D 93597 **(Pages : 3)** **Name.......................................**

Reg. No...................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2020

(CCSS)

Mathematics

MAT 1C 04—DISCRETE MATHEMATICS

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Prove that the number of vertices of odd degree in any graph is even.
2. Define a Hamiltonian graph. Give an example of a Hamiltonian graph with 6 vertices.
3. Prove *or* disprove : Intersection of two chains is a chain.
4. Describe the Hasse diagram of the partially ordered set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ with partial order, 'divides'.
5. Define a finite automaton. Illustrate with an example.
6. Define a grammar G and find the associated L (G).

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 *marks.*

7. Prove that a connected graph G with at least two vertices contains at least two vertices that are not cut vertices.
8. If G is Hamiltonian, then prove that for every non-empty proper subset S of V, $|\omega(G - S)| \leq |S|$.
9. If the degree of each vertex of a graph G is an even positive integer, prove that G is an edge-disjoint union of cycles.

Turn over

10. Let $(X \leq)$ be a poset. Prove that X can be expressed as the union of maximal chains.

11. Write in disjunctive normal forms :

(i) $x_1' x_2 (x_1' + x_2 + x_1 x_3)$; and (ii) $(xy + x'y + x'y')' (x + y)$.

12. (a) If $\Sigma = \{a, b\}$ and $L = \{aa, bb\}$, using set notation describe $\bar{L}$.

(b) Prove that $(L_1 L_2)^R = L_2^R L_1^R$.

13. Show that $L = \{a^n : a \geq 4\}$ is regular.

14. Find a dfa that recognize the set of all strings on $\Sigma = \{a, b\}$ starting with the prefix *ba*.

(5 × 4 = 20 marks)

Part C

Answer **either** *A* **or** *B of each of the following* **three** *questions.*
Each question carries 16 *marks.*

15. A (a) Prove that the following statements are equivalent.

(i) G has exactly one cycle ;

(ii) G is connected and $n = m$;

(iii) For some edge e of G, $G - e$ is a tree ; and

(iv) G is connected and the set of edges of G that are not cut edges forms a cycle.

(b) For a simple graph G with n vertices, $n \geq 2$, which is complete, prove that $k(G) = n - 1$.

B. (a) Prove that K_5 and $K_{3,3}$ are non-planar.

(b) Describe the common features of K_5 and $K_{3,3}$.

16. A. Let X be a finite set and $\leq$ be a partial order on X. Define a binary relation R on X by xRy if and only if y covers x. Prove that $\leq$ is the smallest order relation on X containing R.

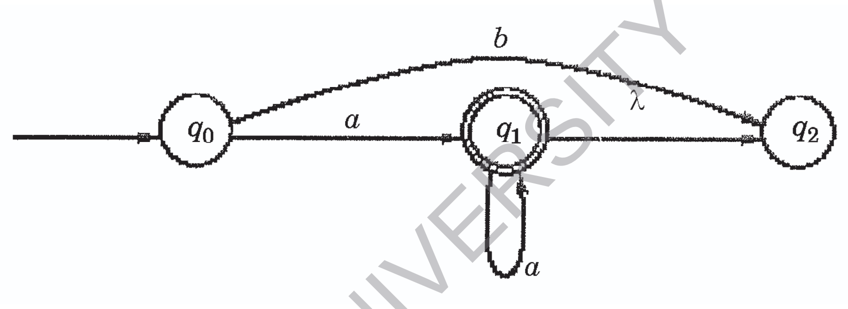
B. (a) Given integers $0 \le r_1 < r_2 < ... < r_k \le n$, prove that there exists one and only one symmetric function of n Boolean variables whose characteristic numbers are $r_1, r_2, ..., r_k$.

(b) Prove that the set of all Boolean functions of n Boolean variables $x_1, x_2, ..., x_n$ is a subalgebra of the Boolean algebra of all Boolean functions of these variables.

17. A. (a) For every nfa with arbitrary number of final states, prove that there exists an equivalent nfa with only one final state.

(b) Show that if L is regular, then L^R is regular.

B. Convert the nfa given by the following transition graph into an equivalent dfa.



(3 × 16 = 48 marks)

D 93594 **(Pages : 4)** **Name.....................................**

Reg. No.................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2020

(CCSS)

Mathematics

MAT 1C 01—ALGEBRA—I

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Show that (1, 2) is a generator of the cyclic group $\mathbb{Z}_2 \times \mathbb{Z}_3$.
2. Describe an isometry of the plane that fixes only one point.
3. Let G = S_3 act on the set {1, 2, 3} by $\sigma x = \sigma(x)$ for $\sigma \in S_3$ and $x \in X$. Find the isotropy group G_x for $x = 1$.
4. Let D be an integral domain and F = D × D / ~ be the field of quotients of D where the notations are the usual ones. Show that for $a \neq b$, $(a, 1)$ and $(b, 1)$ are not ~–related.
5. Verify whether $x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$.
6. Verify whether the ideal generated by 4 is maximal in $\mathbb{Z}_{10}$.
7. Find *irr* $(\alpha, \mathbb{Q})$ where $\alpha = \sqrt{2} + \sqrt{3}$.
8. Verify whether $\mathbb{Q}(\pi)$ is an algebraic extension of $\mathbb{Q}$.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Find all generators of the cyclic group $\mathbb{Z}_3 \times \mathbb{Z}_4$.
10. Let S_n be the symmetric group on n symbols and A_n be the alternating group. Show that S_n/A_n is isomorphic to $\mathbb{Z}_2$.

Turn over

11. Let X be a G-set and $x \in X$. Show that $G_x = (g \in G : gx = x)$ is a subgroup of G.

12. Describe the field of quotients of the integral domain Z_2.

13. Prove that $x^5 + 6x^3 + 9x + 3$ is irreducible over the rationals.

14. Show that doubling the cube by straight edge and compass is an impossible construction.

(4 × 4 = 16 marks)

Part C

Answer **either** *part A* **or** *part B of each of the* **four** *questions.*
Each question carries 12 *marks.*

15. (A) (a) Let m and n be relatively prime. Show that $\mathbb{Z}_m \times \mathbb{Z}_n$. is isomorphic to $\mathbb{Z}_{mn}$.

(b) Show that the group $\mathbb{Z} \times \mathbb{Z}_2$ is not cyclic.

(c) Show that the direct product $G_1 \times G_2$ of groups G_1 and G_2 contains a subgroup isomorphic to G_1 and a subgroup isomorphic to G_2.

(B) Let H be a subgroup of a group G. Show that the following are equivalent.

(a) $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

(b) $gHg^{-1} = H$ for all $g \in G$.

(c) $gH = Hg$ for all $g \in G$.

16. (A) (a) Let X be a G-set. For each $g \in G$ let $\sigma_g : X \to X$ be defined by $\sigma_g(x) = gx$ for all $x \in X$. Show that :

(i) σ_g is a permutation of X.

(ii) $\phi : G \to S_X$ defined by $g \mapsto \sigma_g$ is a homomorphism where S_X is the group of all permutations of X.

(b) Give an example of a G-set where G = $\mathbb{Z}_4$.

(B) (a) Let G be a finite group and X be a finite G-set. Let r be the number of orbits in X. For $g \in G$ let $X_g = \{x \in X : gx = x\}$. Show that

(i) $\sum_{g \in G} |X_g| = \sum_{x \in X} |G_x|$ where G_x is the isotropy group of x.

(ii) $r|G| = \sum_{g \in G} |X_g|$.

(b) Give a non trivial action of $\mathbb{Z}_3$ on $X = \{1, 2, 3\}$ and describe X_g for each $g \in \mathbb{Z}_3$.

17. (A) (a) State and prove division algorithm in the polynomial ring F [x] where F is a field.

(b) Show that $a \in F$ is a zero of $f(x) \in F[x]$ if and only if $(x - a)$ is a factor of $f(x)$.

(B) (a) Let R be a ring and N be an ideal of R. Show that $\eta : R \to R/N$ defined by $a \mapsto a + N$ for $a \in R$ is a ring homomorphism.

(b) Let $\phi : R \to R'$ be a ring homomorphism. Show that :

(i) $\mathrm{Ker}\phi$ is an ideal of R.

(ii) $R / \mathrm{Ker}\phi$ is isomorphic to ϕ (R).

18. Let E be an extension of a field F and $\alpha \in E$.

(a) Prove that if the evaluation homomorphism $\phi_\alpha : F[x] \to E$ is one to one then α is not algebraic over F.

(b) Show that if α is algebraic over F then there exists a polynomial $p(x) \in F[x]$ satisfying the following.

(i) $p(\alpha) = 0$.

(ii) If $f(x) \in F[x]$ and $f(\alpha) = 0$ then $p(x)$ divides $f(x)$.

(B) (a) Let E be a finite extension of degree n of a finite field F of q elements. Show that the number of elements of E is q^n.

(b) Prove that a finite field of p^n elements exists for each prime p and each natural number n.

($4 \times 12 = 48$ marks)

D 93595 **(Pages : 4)** **Name.......................................**

Reg. No..................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2020

(CCSS)

Mathematics

MAT 1C 02—LINEAR ALGEBRA

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Show that if V is a finite-dimensional vector space, then any *two* bases of V have the same number of elements.

2. Define linearly dependent subset of a vector space V and prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

3. Let $\beta = (\alpha_1, \alpha_2, \alpha_3)$ be the basis of C^3 defined by $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 1, 1), \alpha_3 = (2, 2, 0)$. Find the dual basis of β.

4. Let F be a field and let f be the linear functional on F^2 defined by $f(x_1, x_2) = ax_1 + bx_2$. If T is the linear operator defined by $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$, then find $T^t f$, where T^t denotes the transpose of T.

5. Show that similar matrices have the same characteristic polynomial.

6. Let P be the linear operator on R^2 defined by $P(x_1, x_2) = (x_1, 0)$. What is the minimal polynomial for P.

7. Let V be a real vector space and E be a projection on V. Prove that $(I + E)$ is invertible and find $(I + E)^{-1}$.

8. Prove that an orthogonal set of non-zero vectors is linearly independent.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Is the vector $(3, -1, 0, -1)$ in the subspace of R^4 spanned by the vectors $(2, -1, 3, 2), (-1, 1, 1, -3)$ and $(1, 1, 9, -5)$? Justify your answer.

10. Let V and W be vector spaces over the field F and let U be an isomorphism from V onto W. Prove that $T \mapsto UTU^{-1}$ is an isomorphism of $L(V, V)$ onto $L(W, W)$.

11. Let V be a finite-dimensional vector space over the field F and let W be a subspace of V. Show that $\dim W + \dim W^{o} = \dim V$.

12. Let W be an invariant subspace for a linear operator T on a vector space V. Prove that :
    (i) The characteristic polynomial for T_W divides the characteristic polynomial for T.
    (ii) The minimal polynomial for T_W divides the minimal polynomial for T.

13. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$. Is A similar over the field of real numbers to a triangular matrix ? Justify your answer.

14. Let W be a finite dimensional subspace of an inner product space V and let E be the orthogonal projection of V on W. Show that the mapping $\beta \mapsto \beta - E\beta$ is the orthogonal projection of V onto $W^{\perp}$.

(4 × 4 = 16 marks)

Part C

Answer **either** *A* **or** *B of each of the following questions.*
Each question carries 12 *marks.*

15. A (a) Show that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$ is finite dimensional and $\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2)$.

   (b) Let V be the vector space of all 2 × 2 matrices over R. Let W_1 be the subspace of the matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and W_2 be the subspace of the matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$. Find the dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

B (a) Let V be an n-dimensional vector space over the field F and Let β and β' be two ordered bases of V. Show that there is a unique $n \times n$ invertible matrix P with entries in F such that :

(i) $[\alpha]_\beta = P[\alpha]_{\beta'}$; and (ii) $[\alpha]_{\beta'} = P^{-1}[\alpha]_\beta$

for every vector α in V.

(b) Show that if A is an $m \times n$ matrix with entries in the field F, then row rank (A) = column rank (A).

16. A (a) Let T be a linear transformation from V into W. Show that T is non-singular iff T carries each linearly independent subset of V onto a linearly independent subset of W.

(b) Let V be a finite-dimensional vector space over the field F. For each vector α in V define $L_\alpha(f) = f(\alpha), f \in V^*$. Show that the mapping $\alpha \mapsto L_\alpha$ is an isomorphism of V onto V**.

B (a) Let W be the subspace of R^5 which is spanned by the vectors $\alpha_1 = (2, -2, 3, 4, -1), \alpha_2 = (0, 0, -1, -2, 3)\ \alpha_3 = (-1, 1, 2, 5, 2), \alpha_4 = (1, -1, 2, 3, 0)$. Find a basis for W°.

(b) Let W_1 and W_2 be subspaces of a finite dimensional vector space V. Show that

$(W_1 \cap W_2)^\circ = W_1{}^\circ + W_2{}^\circ$.

17. A Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for R^3, each vector of which is a characteristic vector of T.

B Let T be a linear operator on a finite dimensional vector space V. Show that if f is the characteristic polynomial for T, then $f(T) = 0$.

18. A (a) Let $W_1, W_2, \ldots, W_k$ be subspaces of a finite-dimensional vector space V such that $V = W_1 \oplus \ldots \oplus W_k$. Show that there exists k linear operators $E_1, E_2, \ldots, E_k$ on V such that :

(i) each E_i is a projection ;

(ii) $E_i E_j = 0$ if $i \neq j$;

(iii) $I = E_1 + E_2 + \ldots + E_k$; and

(iv) The range of E_i is W_i.

(b) Let V be a finite-dimensional vector space and let W_1 be any subspace of V. Prove that there is a subspace W_2 of V such that $V = W_1 \oplus W_2$.

B Let W be the subspace of R^2 with the standard inner product. Let E be the orthogonal projection of R^2 onto W. Find :

(a) A formula for $E(x_1, x_2)$;

(b) The matrix of E in the standard ordered basis ;

(c) $W^{\perp}$; and

(d) An orthonormal basis in which E is represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(4 × 12 = 48 marks)

D 93598 **(Pages : 4)** **Name.................................**

Reg. No..............................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2020

(CCSS)

Mathematics

MAT 1C 05—NUMBER THEORY

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Define completely multiplicative function and give an example of a multiplicative function which is not completely multiplicative.

2. If P is an odd positive integer, prove that $(-1/P) = (-1)^{(P-1)/2}$.

3. For all $x \geq 1$, prove that :

$$\sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right] = 1.$$

4. For $n \geq 1$, prove that the n^{th} prime p_n satisfies the inequality :

$$\frac{1}{6} n \log n < p_n.$$

5. Describe briefly about frequency analysis in cryptography.

6. What is meant by 'hash functions' in cryptography ?

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 *marks.*

7. If $n \geq 1$, prove that :

$$\phi(n) = \sum_{d/n} \mu(d) \cdot \frac{n}{d}.$$

Turn over

8. If $d(n)$ denotes the number of positive divisors of an integer n, then prove that :

$$\prod_{t/n} t = n^{d(n)/2}.$$

9. Determine whether 2/9 is a quadratic residue or non-residue mod 383.

10. For $x \geq 1$, prove that :

$$\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + G(s) + O\left(x^{-s}\right) \text{ if } \begin{matrix} s > 0 \\ s \neq 1 \end{matrix}.$$

11. Define the Chebyshev's functions $\Psi(x)$ and $I(x)$ and obtain a relation connecting them.

12. Prove that $\lim_{x \to \infty} \left(\frac{M(x)}{x} - \frac{H(x)}{x \log x} \right) = 0.$

13. Find the inverse of the matrix $\begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix}$ mod 29.

14. Describe briefly about RSA cryptosystem.

(5 × 4 = 20 marks)

Part C

Answer **either** A **or** B *of each questions.*
Each question carries 16 *marks.*

15. (A) (a) If f and g are multiplicative functions, prove that their Dirichlet product $f * g$ is also a multiplicative function.

(b) If f is an arithmetical function with $f(1) \neq 0$, prove that there is a unique arithmetical function f^{-1} with :

$$f * f^{-1} = f^{-1} * f = I.$$

where I is the identify function.

(B) (a) For every odd prime p, prove that :

$$(2/p) = (-1)^{(p^2-1)/8}.$$

(b) If p and q are distinct odd primes, then prove that :

$$(p/q)(q/p) = (-1)^{(p-1)(q-1)/4}.$$

16. (A) (a) State and prove Euler's summation formula.

(b) For $x \geq 2$, prove that :

$$\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x)$$

where the sum is extended over all primes $\leq x$.

(B) (a) If $0 < a < b$, prove that there exists an x_0 such that :

$$\pi(ax) < \pi(bx) \text{ if } x \geq x_0.$$

(b) Prove that the prime number theorem implies $\lim_{x \to \infty} \frac{M(x)}{x} = 0$.

17. (A) (a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(Z/NZ)$ and let $D = ad - bc$. Prove that the following are equivalent.

(i) g.c.d. (D, N) = 1 ;

(ii) A has an inverse matrix ;

(iii) If x and y are not both 0 in (Z/NZ), then $A\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;

(iv) A gives a 1 – to –1 correspondence of $(Z/NZ)^2$ with itself.

(b) The message "!IWGVIEX! ZRADRYD", which was sent using a linear enciphering transformation of digraph vectors in a 29-letter alphabet, in which A – Z have numerical equivalents 0 – 25, blank = 26, ? = 27, ! = 28 was intercepted. It is known that the last five letters of the plain-text are the sender's signature "MARIA" :

(i) Find the deciphering matrix and read the message.

(ii) Find the enciphering matrix and send the following reply in code : "DAMN FOG ! JO".

(B) (a) Briefly describe about digraph transformations.

(b) Suppose that both plaintexts and ciphertexts consist of trigraph message units, but while plain texts are written in the 27-letter alphabet (Consisting of A – Z and blank = 26), cipher texts are written in the 28-letter alphabet obtained by adding the symbol "/" (with numerical equivalent 27) to the 27 letter alphabet. It is required that each user A choose n_A between 27^3 and 28^3 so that a plaintext trigraph in the 27-letter and then $C = P^{eA} \bmod n_A$ corresponds to a ciphertext trigraph in the 28-letter alphabet. If the deciphering key is $K_D = (n, d) = (21583, 20787)$, decipher the message "YSNAUOZHXXH" (one blank at the end).

(3 × 16 = 48 marks)

D 93596 **(Pages : 4)** **Name..................................**

Reg. No...............................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2020

(CCSS)

Mathematics

MAT 1C 03—REAL ANALYSIS—I

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Let Y be an open subset of a metric space X. If a subset E of Y is open relative to Y, then prove that E is open in X.

2. If E is an infinite subset of a compact set K, then prove that E has a limit point in K.

3. Prove that composition of continuous maps is continuous.

4. Let f be a bijective continuous map from a metric space X onto a metric space Y. Is f^{-1} continuous ? Justify your answer.

5. Suppose $f \geq 0,$ is continuous on $[a, b]$ and $\int_a^b f(x)\,dx = 0.$ Prove that $f(x) = 0$ for all $x \in [a, b]$.

6. Let f_1, f_2 be bounded real functions and α be monotonic increasing real function on $[a, b]$. If f_1, f_2 are Riemann-Stieltjes integrable with respect to α on $[a, b]$, then prove that $f_1 + f_2$ is Riemann-Stieltjes integrable with respect to α on $[a, b]$.

7. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, then prove that $\{f_n + g_n\}$ converge uniformly on E.

8. Let $\{f_n\}$ be a uniformly bounded sequence of continuous functions on a compact set E. Does $\{f_n\}$ has a convergent subsequence ? Justify your answer.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let X be a metric space and $E \subset X$. If p is a limit point of E, then prove that every neighborhood of p contains infinitely many points of E.

Turn over

10. Prove that closed subsets of compact sets are compact.

11. Prove that continuous image of a connected metric space is connected.

12. Let f be differentiable in (a, b). If $f'(x) \geq 0$ for all $x \in (a, b)$, then prove that f is monotonically increasing.

13. Let f be a bounded function and α be a monotonic increasing function on $[a, b]$. If P^* is a refinement of P, then prove that :

$$U(P^*, f, \alpha) \leq U(P, f, \alpha).$$

14. Let $\mathcal{C}(X)$ denote the set of all complex valued, continuous, bounded functions defined on a metric space X. Prove that $\mathcal{C}(X)$ is a complete metric space with respect to the metric.

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

(4 × 4 = 16 marks)

Part C

Answer A **or** *B of the following questions.*
Each question carries 12 *marks.*

Unit I

15. (A) (a) Let $\{F_\alpha\}$ be a collection of closed subsets of a metric space X. Prove $\bigcap_\alpha F_\alpha$ is closed.

(b) Prove that every bounded infinite subset of $\mathbb{R}^k$ has a limit point.

(B) (a) If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}^1$ such that $I_n \supset I_{n+1}$, n = 1, 2, 3, ..., then prove that $\bigcap_{n=1}^{\infty} I_n$ is not empty.

(b) Let P be a non-empty perfect set in $\mathbb{R}^k$. Prove that P is uncountable.

UNIT II

16. (A) (a) Let f be a continuous real function on a compact metric space X. Prove that f attains its maximum and its minimum on X.

(b) Let f be monotonic on an open interval (a, b). Prove that the set of points of (a, b) at which f is discontinuities is at most countable.

(B) (a) Prove that continuous image of a compact metric space is compact.

(b) Let f and g be continuous real functions on $[a, b]$ which are differentiable in (a, b). Prove that there is a point $x \in (a, b)$ at which :

$$[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x).$$

UNIT III

17. (A) (a) Let f be a bounded, monotonic real function and α be a continuous, monotonic increasing real function on $[a, b]$. Prove that f is Riemann-Stieltjes integrable with respect to α on $[a, b]$.

(b) For $1 < s < \infty$, define :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Prove that $\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{x - [x]}{x^{s+1}}\, dx.$

(B) (a) Let f be Riemman integrable on $[a, b]$. For $a \le x \le b$, let

$$F(x) = \int_a^x f(t)\, dt.$$

Prove that F is continuous on $[a, b]$.

(b) Let γ be a curve on $[a, b\}$. If γ' is continuous on $[a, b]$, then prove that γ is rectifiable and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

UNIT IV

18. (A) (a) Let $\{f_n\}$ be a sequence of functions differentiable on $[a, b]$ and $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f'_n\}$ converges uniformly on $[a, b]$, then prove that $\{f_n\}$ converges uniformly on $[a, b]$ to a function f and

$$f'(x) = \lim_{x \to \infty} f'_n(x)$$

for all $x \in [a, b]$.

(b) If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then prove that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

(B) (a) Prove that there exists a real continuous function on the real line which is nowhere differentiable.

(b) Let K be compact and $f_n \in \mathcal{C}(K)$ for $n = 1, 2, 3, \ldots$. If $\{f_n\}$ is pointwise bounded and equicontinuous on K, then prove that $\{f_n\}$ contains a uniformly convergent subsequence.

(4 × 12 = 48 marks)