	1	0.5472	1	0.2263	0.924	0.5270	0.90	0.2329
	6	PT2	1	0.5760	0.98	0.1852	0.948	0.5616	0.934	0.2247
		PT2H	1	0.8991	1	0.5063	0.961	0.9095	0.921	0.5377
		APT2	1	0.5539	1	0.2295	0.949	0.5615	0.935	0.2267
(50,40)	7	PT2	1	0.4519	1	0.2292	0.952	0.5227	0.95	0.1911
		PT2H	1	0.8522	1	0.4457	0.956	0.8693	0.929	0.4698
		APT2	1	0.5170	1	0.1905	0.951	0.5208	0.926	0.1928
	8	PT2	1	0.5035	0.99	0.2197	0.951	0.5171	0.95	0.1916
		PT2H	1	0.8535	1	0.4470	0.955	0.8629	0.935	0.4715
		APT2	1	0.5151	1	0.1919	0.945	0.5201	0.939	0.1915
(50,45)	9	PT2	1	0.4593	1	0.1986	0.951	0.5014	0.946	0.1806
		PT2H	1	0.8445	1	0.4242	0.955	0.8448	0.942	0.4451
		APT2	1	0.4987	1	0.1799	0.957	0.5001	0.943	0.1816
	10	PT2	1	0.5497	1	0.1660	0.951	0.4994	0.946	0.1807
		PT2H	1	0.8431	1	0.4248	0.957	0.8422	0.937	0.4457
		APT2	1	0.4913	1	0.1829	0.952	0.4991	0.954	0.1803

Table 5.6: Average biases and MSEs for Bayesian estimates (Est) of $\alpha = 4$ and $\lambda = 3$

(n,m)	R	Method	L_{SE}			L_{LL} (d=-1)			L_{LL} (d=1)		
			Est	Bias	Risk	Est	Bias	Risk	Est	Bias	Risk
(30,20)	1	PT2 (α)	3.9883	0.0116	0.3133	4.1519	0.1635	0.1612	3.8405	-0.1478	0.1757
		(λ)	2.9821	0.0178	0.0157	2.9897	0.0075	0.0081	2.9743	-0.0078	0.0076
		PT2H	3.9822	0.0177	0.4637	4.2354	0.2532	0.2358	3.7739	-0.2083	0.2832
			3.0939	-0.0939	0.0425	3.1103	0.0163	0.0202	3.0765	-0.0173	0.0226
		APT2	3.9897	0.0102	0.0891	4.0357	0.046	0.0446	3.9465	-0.0432	0.0466
			3.076	-0.076	0.0168	3.0814	0.0054	0.008	3.0704	-0.0055	0.0088
	2	PT2	3.9923	0.0076	0.2893	4.1393	0.1469	0.1518	3.852	-0.1403	0.157
			2.9613	0.0386	0.017	2.9689	0.0075	0.0091	2.9533	-0.008	0.008
		PT2H	3.9074	0.0925	0.4392	4.129	0.2215	0.2548	3.7018	-0.2056	0.2302
			3.0433	-0.0433	0.0397	3.0615	0.0182	0.02001	3.0235	-0.0197	0.0201
		APT2	3.9997	0.0002	0.0876	4.0435	0.04375	0.0447	3.956	-0.0437	0.0447
			3.0627	-0.0627	0.0138	3.0676	0.0048	0.0066	3.0577	-0.005	0.0072
(30,25)	3	PT2	3.7579	0.242	0.2989	3.8827	0.1247	0.1871	3.6429	-0.115	0.1313
			3.0093	-0.0093	0.0126	3.0155	0.0062	0.0063	3.003	-0.0063	0.0063
		PT2H	3.5394	0.4605	0.5754	3.7358	0.1964	0.4118	3.3727	-0.1666	0.2284
			3.0372	-0.0372	0.0292	3.0511	0.0138	0.0142	3.0232	-0.0139	0.0152
		APT2	3.8935	0.1064	0.1025	3.9395	0.0459	0.057	3.8485	-0.045	0.0477
			3.033	-0.033	0.0127	3.0387	0.0057	0.0063	3.027	-0.0059	0.0064
	4	PT2	3.8345	0.1654	0.3371	3.9927	0.1582	0.2048	3.6849	-0.1496	0.1582
			2.9982	0.0017	0.0107	3.0035	0.0053	0.0055	2.9927	-0.0054	0.0053
		PT2H	3.5382	0.4617	0.5787	3.7326	0.1944	0.419	3.3682	-0.1699	0.2271
			3.0206	-0.0206	0.0359	3.0377	0.017	0.0184	3.0021	-0.0184	0.0178
		APT2	3.8858	0.1141	0.1138	3.9376	0.0517	0.0631	3.8368	-0.049	0.0536
			3.0335	-0.0335	0.0129	3.0392	0.0057	0.0065	3.0273	-0.0061	0.0065
(50,30)	5	PT2	3.6843	0.3156	0.281	3.7774	0.0931	0.1824	3.5957	-0.0885	0.1161
			3.1005	-0.1005	0.0202	3.1056	0.005	0.0095	3.0954	-0.005	0.0108
		PT2H	3.8737	0.1262	0.3153	4.0257	0.152	0.1891	3.7258	-0.1478	0.1523
			3.1001	-0.1001	0.0293	3.1094	0.0092	0.014	3.09	-0.0101	0.0155
		APT2	3.8374	0.1625	0.1076	3.8783	0.0408	0.062	3.7974	-0.04	0.0479
			2.9859	0.014	0.0119	2.9916	0.0057	0.0062	2.9798	-0.006	0.0057
	6	PT2	3.7166	0.2833	0.2906	3.8264	0.1097	0.1855	3.6154	-0.1011	0.124
			3.0169	-0.0169	0.0115	3.0224	0.0054	0.0057	3.0112	-0.0057	0.0057
		PT2H	3.9128	0.0871	0.3089	4.0628	0.15	0.1832	3.7606	-0.1521	0.152
			3.1219	-0.1219	0.0318	3.1301	0.0081	0.015	3.113	-0.0088	0.017
		APT2	3.8692	0.1307	0.0807	3.9018	0.0325	0.0448	3.8382	-0.031	0.0372
			2.9866	0.0133	0.0073	2.9901	0.0035	0.0038	2.9829	-0.0036	0.0035
(50,40)	7	PT2	3.4683	0.5316	0.4514	3.5581	0.0897	0.3113	3.3886	-0.0797	0.1744
			3.0293	-0.0293	0.0099	3.0338	0.0045	0.0049	3.0247	-0.0045	0.005
		PT2H	4.122	-0.122	0.3524	4.3021	0.1801	0.1596	3.963	-0.1589	0.2308
			3.1877	-0.1877	0.0484	3.1942	0.0065	0.0221	3.181	-0.0066	0.0266
		APT2	3.6772	0.3227	0.2787	3.7676	0.0904	0.179	3.5932	-0.0839	0.1154
			3.0494	-0.0494	0.0105	3.0534	0.004	0.0051	3.0453	-0.004	0.0054
	8	PT2	3.4687	0.5312	0.4181	3.5378	0.069	0.288	3.4015	-0.0672	0.1611
			3.0579	-0.0579	0.0111	3.0618	0.0038	0.0053	3.0541	-0.0038	0.0058
		PT2H	4.1509	-0.1509	0.3647	4.334	0.1831	0.1573	3.9935	-0.1573	0.2457
			3.1878	-0.1878	0.0486	3.1943	0.0065	0.0223	3.1809	-0.0068	0.0266
		APT2	3.6716	0.3283	0.2814	3.7619	0.0903	0.1811	3.5882	-0.0834	0.1165
			3.0554	-0.0554	0.0108	3.0593	0.0038	0.0052	3.0515	-0.0038	0.0056
(50,45)	9	PT2	4.1778	-0.1778	0.2209	4.2729	0.095	0.0972	4.084	-0.0938	0.1359
			3.0097	-0.0097	0.0052	3.0123	0.0026	0.0026	3.0071	-0.0025	0.0026
		PT2H	3.4874	0.5125	0.49	3.6093	0.1219	0.3443	3.3811	-0.1063	0.1891
			3.1277	-0.1277	0.0322	3.1356	0.0079	0.0149	3.1196	-0.008	0.0175
		APT2	3.66	0.3399	0.3022	3.7555	0.0955	0.1987	3.569	-0.0909	0.1231
			3.187	-0.187	0.0414	3.1903	0.0032	0.0191	3.1838	-0.0032	0.0225
	10	PT2	4.1511	-0.1511	0.229	4.2549	0.1037	0.1021	4.0502	0.1009	0.1392
			3.0162	-0.0162	0.0061	3.0192	0.0029	0.003	3.0132	-0.0029	0.0031
		PT2H	3.5225	0.4774	0.4476	3.6374	0.1148	0.3131	3.4174	-0.1051	0.1733
			3.1404	-0.1404	0.0355	3.1482	0.0078	0.0164	3.1324	-0.008	0.0193
		APT2	3.6361	0.3638	0.2787	3.7099	0.0737	0.183	3.5637	-0.0724	0.112
			3.1967	-0.1967	0.0482	3.2013	0.0046	0.0221	3.1918	-0.0048	0.0263

Table 5.7: Average biases and MSEs for Bayesian estimates (Est) of $\alpha = 1$ and $\lambda = 0.5$

(n,m)	R	Method	L_{SE}			L_{LL} (d=-1)			L_{LL} (d=1)		
			Est	Bias	Risk	Est	Bias	Risk	Est	Bias	Risk
(30,20)	1	PT2	1.0229	-0.0229	0.0327	1.0393	0.0163	0.0158	1.0071	-0.0158	0.0171
			0.512	-0.012	0.0045	0.5142	0.0022	0.0022	0.5098	-0.0021	0.0023
		PT2H	0.9843	0.0156	0.057	1.0147	0.0303	0.0278	0.9574	-0.0268	0.0304
			0.5476	-0.0476	0.0288	0.561	0.0133	0.0137	0.5344	-0.0132	0.0152
		APT2	1.0366	-0.0366	0.0242	1.048	0.0114	0.0117	1.0251	-0.0114	0.0126
			0.5552	-0.0552	0.0111	0.5592	0.004	0.0053	0.5511	-0.004	0.0058
	2	PT2	1.028	-0.028	0.0362	1.0461	0.018	0.0174	1.0106	-0.0174	0.0191
			0.4912	0.0087	0.0053	0.4938	0.0026	0.0027	0.4885	-0.0026	0.0026
		PT2H	0.9509	0.049	0.0727	0.9872	0.0362	0.0375	0.9169	-0.034	0.0363
			0.5135	-0.0135	0.0356	0.5313	0.0177	0.0175	0.4959	-0.0175	0.0182
		APT2	0.9939	0.006	0.0269	1.0078	0.0139	0.0132	0.9808	-0.013	0.0139
			0.5174	-0.0174	0.0064	0.5205	0.003	0.0031	0.5144	-0.003	0.0032
(30,25)	3	PT2	0.9422	0.0177	0.0257	0.9538	0.0115	0.0133	0.9313	-0.0109	0.0125
			0.5107	-0.0107	0.0057	0.5134	0.0027	0.0028	0.5078	-0.0028	0.0028
		PT2H	0.9786	0.0213	0.05	1.0042	0.0256	0.025	0.9546	-0.024	0.0256
			0.6147	-0.1147	0.0345	0.6254	0.0106	0.0159	0.604	-0.0106	0.0189
		APT2	0.8733	0.1266	0.0299	0.8804	0.007	0.0161	0.8665	-0.0068	0.0139
			0.4958	0.0041	0.0043	0.498	0.0021	0.0022	0.4936	-0.0021	0.0021
	4	PT2	0.9624	0.0175	0.0262	0.9756	0.0131	0.013	0.9506	-0.0117	0.0134
			0.5049	-0.0049	0.004	0.5069	0.002	0.002	0.5029	-0.002	0.002
		PT2H	0.96	0.0399	0.0566	0.9884	0.0283	0.0289	0.9334	-0.0266	0.0283
			0.5861	-0.0861	0.0329	0.5988	0.0126	0.0154	0.5733	-0.0127	0.0177
		APT2	0.8513	0.1486	0.0376	0.8593	0.0079	0.0204	0.8437	-0.0076	0.0174
			0.5016	-0.0016	0.0051	0.5042	0.0025	0.0025	0.4991	-0.0025	0.0025
(50,30)	5	PT2	0.9374	0.0625	0.0245	0.9478	0.0103	0.0129	0.9272	-0.0102	0.0116
			0.5702	-0.0702	0.0096	0.5726	0.0023	0.0046	0.5679	-0.0023	0.005
		PT2H	0.9953	0.0046	0.0323	1.0121	0.0168	0.0157	0.9797	-0.0155	0.0169
			0.6086	-0.1086	0.0284	0.6169	0.0082	0.0132	0.6003	-0.0083	0.0153
		APT2	0.853	0.1469	0.0328	0.8586	0.0056	0.0178	0.8474	-0.0055	0.0151
			0.4595	0.0404	0.0052	0.4613	0.0018	0.0027	0.4576	-0.0018	0.0025
	6	PT2	0.9199	0.08	0.0216	0.9277	0.0077	0.0113	0.9124	-0.0074	0.0103
			0.5321	-0.0321	0.0047	0.534	0.0018	0.0023	0.5303	-0.0018	0.0024
		PT2H	1.0704	-0.0704	0.0442	1.0902	0.0198	0.0206	1.051	-0.0193	0.024
			0.6714	-0.1714	0.0476	0.6803	0.0089	0.0217	0.662	-0.0093	0.0262
		APT2	0.8667	0.1332	0.0287	0.8723	0.0055	0.0154	0.8613	-0.0054	0.0133
			0.4749	0.025	0.0042	0.4767	0.0017	0.0021	0.4731	-0.0017	0.002
(50,40)	7	PT2	0.8621	0.1378	0.0219	0.8687	0.0066	0.0172	0.8557	-0.0063	0.0148
			0.552	-0.052	0.0066	0.554	0.0019	0.0031	0.55	-0.0019	0.0034
		PT2H	0.9975	0.0024	0.0322	1.014	0.0164	0.0159	0.9817	-0.0158	0.0165
			0.5926	-0.0926	0.0221	0.5994	0.0067	0.0103	0.5858	-0.0067	0.0118
		APT2	0.9885	0.0114	0.0156	0.9964	0.0079	0.0078	0.9809	-0.0076	0.0079
			0.5753	-0.0753	0.0088	0.5769	0.0015	0.0042	0.5737	-0.0015	0.0046
	8	PT2	0.865	0.1349	0.0299	0.8709	0.0059	0.0161	0.8592	-0.0057	0.0139
			0.5473	-0.0473	0.0057	0.5491	0.0017	0.0027	0.5456	-0.0017	0.003
		PT2H	1.0417	-0.0417	0.0391	1.0608	0.019	0.0185	1.0234	-0.0182	0.0209
			0.6435	-0.1435	0.0341	0.6503	0.0068	0.0156	0.6369	-0.0066	0.0186
		APT2	0.9894	0.0105	0.0142	0.9966	0.0071	0.0071	0.9824	-0.007	0.0071
			0.5685	-0.0685	0.0078	0.57	0.0015	0.0037	0.5669	-0.0015	0.004
(50,45)	9	PT2	1.0464	-0.0464	0.0185	1.0548	0.0084	0.0086	1.0384	-0.0079	0.0099
			0.5048	-0.0048	0.0019	0.5058	0.0009	0.00096	0.5039	-0.0009	0.00096
		PT2H	0.9033	0.0966	0.0316	0.917	0.0137	0.0198	0.8897	-0.0135	0.017
			0.6341	-0.1341	0.033	0.6415	0.0074	0.0152	0.6265	-0.0076	0.0179
		APT2	0.9464	0.0535	0.0155	0.9527	0.0063	0.0081	0.9401	-0.0062	0.0074
			0.655	-0.155	0.0287	0.6573	0.0023	0.0134	0.6527	-0.0023	0.0154
	10	PT2	1.0443	-0.0443	0.0191	1.0532	0.0089	0.0089	1.036	-0.0082	0.0103
			0.5105	-0.0105	0.0022	0.5115	0.001	0.001	0.5094	-0.001	0.0011
		PT2H	0.8892	0.1107	0.03	0.9033	0.014	0.0217	0.8755	-0.0136	0.0185
			0.6199	-0.1199	0.0302	0.6278	0.0078	0.0139	0.612	-0.0079	0.0164
		APT2	0.955	0.0449	0.0155	0.9618	0.0067	0.0081	0.9483	-0.0067	0.0074
			0.6453	-0.1453	0.0253	0.6474	0.002	0.0118	0.6432	-0.002	0.0135

Table 5.8: Coverage probability and average confidence length for HPDC across censoring schemes.

(n,m)	R	Scheme	α=4, λ=3				α=1, λ=0.5			
			α		λ		α		λ	
			CP	ACL	CP	ACL	CP	ACL	CP	ACL
(30,20)	1	PT2	95.1	2.0107	95.5	0.5215	95.1	0.6627	95.1	0.2661
		PT2H	95.1	2.5967	95.6	0.652	95.1	0.8875	95.1	0.6169
		APT2	95.1	1.1525	95.6	0.3993	95.1	0.5959	95.2	0.3387
	2	PT2	95.1	1.9776	95.8	0.4765	95.1	0.6374	95.3	0.2637
		PT2H	95.1	2.5102	95.1	0.696	95.2	0.9578	95.2	0.6752
		APT2	95.1	1.1643	95.5	0.3892	95.2	0.6628	95.1	0.2756
(30,25)	3	PT2	95.3	1.8594	95.2	0.4511	95.1	0.5907	95.1	0.2717
		PT2H	95.1	2.2921	95.2	0.586	95.2	0.8829	95.1	0.5213
		APT2	95.2	1.1095	95.4	0.3723	95.4	0.4561	95.2	0.2483
	4	PT2	95.1	1.144	95.7	0.3815	95.1	0.5814	95.3	0.1971
		PT2H	95.1	2.3201	95.4	0.6708	95.6	0.8762	95.2	0.5889
		APT2	95.2	1.1894	95.6	0.3568	95.5	0.4626	95.1	0.298
(50,30)	5	PT2	95.1	1.6906	95.4	0.3983	95.1	0.544	95.4	0.2449
		PT2H	95.1	2.0278	95.2	0.4887	95.2	0.6388	95.1	0.502
		APT2	95.1	1.0696	95.1	0.3269	95.6	0.4329	95.3	0.2244
	6	PT2	95.1	1.8639	95.1	0.4074	95.1	0.4269	95.1	0.2228
		PT2H	95.1	2.1437	95.2	0.4512	95.8	0.7129	95.6	0.5024
		APT2	95.3	0.9921	95.3	0.2922	95.2	0.4049	95.1	0.2396
(50,40)	7	PT2	95.1	1.6179	95.1	0.3519	95.1	0.4361	95.1	0.237
		PT2H	95.1	2.2642	95.6	0.4556	95.1	0.6557	95.1	0.4181
		APT2	95.1	1.5596	95.1	0.3567	95.1	0.4884	95.4	0.2132
	8	PT2	95.1	1.4967	95.3	0.3281	95.3	0.4101	95.2	0.2283
		PT2H	95.1	2.1904	95.4	0.4473	95.2	0.6925	95.1	0.4381
		APT2	95.1	1.6012	95.2	0.303	95.1	0.4766	95.4	0.2147
(50,45)	9	PT2	95.1	1.4052	95.1	0.2712	95.2	0.455	95.6	0.1628
		PT2H	95.1	1.8486	95.8	0.4533	95.1	0.6276	95.1	0.4631
		APT2	95.1	1.5994	95.7	0.2895	95.1	0.3879	95.1	0.2443
	10	PT2	95.1	1.4202	95.5	0.2804	95.2	0.5095	96	0.1638
		PT2H	95.2	1.7498	95.1	0.4736	95.1	0.6248	95.1	0.4396
		APT2	95.1	1.4254	95.5	0.3398	95.6	0.3647	95.4	0.2578

Table 5.9: Measurements of burr formation in L-shaped iron sheet jobs (in millimeters)

0.04	0.02	0.06	0.12	0.14	0.08	0.22	0.12	0.08	0.26
0.24	0.04	0.14	0.16	0.08	0.26	0.32	0.28	0.14	0.16
0.24	0.22	0.12	0.18	0.24	0.32	0.16	0.14	0.08	0.16
0.24	0.16	0.32	0.18	0.24	0.22	0.16	0.12	0.24	0.06
0.02	0.18	0.22	0.14	0.06	0.04	0.14	0.26	0.18	0.16

(7) It appears that HPDC is a better method for producing confidence intervals, as it consistently produces intervals of the least length.

(8) The PT2H method often yields estimates with higher bias and greater MSE or risk, and it also tends to generate confidence intervals with longer lengths compared to PT2 and APT2 methods.

5.8 Data Analysis

In this section, we compare three censoring methods - PT2, PT2H, and APT2 - based on their effectiveness in the EP distribution. To achieve this, we use real data from Dasgupta (2011), which consists of 50 observations.The dataset pertains to the measurements of burrs formed during the piercing operation on L-shaped iron sheet jobs, and these measurements are in millimeters. The dataset is provided in the Table 5.9. Through this analysis, we aim to demonstrate the superiority of one method over the others and provide valuable insights for future research in the field.

In order to evaluate the goodness of fit for the model under consideration, we have considered the KS test, AD test, CVM test, AIC, and BIC. Based on the KS test, the dataset shows a D statistic of 0.0885 and a p-value of 0.8279. The MLEs for the parameters α and λ were found to be 1.6115 and 4.1480, respectively. Furthermore, the TTT transform plot, which is shown in Figure 5.3 and appears to have a concave shape, suggests that the failure rate of the dataset is increasing. The failure rate curve of the data set is also provided in Figure 5.4, further validating the EP distribution modelling as an appropriate choice for this type of data.

To check the effectiveness of PT2, PT2H, and APT2, artificial data was constructed using the complete data for the following schemes.

$$\textit{Scheme}\,1:\ n = 50;\ m = 40;\ T = 0.26;\ R = (1_{10}, 0_{30})$$

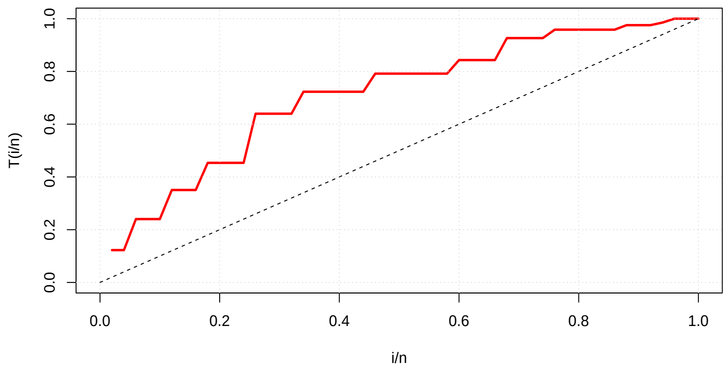


Figure 5.3: The TTT transform plot for the real data

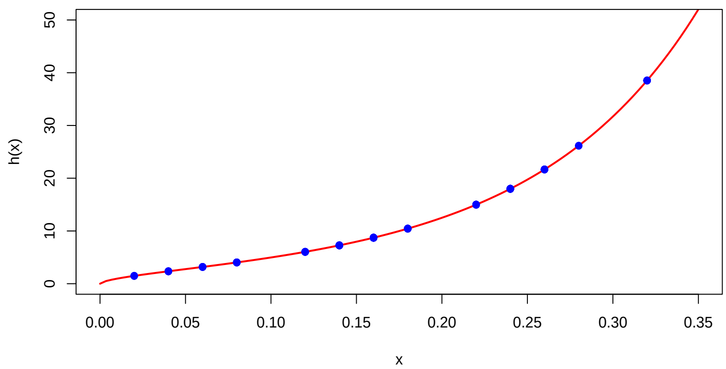


Figure 5.4: The failure rate curve for the real data

$$Scheme\,2:\ n = 50;\ m = 40;\ T = 0.26;\ R = (1_5, 0_{30}, 1_5)$$

$$Scheme\,3:\ n = 50;\ m = 45;\ T = 0.26;\ R = (0_{40}, 1_5)$$

Initially, we conducted a comparison between the EP distribution and both the Weibull (W) distribution and the exponentiated exponential (EE) distribution by Gupta and Kundu (2001) within scheme 1 and the results are presented in Table 5.10. Based on the results, it can be confirmed that the EP distribution outperforms the W and EE distributions due to its larger p-values and smaller statistic values. Additionally, when comparing the AIC and BIC, the EP distribution has the lowest values. Moreover, PT2 and APT2 perform better than PT2H since they have larger p-values and smaller statistic values in most cases. It is worth noting that PT2 and APT2 perform similarly as they construct the same sample under scheme 1.

Table 5.11 presents the results from the remaining two schemes, indicating that APT2 outperforms PT2 and PT2H for the EP distribution in the given dataset. Table 5.12 includes the values of MLE, Bayesian estimates, and CL based on ACI, BOOT-P, and HPDC. The estimates obtained by MLE are less deviated from the estimates of the complete sample when compared to Bayesian estimates. Among the confidence intervals, HPDC outperforms other confidence intervals by consistently producing the shortest length. Figure 5.5 shows the trace plot of the Bayesian estimates for the EP distribution parameters based on real data.

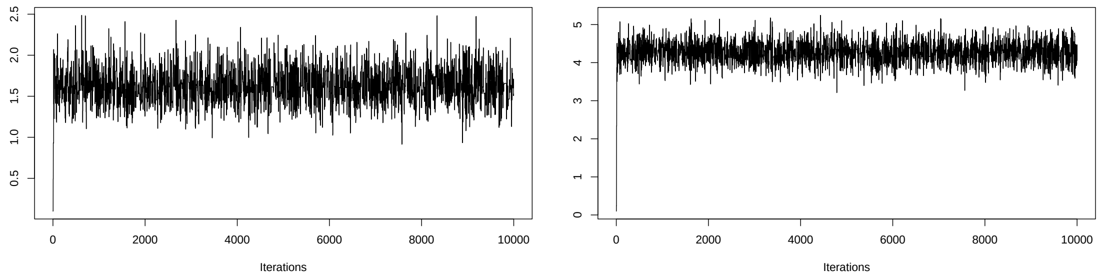


Figure 5.5: Traceplot of Bayesian estimates for the EP distribution parameters α (left) and λ (right) based on real data

5.9 Summary

In this study, we examined parameter estimation for the EP distribution using PT2, PT2H, and APT2 censoring methods. We employed the MLE and Bayesian approaches for point estimation and considered the ACI, BOOT-P, and HPDC methods for interval estimation of unknown parameters. Through Monte Carlo simulation, we assessed the performance of parameter estimates under the three censoring methods. Our findings indicate that, in each censoring method, the estimated values get closer to the true values as the sample size (n) and effective failure numbers (m) increase. Additionally, the ACL value decreases with an increase in m. Notably, PT2 and APT2 tend to outperform PT2H, as the PT2H method often yields estimates with higher bias and greater MSE or risk, and it also generates confidence intervals with longer lengths compared to the PT2 and APT2 methods. PT2 may be considered when prioritizing a fixed experimental time over accurate estimates. Finally, we compared the EP distribution to popular lifetime distributions using real data, revealing that EP outperforms the other models.

Table 5.10: Goodness-of-Fit Test Values for EP, W, and EE Models under Scheme 1.

Censoring Method	Model	KS (p-value)	AD (p-value)	CVM (p-value)	AIC	BIC
PT2	EP	0.1159 0.6555	0.5610 0.6845	0.0785 0.7034	-87.8153	-84.4375
	W	0.1183 0.6300	0.7759 0.4979	0.0950 0.6124	-84.4009	-81.0231
	EE	0.1313 0.4951	1.0060 0.3536	0.1422 0.4158	-78.4180	-75.0403
PT2H	EP	0.1914 0.1538	1.8010 0.1188	0.2927 0.1416	-59.7987	-56.6880
	W	0.1967 0.1332	1.6399 0.1465	0.2346 0.2096	-58.9773	-55.8666
	EE	0.2198 0.0679	1.6519 0.1443	0.2261 0.2225	-56.7902	-53.6795
APT2	EP	0.1159 0.6554	0.5611 0.6845	0.0785 0.7034	-87.8153	-84.4375
	W	0.1183 0.6299	0.7756 0.4981	0.0950 0.6124	-84.4009	-81.0231
	EE	0.1314 0.4950	1.0059 0.3537	0.1422 0.4158	-78.4180	-75.0403

Among the censoring methods, the APT2 method showed better performance for the EP distribution when compared to the other two methods, PT2 and PT2H.

Table 5.11: Goodness-of-Fit Test Values for EP distribution under Scheme 2 and 3.

Censoring Scheme	**Censoring Method**	**KS** (p-value)	**AD** (p-value)	**CVM** (p-value)	**AIC**	**BIC**
Scheme 2	PT2	0.1655 0.2234	1.2968 0.2333	0.2182 0.2352	-68.3387	-64.9609
	PT2H	0.2256 0.0463	2.6756 0.0404	0.4696 0.0470	-56.7297	-53.5079
	APT2	0.1460 0.3610	1.2038 0.2658	0.1762 0.3196	-74.1881	-70.8103
Scheme 3	PT2	0.1619 0.1888	1.2833 0.2377	0.2185 0.2347	-83.5928	-79.9795
	PT2H	0.2157 0.0402	2.5621 0.0463	0.4493 0.0532	-70.8469	-67.3716
	APT2	0.1501 0.2625	1.1473 0.2882	0.1767 0.3183	-89.0750	-85.4616

Table 5.12: MLEs, Bayesian estimates, and confidence lengths under ACI, BOOT-P, and HPDC for real data

Censoring Scheme	Censoring Method		MLE	Bayesian Estimates	CL		
					ACI	BOOT-P	HPDC
Scheme 1	PT2	(α)	1.6570	1.6009	0.8439	0.8435	0.8352
		(λ)	4.2744	4.2849	0.9886	0.9994	0.9625
	PT2H		1.5409	1.5074	0.8805	0.9321	0.7048
			4.1010	4.0483	1.1877	1.1251	1.1364
	APT2		1.6571	1.6264	0.8439	0.8562	0.7331
			4.2745	4.2178	0.9886	0.9954	0.9921
Scheme 2	PT2		1.4529	1.3789	0.7818	0.8883	0.7322
			3.9032	4.4554	1.0849	1.0694	1.2038
	PT2H		1.4583	1.4171	0.8489	0.9250	0.8326
			3.9115	3.8609	1.2005	1.0877	1.2043
	APT2		1.5914	1.5586	0.8862	0.8875	0.7084
			4.1278	4.0987	1.0565	1.0231	1.0218
Scheme 3	PT2		1.5188	1.3993	0.7770	0.9170	0.7266
			4.0283	4.5592	0.9992	1.0037	1.0844
	PT2H		1.5258	1.5002	0.8464	0.9446	0.8349
			4.0266	4.0101	1.0873	1.0608	1.2004
	APT2		1.6548	1.6286	0.8042	0.8123	0.8795
			4.2093	4.1625	1.0458	1.0232	0.9667

CHAPTER 6

Generalized Lindley Distribution Under Joint Adaptive Progressive Type-II Censoring

6.1 Introduction

In comparative research, evaluating and comparing the performance of different populations or production lines is a common objective. To facilitate this, researchers often use joint censoring schemes, which allow for simultaneous analysis across multiple groups. Recently, there has been growing interest in utilizing these schemes to examine the longevity of items produced by different manufacturers, particularly through jointly censored testing methods. Bhattacharyya & Mehrotra (1981) were among the first to discuss the challenges of comparing products from distinct production lines using such schemes. Later, Bhattacharyya (1995) presented a thorough review of advancements in this area, addressing both parametric and nonparametric methods within the framework of joint Type-II censoring.

Rasouli & Balakrishnan (2010) introduced a joint progressive Type-II censoring scheme and applied it to two exponential samples. This experiment involves selecting two samples, of sizes n_1 and n_2, from two distinct populations, with a total sample size $N = n_1 + n_2$. The experiment proceeds by fixing the value m and defining R as

$(R_1, R_2, ..., R_m)$. The experiment concludes once m failure times are observed from the combined samples. Upon the first failure, whether from Sample 1 or Sample 2, s_1 units are removed from Sample 1 and q_1 units from Sample 2, where $R_1 = s_1 + q_1$. Similarly, at the second failure, s_2 units are removed from Sample 1 and q_2 units from Sample 2, with $R_2 = s_2 + q_2$. This process continues until the mth failure, at which point all remaining units are removed from the experiment. The sequence $R = (R_1, R_2, ..., R_m)$ can be decomposed into $S + Q = (s_1, s_2, ..., s_m) + (q_1, q_2, ..., q_m)$. Notably, if $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = N - m$, this censoring scheme corresponds to the joint Type-II censoring scheme proposed by Balakrishnan & Rasouli (2008).

Sultana et al. (2021) introduced a new joint adaptive Type-II progressive censoring (JAPT) scheme for independent samples from two populations, assuming exponential lifetimes. They derived the MLEs and their exact distributions, constructed approximate confidence intervals, and considered Bayesian inference under a Beta-Gamma prior. Additionally, they utilized the variable neighborhood search method to determine the optimal censoring scheme within a Bayesian framework, demonstrating the effectiveness of their approach through simulations and real data application. Based on this, Sultana et al. (2023) extended the JAPT scheme to the three-parameter Weibull distribution, estimating its parameters using both ML estimation and Bayesian methods. They also established connections between the balanced JAPT scheme proposed by Sultana et al. (2021) and the BJPT censoring scheme introduced by Mondal & Kundu (2019).

This chapter focuses on developing inferential methods for the generalized Lindley (GL) distribution, introduced by Nadarajah et al. (2011), under the framework of the JAPT censoring scheme. The GL distribution, an extension of the Lindley distribution proposed by Lindley (1958), introduces a shape parameter (α) that enhances its flexibility for modelling different data types. As a mixture of exponential and gamma distributions, it effectively handles both random and age-dependent failures, making it suitable for applications in survival analysis, actuarial science, queuing theory, reliability engineering, and environmental studies. Its growing popularity, as noted by Ghitany et al. (2008), stems from its ability to model lifetime data with complex patterns.

The GL distribution is particularly valued for its capacity to represent different hazard rate behaviors, including increasing, decreasing, and bathtub-shaped forms. It serves as a versatile alternative to distributions like gamma, lognormal, Weibull, and exponentiated exponential, especially for modelling bathtub-shaped data. In a parallel system where component failure times follow a Lindley distribution, the probability of system failure before a given time is captured by the GL distribution. Retaining the simplicity of the Lindley distribution while offering greater flexibility, the GL distribution is highly effective for capturing complex data structures.

The PDF of the GL distribution is defined as:

$$f(x) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right]^{\alpha-1}\exp(-\lambda x); x > 0,\ \alpha, \lambda > 0,$$

where λ is the scale parameter and α is the shape parameter. The CDF is expressed as:

$$F(x; \alpha, \lambda) = \left(1 - \frac{1+\lambda+\lambda x}{(1+\lambda)}e^{-\lambda x}\right)^{\alpha};\ x > 0,\ \alpha, \lambda > 0.$$

In this context, Singh et al. (2013) addressed estimation and prediction problems for the GL distribution under progressive Type-II censoring with Beta-Binomial removals. In a subsequent study, Kumar Singh et al. (2016) explored estimation and prediction for the GL distribution using Type-I hybrid censoring scheme, deriving one and two-sample predictive posteriors for future order statistics. The GL distribution has also been extensively studied by other researchers, including Merovci and Elgarhy (2013), Merovci and Sharma (2014), Bhati et al. (2015), and Korkmaz et al. (2018).

The objective of this chapter is to develop statistical inference for the GL distribution using the JAPT censoring scheme. Section 6.2 provides a detailed description of the JAPT censoring scheme. In Section 6.3, we derive the MLEs for the model parameters and construct their asymptotic confidence intervals using the Fisher information matrix. Section 6.4 presents the Bayesian estimates and HPD credible intervals obtained through the importance sampling technique. The bootstrap confidence intervals are discussed in Section 6.5. Section 6.6 includes a simulation study to assess the performance of the estimators, and Section 6.7 demonstrates the application of the proposed methods using a real dataset. Finally, the chapter is summarized in Section 6.8.

6.2 Joint Adaptive Type-II Progressive Censoring Scheme

Consider a lifetime experiment involving two independent production lines, where samples are taken from each line with sample sizes n_1 and n_2, respectively. Suppose the sample from the first production line follows a distribution with CDF $F(\cdot;\alpha_1,\lambda_1)$ and PDF $f(\cdot;\alpha_1,\lambda_1)$, while the sample from the second production line follows a distribution with CDF $G(\cdot;\alpha_2,\lambda_2)$ and PDF $g(\cdot;\alpha_2,\lambda_2)$. The JAPT censoring scheme is outlined as follows.

Let m be a fixed integer representing the number of failures to be observed and T denote a specified time point, serving as a predetermined cut-off for the duration of the experiment. We define $(R_1, R_2, ..., R_m)$ as a sequence of non-negative integers that are selected in advance, ensuring that the sum of these integers equals $N-m$, where $N = n_1 + n_2$. Additionally, we define two vectors, $S = (s_1, s_2, ..., s_m)$ and $Q = (q_1, q_2, ..., q_m)$, where s_i denotes the number of units removed from the first sample at the ith stage, and q_i denotes the number of units removed from the second sample at the ith stage, for $i = 1, 2, ..., m$. Therefore, $R_i = s_i + q_i$. In this life testing experiment using two independent samples concurrently, $Z_{1:N}$ denotes the failure time of the first unit from the entire sample. Upon its failure, we remove the failed item and randomly remove R_1 observations from the sample. At this stage, the removals from the first sample are represented by s_1, and the removals from the second sample are represented by q_1, ensuring that $R_1 = s_1 + q_1$. Subsequently, at the time $Z_{2:N}$ of the second failure, we remove the failed item and randomly remove $R_2 = s_2 + q_2$ observations from the remaining $(N - R_1 - 1)$ items, where s_2 and q_2 represent the number of items removed from the first and second samples, respectively. This process continues until the mth failure occurs or until the time point T is reached. If the mth failure occurs at $Z_{m:N}(< T)$, the experiment terminates at $Z_{m:N}$. However, if only J failures $(J < m)$ occur before T, no further observations are randomly removed after $Z_{J+1:N}$ until the mth failure at $Z_{m:N}$, at which point the experiment terminates by removing all remaining items from the sample. In this case, $Z_{J:N} < T < Z_{J+1:N}$ for $J = 1, 2, \ldots, m$, where $Z_{0:N} \equiv 0$ and $Z_{m+1:N} \equiv \infty$. Consequently, $R_{J+1} = R_{J+2} = \ldots = R_{m-1} = 0$ and $R_m = N - m - \sum_{i=1}^{J} R_i$. Therefore, the progressive censoring scheme is given by $(R_1, R_2, \ldots, R_{J^*}, 0, 0, \ldots, N - m - \sum_{i=1}^{J} R_i)$, where $J^* = \max\{J : Z_{J:N} < T\}$. The schematic representation of the JAPT experiment terminating before time T and after time T is shown in Figures 6.1 and 6.2, respectively.

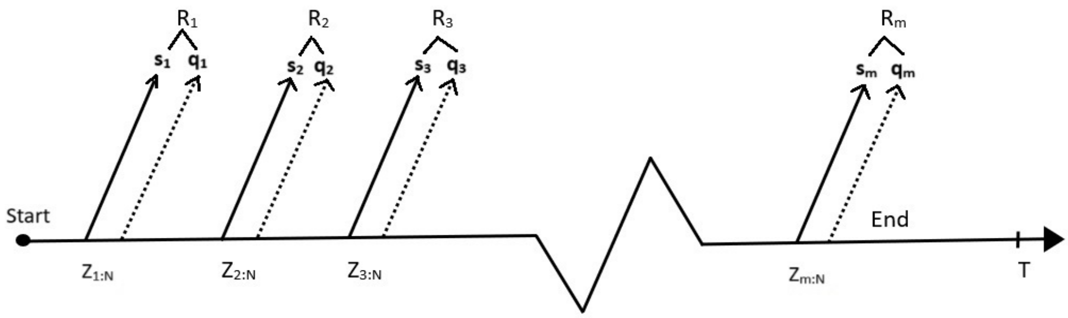


Figure 6.1: Illustration of the JAPT experiment terminating before time T, with the solid line representing the first sample and the dotted line representing the second sample.

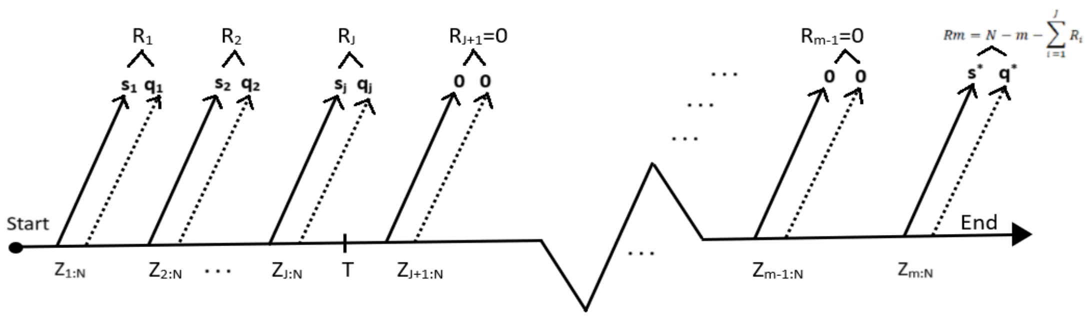


Figure 6.2: Illustration of the JAPT experiment terminating after time T, with the solid line representing the first sample and the dotted line representing the second sample.

Let D_i $(i = 1, 2, ..., m)$ be either 1 or 0, indicating whether Z_i is from sample 1 or sample 2, respectively. Thus, for a given experiment time T and a dataset of size m, the ordered lifetimes $\{(Z_1, D_1), (Z_2, D_2), ..., (Z_J, D_J), (Z_{J+1}, D_{J+1}), ..., (Z_m, D_m)\}$ obtained from a joint sample of size n are referred to as a JAPT censored sample.

Then the likelihood function of the JAPT censored sample can be written as:

$$L(Z, D) = A_J \left[\prod_{i=1}^{m} [f(z_i)]^{D_i} [g(z_i)]^{1-D_i} \right] \left[\prod_{i=1}^{J} [\bar{F}(z_i)]^{s_i} [\bar{G}(z_i)]^{q_i} \right] \left[\bar{F}(z_m) \right]^{s^*} \left[\bar{G}(z_m) \right]^{q^*}, \tag{6.2.1}$$

where,

$$A_J = \prod_{i=1}^{m} \left[N - i + 1 - \sum_{k=1}^{\min\{i-1, J\}} R_k \right],$$

$$s^* = \begin{cases} \sum_{i=J+1}^{m} s_i & ;\text{for } J < m, \\ 0 & ;\text{otherwise} \end{cases}, \qquad q^* = \begin{cases} \sum_{i=J+1}^{m} q_i & ;\text{for } J < m, \\ 0 & ;\text{otherwise.} \end{cases}$$

For $J = m$, this method corresponds to the joint progressive Type-II censoring scheme defined by Rasouli and Balakrishnan (2010).

Suppose the sample from the first production line is $(X_1, X_2, \ldots, X_{n_1})$, which follows the GL distribution, $GL(x; \alpha_1, \lambda_1)$. Similarly, let the sample from the second production line be $(Y_1, Y_2, \ldots, Y_{n_2})$, follows $GL(y; \alpha_2, \lambda_2)$. Both samples are independently and identically distributed (i.i.d.). Then, the PDF and CDF of both samples are respectively given by:

$$f(x) = \frac{\alpha_1 \lambda_1^2}{1+\lambda_1}(1+x)\left[1 - \frac{1+\lambda_1+\lambda_1 x}{1+\lambda_1}\exp(-\lambda_1 x)\right]^{\alpha_1 - 1} \exp(-\lambda_1 x); x > 0,\ \alpha_1, \lambda_1 > 0 \tag{6.2.2}$$

$$F(x) = \left[1 - \frac{1+\lambda_1+\lambda_1 x}{1+\lambda_1}\exp(-\lambda_1 x)\right]^{\alpha_1} ; x > 0,\ \alpha_1, \lambda_1 > 0 \tag{6.2.3}$$

and

$$g(y) = \frac{\alpha_2 \lambda_2^2}{1+\lambda_2}(1+y)\left[1 - \frac{1+\lambda_2+\lambda_2 y}{1+\lambda_2}\exp(-\lambda_2 y)\right]^{\alpha_2 - 1} \exp(-\lambda_2 y); y > 0,\ \alpha_2, \lambda_2 > 0 \tag{6.2.4}$$

$$G(y) = \left[1 - \frac{1+\lambda_2+\lambda_2 y}{1+\lambda_2}\exp(-\lambda_2 y)\right]^{\alpha_2} ; y > 0,\ \alpha_2, \lambda_2 > 0. \tag{6.2.5}$$

6.3 Maximum Likelihood Estimation

This section focuses on the MLE of the unknown parameters of the JAPT model. Building on the discussion from the previous section, the joint likelihood function of the parameters can be rewritten as follows:

$$\begin{aligned} L(\Theta|Z, D) = A_J \prod_{i=1}^{m} & \left(\frac{\alpha_1 \lambda_1^2}{1+\lambda_1}(1+z_i) V^{\alpha_1 - 1}(z_i, \lambda_1) e^{-\lambda_1 z_i}\right)^{D_i} \times \\ & \left(\frac{\alpha_2 \lambda_2^2}{1+\lambda_2}(1+z_i) V^{\alpha_2 - 1}(z_i, \lambda_2) e^{-\lambda_2 z_i}\right)^{1-D_i} \\ & \prod_{i=1}^{J} \left(1 - V^{\alpha_1}(z_i, \lambda_1)\right)^{s_i} \left(1 - V^{\alpha_2}(z_i, \lambda_2)\right)^{q_i} \times \\ & \left[1 - V^{\alpha_1}(z_m, \lambda_1)\right]^{s^*} \left[1 - V^{\alpha_2}(z_m, \lambda_2)\right]^{q^*}, \end{aligned} \tag{6.3.1}$$

where $\Theta = (\alpha_1, \lambda_1, \alpha_2, \lambda_2)$ represents the parameter vector and

$$V(z_i, \lambda) = 1 - \frac{1 + \lambda + \lambda z_i}{1 + \lambda} e^{-\lambda z_i}. \tag{6.3.2}$$

The associated log-likelihood of (6.3.1) is given by,

$$\begin{aligned}
\log \mathrm{L}(\Theta|Z, D) &= w_1 \log \alpha_1 + w_2 \log \alpha_2 + 2w_1 \log \lambda_1 - w_1 \log(1 + \lambda_1) + 2w_2 \log \lambda_2 \\
&- w_2 \log(1 + \lambda_2) + \sum_{i=1}^{m} \log(1 + z_i) + \sum_{i=1}^{m} D_i(\alpha_1 - 1) \log V(z_i, \lambda_1) - \lambda_1 \sum_{i=1}^{m} D_i z_i \\
&+ \sum_{i=1}^{m} (1 - D_i)(\alpha_2 - 1) \log V(z_i, \lambda_2) - \lambda_2 \sum_{i=1}^{m} (1 - D_i) z_i + \sum_{i=1}^{J} s_i \log\left(1 - V^{\alpha_1}(z_i, \lambda_1)\right) \\
&+ \sum_{i=1}^{J} q_i \log\left(1 - V^{\alpha_2}(z_i, \lambda_2)\right) + s^* \log\left[1 - V^{\alpha_1}(z_m, \lambda_1)\right] + q^* \log\left[1 - V^{\alpha_2}(z_m, \lambda_2)\right],
\end{aligned} \tag{6.3.3}$$

where $w_1 = \sum_{i=1}^{m} D_i$ and $w_2 = m - \sum_{i=1}^{m} D_i$.

By computing the first partial derivatives of the log-likelihood function (6.3.3) with respect to each parameter, we obtain the following equations:

$$\frac{\partial \log L}{\partial \alpha_1} = \frac{w_1}{\alpha_1} + \sum_{i=1}^{m} D_i \log V(z_i, \lambda_1) - \sum_{i=1}^{J} \frac{s_i V^{\alpha_1}(z_i, \lambda_1) \log V(z_i, \lambda_1)}{1 - V^{\alpha_1}(z_i, \lambda_1)} - \frac{s^* V^{\alpha_1}(z_m, \lambda_1) \log V(z_m, \lambda_1)}{1 - V^{\alpha_1}(z_m, \lambda_1)} \tag{6.3.4}$$

$$\frac{\partial \log L}{\partial \alpha_2} = \frac{w_2}{\alpha_2} + \sum_{i=1}^{m} (1 - D_i) \log V(z_i, \lambda_2) - \sum_{i=1}^{J} \frac{q_i V^{\alpha_2}(z_i, \lambda_2) \log V(z_i, \lambda_2)}{1 - V^{\alpha_2}(z_i, \lambda_2)} - \frac{q^* V^{\alpha_2}(z_m, \lambda_2) \log V(z_m, \lambda_2)}{1 - V^{\alpha_2}(z_m, \lambda_2)} \tag{6.3.5}$$

$$\begin{aligned}
\frac{\partial \log L}{\partial \lambda_1} &= \frac{w_1(\lambda_1 + 2)}{\lambda_1(\lambda_1 + 1)} + (\alpha_1 - 1) \sum_{i=1}^{m} D_i \frac{V'(z_i, \lambda_1)}{V(z_i, \lambda_1)} - \sum_{i=1}^{m} D_i z_i - \alpha_1 \sum_{i=1}^{J} \frac{s_i V'(z_i, \lambda_1) V^{\alpha_1 - 1}(z_i, \lambda_1)}{1 - V^{\alpha_1}(z_i, \lambda_1)} \\
&- \frac{s^* \alpha_1 V^{\alpha_1 - 1}(z_m, \lambda_1) V'(z_m, \lambda_1)}{1 - V^{\alpha_1}(z_m, \lambda_1)}
\end{aligned} \tag{6.3.6}$$

$$\begin{aligned}
\frac{\partial \log L}{\partial \lambda_2} &= \frac{w_2(\lambda_2 + 2)}{\lambda_2(\lambda_2 + 1)} + (\alpha_2 - 1) \sum_{i=1}^{m} (1 - D_i) \frac{V'(z_i, \lambda_2)}{V(z_i, \lambda_2)} - \sum_{i=1}^{m} (1 - D_i) z_i - \alpha_2 \sum_{i=1}^{J} \frac{q_i V'(z_i, \lambda_2) V^{\alpha_2 - 1}(z_i, \lambda_2)}{1 - V^{\alpha_2}(z_i, \lambda_2)} \\
&- \frac{q^* \alpha_2 V^{\alpha_2 - 1}(z_m, \lambda_2) V'(z_m, \lambda_2)}{1 - V^{\alpha_2}(z_m, \lambda_2)}
\end{aligned} \tag{6.3.7}$$

As indicated by equations (6.3.4-6.3.7), an analytical solution for the MLEs of the parameters are not feasible. Consequently, the `nlm` function in R can be employed to numerically implement an iterative Newton-Raphson method to obtain the desired MLE of the parameters for any given data set.

6.3.1 Asymptotic Confidence Interval

ACIs are crucial in statistical inference, particularly with large sample sizes. These intervals rely on the asymptotic properties of estimators, which typically approximate a normal distribution as the sample size increases. A fundamental element in constructing these intervals is the Fisher information matrix, which measures the amount of information that an observable random variable provides about an unknown parameter. The Fisher information matrix is defined as the negative expectation of the second partial derivative of the log-likelihood function. In some cases, calculating this expectation can be challenging. Consequently, a practical alternative is to use the observed information matrix in place of the Fisher information matrix. The observed information matrix $\mathcal{I}(\Theta)$ for the parameter vector $\Theta = (\alpha_1, \lambda_1, \alpha_2, \lambda_2)$ is given by:

$$\mathcal{I}(\Theta) = -\begin{bmatrix} \frac{\partial^2 l}{\partial \alpha_1^2} & \frac{\partial^2 l}{\partial \alpha_1 \partial \lambda_1} & \frac{\partial^2 l}{\partial \alpha_1 \partial \alpha_2} & \frac{\partial^2 l}{\partial \alpha_1 \partial \lambda_2} \\ \frac{\partial^2 l}{\partial \lambda_1 \partial \alpha_1} & \frac{\partial^2 l}{\partial \lambda_1^2} & \frac{\partial^2 l}{\partial \lambda_1 \partial \alpha_2} & \frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_2} \\ \frac{\partial^2 l}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 l}{\partial \alpha_2 \partial \lambda_1} & \frac{\partial^2 l}{\partial \alpha_2^2} & \frac{\partial^2 l}{\partial \alpha_2 \partial \lambda_2} \\ \frac{\partial^2 l}{\partial \lambda_2 \partial \alpha_1} & \frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 l}{\partial \lambda_2 \partial \alpha_2} & \frac{\partial^2 l}{\partial \lambda_2^2} \end{bmatrix}$$

The MLE of the parameters follows a bivariate normal distribution with mean $\hat{\Theta} = (\hat{\alpha_1}, \hat{\lambda_1}, \hat{\alpha_2}, \hat{\lambda_2})$ and variance-covariance matrix $\mathcal{I}^{-1}(\hat{\Theta}) = [I_{ij}]$, $i = 1, ..., 4$, and $j = 1, ..., 4$. The $100(1-\delta)\%$ approximate confidence interval for a given confidence level δ are,

$$\begin{cases} \hat{\alpha_1} \pm Z_{\frac{\delta}{2}} \sqrt{I_{11}} \quad \text{and } \hat{\lambda_1} \pm Z_{\frac{\delta}{2}} \sqrt{I_{22}} \\ \hat{\alpha_2} \pm Z_{\frac{\delta}{2}} \sqrt{I_{33}} \quad \text{and } \hat{\lambda_2} \pm Z_{\frac{\delta}{2}} \sqrt{I_{44}}, \end{cases} \tag{6.3.8}$$

where I_{ii} represents the diagonal elements of $\mathcal{I}^{-1}(\hat{\Theta})$.

6.4 Bayesian Inference

This section focuses on deriving Bayesian estimates for the unknown model parameters and establishing credible intervals using the JAPT censored data. Bayesian estimation serves as a strong alternative to classical methods by considering unknown parameters as random variables and combining prior information with observed data. In this framework, prior knowledge about the unknown parameter is essential before making inferences using the likelihood function. This prior information can vary in completeness depending on the chosen prior distribution. A non-informative prior reflects minimal or no prior knowledge about the parameter, whereas an informative prior offers substantial information to help quantify the uncertainty associated with the parameter. The choice of prior is crucial, as it should not heavily influence model estimates or model selection. It is important to note that there is no universally accepted method for selecting an appropriate prior in Bayesian analysis.

Our approach involves assuming independent gamma priors for all the unknown parameters α_1, λ_1, α_2, and λ_2. The gamma distribution is a popular choice as a prior due to its maximum entropy properties, its ability to constrain random variables to positive values, and its flexibility in modelling various shapes of the PDF. It includes special cases such as the exponential and chi-square distributions and has a log-concave density function over the interval $(0, \infty)$. An example of a gamma prior is Jeffreys' prior, which is widely used in Bayesian analysis. In lifetime models, the gamma distribution is often used as an informative prior because of its simplicity and computational convenience. Given its applicability, the gamma prior has been frequently employed in Bayesian estimation for the GL distribution, as seen in studies by Singh et al. (2014) and Kumar Singh et al. (2016), among others. Consequently, the assumed gamma priors for the unknown parameters are defined as follows:

$$\pi^*(\Theta_i) \propto \Theta^{a_i-1}e^{-b_i\Theta_i}; \Theta > 0; a_i, b_i > 0; i = 1, .., 4. \tag{6.4.1}$$

Thus the joint prior distribution is given by

$$\pi^*(\alpha_1, \lambda_1, \alpha_2, \lambda_2) \propto \alpha_1^{a_1-1}\lambda_1^{a_2-1}\alpha_2^{a_3-1}\lambda_2^{a_4-1}e^{-b_1\alpha_1-b_2\lambda_1-b_3\alpha_2-b_4\lambda_2}. \tag{6.4.2}$$

The posterior density function can be derived from Eq. (6.3.1) and (6.4.2) is expressed as:

$$\pi(\alpha_1, \lambda_1, \alpha_2, \lambda_2|Z) = \frac{\alpha_1^{w_1+a_1-1}\alpha_2^{w_2+a_3-1}\lambda_1^{2w_1+a_2-1}\lambda_2^{2w_2+a_4-1}}{(1+\lambda_1)^{w_1}(1+\lambda_2)^{w_2}} \exp\{-b_1\alpha_1 - b_2\lambda_1 -$$
$$b_3\alpha_2 - b_4\lambda_2 + (\alpha_1 - 1)\sum_{i=1}^{m} D_i \log V(z_i, \lambda_1) + (\alpha_2 - 1)\sum_{i=1}^{m}(1 - D_i)\log V(z_i, \lambda_2) +$$
$$\sum_{i=1}^{J} s_i \log[1 - V^{\alpha_1}(z_i, \lambda_1)] + \sum_{i=1}^{J} q_i \log[1 - V^{\alpha_2}(z_i, \lambda_2)] + s^* \log[1 - V^{\alpha_1}(z_m, \lambda_1)] +$$

$$q^* \log\left[1 - V^{\alpha_2}(z_m, \lambda_2)\right] + \sum_{i=1}^{m} \log(1+z_i) - \lambda_1 \sum_{i=1}^{m} D_i z_i - \lambda_2 \sum_{i=1}^{m} (1-D_i) z_i \Bigg\}. \tag{6.4.3}$$

The joint posterior distribution (6.4.3) demonstrates that for high-dimensional cases, both the posterior distribution and the associated Bayes estimators cannot be simplified into a closed form. Thus, the MCMC method is employed to obtain the Bayes estimators of the model parameters. In the next subsection, we provide a detailed discussion of the MCMC method.

6.4.1 Markov Chain Monte Carlo Method

Given the complexity of the joint posterior distribution presented in Eq. (6.4.3), it becomes evident that analytical solutions for the posterior distributions and corresponding Bayes estimators are infeasible, especially in high-dimensional settings. As a result, numerical methods must be employed to approximate these quantities. Among the various available methods, the MCMC approach stands out as a powerful tool for sampling from complex posterior distributions. This method enables the approximation of the posterior distributions and subsequently allows for the accurate estimation of the Bayes estimators. Here, we provide a comprehensive overview of the MCMC technique, outlining its application within the context of our study.

The posterior distribution (6.4.3) simplifies to four conditional gamma distributions, as described in Eq. (6.4.4) through (6.4.7), along with the associated weight function (6.4.8) of the model parameters. Thus, the importance sampling technique, a subset of MCMC methods, can be used to estimate the Bayesian estimates using the Algorithm 1. The conditional posterior distributions derived from Eq. (6.4.3) are given as follows.

$$\pi_1(\alpha_1|\lambda_1, Z) \propto \alpha_1^{w_1+a_1-1} \exp\left\{-\alpha_1 \left(b_1 - \sum_{i=1}^{m} D_i \log V(z_i, \lambda_1)\right)\right\} \tag{6.4.4}$$

$$\pi_2(\lambda_1|\alpha_1, Z) \propto \lambda_1^{2w_1+a_2-1} \exp\left\{-\lambda_1 (b_2 + \sum_{i=1}^{m} D_i z_i)\right\} \tag{6.4.5}$$

$$\pi_3(\alpha_2|\lambda_2, Z) \propto \alpha_2^{w_2+a_3-1} \exp\left\{-\alpha_2 \left(b_3 - \sum_{i=1}^{m} (1-D_i) \log V(z_i, \lambda_2)\right)\right\} \tag{6.4.6}$$

$$\text{and } \pi_4(\lambda_2|\alpha_2, Z) \propto \lambda_2^{2w_2+a_4-1} \exp\left\{-\lambda_2 \left(b_4 + \sum_{i=1}^{m} (1-D_i) z_i\right)\right\}. \tag{6.4.7}$$

The associated weight function is,

$$\Pi(\alpha_1,\lambda_1,\alpha_2,\lambda_2|Z) = \exp\Bigg\{-\sum_{i=1}^{m} D_i \log V(z_i,\lambda_1) - \sum_{i=1}^{m}(1-D_i)\log V(z_i,\lambda_2) + \sum_{i=1}^{J} s_i \log\Big(1 - V^{\alpha_1}(z_i,\lambda_1)\Big) + \sum_{i=1}^{J} q_i \log\Big(1-(V^{\alpha_2}(z_i,\lambda_2))\Big) + s^*\log\Big(1-V^{\alpha_1}(z_m,\lambda_1)\Big) + q^*\log\Big(1-V^{\alpha_2}(z_m,\lambda_2)\Big)\Bigg\} \div (1+\lambda_1)^{w_1}(1+\lambda_2)^{w_2}\left(b_1 - \sum_{i=1}^{m} D_i \log V(z_i,\lambda_1)\right)^{w_1+a_1} \left(b_3 - \sum_{i=1}^{m}(1-D_i)\log V(z_i,\lambda_2)\right)^{w_2+a_3}. \tag{6.4.8}$$

Algorithm 1 Importance Sampling Technique

1: Start with the initial value $\Theta^0 = (\hat{\alpha_1}, \hat{\lambda_1}, \hat{\alpha_2}, \hat{\lambda_2})$, and set $j = 1$.
2: Generate $\alpha_k^{(j)}, k = 1, 2$, from gamma distributions (6.4.4) and (6.4.6).
3: Generate $\lambda_k^{(j)}, k = 1, 2$, from gamma distributions (6.4.5) and (6.4.7).
4: Calculate the value of associated weight $\Pi(\alpha_1, \lambda_1, \alpha_2, \lambda_2|Z)$, set $j = j + 1$.
5: Repeat the steps 2-4 MC times.
6: Under squared error loss function the Bayes estimate of any function $\Psi(\Theta)$ can be expressed as:

$$\hat{\Psi}(\Theta) = E(\Psi(\Theta)|T) = \frac{\frac{1}{MC-MC^*}\sum_{i=MC^*+1}^{MC}\Psi(\alpha_1^{(i)},\lambda_1^{(i)},\alpha_2^{(i)},\lambda_2^{(i)})\Pi(\alpha_1^{(i)},\lambda_1^{(i)},\alpha_2^{(i)},\lambda_2^{(i)}|Z)}{\frac{1}{MC-MC^*}\sum_{i=MC^*+1}^{MC}\Pi(\alpha_1^{(i)},\lambda_1^{(i)},\alpha_2^{(i)},\lambda_2^{(i)}|Z)} \tag{6.4.9}$$

where the integer MC^* denotes the number of iterations required to achieve the stationary posterior distribution.
7: Moreover, the associated posterior variance $V(\Psi(\Theta))$ is given by,

$$V(\Psi(\Theta)) = \frac{\frac{1}{MC-MC^*}\sum_{i=MC^*+1}^{MC}(\Psi^{(i)}(\Theta)-\hat{\Psi}(\Theta))^2\Pi(\alpha_1^{(i)},\lambda_1^{(i)},\alpha_2^{(i)},\lambda_2^{(i)}|Z)}{\frac{1}{MC-MC^*}\sum_{i=MC^*+1}^{MC}\Pi(\alpha_1^{(i)},\lambda_1^{(i)},\alpha_2^{(i)},\lambda_2^{(i)}|Z)}. \tag{6.4.10}$$

6.4.2 HPD Credible Interval

The credible intervals of the Bayes estimates are derived using the concept HPD intervals, as proposed by Chen & Shao (1999). The procedures for constructing HPD credible intervals for the model parameters are outlined in Algorithm 2.

6.5 Bootstrap Confidence Intervals

In this section, we explore the application of the bootstrap technique for estimating confidence intervals. By re-sampling from the pseudo-population, we can estimate

Algorithm 2 HPD Credible Interval

1: Arrange the MCMC sample $\Theta^{(i)} = (\alpha_1^{(i)}, \lambda_1^{(i)}, \alpha_2^{(i)}, \lambda_2^{(i)})$, $i = MC^* + 1, MC^* + 2, ..., MC$ obtained via the importance sampling method in ascending order, denoted as $\Theta_{j(i)}, j = 1, ..., 4$.

2: Calculate the associated weight h_i

$$h_i = \frac{\Pi(\alpha_1^{(i)}, \lambda_1^{(i)}, \alpha_2^{(i)}, \lambda_2^{(i)}|Z)}{\sum_{i=MC^*+1}^{MC} \Pi(\alpha_1^{(i)}, \lambda_1^{(i)}, \alpha_2^{(i)}, \lambda_2^{(i)}|Z)}. \tag{6.4.11}$$

3: Next, denote the ordered value $h_{(i)}$ as corresponding to the value $\Theta_{j(i)}$.

4: Estimate the δ-quantile marginal posterior of $\Theta_{j(i)}$ as

$$\tilde{\Theta}_j(\delta) = \begin{cases} \Theta_{j(1)}, & \text{if } \delta = 0, \\ \Theta_{j(k)}, & \text{if } \sum_{i=1}^{k-1} h_{(i)} < \delta < \sum_{i=1}^{k} h_{(i)}. \end{cases}$$

5: Then the $100(1-\delta)\%$ credible interval of Θ is given by $(\tilde{\Theta}_j(\frac{\delta}{2}), \tilde{\Theta}_j(1-\frac{\delta}{2}))$.

any quantity of interest. This method is particularly advantageous for statistical inference when dealing with small sample sizes. Specifically, we employ the percentile parametric bootstrap technique to construct confidence intervals for model parameters. For a detailed explanation of BOOT-P methods, refer to Efron (1982). The Algorithm 3 provides the necessary steps to compute the BOOT-P confidence intervals from JAPT censored data.

6.6 Simulation Study

In this section, a simulation study is conducted to evaluate the performance of the MLE and Bayes estimates derived earlier. The study explores the optimal test time T and examines the impact of varying censoring schemes on the results. Evaluation criteria include the mean estimate and MSE for point estimates, as well as confidence intervals (CI) assessed using ACL and CP. The simulation considers various sample sizes: $(n_1, n_2, m) = (50, 50, 80)$, $(50, 50, 90)$, $(70, 50, 90)$, and $(70, 50, 110)$, with parameter sets $(\alpha_1, \lambda_1, \alpha_2, \lambda_2) = (3, 6, 5, 9)$ and $(5, 4, 2, 3)$. Hyperparameters are selected for the informative case to ensure that the true parameter is close to the prior mean, reflecting practical scenarios, while for the non-informative case, the hyperparameters are set to 0. The methods' performance is analyzed over two different experiment time limits, 0.6 and 1, to observe how time duration affects the accuracy and precision of parameter estimates. The three censoring schemes used

Algorithm 3 BOOT-P Confidence Intervals

1: From the data set $\{(Z_1, D_1), (Z_2, D_2), ..., (Z_J, D_J), (Z_{J+1}, D_{J+1}), ..., (Z_m, D_m)\}$ compute the parameters estimate $\hat{\Theta} = (\hat{\alpha_1}, \hat{\lambda_1}, \hat{\alpha_2}, \hat{\lambda_2})$.
2: Generate Sample 1 from GL$(\hat{\alpha}_1, \hat{\lambda}_1)$ of size n_1 and Sample 2 from GL$(\hat{\alpha}_2, \hat{\lambda}_2)$ of size n_2, ensuring that the joint sample size is $n = n_1 + n_2$.
3: Using the algorithm provided by Balakrishnan & Sandhu (1995), generate a PT2 censored sample of size m from the combined sample of size n.
4: Construct a random Type-II progressive censored sample of size m for the given time point T. Present the sample as $\{Z_1, Z_2, ..., Z_J\}$, $J \leq m$.
5: As described by Ng et al. (2009), create two samples of size $(m - J)$ from the truncated distribution and select the smallest $(m - J)$ values from the combined sample. Thus, the bootstrap JAPT sample $\{(Z_1^*, D_1^*), (Z_2^*, D_2^*), ..., (Z_J^*, D_J^*), (Z_{J+1}^*, D_{J+1}^*), ..., (Z_m^*, D_m^*)\}$ is acquired.
6: Based on the bootstrap sample calculate $\hat{\Theta}^* = (\hat{\alpha_1}^*, \hat{\lambda_1}^*, \hat{\alpha_2}^*, \hat{\lambda_2}^*)$.
7: Repeat the steps 2-6 M times, we get $\hat{\Theta}^{*(i)} = (\hat{\alpha_1}^{*(i)}, \hat{\lambda_1}^{*(i)}, \hat{\alpha_2}^{*(i)}, \hat{\lambda_2}^{*(i)}), i = 1, 2, ..., M$.
8: Arrange the bootstrap sample estimates in ascending order.
9: The bootstrap estimate is given by $\hat{\Theta}_{iM}^* = \frac{1}{M}\sum_{i=1}^{M} \hat{\Theta}_i^{*(i)}$
10: The CDF of $\hat{\Theta}_i^*$ is expressed as $F(\hat{\Theta}_i^*) = P(\hat{\Theta}_i^* \leq x)$. The approximate $100(1-\delta)$ BOOT-P confidence intervals are given by $(\hat{\Theta}_i^*(\frac{\delta}{2}), \hat{\Theta}_i^*(1 - \frac{\delta}{2}))$.

are: $(R_1, R_2, ..., R_m) = (0_{2m-N}, 1_{N-m})$, $(0_{m-1}, N - m)$, and $(1_{N-m}, 0_{2m-N})$, where $(0_3, 1_2)$ represents $(0, 0, 0, 1, 1)$. The complete set of censoring schemes used in the study is given in Table 6.1.

1. Initially we generate two samples of sizes n_1 and n_2 from $GL(\alpha_1, \lambda_1)$ and $GL(\alpha_2, \lambda_2)$ respectively.

2. Randomly select a PT2 censored sample of size m from a combined sample of size $n_1 + n_2$ using the algorithms developed by Balakrishnan & Sandhu (1995).

3. Select the sample $\{(Z_1, D_1), (Z_2, D_2), ..., (Z_J, D_J)\}$ based on the time duration T.

4. If $J < m$, generate truncated samples from GL distribution with size $J - m$.

5. Select the least $(m - J)$ observations and update JAPT censored sample $\{(Z_1, D_1), (Z_2, D_2), ..., (Z_J, D_J), (Z_{J+1}, D_{J+1}), ..., (Z_m, D_m)\}$.

6. Calculate w_1 and w_2.

Table 6.1: Censoring schemes employed in the simulation study

Sl. No.	**Sample Size** (n_1, n_2, m)	**Censoring Scheme** R
1	$(50, 50, 80)$	$(0_{60}, 1_{20})$
2	,,	$(0_{79}, 20)$
3	,,	$(1_{20}, 0_{60})$
4	$(50, 50, 90)$	$(0_{80}, 1_{10})$
5	,,	$(0_{89}, 10)$
6	,,	$(1_{10}, 0_{80})$
7	$(70, 50, 90)$	$(0_{60}, 1_{30})$
8	,,	$(0_{89}, 30)$
9	,,	$(1_{30}, 0_{60})$
10	$(70, 50, 110)$	$(0_{100}, 1_{10})$
11	,,	$(0_{109}, 10)$
12	,,	$(1_{10}, 0_{100})$

7. Implement the Newton-Raphson method to obtain the MLEs of the model parameters α_1, λ_1, α_2, and λ_2 from the nonlinear equations (6.3.4-6.3.7).

8. Calculate ACI from (6.3.8) and BCI from Algorithm 3.

9. Compute Bayesian estimates (informative and non-informative) and HPD intervals using Algorithms 1 and 2.

10. Calculate the Est, MSE and results are outlined in Tables 6.2 and 6.3.

11. Calculate the CP, ACL and results are outlined in Tables 6.4 and 6.5.

The key findings from the simulation study are summarized as follows.

1. The Bayes estimates obtained with an informative prior are noticeably better than those with a non-informative prior and the MLEs.

2. The Bayes estimates obtained with an informative prior exhibit smaller MSE compared to the estimates obtained with a non-informative prior and the MLEs, with the MSE decreasing as the sample size increases.

3. As the time T increases, the MSE of all estimates decreases, indicating improved accuracy and precision in the estimates.

4. Most of the time, the coverage probability remains at its nominal level for ACI, HPD (both informative and non-informative), and Bootstrap intervals. Meanwhile, the value of ACL is shorter for HPD intervals with an informative prior compared to other confidence intervals.

5. As the observed sample size increases, the ACL decreases because larger samples yield more precise estimates.

6.7 Real Data Application

In this section, we provide a detailed demonstration of the theoretical findings by applying them to a real-world data example. For this purpose, we utilize adaptive progressive censored data derived from real data sets that document the waiting time of customers before service at two banks, A and B. These data sets were initially analyzed in the study by Ghitany et al. (2008). The complete data, with sample sizes $n_1 = 100$ for Bank A and $n_2 = 60$ for Bank B, are presented in Table 6.6. By using these real-world examples, we aim to illustrate the practical relevance and accuracy of our theoretical findings in a tangible and meaningful way.

To ensure the validity of the data for the GL distribution, we analyze the survival plot, which shows both the fitted and empirical survival functions, as illustrated in Figures 6.3 and 6.4, indicating that the data sets align with the GL distribution. Additionally, we assessed the goodness-of-fit of the proposed model using the KS test, the AD test, and the CVM test, and compared it with the Lindley distribution. Based on our analysis, we can identify the superior model for these data sets. The comparison results, summarized in Table 6.7, indicate that the GL distribution demonstrates higher p-values and lower statistic values across all tests, suggesting a better fit for the GL model.

Thus, the JAPT censored data generated from the joint waiting times for $m = 140$, $T = 30$, and $R = (0_{120}, 1_{20})$ is as follows: (0.1, 0), (0.2, 0), (0.3, 0), (0.7, 0), (0.8, 1), (0.8, 0), (0.9, 0), (1.1, 0), (1.2, 0), (1.3, 1), (1.5, 1), (1.8, 1), (1.8, 0), (1.9, 1), (1.9, 0), (1.9, 0), (2, 0), (2.1, 1), (2.2, 0), (2.3, 0), (2.3, 0), (2.3, 0), (2.5, 0), (2.6, 1), (2.6, 0), (2.7, 1), (2.7, 0), (2.7, 0), (2.9, 1), (2.9, 0), (3.1, 1), (3.1, 0), (3.1,

Table 6.2: Estimates and MSEs for $\Theta = (3, 6, 5, 9)$ across different estimation techniques

		MLE				Bayesian				Bayesian (non informative)			
CS	T	α_1 (MSE)	λ_1 (MSE)	α_2 (MSE)	λ_2 (MSE)	α_1 (MSE)	λ_1 (MSE)	α_2 (MSE)	λ_2 (MSE)	α_1 (MSE)	λ_1 (MSE)	α_2 (MSE)	λ_2 (MSE)
1	0.6	3.3558	6.2865	5.414	9.138	3.1234	6.2086	4.9833	8.9933	3.5633	6.7004	5.0201	8.8343
		1.2101	1.1208	3.0893	1.777	0.4356	0.524	0.6584	0.5755	2.1065	1.7734	3.2143	2.1853
	1	3.3701	6.2614	5.6002	9.3107	3.0831	6.1373	4.9291	8.9989	3.3995	6.5301	5.2091	8.9675
		1.0038	1.1164	3.0449	1.6222	0.4053	0.507	0.6303	0.5641	1.7536	1.7067	3.1161	2.1411
2	0.6	3.3733	6.2752	5.6221	9.3181	3.1102	6.1363	4.994	9.0351	3.5238	6.7299	4.9554	8.8776
		1.2303	1.1154	3.7966	1.865	0.5161	0.4676	0.7157	0.5156	1.708	1.8198	2.8951	2.3001
	1	3.3733	6.2752	5.6221	9.3181	3.1102	6.1363	4.994	9.035	3.5238	6.7299	4.9554	8.8776
		1.2303	1.1154	3.7966	1.865	0.5161	0.4676	0.7157	0.5156	1.7081	1.8198	2.8951	2.3001
3	0.6	3.9308	7.2499	4.7838	8.293	3.213	6.4952	4.9718	8.718	3.8104	7.1545	4.4432	8.0095
		2.2703	2.3955	1.7338	1.6385	0.5589	0.7856	0.664	0.6161	2.3298	2.9044	2.2995	2.6916
	1	3.4205	6.4203	5.2389	8.9134	3.0747	6.1007	5.0693	8.8854	3.4443	6.4335	4.8433	8.6319
		1.0951	0.9777	1.2042	1.4998	0.4405	0.5566	0.4382	0.6114	1.4458	1.4737	2.1189	2.0841
4	0.6	3.2771	6.2314	5.4579	9.0863	3.1048	6.1261	5.0641	8.9175	3.5336	6.5404	4.9606	8.6437
		0.9644	0.8651	2.4309	1.1365	0.3643	0.4482	0.5718	0.5127	1.8871	1.6572	2.2742	1.8965
	1	3.2714	6.216	5.6216	9.2893	3.0474	5.9937	5.0784	8.9344	3.4459	6.4062	5.0234	8.8693
		0.9037	0.8181	2.4549	1.0448	0.3115	0.3786	0.5217	0.4692	1.4777	1.331	2.1137	1.7654
5	0.6	3.2766	6.1843	5.5346	9.2545	3.0254	5.9743	5.0192	8.9187	3.4337	6.4407	5.0401	8.7807
		1.0042	0.9986	3.2192	1.6168	0.4357	0.468	0.6654	0.5621	1.4334	1.4504	2.486	1.9184
	1	3.3063	6.2165	5.6115	9.3268	3.0831	6.0344	5.0266	8.9192	3.5199	6.4794	5.0388	8.9046
		1.008	0.9337	3.1991	1.5265	0.4198	0.4635	0.6199	0.5542	1.3971	1.4266	2.2667	1.0727
6	0.6	3.9268	7.2173	4.7866	8.3246	3.256	6.6296	4.9322	8.7483	3.9215	7.1716	4.5549	8.1725
		2.0956	2.1686	1.6806	1.4999	0.5359	0.6279	0.6551	0.5805	1.9215	2.5984	2.007	2.0073
	1	3.3979	6.3827	5.271	8.9252	3.0775	6.124	5.0482	8.9006	3.2985	6.312	4.86	8.6511
		1.0596	0.8955	1.1091	0.7972	0.4506	0.5101	0.6127	0.5632	1.2448	1.1955	1.0748	1.6693
7	0.6	3.2019	6.1564	5.4969	9.1368	3.1135	6.3391	4.9309	8.9215	3.5693	7.008	4.974	8.9406
		0.6951	0.7661	3.6008	1.7374	0.3795	0.4942	0.6569	0.6088	1.5572	2.1396	2.6433	2.0315
	1	3.2524	6.1799	5.6182	9.3412	3.1069	6.2698	4.9495	8.9499	3.361	6.7459	5.1856	8.9859
		0.6287	0.6702	3.4471	1.687	0.3614	0.4739	0.5976	0.6032	0.9988	1.4182	2.4893	1.7312
8	0.6	3.2351	6.1789	5.6321	9.2979	3.1117	6.4273	4.9069	9.0349	3.4928	7.0137	4.9662	9.0745
		0.8251	0.8504	3.6371	1.8049	0.3823	0.5743	0.7099	0.6118	1.2951	1.8536	2.4017	1.9412
	1	3.2351	6.1789	5.6321	9.2979	3.1117	6.4273	4.9069	9.0349	3.4928	7.0137	4.9662	9.0745
		0.8251	0.8504	3.6371	1.8049	0.3823	0.5743	0.7099	0.6118	1.2951	1.8536	2.4017	1.9412
9	0.6	3.7993	7.1591	4.5548	7.9634	3.2922	6.6504	4.9736	8.6158	3.8898	7.2298	4.5455	8.0331
		1.511	1.9206	1.7371	2.2035	0.5347	0.9657	0.6904	0.7141	2.0939	2.5709	2.1407	2.6735
	1	2.3549	6.3910	5.1625	8.8065	3.1296	6.1916	5.0325	8.8399	3.1488	6.2806	4.9535	8.6928
		0.7959	0.8403	1.5093	1.2797	0.4572	0.5494	0.6899	0.6834	0.8071	1.0883	1.3807	2.6418
10	0.6	3.239	6.2459	5.1428	8.8217	3.0632	6.1506	4.968	8.8459	3.343	6.4037	4.8277	8.4806
		0.6429	0.6209	2.4359	1.4451	0.3572	0.4768	0.6427	0.5321	1.2087	0.9527	1.8739	1.6972
	1	3.1853	6.1687	5.5646	9.2432	3.07	6.0492	5.0446	8.9848	3.3028	6.2782	5.0194	8.9032
		0.517	0.5943	2.2089	1.3643	0.3492	0.4234	0.6403	0.5369	0.9736	0.9134	1.7654	1.4479
11	0.6	3.1206	6.0701	5.4382	9.0783	3.026	6.0504	4.9851	8.8409	3.2358	6.3044	5.0292	8.6327
		0.558	0.6394	3.2884	1.7642	0.3328	0.3919	0.6452	0.5953	0.8985	0.9251	2.0515	1.628
	1	3.1954	6.1515	5.5817	9.291	3.0652	6.0428	4.9926	8.9817	3.2977	6.3376	4.9083	8.7275
		0.5307	0.5704	3.1184	1.6048	0.3071	0.4526	0.6236	0.5468	0.7304	0.895	2.0317	1.6258
12	0.6	3.8617	7.2301	4.4294	7.8788	3.3075	6.624	4.9822	8.5095	3.7545	7.0764	4.3558	7.6608
		1.4599	1.491	1.5666	2.1004	0.5347	0.8309	0.6625	0.7022	1.6572	1.9357	1.9044	2.0507
	1	3.3272	6.3533	5.0624	8.7391	3.1021	6.1517	4.922	8.7983	3.2985	6.2606	4.5736	8.3858
		0.6695	0.6392	1.159	1.9065	0.3709	0.4318	0.6951	0.5701	0.993	0.8757	1.9011	2.0346

Table 6.3: Estimates and MSEs for $\Theta = (5, 4, 2, 3)$ across different estimation techniques

CS	T	MLE α_1 (MSE)	MLE λ_1 (MSE)	MLE α_2 (MSE)	MLE λ_2 (MSE)	Bayesian α_1 (MSE)	Bayesian λ_1 (MSE)	Bayesian α_2 (MSE)	Bayesian λ_2 (MSE)	Bayesian (non informative) α_1 (MSE)	Bayesian (non informative) λ_1 (MSE)	Bayesian (non informative) α_2 (MSE)	Bayesian (non informative) λ_2 (MSE)
1	0.6	5.8681	4.3055	2.3392	3.2702	4.9779	4.0928	2.4114	3.4693	5.2787	4.1641	2.5385	3.6405
		1.8647	0.4225	0.4818	0.2705	0.7181	0.3832	0.5245	0.3986	3.3515	0.4299	0.8929	0.6371
	1	5.4846	4.0725	2.1756	3.1038	4.8369	3.9207	2.2696	3.3499	4.822	3.8716	2.4177	3.5143
		1.8174	0.3401	0.3572	0.2501	0.7177	0.1786	0.4102	0.2992	2.4718	0.4079	0.7512	0.569
2	0.6	5.2817	4.3162	2.3176	3.2193	4.9882	4.0967	2.3043	3.4254	5.3999	4.2776	2.4752	3.5879
		1.5478	0.4945	0.5065	0.2431	0.7808	0.1748	0.3282	0.3387	2.9988	0.3807	0.7576	0.5581
	1	5.0397	4.1107	2.1349	3.0711	4.7325	3.8837	2.1685	3.3137	5.0137	3.9854	2.4689	3.5406
		1.0333	0.4295	0.3621	0.2183	0.7642	0.1697	0.2695	0.2687	2.9009	0.2497	0.6816	0.5573
3	0.6	5.8171	5.6587	3.2769	4.3348	5.5957	4.5656	2.6952	4.0043	6.4123	4.7605	3.1248	4.3304
		1.7444	3.1918	2.8369	2.0261	1.1337	0.4682	0.9807	1.2721	4.9183	0.8083	2.1872	2.1881
	1	5.2168	4.3004	2.4984	3.5541	5.0527	4.0183	2.4035	3.5591	5.0419	3.9321	2.4783	3.6396
		1.0366	0.3368	0.6714	0.4926	0.7314	0.1416	0.6019	0.5899	1.8803	0.2272	0.9228	0.8406
4	0.6	5.2806	4.2684	2.8071	3.7793	5.4317	4.3911	2.3621	3.3411	5.1961	4.1107	2.3117	3.4692
		1.1627	0.493	0.2457	0.1997	0.9505	0.3056	0.4681	0.3611	2.8194	0.4003	0.4766	0.5624
	1	5.1952	4.0805	2.1617	3.0976	4.9688	3.9378	2.2609	3.2615	4.8057	3.8075	2.2488	3.3635
		0.5712	0.2745	0.2389	0.1116	0.7051	0.1746	0.3909	0.2621	1.8915	0.3219	0.4683	0.4017
5	0.6	5.1906	4.0933	2.1369	3.1312	5.1174	4.0171	2.2911	3.3978	5.2955	4.1757	2.3016	3.3263
		1.3963	0.3551	0.4527	0.1165	0.6518	0.0603	0.2585	0.3131	2.0046	0.2901	0.6859	0.4554
	1	5.1604	3.9941	2.0852	3.0104	4.9433	3.899	2.2076	3.1975	4.704	3.812	2.2654	3.2832
		1.4086	0.2731	0.2813	0.1061	0.6361	0.0246	0.2061	0.2119	1.0923	0.1435	0.4957	0.3482
6	0.6	5.7725	5.4802	3.2862	4.3641	5.5833	4.5521	2.7698	4.0659	6.1287	4.6601	3.1214	4.2372
		1.0884	2.2337	2.6016	2.0187	1.0706	0.4241	0.7852	1.1165	3.2467	0.6166	1.5575	1.0133
	1	5.2809	4.3262	2.5301	3.5678	5.0478	3.9663	2.3846	3.4581	4.7799	3.8976	2.6452	3.6517
		0.8909	0.3081	0.6994	0.4787	0.6285	0.1043	0.5218	0.4274	1.6472	0.1744	1.1886	0.7564
7	0.6	5.8099	3.9634	2.3038	3.1515	4.7188	3.9969	2.3898	3.4282	4.6062	3.9916	2.5741	3.6174
		2.1221	0.2269	0.4732	0.2334	0.7491	0.1654	0.4551	0.3617	2.0625	0.3354	0.796	0.6067
	1	5.3892	4.0708	2.1489	3.0851	4.7495	3.9487	2.2305	3.4825	4.7672	3.9689	2.411	3.6459
		2.6164	0.2541	0.3161	0.2441	0.7447	0.1454	0.3908	0.3546	1.7001	0.2595	0.6383	0.5826
8	0.6	4.7171	3.8312	2.2361	3.0481	4.7023	4.008	2.3961	3.4061	4.5555	3.9459	2.4662	3.5603
		1.7299	0.2343	0.4197	0.1917	0.7171	0.1491	0.5042	0.3093	2.5959	0.3552	0.6724	0.5044
	1	5.4894	4.1085	2.1997	3.1202	4.7391	3.9227	2.2609	3.2475	4.7277	3.9362	2.431	3.721
		1.0741	0.1808	0.3916	0.1778	0.6498	0.1229	0.3849	0.2983	1.6374	0.2614	0.6098	0.4911
9	0.6	5.6251	5.6283	3.2667	4.1968	5.5948	4.6106	2.7633	3.9179	6.2771	4.7731	3.1782	4.2225
		2.1377	2.9541	2.5486	1.6322	1.0731	0.4896	1.0841	1.0835	3.4773	0.7733	2.4018	1.8434
	1	6.2251	4.4078	2.4421	3.4274	4.9579	4.0419	2.3685	3.4213	5.0359	3.9913	2.3884	3.5386
		4.6302	0.3583	0.5567	0.3518	0.4919	0.1085	0.4581	0.4249	1.1848	0.1948	0.6891	0.6246
10	0.6	5.1387	4.1755	2.9808	3.0244	5.4455	4.1964	2.7862	3.8663	5.8012	4.2691	3.0289	3.1832
		1.1983	0.1558	0.3071	0.1511	0.6221	0.1482	0.4247	0.3598	1.886	0.2673	1.0159	0.4097
	1	5.0419	4.0775	2.1709	3.0755	4.9218	3.9745	2.1977	3.2457	4.6712	3.8417	2.2709	3.3309
		0.6171	0.1014	0.2417	0.1258	0.6141	0.1075	0.2743	0.2302	1.3156	0.2073	0.4727	0.3569
11	0.6	5.2291	5.2018	2.9587	3.8087	5.4542	4.5002	2.6721	3.6946	5.7754	4.5392	2.9962	3.8963
		1.8793	1.7046	1.6049	0.8227	0.9135	0.3674	0.8916	0.6932	2.0333	0.4276	1.9157	1.0781
	1	5.1661	4.0512	2.1061	3.0183	4.9121	3.9225	2.2052	3.2199	4.6619	3.8159	2.2377	3.2554
		1.8088	0.1714	0.2816	0.1711	0.587	0.114	0.2995	0.2127	1.1379	0.1857	0.4957	0.2756
12	0.6	5.2987	4.7341	3.1381	4.2743	5.4996	4.5822	2.8077	3.9111	5.7926	4.5454	3.0643	4.1239
		1.6901	2.2746	1.2528	1.0234	0.8998	0.4147	1.0188	1.0211	1.9181	0.3881	1.6902	1.5096
	1	5.1769	4.4031	2.4802	3.4637	4.943	4.0047	2.3632	3.3787	4.572	3.8414	2.5354	3.5028
		0.7212	0.3009	0.6351	0.3559	0.5152	0.0813	0.4901	0.3447	0.8572	0.1142	0.9054	0.5113

Table 6.4: CP and corresponding ACL values for $\Theta = (3, 6, 5, 9)$ across different confidence intervals

CS	T	ACI				Bayesian (informative)				Bayesian (non informative)				BOOT-P			
		CP(α_1) ACL	CP(λ_1) ACL	CP(α_2) ACL	CP(λ_2) ACL	CP(α_1) ACL	CP(λ_1) ACL	CP(α_2) ACL	CP(λ_2) ACL	CP(α_1) ACL	CP(λ_1) ACL	CP(α_2) ACL	CP(λ_2) ACL	CP(α_1) ACL	CP(λ_1) ACL	CP(α_2) ACL	CP(λ_2) ACL
1	T=0.6	0.952 3.5792	0.947 3.8668	0.954 6.1674	0.941 4.864	1 2.1354	0.951 2.6515	0.99 2.5371	1 2.5844	1 3.3012	0.37 3.5753	0.93 3.5021	0.93 3.4684	0.956 4.5959	0.962 4.0342	0.932 6.8305	0.987 5.9076
	T=1	0.954 3.4886	0.938 3.8335	0.961 6.0128	0.947 4.7314	0.951 2.1244	0.951 2.6269	0.951 2.5264	0.951 2.4938	1 3.1647	0.42 3.5197	0.98 3.4961	0.94 3.4596	0.987 4.1933	0.963 3.927	0.954 6.2446	0.975 5.0961
2	T=0.6	0.963 3.6128	0.952 3.8686	0.963 6.3245	0.94 4.9	1 2.2021	0.95 2.71	0.99 2.5772	1 2.5996	1 3.5137	0.19 3.7139	0.97 3.6371	0.96 3.5487	0.974 4.1219	0.998 3.9187	0.993 7.5876	1 4.9288
	T=1	0.963 3.6128	0.952 3.8686	0.963 6.3245	0.94 4.9	1 2.2021	0.95 2.71	0.99 2.5772	1 2.5996	1 3.5137	0.19 3.7139	0.97 3.6371	0.96 3.5487	0.98 4.1037	1 3.9111	0.98 7.5082	0.967 4.8359
3	T=0.6	0.987 4.1346	0.883 4.1318	0.901 5.1496	0.885 4.4968	1 2.0563	0.99 2.3406	0.95 2.4433	0.9 2.3542	1 2.7516	0.95 2.9611	0.97 2.8125	0.97 2.9174	0.998 5.41	1 3.8056	1 5.9043	0.996 4.117
	T=1	0.971 3.4463	0.961 3.6586	0.923 4.8822	0.902 4.3912	1 1.9511	1 2.1699	0.991 2.3197	0.91 2.2306	1 2.517	1 2.6524	0.97 2.6431	0.97 2.8688	0.934 4.2928	0.987 3.6063	0.964 5.6769	1 4.0661
4	T=0.6	0.961 3.3044	0.95 3.5334	0.942 5.9182	0.921 4.5824	1 2.0974	0.95 2.3827	0.999 2.4988	0.985 2.3836	1 2.961	0.96 3.0164	0.98 3.1654	0.97 3.0311	1 3.5638	0.974 3.4334	1 6.168	0.968 4.5041
	T=1	0.962 3.2721	0.952 3.4959	0.971 5.1257	0.951 4.4736	1 2.0542	0.928 2.3405	0.965 2.3066	0.9 2.1984	1 2.8929	0.97 2.934	0.98 3.0196	0.94 3.0242	0.912 3.497	0.932 3.2237	0.97 5.8616	0.964 3.9515
5	T=0.6	0.952 3.2929	0.925 3.4896	0.958 6.0029	0.939 4.6542	1 2.0797	0.96 2.3831	0.98 2.5006	0.951 2.3959	1 2.9676	0.91 3.0084	0.98 3.2675	0.94 3.108	1 3.9352	0.968 3.6964	1 6.7653	0.962 4.6991
	T=1	0.953 3.2192	0.941 3.3931	0.968 5.9065	0.944 4.5864	1 2.0012	0.951 2.3816	0.96 2.4047	0.9 2.3944	1 2.8948	0.98 3.0029	0.89 3.2487	0.94 3.0278	0.989 3.6543	0.999 3.5533	1 6.6191	0.968 4.5266
6	T=0.6	0.979 4.0135	0.87 3.8673	0.912 5.0071	0.879 4.2668	1 2.0516	0.999 2.2337	0.958 2.3898	0.89 2.2676	1 2.7268	0.98 2.7073	0.96 2.7934	0.94 2.7731	0.968 5.1651	0.974 3.5709	0.956 5.4867	1 3.8073
	T=1	0.971 3.3275	0.956 3.4279	0.927 4.7669	0.901 3.6289	1 1.9234	0.971 2.0604	0.95 2.1717	0.89 2.1242	1 2.3666	0.96 2.4548	0.95 2.6786	0.98 2.6491	0.974 3.9286	0.987 3.3998	0.968 5.7258	1 3.7797

Table 6.4: (continued)

CS	T	ACI CP(α_1) ACL	ACI CP(λ_1) ACL	ACI CP(α_2) ACL	ACI CP(λ_2) ACL	Bayesian (informative) CP(α_1) ACL	Bayesian (informative) CP(λ_1) ACL	Bayesian (informative) CP(α_2) ACL	Bayesian (informative) CP(λ_2) ACL	Bayesian (non informative) CP(α_1) ACL	Bayesian (non informative) CP(λ_1) ACL	Bayesian (non informative) CP(α_2) ACL	Bayesian (non informative) CP(λ_2) ACL	BOOT-P CP(α_1) ACL	BOOT-P CP(λ_1) ACL	BOOT-P CP(α_2) ACL	BOOT-P CP(λ_2) ACL
7	T=0.6	0.952 2.8963	0.943 3.3299	0.957 5.2874	0.948 3.9783	1 1.9701	1 2.4953	1 2.551	1 2.5338	1 2.8233	0.92 3.2475	0.97 3.5237	0.95 3.3371	1 3.4426	0.989 3.3912	0.947 5.451	0.985 5.0062
	T=1	0.966 2.6421	0.947 3.3225	0.964 5.1312	0.946 3.0644	1 1.966	0.987 2.456	1 2.4615	0.961 2.4444	1 2.7339	0.98 3.1772	0.97 3.4407	0.97 3.2285	0.968 3.3744	0.969 3.373	0.989 5.3521	0.99 5.0054
8	T=0.6	0.964 2.9577	0.946 3.3595	0.959 6.4215	0.944 4.9865	1 2.0351	0.99 2.5971	1 2.5946	0.964 2.5725	1 3.0867	0.94 3.4421	0.99 3.7951	0.98 3.4138	1 3.0738	0.986 3.2743	0.938 5.5985	0.974 4.9594
	T=1	0.964 2.9577	0.946 3.3595	0.959 6.4215	0.944 4.9865	1 2.0351	0.99 2.5971	1 2.5946	0.964 2.5725	1 3.0867	0.94 3.4421	0.99 3.795	0.98 3.4138	0.98 3.0738	0.969 3.2743	1 5.5985	0.981 4.9594
9	T=0.6	0.966 3.3848	0.854 3.5761	0.871 4.9057	0.803 4.4476	1 1.875	0.95 2.184	0.85 2.4493	0.897 2.3745	1 2.4106	0.93 2.6123	0.97 2.9194	0.93 3.4284	0.983 3.9971	0.986 3.1825	1 5.0312	0.997 3.8458
	T=1	0.965 2.8732	0.939 3.1816	0.905 4.8253	0.893 4.0309	0.98 1.7627	0.98 2.0088	0.97 2.3098	0.89 2.2941	0.99 2.0542	0.98 2.3561	0.94 2.1021	0.95 3.3074	0.98 3.0757	0.963 3.1295	0.924 5.0055	1 3.4283
10	T=0.6	0.953 2.7136	0.955 2.9363	0.928 5.2471	0.925 3.4227	1 1.8798	0.9 2.084	0.954 2.4226	0.97 2.3205	1 2.3834	0.96 2.4476	0.97 3.0068	0.94 2.8897	0.938 2.8936	1 2.847	0.97 4.8844	0.964 4.4073
	T=1	0.962 2.631	0.946 2.8765	0.965 5.0076	0.948 3.3127	1 1.8539	0.89 2.0175	0.974 2.4716	0.899 2.2788	1 2.3274	0.96 2.3689	0.94 2.9268	0.91 2.398	0.952 2.7954	0.952 2.6722	0.953 4.324	0.963 4.2549
11	T=0.6	0.943 2.5875	0.935 2.8594	0.935 5.8445	0.916 4.5304	1 1.8607	0.97 2.0647	0.9 2.4565	0.894 2.3424	1 2.3852	0.94 2.4509	0.96 3.1485	0.94 2.943	1 3.1004	0.987 2.9875	0.958 5.2561	0.987 4.6912
	T=1	0.966 2.4437	0.946 2.7684	0.954 5.7334	0.939 4.4319	1 1.7693	0.947 2.0349	0.942 2.4567	0.968 2.389	1 2.3166	0.96 2.4168	0.94 3.1104	0.97 2.6245	0.965 2.9684	0.947 2.8633	0.963 5.1662	0.954 4.6654
12	T=0.6	0.956 3.2972	0.77 3.2509	0.839 4.4754	0.773 3.9836	1 1.8342	1 1.9461	0.912 2.3966	0.987 2.1796	1 2.2667	0.94 2.283	0.97 2.6516	0.97 2.5475	0.989 3.8125	0.986 2.9625	0.977 4.8944	0.984 3.4749
	T=1	0.966 2.71892	0.951 2.8615	0.89 4.3774	0.884 3.52	0.99 1.7087	0.99 1.795	0.951 2.2172	1 2.101	0.99 1.9901	0.98 2.0194	0.97 2.4399	0.94 2.3583	1 2.9699	0.985 2.7473	1 4.461	0.938 3.8741

Table 6.5: CP and corresponding ACL values for $\Theta = (5, 4, 2, 3)$ across different confidence intervals

CS	T	ACI				Bayesian (informative)				Bayesian (non informative)				BOOT-P			
		CP(α_1) ACL	CP(λ_1) ACL	CP(α_2) ACL	CP(λ_2) ACL	CP(α_1) ACL	CP(λ_1) ACL	CP(α_2) ACL	CP(λ_2) ACL	CP(α_1) ACL	CP(λ_1) ACL	CP(α_2) ACL	CP(λ_2) ACL	CP(α_1) ACL	CP(λ_1) ACL	CP(α_2) ACL	CP(λ_2) ACL
1	T=0.6	0.975 4.2807	0.949 2.2771	0.976 2.2064	0.962 1.9137	0.98 2.7772	1 1.5329	0.99 1.9778	0.987 1.7274	0.97 3.6968	1 1.6904	0.98 2.3828	0.998 1.934	1 4.5612	0.984 2.2533	0.938 2.9133	0.967 4.8402
	T=1	0.963 3.1532	0.94 2.1574	0.963 2.0611	0.957 1.8763	0.92 2.667	0.95 1.4394	1 1.8856	0.987 1.7178	0.91 3.3702	0.99 1.5632	1 2.2697	0.98 1.8832	0.989 4.3305	1 2.1602	0.968 2.2957	0.932 4.6585
2	T=0.6	0.973 4.2724	0.95 2.2733	0.968 2.1858	0.959 1.8904	0.98 2.788	1 1.5318	1 1.9084	0.963 1.7407	0.94 3.6132	0.99 1.6558	0.98 2.3817	0.94 1.9402	1 5.4138	1 2.2261	0.96 2.5798	0.984 4.7581
	T=1	0.934 3.6797	0.911 2.1644	0.949 2.0175	0.917 1.8626	0.96 2.7262	0.97 1.466	1 1.8656	0.996 1.7379	0.91 3.4955	0.93 1.5842	0.97 2.2357	0.94 1.9376	0.963 5.2273	0.974 2.4321	0.948 2.2898	1 4.0255
3	T=0.6	0.927 4.2764	0.999 2.6881	0.993 3.1381	0.991 2.306	0.99 2.8167	0.99 1.4346	0.98 1.9096	0.961 1.6532	0.95 3.5535	0.96 1.5839	0.96 2.2979	0.91 1.7969	0.948 4.6198	0.987 2.9094	1 3.3324	0.99 4.1345
	T=1	0.988 3.2738	0.967 2.1715	0.975 2.3424	0.917 2.036	0.972 2.5295	0.974 1.2687	1 1.7311	0.989 1.5227	1 2.9172	0.94 1.3335	0.98 1.9454	0.94 1.6426	0.98 3.9267	1 1.9592	0.969 2.9842	0.978 4.0622
4	T=0.6	0.964 4.2538	0.609 2.3675	0.913 2.1952	0.735 1.8847	0.97 2.7301	0.99 1.3979	0.97 1.9047	0.981 1.5929	1 3.3831	0.98 1.4816	0.97 2.2783	0.99 1.7228	0.992 4.2987	0.993 2.1571	0.999 2.5743	1 3.8214
	T=1	0.966 4.1532	0.956 2.0058	0.953 1.9728	0.953 1.7546	0.981 2.5496	0.951 1.267	1 1.7968	0.953 1.4978	1 3.0132	0.98 1.3526	0.94 2.0258	0.97 1.6489	1 4.1596	1 1.989	1 2.1656	0.998 3.7141
5	T=0.6	0.961 4.1548	0.604 2.179	0.935 2.113	0.782 1.6673	0.96 2.7253	0.99 1.4005	0.95 1.8296	0.986 1.5776	0.97 3.4268	0.94 1.4934	1 2.3001	0.98 1.724	0.996 5.7182	0.992 2.1471	1 2.4385	0.987 3.8389
	T=1	0.94 3.8541	0.942 1.9818	0.953 1.8974	0.949 1.5204	0.85 2.5582	0.958 1.2599	1 1.7991	0.974 1.5019	0.97 2.9486	1 1.3485	0.97 2.1034	0.98 1.6476	1 5.4019	1 1.9239	0.948 2.0751	1 3.6471
6	T=0.6	0.932 3.4925	0.908 2.5418	0.766 3.0495	0.926 2.1853	0.98 2.7624	0.98 1.3645	0.98 1.8826	0.972 1.5414	0.91 3.3955	0.95 1.4575	0.96 2.2068	1 1.6624	0.991 4.4231	0.989 2.7723	0.947 3.0729	1 4.0263
	T=1	0.991 3.1146	0.966 2.0575	0.969 2.3184	0.898 1.9254	0.965 2.4598	0.957 1.1785	1 1.6905	0.99 1.3782	1 2.7811	0.98 1.2589	0.94 1.9602	0.97 1.4835	0.989 4.1229	0.947 1.8408	0.948 3.0311	0.968 3.6657

Table 6.5: (continued)

CS	T	ACI				Bayesian (informative)				Bayesian (non informative)				BOOT-P			
		CP(α_1)	CP(λ_1)	CP(α_2)	CP(λ_2)	CP(α_1)	CP(λ_1)	CP(α_2)	CP(λ_2)	CP(α_1)	CP(λ_1)	CP(α_2)	CP(λ_2)	CP(α_1)	CP(λ_1)	CP(α_2)	CP(λ_2)
		ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL
7	T=0.6	0.932 4.1861	0.936 1.9372	0.967 2.1526	0.956 1.8522	0.94 2.5897	1 1.4441	0.98 1.9591	0.03 1.636	1 3.2221	0.98 1.4749	0.92 2.2158	0.97 1.7178	1 4.3196	0.99 1.8651	1 2.6405	0.968 2.7239
	T=1	0.95 4.07	0.945 1.908	0.961 2.0575	0.957 1.5194	0.979 2.4863	1 1.3422	1 1.8815	0.984 1.5375	0.99 2.9674	0.94 1.4244	1 2.1364	0.95 1.6281	0.984 2.2975	0.969 1.6167	1 2.2961	0.981 1.8826
8	T=0.6	0.883 4.6364	0.934 1.8921	0.959 2.0821	0.969 1.8047	0.92 2.5966	1 1.4432	0.98 1.9736	0.962 1.7378	0.99 3.2344	1 1.5589	0.96 2.3714	0.98 1.9393	0.932 4.3771	0.935 1.7273	1 2.3597	0.948 2.6236
	T=1	0.964 4.5695	0.941 1.701	0.958 2.0031	0.94 1.7476	0.92 2.5688	0.99 1.3584	0.99 1.9187	0.991 1.675	1 3.1419	0.96 1.4781	0.97 2.1749	0.93 1.6003	1 3.4381	0.99 1.6555	0.989 2.1183	0.978 1.9626
9	T=0.6	0.811 4.7139	0.193 2.4338	0.767 3.0352	0.414 2.1874	0.91 2.6048	1 1.3625	1 1.8754	0.992 1.5899	0.97 3.1602	0.98 1.4569	1 2.3197	0.94 1.7378	1 4.2214	0.999 2.5057	1 2.7473	1 2.072
	T=1	0.983 4.2834	0.935 1.9684	0.986 2.2799	0.955 2.001	1 2.295	0.947 1.1634	1 1.7192	0.995 1.5077	1 2.6125	0.98 1.2175	0.97 1.9007	0.95 1.6484	0.996 4.1792	0.99 1.7505	1 2.2953	0.921 1.7226
10	T=0.6	0.969 4.3911	0.911 2.0904	0.932 2.6367	0.941 1.8683	0.987 2.4484	0.96 1.2376	0.97 1.8257	0.908 1.4576	0.98 2.9488	0.97 1.3171	0.94 2.1691	1 1.5313	0.98 3.7821	0.971 1.7317	0.934 2.6154	1 1.8424
	T=1	0.958 4.1688	0.956 1.6923	0.957 1.9529	0.962 1.7049	0.961 2.2283	0.974 1.0982	1 1.7311	0.948 1.4604	1 2.4827	0.96 1.1288	0.98 1.9868	0.99 1.5124	0.989 3.3944	0.964 1.6066	0.968 2.0289	1 1.5895
11	T=0.6	0.963 4.5079	0.978 1.9007	0.948 1.9059	0.975 1.7591	0.989 2.4611	0.98 1.2375	0.96 1.7755	0.11 1.4535	0.99 2.9516	1 1.3169	0.96 2.1685	0.98 1.5519	0.997 4.1007	0.998 1.4113	0.98 2.1695	0.999 1.8711
	T=1	0.956 4.3546	0.972 1.6879	0.95 1.8843	0.969 1.683	0.948 2.2358	0.921 1.0884	0.938 1.7415	0.971 1.4462	1 2.5215	0.96 1.1286	0.98 1.9656	0.99 1.56	1 3.3013	1 1.3949	1 2.0991	1 1.5679
12	T=0.6	0.935 4.705	0.963 2.2299	0.949 3.0156	0.914 2.0008	0.987 2.4361	0.97 1.2132	0.95 1.7842	0.947 1.4229	0.98 2.9565	0.94 1.2726	1 2.2087	0.97 1.5013	0.971 3.4008	0.961 2.3273	0.993 2.1069	0.974 1.9496
	T=1	0.988 4.1186	0.934 1.7734	0.965 2.2223	0.93 1.8271	0.961 2.1668	0.971 1.0404	1 1.6745	0.982 1.326	1 2.3569	0.98 1.0673	0.96 1.8841	0.92 1.4101	0.921 2.4448	0.932 1.5723	0.951 1.791	0.963 1.5346

Table 6.6: Waiting times for customers before service in banks

Bank A									
0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7
2.9	3.1	3.2	3.3	3.5	3.6	4.0	4.1	4.2	4.2
4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3
6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0
8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5
11.9	12.4	12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9
14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0
19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5
Bank B									
0.1	0.2	0.3	0.7	0.9	1.1	1.2	1.8	1.9	2.0
2.2	2.3	2.3	2.3	2.5	2.6	2.7	2.7	2.9	3.1
3.1	3.2	3.4	3.4	3.5	3.9	4.0	4.2	4.5	4.7
5.3	5.6	5.6	6.2	6.3	6.6	6.8	7.3	7.5	7.7
7.7	8.0	8.0	8.5	8.5	8.7	9.5	10.7	10.9	11.0
12.1	12.3	12.8	12.9	13.2	13.7	14.5	16.0	16.5	28.0

Table 6.7: Goodness-of-fit test results for the GL and Lindley distributions

	Model	Estimates	KS test		AD test		CVM test	
			Statistic	p-value	Statistic	p-value	Statistic	p-value
Data 1	GL	$\hat{\alpha}_1$=1.2772 $\hat{\lambda}_1$=0.2107	0.0502	0.962	0.2589	0.9654	0.0421	0.9227
	Lindley	$\hat{\lambda}$=0.1865	0.0676	0.7495	0.4863	0.7604	0.0581	0.8265
Data 2	GL	$\hat{\alpha}_2$=0.9267 $\hat{\lambda}_2$=0.2689	0.0683	0.942	0.2575	0.9662	0.0404	0.9325
	Lindley	$\hat{\lambda}$=0.2797	0.0797	0.8401	0.30549	0.9337	0.0495	0.8807

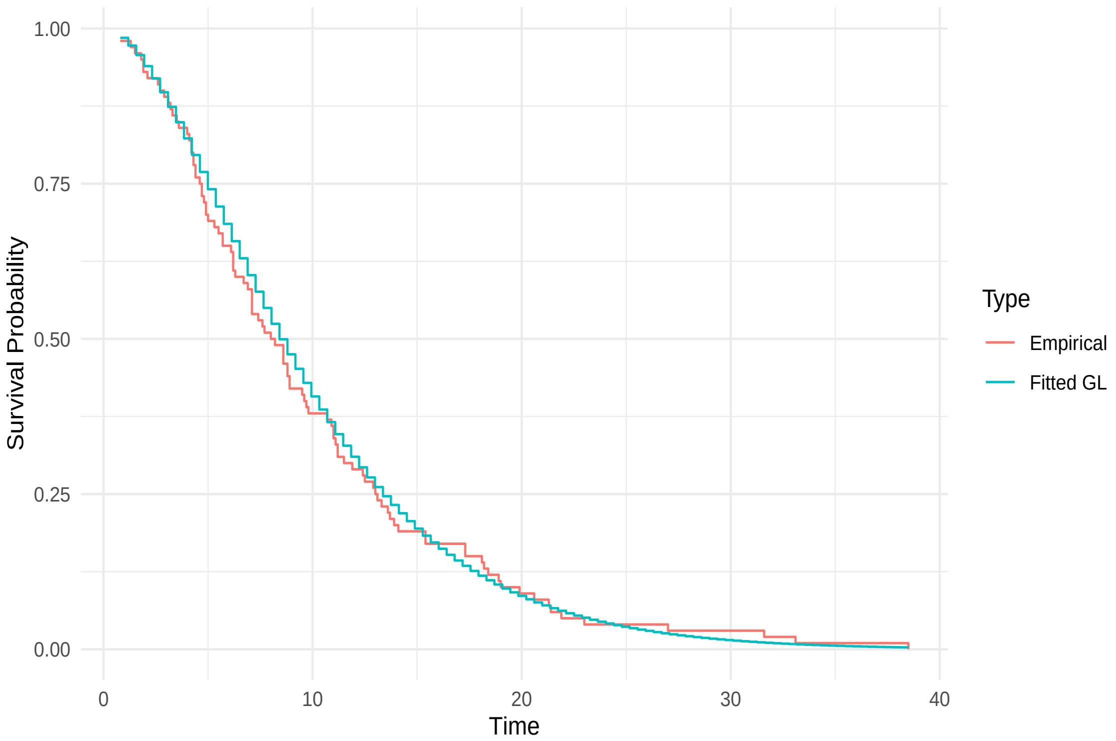


Figure 6.3: Survival probability plot for customer waiting times at Bank A

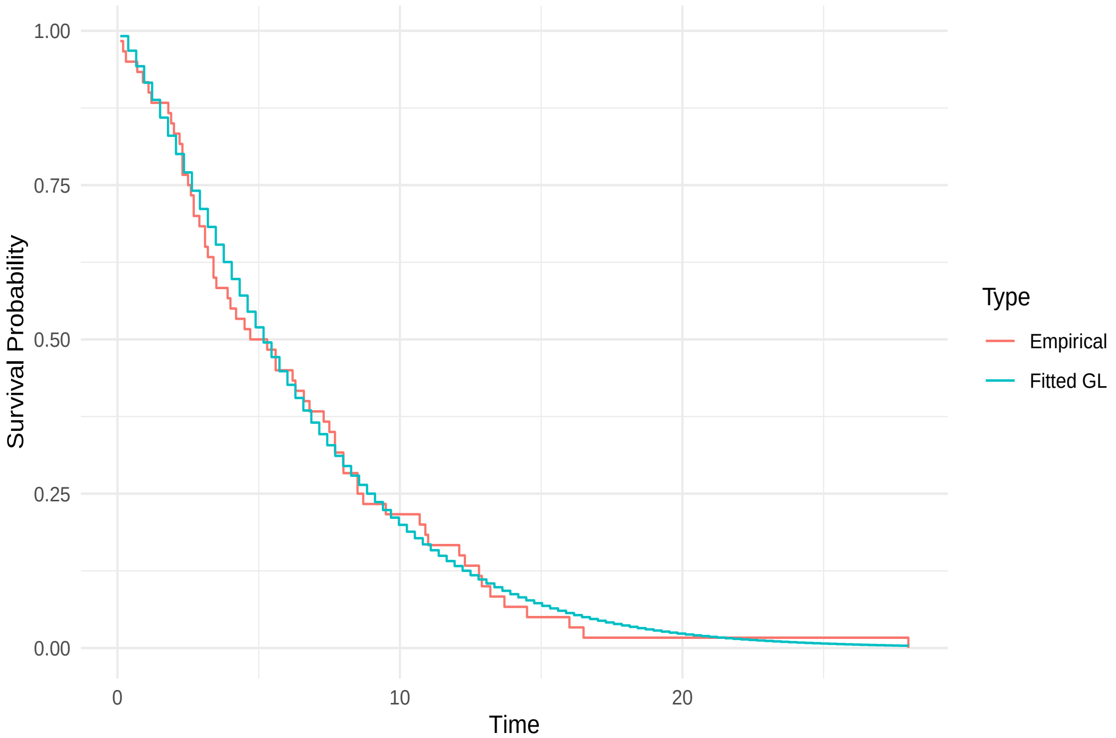


Figure 6.4: Survival probability plot for customer waiting times at Bank B

0), (3.2, 1), (3.2, 0), (3.3, 1), (3.4, 0), (3.4, 0), (3.5, 1), (3.5, 0), (3.6, 1), (3.9, 0), (4, 1), (4, 0), (4.1, 1), (4.2, 1), (4.2, 0), (4.2, 0), (4.3, 1), (4.3, 0), (4.4, 1), (4.4, 0), (4.5, 0), (4.6, 1), (4.7, 1), (4.7, 0), (4.7, 0), (4.8, 1), (4.9, 1), (4.9, 0), (5, 1), (5.3, 1), (5.3, 0), (5.5, 1), (5.6, 0), (5.6, 0), (5.7, 1), (5.7, 0), (6.1, 1), (6.2, 1), (6.2, 0), (6.2, 0), (6.2, 0), (6.3, 1), (6.3, 0), (6.6, 0), (6.7, 1), (6.8, 0), (6.9, 1), (7.1, 1), (7.1, 0), (7.1, 0), (7.1, 0), (7.3, 0), (7.4, 1), (7.5, 0), (7.6, 1), (7.7, 1), (7.7, 0), (7.7, 0), (8, 1), (8, 0), (8, 0), (8.2, 1), (8.5, 0), (8.5, 0), (8.6, 1), (8.6, 0), (8.6, 0), (8.7, 0), (8.8, 1), (8.8, 0), (8.9, 1), (8.9, 0), (9.5, 1), (9.5, 0), (9.6, 1), (9.7, 1), (9.8, 1), (10.7, 1), (10.7, 0), (10.9, 1), (10.9, 0), (11, 1), (11, 0), (11, 0), (11.1, 1), (11.2, 1), (11.2, 0), (11.5, 1), (11.9, 1), (12.1, 0), (12.3, 0), (12.5, 1), (12.8, 0), (12.9, 1), (12.9, 0), (13, 1), (13.1, 1), (13.3, 1), (13.6, 1), (13.7, 1), (13.7, 0), (13.9, 1), (14.5, 0), (15.4, 0), (16.5, 0), (17.3, 0), (19, 1), (20.6, 1).

Next, we calculate the point estimates and interval estimates of the parameters using the above JAPT censored data. We employ non-informative priors and set the hyperparameter values to 0.0001 because there is no prior knowledge about the model parameters available. The Bayes estimates are derived using the importance sampling algorithm, utilizing $MC = 8,000$ iterations. The estimates, along with their corresponding standard errors and 95% confidence intervals with their respective lengths, are presented in Table 6.8. The Bayes estimates have smaller standard errors compared to the MLEs, and the confidence intervals from the ACI, HPD, and Bootstrap methods have similar performance. The trace plots presented in Figure 6.5 demonstrate that the Markov Chains for all parameters exhibit similar variations around their central values. Moreover, the density plot in Figure 6.6 displays symmetric and unimodal shapes, identifying the modes of the posterior distributions of the model parameters, which are the values most strongly supported by the data and the selected priors.

6.8 Summary

The GL distribution is a powerful tool for analyzing lifetime data, especially when using the JAPT censoring scheme. This method effectively handles censored data by adjusting the stopping time based on observed failures. The GL distribution is particularly useful for modelling both random failures and failures that occur due to ageing. It is especially suitable for reliability engineering, where systems

Table 6.8: Estimates (Est), standard errors (SE), Confidence Intervals (CI), and confidence lengths (CL) across different estimation methods

Parameters	MLE		Bayesian		BOOT-P
	Est SE	ACI CL	Est SE	HPD CL	CI CL
$\hat{\alpha_1}$	1.2497 0.2339	(0.7912, 1.7081) 0.9168	1.34404 0.2114	(0.9397, 1.9125) 0.9728	(1.0582, 1.6044) 0.5462
$\hat{\lambda_1}$	0.2 0.0262	(0.1486, 0.2513) 0.1027	0.222071 0.0179	(0.21703, 0.3114) 0.0943	(0.172, 0.2439) 0.0719
$\hat{\alpha_2}$	1.086 0.1759	(0.7411, 1.4309) 0.6897	1.0878 0.1704	(0.8888, 1.5612) 0.6724	(0.8688, 1.5823) 0.7135
$\hat{\lambda_2}$	0.2835 0.0322	(0.2205, 0.3466) 0.1261	0.2931 0.0252	(0.2742, 0.3787) 0.1045	(0.2343, 0.3468) 0.1125

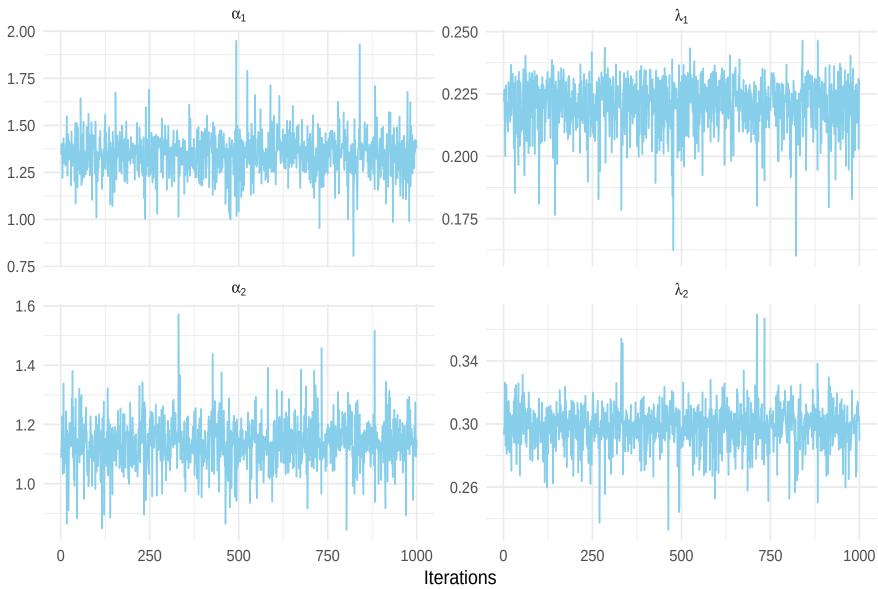


Figure 6.5: Trace plots for the posterior distributions of the model parameters

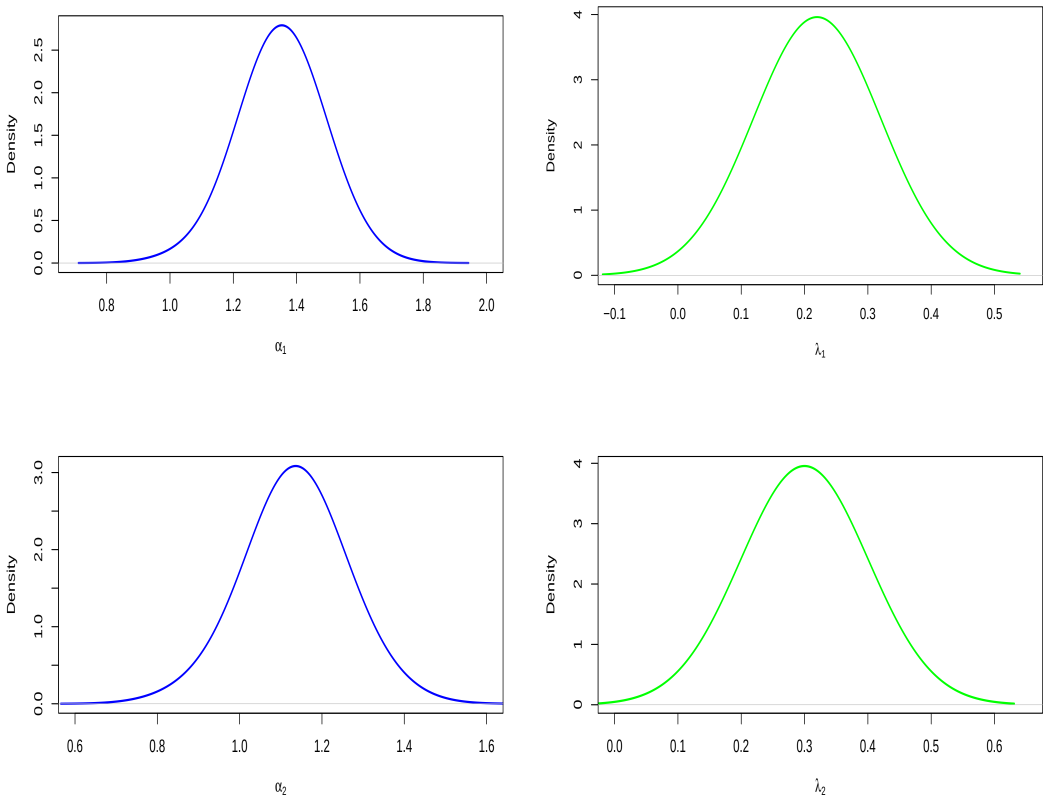


Figure 6.6: Density plots for the posterior distributions of the model parameters

experience early random failures followed by wear and ageing, and for medical survival analysis, where data may include sudden deaths or age-related diseases. Moreover, the GL distribution can represent various hazard rate patterns, such as increasing, decreasing, and bathtub-shaped rates, making it a versatile alternative to distributions like gamma, lognormal, Weibull, and exponentiated exponential, particularly for bathtub-shaped data.

In this study, the distribution parameters were estimated using MLE and bootstrap methods, along with Bayesian techniques that employed importance sampling algorithms for posterior estimation. Our results, supported by real data analysis and Monte Carlo simulations, demonstrate the efficacy of these methods in accurately estimating GL distribution parameters. These approaches are adaptable to other lifetime distributions and can be extended to scenarios involving more than two production lines, indicating their broad applicability and potential for future research.

CHAPTER 7

Type I-Type II Mixture Censoring Scheme for Lifetime Data Analysis

7.1 Introduction

Effective censoring schemes are crucial in the field of statistical analysis, as they play a vital role in accurately evaluating and deriving significant insights from censored data. Censoring, a common phenomenon in many fields such as survival analysis, reliability engineering, and clinical trials, occurs when complete information about the event of interest is not available for some observations. Traditional censoring schemes (Type-I and Type-II), while widely used, may not capture the full potential of the available data or adequately handle complex scenarios.

In this chapter, we delve into a new censoring scheme that promises to advance statistical inference techniques and unlock new avenues of knowledge. This innovative approach challenges conventional methodologies, offering researchers a fresh perspective on handling censored data. By introducing a novel framework for censoring, we aim to bridge the gap between existing methods and the increasing complexity of real-world datasets, enabling more accurate and insightful analysis.

The conventional censoring schemes, Type-I and Type-II, exhibit distinct features. The Type-I censoring scheme involves running the experiment until a fixed time point T is reached, while the Type-II censoring scheme allows the experiment to continue until the occurrence of the mth failure. These schemes lack complete event time information for censored observations. This loss of information can limit the precision and accuracy of statistical analysis, potentially leading to biased estimates or reduced statistical power.

Many of the censoring schemes discussed earlier fail to ensure both a maximum possible number of failures and a time constraint simultaneously. If there is additional time available for the continuation of the present experiment after the mth failure, we can utilize this supplementary time to collect more failure data if available. To address this case, a new censoring scheme named the Type I-Type II (T1-T2) mixture censoring scheme has been developed. This scheme guarantees that a minimum of m and a maximum of n failures occur within a supplementary time S provided after the m-th failure. In this setup, an experiment is conducted across n units, with the effective sample size m fixed prior to the experiment, similar to the Type II censoring scheme. After the occurrence of the mth failure, if sufficient time is available, the experiment is extended for a random duration S, associated with the lifetime variable. The test is then terminated at a time $T^* = \min\{X_{n:n}, X_{m:n} + S\}$. Here, the number of observed failures r and the experiment duration T^* are random variables. When $S = 0$, the proposed method reduces to the conventional Type II censoring method.

The remaining sections of the chapter are arranged as follows. In the Section 7.2, a description of the introduced censoring scheme using Weibull distribution is given. The Sections 7.3 and 7.4 are the attempts to derive expressions for the expected duration of the experiment and the Fisher information matrix, respectively. The parameter estimation using ML method and Bayes method are given in Sections 7.5 and 7.6, respectively. A simulation study is performed in Section 7.7. In Section 7.8, a real data is used to demonstrate the findings, and finally the chapter is concluded in Section 7.9.

7.2 Model Description

Consider n units undergoing a life-testing experiment, with their ordered lifetimes denoted by $X_{1:n}, X_{2:n}, X_{3:n}, ..., X_{n:n}$. Following the mth failure, a supplementary time S is allocated. If $r\ (m \le r \le n)$ denotes the number of failures occur before

T^*, where $T^* = \min\{X_{n:n}, X_{m:n} + S\}$. Then the available data will be of the form:

$$\begin{aligned}
&\text{Case I } : \{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\} \quad \text{if} \quad T^* = X_{n:n} \\
&\text{Case II} : \{X_{1:n}, X_{2:n}, \ldots, X_{r:n}\} \quad \text{if} \quad T^* = X_{m:n} + S.
\end{aligned}$$

The forms of likelihood function corresponding to both cases are respectively,

$$L(x_{i:n};\theta) \propto \begin{cases} \prod_{i=1}^{n} f(x_{i:n};\theta) & if \quad T^* = X_{n:n} \\ \prod_{i=1}^{r} f(x_{i:n};\theta)\left\{1 - F(x_{m:n}+s)\right\}^{n-r} & if \quad T^* = X_{m:n} + S. \end{cases}$$

In the fields of life-testing and reliability, the Weibull distribution is one of the most often used lifetime model. Based on the above expression, the log-likelihood function for the Weibull distribution with PDF,

$$f(x;\gamma,\delta) = \gamma\delta x^{\gamma-1} e^{-\delta x^{\gamma}};\ x,\delta,\gamma > 0, \tag{7.2.1}$$

and CDF,

$$F(x;\gamma,\delta) = 1 - e^{-\delta x^{\gamma}};\ x,\delta,\gamma > 0, \tag{7.2.2}$$

is obtained as,

$$\log L(\theta) \propto \begin{cases} n\log\gamma + n\log\delta + (\gamma-1)\sum_{i=1}^{n}\log x_i - \delta\sum_{i=1}^{n} x_i^{\gamma} & ; \quad r = n \\ \\ r\log\gamma + r\log\delta + (\gamma-1)\sum_{i=1}^{r}\log x_i - \delta\sum_{i=1}^{r} x_i^{\gamma} - \delta(n-r)(x_m+s)^{\gamma} & ; \quad r = m, m+1, ..., n-1. \end{cases}$$

In this context, the number of failures, denoted by r, taking the value of n in case I and ranging from m to $n-1$ in case II.

7.3 Expected Duration of Experiment

Let K be the random variable representing the number of failures observed and T^* be the random variable representing the duration of experiment under T1-T2 censoring scheme.

Suppose S be the random time given after m^{th} failure, which is the additional random time given by the experimenter to complete the experiment. The conditional distribution of K given that $S = s$ is,

$$P(K = r|S = s) = \int_x P(X_{r:n} \le X_{m:n} + S \le X_{n:n}|S = s, X_{m:n} = x) f_{m:n}(x)dx,$$

$r = m, m+1, ..., n-1$, where $f_{m:n}$ is the PDF of $X_{m:n}$. Let $g(s)$ be the PDF of the random duration S. Then,

$$P(K = r) = \int_s \int_x P(X_{r:n} \leq X_{m:n} + S | S = s, X_{m:n} = x) f_{m:n}(x) g(s) dx ds.$$

$r = m, m+1, ..., n-1$. Now consider $r = n$,

$$\begin{aligned} P(K = n | S = s) =& P(X_{n:n} \leq X_{m:n} + S | S = s) \\ =& \int_x P(X_{n:n} \leq X_{m:n} + S | S = s, X_{m:n} = x) f_{m:n}(x) dx \end{aligned}$$

$$P(K = n) = \int_s \int_x P(X_{n:n} \leq X_{m:n} + S | S = s, X_{m:n} = x) f_{m:n}(x) g(s) dx ds.$$

Therefore, the expected value of K is,

$$E[K] = \sum_r r P(K = r)$$

To find the expected duration of the experiment, consider

$$\begin{aligned} E[T^* | S] =& E[\min\{X_{n:n}, X_{m:n} + S\} | S] \\ =& E\left[X_{m:n} + S | S\right] P\left[X_{m:n} + S \leq X_{n:n} | S\right] + \\ & E\left[X_{n:n} | S\right] P\left[X_{n:n} < X_{m:n} + S | S\right]. \end{aligned}$$

Moreover,

$$\begin{aligned} E[T^* | S = s] =& E[min\{X_{n:n}, X_{m:n} + S\} | S = s] \\ =& \int_0^\infty P(min\{X_{n:n}, X_{m:n} + S\} \geq x | S = s) dx \\ =& \int_0^\infty P(X_{n:n} \geq x, X_{m:n} + S \geq x | S = s) dx. \end{aligned}$$

$$E[T^*] = E(E[T^* | S = s]) = \int_0^\infty \left(\int_0^\infty P(X_{n:n} \geq x, X_{m:n} + S \geq x | S = s) dx \right) g(s) ds$$

Lemma 7.3.1. *Assume that T^+ represents the censoring time obtained by multiplying some constant α to T. Then, $E[T^+] = \alpha E[T^*]$.*

The proof is trivial.

7.4 Fisher Information

Effron & Johnstone (1990) shown that the Fisher information about its parameters of a continuous random variable can be expressed in terms of hazard function of the random variable. Park & Balakrishnan (2009) derived Fisher information in different hybrid censoring schemes. Based on this, we can derive Fisher information in the T1-T2 censoring scheme. Let $P(r)$ be the probability mass function of K and $q(t^*)$ be the PDF of T^*,

$$I(\gamma, \delta) = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}$$

where

$$\begin{aligned}
I_{11} &= \sum_{r=m}^{n} \int_0^{\infty} \int_0^{t^*} \left\{ \frac{\partial}{\partial \gamma} \log h(t) \right\}' \left\{ \frac{\partial}{\partial \gamma} \log h(t) \right\} \sum_{i=1}^{r} f_{i:n}(t; \gamma, \delta) q(t^*) P(r) dt dt^* \\
&= \sum_{r=m}^{n} \int_0^{\infty} \int_0^{t^*} \left\{ \frac{1}{\gamma} + \log t \right\}^2 \sum_{i=1}^{r} f_{i:n}(t; \gamma, \delta) q(t^*) P(r) dt dt^* \\
I_{12} &= \sum_{r=m}^{n} \int_0^{\infty} \int_0^{t^*} \left\{ \frac{\partial}{\partial \gamma} \log h(t) \right\}' \left\{ \frac{\partial}{\partial \delta} \log h(t) \right\} \sum_{i=1}^{r} f_{i:n}(t; \gamma, \delta) q(t^*) P(r) dt dt^* \\
&= \frac{1}{\delta} \sum_{r=m}^{n} \int_0^{\infty} \int_0^{t^*} \left\{ \frac{1}{\gamma} + \log t \right\} \sum_{i=1}^{r} f_{i:n}(t; \gamma, \delta) q(t^*) P(r) dt dt^* \\
&= I_{21} \\
I_{22} &= \sum_{r=m}^{n} \int_0^{\infty} \int_0^{t^*} \left\{ \frac{\partial}{\partial \delta} \log h(t) \right\}' \left\{ \frac{\partial}{\partial \delta} \log h(t) \right\} \sum_{i=1}^{r} f_{i:n}(t; \gamma, \delta) q(t^*) P(r) dt dt^* \\
&= \sum_{r=m}^{n} \int_0^{\infty} \int_0^{t^*} \left(\frac{1}{\delta} \right)^2 \sum_{i=1}^{r} f_{i:n}(t; \gamma, \delta) q(t^*) P(r) dt dt^*.
\end{aligned}$$

7.5 Maximum Likelihood Estimation

Suppose $X_{1:n}, X_{2:n}, X_{3:n}, \ldots, X_{n:n}$ be a T1-T2 mixture ordered samples from the Weibull distribution with PDF (7.2.1).

The derivatives of log-likelihood function with respect to γ and δ are respectively given by,

$$\frac{\partial \log L}{\partial \gamma} = \begin{cases} \frac{n}{\gamma} + \sum_{i=1}^{n} \log x_i - \delta \sum_{i=1}^{n} x_i^{\gamma} \log x_i & ; \quad r = n \\ \\ \frac{r}{\gamma} + \sum_{i=1}^{r} \log x_i - \delta \sum_{i=1}^{r} x_i^{\gamma} \log x_i - \delta (n-r)(x_{m:n} + s)^{\gamma} \log(x_{m:n} + s) & ; \quad r = m, m+1, ..., n-1. \end{cases}$$

$$\frac{\partial \log L}{\partial \delta} = \begin{cases} \frac{n}{\delta} - \sum_{i=1}^{n} x_i^{\gamma} & ; \quad r = n \\ \\ \frac{r}{\delta} - \sum_{i=1}^{r} x_i^{\gamma} - (n-r)(x_{m:n}+s)^{\gamma} & ; \quad r = m, m+1, ..., n-1. \end{cases}$$

Now, combining Case I and Case II and equating to zero, the above equations become,

$$\frac{\partial \log L}{\partial \gamma} = \frac{w}{\gamma} + \sum_{i=1}^{w} \log x_i - \delta P = 0 \tag{7.5.1}$$

$$\frac{\partial \log L}{\partial \delta} = \frac{w}{\delta} - Q = 0 \tag{7.5.2}$$

where $w = n$, $P = \sum_{i=1}^{n} x_i^{\gamma} \log x_i$ and $Q = \sum_{i=1}^{n} x_i^{\gamma}$ for case I and $w = r$, $P = \sum_{i=1}^{r} x_i^{\gamma} \log x_i + (n-r)(x_{m:n}+s)^{\gamma} \log(x_{m:n}+s)$ and $Q = \sum_{i=1}^{r} x_i^{\gamma} + (n-r)(x_{m:n}+s)^{\gamma}$ for case II.

The MLEs, $\hat{\gamma}$ and $\hat{\delta}$ can be obtained by an iterative scheme proposed by Kundu (2007). For that 7.5.1 and 7.5.2 are arranged to get, $\hat{\delta} = \frac{w}{Q} = v(\gamma)$, and $\hat{\gamma} = \frac{w}{v(\gamma)P - \sum_{i=1}^{w} \log x_i} = u(\gamma)$ where $v(\gamma)$ and $u(\gamma)$ are represents the functions of γ.

Begin with an initial assumption of γ, denoted as $\gamma^{(0)}$. Subsequently, derive $\gamma^{(1)}$ by applying $\gamma^{(0)}$ to the function u, and continue this iterative process to obtain $\gamma^{(n+1)} = u(\gamma^{(n)})$. Terminate the iterative process when $|\gamma^{(n+1)} - \gamma^{(n)}| < \epsilon$, where ϵ is a predefined tolerance limit.

Now, we can verify the existence and uniqueness of the MLEs of the parameters.

Theorem 7.5.1. *Let $g_1(\gamma; \delta, x)$ denote the function on the right-hand side (RHS) of Eq. (7.5.1), with the parameter δ specified. Under this circumstance, there is at least one solution for $g_1(\gamma; \delta, x) = 0$ when $\delta > 0$, and this solution is unique provided that $-\frac{w}{\gamma^2} < \delta P'$, where $w = n$ and $P' = \sum_{i=1}^{n} x_i^{\gamma} (\log x_i)^2$ for Case I and $w = r$ and $P' = \sum_{i=1}^{r} x_i^{\gamma} (\log x_i)^2 + (n-r)(x_{m:n}+s)^{\gamma} (\log(x_{m:n}+s))^2$ for Case II.*

Proof. Given, $g_1(\gamma; \delta, x) = \frac{w}{\gamma} + \sum_{i=1}^{w} \log x_i - \delta P$, where $w = n$ and $P = \sum_{i=1}^{n} x_i^{\gamma} \log x_i$ for Case I and $w = r$ and $P = \sum_{i=1}^{r} x_i^{\gamma} \log x_i + (n-r)(x_{m:n}+s)^{\gamma} \log(x_{m:n}+s)$ for Case II.

Consider $\lim_{\gamma \to 0} g_1(\gamma; \delta, x) = \infty$. Also $\lim_{\gamma \to \infty} g_1(\gamma; \delta, x) = -\infty$. Hence, there is at least one solution where $\hat{\gamma}$ belongs to the interval $(0, \infty)$ such that $g_1(\gamma; \delta, x) = 0$. Furthermore, the solution will be unique when $\frac{\partial g_1(\gamma; \delta, x)}{\partial \gamma} < 0$, where $\frac{\partial g_1(\gamma; \delta, x)}{\partial \gamma} = -\frac{w}{\gamma^2} - \delta P'$. Here, $w = n$ and $P' = \sum_{i=1}^{n} x_i^{\gamma} (\log x_i)^2$ for Case I and $w = r$ and

$P' = \sum_{i=1}^{r} x_i^{\gamma} (\log x_i)^2 + (n-r)(x_{m:n}+s)^{\gamma}(\log(x_{m:n}+s))^2$ for Case II. $\square$

Theorem 7.5.2. *Let $g_2(\delta;\gamma,x)$ denote the function on the RHS of Eq. (7.5.2), with the parameter γ specified. Under this circumstance, there is at least one solution for $g_2(\delta;\gamma,x) = 0$ when $\gamma > 0$, and this solution is unique provided that $-\frac{w}{\delta^2} < 0$, where $w = n$ for Case I and $w = r$ for Case II.*

Proof. Given, $g_2(\delta;\gamma,x) = \frac{w}{\delta} - Q$, where $w = n$ and $Q = \sum_{i=1}^{n} x_i^{\gamma}$ for case I and $w = r$ and $Q = \sum_{i=1}^{r} x_i^{\gamma} + (n-r)(x_{m:n}+s)^{\gamma}$ for case II.

Consider $\lim_{\delta\to 0} g_2(\delta;\gamma,x) = \infty$. Also $\lim_{\delta\to\infty} g_2(\delta;\gamma,x) = -\infty$. Hence, there is at least one solution where $\hat{\delta}$ belongs to the interval $(0,\infty)$ such that $g_2(\delta;\gamma,x) = 0$. Furthermore, the solution will be unique when $\frac{\partial g_2(\delta;\gamma,x)}{\partial\delta} < 0$, where $\frac{\partial g_1(\delta;\gamma,x)}{\partial\delta} = -\frac{w}{\delta^2}$. Here, $w = n$ for Case I and $w = r$ for Case II. $\square$

7.6 Bayes Estimation

In this section, we explore the Bayesian estimation of the unknown parameters γ and δ within the context of a T1-T2 censoring scheme. Here we assume that the prior distributions of γ and δ are follows independent $Gamma(\alpha_1,\beta_1)$ and $Gamma(\alpha_2,\beta_2)$, respectively, where $\alpha_1,\alpha_2,\beta_1,\&\,\beta_2$ are the hyper parameters. When both γ and δ are unknown, their conjugate prior does not exist. Therefore, gamma priors can be used in these circumstances due to their flexibility and ability to account for non-informative priors. Researchers have demonstrated this approach in Bayesian analysis of the Weibull distribution, as evidenced in works such as those by Berger & Sun (1993) and Kundu (2009). Thus the independent prior distributions of γ and δ are,

$$\pi_1(\gamma) \propto \gamma^{\alpha_1-1} e^{-\beta_1\gamma};\ \gamma \geq 0,\ \alpha_1,\beta_1 > 0,$$

$$\pi_2(\delta) \propto \delta^{\alpha_2-1} e^{-\beta_2\delta};\ \delta \geq 0, \alpha_2,\beta_2 > 0.$$

The joint prior distribution of the parameters γ and δ is expressed as,

$$\pi(\gamma,\delta) \propto \gamma^{\alpha_1-1}\delta^{\alpha_2-1} e^{-\beta_1\gamma-\beta_2\delta};\ \gamma,\delta \geq 0, \alpha_1,\alpha_2,\beta_1,\beta_2 > 0. \tag{7.6.1}$$

Using Bayes Theorem, the joint posterior density function of γ and δ is given by

$$\pi(\gamma,\delta|\underline{X}) \propto l(\gamma,\delta|\underline{X})\pi(\gamma,\delta). \tag{7.6.2}$$

According to Eq. (7.6.2), the joint posterior density of γ and δ, given the data, is

$$\pi(\gamma, \delta|\underline{X}) \propto \gamma^{\alpha_1+w-1}\delta^{\alpha_2+w-1}\prod_{i=1}^{w} x_i^{\gamma-1} e^{-\delta\left[\beta_2+\sum_{i=1}^{w} x_i^{\gamma}+(n-w)(x_{m:n}+s)^{\gamma}\right]-\beta_1\gamma} \tag{7.6.3}$$

where w is the number of observed failures, x_i are the observed failure times, $x_{m:n}$ represents the failure time of mth observation among n total observations, and s is the supplementary time.

7.6.1 Estimation based on symmetric and asymmetric loss functions

To determine the Bayes estimates, we take into account both symmetric and asymmetric loss functions. For the symmetric loss function, the squared error loss function (L_{SE}) is considered. The L_{SE} can be defined as,

$$L_{SE}\left(g(\theta), \hat{g}(\theta)\right) = \left(g(\theta) - \hat{g}(\theta)\right)^2, \tag{7.6.4}$$

where $\hat{g}(\theta)$ is the estimate of $g(\theta)$.
For the asymmetric loss function, the LINEX loss function (L_{LL}) is considered. The L_{LL} can be defined as,

$$L_{LL}\Big(g(\theta), \hat{g}(\theta)\Big) = e^{d(g(\theta)-\hat{g}(\theta))} - d\left(g(\theta) - \hat{g}(\theta)\right) - 1, \quad d \neq 0, \tag{7.6.5}$$

where d is the loss parameter. For the LINEX loss function, the Bayes estimate $\hat{\theta}_{LL}$ can be calculated as,

$$\hat{\theta}_{LL} = -\frac{1}{d}\log\left\{E_\theta\left(e^{-d\theta}\right)\right\}, \tag{7.6.6}$$

where the expectation exists and finite.

7.6.2 M-H Algorithm

Based on Eq. (7.6.3) the conditional distributions of the unknown parameters γ and δ can be derives as follows:

$$\pi(\gamma|\delta, \underline{X}) \propto \gamma^{\alpha_1+w-1}\prod_{i=1}^{w} x_i^{\gamma-1} e^{-\delta\left[\sum_{i=1}^{w} x_i^{\gamma}+(n-w)(x_{m:n}+s)^{\gamma}\right]-\beta_1\gamma} \tag{7.6.7}$$

$$\pi(\delta|\gamma, \underline{X}) \propto \text{Gamma}\left(\alpha_2 + w, \beta_2 + \sum_{i=1}^{w} x_i^{\gamma} + (n-w)(x_{m:n} + s)^{\gamma}\right). \tag{7.6.8}$$

As the conditional distribution of γ lacks an explicit form, we need to utilize the M-H technique to calculate Bayes estimates and to construct HPD credible intervals. The M-H algorithm proposed by Metropolis et al. (1953) and Hasting (1970) is used to generate Bayes estimates of γ and δ. This method can be extremely helpful for constructing posterior samples using random proposal distributions when the posterior density is operationally infeasible. Based on the constructed posterior samples, bayesian inference can be drawn regarding the unknown parameters γ and δ. The following steps can be used to construct the posterior samples.

1. Choose an initial value (γ^0, δ^0), and let $\sigma = \gamma^{i-1}$.
2. Proposing a new point γ^i from $\pi_1(\gamma|\delta^0, \underline{X})$ from the proposing distribution $g(\gamma) \equiv N(\gamma^0 1), \gamma > 0$.
3. Generate a random sample u from Uniform(0,1) and a random sample ψ from the proposal distribution. If $u \leq \frac{\pi(\psi)g(\sigma)}{\pi(\sigma)g(\psi)}$, then set $\gamma = \psi$. Here, the acceptance and rejection probabilities of ψ are $min\left\{1, \frac{\pi(\psi)g(\sigma)}{\pi(\sigma)g(\psi)}\right\}$ and $1 - min\left\{1, \frac{\pi(\psi)g(\sigma)}{\pi(\sigma)g(\psi)}\right\}$, respectively.
4. Generate δ^i's from Gamma$(\alpha_2 + w, \beta_2 + \sum_{i=1}^{w} x_i^\gamma + (n-w)(x_{m:n} + s)^\gamma)$, as in step 3.
5. Repeat steps 2 to 4, N times and obtained $(\gamma_1, \delta_1), (\gamma_2, \delta_2), ..., (\gamma_N, \delta_N)$.

To ensure convergence and eliminate the impact of initial value selection, the first M estimates, known as the burn-in period, are excluded. The remaining sample, consisting of γ^i and δ^i with i ranging from $M+1$ to N, where N is sufficiently large, constitutes an estimated posterior sample that can be utilized for bayesian inference. Based on Figure 7.1, these marginal posterior distributions exhibit similarities to the normal distribution, leading us to consider the normal distribution as a suitable proposal distribution. After obtaining the posterior samples through the aforementioned algorithm, we compute the Bayes estimates, as well as assess bias and risk, considering both squared loss function and LINEX loss function.

The $100(1-\alpha)\%$ HPD credible interval of the unknown parameter γ can be constructed from the ordered samples $(\gamma_{(M+1)}, \gamma_{(M+2)}, ..., \gamma_{(N)})$ by selecting the smallest interval, and similarly for δ.

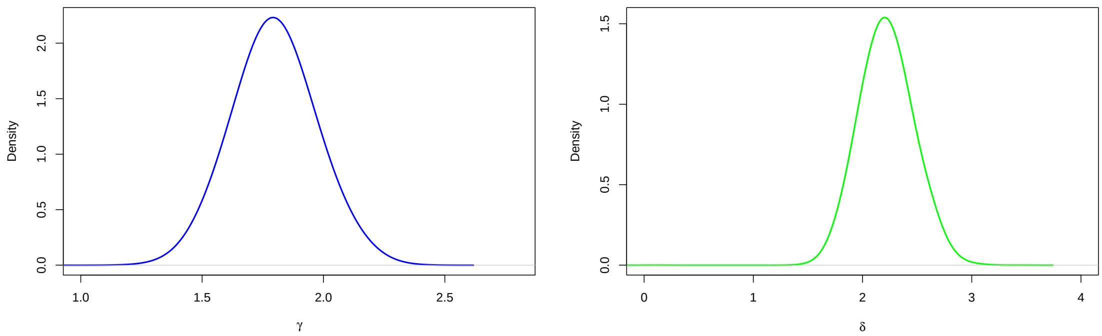


Figure 7.1: Marginal posterior density functions for γ and δ

7.7 Simulation Study

This section is dedicated to presenting experimental results, primarily focused on examining the behavior of various methods under different sample sizes and censoring schemes for various values of parameters of $f(x; \gamma, \delta)$ under T1-T2 mixture censoring scheme. The statistical software R is used to perform the computations.

A MCMC approach is used to generate samples for various values of (n, m) and assign a value to S in order to assess the performance of MLE. Without loss of generality, we choose two different values for the parametres, $(\gamma, \delta) = (1, 1)$ and (1.5, 2). The censoring schemes $(n, m) =$ (100, 100), (100, 90), (100, 85), (60, 60), (60, 55), (60, 50), (60, 45), (30, 30), (30, 25), (30, 20), (30, 15), (15, 15), (15, 12), (15, 10), and (15, 7) are chosen for the supplymentary times $S = 0.1$ and $S = 0.2$ with 1000 replications. Bias and MSE are computed for the unknown parameters in order to examine the effectiveness of MLE, and the results are given in the Table 7.1. As can be seen from Table 7.1, the MLE of parameters performs pretty satisfactorily with respect to bias and MSE. The MSE value is declining as sample size increases, indicating that the MLEs of the unknown parameters are consistent. Moreover, we can observe that the MSE value obtained for S equal to 0.2 is smaller compared to the MSE value obtained for S equal to 0.1. As the supplementary time increases, there is a greater chance of more failures occurring, which, in turn, leads to more precise statistical results. The ACL and CP based on 95% confidence interval are determined and the results are given in Table 7.4 and 7.5. As the sample size grows, the value of ACL gets shorter, and the CP remains at its nominal level.

The Bayes estimates for squared error and the LINEX loss functions are derived using the M-H algorithm. Under LINEX, the arbitrarily chosen values of the loss parameter d are -1 and 1. For the prior distribution, hyperparameters are chosen in such a way that the corresponding prior means are somewhat close to the actual parameter values. Hence, the selected values are $(\alpha_1 = 1, \beta_1 = 1, \alpha_2 = 1, \beta_2 = 1)$ and $(\alpha_1 = 2.25, \beta_1 = 1.5, \alpha_2 = 4, \beta_2 = 2)$ (see, Kundu (2008)). Tables 7.2-7.3 show the estimated bias and risk for various censoring schemes under the aforementioned loss functions. When we compare the bias and risk of each estimate, it is clear that the estimates obtained under the LINEX loss function perform better. The values of bias and risk are decreasing with the increase in the effective sample size. Tables 7.4 and 7.5 displays the ACL and CP based on the 95% HPD credible interval (HPD). As the effective sample size rises, the ACL of HPD credible interval becomes narrower. Meanwhile, the nominal level of the CP is maintained. Comparing ACL of HPD with MLE, it is clear that ACL of HPD performs better for small sample sizes while they both have values that are nearly equivalent for higher sample sizes.

7.8 Real Data Example

For the real-data application, we use the data from Hinkley (1977), which consists of 30 values of March precipitation readings (in inches) for Minneapolis/St. Paul. The data set is provided in Table 7.6. First, it must be verified that this data set can be used to analyze the Weibull distribution. Also, for the sake of comparison, Lindley and IW distributions are also taken into consideration. We looked at the KS test to determine the quality of fit. In order to confirm the data's suitability, other criteria are also taken into account, including the AIC, AICC, BIC, the Hannan-Quinn criterion (HQC), and negative log-likelihood (NLL). The distribution with the smallest estimated values of the KS statistic, AIC, AICC, BIC, HQC, and NLL and a comparatively high p-value is the better lifetime model. Table 7.7 indicates that the Weibull model fits the data far better than the other models. Additionally, we provide fitted density, empirical distribution function, P-P plot and Q-Q plot in Figure 7.2. Thus, Weibull distribution is used to draw conclusions about the data under consideration.

In order to analyze the data set under the T1-T2 mixture censoring scheme, we

Table 7.1: Average values of the biases and MSEs for the maximum likelihood estimates of $(\gamma,\ \delta) = (1,\ 1)$ and $(1.5,\ 2)$ under different censoring schemes.

Censoring Scheme	γ=1, δ=1				γ=1.5, δ=2			
	Bias(γ)	Bias(δ)	MSE(γ)	MSE(δ)	Bias(γ)	Bias(δ)	MSE(γ)	MSE(δ)
n=100, m=100, S=0.1	0.0101	0.0077	0.0064	0.0115	0.0151	0.0291	0.0144	0.0466
n=100, m=90, S=0.1	0.0112	0.0096	0.0077	0.0122	0.0152	0.0329	0.0168	0.0550
n=100, m=85, S=0.1	0.0122	0.0109	0.0083	0.0126	0.0169	0.0370	0.0178	0.0582
n=100, m=80, S=0.1	0.0125	0.0118	0.0089	0.0131	0.0174	0.0401	0.0188	0.0640
n=60, m=60, S=0.1	0.0189	0.0080	0.0111	0.0183	0.0284	0.0415	0.0251	0.0751
n=60, m=55, S=0.1	0.0219	0.0109	0.0135	0.0192	0.0306	0.0489	0.0294	0.0865
n=60, m=50, S=0.1	0.0264	0.0153	0.0161	0.0205	0.0365	0.0641	0.0333	0.1009
n=60, m=45, S=0.1	0.0306	0.0210	0.0192	0.0237	0.0397	0.0760	0.0386	0.1240
n=30, m=30, S=0.1	0.0472	0.0284	0.0274	0.0454	0.0709	0.1234	0.0616	0.2104
n=30, m=25, S=0.1	0.0577	0.0436	0.0387	0.0576	0.0814	0.1659	0.0799	0.3046
n=30, m=20, S=0.1	0.0805	0.0811	0.0568	0.0988	0.1008	0.2485	0.1069	0.5430
n=30, m=15, S=0.1	0.1105	0.1468	0.0853	0.1876	0.1388	0.4397	0.1579	1.3586
n=15, m=15, S=0.1	0.0992	0.0558	0.0722	0.1133	0.1488	0.2633	0.1625	0.6763
n=15, m=12, S=0.1	0.1294	0.1062	0.1107	0.1910	0.1739	0.3879	0.2191	1.2546
n=15, m=10, S=0.1	0.1640	0.1719	0.1501	0.2932	0.2090	0.5749	0.2843	2.4203
n=15, m=7, S=0.1	0.2473	0.4116	0.2840	1.2957	0.2950	1.1548	0.4437	8.1042
n=100, m=100, S=0.2	0.0101	0.0077	0.0064	0.0115	0.0151	0.0291	0.0144	0.0466
n=100, m=90, S=0.2	0.0104	0.0090	0.0076	0.0120	0.0126	0.0281	0.0158	0.0524
n=100, m=85, S=0.2	0.0108	0.0097	0.0080	0.0124	0.0142	0.0313	0.0168	0.0548
n=100, m=80, S=0.2	0.0124	0.0114	0.0086	0.0128	0.0149	0.0333	0.0177	0.0585
n=60, m=60, S=0.2	0.0189	0.0080	0.0111	0.0183	0.0284	0.0415	0.0251	0.0751
n=60, m=55, S=0.2	0.0209	0.0101	0.0134	0.0191	0.0275	0.0426	0.0281	0.0817
n=60, m=50, S=0.2	0.0246	0.0139	0.0153	0.0202	0.0325	0.0537	0.0305	0.0904
n=60, m=45, S=0.2	0.0279	0.0178	0.0179	0.0222	0.0345	0.0620	0.0335	0.1050
n=30, m=30, S=0.2	0.0472	0.0284	0.0274	0.0454	0.0709	0.1234	0.0616	0.2104
n=30, m=25, S=0.2	0.0546	0.0399	0.0367	0.0542	0.0742	0.1435	0.0741	0.2639
n=30, m=20, S=0.2	0.0687	0.0619	0.0496	0.0748	0.0817	0.1811	0.0886	0.3714
n=30, m=15, S=0.2	0.0930	0.1071	0.0704	0.1271	0.1000	0.2584	0.1152	0.5804
n=15, m=15, S=0.2	0.0992	0.0558	0.0722	0.1133	0.1488	0.2633	0.1625	0.6763
n=15, m=12, S=0.2	0.1197	0.0896	0.1032	0.1608	0.1536	0.3106	0.1910	0.8809
n=15, m=10, S=0.2	0.1428	0.1323	0.1300	0.2247	0.1660	0.3807	0.2210	1.1855
n=15, m=7, S=0.2	0.1901	0.2315	0.1960	0.4099	0.2008	0.5661	0.2876	2.1479

Table 7.2: Average values of the biases and Risks for the Bayesian estimates of parameters $(\gamma,\, \delta) = (1,\, 1)$ under different censoring schemes.

γ=1, δ=1	SE				LL							
	Bias		Risk		Bias				Risk			
					v=-1		v=1		v=-1		v=1	
	γ	δ	γ	δ	γ	δ	γ	δ	γ	δ	γ	δ
n=100, m=100, S=0.1	-0.1807	0.0245	0.0402	0.0107	0.0038	0.0051	-0.0037	-0.0049	0.0185	0.0054	0.0219	0.0053
n=100, m=90, S=0.1	-0.2151	0.0367	0.0544	0.0157	0.0041	0.0075	-0.0041	-0.0069	0.0248	0.0078	0.0299	0.0080
n=100, m=85, S=0.1	-0.1917	0.0409	0.0457	0.0145	0.0044	0.0066	-0.0045	-0.0062	0.0210	0.0073	0.0249	0.0072
n=100, m=80, S=0.1	-0.1923	0.0264	0.0458	0.0128	0.0044	0.0062	-0.0045	-0.0059	0.0210	0.0065	0.0251	0.0064
n=60, m=60, S=0.1	-0.1385	-0.0012	0.0270	0.0154	0.0039	0.0077	-0.0040	-0.0076	0.0126	0.0076	0.0145	0.0078
n=60, m=55, S=0.1	-0.1854	-0.0077	0.0498	0.0223	0.0077	0.0112	-0.0077	-0.0111	0.0226	0.0111	0.0276	0.0113
n=60, m=50, S=0.1	-0.1912	-0.0158	0.0551	0.0243	0.0094	0.0122	-0.0092	-0.0118	0.0248	0.0118	0.0309	0.0126
n=60, m=45, S=0.1	-0.1458	0.0050	0.0406	0.0263	0.0098	0.0135	-0.0095	-0.0127	0.0184	0.0129	0.0225	0.0136
n=30, m=30, S=0.1	0.0065	0.0269	0.0206	0.0409	0.0103	0.0203	-0.0102	-0.0200	0.0103	0.0211	0.0103	0.0203
n=30, m=25, S=0.1	0.0175	0.0774	0.0257	0.0404	0.0126	0.0175	-0.0128	-0.0170	0.0133	0.0216	0.0126	0.0192
n=30, m=20, S=0.1	-0.1085	-0.0158	0.0527	0.0525	0.0209	0.0273	-0.0201	-0.0251	0.0239	0.0252	0.0297	0.0282
n=30, m=15, S=0.1	-0.2812	-0.3702	0.1647	0.3088	0.0456	0.0999	-0.0408	-0.0768	0.0675	0.1160	0.1053	0.2299
n=15, m=15, S=0.1	-0.1103	0.0076	0.0669	0.0664	0.0286	0.0355	-0.0261	-0.0313	0.0296	0.0321	0.0388	0.0359
n=15, m=12, S=0.1	-0.1113	-0.0013	0.0768	0.0744	0.0334	0.0392	-0.0311	-0.0355	0.0343	0.0361	0.0444	0.0400
n=15, m=10, S=0.1	-0.1843	-0.0774	0.1151	0.0930	0.0417	0.0446	-0.0396	-0.0425	0.0496	0.0431	0.0693	0.0524
n=15, m=7, S=0.1	-0.2589	-0.2617	0.1997	0.3329	0.0685	0.1587	-0.0641	-0.1144	0.0819	0.1247	0.1285	0.2608
n=100, m=100, S=0.2	-0.1807	0.0245	0.0302	0.0107	0.0038	0.0051	-0.0037	-0.0049	0.0185	0.0054	0.0219	0.0053
n=100, m=90, S=0.2	-0.2043	0.0181	0.0326	0.0136	0.0055	0.0069	-0.0044	-0.0064	0.0190	0.0067	0.0221	0.0069
n=100, m=85, S=0.2	-0.1738	0.0333	0.0493	0.0147	0.0046	0.0070	-0.0045	-0.0066	0.0187	0.0075	0.0215	0.0073
n=100, m=80, S=0.2	-0.1962	0.0307	0.0589	0.0147	0.0058	0.0071	-0.0052	-0.0067	0.0223	0.0074	0.0269	0.0074
n=60, m=60, S=0.2	-0.1385	-0.0012	0.0270	0.0154	0.0039	0.0077	-0.0040	-0.0076	0.0126	0.0076	0.0145	0.0078
n=60, m=55, S=0.2	-0.1589	-0.0064	0.0364	0.0188	0.0055	0.0095	-0.0056	-0.0093	0.0167	0.0093	0.0198	0.0096
n=60, m=50, S=0.2	-0.1748	-0.0191	0.0397	0.0244	0.0091	0.0117	-0.0087	-0.0113	0.0167	0.0116	0.0197	0.0130
n=60, m=45, S=0.2	-0.1818	0.0203	0.0399	0.0259	0.0079	0.0129	-0.0076	-0.0125	0.0150	0.0130	0.0281	0.0130
n=30, m=30, S=0.2	0.0065	0.0269	0.0206	0.0409	0.0103	0.0103	-0.0102	-0.0110	0.0103	0.0211	0.0103	0.0203
n=30, m=25, S=0.2	0.0480	0.0891	0.0264	0.0450	0.0121	0.0186	-0.0120	-0.0185	0.0138	0.0244	0.0127	0.0210
n=30, m=20, S=0.2	-0.1038	-0.0157	0.0502	0.0429	0.0201	0.0221	-0.0195	-0.0206	0.0229	0.0206	0.0281	0.0228
n=30, m=15, S=0.2	-0.1553	-0.1185	0.0899	0.0976	0.0341	0.0426	-0.0319	-0.0407	0.0392	0.0437	0.0533	0.0563
n=15, m=15, S=0.2	-0.1103	0.0076	0.0669	0.0664	0.0286	0.0355	-0.0261	-0.0313	0.0296	0.0321	0.0388	0.0359
n=15, m=12, S=0.2	-0.1147	0.0149	0.0814	0.0690	0.0348	0.0365	-0.0336	-0.0328	0.0368	0.0339	0.0465	0.0368
n=15, m=10, S=0.2	-0.1636	-0.0467	0.1055	0.0775	0.0399	0.0388	-0.0387	-0.0366	0.0462	0.0366	0.0620	0.0426
n=15, m=7, S=0.2	-0.1570	-0.0917	0.1355	0.1877	0.0585	0.0997	-0.0532	-0.0819	0.0584	0.0819	0.0835	0.1193

Table 7.3: Average values of the biases and Risks for the Bayesian estimates of parameters (γ, δ) = (1.5, 2) under different censoring schemes.

γ=1.5, δ=2	SE				LL							
	Bias		Risk		Bias				Risk			
					v=-1		v=1		v=-1		v=1	
	γ	δ	γ	δ	γ	δ	γ	δ	γ	δ	γ	δ
n=100, m=100, S=0.1	-0.3000	-0.2028	0.1100	0.0931	0.0100	0.0265	-0.0100	-0.0255	0.0483	0.0404	0.0635	0.0549
n=100, m=90, S=0.1	-0.2921	-0.2416	0.1045	0.1167	0.0098	0.0297	-0.0095	-0.0287	0.0459	0.0499	0.0603	0.0700
n=100, m=85, S=0.1	-0.2543	-0.1387	0.0880	0.0796	0.0116	0.0303	-0.0117	-0.0300	0.0389	0.0357	0.0503	0.0455
n=100, m=80, S=0.1	-0.2664	-0.1790	0.0962	0.0828	0.0128	0.0265	-0.0124	-0.0245	0.0421	0.0358	0.0556	0.0491
n=60, m=60, S=0.1	-0.2141	-0.1919	0.0733	0.1100	0.0138	0.0381	-0.0136	-0.0353	0.0325	0.0470	0.0419	0.0667
n=60, m=55, S=0.1	-0.2368	-0.2132	0.0895	0.1295	0.0170	0.0446	-0.0165	-0.0399	0.0391	0.0541	0.0521	0.0809
n=60, m=50, S=0.1	-0.2476	-0.2585	0.1019	0.1762	0.0208	0.0573	-0.0197	-0.0526	0.0438	0.0724	0.0603	0.1128
n=60, m=45, S=0.1	-0.2355	-0.2172	0.0947	0.1581	0.0200	0.0594	-0.0192	-0.0524	0.0410	0.0653	0.0556	0.1015
n=30, m=30, S=0.1	0.0164	0.1118	0.0410	0.1105	0.0209	0.0507	-0.0199	-0.0475	0.0205	0.0608	0.0209	0.0526
n=30, m=25, S=0.1	-0.0041	0.1723	0.0545	0.1293	0.0277	0.0510	-0.0267	-0.0485	0.0270	0.0748	0.0283	0.0581
n=30, m=20, S=0.1	-0.0830	-0.0682	0.0763	0.2182	0.0363	0.1183	-0.0331	-0.0975	0.0344	0.0980	0.0437	0.1368
n=30, m=15, S=0.1	-0.1491	-0.2686	0.0986	0.4535	0.0388	0.2268	-0.0375	-0.1657	0.0435	0.1709	0.0576	0.3726
n=15, m=15, S=0.1	-0.1570	-0.0601	0.1141	0.2312	0.0463	0.1195	-0.0433	-0.1079	0.0495	0.1091	0.0684	0.1367
n=15, m=12, S=0.1	-0.0868	0.0384	0.1218	0.2507	0.0584	0.1324	-0.0560	-0.1168	0.0564	0.1294	0.0694	0.1369
n=15, m=10, S=0.1	-0.1774	-0.1020	0.1865	0.3356	0.0838	0.1788	-0.0725	-0.1489	0.0778	0.1500	0.1211	0.2222
n=15, m=7, S=0.1	-0.2125	-0.3120	0.2017	0.6883	0.0800	0.3317	-0.0759	-0.2579	0.0849	0.2593	0.1273	0.5915
n=100, m=100, S=0.2	-0.3000	-0.2028	0.1100	0.0931	0.0100	0.0265	-0.0100	-0.0255	0.0483	0.0404	0.0635	0.0549
n=100, m=90, S=0.2	-0.3016	-0.2259	0.1107	0.1039	0.0099	0.0268	-0.0098	-0.0262	0.0485	0.0449	0.0639	0.0616
n=100, m=85, S=0.2	-0.2929	-0.2453	0.1165	0.1166	0.0106	0.0280	-0.0101	-0.0283	0.0466	0.0502	0.0617	0.0690
n=100, m=80, S=0.2	-0.3172	-0.2655	0.1253	0.1282	0.0124	0.0295	-0.0123	-0.0282	0.0544	0.0543	0.0733	0.0776
n=60, m=60, S=0.2	-0.2141	-0.1919	0.0733	0.1100	0.0138	0.0381	-0.0136	-0.0353	0.0325	0.0470	0.0419	0.0667
n=60, m=55, S=0.2	-0.2224	-0.2001	0.0805	0.1198	0.0156	0.0416	-0.0155	-0.0384	0.0355	0.0508	0.0463	0.0733
n=60, m=50, S=0.2	-0.2549	-0.2446	0.1045	0.1472	0.0208	0.0462	-0.0189	-0.0416	0.0447	0.0609	0.0625	0.0929
n=60, m=45, S=0.2	-0.2654	-0.2651	0.1122	0.1742	0.0213	0.0535	-0.0204	-0.0507	0.0481	0.0721	0.0666	0.1101
n=30, m=30, S=0.2	0.0164	0.1118	0.0410	0.1105	0.0209	0.0507	-0.0199	-0.0475	0.0205	0.0608	0.0209	0.0526
n=30, m=25, S=0.2	0.0215	0.1535	0.0479	0.1240	0.0244	0.0517	-0.0230	-0.0486	0.0240	0.0705	0.0244	0.0567
n=30, m=20, S=0.2	-0.0673	0.0020	0.0682	0.1549	0.0334	0.0807	-0.0304	-0.0741	0.0311	0.0771	0.0387	0.0838
n=30, m=15, S=0.2	-0.0398	0.0052	0.0717	0.2535	0.0361	0.1389	-0.0342	-0.1153	0.0342	0.1229	0.0390	0.1482
n=15, m=15, S=0.2	-0.1570	-0.0601	0.1141	0.2312	0.0463	0.1195	-0.0433	-0.1079	0.0495	0.1091	0.0684	0.1367
n=15, m=12, S=0.2	-0.0064	0.1491	0.0910	0.2526	0.0457	0.1221	-0.0452	-0.1086	0.0460	0.1449	0.0471	0.1225
n=15, m=10, S=0.2	-0.1921	-0.1076	0.1435	0.2756	0.0635	0.1424	-0.0549	-0.1224	0.0639	0.1225	0.0892	0.1764
n=15, m=7, S=0.2	-0.0583	0.0916	0.1439	0.3993	0.0715	0.2239	-0.0688	-0.1702	0.0689	0.2077	0.0903	0.2330

Table 7.4: Confidence length and coverage probability for the parameters (γ, δ) under different censoring schemes.

	γ=1, δ=1				γ=1.5, δ=2			
	Bayesian HPD		ACI		Bayesian HPD		ACI	
	ACL	CP	ACL	CP	ACL	CP	ACL	CP
n=100, m=100, S=0.1	(0.3180, 0.3563)	(0.951, 0.952)	(0.3099, 0.4159)	(0.952, 0.96)	(0.5288, 0.8520)	(0.952, 0.952)	(0.4649, 0.8187)	(0.952, 0.957)
n=100, m=90, S=0.1	(0.3688, 0.4162)	(0.952, 0.951)	(0.3442, 0.4242)	(0.946, 0.954)	(0.5515, 0.9159)	(0.951, 0.951)	(0.5068, 0.8822)	(0.95, 0.947)
n=100, m=85, S=0.1	(0.3686, 0.4303)	(0.953, 0.953)	(0.3601, 0.4315)	(0.954, 0.956)	(0.5738, 0.9019)	(0.952, 0.951)	(0.5270, 0.9228)	(0.947, 0.957)
n=100, m=80, S=0.1	(0.3377, 0.4366)	(0.952, 0.951)	(0.3758, 0.4410)	(0.952, 0.955)	(0.5815, 0.8816)	(0.951, 0.954)	(0.5471, 0.9688)	(0.95, 0.956)
n=60, m=60, S=0.1	(0.3242, 0.4537)	(0.957, 0.952)	(0.4038, 0.5375)	(0.954, 0.948)	(0.6451, 1.0304)	(0.951, 0.953)	(0.6056, 1.0671)	(0.954, 0.959)
n=60, m=55, S=0.1	(0.4955, 0.5992)	(0.953, 0.955)	(0.4425, 0.5468)	(0.954, 0.953)	(0.7269, 1.0754)	(0.952, 0.951)	(0.6526, 1.1389)	(0.955, 0.956)
n=60, m=50, S=0.1	(0.4951, 0.6076)	(0.954, 0.954)	(0.4782, 0.5642)	(0.951, 0.952)	(0.7693, 1.3370)	(0.951, 0.952)	(0.6979, 1.2349)	(0.959, 0.962)
n=60, m=45, S=0.1	(0.5466, 0.6179)	(0.954, 0.951)	(0.5151, 0.5921)	(0.956, 0.961)	(0.7519, 1.2934)	(0.951, 0.951)	(0.7445, 1.3555)	(0.955, 0.966)
n=30, m=30, S=0.1	(0.5174, 0.7871)	(0.953, 0.951)	(0.5891, 0.7750)	(0.947, 0.943)	(0.8079, 1.1309)	(0.951, 0.951)	(0.8836, 1.5953)	(0.947, 0.966)
n=30, m=25, S=0.1	(0.5806, 0.6939)	(0.951, 0.953)	(0.6969, 0.8267)	(0.957, 0.952)	(0.8977, 1.1974)	(0.951, 0.951)	(1.0151, 1.8826)	(0.958, 0.963)
n=30, m=20, S=0.1	(0.8191, 0.8541)	(0.952, 0.953)	(0.8210, 0.9678)	(0.953, 0.951)	(0.9861, 1.7118)	(0.951, 0.951)	(1.1686, 2.4091)	(0.959, 0.968)
n=30, m=15, S=0.1	(1.0720, 1.5067)	(0.951, 0.951)	(0.9854, 1.3004)	(0.948, 0.965)	(1.0137, 2.3050)	(0.951, 0.951)	(1.3746, 3.4663)	(0.954, 0.958)
n=15, m=15, S=0.1	(0.8566, 0.9960)	(0.954, 0.951)	(0.8816, 1.1293)	(0.945, 0.934)	(1.0880, 1.8097)	(0.953, 0.951)	(1.3225, 2.5000)	(0.945, 0.965)
n=15, m=12, S=0.1	(0.9598, 1.0907)	(0.951, 0.952)	(1.0852, 1.2963)	(0.959, 0.94)	(1.3420, 1.7570)	(0.951, 0.951)	(1.5635, 3.2631)	(0.95, 0.962)
n=15, m=10, S=0.1	(1.1078, 1.1124)	(0.951, 0.951)	(1.2523, 1.5622)	(0.961, 0.947)	(1.4652, 2.1629)	(0.951, 0.951)	(1.7649, 4.2863)	(0.952, 0.959)
n=15, m=7, S=0.1	(1.3373, 1.8838)	(0.951, 0.951)	(1.6139, 2.7710)	(0.959, 0.963)	(1.4974, 2.8849)	(0.951, 0.951)	(2.1922, 7.8173)	(0.958, 0.969)

Table 7.5: Confidence length and coverage probability for the parameters (γ,δ) under different censoring schemes.

	$\gamma = 1$, $\delta = 1$				$\gamma = 1.5$, $\delta = 2$			
	Bayesian HPD		ACI		Bayesian HPD		ACI	
	ACL	CP	ACL	CP	ACL	CP	ACL	CP
n=100, m=100, S=0.2	(0.3179, 0.3562)	(0.951, 0.952)	(0.3099, 0.4158)	(0.952, 0.96)	(0.5288, 0.8519)	(0.952, 0.952)	(0.4648, 0.8187)	(0.952, 0.957)
n=100, m=90, S=0.2	(0.3660, 0.4885)	(0.951, 0.952)	(0.3411, 0.4229)	(0.948, 0.954)	(0.5274, 0.9306)	(0.956, 0.952)	(0.4953, 0.8611)	(0.947, 0.952)
n=100, m=85, S=0.2	(0.3942, 0.4954)	(0.952, 0.954)	(0.3553, 0.4288)	(0.953, 0.953)	(0.5649, 0.9012)	(0.956, 0.951)	(0.5108, 0.8890)	(0.951, 0.951)
n=100, m=80, S=0.2	(0.4006, 0.4243)	(0.951, 0.952)	(0.3695, 0.4368)	(0.951, 0.953)	(0.6132, 0.9005)	(0.951, 0.952)	(0.5261, 0.9200)	(0.947, 0.95)
n=60, m=60, S=0.2	(0.3241, 0.4537)	(0.957, 0.952)	(0.4037, 0.5375)	(0.954, 0.948)	(0.6451, 1.0303)	(0.951, 0.953)	(0.6056, 1.0670)	(0.954, 0.959)
n=60, m=55, S=0.2	(0.3853, 0.5344)	(0.952, 0.954)	(0.4389, 0.5454)	(0.953, 0.954)	(0.6768, 1.1031)	(0.954, 0.954)	(0.6394, 1.1143)	(0.956, 0.961)
n=60, m=50, S=0.2	(0.5138, 0.6211)	(0.951, 0.953)	(0.4709, 0.5598)	(0.96, 0.95)	(0.7775, 1.1386)	(0.951, 0.953)	(0.6738, 1.1795)	(0.955, 0.96)
n=60, m=45, S=0.2	(0.4386, 0.6178)	(0.952, 0.953)	(0.5034, 0.5818)	(0.952, 0.956)	(0.7577, 1.2459)	(0.951, 0.951)	(0.7089, 1.2593)	(0.957, 0.962)
n=30, m=30, S=0.2	(0.5174, 0.7870)	(0.953, 0.951)	(0.5890, 0.7749)	(0.947, 0.943)	(0.8078, 1.1308)	(0.951, 0.951)	(0.8836, 1.5953)	(0.947, 0.966)
n=30, m=25, S=0.2	(0.5693, 0.7666)	(0.951, 0.951)	(0.6851, 0.8172)	(0.959, 0.951)	(0.7871, 1.2227)	(0.953, 0.951)	(0.9777, 1.7784)	(0.953, 0.963)
n=30, m=20, S=0.2	(0.8479, 0.7257)	(0.951, 0.951)	(0.7903, 0.9183)	(0.954, 0.949)	(0.8870, 1.4659)	(0.951, 0.953)	(1.0878, 2.0885)	(0.961, 0.97)
n=30, m=15, S=0.2	(0.9756, 1.0953)	(0.953, 0.951)	(0.9230, 1.1366)	(0.959, 0.951)	(1.0329, 1.8774)	(0.951, 0.951)	(1.2294, 2.6186)	(0.953, 0.959)
n=15, m=15, S=0.2	(0.8566, 0.9959)	(0.954, 0.951)	(0.8816, 1.1293)	(0.945, 0.934)	(1.0879, 1.8096)	(0.953, 0.951)	(1.3224, 2.5000)	(0.945, 0.965)
n=15, m=12, S=0.2	(0.9949, 0.9821)	(0.951, 0.951)	(1.0575, 1.2549)	(0.958, 0.939)	(1.1353, 1.8291)	(0.951, 0.951)	(1.4842, 2.9176)	(0.947, 0.964)
n=15, m=10, S=0.2	(1.0483, 1.0680)	(0.952, 0.952)	(1.1943, 1.4344)	(0.955, 0.944)	(1.3200, 1.9196)	(0.951, 0.952)	(1.6201, 3.4056)	(0.959, 0.966)
n=15, m=7, S=0.2	(1.2669, 1.5338)	(0.952, 0.951)	(1.4606, 1.9870)	(0.964, 0.959)	(1.4345, 2.2919)	(0.952, 0.951)	(1.8905, 4.6995)	(0.955, 0.962)

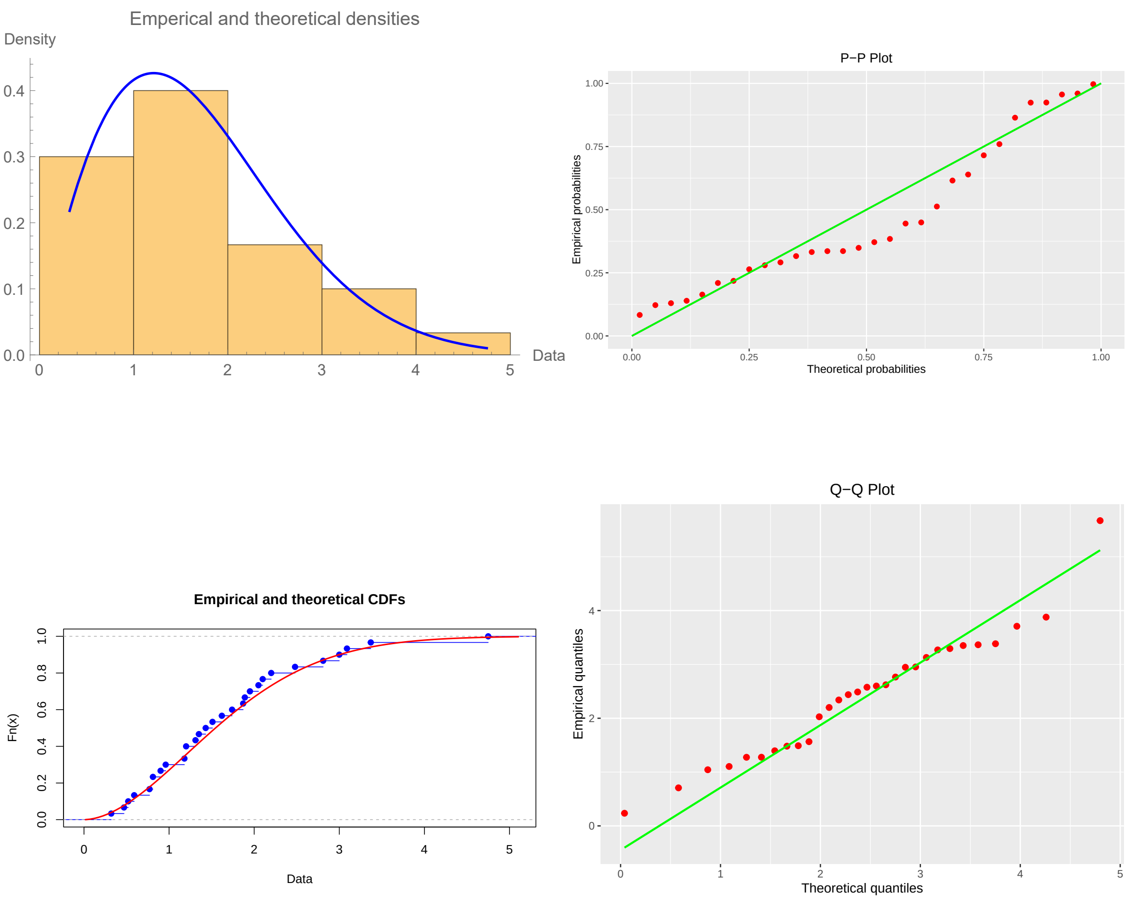
Table 7.6: Data of 30 successive values of March precipitation

0.77	1.74	0.81	1.20	1.95	1.20	0.47	1.43	3.37	2.20
3.00	3.09	1.51	2.10	0.52	1.62	1.31	0.32	0.59	0.81
2.81	1.87	1.18	1.35	4.75	2.48	0.96	1.89	0.90	2.05

Table 7.7: The goodness-of-fit statistics for the data set

Distribution	Estimates	KS	p-value	AIC	AICC	BIC	HQC	NLL
Weibull	1.8089 & 0.3155	0.0689	0.9988	81.2866	81.7310	84.0890	82.1831	77.2866
Lindley	0.9096	0.1882	0.2383	88.2875	88.4303	89.6886	88.7357	86.2875
IW	1.0162 & 1.5496	0.1524	0.4893	154.9602	155.4046	157.7626	155.8567	150.9602

Figure 7.2: Probability plots from the fitted Weibull distribution



created artificial data sets from the real data for the schemes, $(m, S) = (20, 1), (20, 2), (15, 1)$ and $(15, 2)$. For each sample, MLE, the associated ACI, the Bayes estimates under the squared error loss function, and the HPD credible interval are calculated and are provided in the Table 7.8. Bayes estimates are calculated with the assumption that all hyperparameters are zeros. For each censoring scheme, MLE and Bayes estimates are nearly, but not quite, identical. In the case of confidence intervals, the HPD credible interval is shorter than the ACI. For a fixed value of m, interval length decreases with supplymentary time S. The interval length of the scale parameter is shorter than that of the shape parameter.

Table 7.8: Estimate values under different censoring schemes

Scheme	**MLE**		**Bayesian**	
(m, S)	**Estimates**	**ACI**	**Estimates**	**HPD**
(20,1)	$\hat{\gamma}$=1.8461	(1.2618, 2.4304)	$\hat{\gamma}$=1.9131	(1.4346, 2.4902)
	$\hat{\delta}$=0.3099	(0.1307, 0.4891)	$\hat{\delta}$=0.2989	(0.1677, 0.5092)
(20,2)	$\hat{\gamma}$=1.8534	(1.3243, 2.3826)	$\hat{\gamma}$=1.7649	(1.3577, 2.2958)
	$\hat{\delta}$=0.3105	(0.1327, 0.4882)	$\hat{\delta}$=0.3491	(0.1813, 0.5321)
(15,1)	$\hat{\gamma}$=1.9386	(1.2786, 2.5985)	$\hat{\gamma}$=1.887	(1.2267, 2.4044)
	$\hat{\delta}$=0.3042	(0.1259, 0.4825)	$\hat{\delta}$= 0.3269	(0.1614, 0.4939)
(15,2)	$\hat{\gamma}$=1.9174	(1.3644, 2.4704)	$\hat{\gamma}$=1.8477	(1.3800, 2.3001)
	$\hat{\delta}$=0.3031	(0.1272, 0.4791)	$\hat{\delta}$=0.3325	(0.2195, 0.5441)

7.9 Summary

In this chapter, we have introduced the T1-T2 mixture censoring scheme and examined its statistical inference based on the Weibull distribution. The new scheme guarantees a maximum possible number of failures within a given supplementary time. The computational formulas for the expected number of failures and the expected failure time are derived. The Fisher information for the new censoring scheme is presented. The performances of MLE and Bayesian estimates for squared error and LINEX loss function, demonstrate their effectiveness in providing reliable information. A real data is used to illustrate the functionality of the T1-T2 mixture censoring scheme.

CHAPTER 8

Cure fraction modelling Using Exponentiated Weibull Distribution

8.1 Introduction

Survival analysis is a branch of statistics used to study the time it takes for an event to occur, such as the recurrence of an illness or the lifespan of equipment. Imagine tracking cancer patients after chemotherapy to see how long they remain in remission, or studying heart disease patients to understand how long they live after a specific treatment. This type of analysis helps us understand patterns in 'time-to-event' data and make predictions even when we don't have complete information about every case (for example, if some patients are still healthy at the end of the study).

It includes methods like the Kaplan-Meier estimator, which shows survival over time, and the Cox model, which examines how factors like age or treatment affect the likelihood of an event. In some studies, a 'cure fraction' is considered, representing the portion of a group that may never experience the event, such as cancer patients who are considered fully cured. Survival analysis plays a critical role in fields like medicine, engineering, and social sciences, providing insights that guide decision-making and improve outcomes. The survival function, $S(x)$, is a key concept in survival analysis that represents the probability of 'surviving' beyond a specific time x without experiencing the event of interest, such as death, relapse, or equipment failure. Mathematically, it is defined as: $S(x) = P(X > x)$ where X is the random variable representing the time to the event.

Among the numerous applications of survival analysis, cancer research plays a vital role in understanding patient outcomes and improving treatment strategies. Colorectal cancer, one of the most commonly diagnosed cancers worldwide, poses a significant threat to public health. Its survival rates vary depending on factors such as patient demographics, treatment methods, and the stage at diagnosis. For clinicians and policymakers, survival analysis is essential for identifying these factors and estimating the probabilities of survival or recurrence over time.

In this study, we analyze survival time data from the Malabar Cancer Centre (MCC), Thalassery, Kerala, which includes records of 188 colorectal cancer patients who registered for treatment at the center between January 2014 and December 2017. The follow-up data for these patients was retrieved up to August 2023, providing a median follow-up period of 3 years for the cohort. For each patient, key details such as the date of diagnosis, date of the last follow-up, and current status were retrieved from MCC's records. The data was collected systematically, following clear inclusion and exclusion criteria. The inclusion criteria consisted of colorectal cancer patients who had completed treatment at MCC. Patients were excluded if they had incomplete case records, meaning essential information such as diagnosis dates, or follow-up outcomes was missing or unavailable. Follow-up information was gathered up to the most recent available date, and survival time was calculated in months based on these records.

Some patients were still disease-free at the end of the study, suggesting they might be cured, while others had incomplete follow-up (right-censored data). To analyze these survival patterns, we use the exponentiated Weibull distribution-a flexible model that can handle the diverse survival trends observed in the data. This approach allows us to uncover important insights into the survival outcomes of colorectal cancer patients treated at MCC.

The presence of an immune or cured group in a dataset can be identified using a Kaplan-Meier plot of the survival function. If the curve shows a prolonged and stable plateau toward the right end, accompanied by substantial censoring at the extreme end of the plot, it is indicative of the presence of long-term survivors or cured individuals. Also, the TTT-plot is useful for identifying trends in the hazard function before fitting a model to the data. A straight line suggests a constant hazard rate (e.g., exponential distribution), while a convex shape indicates an increasing hazard rate, and a concave shape suggests a decreasing rate. An S-shape corresponds to a bathtub-shaped hazard function, whereas an inverse S-shape reflects an upside-down bathtub hazard. These tools are instrumental in selecting appropriate models for the data. Studies by Corbière et al. (2009) and Martinez et al. (2013) have highlighted this method as a reliable and effective way to assess cure fractions in survival data.

Commonly used parametric models in medical research, such as the Weibull, Gompertz, and log-normal distributions, may not always capture the complexity of survival data. To address this, several advanced models have been developed. Some extend the Weibull distribution, including the exponentiated Weibull by Mudholkar & Srivastava (1993), modified Weibull by Lai et al. (2003a), Weibull-Burr III by Yakubu & Doguwa (2017), and Weibull-Kumaraswamy by Ishaq et al. (2017) distributions. Others, like the generalized Lindley by Nadarajah et al. (2011), generalized X-exponential by Chacko & Deepthi (2019), and exponential power by Smith & Bain (1975) distributions, are independent of the Weibull family but offer greater flexibility for analyzing diverse survival data. These models enhance the ability to model complex lifetime patterns effectively.

Various parametric approaches have been used to model the proportion of immune (cured) individuals. For example, Peng et al. (1998) used the generalized F distribution and Kannan et al. (2010) used the exponentiated-exponential distribution. Other methods include the Burr XII distribution (Shao & Zhou (2004)), generalized modified Weibull distribution (Martinez et al. (2013)), Fréchet distribution (Ramos et al. (2017)), Nadarajah-Haghighi distribution (Usman et al. (2021)), and exponentiated weibull distribution (Martinez et al., (2022)), all of which have been used to model long-term survivors.

This chapter is organized as follows: Section 8.1 provides an introduction to the concept of the cure fraction and describes the colorectal cancer data used in the study. Section 8.2 explores the properties of the proposed model, focusing on the exponentiated Weibull distribution. Section 8.3 derives the expressions for MLE of the model parameters under complete data and the right-censored data. The three distinct cure fraction models are discussed in Section 8.4. Section 8.5

delves into Bayesian estimation techniques and discusses various model selection criteria. Finally, Section 8.6 presents the findings of the study, beginning with a simulation-based evaluation of the effectiveness of the MLEs under the cure fraction model. This is followed by the application of the model to colorectal cancer data, analyzed under different cure fraction models to provide comprehensive insights. Section 8.7 summarizes the chapter, consolidating the key results and conclusions.

8.2 The Exponentiated Weibull Distribution

The three-parameter exponentiated Weibull (EW) distribution is an extension of the Weibull distribution that introduces a second shape parameter. This enhancement allows the EW distribution to effectively model a variety of distribution shapes, including bathtub-shaped, unimodal, and distributions with different hazard rates. This flexibility makes it suitable for use with standard statistical methods.

The EW distribution is especially useful for modelling lifetime data in fields such as reliability, survival analysis, and population studies, as well as for handling extreme value data. It also facilitates the creation of isotonic tests for composite hypotheses related to exponentiality. As a generalization, the EW distribution encompasses several other distributions, including the Weibull, exponentiated Exponential, Burr-type X, Rayleigh, and exponential distributions as special cases. The EW distribution has been widely applied in various fields, such as flood data analysis, human mortality research, cancer survival analysis, excess-of-loss insurance, tree diameter prediction, and earthquake interevent time modelling.

Let the parameter vector of the EW distribution be denoted as $\Theta = (\alpha, \lambda, \theta)^T$. The PDF of the EW distribution, introduced by Mudholkar & Srivastava (1993), for a random variable X is given by

$$f(x;\Theta) = \frac{\alpha\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}\left(1-e^{-(x/\lambda)^{\theta}}\right)^{\alpha-1} e^{-(x/\lambda)^{\theta}};\ x>0, \alpha, \lambda, \theta > 0, \tag{8.2.1}$$

and the CDF is given by

$$F(x;\Theta) = \left(1-e^{-(x/\lambda)^{\theta}}\right)^{\alpha};\ x>0, \alpha, \lambda, \theta > 0. \tag{8.2.2}$$

These functions describe the behavior of the EW distribution, where α controls the additional shape parameter, λ is the scale parameter, and θ determines the shape of the Weibull component.

The EW distribution simplifies to several well-known distributions under specific parameter settings. When both $\alpha = 1$ and $\theta = 1$, it reduces to the Exponential distribution, which is the simplest case. If only $\alpha = 1$, the EW distribution becomes the standard Weibull distribution. Similarly, when $\theta = 1$, it reduces to the exponentiated exponential distribution, and when $\theta = 2$, it simplifies to the exponentiated Rayleigh distribution. These special cases highlight the flexibility of the EW distribution in modelling various types of data by adjusting its parameters.

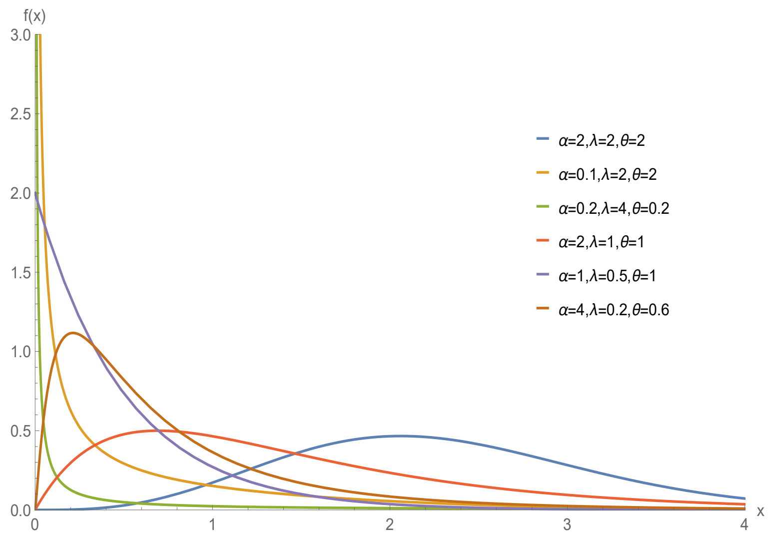


Figure 8.1: Plot of the density function of EW distribution for different parameter values

Pasari & Dikshit (2018) analyzed the behavior of the PDF of the EW distribution under various parameter settings. They noted that when $\theta\alpha \leq 1$, the density function is monotonically decreasing, whereas for $\theta\alpha > 1$, it exhibits a unimodal shape with the mode approximately located at

$$\lambda \left[\frac{2(\theta\alpha - 1)}{\theta(\alpha + 1)} \right]^{1/\theta}.$$

In cases where $\theta\alpha \leq 1$, the mode either occurs at zero (when $\theta < 1$, $\alpha > 1$, and $\theta\alpha \leq 1$) or the PDF does not exhibit a mode (when $\theta < 1$ and $\alpha < 1$). For a fixed value of θ, the EW density becomes increasingly symmetric as α grows larger. Additionally, Jiang and Murthy (1999) provided insights into the behavior of the

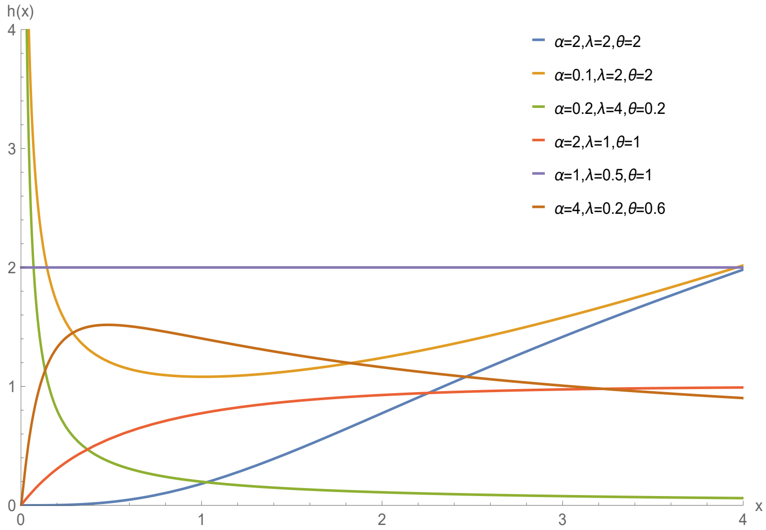


Figure 8.2: Plot of the failure rate function, $h(x)$, of the EW distribution for different parameter values

PDF at $x = 0$:

$$f(0) = \infty \quad \text{if} \quad \theta\alpha < 1, \quad f(0) = \frac{1}{\alpha} \quad \text{when} \quad \theta\alpha = 1, \quad \text{and} \quad f(0) = 0 \quad \text{if} \quad \theta\alpha > 1.$$

They also observed that the combined influence of θ and α on the EW density's shape resembles the effect of the shape parameter θ on the PDF of the two-parameter Weibull distribution.

The hazard rate function and survival function of the EW distribution are given by:

$$h(x;\Theta) = \frac{\alpha\theta x^{\theta-1}\left(1-e^{-(x/\lambda)^{\theta}}\right)^{\alpha-1} e^{-(x/\lambda)^{\theta}}}{\lambda^{\theta}\left(1-\left(1-e^{-(x/\lambda)^{\theta}}\right)^{\alpha}\right)} \tag{8.2.3}$$

and

$$S(x;\Theta) = 1-\left(1-e^{-(x/\lambda)^{\theta}}\right)^{\alpha}. \tag{8.2.4}$$

The EW distribution exhibits a wide variety of hazard rate shapes, including both monotone and non-monotone forms, depending on its shape parameters (θ and α). Specifically, the behavior of the hazard rate function $h(x)$ is influenced by these parameters as follows:

- If $\theta \le 1$ and $\alpha\theta \le 1$, the hazard rate is monotonically decreasing.
- If $\theta \ge 1$ and $\alpha\theta \ge 1$, the hazard rate is monotonically increasing.

- For $\theta < 1$ and $\alpha\theta > 1$, the hazard rate becomes unimodal.

- When $\theta > 1$ and $\alpha\theta < 1$, it exhibits a bathtub shape.

- If $\theta = 1$ and $\alpha = 1$, the hazard rate remains constant.

Figures 8.1 and 8.2 illustrate the diverse shapes of the PDF and hazard function of the EW distribution for various parameter combinations.

8.3 Maximum Likelihood Estimation

In this section, we focus on deriving the MLE for the parameters α, λ, and θ of the EW distribution. Let $x_1, x_2, ..., x_n$ denote the observed sample of size n drawn from the EW distribution. The likelihood function, derived from Eq. (8.2.1), is expressed as follows:

$$L(\Theta) = \prod_{i=1}^{n} \frac{\alpha\theta}{\lambda} \left(\frac{x_i}{\lambda}\right)^{\theta-1} \left(1 - e^{-(x_i/\lambda)^\theta}\right)^{\alpha-1} e^{-(x_i/\lambda)^\theta}. \tag{8.3.1}$$

The corresponding log-likelihood function is,

$$\begin{aligned}\log L(x;\Theta) = n\log\alpha + &n\log\theta - n\theta\log\lambda + (\theta-1)\sum_{i=1}^{n}\log x_i \\ &+ (\alpha-1)\sum_{i=1}^{n}\log\left[1 - e^{-(x_i/\lambda)^\theta}\right] - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}.\end{aligned} \tag{8.3.2}$$

To compute the MLEs, we first derive the log-likelihood function and calculate its derivatives with respect to the parameters α, λ, and θ. These derivatives are then equated to zero to find the parameter estimates. Since the MLEs cannot be derived in closed form, numerical methods are required to solve the resulting system of equations. The derivatives of the log-likelihood function with respect to the parameters are as follows:

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[1 - e^{-(x_i/\lambda)^\theta}\right] \\ \frac{\partial \log L}{\partial \lambda} &= \frac{\theta}{\lambda}\left[-n + (\alpha-1)\sum_{i=1}^{n}\frac{e^{-(x_i/\lambda)^\theta}}{(1 - e^{-(x_i/\lambda)^\theta})}\left(\frac{x_i}{\lambda}\right)^{\theta} - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}\right] \\ \frac{\partial \log L}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^{n}\log(x_i/\lambda) + (\alpha-1)\sum_{i=1}^{n}\frac{\log(x_i/\lambda)e^{-(x_i/\lambda)^\theta}}{\left(1 - e^{-(x_i/\lambda)^\theta}\right)}\left(\frac{x_i}{\lambda}\right)^{\theta} - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}\log\left(\frac{x_i}{\lambda}\right)\end{aligned}$$

R programming can be employed to perform these calculations, using optimization techniques such as the `maxLik()` function to obtain the parameter estimates by maximizing the likelihood.

To obtain the confidence intervals of α, λ and θ, one requires the observed information matrix:

$$I(\Theta) = -\begin{bmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \lambda} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \log L}{\partial \lambda \partial \alpha} & \frac{\partial^2 \log L}{\partial \lambda^2} & \frac{\partial^2 \log L}{\partial \lambda \partial \theta} \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta \partial \lambda} & \frac{\partial^2 \log L}{\partial \theta^2} \end{bmatrix}.$$

Since the MLEs are not in the explicit forms, we check out the asymptotic distribution of the estimates. As $n \to \infty$, $\hat{\Theta}$ converges normally with mean zero and variance $I^{-1}(\Theta)$, where $I^{-1}(\Theta)$ is the variance covariance matrix of Θ, the inverse of observed information matrix.

The asymptotic distribution of the MLE, $\hat{\Theta} = (\hat{\alpha}, \hat{\lambda}, \hat{\theta})$ of $\Theta = (\alpha, \lambda, \theta)$ is, $\sqrt{n}(\hat{\Theta} - \Theta) \to N(0, I^{-1}(\Theta))$. Thus, the asymptotic $100(1-\rho)\%$ confidence intervals of α, λ and θ are $\hat{\alpha} \pm Z_{\frac{\rho}{2}} \sqrt{Var(\hat{\alpha})}$, $\hat{\lambda} \pm Z_{\frac{\rho}{2}} \sqrt{Var(\hat{\lambda})}$ and $\hat{\theta} \pm Z_{\frac{\rho}{2}} \sqrt{Var(\hat{\theta})}$ respectively, where $Z_{\frac{\rho}{2}}$ is the upper $100(\frac{\rho}{2})^{th}$ percentile of standard normal distribution.

In many cases, some data points are right-censored, meaning we don't know the exact survival time for those individuals because they haven't experienced the event by the end of the study. In such cases, the likelihood function is adjusted to account for right-censoring, where the contribution of each censored individual is incorporated by considering the probability that the individual survives beyond the observed time. For a sample of n individuals, the contribution of the i-th individual to the likelihood function, considering the presence of right-censoring, is expressed as:

$$L_i = [f(x_i)]^{C_i} [S(x_i)]^{1-C_i},$$

where, x_i represents the survival time for the i-th patient, while C_i is a binary indicator variable defined as follows:

$$C_i = \begin{cases} 1 & \text{if the event of interest (e.g., death) is observed for the } i\text{-th patient,} \\ 0 & \text{if the survival time is censored (i.e., the event is not observed).} \end{cases}$$

Assume $f(x_i)$ is the density defined in Eq. (8.2.1) and $S(x_i)$ has the form given in Eq. (8.2.4); then, the likelihood under right censoring can be written as:

$$L(\Theta) = \prod_{i=1}^{n} \left[\frac{\alpha\theta}{\lambda} \left(\frac{x_i}{\lambda}\right)^{\theta-1} \left(1 - e^{-(x_i/\lambda)^\theta}\right)^{\alpha-1} e^{-(x_i/\lambda)^\theta} \right]^{C_i} \left[1 - \left(1 - e^{-(x_i/\lambda)^\theta}\right)^{\alpha}\right]^{1-C_i}, \tag{8.3.3}$$

the corresponding log-likelihood function is,

$$\begin{aligned} \log L(\Theta) = {} & \log\alpha \sum_{i=1}^{n} C_i + \log\theta \sum_{i=1}^{n} C_i - \theta\log\lambda \sum_{i=1}^{n} C_i + (\theta-1)\sum_{i=1}^{n} C_i \log x_i - \frac{1}{\lambda^\theta}\sum_{i=1}^{n} C_i x_i^\theta \\ & + (\alpha-1)\sum_{i=1}^{n} C_i \log\left[1 - e^{-(x_i/\lambda)^\theta}\right] + \sum_{i=1}^{n}(1-C_i)\log\left[1 - \left(1 - e^{-(x_i/\lambda)^\theta}\right)^{\alpha}\right]. \end{aligned}$$

To estimate the parameters under right-censoring using the ML method, we begin by deriving the log-likelihood function, $\log L(x, \Theta)$, with respect to the three parameters α, λ, and θ. The resulting equations are then set to zero to determine the MLEs. However, these estimators cannot be expressed in closed form due to the complexity of the equations. Consequently, numerical methods are employed to solve the system of equations.

8.4 Cure Fraction Modelling

Cure fraction models are a class of statistical approaches designed to analyze survival data where a portion of the population is considered immune to the event of interest, often referred to as long-term survivors. These models are particularly relevant in medical research, where certain treatments can result in a cure for some individuals. Several types of cure fraction models have been developed, each addressing different assumptions and data characteristics.

8.4.1 The Mixture Cure Model

The mixture cure model (MCM), introduced by Boag (1949) and Berkson & Gage (1952), is designed to analyze survival data that include two distinct groups: individuals susceptible to the event of interest and long-term survivors who are immune. Often referred to as the standard cure model, it is particularly effective for datasets with long-term survivors. This model assumes the population comprises two subgroups: susceptible individuals who may experience the event with probability $1 - \delta$ and non-susceptible individuals (long-term survivors) who will not, with probability δ. The survival function in the MCM is expressed as

$$S(x) = \delta + (1 - \delta) S_B(x),$$

where the parameter δ denotes the fraction of individuals classified as 'long-term survivors' with respect to the event of interest ($0 < \delta < 1$), and $S_B(x)$ represents the survival function of the baseline model for the susceptible subjects, which is assumed to follow EW distribution with parameters α, λ, and θ, i.e., $S_B(x) \sim EW(\alpha, \lambda, \theta)$. Clearly, as x approaches infinity, the survival function $S(x)$ converges to δ.
The PDF for the lifetime X can be defined as

$$f(x) = \frac{dF(x)}{dx} = (1-\delta) f_B(x),$$

where $f_B(x)$ is the baseline density of susceptible subjects and $F(x) = 1 - S(x)$.

Consider a random sample of size n, denoted by (x_i, C_i) for $i = 1, 2, \ldots, n$, obtained from a cancer dataset. Here, x_i represents the survival time for the i-th patient and C_i is a censoring indicator variable. Thus, the likelihood component corresponding to the i-th cancer patient is expressed as:

$$L_i = [f(x_i)]^{C_i} [S(x_i)]^{1-C_i} = [(1-\delta) f_B(x_i)]^{C_i} [\delta + (1-\delta) S_B(x_i)]^{1-C_i}.$$

The likelihood function for the MCM, using the EW distribution as the baseline model, is given by:

$$\begin{aligned} L(\Theta) = \prod_{i=1}^{n} [f(x_i)]^{C_i} [S(x_i)]^{1-C_i} &= \prod_{i=1}^{n} [(1-\delta) f_B(x_i)]^{C_i} [\delta + (1-\delta) S_B(x_i)]^{1-C_i} \\ &= \prod_{i=1}^{n} \left[(1-\delta) \frac{\alpha\theta}{\lambda} \left(\frac{x_i}{\lambda} \right)^{\theta-1} \left(1 - e^{-(x_i/\lambda)^\theta} \right)^{\alpha-1} e^{-(x_i/\lambda)^\theta} \right]^{C_i} \times \\ &\qquad \left[\delta + (1-\delta) \left[1 - \left(1 - e^{-(x_i/\lambda)^\theta} \right)^{\alpha} \right] \right]^{1-C_i}. \end{aligned}$$

The log-likelihood function of $L(\Theta)$ is differentiated with respect to the parameters α, λ, θ, and δ, and the resulting equations are set to zero to derive the estimating equations. As these equations are non-linear, they cannot be solved analytically and require numerical methods for their solution.

8.4.2 The Non-mixture Cure Model

The non-mixture cure model (NMC) introduced by Yakovlev et al. (1993), offers an alternative to the standard MCM by incorporating an asymptotic property for the cumulative hazard function, which naturally accounts for the cure fraction. Unlike the MCM, which explicitly separates the population into susceptible and cured individuals, the NMC assumes that all individuals have a latent potential for survival. The cure fraction is implicitly incorporated by allowing the survival probability to asymptotically approach a non-zero limit as time progresses toward infinity. NMC is also called bounded cumulative hazard models and promotion time cure models (Tsodikov et al. (2003)). In this approach, the survival function is represented as:

$$S(x) = \delta^{F_B(x)} = e^{\log(\delta) F_B(x)},$$

where δ is a cure parameter within the range $(0, 1)$, and $F_B(x)$ denotes the failure function of the susceptible individuals.
The PDF for the lifetime X is given by

$$f(x) = \frac{dF(x)}{dx} = -\log(\delta) f_B(x) e^{\log(\delta) F_B(x)},$$

where $F(x) = 1 - S(x)$, and the corresponding hazard function $h(x)$ can be written as

$$h(x) = \frac{f(x)}{S(x)} = -\log(\delta) f_B(x).$$

The likelihood function for a sample of n individuals, where the contribution of each individual is represented by the observed data in the NMC, is expressed as follows:

$$L(\Theta) = \prod_{i=1}^{n} [h(x_i)]^{C_i} S(x_i) = \prod_{i=1}^{n} [-\log(\delta) f_B(x_i)]^{C_i} \, e^{\log(\delta) F_B(x_i)} \tag{8.4.1}$$

where C_i is an indicator variable that takes the value 1 if the individual's data is uncensored and 0 if the data is censored. Expanding the terms for the baseline PDF $f_B(x_i)$ and the CDF $F_B(x_i)$, we get:

$$L(\Theta) = \prod_{i=1}^{n} \left[-\log(\delta) \frac{\alpha\theta}{\lambda} \left(\frac{x_i}{\lambda} \right)^{\theta-1} \left(1 - e^{-(x_i/\lambda)^{\theta}} \right)^{\alpha-1} e^{-(x_i/\lambda)^{\theta}} \right]^{C_i} e^{\log(\delta) \left(1 - e^{-(x_i/\lambda)^{\theta}} \right)^{\alpha}}. \tag{8.4.2}$$

8.4.3 Cure Rate Proportional Odds Model

The Cure Rate Proportional Odds Model (CPO), introduced by Gu et al. (2011), incorporates the concept of a cured fraction while allowing for a more flexible approach to hazard ratios over time. Unlike the well-established Cox proportional hazards model, where hazard ratios remain constant, the hazard ratio in the CPO does not converge to one at infinity. This time-dependent behavior of the hazard ratio makes the CPO an essential tool for analyzing survival data with varying risks over time, particularly in the presence of long-term survivors.

The survival function in the CPO model is expressed as

$$S(x) = \frac{1}{1 + \left(\frac{1}{\delta} - 1\right) F_B(x)}, \tag{8.4.3}$$

where $F_B(x) = 1 - S_B(x)$.

Under CPO model, the PDF and hazard function associated with Eq. 8.4.3 are respectively, by

$$f(x) = \frac{\left(\frac{1}{\delta} - 1\right) f_B(x)}{\left[1 + \left(\frac{1}{\delta} - 1\right) F_B(x)\right]^2} \quad \text{and} \quad h(x) = \frac{\left(\frac{1}{\delta} - 1\right) f_B(x)}{1 + \left(\frac{1}{\delta} - 1\right) F_B(x)}.$$

The likelihood function for a sample of n individuals, where the contribution of each individual is represented by the observed data in the CPO model, is expressed as follows:

$$L(\Theta) = \prod_{i=1}^{n} [h(x_i)]^{C_i} S(x_i) = \prod_{i=1}^{n} \frac{\left[(\frac{1}{\delta} - 1) f_B(x_i)\right]^{C_i}}{\left[1 + (\frac{1}{\delta} - 1) F_B(x_i)\right]^{1+C_i}}.$$

Expanding the terms for the baseline PDF $f_B(x_i)$ and the CDF $F_B(x_i)$, we get:

$$L(\Theta) = \prod_{i=1}^{n} \frac{\left[(\frac{1}{\delta} - 1) \frac{\alpha\theta}{\lambda} \left(\frac{x_i}{\lambda}\right)^{\theta-1} \left(1 - e^{-(x_i/\lambda)^\theta}\right)^{\alpha-1} e^{-(x_i/\lambda)^\theta}\right]^{C_i}}{\left[1 + (\frac{1}{\delta} - 1) \left(1 - e^{-(x_i/\lambda)^\theta}\right)^{\alpha}\right]^{1+C_i}}.$$

The cure fraction for each model can be expressed as $\delta = \lim_{x\to\infty} S(x)$, representing the survival probability at infinite time.

8.5 Bayesian Inference

In Bayesian inference, the joint posterior distribution of model parameters is derived by combining the information from the prior distributions of these parameters with the likelihood function derived from observed data. This approach allows us to update prior beliefs about the parameters based on the evidence provided by the data. For a model based on the EW distribution, which incorporates a cure fraction, the parameters of interest are α, λ, θ, and δ. These parameters are subject to the constraints $\alpha > 0$, $\lambda > 0$, $\theta > 0$, and $0 < \delta < 1$, ensuring their validity within the model's probabilistic structure.

To account for these constraints and incorporate prior knowledge, independent prior distributions are assigned to each parameter: α, λ, and θ follow Gamma distributions, while δ follows a Beta distribution. Specifically, the priors are defined as: $\alpha \sim \text{Gamma}(a_1, b_1)$, $\lambda \sim \text{Gamma}(a_2, b_2)$, $\theta \sim \text{Gamma}(a_3, b_3)$, $\delta \sim \text{Beta}(a_4, b_4)$, where a_i and b_i $(i = 1, 2, \ldots, 4)$ are hyperparameters that reflect the shape and scale of the respective distributions. These priors incorporate prior knowledge or assumptions about the parameters' behavior before observing the data. The Gamma distribution has a mean of a/b and variance a/b^2, while the Beta distribution has a mean of $a/(a+b)$ and variance $\frac{ab}{(a+b)^2(a+b+1)}$.

To sample from the joint posterior distribution of the parameters, MCMC methods were employed using the package MCMCpack in R. MCMC is a powerful computational approach for approximating posterior distributions, especially when analytical solutions are intractable. A total of 1,010,000 samples were generated for each parameter, with the first 10,000 iterations discarded as a burn-in period to reduce the influence of initial values and allow the Markov chain to converge to its stationary distribution. The posterior mean, which serves as the Bayesian estimate under a squared error loss function, was calculated from the remaining samples. Furthermore, 95% HPD intervals were constructed to provide the narrowest intervals containing 95% of the posterior probability.

Convergence of the MCMC chains was evaluated using Geweke's convergence diagnostic, which tests for convergence by comparing the means of samples from the early and late portions of the chain. The diagnostic computes a Z-score, reflecting the difference in these means normalized by their pooled standard deviation. Under stationarity, this Z-score follows a standard normal distribution, with values near zero indicating convergence. The Geweke diagnostic was implemented using the `geweke.diag` function from the coda package in R, ensuring a robust and quantitative measure of chain convergence.

The logarithm of the pseudo-marginal likelihood (LPML) is often used as a model diagnostic and selection criterion in bayesian inference (Gelfand & Dey (1994)). The LPML is derived from the conditional predictive ordinates (CPO), which measure the predictive performance of a model by evaluating its ability to predict each observation based on the rest of the data (Geisser & Eddy (1979)). Specifically, the CPO for the i-th observation is given by

$$\text{CPO}_i = \left[\int f(y_i \mid \theta) \pi(\theta \mid \mathbf{y}_{-i}) d\theta \right]^{-1},$$

where $f(y_i \mid \theta)$ is the likelihood for the i-th observation, $\pi(\theta \mid \mathbf{y}_{-i})$ is the posterior distribution of the parameters excluding y_i, and $\mathbf{y}_{-i}$ denotes all data except y_i. The LPML is computed as the sum of the logarithms of the CPOs across all observations:

$$\text{LPML} = \sum_{i=1}^{n} \log(\text{CPO}_i).$$

A higher LPML indicates better model fit, as it reflects stronger predictive capability. In practice, CPO values are often approximated using posterior samples obtained through MCMC methods, leveraging techniques such as the harmonic mean approximation. The LPML provides a comprehensive assessment of a model's predictive adequacy, making it a valuable addition to Bayesian model evaluation.

8.6 Findings

8.6.1 Simulation Study

In this section, we conduct a simulation study to evaluate the performance of the ML method for estimating the parameters $(\alpha, \lambda, \theta, \delta)$ of the EW distribution with a cure fraction. Following the Algorithm 4, $M = 1000$ samples of size n were generated. The MLEs, denoted as $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\theta}$, and $\hat{\delta}$, were obtained using the BFGS optimization method. The estimation accuracy was evaluated using the bias and MSE, defined as follows:
The bias for a parameter $\Theta_i \in \{\alpha, \lambda, \theta, \delta\}$ is calculated as:

$$\text{Bias}(\hat{\Theta}) = \frac{1}{M} \sum_{m=1}^{M} (\hat{\Theta}^{(m)} - \Theta),$$

where $\hat{\Theta}^{(m)}$ is the estimate of Θ for the m-th sample, and Θ is the true parameter value. The MSE for Θ is given by:

$$\text{MSE}(\hat{\Theta}) = \frac{1}{M} \sum_{m=1}^{M} (\hat{\Theta}^{(m)} - \Theta)^2.$$

These measures provide insight into the bias and variability of the estimates derived from the simulation study, offering a robust assessment of the MLE approach.

We conducted the simulation for varying sample sizes: $n = 50$, 100, 250, 500, 750, 1000, under four distinct combinations of parameters $(\alpha, \lambda, \theta, \delta)$. These parameter sets were: (1) $(\alpha = 3, \lambda = 2, \theta = 1.5, \delta = 0.2)$, (2) $(\alpha = 2, \lambda = 5, \theta = 2, \delta = 0.3)$, (3) $(\alpha = 0.5, \lambda = 1, \theta = 0.5, \delta = 0.5)$, and (4) $(\alpha = 1, \lambda = 0.5, \theta = 2, \delta = 0.7)$. The simulations included censored data, providing insights into how the biases, MSEs, and CPs of the parameter estimates vary with sample size. Figures 8.3–8.5 depict these metrics under a nominal 95% confidence level, highlighting the reliability and accuracy of the ML estimation approach across different settings.

Figures 8.3–8.5 illustrate that, across all simulations, the bias values approach zero as the sample size increases, demonstrating the consistency of the ML estimation method. Similarly, the MSE decreases with larger sample sizes, indicating improved estimation precision. For CPs, the results align closely with the nominal confidence level, particularly for higher sample sizes, underscoring the reliability of the method in maintaining accurate interval estimates.

Algorithm 4 Simulating data with a cure fraction in the EW distribution

Step 1: Fix input parameters α, λ, θ, and δ.
Step 2: Generate Uniform Random Samples:

- Generate n random values $u_i \sim \text{Uniform}(0,1)$, $i = 1, 2, \ldots, n$.

Step 3: Transform to Survival Times:

- Compute $x_i' = F^{-1}(u_i; \alpha, \lambda, \theta)$, where

$$F^{-1}(u_i; \alpha, \lambda, \theta) = \lambda \left(-\log\left(1 - u_i^{1/\alpha}\right)\right)^{1/\theta}.$$

Step 4: Generate Cure Status:

- Draw n samples $b_i \sim \text{Bernoulli}(\delta)$, $i = 1, 2, \ldots, n$.
- If $b_i = 1$, set $x_i' = \infty$ (individual is cured).
- Otherwise, retain x_i' as the uncensored lifetime.

Step 5: Introduce Censoring:

- Generate n random samples $c_i \sim \text{Uniform}(0, \max(x_i'))$, where c_i controls the censoring mechanism.

Step 6: Apply Censoring Mechanism:

- For each individual i:
  1. If $x_i' \leq c_i$, set $x_i = x_i'$ and $C_i = 1$ (event observed).
  2. Otherwise, set $x_i = c_i$ and $C_i = 0$ (censored observation).

Step 7: Output:

- Return the dataset (x_i, C_i), where x_i represents the observed time, and C_i is the censoring indicator (1 for event, 0 for censored).

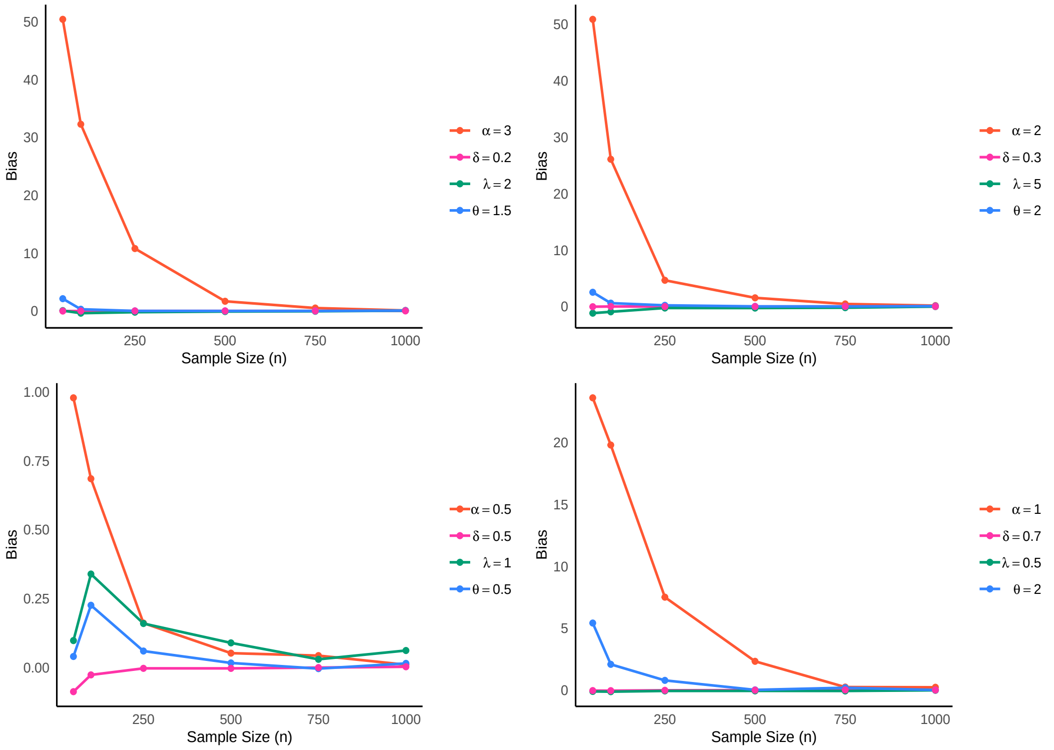


Figure 8.3: Bias estimates for model parameters (α, λ, θ, δ) across varying sample sizes

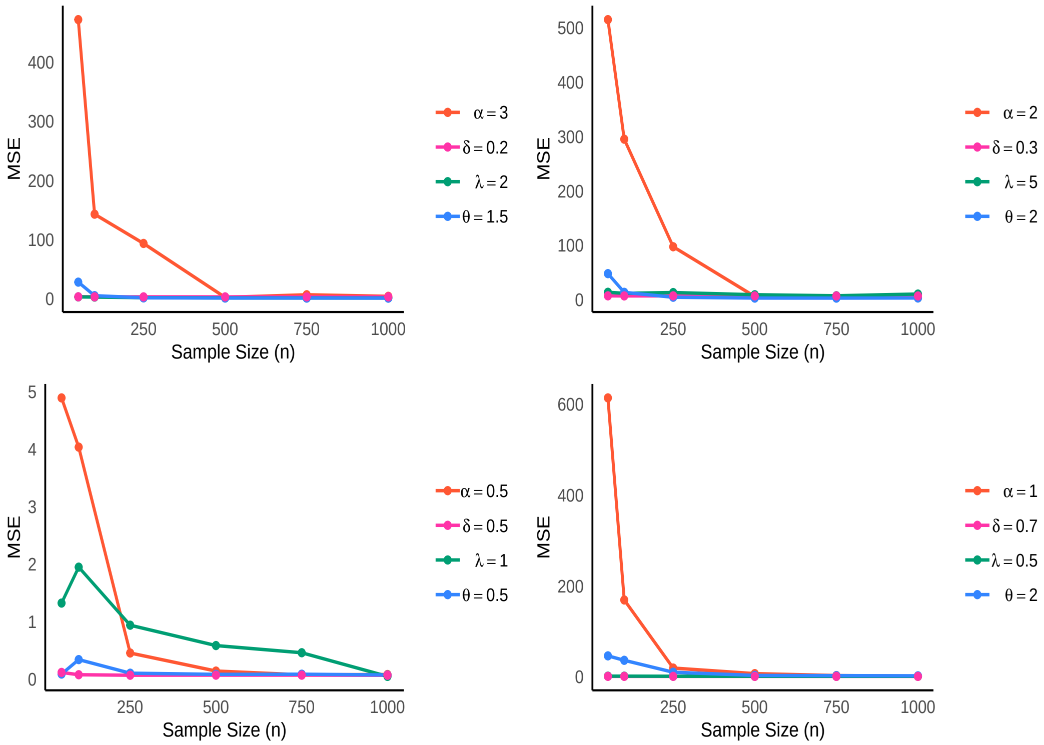


Figure 8.4: MSE estimates for model parameters (α, λ, θ, δ) across varying sample sizes

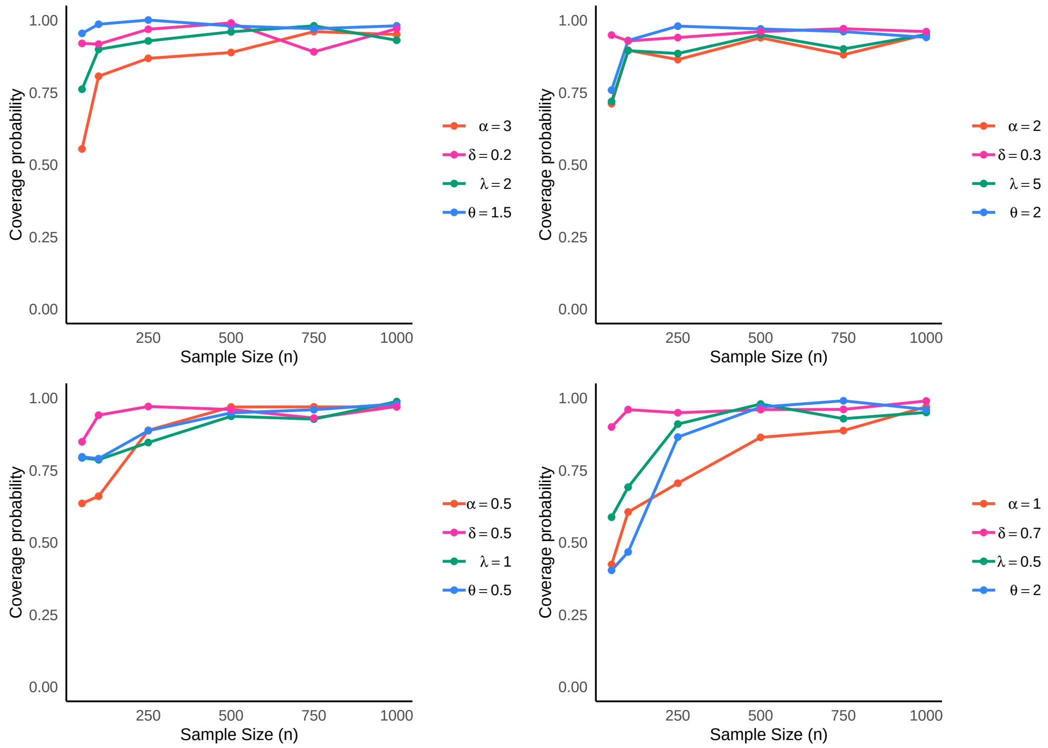


Figure 8.5: Coverage probabilities for the parameters (α, λ, θ, δ) across different sample sizes

8.6.2 Application to Colorectal Cancer Data

This retrospective cohort study analyzed survival time data from 188 colorectal cancer patients treated at MCC, Thalassery. The primary variable of interest was the duration (in months) from the date of diagnosis to death from any cause. Patients who remained alive during the study period were considered censored, with their survival time recorded up to the last follow-up. Approximately one-third of the observations (73 out of 188) were censored. Summary statistics of the dataset, including key descriptive measures of survival times, are presented in Table 8.1 to provide a detailed overview of the data.

Figure 8.6 illustrates the survival function estimated using the Kaplan-Meier method, along with the corresponding TTT plot for the cancer data. The Kaplan-Meier plot exhibits a plateau near the tail, indicating the presence of cured individuals within the study population. Additionally, the S-shape of the TTT plot suggests that a model with a bathtub-shaped hazard rate would be a suitable choice for modelling the data.

First, we fit the scaled uncensored data to various models, including the EW distribution, modified Weibull (MW) distribution (Lai et al. (2003b)), EP distribution (Smith & Bain (1975)), generalized X Exponential (GXE) distribution (Chacko & Deepthi (2019)), and GL distribution (Nadarajah et al. (2011)). The data was scaled to improve numerical

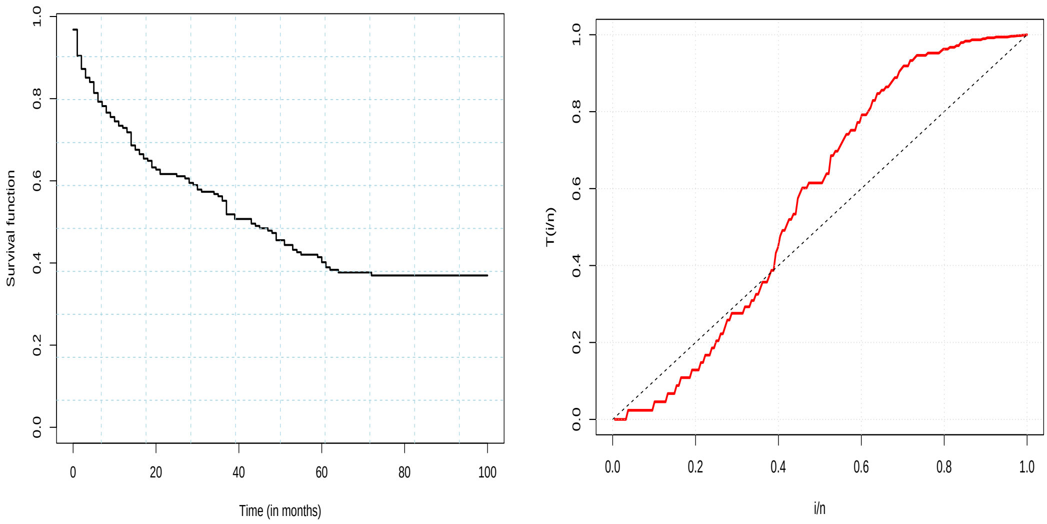


Figure 8.6: Kaplan-Meier survival function (left) and TTT Plot (right) for colorectal cancer data

Table 8.1: Summary statistics of the cancer data

Min	Median	Mean	Max	SD	Skewness	Kurtosis
0.3	37	40.86	100	30.8263	0.1116	1.5079

stability during the fitting process. To assess the effectiveness of each model, we used the KS test statistic along with its corresponding p-value, as well as the AIC. The results, presented in Table 8.2, indicate that the EW model provides a better fit compared to the other models, as it has the lowest KS statistic and AIC values, along with the highest p-value.

Table 8.2: Model fit for uncensored data using KS test and AIC criteria.

Base distributions	Est	KS	p-value	AIC
EW				
$\hat{\alpha}$	**0.1215**	**0.07028**	**0.621**	**-137.126**
$\hat{\lambda}$	**0.57321**			
$\hat{\theta}$	**4.7335**			
MW				
$\hat{\beta}$	1.7543	0.0741	0.5529	-132.971
$\hat{\gamma}$	0.6327			
$\hat{\lambda}$	1.44649			
EP				
$\hat{\alpha}$	0.70053	0.08654	0.3552	-132.214
$\hat{\lambda}$	2.61405			
GXE				
$\hat{\alpha}$	0.74701	0.08646	0.3562	-133.884
$\hat{\lambda}$	3.52999			
GL				
$\hat{\alpha}$	0.78007	0.09551	0.2449	-126.438
$\hat{\lambda}$	4.69553			

Tables 8.3 and 8.4 present the analysis of colorectal cancer data using cure model parameters under different cure fraction models. Table 8.3 summarizes the MLEs and their corresponding SEs. While the AIC values do not clearly indicate a superior model, the smaller SEs for the EW distribution parameters suggest better performance compared to other distributions. Table 8.4 provides Bayesian estimates alongside their SEs, HPD intervals, and LPML values. The HPD intervals effectively cover the Bayesian estimates, and the smaller SEs highlight the precision of the parameter estimates. Based on LPML values, cure fraction models using the EW distribution demonstrate better performance. Although the EW distribution outperforms others in both ML and Bayesian estimation methods, the results do not conclusively identify the best cure model. Furthermore, the Geweke Z-scores in Table 8.5 confirm convergence of the MCMC simulations for the EW distribution, with all absolute Z-scores falling below the critical threshold of 1.96 at the 95% confidence level. Figure 8.7 presents a comparison of survival function estimates derived using the Kaplan-Meier method and the parametric MCM, NMC, and CPO models based on the ML method.

Table 8.3: MLEs, standard errors (SE), and AIC values for various distributions under different cure fraction models.

	Maximum likelihood estimation					
Base distribution	MCM		NMC		CPO	
	Est	SE	Est	SE	Est	SE
EW						
$\hat{\alpha}$	0.1133	0.0105	0.14095	0.01182	0.16749	0.01341
$\hat{\lambda}$	0.6045	0.00084	0.65704	0.00078	0.73533	0.00017
$\hat{\theta}$	5.066	0.00002	4.89241	0.00078	4.84775	0.00041
$\hat{\delta}$	0.3673	0.03642	0.36707	0.03545	0.36945	0.03403
AIC	107.5434		106.6343		107.6002	
MW						
$\hat{\beta}$	1.99979	0.59136	1.3304	0.39872	0.83919	0.2848
$\hat{\gamma}$	0.6824	0.10535	0.70188	0.1088	0.72785	0.11364
$\hat{\lambda}$	0.6871	0.79281	0.92654	0.88287	1.28991	0.96144
$\hat{\delta}$	0.33349	0.06413	0.3327	0.06787	0.33513	0.06647
AIC	108.832		108.6344		108.222	
EP						
$\hat{\alpha}$	0.6396	0.06536	0.70974	0.07326	0.7847	0.06266
$\hat{\lambda}$	2.01423	0.41184	1.28694	0.5935	0.02508	0.0.2457
$\hat{\delta}$	0.32997	0.05138	0.29941	0.08439	0.0263	0.47506
AIC	107.9484		107.3121		106.9357	
GXE						
$\hat{\alpha}$	0.69225	0.08324	0.76159	0.08713	0.7809	0.06259
$\hat{\lambda}$	2.65096	0.56103	1.76918	0.64354	0.00005	0.00012
$\hat{\delta}$	0.34022	0.04313	0.31812	0.05805	0.00212	0.60376
AIC	106.3062		106.5675		107.0102	
GL						
$\hat{\alpha}$	0.67654	0.08839	0.72159	0.08724	0.76976	0.09239
$\hat{\lambda}$	2.9959	0.74444	1.77625	0.73882	0.622	0.84157
$\hat{\delta}$	0.29612	0.05914	0.24862	0.08381	0.14585	0.17215
AIC	107.8334		107.3851		106.9029	

Table 8.4: Bayesian estimates, standard errors (SE), HPD intervals, and LPML values for various distributions under different cure fraction models.

Base distribution	Bayesian estimation								
	MCM			NMC			CPO		
	Est	SE	HPD	Est	SE	HPD	Est	SE	HPD
EW									
$\hat{\alpha}$	0.1070	0.00186	(0.07863, 0.13329)	0.1319	0.00095	(0.10750, 0.16378)	0.1577	0.00096	(0.12781, 0.18525)
$\hat{\lambda}$	0.6422	0.00705	(0.54884, 0.86371)	0.7052	0.03195	(0.60364, 0.99972)	0.7792	0.02758	(0.5951, 0.98731)
$\hat{\theta}$	5.3347	0.02353	(4.85857, 5.63627)	5.2699	0.02221	(4.83983, 5.88706)	5.1695	0.01034	(5.00814, 5.35821)
$\hat{\delta}$	0.3644	0.00612	(0.27168, 0.45471)	0.3619	0.00565	(0.26691, 0.44311)	0.3659	0.00372	(0.28940, 0.46715)
LPML	-52.61295			-52.18041			-52.46955		
MW									
$\hat{\beta}$	1.9418	0.47267	(1.18716, 2.96424)	1.2008	0.44487	(0.33562, 2.20122)	2.1058	0.03852	(2.01153, 2.17143)
$\hat{\gamma}$	0.6745	0.07309	(0.57629, 0.86131)	0.7033	0.06933	(0.53219, 0.85295)	0.9422	0.05824	(0.83556, 1.06771)
$\hat{\lambda}$	0.7014	0.77415	(0.23106, 0.95086)	0.6493	0.75492	(0.33539, 0.94803	0.9493	0.64633	(0.85218, 1.04752)
$\hat{\delta}$	0.3075	0.08017	(0.05625, 0.42811)	0.2746	0.08836	(0.0834, 0.42613)	0.3865	0.03159	(0.32207, 0.44318)
LPML	-53.16729			-53.52109			-55.77984		
EP									
$\hat{\alpha}$	0.6049	0.07867	(0.45321, 0.7659)	0.4938	0.08055	(0.33707, 0.6820)	0.9003	0.07259	(0.7731, 1.07461)
$\hat{\lambda}$	1.5906	0.58364	(0.61754, 2.69352)	2.5437	0.44667	(1.74611, 3.26051)	2.0466	0.01953	(2.0065, 2.08646)
$\hat{\delta}$	0.2499	0.11096	(0.00124, 0.38603)	0.3041	0.06805	(0.14673, 0.42301)	0.3972	0.02299	(0.35518, 0.43913)
LPML	-53.79102			-55.13138			-56.80169		
GXE									
$\hat{\alpha}$	0.685	0.09732	(0.51063, 0.86969)	0.5601	0.0811	(0.39215, 0.73383)	0.8862	0.0738	(0.77162, 1.03154)
$\hat{\lambda}$	2.5793	0.69744	(0.94042, 3.94931)	3.179	0.58565	(1.94666, 4.16514)	1.725	0.0088	(1.70528, 1.74100)
$\hat{\delta}$	0.3183	0.05114	(0.21156, 0.40324)	0.3429	0.04018	(0.25787, 0.41533)	0.3761	0.01588	(0.34739, 0.40866)
LPML	-53.01662			-53.95766			-53.16712		
GL									
$\hat{\alpha}$	0.659	0.0992	(0.47559, 0.87498)	0.5442	0.11274	(0.35682, 0.83111)	0.9606	0.08078	(0.79974, 1.12035)
$\hat{\lambda}$	2.6961	0.926	(1.23248, 4.30786)	3.7284	0.86021	(2.03893, 5.3577)	2.4659	0.17214	(1.98549, 2.70265)
$\hat{\delta}$	0.2365	0.1039	(0.00014, 0.37746)	0.3049	0.05894	(0.1552, 0.39761)	0.3201	0.04059	(0.20854, 0.39229)
LPML	-53.85072			-54.97743			-55.35697		

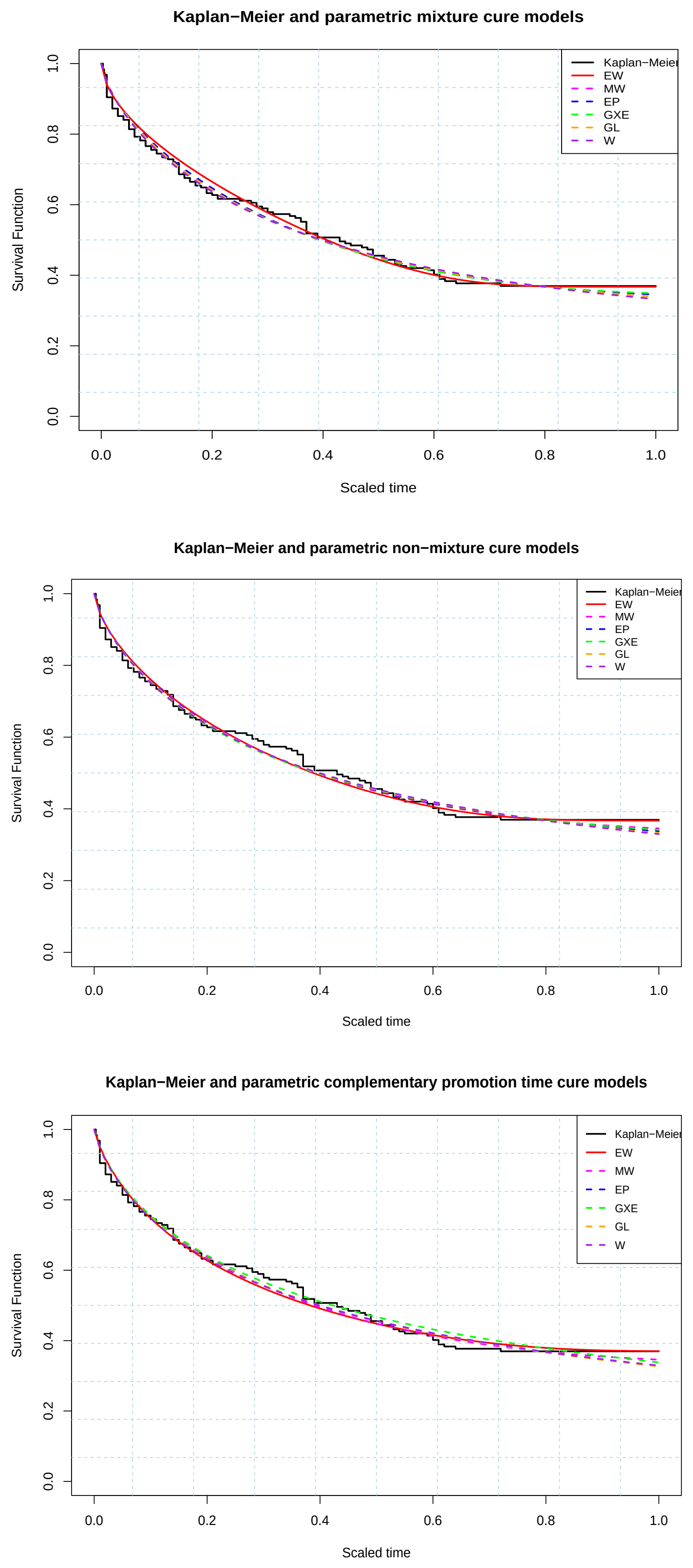


Figure 8.7: Comparison of survival function estimates from Kaplan-Meier and parametric models (MCM, NCM, CPO) using the ML method

Table 8.5: Geweke Z-scores for parameters of the EW distribution under three cure fraction models.

Cure Fraction Model	α	λ	θ	δ
MCM	−0.2486	−1.9160	0.1345	1.4949
NMC	−0.1525	−1.2814	1.6940	1.4380
CPO	0.167	−1.323	1.765	1.757

8.7 Summary

This chapter focuses on analyzing colorectal cancer data from MCC Thalassery using advanced cure fraction models, incorporating right-censored observations. The Kaplan-Meier plot confirms the presence of cured individuals, while the TTT plot indicates a bathtub-shaped failure rate, suggesting the need for flexible hazard function models. The EW distribution proves highly effective in capturing the varying hazard rate behaviors in the data, offering a superior fit compared to other distributions. Using MCM, NMC, and CPO cure fraction models, the analysis reveals complex survival patterns. Parameters are estimated through ML and Bayesian methods, with model performance evaluated using AIC, LPML, and Geweke Z-score values. This study underscores the importance of cure fraction models in understanding cancer survival data and advancing statistical tools for medical research.

CHAPTER 9

Conclusion

The study and analysis of lifetime data are essential in various scientific and technological fields, particularly in reliability engineering, survival analysis, and quality control. Accurately modelling lifetime data permits researchers and practitioners to understand system behaviors, predict future performance, and design effective interventions. Lifetime distributions serve as the foundation for estimating failure times and risk factors, making their selection critical for making meaningful and reliable conclusions.

Modelling lifetime data using various failure rate distributions is especially important as different applications demand distinct behaviors, such as constant, increasing, decreasing, or non-monotonic failure rates. Each failure rate behavior provides insight into underlying processes, such as wear-out phenomena, early-life failures, or mixed failure mechanisms. By exploring and developing flexible distributions, it becomes possible to accurately capture these different behaviors, improving decision-making in fields like medicine, environmental studies, and industrial maintenance.

Chapter 1 lays a comprehensive foundation for the thesis by introducing and examining fundamental concepts essential to the research. Key ageing concepts and properties are examined along with important failure rate distributions and their characteristics. The chapter also delves into the concept of copula models, various censoring methods, and cure fraction modelling, providing a strong foundation for the research. Important model selection criteria are also presented in order to support methodological rigor. The chapter ends with a review of relevant literature, grounding the study in existing knowledge and highlighting areas for further research.

Choosing the right distribution is crucial for understanding data and drawing meaningful conclusions, especially when dealing with complex real-world scenarios. The second chapter introduces the DUS transformation, a powerful tool for developing flexible probability distributions that accurately fit such data without extra parameters. In Chapter 2, the DUS-K distribution has been derived using the DUS transformation with the Kumaraswamy distribution as a baseline, offering a model that can effectively capture a wide range of failure rate behaviors. This distribution, with its comprehensive statistical properties, is shown to be highly practical in various applications, from risk assessment to reliability studies. The application of ML estimation, stress-strength reliability analysis, simulation studies, and real data analysis further validates the utility of the DUS-K distribution in modelling complex lifetime data.

In recent years, there is growing interest in inverse transformations of probability distributions for modelling complex real-world data. Chapter 3 introduced the DUS-IK distribution, developed through the DUS transformation with the IK distribution, showcasing different asymmetries and non-monotonic hazard functions suited for survival and reliability analysis. Characterization properties magnifies theoretical understanding, while stress-strength reliability highlights the practical applications. ML estimation, supported by simulation studies, confirms the precision of parameter estimates by evaluating biases and MSEs. Applications to real data further validate the distribution's practical significance, with its superior fit and performance demonstrated through model selection criteria and the Nikulin-Rao-Robson statistic. This work highlights the importance of characterization in understanding and applying complex distributions, paving the way for future research.

Modelling dependent data is essential in reliability and survival analysis. The BFG-MIW model, developed using the Farlie-Gumbel-Morgenstern copula with a bivariate inverse Weibull distribution, offers a practical approach for analyzing dependent stress-strength reliability. Its statistical properties, including conditional distributions, moment-generating functions, and reliability measures, provide a strong theoretical foundation. The model proves particularly effective for datasets with weak correlations, as demonstrated through its application to kidney patient recurrence times. This study underscores the significance of modelling dependence in data and highlights the broad applicability of the proposed model.

Chapters 5 and 6 explore advanced censoring schemes and statistical methods for lifetime data modelling. Chapter 5 compares the popular Progressive Type-II censoring with Progressive Type-II Hybrid censoring and Adaptive Progressive Type-II censoring, which reduce experimental duration and associated costs. The exponential Power distribution is used to demonstrate the flexibility of these schemes in reliability studies. Chapter 6 further examines the joint Adaptive Progressive Type-II censoring with the generalized Lindley distribution, emphasizing its ability to evaluate the performance of different populations in comparative studies. Both chapters employ ML estimation and Bayesian estimation, validated through simulations and real data, underlining the critical role of these advanced censoring schemes in improving the accuracy, cost-effectiveness, and applicability of reliability analysis.

A novel T1-T2 mixture censoring scheme is introduced, providing an innovative approach to ensure a maximum possible number of failures within a supplementary time frame. The study derives computational formulas for the expected number of failures and failure time, along with presenting the Fisher information for the scheme. The performances of MLEs and Bayesian estimates are evaluated under squared error and LINEX loss functions, demonstrating their reliability and accuracy. Practical applicability is validated through real data analysis, highlighting the relevance of the T1-T2 mixture censoring scheme for lifetime data studies.

In Chapter 8, the study demonstrates the effective application of advanced cure fraction models to colorectal cancer data from MCC Thalassery, incorporating right-censored observations. By utilizing the mixture cure fraction model, non-mixture cure fraction model, and cure rate proportional odds model, the analysis captures the complex survival characteristics of the data. Parameter estimation is conducted using ML method and Bayesian methods, while model performance is evaluated using criteria such as AIC, LPML, and Geweke Z-score values. The EW distribution stands out for its flexibility and superior fit compared to the bathtub models, MW, EP, GXE, and GL. Also, it effectively models the dataset's varying hazard rate behaviors. This chapter highlights the significance of cure fraction models in enhancing the understanding of survival data, providing valuable insights for cancer research and advancing statistical methodologies in survival analysis.

In summary, this thesis advances the modelling and analysis of lifetime data through the development of innovative distributions, advanced censoring schemes, and applications to real-world datasets. By integrating ML and Bayesian methods, the research provides valuable tools for understanding complex survival behaviors and improving reliability analysis. These contributions improve theoretical understanding and provide practical tools for fields like engineering, biostatistics, and cancer research, opening paths for future research in lifetime data analysis.

CHAPTER 10

Recommendations

Drawing from the significant findings of this research, we suggest the following directions for future work:

- Enhancing the T1-T2 mixture censoring scheme incorporating progressive censoring methods.
- Application of copulas to cure fraction models to account for the dependency
- Development of copula-based models for competing risks, where individuals may experience multiple possible events.
- Analysis of lifetime models using joint adaptive progressive Type-II censoring under competing risks models.
- Optimization of censoring time in joint adaptive progressive censoring schemes.
- Application of progressive censoring for dependent or correlated data.
- Combining progressive censoring with cure fraction models to study systems where a portion of the population is 'cured' or immune to failure.

Publications and Presentations

List of published papers

1. Anakha, K. K., & Chacko, V. M. (2020). On Exponential-Weibull Distribution Useful in Reliability and Survival Analysis. *Reliability: Theory & Applications*, 15(1), 20-32. https://cyberleninka.ru/article/n/on-exponential-weibull-distribution-useful-in-reliability-and-survival-analysis

2. Anakha, K. K., & Chacko, V. M. (2020). On Ageing Properties of Lifetime Distributions. *Veritas Journal of Sciences*, 1(1), 107-139. `http://livestom.stthomas.ac.in:8085/index.php/Veritas/article/view/13`

3. Anakha, K. K., & Chacko, V. M. (2021). DUS-Kumaraswamy Distribution: A Bathtub Shaped Failure Rate Model. *International Journal of Statistics and Reliability Engineering*, 8(3), 359-367. https://ijsreg.com/article/view/dus-kumaraswamy-distribution-a-bathtub-shaped-failure-rate-model

4. Nayana, B. M., Anakha, K. K., Chacko, V. M., Aslam, M., & Albassam, M. (2022). A new neutrosophic model using DUS-Weibull transformation with application. *Complex & Intelligent Systems*, 8(5), 4079-4088. `https://link.springer.com/article/10.1007/s40747-022-00698-6`

5. Anakha, K. K., & Chacko, V. M. (2024). Statistical estimation of exponential power distribution on different progressive Type-II censoring schemes. *Journal of the Indian Society for Probability and Statistics*, 25(1), 85-120. `https://link.springer.com/article/10.1007/s41096-023-00172-7`

6. Anakha, K. K., & Chacko, V. M. (2025). On comparative lifetime analysis with the generalized Lindley distribution: insights from joint adaptive progressive Type-II censoring. *Journal of Statistical Computation and Simulation*, 1-28. https://doi.org/10.1080/00949655.2025.2496397

7. Deepthy, G. S., Anakha, K. K., & Nicy Sebastian. (2025). Inference On Partially Observed Competing Risks Models Using Generalized Type-II Hybrid Censoring Scheme. *Austrian Journal of Statistics*, 54(4), 117–135. https://doi.org/10.17713/ajs.v54i4.2049

Papers under review

1. Anakha, K. K., & Chacko, V. M. (2023). Type I–Type II Mixture Censoring Scheme for Lifetime Data Analysis. *Communications in Statistics-Simulation and Computation.*

2. Anakha, K. K., Chacko, V. M., Aslam, M., Sini, K. P., & Deepthy, G. S. (2024). DUS-Inverse Kumaraswamy Distribution: A New Non-monotonic Hazard Rate Model. *Statistics and Applications.*

3. Anakha, K. K.,James, A., and Chacko, V. M. (2025). On a Bivariate Inverse Weibull Distribution Useful in Dependence Reliability Measures. *Austrian Journal of Statistics.*

List of Presentations

1. Anakha K K, presented “Mixture of Exponential and Weibull Distributions” in Two Day National Seminar on Advanced Developments in Statistical Theory and Applications organized by the Department of Statistics, Christ College (Autonomous), Irinjalakuada, in conjunction with the 41st Annual Conference of Kerala Statistical Association during 28-29 February 2020.

2. Anakha K K, presented a paper “A New Bathtub Shaped Failure Rate Distribution and its Application to Aarset Data” in the National Webinar on The Recent Developments in Statistics and Probability-2021 organized by Department of Statistics, St. Thomas College (Autonomous), Thrissur, during 18-19 February 2021.

3. Anakha K K, presented a contributed paper on “DUS-Kumaraswamy Distribution: A New Bathtub Shaped Failure Rate Model” during 23rd Annual Conference of SSCA (online) on Visionary Innovations in Statistical Theory and Applications (VISTA-2021) during 24-28 February, 2021 at ICAR-National Academy of Agricultural Research Management (ICAR-NAARM), Hyderabad.

4. Anakha K K, presented a paper on Comparison of Different Progressive Type-II Censoring Schemes on Exponential Power Data, International Conference on Knowledge Discoveries on Statistical Innovations & Recent Advances in Optimization (ICON-KSRAO), Department of Statistics and Population Research Centre, Andhra University, Visakhapatnam, during 29-30 December 2022.

5. Anakha K K, presented a paper on Statistical Estimation on Different Progressive Type-II Censoring Schemes, International Conference on Applications of Statistics [ICAS-2023], The Department of Statistics, Pachhunga University College Campus, Mizoram University (A Central University) during 2-3 February 2023.

6. Anakha K K, presented a paper on, On a Bivariate Inverse Weibull Distribution Useful in Dependence Reliability Measures, in the Ninth International Conference on “Statistics for Twenty-first Century-2023” (ICST-2023) organized by International Statistics Fraternity (ISF), Department of Statistics and School of Physical and Mathematical Sciences, Univeristy of Kerala, Trivandrum during 15-18 December, 2023.

7. Anakha K K, presented a paper on, Type I-Type II Mixture Censoring Scheme for Lifetime Data Analysis, International Conference on Recent Advances of Probability

and Statistics in Interdisciplinary Research (RAPSIR-2024) organized by Department of Statistics, Faculty of Science, University of Allahabad, Prayagraj, India in Conjunction with 43rd Annual Convention of Indian Society for Probability and Statistics (ISPS) during 06-08 February 2024.

8. Anakha K K, presented a paper on, A New Censoring Scheme for Lifetime Data Analysis, in the National seminar on "Recent Developments and Applications in Statistics and Probability" organized by the Department of Statistics, St. Thomas College (Autonomous), Thrissur, during 14-15 February 2024.

9. Anakha K K, presented a paper on, "On Comparative Lifetime Analysis with the Generalized Lindley Distribution:Insights from Joint Adaptive Progressive Type-II Censoring", in the International Seminar on Applied Statistics & Data Analytics-2025, held during 04-05 March 2025, organized by the Department of Statistics, University of Calicut.

APPENDIX A

Appendix A

Second derivatives of the log-likelihood function with respect to model parameters for each censoring schemes presented in Chapter 5.

PT2

$$\begin{aligned}
\frac{\partial^2 \log L_1}{\partial \alpha^2} =& -\frac{m}{\alpha^2} + \sum_{i=1}^{m} \left[1 - (1 + (\lambda x_i)^{\alpha}) e^{(\lambda x_i)^{\alpha}}\right] (\lambda x_i)^{\alpha} \log^2(\lambda x_i) - \\
& \sum_{i=1}^{m} R_i \left[1 + (\lambda x_i)^{\alpha}\right] (\lambda x_i)^{\alpha} \log^2(\lambda x_i) e^{(\lambda x_i)^{\alpha}} \\
\frac{\partial^2 \log L_1}{\partial \alpha \partial \lambda} =& \frac{m}{\lambda} + \sum_{i=1}^{m} \{\lambda^{\alpha-1} x_i^{\alpha} \left[1 - (1 + R_i) e^{(\lambda x_i)^{\alpha}}\right] (\alpha \log(\lambda x_i) + 1) \\
& - \alpha \lambda^{2\alpha-1} x_i^{2\alpha} (1 + R_i) e^{(\lambda x_i)^{\alpha}} \log(\lambda x_i)\} \\
=& \frac{\partial^2 \log L_1}{\partial \lambda \partial \alpha} \\
\frac{\partial^2 \log L_1}{\partial \lambda^2} =& -\frac{m\alpha}{\lambda^2} + \sum_{i=1}^{m} \alpha x_i^{\alpha} \left[(\alpha - 1) \lambda^{\alpha-2} \left(1 - (1 + R_i) e^{(\lambda x_i)^{\alpha}}\right) - \alpha \lambda^{2\alpha-2} x_i^{\alpha} (1 + R_i) e^{(\lambda x_i)^{\alpha}}\right]
\end{aligned}$$

PT2H

$$\begin{aligned}
\frac{\partial^2 \log L_2}{\partial \alpha^2} =& -\frac{D}{\alpha^2} + \sum_{i=1}^{D} (\lambda x_i)^{\alpha} (\log(\lambda x_i))^2 \left((1 - e^{(\lambda x_i)^{\alpha}}) - (\lambda x_i)^{\alpha} e^{(\lambda x_i)^{\alpha}}\right) \\
& - \sum_{i=1}^{D} R_i (\log(\lambda x_i))^2 (\lambda x_i)^{\alpha} e^{(\lambda x_i)^{\alpha}} ((\lambda x_i^{\alpha} + 1))
\end{aligned}$$

$$
\begin{aligned}
&- (n - D - \sum_{i=1}^{D} R_i)(\lambda T)^\alpha (\log(\lambda T))^2 e^{(\lambda T)^\alpha}((\lambda T)^\alpha + 1) \\
\frac{\partial^2 \log L_2}{\partial\alpha\partial\lambda} =& \frac{D}{\lambda} - \sum_{i=1}^{D} \lambda^{\alpha-1} x_i^\alpha \left(\alpha(\lambda x_i)^\alpha \log(\lambda x_i) e^{(\lambda x_i)^\alpha} - (1 - e^{(\lambda x_i)^\alpha})(\log(\lambda x_i)^\alpha + 1)\right) \\
&- \sum_{i=1}^{D} R_i \lambda^{\alpha-1} x_i^\alpha e^{(\lambda x_i^\alpha)} \left(\alpha(\lambda x_i)^\alpha \log(\lambda x_i) + \log(\lambda x_i)^\alpha + 1\right) \\
&- (n - D - \sum_{i=1}^{D} R_i) \lambda^{\alpha-1} T^\alpha e^{(\lambda T)^\alpha} \left(\alpha(\lambda T)^\alpha \log(\lambda T) + \log(\lambda T)^\alpha + 1\right) \\
&= \frac{\partial^2 \log L_2}{\partial\lambda\partial\alpha} \\
\frac{\partial^2 \log L_2}{\partial\lambda^2} =& - \frac{D\alpha}{\lambda^2} + \alpha \sum_{i=1}^{D} \lambda^{\alpha-2} x_i^\alpha \left((\alpha - 1)(1 - e^{(\lambda x_i)^\alpha}) - \alpha(\lambda x_i)^\alpha e^{(\lambda x_i)^\alpha}\right) \\
&- \alpha \sum_{i=1}^{D} R_i \lambda^{\alpha-2} x_i^\alpha e^{(\lambda x_i)^\alpha} (\alpha(\lambda x_i)^\alpha + \alpha - 1) \\
&- (n - D - \sum_{i=1}^{D} R_i) \alpha \lambda^{\alpha-1} T^\alpha e^{(\lambda T)^\alpha} (\alpha(\lambda T)^\alpha + \alpha - 1)
\end{aligned}
$$

APT2

$$
\begin{aligned}
\frac{\partial^2 \log L_3}{\partial\alpha^2} =& - \frac{m}{\alpha^2} + \sum_{i=1}^{m} (\lambda x_i)^\alpha (\log(\lambda x_i))^2 \left(1 - (1 + (\lambda x_i)^\alpha) e^{(\lambda x_i)^\alpha}\right) \\
&- \sum_{i=1}^{j} R_i (\lambda x_i)^\alpha (\log(\lambda x_i))^2 e^{(\lambda x_i)^\alpha} (1 + (\lambda x_i)^\alpha) \\
&- (n - m - \sum_{i=1}^{j} R_i)(\lambda x_m)^\alpha (\log(\lambda x_m))^2 e^{(\lambda x_m)^\alpha} (1 + (\lambda x_m)^\alpha) \\
\frac{\partial^2 \log L_3}{\partial\alpha\partial\lambda} =& \frac{m}{\lambda} - \sum_{i=1}^{m} x_i^\alpha \lambda^{\alpha-1} \left(\alpha(\lambda x_i)^\alpha \log(\lambda x_i) e^{(\lambda x_i)^\alpha} - (1 - e^{(\lambda x_i)^\alpha}) (\log(\lambda x_i)^\alpha + 1)\right) \\
&- \sum_{i=1}^{j} R_i x_i^\alpha \lambda^{\alpha-1} e^{(\lambda x_i)^\alpha} \left(\alpha(\lambda x_i)^\alpha \log(\lambda x_i) + \log(\lambda x_i)^\alpha + 1\right) \\
&- \left(n - m - \sum_{i=1}^{j} R_i\right) x_m^\alpha \lambda^{\alpha-1} e^{(\lambda x_m)^\alpha} \left(\alpha(\lambda x_m)^\alpha \log(\lambda x_m) + \log(\lambda x_m)^\alpha + 1\right) \\
&= \frac{\partial^2 \log L_3}{\partial\lambda\partial\alpha} \\
\frac{\partial^2 \log L_3}{\partial\lambda^2} =& - \frac{m\alpha}{\lambda^2} + \alpha \sum_{i=1}^{m} x_i^\alpha \lambda^{\alpha-2} \left((\alpha - 1)(1 - e^{(\lambda x_i)^\alpha}) - \alpha(\lambda x_i)^\alpha e^{(\lambda x_i)^\alpha}\right) \\
&- \alpha \sum_{i=1}^{j} R_i x_i^\alpha \lambda^{\alpha-2} e^{(\lambda x_i)^\alpha} \left(\alpha(\lambda x_i)^\alpha + \alpha - 1\right) \\
&- \left(n - m - \sum_{i=1}^{j} R_i\right) \alpha x_m^\alpha \lambda^{\alpha-2} e^{(\lambda x_m)^\alpha} \left(\alpha(\lambda x_m)^\alpha + \alpha - 1\right)
\end{aligned}
$$

Bibliography

Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE transactions on reliability*, *36*(1), 106–108.

Abd AL-Fattah, A. M., El-Helbawy, A. A., & Al-Dayian, G. R. (2017). Inverted kumaraswamy distribution: Properties and estimation. *Pakistan Journal of Statistics*, *33*(1).

Abd Elaal, M. K., & Jarwan, R. S. (2017). Inference of bivariate generalized exponential distribution based on copula functions. *Applied Mathematical Sciences*, *11*(24), 1155–1186.

Abulebda, M., Pandey, A., & Tyagi, S. (2023). On bivariate inverse lindley distribution derived from copula. *Thailand Statistician*, *21*(2), 291–304.

Abulebda, M., Pathak, A. K., Pandey, A., & Tyagi, S. (2022). On a bivariate xgamma distribution derived from copula. *Statistica*, *82*(1), 15–40.

Achcar, J. A., Moala, F. A., Tarumoto, M. H., & Coladello, L. F. (2015). A bivariate generalized exponential distribution derived from copula functions in the presence of censored data and covariates. *Pesquisa Operacional*, *35*, 165–186.

Adamidis, K., & Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics & Probability Letters*, *39*(1), 35–42.

Ahmed, D., Khames, S., & Mokhlis, N. A. (2020). Inference for stress-strength models based on the bivariate general farlie-gumbel-morgenstern distributions. *Journal of Statistics Applications & Probability Letters*, *7*(3), 141–150.

Akdam, N., Kinaci, I., & Saracoglu, B. (2017). Statistical inference of stress-strength reliability for the exponential power (ep) distribution based on progressive type-ii censored samples. *Hacettepe Journal of Mathematics and Statistics*, *46*(2), 239–253.

Almetwally, E. M., Almongy, H. M., Rastogi, M. K., & Ibrahim, M. (2020). Maximum product spacing estimation of weibull distribution under adaptive type-ii progressive censoring schemes. *Annals of Data Science*, *7*(2), 257–279.

Almetwally, E. M., Muhammed, H. Z., & El-Sherpieny, E.-S. A. (2020). Bivariate weibull distribution: properties and different methods of estimation. *Annals of Data Science*, *7*, 163–193.

Al-Moisheer, A. S., Alotaibi, R. M., Alomani, G. A., & Rezk, H. (2020). Bivariate mixture of inverse weibull distribution: properties and estimation. *Mathematical Problems in Engineering*, *2020*, 1–12.

Almuhayfith, F. E. (2024). Statistical inference of comparative generalized inverted exponential populations under joint adaptive progressive type-ii censored samples. *Alexandria Engineering Journal*, *95*, 262–271.

Alsadat, N., Hassan, A. S., Elgarhy, M., Chesneau, C., & El-Saeed, A. R. (2023). Sampling plan for the kavya–manoharan generalized inverted kumaraswamy distribution with statistical inference and applications. *Axioms*, *12*(8), 739.

Aly, H. M., & Abuelamayem, O. A. (2020). Multivariate inverted kumaraswamy distribution: Derivation and estimation. *Mathematical Problems in Engineering*, *2020*(1), 6349523.

Amrutha, M., & Chacko, V. M. (2024). Power generalized dus transformation of inverse kumaraswamy distribution and stress-strength analysis. *Statistics and Applications*, *22*(2), 323–359.

Anakha, K. K., & Chacko, V. M. (2021). Dus-kumaraswamy distribution: A bathtub shaped failure rate model. *International Journal of Statistics and Reliability Engineering*, *8*(3), 359–367.

Ateya, S. F., & Mohammed, H. S. (2017). Statistical inferences based on an adaptive progressive type-ii censoring from exponentiated exponential distribution. *Journal of the Egyptian Mathematical Society*, *25*(4), 393–399.

Bakouch, H. S., Jazi, M. A., Nadarajah, S., Dolati, A., & Roozegar, R. (2014). A lifetime model with increasing failure rate. *Applied Mathematical Modelling*, *38*(23), 5392–5406.

Balakrishnan, N. (2007). Progressive censoring methodology: an appraisal. *Test*, *16*(2), 211–259.

Balakrishnan, N., & Aggarwala, R. (2000). *Progressive censoring: theory, methods, and applications*. Boston: BirkhÄauser.

Balakrishnan, N., Burkschat, M., Cramer, E., & Hofmann, G. (2008). Fisher information based progressive censoring plans. *Computational statistics & data analysis*, *53*(2), 366–380.

Balakrishnan, N., & Cramer, E. (2014). The art of progressive censoring. *Statistics for industry and technology*.

Balakrishnan, N., Cramer, E., & Kundu, D. (2023). *Hybrid censoring know-how: Designs and implementations*. Academic Press.

Balakrishnan, N., & Kundu, D. (2013). Hybrid censoring: Models, inferential results and applications. *Computational Statistics & Data Analysis*, *57*(1), 166–209.

Balakrishnan, N., & Rasouli, A. (2008). Exact likelihood inference for two exponential populations under joint type-ii censoring. *Computational Statistics & Data Analysis*, *52*(5), 2725–2738.

Balakrishnan, N., & Sandhu, R. A. (1995). A simple simulational algorithm for generating progressive type-ii censored samples. *The American Statistician*, *49*(2), 229–230.

Barlow, R., & Proschan, F. (1975). *Statistical theory of reliability and life testing*. Reinhart and Winston, New York:Holt.

Barlow, R. E., & Campo, R. (1975). Total time on test processes and applications to failure data analysis. In *Reliability and fault tree analysis.*

Barlow, R. E., Marshall, A. W., & Proschan, F. (1963). Properties of probability distributions with monotone hazard rate. *The Annals of Mathematical Statistics*, 375–389.

Barlow, R. E., & Proschan, F. (1996). *Mathematical theory of reliability*. SIAM.

Berger, J. O., & Sun, D. (1993). Bayesian analysis for the poly-weibull distribution. *Journal of the American Statistical Association*, *88*(424), 1412–1418.

Bergman, B. (1977). Crossings in the total time on test plot. *Scandinavian Journal of Statistics*, 171–177.

Berkson, J., & Gage, R. P. (1952). Survival curve for cancer patients following treatment. *Journal of the American Statistical Association*, *47*(259), 501–515.

Bhati, D., Malik, M. A., & Vaman, H. J. (2015). Lindley–exponential distribution: properties and applications. *Metron*, *73*, 335–357.

Bhattacharjee, S., & Misra, S. K. (2016). Some aging properties of weibull models. *Electronic Journal of Applied Statistical Analysis*, *9*(2), 297–307.

Bhattacharyya, G. K. (1995). Inferences under two-sample and multi-sample situations. In N. Balakrishnan & A. P. Basu (Eds.), *The exponential distribution: Theory, methods and applications* (pp. 93–118). Newark, NJ: Gordon and Breach.

Bhattacharyya, G. K., & Johnson, R. A. (1974). Estimation of reliability in a multicomponent stress-strength model. *Journal of the American Statistical Association*, *69*(348), 966–970.

Bhattacharyya, G. K., & Mehrotra, K. G. (1981). On testing equality of two exponential distributions under combined type ii censoring. *Journal of the American Statistical Association*, *76*(376), 886–894.

Birnbaum, Z. W. (1956). On a use of the mann-whitney statistic. In *Proceedings of the third berkeley symposium on mathematical statistics and probability* (Vol. 1, pp. 13–17).

Biswas, A., Chakraborty, S., & Mukherjee, M. (2021). On estimation of stress–strength reliability with log-lindley distribution. *Journal of Statistical Computation and Simulation*, *91*(1), 128–150.

Boag, J. W. (1949). Maximum likelihood estimates of the proportion of patients cured by cancer therapy. *Journal of the Royal Statistical Society. Series B (Methodological)*, *11*(1), 15–53.

Bugatekin, A. T. (2017). A new distribution model with two-parameter. *New Trends in Mathematical Sciences*, *5*(2), 80–84.

Burkschat, M. (2008). On optimality of extremal schemes in progressive type ii censoring. *Journal of Statistical Planning and Inference*, *138*(6), 1647–1659.

Burkschat, M., Cramer, E., & Kamps, U. (2006). On optimal schemes in progressive censoring. *Statistics & probability letters*, *76*(10), 1032–1036.

Chacko, V. M., & Deepthi, K. S. (2019). Generalized x-exponential bathtub shaped failure rate distribution. *Journal of the Indian Society of Probability and Statistics*, *20*, 157–171.

Chahkandi, M., & Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate. *Computational Statistics & Data Analysis*, *53*(12), 4433–4440.

Chandra, N., & Pandey, M. (2012). Bayesian reliability estimation of bivariate marshal-olkin exponential stress-strength model. *International Journal of Reliability and Applications*, *13*(1), 37–47.

Chaturvedi, A., & Malhotra, A. (2020). On estimation of stress-strength reliability using lower record values from proportional reversed hazard family. *American Journal of Mathematical and Management Sciences*, *39*(3), 234–251.

Chen, M.-H., & Shao, Q.-M. (1999). Monte carlo estimation of bayesian credible and hpd intervals. *Journal of computational and Graphical Statistics*, *8*(1), 69–92.

Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. *Statistics & Probability Letters*, *49*(2), 155–161.

Chesneau, C. (2024). Investigation of a variable-power fgm-type copula. *Stochastic Models in Probability and Statistics*, 17–36.

Childs, A., Chandrasekar, B., Balakrishnan, N., & Kundu, D. (2003). Exact likelihood inference based on type-i and type-ii hybrid censored samples from the exponential distribution. *Annals of the Institute of Statistical Mathematics*, *55*(2), 319–330.

Clayton, D. G. (1978). A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika*, *65*(1), 141–151.

Cohen, A. C. (1963). Progressively censored samples in life testing. *Technometrics*, *5*(3), 327–339.

Corbière, F., Commenges, D., Taylor, J. M. G., & Joly, P. (2009). A penalized likelihood approach for mixture cure models. *Statistics in medicine*, *28*(3), 510–524.

Cox, D. R. (1972). Regression models and life-tables. *Journal of the Royal Statistical Society: Series B (Methodological)*, *34*(2), 187–202.

Cramer, E., & Iliopoulos, G. (2010). Adaptive progressive type-ii censoring. *Test*, *19*, 342–358.

Cramer, E., & Iliopoulos, G. (2015). Adaptive progressive censoring. In *Ordered data analysis, modeling and health research methods: In honor of hn nagaraja's 60th birthday* (pp. 73–86).

Dasgupta, R. (2011). On the distribution of burr with applications. *Sankhya B*, *73*, 1–19.

Deepthi, K. S., & Chacko, V. M. (2020). An upside-down bathtub-shaped failure rate model using a dus transformation of lomax distribution. In *Stochastic models in reliability engineering* (pp. 81–100). CRC Press.

Deshpande, J. V., & Suresh, R. P. (1990). Non-monotonic ageing. *Scandinavian journal of statistics*, 257–262.

Dey, S., Kumar, D., & Seddik-Ameur, N. (2021). Different classical methods of estimation and chi-squared goodness-of-fit test for unit generalized inverse weibull distribution. *Austrian Journal of Statistics*, *50*(5), 77–100.

Domma, F., & Giordano, S. (2013). A copula-based approach to account for dependence in stress-strength models. *Statistical Papers*, *54*, 807–826.

Dumonceaux, R., & Antle, C. E. (1973). Discrimination between the log-normal and the weibull distributions. *Technometrics*, *15*(4), 923–926.

E, R., Karlin, F., Proschan, & Barlow, S. (1961). Moment inequalities of pólya frequency functions.

Efron, B. (1982). *The jackknife, the bootstrap and other resampling plans.* SIAM.

Efron, B., & Johnstone, I. M. (1990). Fisher's information in terms of the hazard rate. *The Annals of Statistics*, 38–62.

El Barmi, H., Malla, G., & Mukerjee, H. (2021). Estimation of a distribution function with increasing failure rate average. *Journal of Statistical Planning and Inference*, *213*, 179–192.

Elgarhy, M., Haq, M. A. u., & ul Ain, Q. (2018). Exponentiated generalized kumaraswamy distribution with applications. *Annals of Data Science*, *5*, 273–292.

El-Sherpieny, E.-S. A., Muhammed, H. Z., & Almetwally, E. M. (2022). Bivariate chen distribution based on copula function: Properties and application of diabetic nephropathy. *Journal of Statistical Theory and Practice*, *16*(3), 54.

Epstein, B. (1954). Truncated life tests in the exponential case. *The Annals of Mathematical Statistics*, 555–564.

Eyraud, H. (1936). Les principes de la mesure des correlations. *Ann. Univ. Lyon, III. Ser., Sect. A*, *1*(30-47), 111.

Farlie, D. J. (1960). The performance of some correlation coefficients for a general bivariate distribution. *Biometrika*, *47*(3/4), 307–323.

Finkelstein, M. S., & Esaulova, V. (2001). Modeling a failure rate for a mixture of distribution functions. *Probability in the Engineering and Informational Sciences*, *15*(3), 383–400.

Fyffe, D. E., Hines, W. W., & Lee, N. K. (1968). System reliability allocation and a computational algorithm. *IEEE Transactions on Reliability*, *17*(2), 64–69.

Gajjar, A. V., & Khatri, C. G. (1969). Progressively censored samples from log-normal and logistic distributions. *Technometrics*, *11*(4), 793–803.

Garg, M. (2008). On distribution of order statistics from kumaraswamy distribution. *Kyungpook mathematical journal*, *48*(3), 411–417.

Gauthami, P., & Chacko, V. M. (2021). Dus transformation of inverse weibull distribution: An upside-down failure rate model. *Reliability: Theory & Applications*, *16*(2 (62)), 58–71.

Geisser, S., & Eddy, W. F. (1979). A predictive approach to model selection. *Journal of the American Statistical Association*, *74*(365), 153–160.

Gelfand, A. E., & Dey, D. K. (1994). Bayesian model choice: asymptotics and exact calculations. *Journal of the Royal Statistical Society: Series B (Methodological)*, *56*(3), 501–514.

Genest, C., Huang, W., & Dufour, J.-M. (2013). A regularized goodness-of-fit test for copulas. *Journal de la Société française de statistique*, *154*(1), 64–77.

Gharaaghaji, R., Ayatollahi, H., Mohammadpour, M., Fereidoni, J., & Khalkhali, H. R. (2021). Application of mixture cure models in determining the survival of patients with cervical cancer. *International Journal of Cancer Management*, *14*(3).

Ghitany, M. E., Atieh, B., & Nadarajah, S. (2008). Lindley distribution and its application. *Mathematics and computers in simulation*, *78*(4), 493–506.

Ghosh, I., & Hamedani, G. G. (2017). Gamma-kumaraswamy distribution in reliability analysis: properties and applications. *Advances in Statistical methodologies and their application to real problems, edited by Hokimoto T*, *6*, 123–142.

Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. In *Mathematical statistics and probability theory: volume b statistical inference and methods proceedings of the 6th pannonian symposium on mathematical statistics, bad tatzmannsdorf, austria, september 14–20, 1986* (pp. 75–84).

Glaser, R. E. (1980). Bathtub and related failure rate characterizations. *Journal of the American Statistical Association*, *75*(371), 667–672.

Goel, R., & Krishna, H. (2022). Statistical inference for two lindley populations under balanced joint progressive type-ii censoring scheme. *Computational Statistics*, *37*(1), 263–286.

Griffith, W. S. (1982). Representation of distributions having monotone or bathtub-shaped failure rates. *IEEE Transactions on Reliability*, *31*(1), 95–96.

Guess, F., & Proschan, F. (1988). 12 mean residual life: theory and applications. *Handbook of statistics*, *7*, 215–224.

Guilani, P. P., Azimi, P., Niaki, S. T. A., & Niaki, S. A. A. (2016). Redundancy allocation problem of a system with increasing failure rates of components based on weibull distribution: A simulation-based optimization approach. *Reliability Engineering & System Safety*, *152*, 187–196.

Gumbel, E. J. (1958). *Statistics of extremes.* Columbia university press.

Gumbel, E. J. (1960). Bivariate exponential distributions. *Journal of the American Statistical Association*, *55*(292), 698–707.

Gupta, R. C., & Viles, W. (2011). Roller-coaster failure rates and mean residual life functions with application to the extended generalized inverse gaussian model. *Probability in the Engineering and Informational Sciences*, *25*(1), 103–118.

Gupta, R. C., & Warren, R. (2001). Determination of change points of non-monotonic failure rates. *Communications in Statistics-Theory and Methods*, *30*(8-9), 1903–1920.

Gupta, R. D., & Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and weibull distributions. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, *43*(1), 117–130.

Hameed, B. A., Salman, A. N., & Kalaf, B. A. (2020). On estimation of in cased inverse kumaraswamy distribution. *Iraqi Journal of Science*, *61*(4), 845–853.

Haq, M. A. u., Hashmi, S., Aidi, K., Ramos, P. L., & Louzada, F. (2020). Unit modified burr-iii distribution: Estimation, characterizations and validation test. *Annals of Data Science*, 1–26.

Hassan, A. S., Nagy, H. F., Muhammed, H. Z., & Saad, M. S. (2020). Estimation of multicomponent stress-strength reliability following weibull distribution based on upper record values. *Journal of Taibah University for Science*, *14*(1), 244–253.

Hastings, W. K. (1970). Monte carlo sampling methods using markov chains and their applications.

Hemmati, F., & Khorram, E. (2017). On adaptive progressively type-ii censored competing risks data. *Communications in Statistics-Simulation and Computation*, *46*(6), 4671–4693.

Herd, G. R. (1956). *Estimation of the parameters of a population from a multi-*

censored sample. Iowa State University.

Hinkley, D. (1977). On quick choice of power transformation. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, *26*(1), 67–69.

Holland, P. W., & Wang, Y. J. (1987). Dependence function for continuous bivariate densities. *Communications in Statistics-Theory and Methods*, *16*(3), 863–876.

Hollander, M., & Proschan, F. (1984). 27 nonparametric concepts and methods in reliability. *Handbook of statistics*, *4*, 613–655.

Huang, J. S., & Kotz, S. (1984). Correlation structure in iterated farlie-gumbel-morgenstern distributions. *Biometrika*, *71*(3), 633–636.

Huang, J. S., & Kotz, S. (1999). Modifications of the farlie-gumbel-morgenstern distributions. a tough hill to climb. *Metrika*, *49*(2), 135–145.

Ibrahim, J. G., Chen, M.-H., Sinha, D., Ibrahim, J. G., & Chen, M. H. (2001). *Bayesian survival analysis* (Vol. 2). Springer.

Iqbal, Z., Tahir, M. M., Riaz, N., Ali, S. A., & Ahmad, M. (2017). Generalized inverted kumaraswamy distribution: properties and application. *Open Journal of Statistics*, *7*(4), 645–662.

Ishaq, A., Usman, A., Tasi'u, M., Aliyu, Y., & Idris, F. (2017). A new weibull-kumaraswamy distribution: Theory and applications. *Nigerian Journal of Scientific Research*, *16*(2), 158–166.

Jamal, F., Arslan Nasir, M., Ozel, G., Elgarhy, M., & Mamode Khan, N. (2019). Generalized inverted kumaraswamy generated family of distributions: theory and applications. *Journal of Applied Statistics*, *46*(16), 2927–2944.

James, A., & Chandra, N. (2022). Dependence stress-strength reliability estimation of bivariate xgamma exponential distribution under copula approach. *Palestine Journal of Mathematics*, *11*.

James, A., Chandra, N., & Pandey, M. (2022). A copula based stress-strength reliability estimation with lindley marginals. *Journal of Reliability and Statistical Studies*, 341–380.

Jiang, R., & Murthy, D. (1999). The exponentiated weibull family: a graphical approach. *IEEE Transactions on Reliability*, *48*(1), 68–72.

Jiang, R., Murthy, D. N. P., & Ji, P. (2001). Models involving two inverse weibull distributions. *Reliability Engineering & System Safety*, *73*(1), 73–81.

Joe, H. (1997). *Multivariate models and multivariate dependence concepts*. CRC press.

Joe, H. (2005). Asymptotic efficiency of the two-stage estimation method for copula-based models. *Journal of multivariate Analysis*, *94*(2), 401–419.

Johnson, N. L., & Kotz, S. (1975). A vector multivariate hazard rate. *Journal of Multivariate Analysis*, *5*(1), 53–66.

Kannan, N., Kundu, D., Nair, P., & Tripathi, R. (2010). The generalized exponential cure rate model with covariates. *Journal of Applied Statistics*, *37*(10), 1625–1636.

Kavya, P., & Manoharan, M. (2020). On a generalized lifetime model using dus transformation. *Applied Probability and Stochastic Processes*, 281–291.

Kayal, T., Tripathi, Y. M., Dey, S., & Wu, S.-J. (2020). On estimating the reliability in a multicomponent stress-strength model based on chen distribution. *Communications in Statistics-Theory and Methods*, *49*(10), 2429–2447.

Keller, A. Z., Goblin, M. T., & Farnworth, N. R. (1985). Reliability analysis of commercial vehicle engines. *Reliability Engineering*, *10*(1), 15–25.

Khan, M. S., Pasha, G. R., & Pasha, A. H. (2008). Theoretical analysis of inverse weibull distribution. *WSEAS Transactions on Mathematics*, *7*(2), 30–38.

Khatoon, B., Khan, M. J. S., & Noor, Z. (2021). Moments of dual generalized order statistics from inverted kumaraswamy distribution and related inference. *Journal of Statistical Theory and Applications*, *20*(2), 251–266.

Kızılaslan, F., & Nadar, M. (2018). Estimation of reliability in a multicomponent stress–strength model based on a bivariate kumaraswamy distribution. *Statistical Papers*, *59*, 307–340.

Klefsjö, B. (1980). *On some classes of bivariate life distributions*. Umeå universitet.

Kohansal, A. (2019). On estimation of reliability in a multicomponent stress-strength model for a kumaraswamy distribution based on progressively censored sample. *Statistical Papers*, *60*, 2185–2224.

Kohansal, A., & Nadarajah, S. (2019). Stress–strength parameter estimation based on type-ii hybrid progressive censored samples for a kumaraswamy distribution. *IEEE Transactions on Reliability*, *68*(4), 1296–1310.

Korkmaz, M. Ç., Yousof, H. M., Rasekhi, M., & Hamedani, G. G. (2018). The odd lindley burr xii model: Bayesian analysis, classical inference and characterizations. *Journal of data science*.

Koutras, V. P. (2011). Two-level software rejuvenation model with increasing failure rate degradation. In *Dependable computer systems* (pp. 101–115). Springer.

Kumar, D., Singh, U., & Singh, S. K. (2015). A method of proposing new distribution and its application to bladder cancer patients data. *J. Stat. Appl. Pro. Lett*, *2*(3), 235–245.

Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of hydrology*, *46*(1-2), 79–88.

Kumar Singh, S., Singh, U., & Sharma, V. K. (2016). Estimation and prediction for type-i hybrid censored data from generalized lindley distribution. *Journal of Statistics and Management Systems*, *19*(3), 367–396.

Kundu, D. (2007). On hybrid censored weibull distribution. *Journal of Statistical Planning and Inference*, *137*(7), 2127–2142.

Kundu, D. (2008). Bayesian inference and life testing plan for the weibull distribution in presence of progressive censoring. *Technometrics*, *50*(2), 144–154.

Kundu, D. (2009). Bayesian inference and life testing plan for the weibull distribution in presence of progressive censoring. *Quality control and applied statistics*, *54*(1), 99–100.

Kundu, D., & Gupta, A. K. (2017). On bivariate inverse weibull distribution. *Brazilian Journal of Probability and Statistics*, *31*(2), 275–302.

Kundu, D., & Joarder, A. (2006). Analysis of type-ii progressively hybrid censored data. *Computational Statistics & Data Analysis*, *50*(10), 2509–2528.

Kvam, P. H., Singh, H., & Whitaker, L. R. (2002). Estimating distributions with increasing failure rate in an imperfect repair model. *Lifetime data analysis*, *8*, 53–67.

Lai, C. D., & Balakrishnan, N. (2009). *Continuous bivariate distributions*. Springer.

Lai, C. D., & Xie, M. (2006). *Stochastic ageing and dependence for reliability*. Springer Science & Business Media.

Lai, C. D., Xie, M., & Murthy, D. N. P. (2003a). A modified weibull distribution. *IEEE Transactions on Reliability*, *52*(1), 33-37. doi: 10.1109/TR.2002.805788

Lai, C. D., Xie, M., & Murthy, D. N. P. (2003b). A modified weibull distribution. *IEEE Transactions on reliability*, *52*(1), 33–37.

Lambert, P. C. (2007). Modeling of the cure fraction in survival studies. *The Stata Journal*, *7*(3), 351–375.

Langberg, N. A., Leon, R. V., & Proschan, F. (1980). Characterization of nonparametric classes of life distributions. *The Annals of Probability*, *8*(6), 1163–1170.

Lariviere, M. A. (2006). A note on probability distributions with increasing generalized failure rates. *Operations Research*, *54*(3), 602–604.

Lariviere, M. A., & Porteus, E. L. (2001). Selling to the newsvendor: An analysis of price-only contracts. *Manufacturing & service operations management*, *3*(4), 293–305.

Lawless, J. F. (2011). *Statistical models and methods for lifetime data*. John Wiley & Sons.

Lee, L., & Thompson Jr, W. A. (1976). Failure rate—a unified approach. *Journal of Applied Probability*, *13*(1), 176–182.

Leemis, L. M. (1986). Lifetime distribution identities. *IEEE Transactions on Reliability*, *35*(2), 170-174. doi: 10.1109/TR.1986.4335395

Lehmann, E. L. (1966). Some concepts of dependence. *The Annals of Mathematical Statistics*, *37*(5), 1137–1153.

Lemonte, A. J. (2011). Improved point estimation for the kumaraswamy distribution. *Journal of Statistical Computation and Simulation*, *81*(12), 1971–1982.

Lemonte, A. J., Barreto-Souza, W., & Cordeiro, G. M. (2013). The exponentiated kumaraswamy distribution and its log-transform. *Brazilian Journal of Probability and Statistics*, *27*(1), 31-53.

Lieblein, J., & Zelen, M. (1956). Statistical investigation of the fatigue life of deep-groove ball bearings. *Journal of research of the national bureau of standards*, *57*(5), 273–316.

Lindley, D. V. (1958). Fiducial distributions and bayes' theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, 102–107.

Looha, M. A., Pourhoseingholi, M. A., Nasserinejad, M., Najafimehr, H., & Zali, M. R. (2018). Application of a non-parametric non-mixture cure rate model for analyzing the survival of patients with colorectal cancer in iran. *Epidemiology and Health*, *40*.

Maller, R., Resnick, S., Shemehsavar, S., & Zhao, M. (2024). Mixture cure model methodology in survival analysis: Some recent results for the one-sample case. *Statistic Surveys*, *18*, 82–138.

Marshall, A. W., & Proschan, F. (1965). Maximum likelihood estimation for distributions with monotone failure rate. *The annals of mathematical statistics*, *36*(1), 69–77.

Martinez, E. Z., Achcar, J. A., Jácome, A. A. A., & Santos, J. S. (2013). Mixture and non-mixture cure fraction models based on the generalized modified weibull distribution with an application to gastric cancer data. *Computer methods and programs in biomedicine*, *112*(3), 343–355.

Martinez, E. Z., de Freitas, B. C. L., Achcar, J. A., Aragon, D. C., & de Oliveira Peres, M. V. (2022). Exponentiated weibull models applied to medical data in presence of right-censoring, cure fraction and covariates. *Statistics, Optimization & Information Computing*, *10*(2), 548–571.

Maurya, S. K., Kaushik, A., Singh, S. K., & Singh, U. (2017). A new class of exponential transformed lindley distribution and its application to yarn data. *International Journal of Statistics and Economics*, *18*(2), 135–151.

McGilchrist, C. A., & Aisbett, C. W. (1991). Regression with frailty in survival analysis. *Biometrics*, 461–466.

Merovci, F., & Sharma, V. K. (2014). The beta-lindley distribution: Properties and applications. *Journal of applied mathematics*, *2014*(1), 198951.

Merovci, I. E. F., & Elgarhy, M. (2013). A new generalized lindley distribution. *Mathematical Theory and Modeling*, *3*(13), 30-47.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., & Teller, E. (1953). Equation of state calculations by fast computing machines. *The journal of chemical physics*, *21*(6), 1087–1092.

Mi, J. (1995). Bathtub failure rate and upside-down bathtub mean residual life. *IEEE Transactions on reliability*, *44*(3), 388–391.

Mitnik, P. A. (2013). New properties of the kumaraswamy distribution. *Communications in Statistics-Theory and Methods*, *42*(5), 741–755.

Mitnik, P. A., & Baek, S. (2013). The kumaraswamy distribution: median-dispersion re-parameterizations for regression modeling and simulation-based estimation. *Statistical Papers*, *54*, 177–192.

Mitra, M., & Basu, S. K. (1996). On some properties of the bathtub failure rate family of life distributions. *Microelectronics Reliability*, *36*(5), 679–684.

Mohan, R., & Chacko, M. (2021). Estimation of parameters of kumaraswamy-exponential distribution based on adaptive type-ii progressive censored schemes.

Journal of Statistical Computation and Simulation, *91*(1), 81–107.

Mondal, S., & Kundu, D. (2018). A new two sample type-ii progressive censoring scheme. *Communications in Statistics-Theory and Methods*, *48*(10), 2602–2618.

Mondal, S., & Kundu, D. (2020a). Bayesian inference for weibull distribution under the balanced joint type-ii progressive censoring scheme. *American Journal of Mathematical and Management Sciences*, *39*(1), 56–74.

Mondal, S., & Kundu, D. (2020b). A bivariate inverse weibull distribution and its application in complementary risks model. *Journal of Applied Statistics*, *47*(6), 1084–1108.

Morgenstern, D. (1956). Einfache beispiele zweidimensionaler verteilungen. *Mitteilingsblatt fur Mathematische Statistik*, *8*, 234–235.

Mudholkar, G. S., & Srivastava, D. K. (1993). Exponentiated weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, *42*(2), 299–302.

Muhammed, H. Z. (2016). Bivariate inverse weibull distribution. *Journal of Statistical Computation and Simulation*, *86*(12), 2335–2345.

Murthy, V. K., Swartz, G., & Yuen, K. (1973). Realistic models for mortality rates and their estimation. In *Technical reports i and ii.* Department of Biostatistics, University of California at LosAngeles.

Nadar, M., Papadopoulos, A., & Kızılaslan, F. (2013). Statistical analysis for kumaraswamy's distribution based on record data. *Statistical Papers*, *54*, 355–369.

Nadarajah, S. (2005). Reliability for some bivariate gamma distributions. *Mathematical Problems in Engineering*, *2005*(2), 151–163.

Nadarajah, S. (2008). On the distribution of kumaraswamy. *Journal of Hydrology*, *348*(3), 568–569.

Nadarajah, S., Bakouch, H. S., & Tahmasbi, R. (2011). A generalized lindley distribution. *Sankhya B*, *73*, 331–359.

Nagy, H. F., Al-Omari, A. I., Hassan, A. S., & Alomani, G. A. (2022). Improved estimation of the inverted kumaraswamy distribution parameters based on ranked set sampling with an application to real data. *Mathematics*, *10*(21), 4102.

Nair, N. U., Sankaran, P. G., & Balakrishnan, N. (2013). *Quantile-based reliability analysis*. Springer.

Nelsen, R. B. (2006). An introduction to copulas. springer, new york. *MR2197664*.

Nelson, W. B. (2005). *Applied life data analysis*. John Wiley & Sons.

Ng, H. K. T., Kundu, D., & Chan, P. S. (2009). Statistical analysis of exponential lifetimes under an adaptive type-ii progressive censoring scheme. *Naval Research Logistics (NRL)*, *56*(8), 687–698.

Nikulin, M. S. (1973). Chi-square test for continuous distributions with location and scale parameters. *Teoriya Veroyatnostei i ee Primeneniya*, *18*(3), 583–591.

Oakes, D. (1989). Bivariate survival models induced by frailties. *Journal of the American Statistical Association*, *84*(406), 487–493.

Pakdaman, Z., & Ahmadi, J. (2018). Point estimation of the stress–strength reliability parameter for parallel system with independent and non-identical components. *Communications in Statistics-Simulation and Computation*, *47*(4), 1193–1203.

Park, S., & Balakrishnan, N. (2009). On simple calculation of the fisher information in hybrid censoring schemes. *Statistics & Probability Letters*, *79*(10), 1311–1319.

Pasari, S., & Dikshit, O. (2018). Stochastic earthquake interevent time modeling from exponentiated weibull distributions. *Natural hazards*, *90*, 823–842.

Peng, Y., Dear, K. B., & Denham, J. (1998). A generalized f mixture model for cure rate estimation. *Statistics in Medicine*, *17*(8), 813–830. doi: 10.1002/(SICI)1097-0258(19980430)17:8<813::AID-SIM703>3.0.CO;2-0

Pham, H. (2006). *Handbook of reliability engineering*. Springer Science & Business Media.

Popović, B. V., Genç, A. İ., & Domma, F. (2018). Copula-based properties of the bivariate dagum distribution. *Computational and Applied Mathematics*, *37*, 6230–6251.

Qura, M. E., Fayomi, A., Kilai, M., & Almetwally, E. M. (2023). Bivariate power lomax distribution with medical applications. *Plos one*, *18*(3), e0282581.

Rajarshi, S., & Rajarshi, M. B. (1988). Bathtub distributions: A review. *Communications in Statistics-Theory and Methods*, *17*(8), 2597–2621.

Ramos, P. L., Nascimento, D., & Louzada, F. (2017). The long term fr\'echet distribution: Estimation, properties and its application. *arXiv preprint* *arXiv:1709.07593*.

Rao, K. C., & Robson, B. S. (1974). A chi-squabe statistic for goodies-of-fit tests within the exponential family. *Communications in Statistics-Theory and Methods*, *3*(12), 1139–1153.

Rashad, A., Yusuf, M., & Moheb, S. (2019). Approximate bayes estimators of the inverted kumaraswamy distribution parameters based on progressive type-ii censoring scheme. *Journal of Statistics Applications & Probability*, *8*(3), 189–199.

Rasouli, A., & Balakrishnan, N. (2010). Exact likelihood inference for two exponential populations under joint progressive type-ii censoring. *Communications in Statistics—Theory and Methods*, *39*(12), 2172–2191.

Rinne, H. (2008). *The weibull distribution: a handbook*. Chapman and Hall/CRC.

Ross, S. M. (1995). *Stochastic processes*. John Wiley & Sons.

Samanthi, R. G. M., & Sepanski, J. (2019). A bivariate extension of the beta generated distribution derived from copulas. *Communications in Statistics-Theory and Methods*, *48*(5), 1043–1059.

Sarmanov, O. V. (1966). Generalized normal correlation and two-dimensional fréchet classes. In *Doklady akademii nauk* (Vol. 168, pp. 32–35).

Schmiedt, A. B., & Cramer, E. (2024). Generalized ng–kundu–chan model of adaptive progressive type-ii censoring and related inference. *Naval Research Logistics (NRL)*, *71*(3), 389–415.

Schoenberg, I. J. (1951). On pólya frequency functions: I. the totally positive functions and their laplace transforms. *Journal d'Analyse Mathématique*, *1*(1), 331–374.

Shahbaz, M. Q., Shahbaz, S., & Butt, N. S. (2012). The kumaraswamy-inverse weibull distribution. *Shahbaz, MQ, Shahbaz, S., & Butt, NS (2012). The Kumaraswamy–Inverse Weibull Distribution. Pakistan journal of statistics and operation research*, *8*(3), 479–489.

Shao, Q., & Zhou, X. (2004). A new parametric model for survival data with long-term survivors. *Statistics in medicine*, *23*(22), 3525–3543.

Sharma, V. K., & Dey, S. (2019). Estimation of reliability of multicomponent stress-strength inverted exponentiated rayleigh model. *Journal of Industrial and Production Engineering*, *36*(3), 181–192.

Shih, J.-H., Konno, Y., Chang, Y.-T., & Emura, T. (2019). Estimation of a common mean vector in bivariate meta-analysis under the fgm copula. *Statistics*, *53*(3), 673–695.

Singh, S. K., Singh, U., & Sharma, V. K. (2013). Expected total test time and bayesian estimation for generalized lindley distribution under progressively type-ii censored sample where removals follow the beta-binomial probability law. *Applied Mathematics and Computation*, *222*, 402–419.

Singh, S. K., Singh, U., & Sharma, V. K. (2014). Bayesian estimation and prediction for the generalized lindley distribution under asymmetric loss function. *Hacettepe Journal of Mathematics and Statistics*, *43*(4), 661–678.

Sklar, A. (1973). Random variables, joint distribution functions, and copulas. *Kybernetika*, *9*(6), 449–460.

Sklar, M. (1959). Fonctions de répartition à n dimensions et leurs marges. In *Annales de l'isup* (Vol. 8, pp. 229–231).

Smith, R. M., & Bain, L. J. (1975). An exponential power life-testing distribution. *Communications in Statistics-Theory and Methods*, *4*(5), 469–481.

Steffensen, J. F. (1930). Infantile mortality from an actuarial point of view. *Scandinavian Actuarial Journal*, *1930*(2), 272–286.

Sultana, F., Çetinkaya, Ç., & Kundu, D. (2023). Statistical inference for the jointly adaptive progressive type-ii censored weibull distributions. *Journal of Statistical Theory and Practice*, *17*(2), 24.

Sultana, F., Koley, A., Pal, A., & Kundu, D. (2021). On two exponential populations under a joint adaptive type-ii progressive censoring. *Statistics*, *55*(6), 1328–1355.

Thomas, B., & Chacko, V. M. (2021). Power generalized dus transformation of exponential distribution. *International Journal of Statistics and Reliability Engineering*, *8*(3), 359–367.

Thomas, B., & Chacko, V. M. (2023). Power generalized dus transformation in weibull and lomax distributions. *Reliability: Theory & Applications*, *18*(1 (72)), 368–384.

Tripathi, A., Singh, U., & Singh, S. K. (2021). Inferences for the dus-exponential distribution based on upper record values. *Annals of Data Science*, *8*, 387–403.

Tsodikov, A. D., Ibrahim, J. G., & Yakovlev, A. Y. (2003). Estimating cure rates from survival data: an alternative to two-component mixture models. *Journal of the American Statistical Association*, *98*(464), 1063–1078.

Usman, U., Suleiman, S., Arkilla, B. M., & Aliyu, Y. (2021). Nadarajah-haghighi model for survival data with long term survivors in the presence of right censored data. *Pakistan Journal of Statistics and Operation Research*, 695–709.

Vaidyanathan, V. S., & Sharon Varghese, A. (2016). Morgenstern type bivariate lindley distribution. *Statistics, Optimization & Information Computing*, *4*(2), 132–146.

Voinov, V., Pya, N., Shapakov, N., & Voinov, Y. (2013). Goodness-of-fit tests for the power-generalized weibull probability distribution. *Communications in Statistics-Simulation and Computation*, *42*(5), 1003–1012.

Wang, Y., Deng, Y., & Zhou, X.-H. (2024). Causal inference for time-to-event data with a cured subpopulation. *Biometrics*, *80*(2), ujae028.

Wong, K. L. (1988). The bathtub does not hold water any more. *Quality and reliability engineering international*, *4*(3), 279–282.

Wong, K. L. (1991). The physical basis for the roller-coaster hazard rate curve for electronics. *Quality and reliability engineering international*, *7*(6), 489–495.

Wong, K. L., & Lindstrom, D. L. (1988). Off the bathtub onto the roller-coaster curve (electronic equipment failure). In *1988. proceedings., annual reliability and maintainability symposium,* (pp. 356–363).

Wu, S.-J. (2008). Estimation of the two-parameter bathtub-shaped lifetime distribution with progressive censoring. *Journal of Applied Statistics*, *35*(10), 1139–1150.

Xavier, T., & Jose, J. K. (2021). A study of stress-strength reliability using a generalization of power transformed half-logistic distribution. *Communications in Statistics-Theory and Methods*, *50*(18), 4335–4351.

Xie, M. (1989). Some total time on test quantities useful for testing constant against bathtub-shaped failure rate distributions. *Scandinavian journal of statistics*, 137–144.

Xu, J. J. (1996). *Statistical modelling and inference for multivariate and longitudinal discrete response data* (Unpublished doctoral dissertation). University of British Columbia.

Yadav, A. S., Shukla, S., Goual, H., Saha, M., & Yousof, H. M. (2022). Validation of xgamma exponential model via nikulin-rao-robson goodness-of-fit-test under complete and censored sample with different methods of estimation. *Statistics, Optimization & Information Computing*, *10*(2), 457–483.

Yakovlev, A. Y., Asselain, B., Bardou, V. J., Fourquet, A., Hoang, T., Rochefediere, A., & Tsodikov, A. D. (1993). A simple stochastic model of tumor recurrence and its application to data on premenopausal breast cancer. *Biometrie et analyse de donnees spatio-temporelles*, *12*, 66–82.

Yakubu, A., & Doguwa, S. I. (2017). On the properties of the weibull-burr iii distribution and its application to uncensored and censored survival data. *CBN Journal of Applied Statistics*, *8*(2), 91–116.

Yan, W., Li, P., & Yu, Y. (2021). Statistical inference for the reliability of burr-xii distribution under improved adaptive type-ii progressive censoring. *Applied Mathematical Modelling*, *95*, 38–52.

Yousef, M. M., Alsultan, R., & Nassr, S. G. (2023). Parametric inference on partially accelerated life testing for the inverted kumaraswamy distribution based on type-ii progressive censoring data. *Math. Biosci. Eng.*, *20*(2), 1674–1694.

Yu, B., Tiwari, R. C., Cronin, K. A., & Feuer, E. J. (2004). Cure fraction estimation from the mixture cure models for grouped survival data. *Statistics in medicine*, *23*(11), 1733–1747.

Yusuf, M. U., & Bakar, M. R. B. A. (2016). Cure models based on weibull distribution with and without covariates using right censored data. *Indian Journal of Science and Technology*, *9*(28), 1–12.

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