C 21070 **(Pages : 3)** **Name..............................**

Reg. No...........................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2022

(CCSS)

Mathematics

MAT 4E 02—ADVANCED FUNCTIONAL ANALYSIS

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Let X be a normed space over $\mathbb{K}(=\mathbb{R} \text{ or } \mathbb{C})$ and A, B : X $\to$ X be two bounded linear maps. Show that AB is also a bounded linear map and $\|AB\| \le \|A\|\|B\|$.
2. Define the spectral radius $r_\sigma(A)$ of a bounded operator on a normed space X. Show by an example that $r_\sigma(A)$ can be strictly less than $\|A\|$.
3. Give example of a compact operator on an infinite dimensional Banach space with 0 not an eigenvalue of it.
4. State True *or* False. Justify your claim : Every Banach space is a Hilbert space.
5. Let X be an inner product space and $E \subset X$. Define $E^\perp = \{y \in X; \langle x, y\rangle = 0 \text{ for every } x \in X\}$. Show that $E^\perp$ is always a closed subspace.
6. Let $\mathbb{H} = L^2([0,1])$ and define A on $\mathbb{H}$ by $A(x)(t) = tx(t), x \in \mathbb{H}, t \in [0,1]$. Find the adjoint A*.
7. Define Hilbert-Schmidt operators and show that A is Hilbert-Schmidt if and only if A* is Hilbert-Schmidt.
8. Let $A, B \in BL(X)$ with A self-adjoint. Show that AB = 0 if and only if R(A) $\perp$ R(B).

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let X be a normed space and A $\in$ BL (X). Show that A is invertible if and only if A is bounded below and range of A is dense in X.
10. Give an example for a compact operator without eigen values.

Turn over

11. Let X be an inner product space and $x, y \in X$. Show that $|\langle x, y\rangle| \leq \|x\| \|y\|$. Also show that equality holds if and only if x and y are linearly dependent.

12. Show that if A is a normal operator on a Hilbert space, and $\lambda_1 \neq \lambda_2$ are eigenvalues of A, then the corresponding eigen vectors are orthogonal to each other. Show by an example that this need not be true if A is not normal.

13. Let $\{u_1, u_2, \ldots\}$ be a countable orthonormal set in an inner product space X and $x \in X$. Show by example that the identity $x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$ need not hold for all $x \in X$.

14. Show that sum of two self-adjoint operators on a Hilbert space is self-adjoint and the product is self-adjoint if and only if they commute. Give examples for two self-adjoint operators do not commute.

(4 × 4 = 16 marks)

Part C

Answer **either** *Part* (a) **or** (b) *of each of the following questions.*
Each question carries 12 *marks.*

15. (a) (i) Let $A : l^2 \to l^2$ be defined by $A(x(1), x(2), \ldots) = (0, x(1), x(2), \ldots)$; for all $(x(1), x(2), \ldots) \in l^2$. Compute the spectrum $\sigma(A)$, eigen spectrum $\sigma_e(A)$ and approximate eigenspectrum $\sigma_a(A)$.

(ii) Let $A : l^2 \to l^2$ be defined by $A(x(1), x(2), \ldots) = \left(x(1), \frac{x(2)}{2}, \frac{x(3)}{3}, \ldots\right)$; for all $(x(1), x(2), \ldots) \in l^2$. Compute the spectrum $\sigma(A)$, eigenspectrum $\sigma_e(A)$ and approximate eigenspectrum $\sigma_a(A)$.

Or

(b) For $1 \leq p < \infty$, show that the dual $(l^p)'$ of l^p is isometrically isomorphic to l^q, where $\frac{1}{p} + \frac{1}{q} = 1$.

16. (a) Show that the class CL(X) of all compact operators on a Banach space X is a closed two sided ideal in BL(X), the class of all bounded operators on X.

Or

(b) Let X be a normed space and A be a compact operator on X :

(i) Show that every non-zero spectral value of A is an eigenvalue of A.

(ii) Show that $\sigma_a(A) = \sigma(A)$.

17. (a) (i) Introduce orthonormal bases and give two infinite dimensional examples.

(ii) Show that a Hilbert space has a countable orthonormal basis if and only if it is separable.

Or

(b) (i) State and prove Projection theorem.

(ii) State and prove Riesz Representation theorem.

18. (a) (i) Introduce the numerical range $\omega(A)$ of a bounded operator A on a Hilbert space $\mathbb{H}$. Show that neither $\sigma(A)$ nor $\omega(A)$ is contained in the other in general.

(ii) Show that $\omega(A)$ need not be closed.

(iii) Show that $\sigma(A) \subset \overline{\omega(A)}$.

Or

(b) State and prove spectral theorem for compact self-adjoint operators on a Hilbert space.

(4 × 12 = 48 marks)

C 21071 **(Pages : 2)** **Name.............................**

Reg. No...........................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2022

(CCSS)

Mathematics

MAT 4E 04—ALGEBRAIC GRAPH THEORY

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Define a complete graph. Draw K_4.
2. Obtain the line graph of $K_{2,3}$.
3. If $X = C_4$, find Aut (X).
4. Define a system of imprimitivity for a group G.
5. Define vertex transitive graphs with an example.
6. Obtain a maximum matching for Petersen graph.

(6 × 2 = 12 marks)

Part B

Answer any **five** *question.*
Each question carries 4 marks.

7. Define J(v, k, i). For, $u \geq k \geq i$, show that $J(v,k,i) \cong J(v,v-k,v-2k+i)$.
8. Let X and Y be two graphs with minimum valency four. Prove that $X \cong Y$ if and only if $L(X) \cong L(Y)$.
9. State and prove Burnside's Lemma.
10. Let G be a group acting transitively on the set V, and let x be a point of V. Show that there is a one-to-one correspondence between the orbits of G on V × V and the orbits of G_x on V.
11. Prove that the only primitive permutation group on V that contains a transposition is Sym (V).
12. Show that Petersen graph is not a Cayley graph.
13. Prove or disprove. Any two distinct edge atoms are vertex disjoint.
14. If A is an atom and B is a fragment of X, then show that A is a subset of exactly one of B, N(B) and $\overline{B}$.

(5 × 4 = 20 marks)

Turn over

Part C

Answer either A **or** *B of each of the following questions.*
Each question carries 16 *marks.*

Unit I

15. A (i) If X is regular with valency k, then show that L(X) the line graph of X is regular with valency $2k - 2$.

(b) If the line graph of a connected graph X is regular then prove that X is regular or bipartite and semiregular.

B Obtain the Euler's polyhedral formula for planar graphs.

Unit II

16. A Determine the number of isomorphism classes of groups on n vertices.

B State and prove one characterization of primitive permutation groups.

Unit III

17. A (i) Let X be an edge transitive graph with no isolated vertices. If X is not vertex transitive then show that Aut (X) has exactly two orbits and these two orbits are a bipartition of X.

(ii) Prove that the vertex connectivity of a connected edge transitive graph is equal to its minimum valency.

B Let X be a graph on n vertices with connectivity k. Suppose A and B are fragments of X and $A \cap B \neq \theta$. If $|A| \leq |\overline{B}|$, then show that $A \cap B$ is a fragment.

(3 × 16 = 48 marks)

C 21072 **(Pages : 2)** **Name..............................**

Reg. No...........................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2022

(CCSS)

Mathematics

MAT 4E 05—ALGEBRAIC TOPOLOGY

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Define the skeleton of a complex K. Give an example.
2. When will you say that two cycles on a complex are homologous ?
3. What is meant by Betti number of a complex ?
4. Define simplical mapping on polyhedra.
5. True or False. Every simply connected space is contractible. Justify.
6. Determine the fundamental group of Torus.

(6 × 2 = 12 marks)

Part B

Answer any **five** *question.*
Each question carries 4 marks.

7. If K denotes the closure of $\sigma^2 = \langle a_0 a_1 a_2 \rangle$ with vertices ordered by $a_0 < a_1 < a_2$, then find the incidence matrix $\eta(0)$ and $\eta(1)$.
8. Define barycentric co-ordinate and show that it is unique for each point on a simplex.
9. Compute the homology groups and Betti numbers of the 2-sphere S^2.
10. If M denotes the triangulation of the Mobius strip, show that the polyhedron $|M|$ has one hole bounded by 1-simplexes.
11. If x and y are points in a simplex σ. Prove that there exist a vertex v of σ such that $\|x - y\| \le \|x - v\|$.
12. Let X and Y be topological spaces and let M denote the set of all continuous maps f from X into Y. Define a relation $f \sim g$ means that f is homotopic to g. Prove that ~ is an equivalence relation on M.

Turn over

13. Prove that the fundamental group of unit circle is isomorphic to the group of integers under addition.

14. Prove that the fundamental group of punctured n-space $\mathbb{R}^n \setminus \{p\}$ is trivial for $n > 2$.

(5 × 4 = 20 marks)

Part C

*Answer either A **or** B of each of the following questions.*
Each question carries 16 marks.

15. A (a) Let K be the closure of a 2-simplex $\langle a_0 a_1 a_2 \rangle$ with orientation induced by the ordering $a_0 < a_1 < a_2$. Find all Homology groups of K.

(b) Prove that the homology groups of two different orientations of a geometric complex are isomorphic for each dimension.

B (a) Compute the Homology groups of projective plane.

(b) Prove that a set A = $\{a_0, a_1, \ldots, a_k\}$ of points in $\mathbb{R}^n$ is geometrically independent if and only if the set of vectors $\{a_1 - a_0, a_2 - a_0, \ldots, a_k - a_0\}$ is linearly independent.

16. A (a) Prove that there are only five regular, simple polyhedra.

(b) State and ptove Brouwer's Degree theorem.

B (a) Prove that the homology groups of the n-sphere, $n \geq 1$, are

$$H_p\left(S^n\right) = \begin{cases} \mathbb{Z}, & \text{if } p = 0 \text{ or } p = n \\ \{0\}, & \text{if } 0 < p < n \end{cases}$$

(b) Prove that there is a vector field on $S^n, n \geq 1,$ if and only if n is odd.

17. A (a) Prove that in a path connected space X, the fundamental groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic, where $x, y \in X$.

(b) If $F : I \times I \to S^1$ is a homotopy such that F(0, 0) = 1, then show that there is unique covering homotopy $\tilde{F} : I \times I \to \mathbb{R}$ such that $\tilde{F}(0,0) = 0$.

B (a) State and prove the convering path property.

(b) Let X be a topological space. Prove that equivalence of loops is an equivalence relation on the set of loops in X with base point x_0.

(3 × 16 = 48 marks)

C 21073-B **(Pages : 2)** **Name.......................................**

Reg. No..................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2022

(CCSS)

Mathematics

MAT 4E 08—GRAPH THEORY

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 marks.

1. Define vertex cut set and edge cut set of a simple graph G. Illustrate with an example.

2. Define vertex connectivity $\kappa(G)$ of a graph G. Prove that a simple graph G with n vertices $n \geq 2$, is complete if and only if $\kappa(G) = n - 1$.

3. Define chromatic number $\chi(G)$ of a graph G. Prove that $\chi(K_n) = n$.

4. What is a critical graph ? Prove that every critical graph is connected.

(4 × 4 = 16 marks)

Part B

Answer any **two** *questions.*
Each question carries 8 marks.

5. A connected graph G with at least two vertices contains at least two vertices that are not cut vertices : Prove or disprove the statement.

6. Prove that in a critical graph G no vertex cut is a clique.

7. State and prove Interpolation theorem for complete coloring.

(2 × 8 = 16 marks)

Turn over

Part C

Answer A or B of each of the following **two** *questions.*
Each question carries 24 marks.

Unit I

8. a) Prove that the connectivity and edge connectivity of a simple cubic graph are equal.

   b) Prove that for any simple graph G, $2\sqrt{n} \leq \chi\chi^c \leq \left(\frac{(n+1)}{2}\right)^2$, where $n = o(\mathrm{G})$, $\chi^c = \chi\left(\mathrm{G}^c\right)$.

Unit II

9. a) Define flow in a network. Prove that in any network N, the value of any flow f is less than or equal to the capacity of any cut.

   b) Prove that if a graph G admits a complete k-coloring and a complete i-coloring, then, it admits a complete i-coloring for every i between k and l.

(2 × 24 = 48 marks)