C 22767 **(Pages : 7)**

Ph.D. ENTRANCE EXAMINATION, APRIL 2022

MATHEMATICS

Time : Two Hours Maximum : 100 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Consider the following statements :

   (I) Jan is well or Jan is still recovering.

   (II) If Jan is still recovering, then Jan is not well.

   (III) If Jan is well, then Jan is not still recovering.

   Then which among the following are equivalent.

   (a) I and II. (b) I and III.

   (c) II and III. (d) None.

2. Which statement is incorrect if X and Y are the two non-empty relations on a set S :

   (a) If X and Y are transitive, then the intersection of X and Y is also transitive.

   (b) If X and Y are reflexive, then the intersection of X and Y is also reflexive.

   (c) If X and Y are symmetric, then the union of X and Y is not symmetric.

   (d) If X and Y are transitive, then the union of X and Y is not transitive.

3. The Initial value Problem :

   $y' = 3y^{2/3}, y(2) = 0$ has

   (a) Unique solution. (b) Two solutions.

   (c) No solution. (d) Infinitely many solutions.

4. The inverse Laplace transform of $\frac{8}{3s^2+12} + \frac{3}{s^2-49}$ is :

   (a) $\frac{4}{3}\cos(2t) + \frac{3}{7}\cosh(7t)$. (b) $\frac{4}{3}\sin(t) + \frac{3}{7}\sinh(4t)$.

   (c) $\frac{3}{4}\sin(3t) + \frac{3}{7}\sinh(3t)$. (d) $\frac{4}{3}\sin(2t) + \frac{3}{7}\sinh(7t)$.

Turn over

5. The type of singularity of $f(z)=\frac{1+\cos(z)}{(z-\pi)^2}$ at $z=\pi$ is :

(a) Pole of order 2. (b) Pole of order 1.

(c) Removable singularity. (d) Essential singularity.

6. In R with the co-finite topology, consider the following sequences :

(i) {1,2,3...}.

(ii) {1,1, 2,1,3,1...}.

(iii) {1,2,1,2,1,2...}.

Then

(a) All are convergent. (b) All are not convergent

(c) Only (i) convergent. (d) Only (iii) is not convergent.

7. Let W and W' be subspaces of a vector space V such that $W \cup W'$ is also a subspace of V. Then which of the following is not likely to be true ?

(a) $W \cup W' = W$. (b) $W \cup W' = W'$.

(c) $W \cup W' = \phi$. (d) $W = W'$.

8. The nature of the PDE $x^2u_{xx} - 2xyu_{xy} - 3y^2u_{yy} + u_y = 0$ for all values of x, y is :

(a) Parabolic. (b) Hyperbolic.

(c) Elliptic. (d) None of the above.

9. Which among the following pairs of functions is linearly dependent on the given intervals ?

(a) $\sin 4x, \cos 4x$ $(-\infty, \infty)$. (b) $\cos 2x, \sin^2 x$ $(0, \infty)$.

(c) $x^3, x^2|x|$ $[-1,1]$. (d) None of the above

10. Among the following numerical methods, which one is having the highest rate of convergence :

(a) Secant. (b) Muller.

(c) Regula-Falsi. (d) Newton-Raphson.

11. Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H, then :

$\| x \|^2 = \sum | < x, e_i > |^2$.

This equality is known as :

(a) Parseval's identity. (b) Apollonius identity.

(c) Holder's inequality. (d) None of these.

12. How many double transpositions are there in permutation group S_4 as elements.

(a) 3. (b) 4.

(c) 12. (d) 8.

13. Given the two points $[a, f(a)], [b, f(b)]$, the linear Lagrange interpolating polynomial that passes through these two points is given by :

(a) $\frac{x-b}{a-b} f(a) + \frac{x-a}{a-b} f(b)$. (b) $\frac{x}{b-a} f(a) + \frac{x}{b-a} f(b)$.

(c) $f(a) + \frac{f(b)-f(a)}{b-a}(b-a)$. (d) $\frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)$.

14. If $f(x)$ continuous in $[a, b]$ and $f(a) f(b) < 0$, then the equation $f(x) = 0$.

(a) Can have no solution in $[a, b]$.

(b) Can have two solutions in $[a, b]$.

(c) Can have five solutions in $[a, b]$.

(d) Can have six solutions in $[a, b]$.

15. Taylor polynomial of order two at $x = 0$ for the function $\sqrt{1+x}$ is :

(a) $1 + \frac{1}{2}x - \frac{1}{8}x^2$. (b) $1 + \frac{1}{2}x - \frac{1}{4}x^2$.

(c) $2 + \frac{1}{4}x - \frac{1}{8}x^2$. (d) $1 + \frac{1}{8}x - \frac{1}{2}x^2$.

16. Suppose that :

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}, g(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

then

(a) Both series converge uniformly and absolutely on R.

(b) Both series converge uniformly but not absolutely on R.

(c) Both are differentiable on R and f is the derivative of g.

(d) Both are continuous but not differentiable on R.

17. Radius of convergence of the power series $x^2 + x^3 + x^5 + x^7 + x^{11} + \ldots$ is :

(a) 1. (b) 2.

(c) 0. (d) Infinity.

18. In dual simplex method, ———— variables are not required.

(a) Slack. (b) Surplus.

(c) Original. (d) Artificial.

19. The intersection of finite number of closed half spaces in R^n is called :

(a) Convex null. (b) Polyhedral convex set.

(c) Convex cone. (d) Simplex.

20. The eigen functions of symmetric kernel, corresponding to different eigen values are :

(a) Real. (b) Imaginary.

(c) Orthogonal. (d) Equal.

21. Hahn decomposition is unique expect for :

(a) Positive sets. (b) Negative sets.

(c) Null sets. (d) Measurable sets.

22. In a Hausdorrf space, every sequence has :

(a) Many limits. (b) No limit.

(c) At most one limit. (d) Exactly one limit.

23. The convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$ is :

(a) $(f * g)(t) = \frac{1}{2}\left[e^{-t} + \sin(t) - \cos(t)\right]$.

(b) $(f * g)(t) = \frac{1}{4}\left[e^{t} + \sin(t) + \cos(t)\right]$.

(c) $(f * g)(t) = \frac{1}{2}\left[e^{-t} - \sin(t) - \cos(t)\right]$.

(d) $(f * g)(t) = \frac{1}{4}\left[e^{-t} - \sin(t) - \cos(t)\right]$.

24. The integral equation :

$$g(s) = \frac{1}{2} + s - \int_0^1 (s - t) g^2(t)\, dt$$

is

(a) Homogeneous linear Fredholm integral equation of the second kind.

(b) Homogeneous non linear Volterra integral equation of the second kind.

(c) Homogeneous linear Volterra integral equation of the second kind.

(d) Non Homogeneous non linear Fredholm integral equation of the second kind.

25. The k^{th} Betti number of Klein bottle is :

(a) k.

(b) 1 for $k = 0, 1$ and 0 for all other values of k.

(c) 1 for all values of k.

(d) 0 for $k = 0, 1$ and 1 for all other values of k.

(25 × 2 = 50 marks)

Part B

Answer any **ten** *questions.*
Each question carries 5 marks.

26. Determine the range and null space of the linear transformation

$T : R^3 \to R^4$ with $T(x, y, z) = (x - y + z, y - z, x, 2x - 5y + 5z)$.

27. If $\sum a_n, \sum b_n$ are absolutely convergent and define :

$$c_n : \sum_{r=0}^{n} a_r b_{n-r}$$

Then show that $\sum c_n$ is absolutely convergent and $\sum c_n = \sum a_n . \sum b_n$.

28. Let f be a continuous mapping defined from a topological space E to a topological space F. If (p_n) converges to p in E, then show that $(f(p_n))$ converges to $f(p)$ in F.

29. Prove that $\text{lcm}\{a,b\} \times \text{gcd}\{a,b\} = |ab|$.

where : $\text{lcm}(a,b)$ denotes the lowest common multiple of a and b and $\text{gcd}(a, b)$ denotes the greatest common divisor of a and b.

30. Prove that if f is a function that is continuous on the interval $[a, b]$, then f is Riemann-Stieltjes integrable on $[a, b]$.

31. Prove that the implication is left distribution with respect to the disjunction.

32. Find the image of the line passing through origin under the Joukowsky Transform $w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

33. Let (x_n) be a sequence in a normed space X. Show that $x_n \xrightarrow{weakly} x$ implies $\lim \inf_{n \to \infty} \| x_n \| \geq \| x \|$.

34. Let R be a ring. Let M, N be simple (or irreducible) sub modules of R, and let $\varphi : M \to N$ be a homomorphism of R-modules. Prove that either $\varphi = 0$, or φ is an isomorphism.

35. If $\{K_n\}$ be a sequence of nonempty compact sets in R^n satisfying $K_1 \supseteq K_2 \supseteq K_3 \ldots$, then show that $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

36. Prove that it is impossible to trisect an angle with a ruler and compass.

37. Prove that there cannot be a continuous onto function $f : \mathbf{R} \to \mathbf{R}/\mathbf{Q}$.

38. Solve the congruence equation $230x \equiv 1081 (mod\, 12167)$.

39. Apply the Gram-Schmidt orthonormalization process to the following basis for $\mathbf{R}^3$. $B = \{1,1,0\},(1,2,0),(0,1,2)$ and get the orthonormal basis.

40. Show that the continuous image of a Lindelöf space is Lindelöf.

(10 × 5 = 50 marks)