Ph.D. THESIS **MATHEMATICS**

A STUDY ON CLOSELY-CONNECTED VERTICES IN GRAPHS

A thesis

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by

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CERTIFICATE

I hereby certify that the thesis entitled "*A Study on Closely-connected Vertices in Graphs*" is a bonafide record of the original research work carried out by **Ms. Priya K.,** under my guidance for the award of Ph.D. in Mathematics from the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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I hereby declare that the thesis entitled "**A Study on Closely-connected Vertices in Graphs**" is based on the original research work done by me under the supervision of **Dr. Anil Kumar V., Retd. Senior Professor**, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution. The contents of the thesis have undergone plagiarism check using **iThenticate** software at C.H.M.K. Library, University of Calicut, and the similarity index is found to be within the permissible limit. I also declare that the thesis is free from AI generated content.

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ABSTRACT

Graph theory is one of the thriving branches of mathematics. The smooth operation of communication networks is largely dependent on the reliability of communication infrastructure. Fault tolerance is the ability of a system to continue operating as intended despite errors and the main aim is to design fault-tolerant networks by minimizing link failures. In this thesis, emphasizing on the need for fault-tolerant mechanisms for reliable communication between the nodes in a network, a new graph concept called "closely-connected vertices" is introduced and studied in a detailed manner.

A pair of distinct vertices in a finite simple undirected graph is said to be closely-connected if there exists a geodesic linking them preserving the connectivity of the graph. The properties of closely-connected vertices are analysed for various graphs and a variation of domination called cc-domination is formulated. Further, the idea of cc-domination polynomial is developed and a topological index called vertex-connectivity index is studied. Moreover, the ideas of vertex connectivity polynomial and the stability properties of its roots are discussed. The significance of the study is analysed and the concluding chapter specifies certain recommendations of the thesis.

Keywords: closely-connected vertices, cc-domination, cc-domination polynomial, vertex-connectivity index, vertex connectivity polynomial

സംഗ്രഹം

ഗ്രാഫ് സിദ്ധാന്തത്തിൽ ഗ്രാഫ് എന്നത് രണ്ട് പ്രധാന ഘടകങ്ങൾ ഉൾക്കൊള്ളുന്ന ഒരു ഗണിത ഘടനയാണ്. അതിൽ ഒന്നിനെ നോഡുകൾ എന്നും മറ്റേതിനെ നോഡുകളെ തമ്മിൽ യോജിപ്പിക്കുന്ന ലിങ്കുകൾ എന്നും വിളിക്കുന്നു. ഒരു സിസ്റ്റത്തിന് തകരാറുകൾ ഉണ്ടെങ്കിലും അതിന്റെ ഉദ്ദേശിച്ച ജോലി തുടർന്നും നിർവഹിക്കാനുള്ള കഴിവാണ് ഫോൾട്ട്ടോളറൻസ് എന്ന് നിർവചിച്ചിരിക്കുന്നത്. നെറ്റ്‌വർക്കിന്റെ നോഡുകൾ "കൂടുതൽ അടുത്ത് " ബന്ധിപ്പിച്ചിട്ടുണ്ടെങ്കിൽ മാത്രമേ ലിങ്ക് പരാജയങ്ങൾ മൂലം ഉണ്ടാകുന്ന പ്രവചനാതീതമായ തടസ്സങ്ങൾ മുഴുവൻ സിസ്റ്റത്തെയും പരാജയപ്പെടുത്താത്ത വിധത്തിൽ നിയന്ത്രിക്കാൻ കഴിയൂ. അതിനാൽതന്നെ ഈ തിസിസിന്റെ ലക്ഷ്യം ദ്രുതഗതിയിലുള്ള ടോപ്പോളജിക്കൽ മാറ്റങ്ങളെ നേരിടാനും, ശേഷിക്കുന്ന നെറ്റ്‌വർക്കിന് അതിന്റെ ചുമതലകൾ കാര്യക്ഷമമായി നിർവഹിക്കാൻ കഴിയുന്ന ഒരു പുതിയ സ്ഥിരതയിലെത്താനും കഴിയുന്ന നോഡുകളുടെ സവിശേഷതകൾ പഠിക്കുന്നതിലാണ്.

"ക്ലോസ്ലി-കണക്റ്റഡ് വെർട്ടിസസ് " എന്ന പുതിയ ഗ്രാഫ് ആശയം അവതരിപ്പിക്കുകയും വിശദമായി പഠിക്കുകയും ചെയ്യുന്ന എട്ട് അധ്യായങ്ങളാണ് ഈ പ്രബന്ധത്തിൽ ഉള്ളത്. ഗ്രാഫുകളിലെ "ക്ലോസ്ലി-കണക്റ്റഡ്നെസ്സ്" എന്ന ആശയം പരിചയപ്പെടുത്തിയതിനു ശേഷം ഡോമിനേറ്റിംഗ് ഗണത്തിൽ ഗ്രാഫ് കണക്റ്റിവിറ്റി എന്ന ആശയം കൂടി ഉൾപ്പെടുത്തി കൊണ്ടു സിസി- ഡോമിനേഷൻ എന്ന കൂടുതൽ സാമാന്യവൽക്കരിച്ച ആശയം വികസിപ്പിക്കുകയാണ് പിന്നീട് ചെയ്തിരിക്കുന്നത്. ഗ്രാഫ് സിദ്ധാന്തത്തിൽ ഡോമിനേഷൻ പോളിനോമിയൽ എന്നത് അതിൻറെ ഡോമിനേഷൻ ഗണങ്ങളെക്കുറിച്ചുള്ള വിവരങ്ങൾ നൽകുന്ന ഒരു പദമാണ്. ഇതിന് സമാനമായി സിസി-ഡോമിനേഷൻ പോളിനോമിയലിനെ പരിചയപ്പെടുത്തുകയും പഠിക്കുകയും ചെയ്തിട്ടുണ്ട്. ഒരു ഗ്രാഫിന്റെ ദൂരവും കണക്റ്റിവിറ്റിയുമായി ബന്ധപ്പെട്ടിരിക്കുന്ന ടോപ്പോളജിക്കൽ സൂചികയായ വെർട്ടക്സ് കണക്റ്റിവിറ്റി ഇൻഡക്സ് ആണ് അഞ്ചാം അധ്യായത്തിൽ പരാമർശിച്ചിരിക്കുന്നത്. മേൽ പ്രതിപാദിച്ച സൂചികയുമായി ബന്ധപ്പെട്ടു കിടക്കുന്ന വെർട്ടെക്സ് കണക്ട്വിറ്റി പോളിനോമിയലിനെയും അതിന്റെ റൂട്ടുകളെയും കുറിച്ചാണ് ആറാമത്തെ അധ്യായത്തിൽ പഠിച്ചിരിക്കുന്നത്. ഒടുവിൽ പഠനത്തിന്റെ പ്രാധാന്യം വിശകലനം ചെയ്യുകയും, ഉപസംഹാര അദ്ധ്യായം ചില ശുപാർശകൾ വ്യക്തമാക്കുകയും ചെയ്യുന്നു.

സൂചക പദങ്ങൾ : ക്ലോസ്ലി-കണക്റ്റഡ് വെർട്ടിസസ്, സിസി-ഡോമിനേഷൻ, സിസി-ഡോമിനേഷൻ പോളിനോമിയൽ, വെർട്ടക്സ് കണക്റ്റിവിറ്റി ഇൻഡക്സ്, വെർട്ടെക്സ് കണക്ട്വിറ്റി പോളിനോമിയൽ

Contents

List of Symbols

G	A simple finite undirected graph.
$V(G)$	Vertex set of G.
$E(G)$	Edge set of G.
$\|S\|$	Cardinality of the set S.
$d(u,v)$	Distance between u and v.
$\epsilon(v)$	Eccentricity of v.
$\omega(G)$	Number of components of G.
$g(u,v)$	Set of all geodesics linking u and v.
$\Gamma_{cc}(u,v)$	CC-Number of the vertex pair (u,v).
$N(v)$	Neighborhood of the vertex v.
$N_{cc}(v)$	Open cc-neighborhood of the vertex v.
$N_{cc}[v]$	Closed cc-neighborhood of the vertex v.
$deg(v)$	Degree of the vertex v.
$deg_{cc}(v)$	CC-Degree of the vertex v.
$\Delta(G)$	Maximum degree of a vertex in G.
$\Delta_{cc}(G)$	Maximum cc-degree of a vertex in G.

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$\delta(G)$	Minimum degree of a vertex in G.
$\delta_{cc}(G)$	Minimum cc-degree of a vertex in G.
$\alpha_{cc}(G)$	CC-Covering number of G.
$\beta_{cc}(G)$	CC-Independence number of G.
$\overline{G}$	Complement of G.
$\overline{G_{cc}}$	CC-Complement of G.
$ccn(G)$	CC-Neighborhood graph of G.
$\gamma(G)$	Domination number of G.
$\gamma_{cc}(G)$	CC-Domination number of G.
$d(G)$	Domatic number of G.
$d_{cc}(G)$	CC-Domatic number of G.
$D_c(G, i)$	CC$^{(i)}$- Dominating set.
$D[G; x]$	Domination polynomial of G.
$D_c[G; x]$	CC-Domination polynomial of G.
$D_c^{(m)}[G; x]$	m^{th} derivative of $D_c[G; x]$.
$I_{\mathrm{dom}}(G)$	Domination entropy of G.
$I_{ccd}(G)$	CC-Domination entropy of G.
$Q_c(G)$	Closely-connected geodetic communication overlap
$W(G)$	Wiener index of G.
$V_c[G]$	Vertex-connectivity index of G.
$V[G; x]$	Vertex Connectivity Polynomial of G.
$\kappa(G)$	Node connectivity of G.
$\lambda(G)$	Edge connectivity of G.
K_n	Complete graph on n vertices.
P_n	Path on n vertices.

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C_n	Cycle on n vertices.
$K_{m,n}$	Complete bipartite graph.
$K_{n_1,n_2,\ldots,n_m}$	Complete m-partite graph.
$K_{1,n}$	Star graph.
$B_{m,n}$	Bistar graph.
$S(n_1, n_2, \ldots, n_k)$	Starlike tree graph
W_n	Wheel graph .
H_n	Helm.
WB_n	Web graph.
S_n	Shell graph.
BW_n	Bow graph.
BF_n	Butterfly graph.
F_n	Friendship graph.
f_n	Fan graph.
$L_{m,n}$	Lollipop graph.
$B_{m,n,1}$	Generalized barbell graph.
$B_{n,1}$	n-barbell graph.
B_n	Bipartite cocktail party graph.
$W_n^{(m)}$	Windmill graph.
$C_n \odot P_m$	Armed crown.
L_n	Ladder graph.
$L(G)$	Line graph of G.
$S(G)$	Splitting graph of G.
$Sh(G)$	Shadow graph of G.
$H \vee K$	Join of H and K.

$H \circ K$	Corona of H and K.
$H \times K$	Cartesian product of H and K.
$H \wedge K$	Tensor product of H and K.

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Introduction

Graph theory is one of the thriving branches of mathematics. Originating from the modeling and negative resolution of famous Konigsberg bridge problem by Leonhard Euler [12], graph theory has entrenched as one of the best tool to model network systems involved in complex real life problems. The beauty of graph theory relies in its broad range of applications in disciplines from architecture and linguistics to network theory, chemistry, and operational research. Graph theory holds a remarkable place in many areas of applied mathematics by acting as a translator of real-world issues to mathematical models. Many branches of mathematics were inspired by the basic issues with calculation, motion, and measurement. However, the driving force behind the development of graph theory was frequently little more than conundrums designed to test ingenuity rather than to stimulate creativity [8]. It deals with the study of graphs, which are mathematical structures used to model pairwise relations between objects.

One of the important concepts in graph theory is connectivity, which refers to the degree to which nodes in a graph are connected to each other. This is particularly important in network analysis, where it is used to understand how information flows through complex systems. Graph connectivity can be used to develop a mathematical framework for representing networks and thereby to analyze their structures and properties. Overall, graph theory has had a profound impact on our understanding of complex systems and has paved the way for many important technological advancements.

Network performance is influenced by a number of factors, including the number of users and the transmission medium; security concerns include safeguarding data against viruses and unauthorised access. The smooth operation of communication networks is largely dependent on the reliability of communication infrastructure, which ensures that the network has the ability to withstand topological changes. This guarantees that neither the entire network nor sizable sections of it will fail as a result of potential changes and that the remaining network will immediately resume normal operation [35]. The ability of a system to continue operating as intended despite errors is known as fault tolerance and the main aim is to design fault-tolerant networks by minimizing link failures.

In the present work, emphasizing on the need for fault-tolerant mechanisms for reliable communication between the nodes in a network, a new graph concept called "closely-connected vertices" is introduced. Among various branches in graph theory, domination in graphs and graph polynomials are well studied to unveil the structural properties of graphs. The novel concept of closely-connectedness is developed to generalise the adjacency property by incorporating the connectivity aspect and is extended to the areas of domination and graph polynomials to explore more about the structural characteristics of graphs.

An Overview of the Thesis

The thesis comprises of an introductory chapter together with eight chapters in which a new graph concept "**Closely-connected Vertices**" is introduced and studied in a detailed manner. In the introductory chapter, a concise description detailing the motivational facts behind the introduction of the new graph concept is provided. Moreover, a blueprint of the upcoming chapters is furnished.

In **chapter 1**, the terminologies and notations that will appear in the subsequent chapters are detailed. Basic graph theoretic definitions are explained in the first section whereas the second section describes some important named graphs. The third section of this chapter is about various graph operations. Section 1.4 deals with domination related parameters and entropy in graphs. The last section incorporates an introduction to the theory of graph polynomials with certain important results, which are helpful in the study of roots of polynomials and stability properties.

In **chapter 2**, the concept of 'closely-connected vertices' in graphs is introduced through four sections. The necessity for link failure-free distributed systems is emphasised in the introduction, which actually spurred to develop the idea of closely-connected graph vertices. A pair of distinct vertices in a graph is said to be closely-connected if there exists a geodesic linking them preserving the connectivity of the graph. A path P in a graph $G = (V, E)$ is a *cut path* if there exists a graph $G' = (V, E \setminus E')$ for some $E' \subseteq E(P)$ such that $\omega(G') > \omega(G)$.

The *cc-number* of two vertices v_i and v_j in G is defined as

$$\Gamma_{cc}(v_i, v_j) = |\{g \in g(v_i, v_j)|\ g \text{ is not a cut path in } G\}|,$$

where $i \neq j$ and $g(v_i, v_j)$ is the set of all geodesics connecting v_i and v_j in G. The vertices $v_i, v_j \in V$ are *closely-connected* if $\Gamma_{cc}(v_i, v_j) \geq 1$, where $i \neq j$ and the graph G is *closely-connected* if $\Gamma_{cc}(v_i, v_j) \geq 1\ \forall i \neq j$. The second and third sections, respectively, provide descriptions of the related network model and mathematical model of this novel concept. Many new defintions related to closely-connectedness is formulated and each of them is studied for a wide range of graphs in the last section. Moreover, some interesting results on closely-connected vertex pairs are also derived.

Chapter 3 is divided into three sections. The rapidly flourishing variations of graph domination stimulated to develop a more generalised concept of cc-domination in graphs by incorporating graph connectivity instead of mere adjacency. The concepts of cc-dominating sets and cc-domination number of graphs are examined in the second section and the latter is assessed for a few well-known graph classes and few broad findings are established. For a simple finite undirected graph G, a subset C of V is called a *cc-dominating set* if for every vertex $v \in V \setminus C$ there exists a vertex $u \in C$ such that $\Gamma_{cc}(u, v) \geq 1$. The minimum cardinality of a cc-dominating set in G is called the *cc-domination number*, denoted by $\gamma_{cc}(G)$. Further, analogous to the concept of domatic partition in graphs, cc-domatic partition is introduced in the last section, and some conclusions are drawn. The *cc-domatic partition* of a graph G is a partition $\{V_1, \ldots, V_k\}$ of $V(G)$ in which each V_i is a cc-dominating set of G for $i = 1, \ldots, k$. The maximum order of a cc-domatic partition of G is called the *cc-domatic number* of G and is denoted by $d_{cc}(G)$.

In **chapter 4**, a new graph polynomial called cc-domination polynomial is introduced using cc-dominating sets. This novel graph polynomial is introduced as a generalisation of domination polynomial, deduced using dominating sets. The whole chapter is divided into four sections. The first section gives a brief introduction explaining the necessity to formulate the new graph polynomial. In the second section, emphasizing the role of closely-connected vertices in fault tolerance mechanisms, we introduce the $cc^{(i)}$-dominating set and the cc-domination polynomial $D_c[G;x]$ of graphs and some general results are established. For a graph $G = (V, E)$ of order n and for every integer $1 \leq i \leq n$, define the *$cc^{(i)}$-dominating set* of G as;

$$D_c(G,i) = \left\{ (u_1,\ldots,u_i) \in \underbrace{V \times \ldots \times V}_{i \text{ times}} : \forall v \in V \setminus \{u_j\}_{j=1}^{i},\ \exists\ u_j \text{ with } N_{cc}(u_j) = v \right\}.$$

The *cc-domination polynomial* $D_c[G;x]$ of G is defined as:

$$D_c[G;x] = \sum_{i=\gamma_{cc}(G)}^{n} d_c(G,i)x^i,$$

where $d_c(G,i) = |D_c(G,i|$. The third section mainly focusses on computing $D_c[G;x]$ for some special graphs and in the last section the roots of the new graph polynomial are studied in a detailed manner.

In chapter 5, a new topological index is introduced. In literature, most of the proposed topological indices are related either to vertex adjacency relationship or to distance function in a graph. This chapter proposes a topological index that is linked to both the distance and connectivity aspects of a graph. Here, we confine our study to formulate an index by incorporating both the metric and connectivity properties of a graph. Among the two sections, the first section gives a brief introduction emphasizing the need to study vertex-connectivity index in graphs. The vertex-connectivity index is formulated, its properties are discussed and the same is computed for some special graphs in the alternate section. Let $G = (V, E)$ be a graph and u, v be distinct vertices of G. Then, u and v are said to be *vertex-connected* if $\Gamma_{cc}(u, v) = g_{uv}$, where g_{uv} is the cardinality of the set of all geodesics connecting u and v. The graph G is *vertex-connected* if the above condition holds for every vertex pair in G. Define a function $f : V \to \mathbb{R}^+ \cup \{0\}$ given by

$$f(v) = \sum_{\substack{u \in V \setminus \{v\} \\ \Gamma_{cc}(u,v) = g_{uv}}} d(u, v) \quad \forall v \in V.$$

Then, the *vertex-connectivity index* $V_c[G]$ of G is defined as :

$$V_c[G] = \frac{1}{2} \sum_{v \in V} f(v).$$

Chapter 6 is divided into three sections. In the first section, the motivation which lead to introduce the idea of vertex connectivity polynomial is described. In the following section, the concept of vertex connectivity polynomial is defined and studied in a detailed manner by evaluating it for a wide variety of graphs and some important results are obtained. For any pair of distinct vertices (v, w) in a graph $G = (V, E)$, let $g_i(v, w)$ be the set of all geodesics linking v and w such that the deletion of all its edges disconnects the graph G into i components, where $i \in \{1, 2, \ldots, \text{diam(G)+1}\}$. In fact,

$$\bigcup_{i=1}^{\text{diam(G)+1}} g_i(v, w) = g(v, w),$$

where $g(v, w)$ is the set of all geodesics linking the vertices v and w in G. Define for any pair of distinct vertices $(v, w) \in V \times V$,

$$n(v, w) = \max\{i : p(v, w) \in g_i(v, w)\},$$

where $p(v, w)$ is a geodesic connecting v and w. For $i \in \{1, 2, \ldots, \mathrm{diam(G)}+1\}$, the *i-disconnectivity set* D_i of G consisting of unordered pairs of vertices of G is defined as:

$$D_i = \{(v, w) \in V \times V : v \neq w, n(v, w) = i\}$$

The *vertex connectivity polynomial* of G, denoted by $V[G; x]$ is defined as:

$$V[G; x] = \sum_{i=1}^{\mathrm{diam(G)}+1} |D_i| x^i,$$

The final section of this chapter deals about the roots and stability properties of the vertex connectivity polynomial and the same is studied in a detailed manner.

Chapter 7 spot lights the significance of closely-connected vertices in some areas of public interest. In section 7.1, the requirement of close-connectivity for developing robust networks is explained. Certain domains such as digital image processing, military and civilian communications, underwater cable disruptions, PMU placement in power networks and the travelling salesman issue, in which new measures of connectivity could contribute positively are discussed. Additionally, an illustration of the concept of close-connectivity in robust network modelling is provided. In the second section, the concept of close-connectivity is quantified by introducing a new parameter known as closely-connected geodetic communication overlap. Also, an entropy measure called cc-domination entropy is formulated and discussed.

The concluding chapter provides a brief summary of the thesis. Also, this chapter specifically suggests the directions and scopes for further research. It also includes a list of publications, presentations and bibliography.

Chapter 1

PRELIMINARIES

This chapter explores the graph theoretic terminology and notations that will appear in the subsequent chapters. We adopt the basic definitions and notations as in Graph Theory with Applications [9], written by J.A. Bondy and U.S.R. Murty. The chapter is divided into five sections. The first section deals with basic definitions and notations that may appear in the forthcoming chapters. In the second section certain special graphs are familiarised whereas various graph theoretic operations are discussed in the third section. The fourth section incorporates the concepts of graph domination, entropy and related parameters. The last section discusses about graph polynomials and its stability.

1.1 Basic Terminologies

A ***graph*** G is an ordered triple (V, E, ϕ) consisting of a non-empty set V of vertices, a set E of edges, and an incidence function ϕ that associates with each edge of G an unordered pair of vertices of G, notated by $G = (V, E)$. A ***finite graph*** is a graph having finite number of vertices and edges. The number of vertices and number of edges of a finite graph G are called the ***order*** and ***size*** of G respectively. Two or more edges having same end vertices are called ***parallel edges*** and an edge with identical end vertices is called a ***loop***. A ***simple graph*** is a graph having no parallel edges or loops [9].

A pair of ***vertices are adjacent*** if they are incident with a common edge and two ***edges are adjacent*** if they are incident to a common end vertex [9]. Two adjacent vertices are said to be ***neighbors*** of each other and the set of all neighbors of a vertex $v \in V$ is called the ***neighbor set*** of v and is denoted by $N(v)$ [19]. The number of vertices in $N(v)$ is called the ***degree*** of v. Vertices of degree 1 are called ***pendent*** vertices. A graph having all the vertices with same degree is called a ***regular*** graph. The sum of all degrees of vertices in a graph will be always twice the number of edges in it and this equality is known as ***degree sum formula.*** A subset S of the set of vertices of a graph G in which any two distinct vertices are adjacent is called a ***clique*** in G [9].

A ***walk*** in a graph G is an alternating sequence $v_0e_1v_1e_2 \ldots v_{i-1}e_iv_i \ldots v_{n-1}e_nv_n$ of vertices and edges in which the vertices v_{i-1} and v_i are the end points of the edge e_i. The length of a walk is the number of edges in the [9]. A ***path*** is a walk having all the vertices distinct. A path on n vertices is denoted by P_n. A ***trail*** is a walk where all the edges are distinct. A closed trail in which all the vertices are distinct is called a ***cycle***. A cycle of order n is denoted by C_n [19].

A graph $G = (V, E)$ is ***connected*** if for each vertex pair $(u, v) \in V \times V$, there exists a u-v path in G. A graph is said to be ***disconnected*** if it is not connected. A ***component*** of a graph is a maximal connected subgraph. The **node connectivity** $\kappa(G)$ of a graph G is the minimum number of vertices to be deleted to make the graph disconnected, whereas the **edge connectivity** $\lambda(G)$ refers to the minimum number of edges to be deleted to make G disconnected [19]. An edge in G is a ***cut edge*** if its deletion increases the number of components of G [19]. A ***cut vertex*** is a vertex whose removal increases the number of components of G [19]. A graph is ***acyclic*** if it contains no cycles. A connected acyclic graph is called a ***tree*** [9].

The ***distance*** between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest u-v path in G [19]. The ***eccentricity*** of a vertex v in G is defined as the maximum distance from v to a vertex in G, denoted by $\epsilon(v)$ [19]. The maximum distance between any pair of vertices in G is called the ***diameter*** of G [19]. The ***Wiener index*** of a graph $G = (V, E)$ is defined as the sum of all distances between the vertex pairs in G [47]. That is,

$$W(G) = \frac{1}{2} \sum_{u,v \in V} d(u, v).$$

1.2 Special Graphs

A ***complete graph*** K_n is a simple graph of order $n \geq 1$ in which all pairs of vertices are adjacent [9]. A graph is ***bipartite*** if its vertex set can be partitioned into two subsets S and T so that any edge of G has one end vertex in S and the other in T [19]. If each vertex of S is adjacent to every vertex of T, then it is called a ***complete bipartite graph.*** A ***complete m-partite graph*** $K_{n_1,n_2,\ldots,n_m}$ is a graph whose vertex set can be partitioned into m non empty sets V_i ; $i = 1, 2, \ldots, m$ such that every vertex in V_i is adjacent to every vertex in V_j for every $i \neq j$ and $i, j \in \{1, 2, \ldots, m\}$ [9].

A ***wheel graph*** W_n , $n \geq 4$ is defined as the join of the cycle C_{n-1} and K_1. A ***helm*** $H_n, n \geq 4$ is obtained by adding pendent edges to every vertex on the wheel rim. A ***web graph*** $WB_n, n \geq 4$ is obtained by joining the pendent vertices of a helm H_n to form a cycle and then adding a single pendent edge to each vertex of this outer cycle [16].

A ***shell graph*** $S_n, n \geq 3$ is obtained from the cycle graph C_n by adding the edges corresponding to the $n-3$ concurrent chords of the cycle [43]. The vertex at which all the chords are concurrent is called the apex of the shell. A ***bow graph*** B_n is a double shell with same apex in which each shell has any order. A ***butterfly graph*** BF_n is a bow graph along with exactly two pendent edges at the apex [16].

A ***star graph*** is a complete bipartite graph with one of the partite sets having cardinality 1. A ***bistar graph*** $B_{m,n}$ is the union of two star graphs $K_{1,m}$ and $K_{1,n}$ with centres joined together to form a new edge [16]. A ***star like tree graph*** $S(n_1, n_2, \ldots, n_k)$is a graph having only one vertex w of degree greater than 2 such that deletion of w results in a disjoint union of the path graphs $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$ [32] . The ***Lollipop graph*** $L_{m,n}$ is a graph obtained by joining a complete graph K_m to a path P_n with a bridge [24]. The ***n-barbell graph*** $B_{n,1}$ is a graph obtained by connecting two copies of complete graph K_n by a bridge

[43]. The ***generalized barbell graph*** $B_{m,n,1}$ is obtained by joining K_m and K_n by a cut edge. The ***bipartite cocktail party graph*** B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$. The ***windmill graph*** $W_n^{(m)}$ is obtained by taking m copies of K_n with a vertex in common. An ***armed crown*** $C_n \odot P_m$ is a graph obtained by attaching a path P_m to every vertex of the cycle C_n [43].

The ***fan graph*** f_n, $n \geq 2$ is obtained by joining all nodes of P_n to a further node called the center. The ***ladder graph*** L_n, $n \geq 2$ is the product graph $P_2 \times P_n$ [18]. The ***friendship graph*** F_n, $n \geq 2$ is constructed by joining n copies of the cycle graph C_3 with a common vertex, which becomes the universal vertex for the graph [16].

The ***line graph*** $L(G)$ of a simple graph G is obtained by associating a vertex with each edge of the graph G and linking two vertices with an edge in $L(G)$ iff the corresponding edges of G have a vertex in common [7].

1.3 Graph Operations

The **splitting graph** [28] $S(G)$ of a graph G is obtained by adding new vertices v' to G corresponding to each vertex v of G and then joining the vertex v' to all vertices of G adjacent to v in G. The **shadow graph** $Sh(G)$ of a graph G is obtained by taking two copies of G, say G_1 and G_2 and joining each vertex of G_1 to the neighbors of the corresponding vertex of G_2. The **duplication of a vertex** v of a graph G is the graph $G^{'}$ obtained by adding a vertex v' in G with $N(v') = N(v)$ [43].

The **join** [19] of two graphs H and K is the graph $H \vee K$ with vertex set $V(H) \cup V(K)$ and edge set $E(H) \cup E(K) \cup \{uv : u \in V(H), v \in V(K)\}$.
The **corona of two graphs** [13] K and H is formed from one copy of K and $|V(K)|$ copies of H where the i^{th} vertex of K is adjacent to every vertex in the i^{th} copy of H. It is denoted by $K \circ H$.

The **Cartesian product** [19] of two graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$, where the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G \times H$ whenever $u_1 = v_1$ and u_2 adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 adjacent to v_1 in G_1.

The **tensor product** or **conjunction** [19] of two graphs K and H is the graph $K \wedge H$ with vertex set $V(K) \times V(H)$ and the vertices (u, v) and (x, y) are adjacent if and only if $ux \in E(K)$ and $vy \in E(H)$.

1.4 Domination and Entropy of Graphs

A set $S \subseteq V$ of vertices of a graph $G = (V, E)$ of order n is called a ***dominating set*** if every vertex $v \in V$ is either an element of S or is adjacent to an element of S [22]. The ***domination number*** $\gamma(G)$ of G is defined as the minimum cardinality of a dominating set in G. For more details about domination number and their related parameters, we refer to [21], [23], [36] and for variations in ordinary domination [4],[29] and [30] can be referred.

A ***domatic partition*** of a graph G is a partition of $V(G)$, all of whose classes are dominating sets in G. The maximum number of classes of a domatic partition of G is called the ***domatic number*** of G, denoted by $d(G)$. For applications of domatic partition [27],[39] and [49] can be inferred.

The ***domination polynomial*** [3] is defined as $D[G;x] = \sum\limits_{i=1}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of cardinality i in G. For more details on the domination polynomial, [2] and [1] can be referred.

The ***entropy*** of G can be defined using Dehmer's information functional approach [42]. Let $f : S \to \mathcal{R}^+$ be an information functional defined on $S = \{s_1, s_2, \ldots, s_k\}$ such that S is a set of elements of G. Then, the entropy is defined as follows:

$$I_f(G) = -\sum_{i=1}^{k} \left[\frac{f(s_i)}{\sum\limits_{j=1}^{k} f(s_j)} log \left(\frac{f(s_i)}{\sum\limits_{j=1}^{k} f(s_j)} \right) \right],$$

where logarithmic phrases have base 2.

The ***domination entropy*** of a graph G of order n having no isolated vertices is defined as :

$$I_{dom}(G) = -\sum_{i=1}^{n} \left[\frac{d(G,i)}{\gamma_s(G)} log \left(\frac{d(G,i)}{\gamma_s(G)} \right) \right],$$

where $\gamma_s(G)$ is the total number of dominating sets in G.

1.5 Polynomials

A polynomial $f(x_1, \ldots, x_n)$ is said to be ***stable*** [46] with respect to a region $\Omega \subseteq \mathbb{C}^n$ if no zero of f lies in Ω. A ***Hurwitz polynomial*** is a polynomial which is stable with respect to the closed right half plane. Hurwitz polynomials are important in control systems theory, because they represent the characteristic equations of stable linear systems [15]. A polynomial is ***schur stable*** if all its zeros lie in the open unit disk [14].

Let $\mathcal{H}$ be the set of finite graphs of order n and $R[x]$ be the polynomial ring over the real numbers. Then, a ***graph polynomial*** is a function $P : \mathcal{H} \to R[x]$ such that for any two graphs $H_1, H_2 \in \mathcal{H}$, if H_1 is isomorphic to H_2, then $P(H_1) = P(H_2)$. A graph polynomial encodes information about the graph and enables algebraic methods for extracting this information.

Theorem 1.5.1. (Routh-Hurwitz Criteria) *Given a polynomial*

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where the coefficients a_i's are real constants for $i = 1, \ldots, n$, the n Hurwitz matrices using the coefficients a_i of the above polynomial are defined as [14]*:*

$$H_1 = \begin{bmatrix} a_1 \end{bmatrix}, H_2 = \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix}, H_3 = \begin{bmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, \cdots,$$

$$H_n = \begin{bmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

where $a_j = 0$ if $j > n$. All the roots of the polynomial $P(x)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive: $det(H_j) > 0, j = 1, 2, \ldots, n$.

Theorem 1.5.2. *Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0$ be a complex polynomial such that $|a_k| > |a_0| + |a_1| + \ldots + |a_{k-1}| + |a_{k+1}| + \ldots + |a_n|$ for some $0 \leq k \leq n$. Then exactly k zeros of P lie strictly inside the unit circle, and the other $n - k$ zeros of P lie strictly outside the unit circle*[50].

Chapter 2

CLOSELY-CONNECTED VERTICES IN GRAPHS

This chapter is incorporated with four sections to introduce the idea of closely-connected vertices in graphs. The introductory section emphasises the need for link-failure-free distributed systems, which in fact serves as the motivation to formulate the concept of closely-connected vertices in graphs. The corresponding network model and mathematical model are described in the second and third sections, respectively. In the last section, certain fundamental concepts and characteristics of closely-connected vertices are studied, and as a consequence some useful conclusions are drawn. Throughout this work, G *denotes a simple finite undirected graph with vertex set* $V(G)$ *and edge set* $E(G)$ *and* (u, v) *denotes an unordered pair of distinct vertices in* G.

2.1 Introduction

Connectivity plays a fundamental role in digital image processing and analysis. It is extensively used in image filtering and segmentation, image compression and coding, motion analysis, and recognising patterns [44]. Data communication networks are effective and efficient when they integrate performance, reliability, and security. Several factors influence network performance, such as the number of users and the medium of transmission, whereas security concerns include protecting data from unauthorized access and viruses. Communication networks rely heavily on the reliability of their communication infrastructure, as it is crucial for the smooth functioning of the network. The reliability of a network is strongly influenced by the network's capacity to handle topological changes, which ensures that neither the entire network nor significant portions of it will fail as a result of such changes and that the remaining network will resume normal operation immediately. This feature can be achieved through several techniques, and a number of algorithms are already available in the literature regarding this [35]. The main objective is to reduce network down time and keep communication flowing with the least amount of disruption.

Fault tolerance is defined as the capability of a system to keep performing its intended task even in the presence of faults [20]. A characteristic feature that distinguishes distributed systems from single machine systems is the notion of partial failure. Distributed systems must possess the ability to cope up with rapid topological changes and to reach a new steady state so that the remaining network can perform its tasks efficiently. Thus, a distributed routing protocol must be resilient to topological changes and can route packets without relying on a central authority by maintaining a low communication overhead; which ensures that the recovery process is as fast as possible.

In the modern era of cloud computing, fault tolerance mechanisms are indispensable to ensure high availability and authenticity to the users. The faults in the cloud environment may occur due to physical faults, network faults, processor faults, service expiry faults, etc., and so on [40]. Among these, network faults arise mainly due to link failures and can be curtailed only if the nodes of the network are associated ‘more closely’ in such a way that unpredictable disruptions may not fail the whole system. For more details about the fault tolerance in cloud computing systems, [5], [20] and [31] can be referred. Hence, while designing a distributed system, paramount importance should be given to associate nodes ‘more closely’ so that the system will be reliable by automatically recovering from partial failures without seriously affecting its overall performance. Thus, we restrict our focus on studying the properties of those nodes that can cope with rapid topological changes and can quickly reach a new steady state in which the remaining network can perform its tasks efficiently.

2.2 Network Model

A common wired network is used by all hosts for communication. Every host has a fixed broadcast area, and within this close-transmission range, link failures of the network do not disrupt the functioning of the entire system. A pair of hosts that are in communication with one another are referred to as closely-connected neighbors. Moreover, we assume that communication between hosts is bidirectional.

2.3 Mathematical Model

The motivation to introduce cycle graphs C_n in graph theory was to overcome the difficulties that may arise due to the lack of multiple paths between graph vertices. Being free of cut edges, every pair of adjacent vertices in C_n is ‘so close’ in the sense that the deletion of the edge shared by them will not disconnect the whole graph, which is not true for non-adjacent vertices. Thus, the property enjoyed by adjacent vertices in terms of ‘closeness’ does not hold for non-adjacent vertex pairs in C_n. This structural deficiency of C_n serves as a motivation to model this situation mathematically to tolerate link failures in distributed systems.

Let $G = (V, E)$ be a simple finite undirected graph. The nodes of G represent the hosts in the wired network. A pair of nodes (u, v) are said to be in communication if they are within the close-transmission range. That is, there exists a geodesic between u and v whose deletion of edges does not alter the connectivity between the nodes of G.

2.4 Closely-connected Vertices

Definition 2.4.1. A path P in G is a *cut path* if there exists a graph $G^{'} = (V, E \setminus E^{'})$ for some $E^{'} \subseteq E(P)$ such that $\omega(G^{'}) > \omega(G)$.

Remark 2.4.2. It can be easily noted that $E^{'}$ in definition 2.4.1 is an edge cut of

G.

Definition 2.4.3. The *cc-number* of two vertices v_i and v_j in G is defined as

$$\Gamma_{cc}(v_i, v_j) = |\{g \in g(v_i, v_j)|\ g \text{ is not a cut path in } G\}|,$$

where $i \neq j$ and $g(v_i, v_j)$ is the set of all geodesics connecting v_i and v_j in G.

Definition 2.4.4. The vertices $v_i, v_j \in V$ are *closely-connected* if $\Gamma_{cc}(v_i, v_j) \geq 1$, where $i \neq j$.

The graph G is *closely-connected* if $\Gamma_{cc}(v_i, v_j) \geq 1\ \forall i \neq j$.

Definition 2.4.5. The *open cc-neighborhood* of a vertex $u \in V$ is defined as

$$N_{cc}(u) = \{v \in V : \Gamma_{cc}(u, v) \geq 1\}.$$

and the *closed cc-neighborhood* of u is defined as $N_{cc}[u] = N_{cc}(u) \cup \{u\}$.
The cardinality of $N_{cc}(u)$ is called the *cc-degree* of the vertex u, denoted by $deg_{cc}(u)$, in G. The maximum and minimum cc-degree of a vertex in G are given by

$$\Delta_{cc}(G) = \max\ \{|N_{cc}(u)|, u \in V\}$$

$$\delta_{cc}(G) = \min\ \{|N_{cc}(u)|, u \in V\}.$$

The vertex u is said to be *cc-isolated* if $N_{cc}(u) = \phi$. An edge $e = uv$ is a *cc-edge* if $v \in N_{cc}(u)$.

For any non-empty subset S of V, define:

$$N_{cc}(S) = \{u \in V : \exists \text{ a vertex } \ v \in S \text{ such that } \Gamma_{cc}(u, v) \geq 1\} \text{ and}$$

$$N_{cc}[S] = N_{cc}(S) \cup S.$$

Definition 2.4.6. A set $S \subseteq V$ is *cc-independent* if $\Gamma_{cc}(u,v) = 0 \ \forall u, v \in S$. The maximum cardinality of a cc-independent set in G is called the *cc-independence number*, denoted by $\beta_{cc}(G)$.

Definition 2.4.7. A set $S^{'} \subseteq V$ is called a *cc-vertex covering* of G if for every cc-edge $e = uv$ either $u \in S$ or $v \in S$. The minimum cardinality of a cc-vertex covering of G is called the *cc-covering number*, denoted by $\alpha_{cc}(G)$.

Definition 2.4.8. The *cc-complement* of G is the graph $\overline{G_{cc}} = (V, E^{'})$, where $u^{'}v^{'} \in E^{'}$ if and only if $v^{'} \notin N_{cc}(u^{'})$ in G for $u^{'}, v^{'} \in V$.

Example 2.4.9. *In the graph G shown in figure 2.1,*

$\Gamma_{cc}(v_1,v_3) = 0,\ \Gamma_{cc}(v_1,v_5) = 1,\ \Gamma_{cc}(v_1,v_6) = 0,\ \Gamma_{cc}(v_1,v_7) = 0,\ \Gamma_{cc}(v_1,v_8) = 1,$
$\Gamma_{cc}(v_2,v_4) = 0,\ \Gamma_{cc}(v_2,v_5) = 0,\ \Gamma_{cc}(v_2,v_6) = 0,\ \Gamma_{cc}(v_2,v_7) = 0,\ \Gamma_{cc}(v_2,v_8) = 0,$
$\Gamma_{cc}(v_3,v_5) = 1,\ \Gamma_{cc}(v_3,v_6) = 0,\ \Gamma_{cc}(v_3,v_7) = 0,\ \Gamma_{cc}(v_3,v_8) = 1,\ \Gamma_{cc}(v_4,v_6) = 0,$
$\Gamma_{cc}(v_4,v_7) = 0,\ \Gamma_{cc}(v_5,v_7) = 0,\ \Gamma_{cc}(v_5,v_8) = 0,\ \Gamma_{cc}(v_6,v_8) = 0$ *and*
$\Gamma_{cc}(v_i,v_j) = 1$ *if* v_i *and* v_j *are adjacent* $\forall i,j = 1,2,\ldots,8$ *where* $i \neq j$.

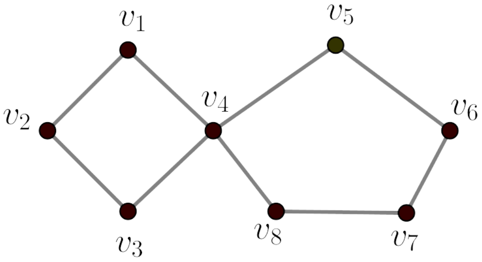

Figure 2.1: The graph G

The open cc-neighborhood of vertices of G and their respective cc-degree are given by:

$N_{cc}(v_1) = \{v_2, v_4, v_5, v_8\},\ deg_{cc}(v_1) = 4.$

$N_{cc}(v_2) = \{v_1, v_3\}$, $deg_{cc}(v_2) = 2$.

$N_{cc}(v_3) = \{v_2, v_4, v_5, v_8\}$, $deg_{cc}(v_3) = 4$.

$N_{cc}(v_4) = \{v_1, v_3, v_5, v_8\}$, $deg_{cc}(v_4) = 4$.

$N_{cc}(v_5) = \{v_1, v_3, v_4, v_6\}$, $deg_{cc}(v_5) = 4$.

$N_{cc}(v_6) = \{v_5, v_7\}$, $deg_{cc}(v_6) = 2$.

$N_{cc}(v_7) = \{v_6, v_8\}$, $deg_{cc}(v_7) = 2$.

$N_{cc}(v_8) = \{v_1, v_3, v_4, v_7\}$, $deg_{cc}(v_8) = 4$.

Therefore, $\Delta_{cc}(G) = 4$ *and* $\delta_{cc}(G) = 2$. *Since* $\delta_{cc}(G) \neq 0$, *none of the vertices of* G *are cc-isolated.*

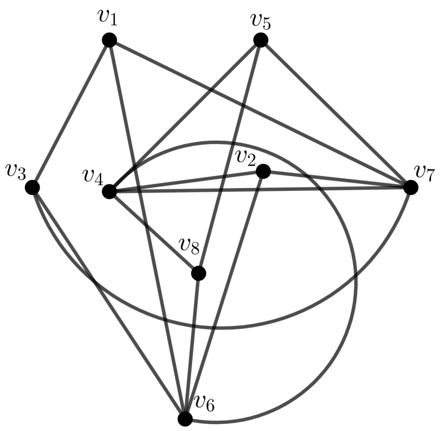

Figure 2.2: The cc-complement graph $\overline{G_{cc}}$

The cc-independent sets of G *are* $\{v_1, v_3\}, \{v_1, v_6\}, \{v_1, v_7\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_6\}, \{v_2, v_7\}, \{v_2, v_8\}, \{v_3, v_6\}, \{v_3, v_7\}, \{v_4, v_6\}, \{v_4, v_7\}, \{v_5, v_7\}, \{v_5, v_8\}, \{v_6, v_8\}, \{v_1, v_3, v_6\}, \{v_1, v_3, v_7\}, \{v_2, v_4, v_6\}$ *and* $\{v_2, v_4, v_7\}$. *Thus,* $\beta_{cc}(G) = 3$. *The minimal cc-vertex coverings of* G *are* $\{v_2, v_4, v_5, v_7\}, \{v_2, v_4, v_6, v_7\}$ *and* $\{v_2, v_4, v_6, v_8\}$. *Since every edge of* G *is a cc-edge, it follows that* $\alpha_{cc}(G) = \alpha(G) = 4$, *where* $\alpha(G)$ *is the vertex-covering number of* G.

Definition 2.4.10. Let G be a graph of order at least two. Then, the *cc-*

neighborhood graph $ccn(G)$ of G is the graph satisfying the following properties:

(i) $V(G) = V(ccn(G))$.

(ii) For $u, v \in V(G)$ with $u \neq v$, u and v are adjacent in $ccn(G)$ iff $\{u, v\}$ is not cc-independent in G.

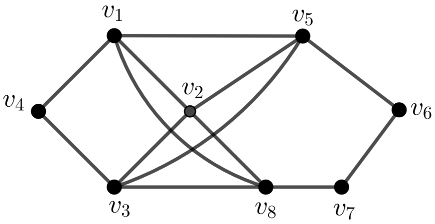

Figure 2.3: The cc-neighbourhood graph $ccn(G)$

The following are immediate consequences of definition 2.4.10 :

(i) $ccn(G) = K_n$ iff $\beta_{cc}(G) = 1$.

(ii) $ccn(G) = \overline{K_n}$ iff G is acyclic.

(iii) If G is connected, then $ccn(G) = \overline{K_n}$ iff G is a tree.

(iv) $ccn(C_n) = C_n \ \forall \ n \geq 3$.

(v) For $k \geq 2$ and $2 \leq n_1 \leq \cdots \leq n_k$,

$$ccn(K_{n_1,\ldots,n_k}) = \begin{cases} K_{n_1,n_2}, & \text{if } k, n_1 = 2 \\ K_{n_1+\cdots+n_k}, & \text{if } k \geq 2 \textit{ and } n_1 > 2. \end{cases}$$

Theorem 2.4.11. *Let $G = (V, E)$ be a graph of order n having no cut edges. Then, $deg(v) \leq deg_{cc}(v) \ \forall v \in V$.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ and without loss of generality, let $v_1 \ldots, v_k$ be the cc-isolated vertices in G. Since G is free of cut edges, these k vertices are in fact isolated in G. Therefore, $deg(v_i) = deg_{cc}(v_i) = 0 \; \forall i = 1, \ldots, k$. Since all other vertices of G belongs to some cycle and since G has no cut edges, it follows that $\Gamma_{cc}(u, v) \geq 1$ for every pair of adjacent vertices $(u, v) \in V \times V$. Thus, $deg_{cc}(v) \geq deg(v) \; \forall v \in V(G)$.

This completes the proof. □

Remark 2.4.12. The converse of the above theorem is not true. For example, consider the graph G_1 shown in the figure 2.4. Here $deg(v_i) \leq deg_{cc}(v_i) \; \forall i = 1, \ldots, 12$, but the graph G_1 has a cut edge.

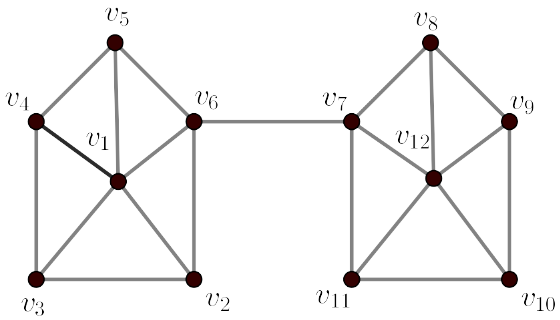

Figure 2.4: The graph G_1

Theorem 2.4.13. *For every non-negative integer $k \neq 1$, there exists a regular graph G with $\delta_{cc}(G) = k$. Moreover, there does not exists a graph G such that $\delta_{cc}(G) = 1$.*

Proof. It follows that

$$\delta_{cc}(K_k) = \begin{cases} k - 1 \; ; \text{ if } k \geq 3 \\ 0 \quad\quad ; \text{ if } k \leq 2 \end{cases}$$

Thus, for every $k \neq 1$, there exists a complete graph G with $\delta_{cc}(G) = k$.

Now let $k = 1$. If possible assume that there exists a graph $G = (V, E)$ with $\delta_{cc}(G) = 1$. Then, there exists adjacent vertices $u, v \in V$ such that $N_{cc}(u) = \{v\}$ and let e be the corresponding edge.

Case(i) e is a cut edge.

Since u and v are adjacent, the deletion of e disconnects G, a contradiction to the assumption that $\Gamma_{cc}(u, v) \geq 1$.

Case(ii) e belongs to a cycle.

In this case, $deg(u) \geq 2$ so that there exists a vertex $w \neq v$ such that $\Gamma_{cc}(u, w) \geq 1$, a contradiction to the fact that $deg_{cc}(u) = 1$.

This completes the proof. □

Corollary 2.4.14. *There exists an infinite family $\mathcal{F}$ of graphs satisfying*

$$\delta(ccn(G)) = \delta_{cc}(ccn(G)) = \Delta_{cc}(ccn(G)) = \Delta(ccn(G)),$$

where $G \in \mathcal{F}$.

Proof. Let $\mathcal{F}$ be the family of closely-connected graphs. Then, for every graph $G \in \mathcal{F}$, $ccn(G) = K_n$ for some positive integer $n > 1$. The result follows from theorem 2.4.13 and the fact $\delta(K_n) = \delta_{cc}(K_n) = \Delta_{cc}(K_n) = \Delta(K_n) = n - 1$. □

Remark 2.4.15. For a graph G,

(i) a vertex $u \in V(G)$ is cc-isolated iff either u is isolated or u has $deg(u)$ cut edges incident to it.

(ii) an edge in G is a cc-edge iff it is not a cut edge.

Theorem 2.4.16. *Let G be a graph of order n. Then, $\Delta_{cc}(G)+\Delta_{cc}(\overline{G_{cc}}) \geq n-1$.*

Proof. Let $v \in V(G)$ be such that $deg_{cc}(v) = \Delta_{cc}(G)$. Then, $n-1-\Delta_{cc}(G)$ vertices are closely-connected to v in $\overline{G_{cc}}$ so that $\Delta_{cc}(\overline{G_{cc}}) \geq n-1-\Delta_{cc}(G)$. That is, $\Delta_{cc}(G)+\Delta_{cc}(\overline{G_{cc}}) \geq n-1$.

This completes the proof. □

Theorem 2.4.17 (CC-Degree Sum Formula). *Let G be a graph of size m with m' cut edges. Then,*

$$\sum_{v\in V(G)} deg_{cc}(v) = 2[m-m']+\sum_{v\in V(G)} |\{w \in V(G) : d(v,w) \geq 2 \ \ and \ \Gamma_{cc}(v,w) \geq 1\}|$$

Proof. We have for every vertex $v \in V(G)$,

$$\begin{aligned}
deg_{cc}(v) &= |\{w \in V(G) : \Gamma_{cc}(v,w) \geq 1\}| \\
&= |\{w \in V(G) : d(v,w) \geq 2,\ \Gamma_{cc}(v,w) \geq 1\}| + deg(v) - v_c \\
\sum_{v\in V(G)} deg_{cc}(v) &= \sum_{v\in V(G)} \Big[|\{w \in V(G) : d(v,w) \geq 2,\ \Gamma_{cc}(v,w) \geq 1\}| + deg(v) - v_c\Big] \\
&= \sum_{v\in V(G)} |\{w \in V(G) : d(v,w) \geq 2,\ \Gamma_{cc}(v,w) \geq 1\}| + \sum_{v\in V(G)} \Big[deg(v) - v_c\Big] \\
&= \sum_{v\in V(G)} |\{w \in V(G) : d(v,w) \geq 2,\ \Gamma_{cc}(v,w) \geq 1\}| + 2[m-m^{'}],
\end{aligned}$$

where v_c is the number of cut edges incident on the vertex v and

$$\sum_{v\in V(G)} deg(v) = 2m$$

follows from the degree sum formula.

This completes the proof. □

Corollary 2.4.18. *Let G be a graph of size m and m' cut edges such that non-adjacent vertices are not closely-connected. Then,*

$$\sum_{u \in V(G)} deg_{cc}(u) = 2[m - m'].$$

Corollary 2.4.19. *Let G be a closely-connected graph of order n and size m. Then,*

$$\sum_{u \in V(G)} |\{v \in V(G) : d(u,v) \geq 2 \ \ and \ \ \Gamma_{cc}(u,v) \geq 1\}| = n(n-1) - 2m.$$

Theorem 2.4.20. *There does not exist a graph G of order $n > 2$ satisfying $ccn(G) = \bar{G}$.*

Proof. If possible there exists a graph G such that $ccn(G) = \bar{G}$.

Case(i) $E(G) = \phi$.

In this case, $G = \bar{K_n}$ so that $ccn(G) = \bar{K_n} = G$, a contradiction since G is not self complementary.

Case(ii) $E(G) \neq \phi$.

Sub case(i): G has only cut edges.

Since $n > 2$, the edge set of $\bar{G}$ is non empty, but $ccn(G) = \bar{K_n}$.

Sub case(ii): G has at least one cc-edge.

Let uv be a cc-edge in G. Then, $\Gamma_{cc}(u,v) \geq 1$ so that u and v are adjacent in $ccn(G) = \bar{G}$, which is not possible.

This completes the proof. □

Corollary 2.4.21. *For a graph G, $ccn(G) = \bar{G}$ iff $G = P_2$.*

Remark 2.4.22. Let $G = (V, E)$ be a cut edge free graph and $u, v \in V$ be such that $d(u, v) = 2$. Then, u and v are closely-connected iff there exists a path $P(u, v)$ such that both the edges of P cannot be characterized by a unique cycle in G.

Theorem 2.4.23. *Let $G = (V, E)$ be a graph and $u, v \in V$ are closely-connected. Then, there exists a path free of cut edges with every intermediate vertex of degree at least 3 in G.*

Proof. Assume that u and v are closely-connected. Then, there exists a path $P(u, v)$ in G such that none of its edges are cut edges. Let w be an intermediate vertex of P so that $deg(w) \geq 2$. If $deg(w) = 2$, then the deletion of the edges incident to w in G increases the number of components of G by isolating w. Therefore, $deg(w) \geq 3$.

This completes the proof. $\square$

Remark 2.4.24. The converse of the above theorem is not true. For example, consider the graph G_2 shown below.

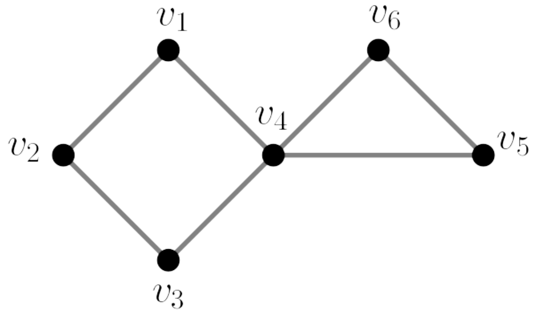

Figure 2.5: The graph G_2

Here, the path $P = v_1v_4v_3$ has the intermediate vertex v_4 with

$$deg(v_4) = deg_{cc}(v_4) = 4,$$

but v_1 and v_3 are not closely-connected.

Theorem 2.4.25. *There does not exist a graph G of order $n \geq 2$ such that $ccn(G)$ is a tree.*

Proof. If possible, let G be a graph such that $ccn(G)$ is a tree and v be a pendent vertex in $ccn(G)$. Then, v is adjacent to a unique vertex u in $ccn(G)$ such that $\Gamma_{cc}(u, v) \geq 1$ and $\Gamma_{cc}(u, w) = 0 \; \forall w \in V(G) \setminus \{u\}$. But, for a vertex to be closely-connected to any other vertex, it must belong to a cycle. Thus v is a part of some cycle in G so that there will be at least two adjacent vertices that are closely-connected to v. Thus, $deg_{cc}(v) \geq 2$ in G, whereas $deg(v) = 1$ in $ccn(G)$, a contradiction to the fact that $deg_{cc}(v)$ in $G = deg(v)$ in $ccn(G)$.
This completes the proof. $\square$

Corollary 2.4.26. *There does not exists a graph G such that $ccn(G)$ has pendent vertices.*

Chapter 3

CC-DOMINATION IN GRAPHS

This chapter is divided into three sections. The first section gives a brief introduction to the necessity of the concept of cc-domination in graphs. The consecutive section explores the concept of cc-dominating sets and cc-domination number of graphs and the latter is evaluated for some well known graph classes. Also, some general results are established. The last section discusses the idea of cc-domatic partition in graphs and certain results are obtained.

3.1 Introduction

Domination is one of the extensively researched branches of graph theory. In ordinary domination, a vertex needs to be either dominated by itself or by its neighbor. But in practice, this model is not always economic. For instance, we have a complicated network system in which control units have to be placed at various places to monitor faults and disturbances in the system. Thus, in order to minimize cost, control units have to be introduced at those vertices that can monitor the vertices ‘closer’ to it rather than adjacent to it. This neccessity ensued in the formulation of the idea of cc-domination in graphs.

3.2 CC-Domination Number of Graphs

In this section, we introduce the concept of *cc-dominating sets* and *cc-domination number* of graphs. Moreover, the *cc-domination number* of some well known graph classes are evaluated and some general results are established.

Definition 3.2.1. Let $G = (V, E)$ be a graph. A subset C of V is called a *cc-dominating set* if for every vertex $v \in V \setminus C$, there exists a vertex $u \in C$ such that $\Gamma_{cc}(u, v) \geq 1$.

The minimum cardinality of a cc-dominating set in G is called the *cc-domination number*, denoted by $\gamma_{cc}(G)$.

For a graph G of order n, the following are some basic properties of $\gamma_{cc}(G)$:

(i) $1 \leq \gamma_{cc}(G) \leq n$.

(ii) $\gamma_{cc}(G) = \gamma(ccn(G))$.

(iii) $\gamma_{cc}(G) = n$ iff G is acyclic.

(iv) If G has no cut edges, then $\gamma_{cc}(G) \leq \gamma(G)$.

(v) If G is acyclic, then $\gamma(G) \leq \gamma_{cc}(G)$.

(vi) If G is connected, then $\gamma_{cc}(G) = n$ iff G is a tree.

Theorem 3.2.2. *Let G be a graph. Then, we have the following:*

(i) *If $ccn(G) = K_n$ for $n \geq 2$, then $\gamma_{cc}(G) = 1$.*

(ii) *For any two-regular graph G, $\gamma_{cc}(G) = \gamma(G)$.*

(iii) *For the complete graph K_n,*

$$\gamma_{cc}(K_n) = \begin{cases} 2, & \text{if } n = 2 \\ 1, & \text{if } n \neq 2. \end{cases}$$

(iv) *For the complete bipartite graph $K_{m,n}$,*

$$\gamma_{cc}(K_{m,n}) = \begin{cases} 2, & \text{if } m, n = 2 \\ 1, & \text{if } m \geq 2, n > 2. \end{cases}$$

(v) *For $n > 2$, $\gamma_{cc}(K_{m_1,\dots,m_n}) = 1$, where $K_{m_1,\dots,m_n}$ is the complete n-partite graph.*

(vi) *For the ladder graph L_n, $\gamma_{cc}(L_n) = 1$, where $n \geq 3$.*

(vii) *For the path graph $P_n, \gamma_{cc}(L(P_n)) = n - 1$, where $n > 1$.*

(viii) *For the cycle graph C_n, $\gamma_{cc}(L(C_n)) = \gamma(C_n)$, where $n \geq 3$.*

(ix) *For the star graph* $K_{1,n}$, $\gamma_{cc}(L(K_{1,n})) = 1$, *where* $n > 2$.

(x) *For the bistar graph* $B_{m,n}$,

$$\gamma_{cc}(L(B_{m,n})) = \begin{cases} 1, & \text{both } m > 1 \text{ and } n > 1 \\ 2, & \text{either } m = 1 \text{ or } n = 1, \text{ not both} \\ 3, & \text{both } m = 1 \text{ and } n = 1. \end{cases}$$

Proof. (i) **Case(i) :** If $n = 2$, then G is either K_2 or $\bar{K_2}$. Since both the graphs are acyclic, the vertices in G are not closely-connected and hence it follows that $ccn(G) = \bar{K_2} \neq K_2$. Thus, the result holds trivially as the hypothesis fails.

Case(ii) : Assume that $n > 2$ and $ccn(G) = K_n$. Then, G is closely-connected so that $\gamma_{cc}(G) = 1$.

(ii) Two-regular graphs are either cycles or disjoint union of cycles. Since for $n \geq 3$, the closely-connected vertices of the cycle graph C_n are precisely those that are adjacent so that $\gamma_{cc}(C_n) = \gamma(C_n)$. Therefore, it directly follows that if G is two-regular, then $\gamma_{cc}(G) = \gamma(G)$.

(iii) The result follows easily from the observation that for $n = 2$, the graph K_2 is acyclic and for $n > 2$, every vertex of K_n cc-dominates all other vertices.

(iv) **Case(i) :** If $m = n = 2$, then $K_{m,n}$ is the two-regular graph C_4 so that $\gamma_{cc}(K_{m,n}) = \gamma(K_{m,n}) = \gamma(C_4) = 2$.

Case(ii) : If $m = 2$ and $n > 2$, then every vertex of degree 2 in $K_{m,n}$ is closely-connected to every other vertex so that $\gamma_{cc}(G) = 1$.

Case(iii) : If $m, n > 2$, then $K_{m,n}$ is closely-connected so that $\gamma_{cc}(G) = 1$.

(v) This follows from the observation that $K_{m_1,\ldots,m_n}$ is closely-connected.

(vi) In L_n for $n \geq 3$, every intermediate vertex cc-dominates all other vertices. Therefore, $\gamma_{cc}(L_n) = 1$.

(vii) For $n > 1$ we have, $L(P_n) = P_{n-1}$. Now, the result follows from the fact that P_{n-1} is a tree.

(viii) The result follows from the observation that the line graph of C_n is itself.

(ix) The result is a consequence of the observation that for $n > 2$, the line graph of $K_{1,n}$ is K_n.

(x) **Case(i) :** If $m = n = 1$, then $B_{m,n}$ is P_4 so that $\gamma_{cc}(L(P_4)) = \gamma_{cc}(P_3) = 3$.

Case(ii) : If $m = 1$ or $n = 1$, but not both, then $L(B_{m,n})$ is $K_{n+1} + e$ so that $\gamma_{cc}(L(B_{m,n})) = 2$.

Case(iii) : If $m, n > 1$, then $L(B_{m,n})$ is a union of K_{m+1} and K_{n+1} with one vertex in common, which is a closely-connected graph. Therefore, $\gamma_{cc}(L(B_{m,n})) = 1$.

This completes the proof. □

Theorem 3.2.3. *Let G be a graph and $u \in V(G)$ be cc-isolated . Then, u belongs to every cc-dominating set of G.*

Proof. Suppose that there exists a cc-dominating set C of G such that $u \notin C$. Since C being a cc-dominating set, there exists a vertex $v \in C$ such that

$\Gamma_{cc}(v,u) \geq 1$, which is a contradiction to the assumption that u is cc-isolated. This completes the proof. ☐

Corollary 3.2.4. *If G has k cc-isolated vertices, then $\gamma_{cc}(G) \geq k$.*

Proof. Let C be a cc-dominating set in G. Then since G has k cc-isolated vertices, it follows from theorem 3.2.3 that $|C| \geq k$. Therefore, $\gamma_{cc}(G) \geq k$. ☐

Theorem 3.2.5. *For a cut edge free graph G, $\gamma_{cc}(G) \leq \gamma(G)$. The equality holds if $N(v) = N_{cc}(v)$ $\forall v \in V(G)$.*

Proof. Being free of cut edges, the adjacent vertices are closely-connected in G so that every dominating set is a cc-dominating set. Now, if $N(v) = N_{cc}(v)$ holds $\forall v \in V(G)$, we can conclude that $\Gamma_{cc}(u,v) \geq 1$ iff u and v are adjacent $\forall u, v \in V(G)$ with $u \neq v$. Thus, every cc-dominating set of G will also be its dominating set so that $\gamma_{cc}(G) = \gamma(G)$.
This completes the proof. ☐

Remark 3.2.6. The converse of the equality in theorem 3.2.5 is not true. That is, $\gamma_{cc}(G) = \gamma(G)$ need not always imply $N(v) = N_{cc}(v)$ $\forall v \in V(G)$.

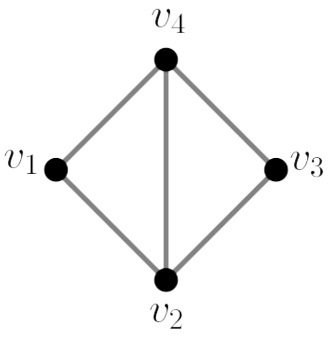

Figure 3.1: The graph G_3

For example, consider the graph G_3 shown in figure 3.1. It is evident that every vertex pair is closely-connected so that $\gamma_{cc}(G) = \gamma(G) = 1$. But, $N(v_1) = \{v_2, v_4\}$ whereas $N_{cc}(v_1) = \{v_2, v_3, v_4\}$ so that $N(v_1) \neq N_{cc}(v_1)$.

Theorem 3.2.7. *Let G be a connected graph in which vertex pairs having common neighbors are closely-connected. Then, $\gamma_{cc}(G) = 1$ if there exists a vertex $v \in V(G)$ such that $N(v) = N_{cc}(v)$ and $\epsilon(v) \leq 2$.*

Proof. Let $v \in V(G)$ be such that $N(v) = N_{cc}(v)$ and $\epsilon(v) \leq 2$. If $G = K_1$, then there is nothing to prove. Otherwise, for any $u \in V(G)$,

Case(i) If u is adjacent to v, then $u \in N(v) = N_{cc}(v)$ so that $\Gamma_{cc}(u, v) \geq 1$.

Case(ii) If u is not adjacent to v, then $d(u, v) = 2$ since $e(v) \leq 2$. Since there exists a path of length 2 in G from u to v, the vertex pair (u, v) shares a common neighbor. Hence, by our assumption it follows that u and v are closely-connected.

Thus, in both the cases, $\Gamma_{cc}(u, v) \geq 1$ and hence $\gamma_{cc}(G) = 1$.
This completes the proof. □

Corollary 3.2.8. *Let G be a connected graph of diameter at most 2 such that the vertex pairs sharing a common neighbor are closely-connected. Then, $\gamma_{cc}(G) = 1$ iff there exists a vertex $v \in V(G)$ such that $N(v) \subseteq N_{cc}(v)$.*

Proof. Assume that $\gamma_{cc}(G) = 1$. Then, there exists a vertex $v \in V(G)$ such that $\Gamma_{cc}(u, v) \geq 1$ $\forall u \in V(G) \setminus \{v\}$. Let $w \in N(v)$. Then, $\Gamma_{cc}(v, w) \geq 1$ so that $N(v) \subseteq N_{cc}(v)$. The proof of the converse part is analogous to that of theorem 3.2.7. □

Corollary 3.2.9. *Let G be either disconnected or having diameter at least 3 such that the vertex pairs sharing a common neighbor in $\bar{G}$ are closely-connected. Then, $\gamma_{cc}(\bar{G}) = 1$ iff there exists a vertex $v \in V(\bar{G})$ such that $N(v) \subseteq N_{cc}(v)$ in $\bar{G}$.*

Proof. This follows from the fact that the diameter of the complement of G is at most 2. ☐

Theorem 3.2.10. *Let $G = (V, E)$ be a graph having no cc-isolated vertices. Then, the complement of every minimal cc-dominating set in G is again a cc-dominating set.*

Proof. Let C be a minimal cc-dominating set in G. If possible, assume that $V \setminus C$ is not a cc-dominating set in G. Then, there exists $u \in C$ such that $\Gamma_{cc}(u, v) = 0 \ \forall v \in V \setminus C$. But, since G has no cc-isolated vertices, there exists some vertex $w \in C \setminus \{u\}$ such that u and w are closely-connected. Thus, $C \setminus \{u\}$ is a cc-dominating set in G, a contradiction to the minimality property of C. This completes the proof. ☐

Corollary 3.2.11. *Let $G = (V, E)$ be a graph of order n such that $\delta_{cc}(G) \geq 1$. Then, $\gamma_{cc}(G) \leq \frac{n}{2}$.*

Proof. Let C be a minimal cc-dominating set of G. Since $\delta_{cc}(G) \geq 1$, G has no cc-isolated vertices so that it follows from theorem 3.2.10 that $V \setminus C$ is a cc-dominating set in G. Therefore, $\gamma_{cc}(G) \leq \min\{|C|, |V \setminus C|\} \leq \frac{n}{2}$. ☐

Theorem 3.2.12. *Let $G = (V, E)$ be a graph. A cc-dominating set C of G is minimal iff for every vertex $u \in C$ one of the following conditions hold:*

1. $N_{cc}(u) \subseteq V \setminus C$.

2. $\exists\, v \in V \setminus C$ *such that* $N_{cc}(v) \cap C = \{u\}$.

Proof. Assume that C is a minimal cc-dominating set in G. Then, for every vertex $u \in C, C \setminus \{u\}$ is not a cc-dominating set in G. That is, there exists $v \in (V \setminus C) \cup \{u\}$ such that $\Gamma_{cc}(v, w) = 0$ is $\forall w \in C \setminus \{u\}$.

Case(i) If $v = u$, then $N_{cc}(u) \subseteq V \setminus C$.

Case(ii) Let $v \in V \setminus C$. Since $\Gamma_{cc}(v, w) = 0$ is $\forall\ w \in C \setminus \{u\}$ and since C is a cc-dominating set in G, it follows that $\Gamma_{cc}(v, u) \geq 1$. Hence it can be concluded that the only vertex in C to which v is closely-connected is u.

Conversely, assume that C is not a minimal cc-dominating set in G. Thus, there exists $u \in C$ such that $C \setminus \{u\}$ is a cc-dominating set of G. That is, $\Gamma_{cc}(u, w) \geq 1$ for some $w \in C \setminus \{u\}$ so that $N_{cc}(u) \cap C \neq \phi$. Also, since every vertex of $V \setminus C$ is closely-connected to some vertex of $C \setminus \{u\}$, there does not exists a vertex $v \in V \setminus C$ satisfying $N_{cc}(v) \cap C = \{u\}$, a contradiction. Hence, C is a minimal cc-dominating set in G.

This completes the proof. □

Theorem 3.2.13. *A graph* $G = (V, E)$ *has a unique minimal cc-dominating set iff the set of all cc-isolated vertices constitutes a cc-dominating set of* G.

Proof. Let C be the unique minimal cc-dominating set in G and let

$$S = \{v \in V : \ v \text{ is cc-isolated}\}.$$

From theorem 3.2.3, it folows that $S \subseteq C$.

Case(i) If $S = C$, then there is nothing to prove.

Case(ii) If $S \neq C$, then $\exists\ u \in C$ such that u is not cc-isolated. That is, $\Gamma_{cc}(u, v) \geq 1$ for some $v \in V$ so that $V \setminus \{u\}$ is a cc-dominating set in G. This proves the existence of a minimal cc-dominating set C' different from C and contained in $V \setminus \{u\}$, a contradiction to the uniqueness of C.

Conversely, assume that the set of S constitutes a cc-dominating set in G. Since the elements of S cannot cc-dominate other vertices of G, it follows that $S = V$. This means that all vertices in G are cc-isolated and hence the only cc-dominating set in G is V.

This completes the proof. □

Corollary 3.2.14. *A graph G has a unique minimal cc-dominating set iff G is acyclic.*

Corollary 3.2.15. *A connected graph G has a unique minimal cc-dominating set iff G is a tree.*

Theorem 3.2.16. *Let G be a cut edge free connected graph of diameter at most 2 and maximum degree $\Delta(G)$. Then, $\gamma_{cc}(G) \leq \Delta(G)$.*

Proof. Let $v \in V(G)$ be such that $deg(v) = \Delta(G)$.

Case(i) If diameter of G is 1, then $G = K_n$ for some n. Hence, the result holds trivially.

Case(ii) If diameter of G is 2, let $V_1(G) = \{u \in V \mid d(u, v) = 1\}$. Since the edges of G are cc-edges, adjacent vertices are closely-connected in G. Thus, $V_1(G)$ itself constitutes a cc-dominating set in G.

Hence, $\gamma_{cc}(G) \leq \Delta(G)$.

This completes the proof. $\square$

Theorem 3.2.17. *For a graph $G = (V, E)$ of order n, $\gamma_{cc}(G) \leq n - \Delta_{cc}(G)$. Further, the equality holds if $\gamma_{cc}(G)$ is either 1 or n.*

Proof. Let $v \in V$ be such that $deg_{cc}(v) = \Delta_{cc}(G)$. Then, $|N_{cc}(v)| = \Delta_{cc}(G)$. Therefore, $V \setminus N_{cc}(v)$ is a cc-dominating set of G so that

$$\gamma_{cc}(G) \leq |V \setminus N_{cc}(v)| = n - \Delta_{cc}(G).$$

If $\gamma_{cc}(G) = 1$, then $\Delta_{cc}(G) = n - 1$. On the other hand, if $\gamma_{cc}(G) = n$, then $\Delta_{cc}(G) = 1$, so that the equality holds in both the cases.

This completes the proof. $\square$

Theorem 3.2.18. *Let G be a connected graph free of cut edges. Then, G has a cc-dominating set whose complement is again a cc-dominating set.*

Proof. Since G is connected, it has a spanning tree, say T. Let $u \in V(G)$ and C be the set of all vertices in T which are at odd unit distance from u. Then, both C and $\bar{C}$ are dominating sets in G. The result follows from the fact that dominating sets in cut edge free graphs are also its cc-dominating sets.

This completes the proof. $\square$

Theorem 3.2.19. *Let $G = (V, E)$ be a disconnected graph with components $G_1, \ldots, G_m$ such that $|V(G_i)| \geq 3$ $\forall i = 1, \ldots, m$. Then, $\gamma_{cc}(\bar{G}) = 1$.*

Proof. Let u be a fixed vertex and v be an arbitrary vertex of the connected graph $\bar{G}$.

Case(i) v is adjacent to u in $\bar{G}$.

Since $|V(G_i)| \geq 3\ \forall i = 1, \ldots, m$, $\bar{G}$ has no cut edges. Hence, $\Gamma_{cc}(u, v) \geq 1$ in $\bar{G}$.

Case(ii) v is not adjacent to u in $\bar{G}$.

Since u and v are adjacent in G, they belong to the same component of G, say G_k. Let $w \in G_j$ for some $j \neq k$. Since $|V(G_i)| \geq 3\ \forall i = 1, \ldots, m$, $\omega(\bar{G}') = \omega(\bar{G})$, where $\bar{G}' = (V, \bar{E} \setminus \{uw, vw\})$. Thus, $\Gamma_{cc}(u, v) \geq 1$.

Hence, it follows that $\{u\}$ is a cc-dominating set in $\bar{G}$ so that $\gamma_{cc}(\bar{G}) = 1$.

This completes the proof. $\square$

Theorem 3.2.20. *Let G be a graph of order n such that $\frac{n}{1+\Delta_{cc}(G)} \leq \gamma_{cc}(G)$. Further, the equality holds if and only if for every minimum cc-dominating set C in G the following conditions are satisfied:*

(i) for any vertex $v \in C$, $deg_{cc}(v) = \Delta_{cc}(G)$.

(ii) C is cc-independent in G.

(iii) every vertex in $V \setminus C$ is closely-connected to exactly one vertex in C.

Proof. Let C be a minimum cc-dominating set in G. Clearly, every vertex in G cc-dominates at most $\Delta_{cc}(G) + 1$ vertices so that

$$n = |N_{cc}[C]| \leq \gamma_{cc}(G)(\Delta_{cc}(G) + 1).$$

Hence, $\frac{n}{1+\Delta_{cc}(G)} \leq \gamma_{cc}(G)$.

Now, if the given conditions are satisfied, $n = \gamma_{cc}(G)\Delta_{cc}(G) + \gamma_{cc}(G)$ so that $\frac{n}{1+\Delta_{cc}(G)} = \gamma_{cc}(G)$. Conversely, if one of the above conditions is not satisfied, then $n < \gamma_{cc}(G)\Delta_{cc}(G) + \gamma_{cc}(G)$, so that the equality fails.

This completes the proof. $\square$

Example 3.2.21. *Consider the cycle graph* C_{3k} $\forall\ k \geq 1$. *Then,*

$$deg_{cc}(v) = 2 = \Delta_{cc}(G), \forall v \in V(C_{3k}).$$

Also, $\gamma_{cc}(C_{3k}) = \gamma(C_{3k}) = k$ *by theorem 3.2.2. Therefore,*

$$\frac{n}{1+\Delta_{cc}(G)} = \frac{3k}{3} = \gamma_{cc}(C_{3k}).$$

Theorem 3.2.22. *Let* $G = (V, E)$ *be a graph. Then, a cc-independent set in* G *is maximal iff it is cc-dominating.*

Proof. Let S be a maximal cc-independent set. If possible, S is not a cc-dominating set. Then, there exists a vertex $v \in V$ such that $\Gamma_{cc}(u, v) = 0\ \forall u \in S$, a contradiction to the fact that S is maximal cc-independent.

Conversely, assume that S is cc-independent and cc-dominating. If S is not maximal cc-independent, then there exists a vertex $w \in V \setminus S$ such that $S \cup \{w\}$ is cc-independent. That is, $\Gamma_{cc}(u, w) = 0\ \forall u \in S$, a contradiction since S is a cc-dominating set.

This completes the proof. □

Lemma 3.2.23. *Every maximal cc-independent set in a graph* G *is its minimal cc-dominating set.*

Proof. Let C be a maximal cc-independent set and $v \in V \setminus C$. If $v \notin N_{cc}(u)$ for every $u \in C$, then $C \cup \{v\}$ is cc-independent, a contradiction to the maximality of C. Therefore, $v \in N_{cc}(u)$ for some $u \in C$ so that C is a cc-dominating set in G. Now, if possible assume that C is not a minimal cc-domination of G. Then, there exists $w \in C$ such that $C \setminus \{w\}$ is a cc-dominating set. That is, there exists a vertex closely-connected to w in C, a contradiction since C is cc-independent. Hence, C is a minimal cc-dominating set in G. □

Theorem 3.2.24. *For any graph G, $\gamma_{cc}(G) \leq \beta_{cc}(G)$.*

Proof. The proof of the theorem is a direct consequence of the lemma 3.2.23. □

Theorem 3.2.25. *Let $G = (V, E)$ be a graph of order n. Then,*

$$\alpha_{cc}(G) + \beta_{cc}(G) = n.$$

Proof. Assume that $S \subseteq V$ be a cc-vertex covering of G of cardinality $\alpha_{cc}(G)$ and $e = uv$ be a cc-edge of G. Since either u or v belongs to S, it follows that $V \setminus S$ is cc-independent in G. Thus,

$$\beta_{cc}(G) \geq |V \setminus S| = n - \alpha_{cc}(G).$$

This completes the proof. □

Theorem 3.2.26. *Let G be a graph of order n having $0 \leq k < n$ cc-isolated vertices and $G^{'}$ be the graph obtained by the deletion of the cc-isolated vertices of G. Then, $\gamma_{cc}(G^{'}) = \gamma_{cc}(G) - k$.*

Proof. Let C be a cc-dominating set of cardinality $\gamma_{cc}(G)$ in G. Then, clearly all the cc-isolated vertices of G belongs to C. Since the deletion of the cc-isolated vertices does not alter the close-connectivity between other vertex pairs in G, but reduces the cc-domination number by k, we get $\gamma_{cc}(G^{'}) = \gamma_{cc}(G) - k$. This completes the proof. □

Corollary 3.2.27. *For the helm H_n, $\gamma_{cc}(H_n) = n$, where $n \geq 4$.*

Corollary 3.2.28. *For the web graph* WB_n*,* $\gamma_{cc}(WB_n) = n$*, where* $n \geq 4$*.*

Proof. Since the deletion of $n-1$ cc-isolated vertices of WB_n results in a closely-connected graph, it follows from theorem 3.2.26 that $\gamma_{cc}(WB_n) = n - 1 + 1$. □

3.3 CC-Domatic Number of Graphs

Analogous to the concept of domatic number in graphs, we introduce the idea of *cc-domatic number* $d_{cc}(G)$ of a graph G and establish some of its properties. Moreover, we obtain some bounds for $d_{cc}(G)$ as well.

Definition 3.3.1. Let G be a graph. The *cc-domatic partition* of G is a partition $\{V_1, \ldots, V_k\}$ of $V(G)$ in which each V_i is a cc-dominating set of G for $i = 1, \ldots, k$. The maximum order of a cc-domatic partition of G is called the cc-domatic number of G and is denoted by $d_{cc}(G)$.

For every graph G, there exists at least one cc-domatic partition of $V(G)$, namely $\{V(G)\}$, so that $d_{cc}(G)$ is well-defined.

Theorem 3.3.2. *Let* G *be a graph. Then,*

$$d_{cc}(G) = \begin{cases} 1; & \textit{if } G \textit{ is acyclic,} \\ n; & \textit{iff } G \textit{ is closely-connected.} \end{cases}$$

Proof. Here we have two cases.

Case(i): If G is acyclic, then every vertex of G is cc-isolated so that $V(G)$ is the only cc-dominating set. Therefore, $d_{cc}(G) = 1$.

Case(ii): If G is closely-connected, then every vertex cc-dominates all other vertices so that the maximum order of a partition of $V(G)$ into cc-dominating sets is n. Hence, $d_{cc}(K_n) = n$.

Conversely, $d_{cc}(G) = n$ implies that every vertex of G is a cc-dominating set. Hence, it follows that G is closely-connected.

This completes the proof. □

Example 3.3.3. *The converse of the above theorem is not true.*

That is, $d_{cc}(G) = 1$ does not always imply that G is acyclic. For example, consider the graph G_4 shown in the figure 3.2. Since the vertex v_5 is cc-isolated, every cc-dominating set of G_4 contains v_5. Hence, it follows that $\{v_1, v_2, v_3, v_4, v_5\}$ is the only cc-domatic partition of G_4. Thus, $d_{cc}(G_4) = 1$, but G_4 is not acyclic.

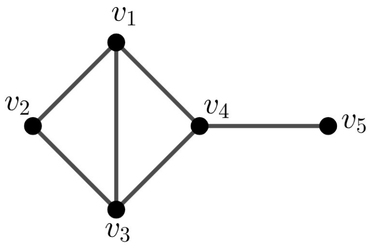

Figure 3.2: The non-acyclic graph G_4

Corollary 3.3.4. *Let G be a graph. If $d_{cc}(G) = |V(G)|$, then $\gamma_{cc}(G) = 1$.*

Proof. The result follows immediately from the fact that G is closely-connected.

□

Remark 3.3.5. The converse of the corollary 3.3.4 is not true. For example, consider the complete bipartite graph $K_{2,n}$, where $n > 2$.

Then, $\gamma_{cc}(K_{2,n}) = 1$ by theorem 3.2.2 . But, the vertices of degree n in $K_{2,n}$ are not closely-connected so that $d_{cc}(G) = n + 1 < |V(K_{2,n})|$.

Corollary 3.3.6. *For the complete graph K_n,*

$$d_{cc}(K_n) = \begin{cases} 1; & \textit{if } n = 2, \\ n; & \textit{otherwise.} \end{cases}$$

Proof. The result easily follows from the theorem 3.3.2 and the fact that K_2 is a tree and K_n is closely-connected for $n \neq 2$. □

Theorem 3.3.7. *For a graph G, $d_{cc}(G) = 1$ iff G has at least one cc-isolated vertex.*

Proof. It can be noted from theorem 3.2.3 that if G has a cc-isolated vertex v, then every cc-dominating set of G contains v. Thus, $d_{cc}(G) = 1$. Conversely, assume that $d_{cc}(G) = 1$ and G has no cc-isolated vertex. Since $\delta_{cc}(G) \geq 1$, it follows from corollary 3.2.11 that $\gamma_{cc}(G) \leq \frac{n}{2}$. Now, for a minimal cc-dominating set C in G, $V \setminus C$ is also a cc-dominating set. Thus, $d_{cc}(G) \geq 2$, a contradiction. Hence, G must have at least one cc-isolated vertex.
This completes the proof. □

Theorem 3.3.8. *We have the following :*

(i) *For any $n \geq 1, d_{cc}(C_{3n}) = 3$ and $d_{cc}(C_{3n+1}) = d_{cc}(C_{3n+2}) = 2$.*

(ii) *For $2 \leq m \leq n$ and $n \geq 3$,*

$$d_{cc}(K_{m,n}) = \begin{cases} n+1, & \textit{if } m = 2 \\ n+m, & \textit{if } m > 2. \end{cases}$$

(iii) *For any graph G, if $N_{cc}(v) = N(v)$ for all $v \in V(G)$, then $d_{cc}(G) = d(G)$.*

Proof. (i) This can be easily deduced from the fact that

$$\gamma_{cc}(C_n) = \begin{cases} \lfloor \frac{n}{3} \rfloor + 1; & \text{if } n \text{ is not a multiple of 3} \\ \lfloor \frac{n}{3} \rfloor; & \text{if } n \text{ is a multiple of 3.} \end{cases}$$

(ii) Let $K_{2,n}$ be the graph with partite sets M and N of cardinalities 2 and n respectively. Then, the vertex pairs of $K_{2,n}$ except the one belonging to M are closely-connected. Thus, $d_{cc}(K_{2,n}) = n + 1$. Now, since it follows from theorem 3.2.2 that $K_{m,n}$ is closely-connected for $m > 2$ and $n \geq 3$, it immediately follows that $d_{cc}(K_{m,n}) = n + m$.

(iii) This follows from the observation that dominating sets and cc-dominating sets of G are exactly the same.

This completes the proof. □

Theorem 3.3.9. *Let G be a graph without cut edges. Then, $d(G) \leq d_{cc}(G)$.*

Proof. This follows from the fact that every dominating set of a cut edge free graph is also a cc-dominating set. □

Theorem 3.3.10. *For any graph G of order n, $d_{cc}(G) \leq \frac{n}{\gamma_{cc}(G)}$. Further, the equality holds if the number of cc-dominating sets of minimum cardinality is $\binom{n}{\gamma_{cc}(G)}$.*

Proof. Let G be a graph of order n with cc-domatic partition $\{C_1, \ldots, C_k\}$. Since $d_{cc}(G) = k$ and $|C_i| \geq \gamma_{cc}(G)$ $\forall i = 1, \ldots, k$,

$$n = \sum_{i=1}^{k} |C_i| \geq k\gamma_{cc}(G).$$

Hence, $d_{cc}(G) \leq \frac{n}{\gamma_{cc}(G)}$.

Now, if the number of cc-dominating sets of cardinality $\gamma_{cc}(G)$ is $\binom{n}{\gamma_{cc}(G)}$, then $|C_i| = \gamma_{cc}(G)\ \forall i = 1, \ldots, k$ so that $d_{cc}(G) = \frac{n}{\gamma_{cc}(G)}$.

This completes the proof. $\square$

Theorem 3.3.11. *Let G be a graph of order n. Then, $d_{cc}(G) \leq \delta_{cc}(G) + 1$. Further, the equality holds if G is the complete graph K_n for $n \neq 2$.*

Proof. Assume that $d_{cc}(G) > \delta_{cc}(G) + 1$ and $C = \{C_1, \ldots, C_k\}$ be a cc-domatic partition of G. Clearly, $k \geq \delta_{cc}(G) + 2$. Let $v \in V(G)$ be such that $deg_{cc}(v) = \delta_{cc}(G)$.

Claim v is not cc-dominated by C_i for some $i = 1, \ldots, k$.

> If possible, $N_{cc}[v] \cap C_i \neq \phi\ \forall i = 1, \ldots, k$. Since $C_i^{'}s$ are mutually disjoint and $|N_{cc}[v]| = \delta_{cc}(G)+1$, the cardinality of the cc-domatic partition C of G must not exceed $\delta_{cc}(G) + 1$, a contradiction. Hence, there exists a member of C, say C_j, such that $N_{cc}[v] \cap C_j = \phi$.

Since C_j cannot cc-dominate v, it cannot be a cc-dominating set in G, a contradiction to our assumption that C is a cc-domatic partition of G. Therefore, $d_{cc}(G) \leq \delta_{cc}(G) + 1$. Now, if $G = K_n$, then $\delta_{cc}(G) = n - 1$ and hence it follows from corollary 3.3.6 that $d_{cc}(G) = \delta_{cc}(G) + 1$.

This completes the proof. $\square$

Theorem 3.3.12. *For any graph G of order n, $d_{cc}(G) \geq \left\lfloor \frac{n}{n-\delta_{cc}(G)} \right\rfloor$.*

Proof. Let C be any subset of $V(G)$ with $|C| \geq n - \delta_{cc}(G)$.

Claim C is a cc-dominating set of G.

If possible, let $v \in \bar{C}$ be such that $N_{cc}(v) \cap C = \phi$. Clearly, $|N_{cc}(v)| \geq \delta_{cc}(G)$. But, since $|C| \geq n - \delta_{cc}(G)$, we get $|N_{cc}(v)| < \delta_{cc}(G)$, a contradiction. Hence, $N_{cc}(v) \cap C \neq \phi$

Now, if we take any $\left\lfloor \frac{n}{n-\delta_{cc}(G)} \right\rfloor$ disjoint subsets each of cardinality $n - \delta_{cc}(G)$, it will be a cc-domatic partition of G. Therefore,

$$d_{cc}(G) \geq \left\lfloor \frac{n}{n - \delta_{cc}(G)} \right\rfloor.$$

This completes the proof. □

Theorem 3.3.13. *Let G be a graph of order n. Then, $d_{cc}(G) + d(\bar{G_{cc}}) \leq n + 1$.*

Proof. It follows from theorem 3.3.11 that $d_{cc}(G) \leq \delta_{cc}(G) + 1$. Also, $d(\bar{G_{cc}}) \leq \delta(\bar{G_{cc}}) + 1$. Thus,

$$d_{cc}(G) + d(\bar{G_{cc}}) \leq \delta_{cc}(G) + \delta(\bar{G_{cc}}) + 2.$$

But, $\delta(\bar{G_{cc}}) = n - 1 - \Delta_{cc}(G)$ so that,

$$d_{cc}(G) + d(\bar{G_{cc}}) \leq \delta_{cc}(G) - \Delta_{cc}(G) + n + 1 \leq n + 1.$$

This completes the proof. □

Theorem 3.3.14. *Let G be a graph of order n. Then,*

$$2 \leq \gamma_{cc}(G) + d_{cc}(G) \leq n + 1.$$

Further, the equality holds if every vertex of G is cc-isolated.

Proof. Since $\gamma_{cc}(G) \geq 1$ and $d_{cc}(G) \geq 1$, the lower bound follows. Now, from theorem 3.2.17 and theorem 3.3.11, it follows that

$$\gamma_{cc}(G) \leq n - \Delta_{cc}(G) \leq n - \delta_{cc}(G) \text{ and } d_{cc}(G) \leq \delta_{cc}(G) + 1.$$

Thus,

$$\gamma_{cc}(G) + d_{cc}(G) \leq n - \delta_{cc}(G) + \delta_{cc}(G) + 1 = n + 1.$$

Now, if every vertex of G is cc-isolated, then $\gamma_{cc}(G) = n$ and $d_{cc}(G) = 1$ by corollary 3.2.4 and theorem 3.3.7 respectively. Hence, $\gamma_{cc}(G) + d_{cc}(G) = n + 1$. This completes the proof. □

Chapter 4

CC-DOMINATION POLYNOMIAL OF GRAPHS

This chapter is divided into four sections. The first section gives a brief introduction explaining the necessity to formulate a new graph polynomial. In the second section, the number of cc-dominating sets of graphs are studied and cc-domination polynomial $D_c[G;x]$ *of a graph* G *is introduced. Also, some general results are established. In the consecutive section,* $D_c[G;x]$ *is evaluated for some special graphs. The final section explores the roots of the graph polynomial discussed in this chapter.*

4.1 Introduction

Graph polynomials are a powerful tool for studying the structure and dynamics of complex networks. They can be used to detect hidden structures in networks and to identify hubs and bottlenecks. Graph polynomials can also be used to create models that can predict the response of networks to changing conditions. The concept of closely-connected vertices has made it possible to introduce the idea of the cc-domination polynomial as a new graph invariant that encodes information about the graph and enables algebraic methods for extracting this information. In this chapter, emphasizing the role of closely-connected vertices in fault tolerance mechanisms, we introduce the cc$^{(i)}$-dominating set and the cc-domination polynomial of graphs.

4.2 CC-Domination Polynomial of Graphs

Definition 4.2.1. Let $G = (V, E)$ be a graph of order n. For every integer $1 \leq i \leq n$, define the *cc$^{(i)}$-dominating set* of G as;

$$D_c(G, i) = \left\{ (u_1, \ldots, u_i) \in \underbrace{V \times \ldots \times V}_{i \text{ times}} : \forall v \in V \setminus \{u_j\}_{j=1}^{i},\ \exists\ u_j \text{ with } N_{cc}(u_j) = v \right\}.$$

Definition 4.2.2. Let G be a graph of order n. Then, the *cc-domination polynomial* $D_c[G; x]$ of G is defined as:

$$D_c[G; x] = \sum_{i=\gamma_{cc}(G)}^{n} d_c(G, i) x^i,$$

where $d_c(G, i) = |D_c(G, i)|$.

The following properties of $D_c[G; x]$ can be easily observed:

(i) $D_c[G;x]$ is a n^{th} degree monic polynomial having no constant term.

(ii) $D_c[G;1] \leq 2^n - 1$.

(iii) $D_c[G;x]$ is a strictly increasing function on $[0, \infty)$.

(iv) $D_c[G;x]$ is non-linear if and only if G is non-trivial.

(v) $deg(D_c[H;x]) \leq deg(D_c[G;x])$, where H is an induced subgraph of G.

(vi) $D_c^{(m)}[G;0] = m!\ d_c(G,m)$, where $\gamma_{cc}(G) \leq m \leq n$.

(vii) If G is free of cut edges, then $d(G,i) \leq d_c(G,i)$ and $d_c(G,n-1) = n$, where $d(G,i)$ is the number of dominating sets of cardinality i in G.

(viii) If G is connected, then $D_c[G;x] = x^n$ iff G is a tree.

(ix) Isomorphic graphs have the same cc-domination polynomial.

Example 4.2.3. *Consider the graph G_5 described in figure 4.1. It can be observed with ease that a single vertex cannot form a cc-dominating set for the entire graph. Therefore, $D_c(G_5,1) = \emptyset$.*

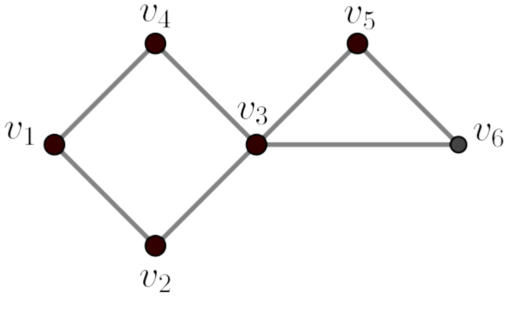

Figure 4.1: The graph G_5

Now, all the vertex pairs except the vertex pairs belonging to the cycle $v_3v_5v_6$ forms a cc-dominating set for G_5. Thus,

$$\begin{aligned}D_c(G_5, 2) &= V(G_5) \times V(G_5) \setminus \{(v_3, v_5), (v_3, v_6), (v_5, v_6)\}\\ &= \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5), (v_1, v_6), (v_2, v_3), (v_2, v_4), (v_2, v_5)\\ &\quad (v_2, v_6), (v_3, v_4), (v_4, v_5), (v_4, v_6)\}\end{aligned}$$

Similar as above, all the vertex triplets except the unique triplet belonging to the cycle $v_3v_5v_6$ forms a cc-dominating set for G_5. Hence,

$$\begin{aligned}D_c(G_5, 3) &= V(G_5) \times V(G_5) \times V(G_5) \setminus \{(v_3, v_4, v_5)\}\\ &= \{(v_1, v_2, v_3), (v_1, v_2, v_4), (v_1, v_2, v_5), (v_1, v_2, v_6), (v_1, v_3, v_4),\\ &\quad (v_1, v_3, v_5), (v_1, v_3, v_6), (v_1, v_4, v_5), (v_1, v_4, v_6), (v_1, v_5, v_6),\\ &\quad (v_2, v_3, v_4), (v_2, v_3, v_5), (v_2, v_3, v_6), (v_2, v_4, v_5), (v_2, v_4, v_6),\\ &\quad (v_2, v_5, v_6), (v_3, v_4, v_5), (v_3, v_4, v_6), (v_3, v_5, v_6)\}.\end{aligned}$$

Now, for $i > 3$, every element of $D_c(G_5, i)$ contains vertices from both the cycles $v_1v_2v_3v_4$ and $v_3v_5v_6$. Therefore, it follows that,
$D_c(G_5, i) = \underbrace{V(G_5) \times \ldots \times V(G_5)}_{i \text{ times}}$.

Hence,

$$D_c[G_5; x] = x^6 + 6x^5 + 15x^4 + 19x^3 + 12x^2.$$

Theorem 4.2.4. *Let G be a graph of order n with $d_c(G, j) = \binom{n}{j}$ for some $j \leq n$. Then,*

$$D_c[G; x] = (x + 1)^n + \sum_{i=0}^{j-1} \left[d_c(G, i) - \binom{n}{i}\right] x^i.$$

Proof. Since every subset of $V(G)$ of cardinality j is a cc-dominating set of G, it follows that $\gamma_{cc}(G) \leq j$.

Claim: $d_c(G,i) = \binom{n}{i}\ \forall i \geq j$.

Proof of the claim : Let H be a subgraph of G with $|V(H)| = i$, where $j < i \leq n$. Then, there exists a subgraph $H^{'}$ of H with $|V(H^{'})| = j$. Hence, by our assumption it follows that $V(H^{'})$ is a cc-dominating set of G. Since $V(H^{'}) \subset V(H)$, we get $V(H)$ is a cc-dominating set of G. Thus, the vertex set of every subgraph of G of order exceeding j constitutes a cc-dominating set of G. Therefore, $d_c(G,i) = \binom{n}{i}\ \forall i \geq j$.

Thus,

$$\begin{aligned} D_c[G;x] &= \sum_{i=\gamma_{cc}(G)}^{j-1} d_c(G,i)x^i + \sum_{i=j}^{n} d_c(G,i)x^i \\ &= \sum_{i=\gamma_{cc}(G)}^{j-1} d_c(G,i)x^i + \sum_{i=j}^{n} \binom{n}{n-i} x^i \\ &= (x+1)^n - \sum_{i=0}^{j-1} \binom{n}{i} x^i + \sum_{i=\gamma_{cc}(G)}^{j-1} d_c(G,i)x^i. \end{aligned}$$

Since $d_c(G,i) = 0$ for $i = 0, 1, \ldots, \gamma_{cc}(G) - 1$, we get

$$\begin{aligned} D_c[G;x] &= (x+1)^n - \sum_{i=0}^{j-1} \binom{n}{i} x^i + \sum_{i=0}^{j-1} d_c(G,i)x^i \\ &= (x+1)^n + \sum_{i=0}^{j-1} \left[d_c(G,i) - \binom{n}{i} \right] x^i. \end{aligned}$$

This completes the proof. □

Example 4.2.5. *In the example 4.1,* $d_c(G_5, j) = \binom{n}{j}$ *only if* $j > 3$. *Therefore, by theorem 4.2.4,*

$$\begin{aligned} D_c[G_5;x] &= (x+1)^6 + \sum_{i=0}^{3} \left[d_c(G,i) - \binom{6}{i} \right] x^i. \\ &= (x+1)^6 + (0-1)x^0 + (0-6)x^1 + (12-15)x^2 + (19-20)x^3 \\ &= x^6 + 6x^5 + 15x^4 + 19x^3 + 12x^2. \end{aligned}$$

Corollary 4.2.6. *Let G be a graph and let $j = inf\,\{i \in \mathbb{N} : \ d_c(G, i) = \binom{n}{i}\}$, where $|V(G)| = n$. Then,*

$$\left|\frac{D_c[G;x]}{f(x)}\right| < 1, \quad \textit{where } f(x) = (x+1)^n - \sum_{i=0}^{\gamma_{cc}(G)-1}\binom{n}{i}x^i.$$

Proof. Since $d_c(G, j) = \binom{n}{j}$, we get $d_c(G, i) < \binom{n}{i}$ $\forall i < j$. Therefore, it follows from theorem 4.2.4 that

$$\begin{aligned} D_c[G;x] &< (x+1)^n - \sum_{i=0}^{j-1}\binom{n}{i}x^i + \sum_{i=\gamma_{cc}(G)}^{j-1}\binom{n}{i}x^i. \\ &= (x+1)^n - \sum_{i=0}^{\gamma_{cc}(G)-1}\binom{n}{i}x^i - \sum_{i=\gamma_{cc}(G)}^{j-1}\binom{n}{i}x^i + \sum_{i=\gamma_{cc}(G)}^{j-1}\binom{n}{i}x^i. \\ &= f(x) \end{aligned}$$

□

Corollary 4.2.7. *Let G be a graph of order n with $d_c(G, \gamma_{cc}(G)) = \binom{n}{\gamma_{cc}(G)}$. Then, $D_c[G;x] = (x+1)^n - \sum_{i=0}^{\gamma_{cc}(G)-1}\binom{n}{n-i}x^i$.*

Proof. The result is an immediate consequence of the substitution $j = \gamma_{cc}(G)$ in theorem 4.2.4. □

Corollary 4.2.8. *A graph G on n vertices is closely-connected iff $D_c[G;x] = (x+1)^n - 1$. In this case, the bound of the number of cc-dominating sets of G is sharp and the corresponding alternating number is -1.*

Proof. Assume that G is closely-connected. Then, it follows from theorem 4.2.4 that $D_c[G;x] = (x+1)^n - 1$. Conversely, if $D_c[G;x] = (x+1)^n - 1$, then $d_c(G, i) = \binom{n}{i}$ $\forall\ 1 \le i \le n$ so that every pair of vertices is closely-connected.

Moreover, for any graph G, the number of cc-dominating sets is given by

$$\begin{aligned} D_c[G;1] &= \sum_{i=\gamma_{cc}(G)}^{n} d_c(G,i) \\ &\leq \sum_{i=1}^{n} \binom{n}{i} \\ &= 2^n - 1 \end{aligned}$$

If G is closely-connected, then $D_c[G;x] = (x+1)^n - 1$ so that $D_c[G;1] = 2^n - 1$. The alternating number of cc-dominating sets of G is the difference of cc-dominating sets of even cardinality and odd cardinality and is given by $D_c[G;-1] = -1$. □

Corollary 4.2.9. *We have the following:*

(i) $D_c[K_n;x] = (x+1)^n - 1$, *where* $n \geq 3$.

(ii) $D_c[F_n;x] = (x+1)^n - 1$, *where* $n \geq 1$.

(iii) $D_c[W_n;x] = (x+1)^n - 1$, *where* $n > 3$.

(iv) $D_c[S_n;x] = (x+1)^n - 1$, *where* $n > 3$.

(v) $D_c[W_n^{(m)};x] = \begin{cases} x^{m+1} & , \text{ if } n = 2 \\ (x+1)^{m(n-1)+1} - 1, & \text{if } n > 2 \end{cases}$,
where $m, n > 1$.

Remark 4.2.10. For a closely-connected graph G of order n, it follows from theorem 4.2.4 that $D_c[G;x] = (x+1)^n - 1$. But, $\gamma_{cc}(G) = 1$ is not at all a sufficient condition to imply that $D_c[G;x] = (x+1)^n - 1$. For example, consider the complete bipartite graph $K_{2,n-2}$, where $n \geq 5$. Then, $\gamma_{cc}(K_{2,n-2}) = 1$, but the vertices of degree $n-2$ in $K_{2,n-2}$ are not closely-connected so that

$$d_c(K_{2,n-2}, i) = \begin{cases} \binom{n}{i} - 2, & \text{if } i = 1, \\ \binom{n}{i}, & \text{if } i \neq 1. \end{cases}$$

Therefore, $D_c[K_{2,n-2}; x] = (x+1)^n - 2x - 1$.

Theorem 4.2.11. *Let $G_1, \ldots, G_m$ be the components of a graph G. Then,*

$$D_c[G; x] = \prod_{i=1}^{m} D_c[G_i; x].$$

Proof. Without loss of generality, let $m = 2$. For $k \geq \gamma_{cc}(G)$, every cc-dominating set of cardinality k in G is obtained from a cc-dominating set of i vertices in G_1 and a cc-dominating set of $k - i$ vertices in G_2 for $i \in \{\gamma_{cc}(G_1), \ldots, |V(G_1)|\}$. Since the number of ways of doing this over all $i = \gamma_{cc}(G_1), \ldots, |V(G_1)|$ is exactly the coefficient of x^k in $D_c[G_1; x]D_c[G_2; x]$, both the sides of the above equation are identical.

This completes the proof. ☐

Theorem 4.2.12. *Let G be a cut edge free graph such that $deg(v) \geq deg_{cc}(v)$ $\forall v \in V(G)$. Then, $D_c[G; x] = D[G; x]$.*

Proof. Since G is free of cut edges, its adjacent vertices are closely-connected so that $deg_{cc}(v) \geq deg(v)$ $\forall v \in V(G)$. That is, $\forall v \in V(G)$, $deg_{cc}(v) = deg(v)$. Thus, the dominating sets and cc-dominating sets of G are the same and hence $D_c[G; x] = D[G; x]$.

This completes the proof. ☐

Corollary 4.2.13. *For the cycle graph C_n,*

$$D_c[C_n;x] = \begin{cases} \sum\limits_{i=0}^{2} \binom{n}{i} x^{n-i}, & \text{if } n = 3,4, \\ x^5 + 5x^4 + 10x^3 + 5x^2, & \text{if } n = 5, \\ x\Big[D_c[C_{n-1};x] + D_c[C_{n-2};x] + D_c[C_{n-3};x]\Big], & \text{if } n \geq 6. \end{cases}$$

Proof. Since $deg(v) = deg_{cc}(v)\ \forall v \in V(C_n)$ with $n \geq 3$, it immediately follows from theorem 4.2.12 that $D_c[C_n;x] = D[C_n;x]$.

Now from [2], it follows that $D[C_n;x] = x\Big[D[C_{n-1};x] + D[C_{n-2};x] + D[C_{n-3};x]\Big]$ with initial values $D[C_1;x] = x, D[C_2;x] = x^2 + 2x$ and $D[C_3;x] = x^3 + 3x^2 + 3x$. From this, the cc-domination polynomial of C_n can be easily deduced. □

Proposition 4.2.14. *Let G be a graph satisfying $D_c[G;x] = D[G;x]$. Then, $\gamma_{cc}(G) = \gamma(G)$.*

Proof. The proof is immediate since $d_c(G, \gamma_{cc}(G))x^{\gamma_{cc}(G)} = d(G, \gamma(G))x^{\gamma(G)}$. □

Remark 4.2.15. The converse of the proposition 4.2.14 is not true.

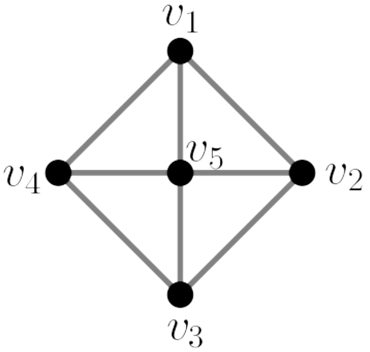

Figure 4.2: The wheel graph W_5

For example, consider the wheel graph W_5 shown in figure 4.2.

Since $deg_{cc}(v_i) = 4\ \forall i = 1, \ldots, 5$ and

$$deg(v_i) = \begin{cases} 4, & \text{if } i = 5 \\ 3, & \text{if } i \neq 5, \end{cases}$$

it follows that $\gamma_{cc}(W_5) = \gamma(W_5) = 1$.

But, $d_c(W_5, 1) = 5$ whereas $d(W_5, 1) = 1$. Hence, $D_c[W_5; x] \neq D[W_5; x]$.

Theorem 4.2.16. *Let G be a graph of order n with $d_c(G, \gamma_{cc}(G)) = \binom{n}{\gamma_{cc}(G)}$ and $G^{'}$ be the graph obtained from G by adding k pendent vertices to $V(G)$. Then,*

$$D_c[G^{'}; x] = x^k \left[(x+1)^n - \sum_{i=0}^{\gamma_{cc}(G)-1} \binom{n}{n-i} x^i \right].$$

Proof. Let $V(G^{'}) = \{v_1, \ldots, v_n, u_1, \ldots, u_k\}$, where $v_i \in V(G)$ and u_j are pendent vertices for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. Clearly, $\gamma_{cc}(G^{'}) = \gamma_{cc}(G) + k$ and $d_c(G^{'}, \gamma_{cc}(G)+k) = d_c(G, \gamma_{cc}(G)$. Also, it can be noted that $d_c(G^{'}, k+i) = d_c(G, i)$ for $i = \gamma_{cc}(G) + 1, \ldots, n$. Therefore,

$$\begin{aligned} D_c[G^{'}; x] &= \sum_{i=k+\gamma_{cc}(G)}^{k+n} d_c(G, i-k) x^i \\ &= x^k \sum_{i=\gamma_{cc}(G)}^{n} d_c(G, i) x^i \\ &= x^k D_c[G; x]. \end{aligned}$$

Now, it evidently follows from theorem 4.2.4 that

$$D_c[G^{'}; x] = x^k \left[(x+1)^n - \sum_{i=0}^{\gamma_{cc}(G)-1} \binom{n}{n-i} x^i \right].$$

This completes the proof. □

Corollary 4.2.17. *For a graph G of order n, $D_c[G \circ K_1; x] = x^n D_c[G; x]$.*

Corollary 4.2.18. *The cc-domination polynomial of* $L(S(n_1, n_2, \dots, n_k))$ *is given by*

$$D_c[L(S(n_1, n_2, \dots, n_k)); x] = x^{\sum\limits_{i=1}^{k} n_i - k}[(x+1)^k - 1]$$

Proof. The line graph of $S(n_1, n_2, \dots, n_k)$ consists of the complete graph K_k with each vertex attached to a path on $n_i - 1$ vertices $\forall i = 1, 2, \dots, k$. Since $\gamma_{cc}(K_k) = 1$ and all other $\sum\limits_{i=1}^{k}(n_i - 1)$ vertices are cc-isolated, we get

$$\begin{aligned}\gamma_{cc}(L(S(n_1, n_2, \dots, n_k))) &= 1 + \sum_{i=1}^{k}(n_i - 1)\\ &= 1 - k + \sum_{i=1}^{k} n_i.\end{aligned}$$

Therefore, from the proof of theorem 4.2.16, it follows that

$$D_c[L(S(n_1, n_2, \dots, n_k)); x] = x^{\sum\limits_{i=1}^{k} n_i - k}[(x+1)^k - 1].$$

□

Theorem 4.2.19. *Let* $G = (V, E)$ *be a graph of order* n *with* k *cc-isolated vertices. Then,*

(i) $k = n - d_c(G, n-1)$.

(ii) $d_c(G, n-2) = \binom{n}{2} - k(n-1) + \binom{k}{2}$.

(iii) $d_c(G, 1) = |\{v \in V : deg_{cc}(v) = n-1\}|$.

Proof. (i) Let $A \subseteq V$ be the set of all cc-isolated vertices in G. Then, for any vertex $v \in V \setminus A$, the set $V \setminus \{v\}$ is a cc-dominating set of G. Therefore, $d_c(G, n-1) = |V(G) \setminus A| = n - k$.

(ii) Let $D \subseteq V$ be a non cc-dominating set of cardinality $n-2$.

Then, $D = V \setminus \{u, v\}$ for $u, v \in V$.

Case(i) At least one of the two vertices u or v are cc-isolated.

Let $u \in V$ be cc-isolated and $v \in V \setminus \{u\}$. Thus, corresponding to every cc-isolated vertex u , there exists $n-1$ vertices in G such that $V \setminus \{u, v\}$ is not a cc-dominating set. Therefore, the total number of cc-dominating sets of cardinality $n-2$ in G is

$$d_c(G, n-2) = \binom{n}{2} - k(n-1) + \binom{k}{2}.$$

.

Case(ii) Both u and v are not cc-isolated.

Since D is a non cc-dominating set, this is possible only if $\Gamma_{cc}(u, v) = 1$ and $\Gamma_{cc}(v, w) \geq 1$ for some $w \in D$. But, this would eventually lead us to conclude that $\Gamma_{cc}(u, y) = 1$ for some $y \in D$, a contradiction to the fact that D is a non cc-dominating set.

(iii) For any vertex $v \in V$, $\{v\}$ is a cc-dominating set iff $\Gamma_{cc}(u, v) \geq 1$ $\forall u \in V \setminus \{v\}$.

This completes the proof. □

Corollary 4.2.20. *Let G be a graph with $D_c[G; x] = [x(x+2)]^n$. Then, G is free of cc-isolated vertices.*

Proof. Let k be the number of cc-isolated vertices in G. Then, from part (i) of theorem 4.2.19, $k = 2n - d_c(G, 2n-1) = 2n - 2n = 0$. □

4.3 CC-Domination Polynomial of Special Graphs

In this section, the cc-domination polynomial of certain special graphs are computed.

Theorem 4.3.1. *Let G be a closely-connected graph of order n and $G^{'}$ be the graph obtained by linking two copies of G through a bridge. Then,*

$$D_c[G^{'};x] = \sum_{i=2}^{2n} d_c(G^{'},i)x^i,$$

where

$$d_c(G^{'},i) = \begin{cases} \binom{n}{\frac{i}{2}}^2 + 2\sum\limits_{k=1}^{\frac{i}{2}-1}\binom{n}{k}\times\binom{n}{i-k}, & \textit{if } i \le n+1 \textit{ is even} \\ 2\sum\limits_{k=1}^{\lfloor\frac{i}{2}\rfloor}\binom{n}{k}\times\binom{n}{i-k}x^i, & \textit{if } 2 \le i \le n+1 \textit{ is odd} \end{cases}$$

and for $i > 1$,

$$d_c(G^{'},n+i) = \begin{cases} \binom{n}{\frac{n+i}{2}}^2 + 2\sum\limits_{k=i}^{\lfloor\frac{n+i}{2}\rfloor-1}\binom{n}{k}\times\binom{n}{n+i-k}, & \textit{if both } n,i \textit{ are odd} \\ \binom{n}{\frac{n+i}{2}}^2 + 2\sum\limits_{k=i}^{\lfloor\frac{n+i}{2}\rfloor-2}\binom{n}{k}\times\binom{n}{n+i-k}, & \textit{if both } n,i \textit{ are even} \\ 2\sum\limits_{k=i}^{\lfloor\frac{n+i}{2}\rfloor}\binom{n}{k}\times\binom{n}{n+i-k}, & \textit{otherwise.} \end{cases}$$

Proof. Since G is closely-connected, $\gamma_{cc}(G^{'}) = 2$. A cc-dominating set of $G^{'}$ of cardinality $i \ge 2$ is obtained by choosing $j > 1$ vertices from one copy of G and $i - j$ vertices from the other copy.

This computation gives the desired result. □

Theorem 4.3.2. *The cc-domination polynomial of the bipartite cocktail party* *graph B_u is given by*

$$D_c[B_u;x] = \begin{cases} x^4, & \textit{if } u=2,\\ x^6+6x^5+15x^4+14x^3+3x^2, & \textit{if } u=3,\\ (x+1)^u-1, & \textit{otherwise.}\end{cases}$$

Proof. Let B_u be the bipartite cocktail party graph with $V(B_u)=\{m_1,\dots,m_u\}\cup\{n_1,\dots,n_u\}$ as shown in figure 4.3.

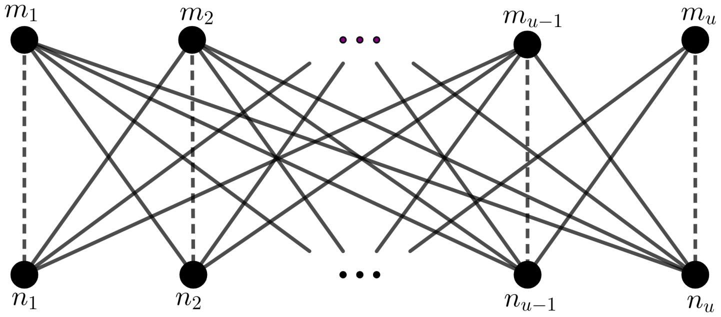

Figure 4.3: The bipartite cocktail party graph B_u

Case(i) For $u=2$, B_u is a disjoint union of two copies of K_2 so that all its vertices are cc-isolated.

Case(ii) For $u=3$, B_u has no singleton cc-dominating sets. The only cc-dominating sets of size 2 are of the form $\{m_i,n_i\}$ for $i=1,2,3$ so that $d_c(B_3,2)=3$. The cc-dominating sets of size 3 are obtained by taking the non-trivial subsets of $V(B_3)$ except those of the form $\{m_i,n_j,n_k\}\cup\{n_i,m_j,m_k\}$ for $i,j,k\in\{1,2,3\}$ with $i\neq j\neq k$. Thus $d_c(B_3,3)=\binom{6}{3}-6=14$. Now, for $i>3$, every subset of the vertex set is a cc-dominating so that $d_c(B_3,i)=\binom{6}{i}$.

Case(iii) It can be readily observed that B_u is closely-connected for $u > 3$. Hence, the result follows immediately from corollary 4.2.8.

This completes the proof. $\square$

Theorem 4.3.3. *Let T be a tree of order n and $v \in V(T)$ be of degree k. If T' is the graph obtained by duplicating the vertex v, then*

$$D_c[T';x] = \begin{cases} xD_c[T;x], & \text{if } k = 1, \\ x^{n-3}D_c[C_4;x], & \text{if } k = 2, \\ x^{n-(k+1)}D_c[K_{2,k};x], & \text{otherwise.} \end{cases}$$

Proof. **Case(i): v is a pendent vertex of T.**

Since $k = 1$, the duplication of v results in a tree graph of order $n + 1$. Therefore, $D_c[T';x] = x^{n+1} = xD_c[T;x]$.

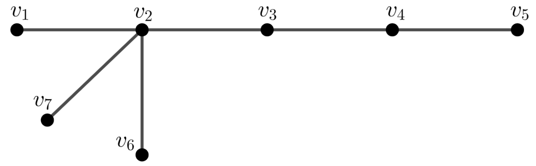

Figure 4.4: The tree graph T

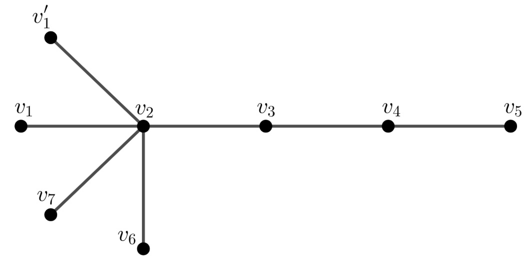

Figure 4.5: The graph $T^{'}$ obtained by duplicating the vertex v_1.

Case(ii): v is a non pendent vertex in T.

If $k = 2$, then T' consists of a cycle C_4 with $n - 3$ vertices attached to it through cut edges. Hence,

$$\begin{aligned} D_c[T'; x] &= x^{n-3} D_c[C_4; x] \\ &= x^{n+1} + 4x^n + 6x^{n-1} \text{ from corollary 4.2.13} \end{aligned}$$

Now if $k > 2$, then the duplication of v produces a complete bipartite graph $K_{2,k}$ with $n - (k+1)$ vertices attached to it through cut edges . Thus,

$$\begin{aligned} D_c[T'; x] &= x^{n-(k+1)} D_c[K_{2,k}; x] \\ &= x^{n-(k+1)}[(x+1)^{n+2} - 2x - 1]. \end{aligned}$$

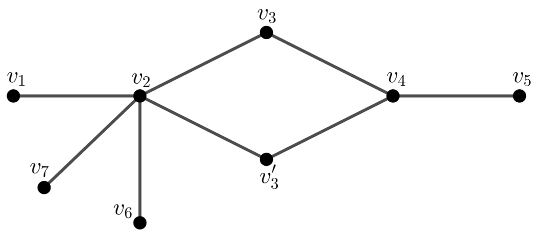

Figure 4.6: The graph T'' obtained by duplicating the vertex v_3

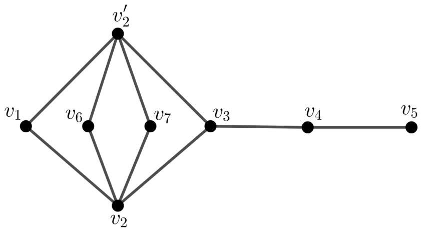

Figure 4.7: The graph T''' obtained by duplicating the vertex v_2

□

Corollary 4.3.4. *For $n > 2$, let $K_{1,n}$ be the star graph with center vertex v and pendent vertices $v_1 \ldots, v_n$. Then,*

$$D_c[K'_{1,n}; x] = \begin{cases} nx + \sum\limits_{i=2}^{n+2} \binom{n+2}{i} x^i, & \text{if } v \text{ is duplicated,} \\ 2nx + \sum\limits_{i=2}^{2n+2} \binom{2n+2}{i} x^i, & \text{if the vertices of } K_{1,n} \text{ are successively duplicated,} \\ x^n[nx + \sum\limits_{i=2}^{n+2} \binom{n+2}{i} x^i], & \text{if the vertices of } K_{1,n} \text{are simultaneously duplicated.} \end{cases}$$

Proof. **Case(i): v is duplicated.**

Since degree of the vertex v is $n > 2$, it follows from theorem 4.3.3 that

$$\begin{aligned} D_c[K'_{1,n}; x] &= D_c[K_{2,n}; x] \\ &= nx + \sum_{i=2}^{n+2} \binom{n+2}{i} x^i. \end{aligned}$$

Case(ii): The vertices of $K_{1,n}$ are successively duplicated.

Let v be duplicated after the duplication of k pendent vertices, where $0 \leq k \leq n$. Then, after duplicating the $k + 1$ vertices including v, we get a complete bipartite graph $K_{2,n+k}$ as shown in figure 4.8. Since the resulting graph has no more pendent vertices, the duplication of the remaining $n - k$ vertices produces a complete bipartite graph $K_{2,2n}$ as in figure 4.9.

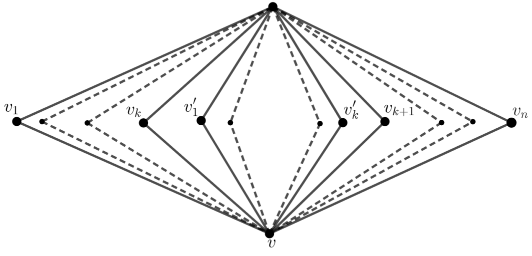

Figure 4.8: The complete bipartite graph $K_{2,n+k}$

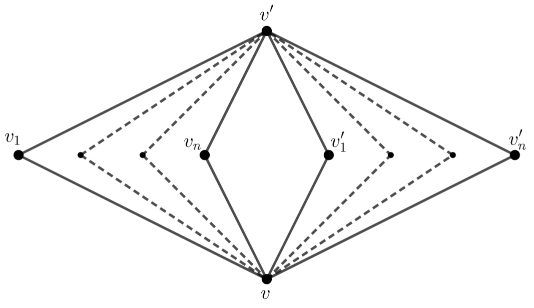

Figure 4.9: The complete bipartite graph $K_{2,2n}$

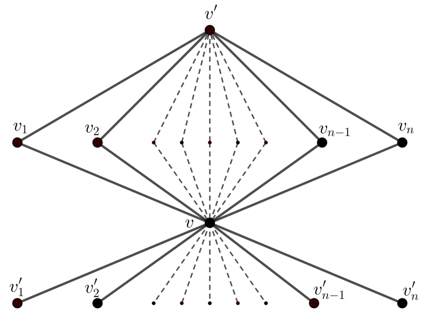

Figure 4.10: The splitting graph of $K_{1,n}$

Case(iii): The vertices of of $K_{1,n}$ are simultaneously duplicated.

In this case, the resulting graph will be the splitting graph of $K_{1,n}$ as shown in the figure 4.10. That is, $S(K_{1,n})$ is a graph of order $2n+2$ with $K_{2,n}$ as an induced subgraph such that n pendent vertices are attached to one of the two vertices on the partite set of cardinality two. Therefore, it follows from theorem 4.2.16 that

$$\begin{aligned} D_c[S(K_{1,n});x] &= x^n D_c[K_{2,n};x] \\ &= x^n\left[nx+\sum_{i=2}^{n+2}\binom{n+2}{i}x^i\right]. \end{aligned}$$

This completes the proof.

□

Theorem 4.3.5. *For $n \geq 3$, the cc-domination polynomial of the graph K'_n obtained by the duplication of one of the vertices of the complete graph K_n is given by*

$$D_c[K'_n; x] = (x+1)^{n+1} - 1.$$

Proof. Let v be the vertex of K_n whose duplication produces the graph K'_n and let v' be the corresponding duplicate vertex. Then, K'_n is the graph in which every vertex pair is adjacent except (v, v'). Thus, $\Gamma_{cc}(u, w) \geq 1 \ \forall u, w \in K'_n$ so that $\gamma_{cc}(K'_n) = 1$. Hence, the result follows immediately from theorem 4.2.4.

This completes the proof. ☐

Theorem 4.3.6. *For every $n > 5$, there exists a graph G of order n such that $D_c[J(G); x]$ has a linear coefficient, where $J(G)$ is the jump graph of G.*

Proof. Let $G = P_n$ be the path graph. Then, $J(P_n)$ is the graph of order $n-1$ with two vertices of degree $n-3$ and all other vertices of degree $n-4$. Since $diam(J(P_n)) = 2$ and being free of cut edges, it can be noted that every single vertex cc-dominates.

This completes the proof. ☐

4.4 CC-Domination Roots of Graphs

In this section, we define the *cc-domination root* of a graph and study some of its properties.

Definition 4.4.1. Let G be a graph with cc-domination polynomial $D_c[G; x]$.

A zero of $D_c[G;x]$ is called a *cc-domination root* of G and the set of all cc-domination roots is defined as the *cc-zero space* of G, denoted by $Z_c(G)$.

Definition 4.4.2. Let G be a graph. Then, the number of distinct real cc-domination roots of G is called the *d_c-number* of G and is denoted by $d_c(G)$.

For a graph G with cc-domination polynomial $D_c[G;x]$, the following properties can be easily observed.

(i) $(0,\infty)$ is a zero-free interval for $D_c[G;x]$.

(ii) 0 is a cc-domination root of G with multiplicity $\gamma_{cc}(G)$.

(iii) $d_c(G) \geq 1$ and $d_c(G) = 1$ if G is acyclic

(iv) $Z_c(G) = \{0\} \iff G$ is acyclic.

(v) Isomorphic graphs have the same d_c-number.

Theorem 4.4.3. *Let G be a graph of order n with exactly two distinct cc-domination roots. Then, $d_c(G) = 2$. In particular, if G has no cc-isolated vertices then $Z_c(G) = \{0,-2\}$.*

Proof. Since 0 is a cc-domination root of G and since complex roots of polynomials occur in conjugate pairs, it follows that both the distinct roots of $D_c[G;x]$ are real. Thus, $d_c(G) = 2$. Now, assume that G has no cc-isolated vertices. Since the multiplicity of 0 as a cc-domination root of G is $\gamma_{cc}(G)$, we have

$$D_c[G;x] = x^{\gamma_{cc}(G)}(x+a)^{n-\gamma_{cc}(G)},$$

where $-a$ is the non-zero cc-domination root of G. Also, $\gamma_{cc}(G) < n$ and $(0,\infty)$ is a zero-free interval of $D_c[G;x]$ so that the coefficient of x^{n-1} in the above equation is $a(n-\gamma_{cc}(G)) > 0$. Since G has no cc-isolated vertices, it follows that $d_c(G, n-1) = \binom{n}{n-1} = n$ so that $n = a(n-\gamma_{cc}(G))$. But $a \geq 2$, since $\gamma_{cc}(G) \geq 1$. Also from corollary 3.2.11 it follows that $\gamma_{cc}(G) \leq \frac{n}{2}$ so that $\frac{n(a-1)}{a} \leq \frac{n}{2}$. Hence, $a \leq 2$. Therefore,

$$D_c[G;x] = x^{\gamma_{cc}(G)}(x+2)^{n-\gamma_{cc}(G)}$$

so that $Z_c(G) = \{0, -2\}$.

This completes the proof. $\square$

Theorem 4.4.4. *Let G be a graph of order n such that every vertex pair in G is closely-connected. Then,*

$$d_c(G) = \begin{cases} 1; \; if \; n \; is \; odd \\ 2; \; if \; n \; is \; even \end{cases}$$

In particular if n is even, then the non-zero real cc-domination root of G is -2 with multiplicity 1.

Proof. Since every vertex pair in G is closely-connected, it immediately follows that $\gamma_{cc}(G) = 1$ and $d_c(G, 1) = n$. Hence, it follows from corollary 4.2.8 that $D_c[G;x] = (x+1)^n - 1$

Case(i) If n is odd, then 1 is the only real n^{th} root of unity so that 0 is the only real cc-domination root of G.

Case(ii) For even n, 1 and -1 are the real n^{th} roots of unity so that 0 and -2 are the real cc-domination roots of G with multiplicity 1.

This completes the proof. $\square$

Theorem 4.4.5. *For $n \geq 5$, the cc-zerospace of $K_{2,n-2}$ is bounded with cardinality n and is given by*

$$Z_c(K_{2,n-2}) = \{-1\} \cup \{e^{\frac{ik2\pi}{n-1}} - 1\}, \text{ where } k = 1, \ldots, n-1.$$

In particular,

$$d_c(K_{2,n-2}) = \begin{cases} 3; & \text{if } n \text{ is odd} \\ 2; & \text{if } n \text{ is even} \end{cases}$$

Proof. Since

$$\begin{aligned} D_c[K_{2,n-2}; x] &= (x+1)^n - x - 1 \\ &= (x+1)[(x+1)^{n-1} - 1], \end{aligned}$$

it is evident that -1 is a zero of the polynomial $D_c[K_{2,n-2}; x]$.

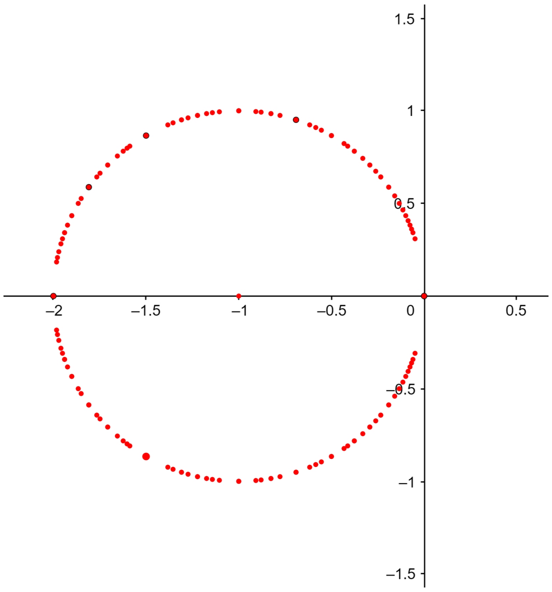

Figure 4.11: The cc-zerospace of $K_{2,n-2}$

Now, since the zeros of $(x+1)^{n-1} - 1$ are precisely the $n - 1^{th}$ roots of unity, it follows that the equation $(x+1)^{n-1} = 1$ has exactly one real root if n is even and two real roots for odd value of n. The figure 4.11 graphically represents the cc-zero space of $K_{2,n-2}$ for $5 \leq n \leq 19$.

This completes the proof. □

Chapter 5

VERTEX-CONNECTIVITY INDEX OF GRAPHS

This chapter is divided into two sections. The first section gives a brief introduction emphasizing the need to study vertex-connectivity index in graphs. In the consecutive section, the vertex-connectivity index is formulated and its properties are studied. Moreover, the same is computed for some special graphs.

5.1 Introduction

Drug discovery research heavily relies on QSAR (quantitative structure-activity relationship) investigations, which describe the connections between biological activity as dependent parameters and physicochemical and structural descriptors as independent parameters. Special graph invariants, often referred to as topological indices are primarily utilised in the computation of QSAR and QSPR (quantitative structure-property relationship) research. The most of the proposed topological indices are related either to vertex adjacency relationship or to distance function in a graph [6]. This chapter deals with a topological index which is linked to both the distance as well as connectivity aspects of a graph.

In this chapter, we confine our study to formulate an index by incorporating both the metric and connectivity properties of a graph. All graphs considered here are connected.

5.2 Vertex-Connectivity Index of Graphs

Definition 5.2.1. Let $G = (V, E)$ be a graph and u, v be distinct vertices of G. Then, u and v are said to be *vertex-connected* if $\Gamma_{cc}(u, v) = g_{uv}$, where g_{uv} is the cardinality of the set of all geodesics connecting u and v.

The graph G is *vertex-connected* if the above condition holds for every vertex pair in G.

Example 5.2.2. *Consider the wheel graph W_6 shown in figure 5.1*

Here,

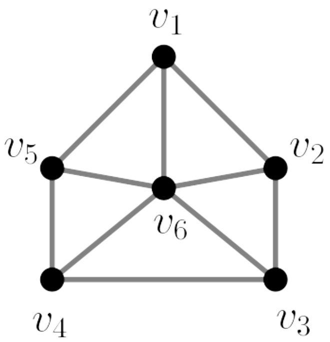

Figure 5.1: The wheel graph W_6

$$\Gamma_{cc}(v_1, v_2) = 1 = g_{v_1v_2}, \quad \Gamma_{cc}(v_1, v_3) = 2 = g_{v_1v_3}, \quad \Gamma_{cc}(v_1, v_4) = 2 = g_{v_1v_4},$$

$$\Gamma_{cc}(v_1, v_5) = 1 = g_{v_1v_5}, \quad \Gamma_{cc}(v_1, v_6) = 1 = g_{v_1v_6}, \quad \Gamma_{cc}(v_2, v_3) = 1 = g_{v_2v_3},$$

$$\Gamma_{cc}(v_2, v_4) = 2 = g_{v_2v_4}, \quad \Gamma_{cc}(v_2, v_5) = 2 = g_{v_2v_5}, \quad \Gamma_{cc}(v_2, v_6) = 1 = g_{v_2v_6},$$

$$\Gamma_{cc}(v_3, v_4) = 1 = g_{v_3v_4}, \quad \Gamma_{cc}(v_3, v_5) = 2 = g_{v_3v_5}, \quad \Gamma_{cc}(v_3, v_6) = 1 = g_{v_3v_6},$$

$$\Gamma_{cc}(v_4, v_5) = 1 = g_{v_4v_5}, \quad \Gamma_{cc}(v_4, v_6) = 1 = g_{v_4v_6}, \quad \Gamma_{cc}(v_5, v_6) = 1 = g_{v_5v_6}.$$

Thus, for every distinct vertex pair (u, v) *of* W_6, $\Gamma_{cc}(u, v) = g_{uv}$, *which shows that the wheel graph* W_6 *is vertex-connected.*

Example 5.2.3. *Consider the graph* $P_2 \times P_4$ *shown in figure 5.2.*

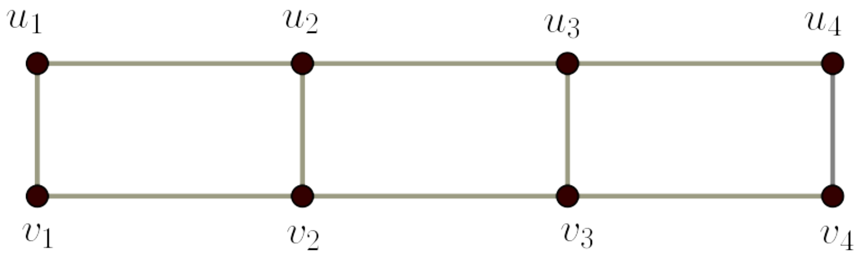

Figure 5.2: The cartesian product $P_2 \times P_4$

It can be easily inferred that the graph is not closely-connected since the deletion of

edges of geodesics linking the vertex pairs (u_1, v_4) *and* (u_4, v_1) *makes the the graph disconnected. But, all other vertex pairs of* $P_2 \times P_4$ *are closely-connected. That is,* $\Gamma_{cc}(u, v) = 1$ *for every distinct vertex pair* $(u, v) \in V(P_2 \times P_4) \setminus \{(u_1, v_4), (u_4, v_1)\}$
For $i \neq j, g_{u_i u_j} = g_{v_i v_j} = 1$ *and*

$$g_{u_i v_j} = \begin{cases} 1; & \text{if } i = j, \\ 2; & \text{if } i = j+1 \text{ or } j = i+1, \\ 3; & \text{if } i = j+2 \text{ or } j = i+2, \\ 4; & \text{if } i = j+3 \text{ or } j = i+3, \end{cases}$$

Thus, $\Gamma_{cc}(u_i, v_j) \neq g_{u_i v_j}$ *for distinct* $i, j \in \{1, 2, 3, 4\}$ *so that the vertex pairs* (u_i, v_j) *are not vertex-connected. This example explicitly shows that not all closely-connected vertices are vertex-connected.*

Example 5.2.4. *Consider the graph* G' *shown in figure 5.3.*

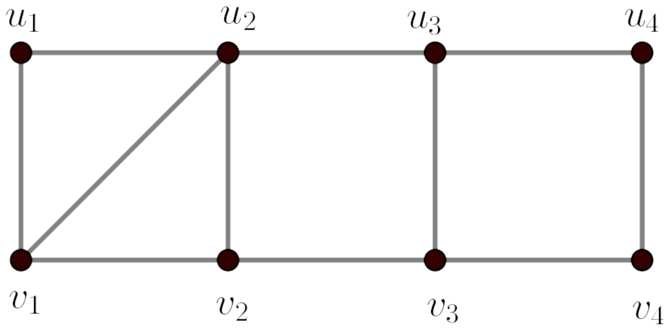

Figure 5.3: The graph G'

It can be easily inferred that the graph G' *is closely-connected.*
That is, $\Gamma_{cc}(u, v) \geq 1$ *for every distinct vertex pair* $(u, v) \in V(G')$.

For $i \neq j, g_{u_iu_j} = g_{v_iv_j} = 1$ *and* $g_{u_2v_1} = g_{u_3v_1} = g_{u_4v_1} = 1$. *Also,*

$$g_{u_iv_j} = \begin{cases} 1; \ \textit{if } i = j, \\ 2; \ \textit{if } i = j+1 \textit{ or } j = i+1, \textit{ where } i \neq 2 \textit{ and } j \neq 1 \\ 3; \ \textit{if } i = j+2 \textit{ or } j = i+2, \textit{ where } i \neq 3 \textit{ and } j \neq 1 \\ 4; \ \textit{if } j = i+3, \end{cases}$$

For $i \neq j$,

$$\Gamma_{cc}(u_i, u_j) = \Gamma_{cc}(v_i, v_j) = 1,$$

$$\Gamma_{cc}(u_2, v_4) = \Gamma_{cc}(u_3, v_4) = \Gamma_{cc}(u_4, v_3) = 1,$$

$$\Gamma_{cc}(u_1, v_4) = 2 \quad \textit{and}$$

$$\Gamma_{cc}(u_iv_j) = \begin{cases} 1; \ \textit{if } i = j, \\ 2; \ \textit{if } i = j+1 \textit{ or } j = i+1, \textit{ where } i, j \neq 3, 4 \\ 3; \ \textit{if } i = j+2 \textit{ or } j = i+2, \textit{ where } i \neq 2 \textit{ and } j \neq 4 \\ 4; \ \textit{if } i = j+3, \end{cases}$$

That is,

$\Gamma_{cc}(u_2, v_4) = 1 \neq 3 = g_{u_2v_4}, \ \Gamma_{cc}(u_3, v_4) = 1 \neq 2 = g_{u_3v_4}$,

$\Gamma_{cc}(u_4, v_3) = 1 \neq 2 = g_{u_4v_3}, \ \Gamma_{cc}(u_1, v_4) = 2 \neq 4 = g_{u_1v_4}$.

Thus, the graph G' *is closely-connected but not vertex-connected.*

Definition 5.2.5. Let $G = (V, E)$ be a graph. Define a function $f : V \to \mathbb{R}^+ \cup \{0\}$ given by

$$f(v) = \sum_{\substack{u \in V \setminus \{v\} \\ \Gamma_{cc}(u,v) = g_{uv}}} d(u, v) \quad \forall v \in V.$$

Then, the *vertex-connectivity index* $V_c[G]$ of G is defined as :

$$V_c[G] = \frac{1}{2}\sum_{v\in V} f(v).$$

Theorem 5.2.6. *Let G be a closely-connected graph such that $g_{uv} = 1$ for every vertex pair (u, v) in G. Then, G is vertex-connected.*

Proof. Let (u, v) be a vertex pair in G. Since G is closely-connected, $\Gamma_{cc}(u, v) \geq 1$. But, since $g_{uv} = 1$ and $\Gamma_{cc}(u, v) \leq g_{uv}$, we get $\Gamma_{cc}(u, v) \leq 1$. Therefore, $\Gamma_{cc}(u, v) = 1 = g_{uv}$. That is, G is vertex-connected.

This completes the proof. □

Remark 5.2.7. The converse of theorem 5.2.6 is not true. For example, consider the wheel graph W_6 shown in figure 5.1. Clearly W_6 is vertex-connected, but for the vertex pair (v_1, v_3), $g_{v_1v_3} \neq 1$.

Theorem 5.2.8. *Let $G = (V, E)$ be a graph of size m such that closely-connected vertices are precisely those that are adjacent. Then, $V_c[G] = m$.*

Proof. Let u, v be adjacent vertices in G. Then, $\Gamma_{cc}(u, v) \geq 1$. Since G is simple, there will be no parallel edges between u and v so that $g_{uv} = 1$. But, since $\Gamma_{cc}(u, v) \leq g_{uv}$, it follows that $\Gamma_{cc}(u, v) = 1 = g_{uv}$. Since none of the non-adjacent vertices of G are closely-connected, it can be concluded that the vertex-connected vertices in G are precisely those that are adjacent. Therefore, by degree-sum formula,

$$V_c[G] = \sum_{u\in V} \frac{1}{2} d(u) = m,$$

where $d(u)$ denotes the degree of the vertex u in G.

This completes the proof. □

Remark 5.2.9. If G is closely-connected, then $|E(G)| \leq V_c[G] \leq W[G]$. In particular if G is also vertex-connected, then $V_c[G] = W[G]$, where $W[G]$ is the Wiener index of G.

Theorem 5.2.10. *There exists a closely-connected graph G_n of order $n > 4$ satisfying $|E(G_n)| < V_c[G_n] < W[G_n]$. Moreover, $V_c[G_n] = \frac{2n^2-7n+2}{2}$*

Proof. Consider the cycle graph C_n on $n > 4$ labelled vertices $v_1, v_2, \ldots, v_n$ in the anticlockwise direction and let G_n be the graph obtained by drawing edges between the non-adjacent vertices $(v_1, v_3), (v_1, v_4), \ldots, (v_1, v_{n-2})$ of C_n. It can be easily verified that G_n is closely-connected. Moreover, the vertex pair (v_1, v_{n-1}) is closely-connected but not vertex-connected as the removal of the edges of the geodesic $v_1 v_n v_{n-1}$ disconnects the graph by isolating the vertex v_n. Thus, the graph G is not vertex-connected so that $V_c[G_n] < W[G_n]$.

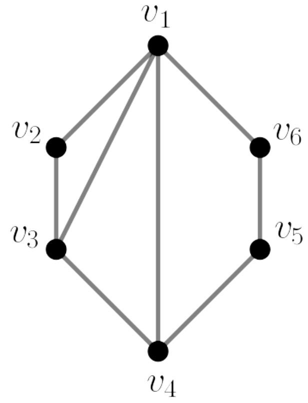

Figure 5.4: The graph G_6

Now by the remark 5.2.9, $|E(G_n)| \leq V_c[G_n]$ and we have $|E(G_n)| = 2n - 4$. Clearly, the vertices v_2 and v_n are non-adjacent but vertex-connected since the only geodesic joining them is $v_2 v_1 v_n$, which is not a cut path in G_n. Therefore,

$V_c[G_n] \geq 2n-4+2 > |E(G_n)|$. Thus, $|E(G_n)| < V_c[G_n] < W[G_n]$.

Now we have,

$$\begin{aligned}
f(v_1) &= \underbrace{1+1+\cdots+1}_{n-2 \text{ times}} && = n-2 \\
f(v_2) &= 1+1+\underbrace{2+\cdots+2}_{n-4 \text{ times}} && = 2+2(n-4) \\
f(v_3) &= 1+1+1+\underbrace{2+\cdots+2}_{n-5 \text{ times}} && = 3+2(n-5) \\
&\vdots \qquad\qquad\qquad \vdots && \vdots \\
f(v_{n-4}) &= 1+1+1+\underbrace{2+\cdots+2}_{n-5 \text{ times}} && = 3+2(n-5) \\
f(v_{n-3}) &= 1+1+1+\underbrace{2+\cdots+2}_{n-4 \text{ times}} && = 3+2(n-4) \\
f(v_{n-2}) &= 1+1+1+\underbrace{2+\cdots+2}_{n-5 \text{ times}} && = 3+2(n-5) \\
f(v_{n-1}) &= 1+1+2+\underbrace{3+\cdots+3}_{n-6 \text{ times}} && = 4+3(n-6) \\
f(v_n) &= 1+1+\underbrace{2+\cdots+2}_{n-4 \text{ times}} && = 2+2(n-4)
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_c[G_n] &= \frac{1}{2}\sum_{v\in V(G_n)} f(v) \\
&= \frac{1}{2}[f(v_1)+f(v_2)+\cdots+f(v_n)] \\
&= \frac{1}{2}[f(v_1)+f(v_{n-3})+f(v_{n-1})+2f(v_n)+(n-5)\times f(v_{n-2})] \\
&= \frac{1}{2}[n-2+2n-5+3n-14+4n-12+2n^2-17n+35] \\
&= \frac{1}{2}[2n^2-7n+2]
\end{aligned}$$

This completes the proof. ☐

Theorem 5.2.11. *For a graph G, the following holds:*

(i) *$W[G] \geq V_c[G]$, where $W[G]$ is the Wiener index of G. In particular if G has cut edges, then $W[G] > V_c[G]$ and equality holds if the vertex connectivity polynomial $V[G;x]$ of G is linear.*

(ii) *G is acyclic iff $V_c[G] = 0$.*

Proof. (i) Since all the vertices of G need not be vertex-connected, it follows from the definition that $W[G] \geq V_c[G]$. Now, if G has at least one cut edge, the end vertices of cut edges are not vertex-connected. Thus, all vertex pairs of G are not vertex-connected and hence $W[G] > V_c[G]$. The linearity of $V[G;x]$ implies that all pairs of vertices of G are vertex-connected so that $W[G] = V_c[G]$ holds in this case.

(ii) If G is acyclic, then all its edges are cut edges so that none of the vertex pairs are vertex-connected. Therefore, $V_c[G] = 0$. Conversely, if G is not acyclic, then there exists at least one edge of G as a part of some cycle. Since it is not a cut edge, its deletion does not disconnect the graph. Thus, the end vertices of this edge are vertex-connected so that $V_c[G] \geq 1$. That is, $V_c[G] \neq 0$.

This completes the proof. ☐

Theorem 5.2.12. *The following are the vertex-connectivity index of some important graphs:*

(i) *For $n \geq 3$, $V_c[C_n] = n$.*

(ii) *For* $m, n \geq 2$ *and* $m \leq n$,

$$V_c[K_{m,n}] = \begin{cases} 4 & \textit{if} \;\; m = n = 2, \\ n(n+1), & \textit{if} \;\; m = 2, n > 2, \\ mn + m(m-1) + n(n-1), & \textit{if} \;\; m > 2. \end{cases}$$

(iii) *Let* G_1, G_2 *be two graphs and let* $G_1 \vee G_2$ *be their join. Then,*

$$V_c[G_1 \vee G_2] = W[G_1 \vee G_2].$$

Proof. (i) Since the vertex-connected vertices of C_n are exactly the vertices that are adjacent in C_n , it immediately follows that

$$V_c[C_n] = \text{Number of edges of } C_n = n.$$

(ii) Let M and N be the bipartite sets of vertices of $K_{m,n}$ with cardinalities m and n respectively. We consider three cases:

Case(i) $m, n = 2$.

Since $K_{2,2}$ is C_4, the result follows trivially from (i).

Case(ii) $m = 2, n > 2$.

Since the linear coefficient of the vertex connectivity polynomial of $K_{2,n}$ is $\binom{n+2}{2} - 1$ and since the unique vertex pair in the partite set M is not vertex-connected, it follows that there are $2n$ vertex pairs at unit distance and $\binom{n}{2}$ vertex pairs at twice the unit distance which are vertex-connected. Therefore,

$$V_c[K_{2,n}] = 2n \times 1 + \binom{n}{2} \times 2 = n(n+1).$$

Case(iii) $m > 2$.

In this case, the linear coefficient of the vertex connectivity polynomial is $\binom{m+n}{2}$ so that all the vertex pairs are vertex-connected. It can be observed that there are mn vertex pairs at unit distance and $\binom{m}{2}$ + $\binom{n}{2}$ vertex pairs at twice the unit distance in $K_{m,n}$. Therefore,

$$\begin{aligned} V_c[K_{m,n}] &= mn \times 1 + 2\left[\binom{m}{2} + \binom{n}{2}\right] \\ &= mn + m(m-1) + n(n-1). \end{aligned}$$

(iii) This follows from the observation that $G_1 \vee G_2$ is vertex-connected.

This completes the proof. □

Remark 5.2.13. Note that for a cut edge free graph G, $V_c[G]$ and $W[G]$ need not be equal. For example, not every vertex pair of the cycle graph C_n is vertex-connected for $n \geq 4$ so that $V_c[C_n] \neq W[C_n]$.

Theorem 5.2.14. *Let $G = (V, E)$ be a graph and let $G' = (V', E')$ be the graph obtained by adjoining pendent vertices to the vertices of G through bridges. Then, $V_c[G'] = V_c[G]$.*

Proof. Since the new pendent vertices of G' are adjoined to the vertices of G through bridges, the deletion of any of these bridges makes the graph disconnected. Hence, the newly adjoined vertices are neither mutually vertex-connected nor vertex-connected with any of the vertices of G so that $V_c[G'] = V_c[G]$. This completes the proof. □

Corollary 5.2.15. *Let $G = (V, E)$ be a graph and let $G' = (V', E')$ be the graph obtained by adjoining acyclic graphs to the vertices of G through bridges. Then, $V_c[G'] = V_c[G]$.*

Proof. The proof follows immediately from the fact that none of the newly adjoined acyclic graphs increases the number of vertex-connected vertices in G' from that of G. □

Corollary 5.2.16. *For $m \geq 3$, let $A_m = P_2 \times C_m$ and $A = A_m \odot K_{1,n}$ be the generalized n-crown obtained by introducing n new pendent edges at each vertex of the outermost C_m in A_m [26]. Then,*

$$V_c[A] = \begin{cases} \frac{m(m^2+2m-1)}{2}, & \textit{if } m \textit{ is odd,} \\ \frac{m^2(m+2)}{2}, & \textit{if } m \textit{ is even.} \end{cases}$$

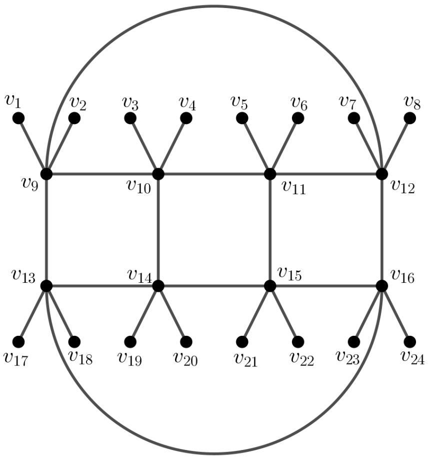

Figure 5.5: The generalized 2-crown $A_4 \odot K_{1,2}$

Proof. Since A is constructed by adjoining n new pendent vertices at each vertex of the outermost C_m in A_m, it follows from theorem 5.2.14 that $V_c[A] = V_c[A_m]$.

But, since A_m being a vertex-connected graph, it follows from the remark 5.2.9 that $V_c[A_m] = W[A_m]$. In [26], it has been proved that,

$$W[A_m] = \begin{cases} \frac{m(m^2+2m-1)}{2}, & \text{if } m \text{ is odd,} \\ \frac{m^2(m+2)}{2}, & \text{if } m \text{ is even.} \end{cases}$$

This completes the proof. □

Theorem 5.2.17. *Let G be a graph of diameter 2 with n vertices and m edges among which n_1 pairs of vertices are vertex-connected and m_1 are cut edges. Then,*

$$V_c[G] = 2n_1 - m + m_1.$$

Proof. Out of the n_1 vertex-connected vertex pairs, there are $m-m_1$ edges linking adjacent vertex-connected vertices and the remaining are at two unit distance. Therefore, $V_c[G] = m - m_1 + 2[n_1 - (m - m_1)]$.
This completes the proof. □

Theorem 5.2.18. *Let $G_1, G_2, \ldots G_n$ be connected graphs and let G be the connected graph obtained by adjoining G_i and G_{i+1} through a bridge, where $i = 1, \ldots, n-1$. Then,*

$$V_c[G] = \sum_{i=1}^{n} V_c[G_i].$$

Proof. For $i = 1, \ldots, n-1$ and $j = 1, 2, \ldots, n-i$, it can be observed that every path linking the vertices of G_i and G_{i+j} incorporates j bridges mentioned in the construction of G. Thus, the addition of bridges neither alters the vertex-connected vertices of G_i nor produces new vertex-connected vertices in G for every $i = 1, \ldots, n-1$.. Therefore, $V_c[G] = \sum_{i=1}^{n} V_c[G_i]$.
This completes the proof. □

Theorem 5.2.19. *Let G_1 and G_2 be two non-trivial graphs of orders n_1, n_2 and sizes m_1, m_2 respectively. If G_1 is vertex-connected, then $G_1 \circ G_2$ is vertex-connected and*

$$V_c[G_1 \circ G_2] = (n_2 + 1)^2 V_c[G_1] + n_1(n_2^2 - m_2) + n_1 n_2 (n_1 - 1)(n_2 + 1).$$

Proof. Let $G = G_1 \circ G_2$. Since G_1 is vertex-connected, the vertex pairs in G belonging to G_1 remains vertex-connected. The i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2 and since G_2 is a non-trivial connected graph, the deletion of the edge between the i^{th} vertex of G_1 and the k^{th} vertex of G_2 in the i^{th} copy of G_2 does not make G disconnected, where $i \in \{1, \ldots, n_1\}$ and $k \in \{1, \ldots, n_2\}$. The geodesic linking the i^{th} vertex of G_1 and the k^{th} vertex of G_2 in the j^{th} copy of G_2 is through the i-j path in G_1 followed by the j-k edge, where $i, j \in \{1, \ldots, n_1\}, i \neq j$ and $k \in \{1, \ldots, n_2\}$. Since G_1 is vertex-connected, we get that the i^{th} vertex of G_1 and the k^{th} vertex of G_2 in the j^{th} copy of G_2 are also vertex-connected. Now, since every vertex pair belonging to the i^{th} copy of G_2 in G is connected through the i^{th} vertex of G_1 and also through the vertices of G_2, they also remain vertex-connected. Thus, G is vertex-connected and hence it follows that $V_c[G] = W[G]$. In [48], the wiener index of corona of two graphs has been proved as $W[G] = (n_2+1)^2 W[G_1] + n_1(n_2^2 - m_2) + n_1 n_2 (n_1 - 1)(n_2 + 1)$. This completes the proof. ☐

Theorem 5.2.20. *For odd* $n \geq 3$,

$$V_c[P_3 \wedge C_n] = 14n.$$

Proof. The tensor product $P_3 \wedge C_n$ is a connected (2,4) biregular graph with $3n$ vertices when n is odd [26] . It is a connection of n cycles with 4 vertices and each cycle has one common vertex with the other cycle. That is, it has an outer cycle of length $2n$ and the remaining n vertices together with n alternate vertices of the outer cycle again constitutes a cycle of length $2n$.

Thus, all the vertices of $P_3 \wedge C_n$ except those belonging to both the $2n$ cycles are of degree 2. Now, consider the outer cycle of $P_3 \wedge C_n$. Every vertex of

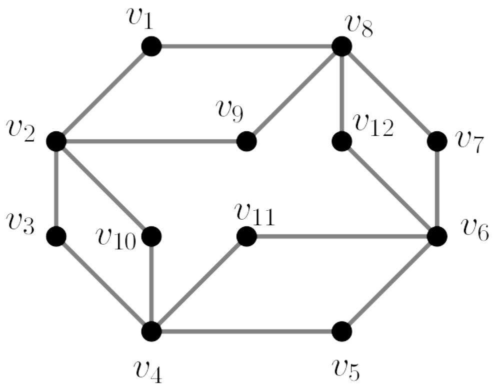

Figure 5.6: $P_3 \wedge C_4$

degree 2 along with any other vertex at a distance greater than 2 on the outer cycle disconnects the graph and the vertices of degree 4 are vertex-connected only with its adjacent vertices on the outer cycle. Thus, there are $2n$ vertex-connected vertex pairs at a unit distance and n vertex-connected vertex pairs at twice the unit distance on the outer cycle. Therefore, on the outer cycle, sum of all distances between the vertex-connected vertices is given by

$$S_o = 2n \times 1 + n \times 2 = 4n.$$

Now, every vertex of degree 2 on the outer cycle is vertex-connected to exactly 3 vertices at twice the unit distance on the inner cycle whereas every vertex of degree 4 on the outer cycle is vertex-connected to exactly 2 vertices at a unit distance on the inner cycle. Thus, sum of all distances between the vertex-connected vertices on the outer cycle and the inner cycle is given by,

$$S_{oi} = 3 \times 2 \times n + 2 \times 1 \times n = 8n.$$

Now, all the n vertices of degree 2 on the inner cycle are vertex-connected

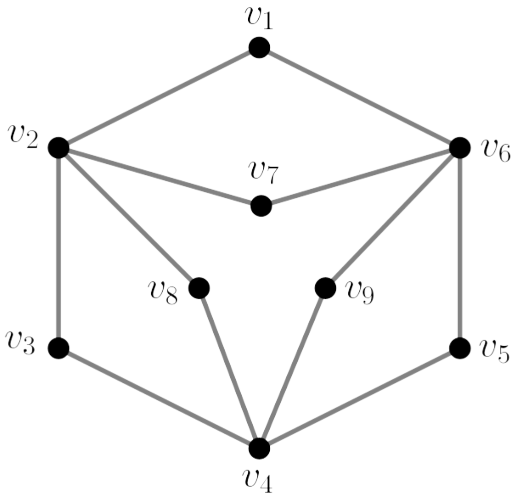

Figure 5.7: A component of $P_3 \wedge C_6$

with the vertices located at twice the unit distance on the inner cycle. Therefore, sum of all distances between the vertex-connected vertices on the inner cycle is given by

$$S_i = 2 \times n = 2n.$$

Hence,

$$V_c[P_3 \wedge C_n] = S_o + S_{oi} + S_i = 4n + 8n + 2n = 14n.$$

This completes the proof. □

Theorem 5.2.21. *For even $n > 4$, let G be one of the components of the graph $P_3 \wedge C_n$. Then, $V_c[G] = 7n$.*

Proof. The tensor product $P_3 \wedge C_n$ is a disconnected graph of order $3n$ [26]. In this case, both the components of $P_3 \wedge C_n$ are $(2, 4)$ biregular with $\frac{3n}{2}$ vertices among which $\frac{n}{2}$ of degree 4 and n are of degree 2. For vertices having degree 4, it can be observed that only the adjacent vertices are vertex-connected and for degree 2 vertices vertex-connected vertices are those located at at most two units distance. Therefore,

$$V_c[G] = \frac{1}{2}\left[\frac{n}{2} \times 1 \times 4 + n \times 1 \times 2 + n \times 2 \times 5\right] = 7n.$$

This completes the proof. □

Theorem 5.2.22. *For $n > 1$,*

$$V_c[P_2 \times P_n] = \frac{4n^3 + 56n - 144}{12}$$

Proof. In $P_2 \times P_n$, there are 4 vertices of degree 2 and $2n - 4$ vertices of degree 3. The vertices of degree 2 are vertex-connected with those vertices of $P_2 \times P_n$ which lie in the same row or column. Therefore, sum of all distances between the vertices which are vertex-connected with degree 2 vertices is given by

$$\begin{aligned} S_2 &= 4[1 + 1 + 2 + \cdots + n - 1] \\ &= 2n^2 - 2n + 4. \end{aligned}$$

The four vertices of degree 3 adjacent to the degree 2 vertices are vertex-connected only with vertices in the same row or column and with the unique vertex at two unit distance with degree greater than two. The remaining degree 3 vertices are vertex-connected with all vertices in the same row or column and with vertices at two unit distance in the alternate row.

Case(i) : n is odd.

Let S_3' be the sum of all distances between the vertices that are vertex-connected to the mid vertices $v_{\frac{n+1}{2}}$, $u_{\frac{n+1}{2}}$ and S_3'' be the sum of all distances between the vertices that are vertex-connected to the remaining degree 3 vertices. Then,

$$\begin{aligned} S_3^{'} &= 2\Bigg\{2\Bigg[1+2+\cdots+\frac{n-1}{2}\Bigg]+5\Bigg\} \\ &= \frac{n^2+19}{2}. \end{aligned}$$

$$\begin{aligned}
S_{v_{\frac{n-1}{2}}} &= 2\Big(1+2+\cdots+\frac{n-3}{2}\Big)+\frac{n-1}{2}+\frac{n+1}{2}+5 \\
&= \frac{(n-3)(n-1)}{4}+n+5 \\
&= \frac{n^2+23}{4} \\
S_{v_{\frac{n-3}{2}}} &= 2\Big(1+2+\cdots+\frac{n-5}{2}\Big)+\frac{n-3}{2}+\frac{n-1}{2}+\frac{n+1}{2}+\frac{n+3}{2}+5 \\
&= \frac{(n-5)(n-3)}{4}+2n+5 \\
&= \frac{n^2+35}{4} \\
S_{v_{\frac{n-5}{2}}} &= 2\Big(1+2+\cdots+\frac{n-7}{2}\Big)+3n+5 \\
&= \frac{(n-7)(n-5)}{4}+3n+5 \\
&= \frac{n^2+55}{4} \\
\cdots &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
S_{v_{\frac{n-(n-6)}{2}}} &= 2\Big(1+2+\cdots+\frac{n-(n-4)}{2}\Big)+(\frac{n-5}{2})n+5 \\
&= \frac{(n-(n-4))(n-(n-6))}{4}+\frac{n^2-5n+10}{2} \\
&= \frac{n^2+20+(n-6)(n-4)}{4} \\
&= \frac{n^2-5n+22}{2} \\
S_{v_{\frac{n-(n-4)}{2}}} &= (1+1+2+\cdots+n-2)+3 \\
&= \frac{n^2-3n+10}{2}
\end{aligned}$$

Thus,

$$\begin{aligned}
S_3^{''} &= 4\Big[S_{v_{\frac{n-1}{2}}} + S_{v_{\frac{n-3}{2}}} + \cdots + S_{v_{\frac{n-(n-4)}{2}}}\Big]\\
&= 4\Big[\frac{n^2+20+(1\times 3)+\cdots+n^2+20+(n-6)(n-4)}{4} + \frac{n^2-3n+10}{2}\Big]\\
&= \Big[\frac{(n-5)}{2}(n^2+20)+(1\times 3)+\cdots+(n-6)(n-4)+2(n^2-3n+10)\Big]\\
&= \frac{n^3-n^2+8n-60}{2} + \sum_{i=1}^{\frac{n-5}{2}}(2k-1)(2k+1)\\
&= \frac{n^3-n^2+8n-60}{2} + \sum_{i=1}^{\frac{n-5}{2}}(4k^2-1)\\
&= \frac{n^3-n^2+8n-60}{2} + 4\sum_{i=1}^{\frac{n-5}{2}}k^2 - \sum_{i=1}^{\frac{n-5}{2}}1\\
&= \frac{n^3-n^2+8n-60}{2} + 4\sum_{i=1}^{\frac{n-5}{2}}k^2 - \sum_{i=1}^{\frac{n-5}{2}}1\\
&= \frac{n^3-n^2+8n-60}{2} + \frac{(n-5)(n-4)(n-3)}{6} - \Big(\frac{n-5}{2}\Big)\\
&= \frac{4n^3-15n^2+68n-225}{6}
\end{aligned}$$

Therefore, the sum of all distances between the vertices that are vertex-connected with degree 3 vertices is given by

$$\begin{aligned}
S_3 &= S_3^{'} + S_3^{''}\\
&= \frac{n^2+19}{2} + \frac{4n^3-15n^2+68n-225}{6}\\
&= \frac{4n^3-12n^2+68n-168}{6}
\end{aligned}$$

Thus,

$$\begin{aligned}
V_c[P_2\times P_n] &= \frac{1}{2}[S_2+S_3]\\
&= \frac{1}{2}\left[2n^2-2n+4+\frac{4n^3-12n^2+68n-168}{6}\right]
\end{aligned}$$

$$= \frac{4n^3 + 56n - 144}{12}$$

Case(ii) : n is even.

If n is even, then the sum of all distances between the vertices which are vertex-connected with degree 3 vertices is given by

$$\begin{aligned}
S_{v_{\frac{n}{2}}} &= 2\Big(1 + 2 + \cdots + \frac{n-2}{2}\Big) + \frac{n}{2} + 5\\
&= \frac{n^2}{4} + 5\\
S_{v_{\frac{n-2}{2}}} &= 2\Big(1 + 2 + \cdots + \frac{n-4}{2}\Big) + \frac{3n}{2} + 5\\
&= \frac{n^2}{4} + 5 + 1\cdot 2\\
S_{v_{\frac{n-4}{2}}} &= 2\Big(1 + 2 + \cdots + \frac{n-6}{2}\Big) + \frac{5n}{2} + 5\\
&= \frac{n^2}{4} + 5 + 2\cdot 3\\
\cdots &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
S_{v_{\frac{n-(n-4)}{2}}} &= 2\Big(1 + 2 + \cdots + \frac{n-(n-2)}{2}\Big) + \frac{(n-3)n}{2} + 3
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_3 &= 4\Bigg[S_{v_{\frac{n}{2}}} + S_{v_{\frac{n-2}{2}}} + \cdots + S_{v_{\frac{n-(n-4)}{2}}}\Bigg]\\
&= 4\Bigg[\frac{n^2}{4} + 5 + \frac{n^2}{4} + 5 + 1\cdot 2 + \cdots + \frac{n^2}{4} + 5 + \Big(\frac{n-6}{2}\Big)\Big(\frac{n-4}{2}\Big) + \frac{n^2 - 3n + 10}{2}\Bigg]\\
&= 4\Bigg[\Big(\frac{n^2}{4} + 5\Big)\Big(\frac{n-4}{2}\Big) + \frac{n^2 - 3n + 10}{2} + \sum_{k=1}^{\frac{n-6}{2}} k(k+1)\Bigg]\\
&= \frac{n^3 + 8n - 40}{2} + \frac{n^3 - 12n^2 + 44n - 48}{6}\\
&= \frac{4n^3 - 12n^2 + 68n - 168}{6}
\end{aligned}$$

Therefore,

$$V_c[P_2 \times P_n] = \frac{1}{2}[S_2 + S_3]$$

$$\begin{aligned}
&= \frac{1}{2}\left[2n^2 - 2n + 4 + \frac{4n^3 - 12n^2 + 68n - 168}{6}\right] \\
&= \frac{4n^3 - 6n^2 - 4n + 24}{6} \\
&= \frac{4n^3 + 56n - 144}{12}
\end{aligned}$$

This completes the proof. □

Theorem 5.2.23. *For* $m, n > 2$*,*

$$V_c[P_m \times P_n] = W[P_m \times P_n] - [2m^2n + 2mn^2 - 4mn - 2m^2 - 2n^2 + 4].$$

Proof. In $P_m \times P_n$, there are 4 vertices of degree 2, $2(m+n-4)$ vertices of degree 3 and $(m-2)(n-2)$ vertices of degree 4. Every vertex of degree 4 is vertex-connected with all other vertices of $P_m \times P_n$.

Let us first consider the vertices of degree 2 in $P_m \times P_n$. A vertex of degree 2 is vertex-connected with all other vertices except those $(m+n-3)$ border vertices which does not belong to its row or column. Therefore, sum of all distance between the vertices which are not vertex-connected with degree 2 vertices is given by

$$\begin{aligned}
S_2' &= 4\left[(n + \ldots + n + m - 3) + (m + \ldots + m + n - 3) + (m + n - 2)\right] \\
&= 4\left[\frac{m-2}{2}(2n + m - 3) + \frac{n-2}{2}(2m + n - 3) + m + n - 2\right] \\
&= 8mn + 2m^2 + 2n^2 - 14m - 14n + 16.
\end{aligned}$$

The vertices of degree 3 in the first and m^{th} columns are vertex-connected with all other vertices except the border vertices of first and n^{th} rows which does not belong to their respective columns. Similarly, the vertices of degree 3 in the first and n^{th} rows are vertex-connected with all other vertices except the border vertices of first and m^{th} columns which does not belong to their respective rows. Now we consider 4 cases.

Case 1: m and n are even.

The sum of all distance between the vertices which are not vertex-connected with degree 3 vertices is given by

$$S^{'}_{31} = 4\left[\sum_{j=2}^{\frac{n}{2}}\sum_{i=j}^{j+m-2} i + \sum_{j=\frac{n}{2}+1}^{n-1}\sum_{i=j}^{j+m-2} i + \sum_{j=2}^{\frac{m}{2}}\sum_{i=j}^{j+n-2} i + \sum_{j=\frac{m}{2}+1}^{m-1}\sum_{i=j}^{j+n-2} i\right].$$

Observe that,

$$\sum_{j=2}^{\frac{n}{2}}\sum_{i=j}^{j+m-2} i = \frac{(m-1)(n-2)(2m+n)}{8},$$

$$\sum_{j=2}^{\frac{m}{2}}\sum_{i=j}^{j+n-2} i = \frac{(n-1)(m-2)(2n+m)}{8},$$

$$\sum_{j=\frac{n}{2}+1}^{n-1}\sum_{i=j}^{j+m-2} i = \frac{(m-1)(n-2)(2m+3n-4)}{8},$$

$$\sum_{j=\frac{m}{2}+1}^{m-1}\sum_{i=j}^{j+n-2} i = \frac{(n-1)(m-2)(2n+3m-4)}{8}.$$

Therefore,

$$\begin{aligned} S^{'}_{31} &= 4[\frac{(m-1)(n-2)}{8}(4m+4n-4) + \frac{(n-1)(m-2)}{8}(4m+4n-4)] \\ &= 2(m+n-1)(2mn-3m-3n+4) \\ &= 4m^2n + 4n^2m - 16mn - 6m^2 - 6n^2 + 14m + 14n - 8. \end{aligned}$$

Case 2: m is odd and n is even

The sum of all distance between the vertices which are not vertex-connected with degree 3 vertices is given by

$$S^{'}_{32} = 4\left[\sum_{j=2}^{\frac{n}{2}}\sum_{i=j}^{j+m-2} i + \sum_{j=\frac{n}{2}+1}^{n-1}\sum_{i=j}^{j+m-2} i + \sum_{j=2}^{\frac{m-1}{2}}\sum_{i=j}^{j+n-2} i\right.$$

$$+\sum_{j=\frac{m+1}{2}+1}^{m-1}\sum_{i=j}^{j+n-2} i+\sum_{i=\frac{m+1}{2}}^{\frac{m+1}{2}+n-2} i\Bigg].$$

We have,

$$\sum_{j=2}^{\frac{m-1}{2}}\sum_{i=j}^{j+n-2} i=\frac{(n-1)(m-3)(2n+m-1)}{8}.$$

$$\sum_{j=\frac{m+1}{2}+1}^{m-1}\sum_{i=j}^{j+n-2} i=\frac{(n-1)(m-3)(2n+3m-3)}{8}$$

$$\sum_{i=\frac{m+1}{2}}^{\frac{m+1}{2}+n-2} i=\frac{(m+n-1)(n-1)}{2}.$$

Therefore,

$$S'_{32}=4m^2n+4mn^2-16mn-6m^2-6n^2+14m+14n-8.$$

Case 3: m is even and n is odd

The sum of all distance between the vertices which are not vertex-connected with degree 3 vertices is given by

$$S'_{33}=4\Bigg[\sum_{j=2}^{\frac{m}{2}}\sum_{i=j}^{j+n-2} i+\sum_{j=\frac{m}{2}+1}^{m-1}\sum_{i=j}^{j+n-2} i+\sum_{j=2}^{\frac{n-1}{2}}\sum_{i=j}^{j+m-2} i$$
$$+\sum_{j=\frac{n+1}{2}+1}^{n-1}\sum_{i=j}^{j+m-2} i+\sum_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+m-2} i\Bigg].$$

Therefore, similar to case 2, we obtain

$$S'_{33}=4n^2m+4nm^2-16mn-6m^2-6n^2+14m+14n-8.$$

Case 4: m and n are odd

The sum of all distance between the vertices which are not vertex-connected with degree 3 vertices is given by

$$S'_{34}=4\Bigg[\sum_{j=2}^{\frac{m-1}{2}}\sum_{i=j}^{j+n-2} i+\sum_{j=\frac{m+1}{2}+1}^{m-1}\sum_{i=j}^{j+n-2} i++\sum_{j=2}^{\frac{n-1}{2}}\sum_{i=j}^{j+m-2} i$$

$$+\sum_{j=\frac{n+1}{2}+1}^{n-1}\sum_{i=j}^{j+m-2} i+\sum_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+m-2} i+\sum_{i=\frac{m+1}{2}}^{\frac{m+1}{2}+n-2} i\Bigg].$$

$$= 2(m+n-1)[(n-1)(m-3)+(m-1)(n-3)+n-1+m-1]$$

$$= 2(m+n-1)[2mn-3m-3n+4]$$

$$= 4m^2n+4mn^2-16mn-6m^2-6n^2+14m+14n-8.$$

That is, in all the above cases, the sum of all distances between the vertices which are not vertex-connected with degree 3 vertices is the same and is given by

$$S_3^{'} = 4m^2n+4mn^2-16mn-6m^2-6n^2+14m+14n-8.$$

Therefore, the sum of all distances between the vertices which are not vertex-connected with either degree 2 vertices or degree 3 vertices is given by

$$S_2^{'}+S_3^{'} = 4m^2n+4mn^2-8mn-4m^2-4n^2+8.$$

Hence,

$$\begin{aligned} V_c[P_m\times P_n] &= W[P_m\times P_n]-\frac{1}{2}[S_2^{'}+S_3^{'}] \\ &= W[P_m\times P_n]-\frac{1}{2}[4m^2n+4mn^2-8mn-4m^2-4n^2+8] \\ &= W[P_m\times P_n]-[2m^2n+2mn^2-4mn-2m^2-2n^2+4]. \end{aligned}$$

This completes the proof. □

Chapter 6

VERTEX CONNECTIVITY POLYNOMIAL OF GRAPHS

This chapter is divided into three sections. In the first section, the motivation which lead to introduce the idea of vertex connectivity polynomial is described. In the following section, the concept of vertex connectivity polynomial is defined and studied in a detailed manner by evaluating it for a wide variety of graphs. Also, some important results are obtained. The concluding section of this chapter deals about the roots and stability properties of the vertex connectivity polynomial and the same is studied in detail.

6.1 Introduction

Chemical graph theory [45] is a branch of mathematical chemistry in which the connectivity in a chemical graph is analysed. Chemical graph serves as a convenient model for any real or abstract chemical system and topological index is a number that can be used to characterize the graph of a molecule. The first distance-based topological index was the Wiener index introduced by Wiener in 1947 [47]. Later, in 1988, Hosoya [25] introduced the Wiener polynomial as a counting polynomial, which is strongly connected to the Wiener index. Nowadays, the Wiener polynomial is known by the name Hosoya polynomial. Thus, the introduction of Hosoya polynomial has turned out to be a gateway to explore the interconnections between topological indices and graph polynomials. This served as a motivation to introduce the vertex connectivity polynomial of connected graphs, which in fact reveals both the metric and connectedness properties of vertex pairs in it.

6.2 Vertex Connectivity Polynomial of Graphs

Definition 6.2.1. Let $G = (V, E)$ be a graph of order n. Then, for any pair of distinct vertices $(v, w) \in V \times V$, let $g_i(v, w)$ be the set of all geodesics linking v and w such that the deletion of all its edges disconnects the graph into i components, where $i \in \{1, 2, \dots, \text{diam(G)+1}\}$. In fact,

$$\bigcup_{i=1}^{\text{diam(G)+1}} g_i(v, w) = g(v, w),$$

where $g(v, w)$ is the set of all geodesics linking the vertices v and w in G.

Define for any pair of distinct vertices $(v, w) \in V \times V$,

$$n(v, w) = \max\{i : p(v, w) \in g_i(v, w)\},$$

where $p(v, w)$ is a geodesic connecting v and w.

Definition 6.2.2. Let $G = (V, E)$ be a graph on n vertices. Then, for $i \in \{1, 2, \ldots, \text{diam(G)+1}\}$, the *$i$-disconnectivity set D_i* of G consisting of unordered vertex pairs in G is defined as:

$$D_i = \{(v, w) \in V \times V : v \neq w, n(v, w) = i.\}$$

For vertices $v, w \in V$, if $(v, w) \in D_i$ for some $i > 1$, then we say that the vertex pair (v, w) disconnects the graph G into i components.

If $D_i = \emptyset$ for some $i > 1$, then $|D_j| = 0 \ \forall j = i, i + 1, \ldots, \text{diam(G)+1}$.

Definition 6.2.3. Let G be a graph of order n. Then, the *vertex connectivity polynomial* of G , denoted by $V[G; x]$ is defined as:

$$V[G; x] = \sum_{i=1}^{\text{diam(G)+1}} |D_i| x^i,$$

The following properties of $V[G; x]$ can be easily observed:

(i) $V[G; x]$ is a polynomial of degree at most diam(G)+1.

(ii) $V[G; 1] = \sum_{i=1}^{\text{diam(G)+1}} |D_i| = \binom{n}{2}$

(iii) $V[G; x]$ is a polynomial with each coefficient a positive integer less than or equal to $\binom{n}{2}$.

(iv) If G is a tree, then $V[G; x]$ will have no linear coefficient and the coefficient of x^2 will be the number of non-zero entries in the adjacency matrix of G.

(v) If $V[G;x]$ is a linear polynomial, then G cannot be disconnected by any pair of its vertices.

Theorem 6.2.4. *For the complete graph K_n,*

$$V[K_n;x] = \binom{n}{2}x, \text{ where } n \geq 3.$$

Proof. We know that any two vertices in K_n are adjacent and since every edge is a part of some cycle, the deletion of a single edge could not disconnect the graph. Thus all the $\binom{n}{2}$ vertex pairs belongs to D_1 and hence D_i is empty $\forall i > 1$.
This completes the proof. □

Theorem 6.2.5. *If C_n is the cycle graph on n vertices, then for $n \geq 3$*

$$V[C_n;x] = \begin{cases} \sum_{i=1}^{\frac{n-1}{2}} nx^i, & \text{if } n \text{ is odd,} \\ \sum_{i=1}^{\frac{n}{2}-1} nx^i + \frac{n}{2}x^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Observe that the diameter of C_n is $\lfloor \frac{n}{2} \rfloor$ if n is odd and is $\frac{n}{2}$ otherwise. Since adjacent vertices of C_n do not disconnect it and the vertex pairs separated by a distance i, where $i > 1$ disconnect the graph into i components, the proof of the theorem follows easily. □

Theorem 6.2.6. *For the friendship graph F_n,*

$$V[F_n;x] = \binom{2n+1}{2}x, \text{ where } n > 1.$$

Proof. Since the vertices in the same copy of C_3 are adjacent and those in different copies are separated by a path of length 2, none of the vertex pairs disconnects the graph. Therefore, $|D_1| = \binom{2n+1}{2}$.
This completes the proof. □

Theorem 6.2.7. *Let G be a tree on n vertices. Then, for $i \geq 1$, the coefficient of x^{i+1} in $V[G;x]$ is precisely the number of paths of length i in the graph G.*

Proof. Since G is a tree, every edge is a cut edge so that if any two vertices of G are linked by a path of length i, then the deletion of its edges makes the graph disconnected into $i+1$ components.

This completes the proof. □

Corollary 6.2.8. *For the path graph P_n,*

$$V[P_n;x] = \sum_{i=1}^{n-1}(n-i)x^{i+1}, \text{ where } n \geq 2.$$

Proof. Observe that in P_n, there are $n-i$ paths of length i for $i = 1, 2, \ldots, n-1$ and hence $n-i$ vertex pairs belongs to D_{i+1} for $i = 1, 2, \ldots, n-1$. □

Corollary 6.2.9. *For the star graph $K_{1,n}$,*

$$V[K_{1,n};x] = \binom{n}{2}x^3 + nx^2, \text{ where } n \geq 2.$$

Proof. In $K_{1,n}$, there are n pairs of adjacent vertices and $\binom{n}{2}$ paths of length 2. Hence by theorem 6.2.7 , $V[K_{1,n};x] = \binom{n}{2}\ x^3 + nx^2$. □

Corollary 6.2.10. *For the bistar graph $B_{m,n}$,*

$$V[B_{m,n};x] = mnx^4 + \frac{m(m+1)+n(n+1)}{2}x^3 + (m+n+1)x^2, \text{ where } m,n \geq 2.$$

Proof. Since a bistar graph is a tree, D_1 is empty. Note that in $B_{m,n}$, there are $m+n+1$ cut edges and hence $m+n+1$ vertex pairs belongs to D_2. Since the vertex pairs sharing a single common neighbor disconnects the graph into three components, cardinality of D_3 is $\binom{m}{2} + \binom{n}{2} + m + n$. Also, the paths of length

3 disconnects the graph into 4 components and since there are mn such paths, cardinality of D_4 is mn. Since diameter of $B_{m,n}$ is 3, D_i is empty $\forall i > 4$.
This completes the proof. $\square$

Theorem 6.2.11. *For the lollipop graph $L_{m,n}$,*

$$V[L_{m,n};x] = \sum_{i=2}^{n+1}(m+n-i+1)x^i + \binom{m}{2}x, \text{ where } m \geq 3 \text{ and } n \geq 1.$$

Proof. Observe that in $L_{m,n}$, there are $\binom{m}{2}$ vertex pairs as part of K_m and hence belongs to D_1. Let v be the vertex of K_m attached to the bridge. Then, for $i = 1, 2 \ldots, n$, the vertex of P_n at a distance i from v contributes m vertex pairs to D_{i+1} and one vertex pair each to D_j for $j = 2, 3, \ldots, i$. Thus, we get there are $m+n-i+1$ vertex pairs belonging to D_i, where $2 \leq i \leq n+1$.
This completes the proof. $\square$

Theorem 6.2.12. *For the complete bipartite graph $K_{m,n}$, $m, n \geq 2, m \leq n$;*

$$V[K_{m,n};x] = \begin{cases} 4x + 2x^2, & \text{if } m,n = 2, \\ [\binom{n+2}{2} - 1]x + x^2, & \text{if } m = 2, n > 2, \\ \binom{m+n}{2}x, & \text{if } m > 2. \end{cases}$$

Proof. Let M and N be the bipartite sets of vertices of $K_{m,n}$ with cardinalities m and $n, m \leq n$ respectively. We consider three cases:

Case(i) $m, n = 2$

$K_{2,2}$ is C_4 and hence by theorem 6.2.5, $V[K_{2,2};x] = 4x + 2x^2$

Case(ii) $m = 2, n > 2$

In $K_{2,n}, n > 2$, there are $2n$ pairs of adjacent vertices which does not disconnect the graph. Since degree of each vertex in the set M is $n > 2$, the $\binom{n}{2}$ vertex pairs of N also does not disconnect $K_{2,n}$. Since the vertices of N are of degree 2, the remaining vertex pair of M disconnects the graph into exactly two components by isolating one of the vertices of N. Thus, there are $\binom{n+2}{2}-1$ vertex pairs belonging to D_1 and a single vertex pair in D_2.

Case(iii) $m > 2$

In this case, all the vertices are having degree at least 3 and the distance between any two vertices is at most 2 so that none of the $\binom{m+n}{2}$ vertex pairs could disconnect the graph.

This completes the proof. □

Theorem 6.2.13. *Let $k > 2$. Then for $n_1, n_2, \ldots n_k \geq 2$,*

$$V[K_{n_1,n_2,\ldots,n_k}; x] = \binom{\sum_{i=1}^{k} n_i}{2} x, \text{ where } n_i \leq n_{i+1}, 1 \leq i \leq k-1.$$

Proof. Let $N_1, N_2, \cdots N_k$ be the partite sets of vertices of $K_{n_1,n_2,\cdots,n_k}$ with cardinalities $n_1, n_2, \cdots, n_k$ respectively. Since $n_i \geq 2$ for all i and since the degree of a vertex in N_i is $\sum_{j \neq i} n_j$, it is trivial that the degree of every vertex of $K_{n_1,n_2,\ldots,n_k}$ is at least 4. It can be observed that every edge of the required graph is part of some cycle and any two vertices are connected by a path of length at most 2 so that no vertex pair could disconnect the graph by isolating vertices.

This completes the proof. □

Theorem 6.2.14. *Let G be a graph of order n and $(v, w) \in V \times V$. Let $p(v, w)$ be a geodesic having at least one intermediate vertex of degree two. Then, the vertex* *pair (v, w) disconnects the graph G. Moreover, if $N_{vw}(p)$ denotes the number of intermediate vertices of degree two in the path $p(v, w)$, then*

$$\omega(G_{vw}) \geq 1 + N_{vw}(p),$$

where $\omega(G_{vw})$ is the number of components to which the graph G has been disconnected by the vertex pair (v, w).

Proof. If the vertices are adjacent, then the result holds trivially because of the absence of intermediate vertices between them. Now let (v, w) be a non-adjacent vertex pair in G and $p(v, w)$ be a geodesic in G connecting them. If u is an intermediate vertex of degree two in $p(v, w)$, then the deletion of all edges of $p(v, w)$ makes the vertex u isolated and hence the graph becomes disconnected. Thus, it can be concluded that, on the deletion of the edges of the path $p(v, w)$, each intermediate vertex of degree two in $p(v, w)$ increases the number of components of G exactly by one and hence $\omega(G_{vw}) \geq 1 + N_{vw}(p)$ holds.
This completes the proof. □

Theorem 6.2.15. *Let G_1, G_2 be two disjoint graphs with vertex sets $\{v_1, \dots v_n\}$ and $\{u_1, \dots u_m\}$ respectively and let G be the graph obtained by joining the vertex v_1 of G_1 with u_1 of G_2 by an edge e. Then,*

$$V[G; x] = V[G_1; x] + V[G_2; x] + [|C_{1i}| \cdot |C_{2j}|]x^{i+j} + |C_{1i}|x^{i+1} + |C_{2j}|x^{j+1} + x^2,$$

where $C_{1i} = \{v_k : (v_1, v_k)$ disconnects the graph G_1 into i components$\}$ and $C_{2j} = \{u_k : (u_1, u_k)$ disconnects the graph G_2 into j components$\}$, where $i = 1, 2, \dots, diam(G_1) + 1$ and $j = 1, 2, \dots, diam(G_2) + 1$.

Proof. It is obvious that (v_i, v_j) and (u_k, u_l) disconnects G in the same manner as it is in G_1 and G_2 for $i, j = 1, 2, \dots n, i < j$ and $k, l = 1, 2, \dots m, k < l$. Since

v_1 and u_1 are adjacent, (v_l, u_k) disconnects the graph into $i+j$ components, and hence belongs to D_{i+j} in G, where $(v_1, v_l) \in D_i$ in G_1 and $(u_1, u_k) \in D_j$ in G_2 for $l = 2, 3, \ldots n$ and $k = 2, 3, \ldots m$. It can also be observed that if $v_k \in C_{1i}$ for some $k = 2, 3, \ldots n$, then (v_k, u_1) disconnects G into $i+1$ components and hence those $|C_{1i}|$ vertex pairs belongs to D_{i+1}, $i = 1, 2, \ldots$,diam(G_1)+1. Similar argument follows for the vertices belonging to C_{2j} also, where $j = 1, 2, \ldots$,diam(G_2)+1. This completes the proof. □

Corollary 6.2.16. *For the n-barbell graph* $B_{n,1}$,

$$V[B_{n,1}; x] = n^2x^2 + n(n-1)x, \ n \geq 3.$$

Proof. We have, $V[K_n; x] = \binom{n}{2}x$ and since any two vertices of K_n are adjacent, $|C_{11}| = |C_{21}| = n - 1$. Therefore, it follows from theorem 6.2.15 that,

$$V[B_{n,1}; x] = n^2x^2 + n(n-1)x.$$

□

Theorem 6.2.17. *Let* G_1 *and* G_2 *be two graphs with vertices* $\{v_1, \ldots v_n\}$ *and* $\{u_1, \ldots u_m\}$ *respectively, where* $n, m \geq 2$. *Then,*

$$V[G_1 \vee G_2; x] = \binom{n+m}{2}x.$$

Proof. Since all the n(or m) vertices of G_1(or G_2) are adjacent to every vertex of G_2(or G_1) and since G_2(or G_1) is connected, there exists at least two paths, one within G_2(or G_1) and the other traversing through the vertices of G_1(or G_2) such that no vertex pair of G_2(or G_1) can disconnect $G_1 \vee G_2$. Similar argument follows for the vertex pairs (v_i, u_j) also, where $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots m$.

Thus, in short, no vertex pair of $G_1 \vee G_2$ could disconnect it.

This completes the proof. □

Corollary 6.2.18. *For the wheel graph W_n and shell graph S_n, where $n > 3$,*

$$V[W_n; x] = V[S_n; x] = \tbinom{n}{2} x.$$

Corollary 6.2.19. *For the helm H_n where $n > 3$, we have*

$$V[H_n; x] = \binom{n-1}{2} x^3 + n(n-1)x^2 + \binom{n}{2} x.$$

Proof. Since none of the vertex pairs of W_n disconnects it, by corollary 6.2.18, each of the $n-1$ pendent vertices of H_n with every vertex on W_n disconnects the graph into exactly two components and the $\binom{n-1}{2}$ pendent vertex pairs disconnects the graph into three components. □

Corollary 6.2.20. *For the web graph WB_n where $n > 3$, we have*

$$V[WB_n; x] = \tbinom{n-1}{2} x^3 + (2n-1)(n-1)x^2 + \tbinom{2n-1}{2} x.$$

Proof. Here, it can be observed that each of the pendent vertices of WB_n along with the vertices of H_n disconnects the graph into two components and any two pendent vertices together disconnects WB_n into three components by isolating themselves. Also, all the remaining vertex pairs do not disconnect the graph due to the existence of multiple paths connecting them. □

Corollary 6.2.21. *For the bow graph BW_N where $N > 4$,*

$$V[BW_N; x] = \tbinom{N}{2} x.$$

Proof. Let the bow graph BW_N include the shells S_m and S_n. By corollary 6.2.18, it is evident that no vertex pair in S_m and S_n disconnects the respective graphs so that it follows that there doesn′t exist a vertex pair in BW_N which makes the graph disconnected. □

Corollary 6.2.22. *For the the butterfly graph BF_n with $n > 4$ vertices,*

$$V[BF_n;x] = x^3 + (2n-4)x^2 + \binom{n-2}{2}x.$$

Proof. Here, the pendent vertices along with any of the vertex of the underlying bow graph disconnects BF_n into two components and both the pendent vertices together disconnects the graph into three components. The result now follows directly from corollary 6.2.21 . □

Theorem 6.2.23. *Let G and $G \circ K_1$ be as defined above. Then $(v_j, u_j) \in D_2$, $\forall j = 1, 2, \ldots, n$. If $(v_j, v_k) \in D_i$, then $(v_j, u_k) \in D_{i+1}$ and $(u_j, u_k) \in D_{i+2}$ $\forall j, k = 1, 2, \ldots, n, j < k$ and $\forall i = 1, 2, \ldots,$diam(G).*

Proof. By the very construction of $G \circ K_1$, it is clear that v_j and u_j are linked by a cut edge $\forall j = 1, 2, \ldots, n$ and hence $(v_j, u_j) \in D_2$. Now suppose that $(v_j, v_k) \in D_i$, where $j, k \in \{1, 2, \ldots, n\}$ and $i \in \{1, 2, \ldots,\text{diam}(G)\}$. Since (v_j, u_j) and (v_k, u_k) together disconnects $G \circ K_1$ into three components, in which one of the components is G and since (v_j, v_k) disconnects G into i components, it can be observed that $(v_j, u_k) \in D_{i+1}$ and $(u_j, u_k) \in D_{i+2}$.

This completes the proof. □

Corollary 6.2.24. *For $n \geq 3$, we have*

$$V[P_n \circ K_1;x] = \sum_{i=3}^{n-1}(3i-2)x^{n+3-i} + x^{n+2} + 4x^{n+1} + 2(n-1)x^3 + nx^2 + V[P_n;x].$$

Proof. Let $\{v_1, v_2, \dots v_n, u_1, u_2, \dots u_n\}$ be the vertices of $P_n \circ K_1$ taken in order. Then, we know that $(v_j, v_k) \in D_{k-j+1}$ for $j, k \in \{1, 2, \dots n\}, j < k$. Then, by theorem 6.2.23, $(v_j, u_j) \in D_2$, $(v_j, u_k) \in D_{k-j+2}$ and $(u_j, u_k) \in D_{k-j+3}$, where $j, k \in \{1, 2, \dots n\}, j < k$. From this it can be observed that,
$|D_i| = |D_i|$ in $P_n + (i-1)(n-i+2)$ for $i = 2, 3$ and
$|D_i| = |D_i|$ in $P_n + n - i + 3 + 2(n-i+2)$ for $i = 4, 5, \dots, n$. Also, since $P_n \circ K_1$ is a tree with diameter $n+1$, the number of paths of length $n+i$ for $i = 0, 1$ in it can be easily calculated as 4 and 1 respectively. ☐

Corollary 6.2.25. *For $n \geq 3$, we have*

$$V[C_n \circ K_1; x] = \begin{cases} V[C_n; x] + 2n\sum_{j=3}^{[\frac{n}{2}]+1} x^j + n\sum_{j=3}^{[\frac{n}{2}]+2} x^j + 3nx^2, & \textit{if } n \textit{ is odd.} \\ (1+x^2)V[C_n; x] + 2n\sum_{j=3}^{[\frac{n}{2}]} x^j + +nx^{[\frac{n}{2}]+1} + 3nx^2, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Let $\{v_1, v_2, \dots v_n, u_1, u_2, \dots u_n\}$ be the vertices of $C_n \circ K_1$ taken in order. It is trivial that all the vertexpairs as part of C_n disconnects $C_n \circ K_1$ in the same way as it is in C_n. It can be easily inferred that $(v_j, v_k) \in D_i$ if $k - j = i$ or $n - i$ for $j, k \in \{1, 2, \dots n\}, j < k$ and $i = 1, 2, \dots [\frac{n}{2}]$. Thus if n is odd, corresponding to each u_k, where $k = 1, 2, \dots, n$, there exists 3 vertices as part of C_n such that u_k with those vertices disconnects the graph into two components and 2 vertices each as part of C_n such that those vertex pairs disconnects the graph into i components, where $i = 3, \dots, [\frac{n}{2}] + 1$. Now, in the case when n is even, each u_k along with three of the vertices of C_n disconnects the graph into two components separately and there exists 2 vertices each as part of C_n such that each of those vertices along with u_k makes the graph disconnected into i components, where $i = 3, \dots, [\frac{n}{2}], k = 1, 2, \dots, n$. Also, there exists a unique vertex of C_n such that u_k along with that vertex disconnects the graph into $[\frac{n}{2}] + 1$ components. Hence in both cases, it can also be concluded that if $(u_j, v_k) \in D_i$, then $(u_j, u_k) \in D_{i+1}$

for $i = 2, \dots, [\frac{n}{2}] + 2$. From these inferences, cardinality of $D_i^{'}s$ of $C_n \circ K_1$ can be calculated directly. □

6.3 Roots and Stability of Vertex Connectivity Polynomial

The study of a graph polynomial is worthy only if it succeeds in predicting the behavior of some stable physical systems. Thus, the introduction of the concept of vertex connectivity polynomial of graphs triggered the need to study the nature of its roots as well as the stability properties of the same for various graph classes. In this section, certain results about the nature of roots, stability and schur stability of the vertex connectivity polynomial are discussed.

Definition 6.3.1. Let G be a graph. Then, the roots of the vertex connectivity polynomial $V[G; x]$ are called the vertex connectivity roots of G.

The following are some of the observations about the vertex connectivity roots of graphs.

(i) $(0,\infty)$ is a zero-free interval of the vertex connectivity polynomial $V[G; x]$ for any graph G.

(ii) If G is a tree, then it has $diam(G) + 1$ vertex connectivity roots counting multiplicities.

Theorem 6.3.2. *Let G be a graph. Then, zero is a vertex connectivity root of G of multiplicity greater than one iff G is a tree.*

Proof. Assume that zero is a vertex connectivity root of G of multiplicity $k > 1$. Then, $V[G; x] = x^k g(x)$, a polynomial of degree greater than or equal to 2. This means that every vertex pair of G disconnects the graph into at least two components so that none of its edges belongs to a cycle. Hence, it follows that G is a tree. Conversely, if G is a tree, all its edges being cut edges, none of the vertex pairs are vertex-connected. That is, $V[G; x]$ is a polynomial lacking a linear coefficient.

This completes the proof. □

Theorem 6.3.3. *Let G be a graph of order n. Then, zero is the only vertex connectivity root of G iff either G is vertex-connected or every vertex pair of G disconnects the graph into exactly two components.*

Proof. Assume that zero is the only vertex connectivity root of G. Then, $V[G; x] = rx^k$, where $r, k > 0$. But, $V[G; 1] = \sum_{i=1}^{\text{diam(G)}+1} |D_i| = \binom{n}{2}$ so that $r = \binom{n}{2}$. That is, $V[G; x] = \binom{n}{2}x^k$.

Case (i): If $k = 1$, then $V[G; x] = \binom{n}{2}x$ so that G is vertex-connected.

Case (ii): If $k > 1$, then all the $\binom{n}{2}$ vertex pairs disconnects the graph into k components. But if a vertex pair disconnects G into k components, then corresponding to that there exists a vertex pair disconnecting the graph into two components. Thus, it follows that $V[G; x] = \binom{n}{2}x^2$.

Conversely, if either G is vertex-connected or every vertex pair of G disconnects the graph into exactly two components, then we get $V[G; x]$ as $\binom{n}{2}x$ and $\binom{n}{2}x^2$ respectively. In both the cases, it follows that zero is the only vertex connectivity root of G .

This completes the proof. □

Corollary 6.3.4. *Let G be a non-vertex-connected graph of order n. Then, zero is the only vertex connectivity root of G iff $G = P_2$.*

Proof. Let zero be the only vertex connectivity root of G. Since G is not vertex-connected, from theorem 6.3.3 it follows that G is a tree and $V[G; x] = \binom{n}{2}x^2$. This means that all the $\binom{n}{2}$ vertex pairs disconnects G into two components. Now if possible assume $n > 2$. Then, since every edge of G is a cut edge, we can find a vertex pair at 2 units distance which disconnects G into three components. Therefore, $n = 2$ and $G = P_2$.

This completes the proof. □

Theorem 6.3.5. *Let G be a graph and n_1, n_2 are respectively be the number of vertex-connected vertex pairs and the number of vertex pairs which disconnects G, where $n_1 > n_2$. If n_1 is prime, then G lacks non-zero real vertex connectivity roots.*

Proof. Let $V[G; x] = xg(x)$, where $g(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + n_1$, $\sum_{i=1}^{n} a_i = n_2$, for some n. Clearly, all the zeros of $g(x)$ are non-zero. Let β be a zero of $g(x)$ such that $|\beta| \leq 1$. Then,

$$g(\beta) = a_n\beta^n + a_{n-1}\beta^{n-1} + \dots a_1\beta + n_1 = 0.$$

That is,

$$|n_1| = |a_n\beta^n + a_{n-1}\beta^{n-1} + \dots a_1\beta| \leq |a_n| + |a_{n-1}| + \dots |a_1| = n_2,$$

a contradiction. Therefore, $|\beta| > 1$.

Now, if possible, assume that β is real. Then, $g(x)$ is reducible over the set of all real numbers. That is,

$$g(x) = (x - \beta)h(x),$$

where $h(x)$ is a non constant integer polynomial.

Now,

$$n_1 = g(0) = -\beta h(0).$$

Since n_1 is prime and since $|\beta| > 1$, $|h(0)|$ must be 1. Let $\beta_1, \ldots, \beta_{n-1}$ be the zeros of $h(x)$ and a be its leading coefficient. Then,

$$|\beta_1 \cdots \beta_{n-1}| = \frac{1}{|a|} \leq 1.$$

But $\beta_1, \ldots, \beta_{n-1}$ are the zeros of $g(x)$ also so that $|\beta_i| > 1 \; \forall i = 1, \ldots, n-1$, a contradiction. Therefore, $g(x)$ has no real zeros and hence G lacks non-zero real vertex connectivity roots.

This completes the proof. □

Corollary 6.3.6. *Let G be a graph and n_1, n_2 be respectively the number of vertex-connected vertex pairs and the number of vertex pairs which disconnects G, where $n_1 > n_2$. Then, all the non-zero vertex connectivity roots of G lie strictly outside the unit circle.*

Theorem 6.3.7. *Let G be a graph with $|D_i| = m \; \forall i \leq n$, where $2 \leq n \leq diam(G) + 1$. Then, all the non-zero vertex connectivity roots of G are distinct and are of unit modulus. In addition, if G is a tree of diameter p, where p is an odd prime, then G lacks non-zero real vertex connectivity roots.*

Proof. Without loss of generality, let G be a graph which is not a tree.

Since $|D_i| = m$, $\forall\ i = 1, 2, \ldots, n$,

$$V[G; x] = m\sum_{i=1}^{n} x^i = mx[1 + x + \ldots + x^{n-1}].$$

The zeros of the polynomial $1 + x + \cdots + x^{n-1}$ are the n^{th} roots of unity except 1 and hence it follows that the vertex connectivity roots of G are distinct. Now, if G is a tree, then $V[G; x]$ is a polynomial of degree $diam(G) + 1$. That is,

$$V[G, x] = m \sum_{i=2}^{diam(G)+1} x^i = mx^2[1 + x + \cdots + x^{diam(G)-1}].$$

Since $diam(G) = p$, the polynomial $1 + x + \cdots + x^{p-1}$ is irreducible over the set of all rationals and hence it has no rational roots. But, the only possible real $p - 1^{th}$ root of unity is -1, which is rational. Therefore, it can be concluded that G is free of non-zero real vertex connectivity roots.

This completes the proof. □

Corollary 6.3.8. *If n is odd, then all the non-zero vertex connectivity roots of C_n lies on the unit circle. In addition, if $\frac{n-1}{2}$ is even, then C_n has exactly two real vertex connectivity roots.*

Proof. This follows from the fact that $V[C_n; x] = \sum_{i=1}^{\frac{n-1}{2}} nx^i$ for odd n and -1 is a k^{th} root of unity if k is even. □

Theorem 6.3.9. (i) *Let G_1 and G_2 be two disjoint graphs. If a is a vertex connectivity root of both G_1 and G_2, then a is also a vertex connectivity root of $G_1 \vee G_2$.*

(ii) *Let G be a vertex-connected graph of order n and let G' be the graph obtained by adjoining one pendent vertex to m distinct vertices of $G, m \leq n$. Then,* *the vertex connectivity roots of G' are the roots of the polynomial*

$$V[G;x] + nmx^2 + \binom{m}{2}x^3.$$

.

Proof. (i) This follows from the fact that $V[G_1 \vee G_2; x] = V[G_1;x] + V[G_2;x]$.

(ii) Since G is vertex-connected, the newly adjoined vertices along with every vertex of G disconnects G' into exactly two components and mutually disconnects G' into three components.

Therefore, $V[G';x] = V[G;x] + nmx^2 + \binom{m}{2}x^3$.

This completes the proof. □

Theorem 6.3.10. *Let G be a graph of order n and diameter d. Then, any vertex connectivity root β of G either has non-positive real part or satisfies*

$$|\beta| < \frac{1 + \sqrt{1 + 2n(n-1)}}{2}.$$

Proof. Let β be a non-zero vertex connectivity root of G. If $|\beta| \leq 1$, then the result holds trivially since $\frac{1+\sqrt{1+2n(n-1)}}{2} > 1$.

Case(i) : Let G be a tree. Then, the degree of $V[G;x]$ is $d+1$.

We have, $V[G;x] = x^2 g(x)$, where $g(x) = a_{d-1}x^{d-1} + \ldots + a_1 x + a_0$. Since $\sum_{i=0}^{d-1} a_i = \binom{n}{2}$, $|a_i| \leq \binom{n}{2} \forall i = 0, 1, \ldots, d-1$. Now, if $|\beta| > 1$ and $Re(\beta) > 0$, then

$$\begin{aligned}\frac{|g(\beta)|}{|\beta^{d-1}|} &= \left| a_{d-1} + \frac{a_{d-2}}{\beta} + \ldots + \frac{a_1}{\beta^{d-2}} + \frac{a_0}{\beta^{d-1}} \right| \\ &\geq \left| a_{d-1} + \frac{a_{d-2}}{\beta} \right| - \frac{n(n-1)}{2} \left[\frac{1}{|\beta^2|} + \ldots + \frac{1}{|\beta^{d-1}|} \right]\end{aligned}$$

$$> Re(a_{d-1}) + a_{d-2}Re\Big(\frac{1}{\beta}\Big) - \frac{n(n-1)}{2}\left[\frac{1}{|\beta^2| - |\beta|}\right]$$

$$\geq 1 - \frac{n(n-1)}{2}\left[\frac{1}{|\beta^2| - |\beta|}\right], \text{since } a_{d-1}, a_{d-2} \geq 1 \textit{ and } Re(\beta) > 0.$$

$$= \frac{|\beta^2| - |\beta| - \frac{n(n-1)}{2}}{|\beta^2| - |\beta|}$$

$$\geq 0,$$

whenever $|\beta^2| - |\beta| - \frac{n(n-1)}{2} \geq 0$.
That is, if $|\beta| \geq \frac{1+\sqrt{1+4\frac{n(n-1)}{2}}}{2} = \frac{1+\sqrt{1+2n(n-1)}}{2}$ and $Re(\beta) > 0$, then $\frac{|g(\beta)|}{|\beta^{d-1}|} > 0$ so that β cannot be a zero of $g(x)$.

Case(ii) Let G be a graph which is not a tree. Then, at least one edge of G is a part of some cycle so that the linear coefficient of $V[G;x]$ is non-zero. Let $V[G;x] = xh(x)$, where $h(x) = a_{m-1}x^{m-1} + \ldots + a_1x + a_0$. Here also, $\sum_{i=0}^{m-1} a_i = \binom{n}{2}, |a_i| \leq \binom{n}{2} \forall i = 0, 1, \ldots, m-1$, so that similar calculation as in case(i) yields that if $|\beta| \geq \frac{1+\sqrt{1+2n(n-1)}}{2}$ and $Re(\beta) > 0$, then β cannot be a zero of $g(x)$. .

Therefore, from the above two cases we get that every vertex connectivity root of the graph G either has non-positive real part or has modulus strictly less than $\frac{1+\sqrt{1+2n(n-1)}}{2}$.

This completes the proof. □

Theorem 6.3.11. *Let G be a tree with diameter $d \leq 2$. Then, all the non-zero vertex connectivity roots of G lie in the left half plane. In addition, if the leading coefficient of $V[G;x]$ dominates, then the polynomial is schur stable.*

Proof. **Case(i):** If $d = 1$, then $G = P_2$ so that $V[G; x] = x^2$. Thus, the result holds trivially.

Case(ii): If $d = 2$, then $V[G; x] = x^2(ax+b)$, a cubic polynomial where $a, b > 0$. Thus, the only non-zero zero $x = \frac{-b}{a}$ of $V[G; x]$ lies in the left half plane. Now, if the leading coefficient of $V[G; x]$ dominates, then $|\frac{b}{a}| < 1$ and thus all the vertex connectivity roots of G lies inside the unit circle. Hence, $V[G; x]$ is schur stable.

This completes the proof. □

Corollary 6.3.12. *For the star graph $K_{1,n}$, $V[K_{1,n}; x]$ is schur stable iff $n > 3$.*

Theorem 6.3.13. *Every vertex connectivity root of the path graph P_n lie in the closed left half plane iff $n \leq 5$*

Proof. We have,

$$V[P_n; x] = \sum_{i=1}^{n-1} (n-i)x^{i+1}.$$

Thus, the result holds trivially for $n = 2$ and $n = 3$.

Now for $n \geq 4$, let $V[P_n; x] = x^2 g(x)$.

When $n = 4$, $g(x) = x^2 + 2x + 3$ so that

$$H_1 = \begin{bmatrix} 2 \end{bmatrix}, H_2 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

are having positive determinants.

If $n = 5$, then $g(x) = x^3 + 2x^2 + 3x + 4$ so that all the Hurwitz matrices

$$H_1 = \begin{bmatrix} 2 \end{bmatrix}, H_2 = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 0 & 5 & 4 \end{bmatrix}$$

have positive determinant.

For $n = 6$, $g(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$.

$$H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

so that $det(H_3) < 0$.

For $n \geq 7$,

$$H_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 6 & 5 & 4 \end{bmatrix}$$

so that $det(H_3) = 0$.

Thus, for $n \geq 6$, $g(x)$ is not stable so that all the vertex connectivity roots of P_n does not belong to the closed left half plane.

This completes the proof. $\square$

Theorem 6.3.14. *The vertex connectivity roots of the bistar graph $B_{m,m}$ always lie in the closed left half plane and are all real iff $m \geq 7$. Also, for all $m \geq 2$, zero is the only vertex connectivity root of multiplicity > 1.*

Proof. We have,

$$V[B_{m,m}; x] = x^2[m^2x^2 + m(m+1)x + (2m+1)].$$

The zeros of the polynomial $g(x) = m^2x^2 + m(m+1)x + (2m+1)$ are

$$\begin{aligned} x &= \frac{-m(m+1) \pm \sqrt{m^2(m+1)^2 - 4m^2(2m+1)}}{2m^2} \\ &= \frac{-(m+1) \pm \sqrt{m^2 - 6m - 3}}{2m}. \end{aligned}$$

Since $\sqrt{m^2 - 6m - 3} < \sqrt{m^2} < m + 1$, all the zeros of $g(x)$ lie in the left half plane. The zeros of $g(x)$ are real iff $m^2 - 6m - 3 \geq 0$ iff $m \geq 3 + \sqrt{12}$ or $m \leq 3 - \sqrt{12}$. Since m is a positive integer and since $6 < 3 + \sqrt{12} < 7$, we get that the vertex connectivity roots of $B_{m,m}$ are all real iff $m \geq 7$. Now, since $m^2 - 6m - 3$ is never zero for any integer m, all the zeros of $g(x)$ are distinct. This completes the proof. □

Corollary 6.3.15. *For $m \geq 3$, $V[B_{m,m}; x]$ is schur stable.*

Proof. In the proof of theorem 6.3.14, the zeros of $g(x)$ are $\frac{-(m+1)\pm\sqrt{m^2-6m-3}}{2m}$. It can be observed that $|\frac{-(m+1)\pm\sqrt{m^2-6m-3}}{2m}| < 1$ for $m \geq 3$ so that all the vertex connectivity roots of $B_{m,m}$ lie inside the unit circle. □

Theorem 6.3.16. (i) *For $n > 3$, the helm H_n is not schur stable.*

(ii) *For $n > 3$, the web graph WB_n is not schur stable. But, all the vertex connectivity roots of WB_n are real and belong to the closed left half plane.*

(iii) *For $n > 4$, the butterfly graph BF_n is not schur stable. But, all the vertex connectivity roots of BF_n are real and belong to the closed left half plane.*

Proof. (i) We have,

$$V[H_n; x] = \binom{n-1}{2} x^3 + n(n-1)x^2 + \binom{n}{2} x.$$

Here, the quadratic coefficient $n(n-1) > \binom{n-1}{2} + \binom{n}{2}$ so that from theorem 1.5.2 it follows that only two zeros of $V[H_n; x]$ lie inside the unit circle and one zero outside the unit circle. Hence, H_n is not schur stable.

(ii) We have, $V[WB_n; x] = \binom{n-1}{2} x^3 + (2n-1)(n-1)x^2 + \binom{2n-1}{2} x$.
That is, $V[WB_n; x] = \binom{n-1}{2} x[x^2 + \frac{2(2n-1)}{n-2} x + \frac{2(2n-1)}{n-2}]$. Since the discriminant

of the quadratic polynomial $p(x) = x^2 + \frac{2(2n-1)}{n-2}x + \frac{2(2n-1)}{n-2}$ is $\frac{12(2n-1)}{(n-2)^2} > 0$, it follows that all the vertex connectivity roots of WB_n are real. The remaining results follows easily from the fact that the zeros of $p(x)$ are exactly $\frac{-2(2n-1)\pm\sqrt{12(2n-1)}}{2(n-2)}$.

(iii) We have,

$$V[BF_n; x] = x^3 + (2n-4)x^2 + \binom{n-2}{2}x.$$

For $n = 5, 6, 7$, the non-zero vertex connectivity roots of BF_n are exactly $-3 \pm \sqrt{6}, -4 \pm \sqrt{10}$ and $-5 \pm \sqrt{15}$ respectively so that in all those cases the graph is not schur stable. Now for $n > 7$, the linear coefficient $\binom{n-2}{2} > 1 + 2n - 4$ and hence by theorem 1.5.2, only one zero of $V[BF_n; x]$ lie inside the unit circle and the other two zeros lie outside the unit circle. Hence for $n > 4$, BF_n lacks schur stability. Now, since $2n - 4$ is greater than the magnitude of the square of the discriminant of the quadratic factor in $V[BF_n; x]$ and since the discriminant is positive, it follows that all the non-zero vertex connectivity roots of BF_n belong to the left half plane.

This completes the proof. □

Chapter 7

SIGNIFICANCE OF CLOSELY-CONNECTED VERTICES IN GRAPHS

This chapter is divided into two sections and is intended to discuss the significance of closely-connectedness introduced and studied in the previous chapters. In the first section, network robustness is taken into account, in which military and civilian communications, disruption of undersea cables, digital image processing, placement of PMU'S in power networks and travelling salesman problem are discussed. Also, the idea of close-connectivity in modelling robust networks is also illustrated. The second section discusses about the quantification of close-connectivity and a parameter called closely-connected communication overlap is developed. Moreover, a new entropy measure cc-domination entropy is introduced and studied.

7.1 Network Robustness

In highly interconnected systems, the failure of one part can trigger a chain reaction, leading to widespread cascading failures. Fault tolerance helps contain these failures, localizing their impact and preventing them from spreading through the entire system. A well-designed fault-tolerant system not only survives failures but also detects and recovers from them automatically, improving overall efficiency and reducing the need for human intervention. In general, fault tolerance is essential for the stability, security, and resilience of complex networks, making them more reliable and capable of handling unforeseen challenges. In the present work, closely-connected vertices in graphs are introduced, emphasizing on the need for fault tolerance mechanisms in sophisticated networks and this chapter discusses the relevance of closely-connected vertices in various disciplines and some parameters of practical interest are formulated. In this section, various fields in which robust networks are unavoidable are analysed and a comparison of node connectivity and link connectivity with close-connectivity is examined.

7.1.1 Military and Civilian Communications

The communication infrastructure is becoming more and more vulnerable for both military and civilian communications. Terrorist attacks pose a threat to civilian areas, while the growing trend in the military domain is to see communication networks as high-value targets [11]. Because of this predicament, it is vitally crucial to address network robustness, or the network's ongoing capacity to function in the face of an attack. Hence, it is imperative for communication network designers to consider the possibility of network attacks, some of which may cause harm. Strong networks are resilient to these kinds of losses and disruptions. Recent incidents have explicitly showcased the vulnerability of communication systems. For example, the Solar winds cyberattack in 2020 compromised numerous government and private sector networks, including those of major defense contractors. Additionally, the hacking of cell phone networks in several countries have exposed the potential for malicious actors to intercept and manipulate sensitive communications.

7.1.2 Disruption of Undersea Cables

Disruptions to undersea cables can cause significant internet blackouts and disrupt global communication. These cables are the backbone of the internet, and are often funded by tech companies like Google, Microsoft, Amazon, and Meta. Disruptions to submarine cables can affect businesses, governments, and individuals. In 2024, the Houthi strikes in the Red Sea damaged four of the more than 15 submarine cables in the region, affecting 25% of traffic flow between Asia, Europe, and the Middle East. This has forced internet providers to reroute up to a quarter of traffic.

Undersea cables play a significant role in ensuring global internet connectivity, enabling global communication and commerce and maintaining vital infrastructure links between continents. Undersea cable failures has a sounding economic impact on industries that rely heavily on a stable internet connection, such as finance, e-commerce, and telecommunication. Moreover, it may seriously affect the international relations and diplomatic efforts between countries. Recently, several undersea communication cables in the Red Sea have been cut, affecting 25% of data traffic flowing between Asia and Europe. That is why it is crucial for countries and companies to invest in the maintenance and protection of undersea cables to ensure uninterrupted and reliable internet connectivity worldwide.

In recent years, there has been a growing recognition of the need to protect and strengthen undersea cables against natural disasters, sabotage, and other potential threats. The future outlook for undersea cables includes advancements in technology that may improve their resilience and efficiency in the years to come. As technology continues to evolve and global connectivity becomes increasingly vital, the resilience of undersea cables will remain a key focus for governments, telecommunication companies, and other stakeholders in the digital age. The following [10],[17],[34] and [41] can be referred for more details about undersea communication system.

7.1.3 Digital Image Processing

In digital image processing, connectivity refers to the way pixels or regions in an image are connected or related to each other. By analyzing connectivity, it becomes possible to extract meaningful information from images and perform various operations for image enhancement and understanding. For example, in image segmentation, connectivity helps in identifying and separating different objects or regions based on their spatial relationships within the image. Additionally, connectivity is crucial for motion analysis as it enables tracking and understanding the movement of objects or features across consecutive frames of a video. Thus, a more rigorous understanding about connectivity of an image is necessarily required to formulate algorithms that can more accurately identify and analyze objects, patterns, and structures within the image. The notion of close-connectivity can be broadened suitably and experimented as a substitute for gaining more crucial information from an image.

7.1.4 Placement of PMU's in Power Networks : Monitoring Power Networks

The Phasor Measurement Unit (PMU) is a crucial component for monitoring, safeguarding, and controlling the power network. PMU provides real-time, synchronised computations of voltage, current, and frequency phasor data. But, it is not practical to install PMU in each node for the purpose of monitoring the power network, both in terms of cost and in terms of huge data management. Thus, placing PMU's at selected bus locations in the power system is one of the efficient methods to monitor the power system. It is therefore imperative to minimize the number of PMU's in the node network while maintaining the ability to monitor and observe the system.

Rules for placing PMU's in power networks

A power system is said to be observed if all the states can be determined by a set of PMU's according to the following rules [33]:

1. Assign a state of current phase measurement to each branch incident to a bus provided with a PMU (also, assign a state of voltage measurement to each bus located with a PMU);

2. Assign a state of voltage measurement to every bus incident to a branch

with known current and the other end bus with known voltages by using Ohm's law;

3. Assign a state of current phase measurement to each branch connecting two buses with a known voltage by using Ohm's law;

4. Assign a state of current phase measurement to a branch whose current can be inferred by using Kirchhoff's current law.

PMU placement : New model

In recent decades, several academics have presented different approaches to track the ideal placement of PMU's. The idea of power domination is one such approach developed by modelling the problem by considering the branches incident to bus stations as the edges incident to a vertex in a graph. In other words, adjacency relation among graph vertices is taken into account for developing the idea of power domination.

This can be more generalized by substituting the closely-connected property of vertices instead of adjacency and geodesics instead of edges except for cut paths. This model can further reduce the number of PMU's and thereby minimize the cost of installation than the power domination model.

7.1.5 Travelling Salesman Problem

In our day-to-day life, there may usually arise situations in which time-bound visits have to be made at different places in a single journey. All the intermediate places between the source and the final destination can be equally important so that none of them can be avoided. The common practice in such situations is to use the shortest routes to reach the intermediate places between the source and the destination. But, there may arise situations in which the shortest routes cause trouble due to potholes, road maintenance works, heavy traffic, and waiting time in signals. To avoid these, the shortest route between different destinations cannot always be preferable, and we prefer routes that optimize time and cost of fuel. This is possible only if an alternate path exists between the source and the destination connecting all the intermediate places to achieve our purpose, even if the shortest route is not accessible. Mathematically, this is the idea of a closely-connected graph in which the whole graph remains connected eventhough the edges between the source and destination are deleted.

7.1.6 Modelling Robust Communication Networks

A natural way to model the topology of a communication network is as an undirected graph consisting of nodes and links [11]. For the purpose of analysing topology, we ignore any variation in the type of links. Robustness of the topology will come from the presence of alternate paths, which ensure that communication remains possible in spite of damage to the network. Apart from the conventional measures of connectivity such as node connectivity and link connectivity, the notion of close-connectivity can also be taken into account to evaluate the robustness and fault tolerance of networks as it always preserves graph connectivity.
Illustration : Here $\kappa(H) = 1$ and $\lambda(H) = 1$. That is, deletion of either one vertex or one edge is sufficient to disconnect G.
Thus, node connectivity and link connectivity gives a measure of vulnerability of the above network. But, $\gamma_{cc}(H) = 4$, which means that a minimum of four

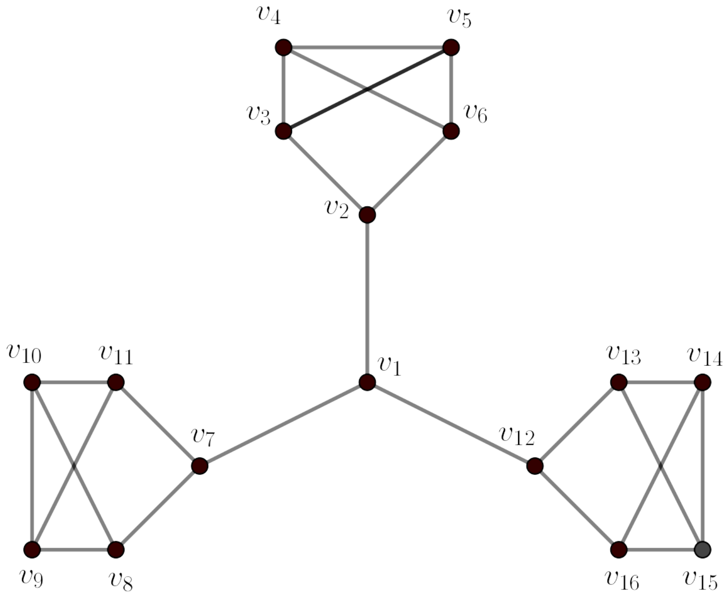

Figure 7.1: The graph H.

vertices are necessarily required to be strengthened in the network to maintain a continuous flow of information to all other nodes without disruption. Thus, close-connectivity parameters can be in fact used to measure the strength of the network required for communication flow without disruption.

7.2 Quantification of Close-connectivity

In this section, two numerical measures are formulated to quantify the concept of close-connectivity, keeping in mind that these parameters can be analysed to assist various areas of practical importance futuristically.

7.2.1 Closely-connected Geodetic Communication Overlap.

Definition 7.2.1. Let $C_g(G)$ be the set of all geodesics in a graph G. An incidence mapping is the mapping $\eta : V(G) \times C_g(G) \to \{0, 1\}$ defined by

$$\eta(w,g) = \begin{cases} 1 \text{ ; if } w \text{ is contained in } g. \\ 0 \text{ ; otherwise,} \end{cases}$$

for $w \in V(G)$ and $g \in C_g(G)$ [37].

Definition 7.2.2. A mapping $\varphi : V(G) \times V(G) \to C_g(G)$ is said to be a closely-connected geodesic assignment of G, if for any distinct vertices $u, v \in V(G)$, $\varphi(u, v)$ is a $u - v$ geodesic except for cut paths.

Definition 7.2.3. Let G be a graph and $F_c(G)$ be the set of all closely-connected geodesic assignments in G. The closely-connected geodetic communication overlap $Q_c(w)$ of a vertex w in G is defined as

$$Q_c(w) = \min_{\varphi \in F_c(G)} \left\{ \sum_{u,v \in V(G)} \eta(w, \varphi(u, v)) \right\}$$

and the closely-connected geodetic communication overlap $Q_c(G)$ of the graph G is defined as

$$Q_c(G) = \max_{w \in V(G)} Q(w).$$

7.2.2 CC-Domination Entropy of Graphs

Determining the entropy of relational structures, such as complex networks or graphs, tackles a wide range of issues, including social network analysis, communications routing, protein interactions, identification of important players in transaction networks, vulnerability of water distribution networks, and city traffic studies. An extensive body of research has been established that gives a variety of information entropy measures, in light of the significant variations in routes and flow types that these diverse networks follow. Each entropy measure provides different characteristics of a network or its components. Moreover, not all metrics are applicable to all networks [38].

In this subsection, we introduce a new graph entropy measure, based on the cc-dominating sets of graphs.

Definition 7.2.4. Let G be a graph of order n without cc-isolates. The information functional g is defined as $g := d_c(G,i)$. Then, the *cc-domination entropy* $I_{ccd}(G)$ of G is defined as :

$$I_{ccd}(G) = I_g(G) = -\sum_{i=\gamma_{cc}(G)}^{n} \frac{d_c(G,i)}{\gamma^t_{cc}(G)} log\left(\frac{d_c(G,i)}{\gamma^t_{cc}(G)}\right),$$

where the logarithm is of base 2 and $\gamma^t_{cc}(G)$ is the total number of cc-dominating sets in G.

Theorem 7.2.5. *Let G be a graph. Then, $I_{ccd}(G) = 0$ iff G is acyclic.*

Proof. Assume that G is acyclic. Then, $d_c(G,n) = 1$ and $d_c(G,i) = 0$ $\forall i < n$ so that $\gamma^t_{cc}(G) = 1$. Therefore, $I_{ccd}(G) = 0$.

Conversely, let $I_{ccd}(G) = 0$. Since $d_c(G,i) \leq \gamma^t_{cc}(G)$ and $\gamma^t_{cc}(G) \geq 1$, it follows that $log\left(\frac{d_c(G,i)}{\gamma^t_{cc}(G)}\right) \leq 0$ $\forall i = 1, \ldots, n$. Therefore,

$$\begin{aligned} I_{ccd}(G) = 0 &\Longleftrightarrow d_c(G,i) \cdot log\left(\frac{d_c(G,i)}{\gamma^t_{cc}(G)}\right) = 0 \\ &\Longleftrightarrow d_c(G,i) = 0 \text{ or } log\left(\frac{d_c(G,i)}{\gamma^t_{cc}(G)}\right) = 0 \ \forall i = 1, \ldots, n. \end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow log\Big(\frac{d_c(G,i)}{\gamma_{cc}^t(G)}\Big) = 0 \ \forall i \geq \gamma_{cc}(G)\\
&\Longleftrightarrow \frac{d_c(G,i)}{\gamma_{cc}^t(G)} = 1 \ \forall i \geq \gamma_{cc}(G)\\
&\Longleftrightarrow d_c(G,i) = \gamma_{cc}^t(G) \ \forall i \geq \gamma_{cc}(G)\\
&\Longleftrightarrow d_c(G,i) = \sum_{i=\gamma_{cc}(G)}^{n} d_c(G,i) \ \forall i \geq \gamma_{cc}(G)\\
&\Longleftrightarrow \gamma_{cc}(G) = n.
\end{aligned}$$

This completes the proof. □

Theorem 7.2.6. *For a closely-connected graph G of order n,*

$$I_{ccd}(G) = log(2^n - 1) - \frac{1}{2^n - 1}\sum_{i=1}^{n}\binom{n}{i} log\binom{n}{i}.$$

Proof. Since $D_c[G;x] = (x+1)^n - 1$, it follows that $\gamma_{cc}^t(G) = \sum_{i=1}^{n}\binom{n}{i} = 2^n - 1$. Therefore,

$$\begin{aligned}
I_{ccd}(G) &= -\sum_{i=1}^{n}\frac{\binom{n}{i}}{2^n-1} log\left(\frac{\binom{n}{i}}{2^n-1}\right)\\
&= log(2^n - 1) - \frac{1}{2^n - 1}\sum_{i=1}^{n}\binom{n}{i} log\binom{n}{i}.
\end{aligned}$$

This completes the proof. □

Theorem 7.2.7. *For $n \geq 5$,*

$$I_{ccd}(K_{2,n-2}) = log(2^n - 3) - \frac{(n-2)log(n-2)}{2^n - 3} - \frac{1}{2^n - 3}\sum_{i=2}^{n}\binom{n}{i} log\binom{n}{i}.$$

Proof. Since $D_c[G;x] = (x+1)^n - 2x - 1$, it follows that

$$\gamma_{cc}^t(G) = \sum_{i=1}^{n}\binom{n}{i} - 2 = 2^n - 3.$$

Therefore,

$$
\begin{aligned}
I_{ccd}(K_{2,n-2}) &= -\sum_{i=\gamma_{cc}(G)}^{n} \frac{d_c(G,i)}{\gamma_{cc}^{t}(G)} log\left(\frac{d_c(G,i)}{\gamma_{cc}^{t}(G)}\right) \\
&= log(2^n-3) - \frac{1}{2^n-3}\sum_{i=1}^{n} d_c(G,i) log(d_c(G,i)) \\
&= log(2^n-3) - \frac{(n-2)log(n-2)+nlogn}{2^n-3} - \frac{1}{2^n-3}\sum_{i=2}^{n-2}\binom{n}{i} log\binom{n}{i}
\end{aligned}
$$

This completes the proof. □

Chapter 8

CONCLUSION AND RECOMMENDATIONS

A synopsis of the thesis and some recommendations that aid in further exploration of the subject are included in this chapter.

8.1 Summary of the Thesis

In this thesis, a new graph concept ‘closely-connectedness’ between graph vertices has been introduced and studied in a detailed manner. The motivation to introduce this novel concept is to increase fault tolerance in networks by minimizing link failures. The necessity of link-failure-free distributed systems is emphasised, which actually spurred the development of the idea of closely-connected vertices in graphs. The related network model and mathematical model are discussed. Moreover, several basic ideas and traits of closely-connected vertices are studied to draw insightful conclusions. This new concept of closely-connectedness has been further developed to formulate a new variation called cc-domination for the pre-existing idea of domination in graphs. The concepts of cc-dominating sets and cc-domination number of graphs are discussed and the latter is assessed for a few well-known graph classes. Additionally, certain general findings have been made. The concept of cc-domatic partition in graphs is studied and some specific findings are drawn.

An effective tool for examining the dynamics and structure of intricate networks is a graph polynomial. They can be used to find hubs and bottlenecks in networks as well as to uncover hidden structures. Additionally, graph polynomials aid to forecast networks in accordance with shifting circumstances. A new graph polynomial called cc-domination polynomial has been introduced and its properties are studied. The thesis provides explicit formulae to find cc-domination polynomial of some well known graphs. Additionally, a few broad findings are established. The roots of the cc-domination polynomial that was presented in the earlier sections are also examined.

Special graph invariants, often referred to as topological indices are primarily utilised in the computation of QSAR and QSPR research. A new index is developed by combining both the connectivity and metric features of a graph. The idea of vertex-connectivity index is formulated by taking the sum of all distances between all vertex-connected vertices in a graph and the same is calculated for some graphs. Further, the idea of vertex connectivity polynomial is formulated and the same has been computed for certain graphs. The roots and stability characteristics of the vertex connectivity polynomial are also covered in detail.

Finally, the significance of closely-connected vertices is discussed. Network robustness is examined which includes the domains of digital image processing, military and civilian communications, underwater cable disruptions, PMU placement in power networks, and the travelling salesman issue. Additionally, an illustration of the concept of close-connectivity in robust network modelling is provided. The quantification of close-connectivity is covered, where a parameter known as closely-connected geodetic communication overlap is created. A novel entropy measure called cc-domination entropy is presented and calculated for some graphs. The intention to develop the above two parameters at the end of the thesis is to act as a gateway for developing the idea of close-connectivity quantitatively to solve real life problems.

8.2 Recommendations

1. The properties of maximal closely-connected induced subgraphs of a graph can be studied and a variation of domination similar to connected domination can be introduced.

2. Identify and characterize the graph classes having same cc-domination polynomial.

3. Identify and characterize the graph classes having same vertex connectivity polynomial.

4. Check whether the collection of cc-domination polynomials $D_c[G;x]$ of all simple finite undirected graphs G is dense over the set of all functions satisfying the property $\left|\frac{D_c[G;x]}{f(x)}\right| < 1$.

5. Characterize the closely-connected graphs G for which $D_c[\bar{G};x] = x^n$.

6. Explore the formulae for cc-domination polynomial and vertex connectivity polynomial of graphs obtained from various graph operations.
7. Characterize the polynomials over the set of integers which may be the cc-domination polynomial of some simple finite graphs.
8. Characterize the polynomials over the set of integers which may be the vertex connectivity polynomial of some simple finite graphs.
9. Calculate the closely-connected geodetic communication overlap and cc-domination entropy of various graph classes and study about its real life applications.

Bibliography

[1] S. Alikhani. On the domination polynomial of some graph operations. *International Scholarly Research Notices*, 2013(1):146595, 2013.

[2] S. Alikhani and Y. Peng. Dominating sets and domination polynomials of cycles. *arXiv:0905.3268*, 2009.

[3] S. Alikhani and Y. Peng. Introduction to domination polynomial of a graph. *arXiv preprint arXiv:0905.2251*, 2009.

[4] A. Alwardi, R. Rangarajan, and A. Alqesmah. On the injective domination of graphs. *Palestine Journal of Mathematics*, 7(1):202–210, 2018.

[5] Z. Amin, H. Singh, and N. Sethi. Review on fault tolerance techniques in cloud computing. *International Journal of Computer Applications*, 116(18):11–17, 2015.

[6] J Baskar Babujee and S Ramakrishnan. Topological indices and new graph structures. *Applied Mathematical Sciences*, 6(108):5383–5401, 2012.

[7] Lowell W Beineke and Jay S Bagga. *Line graphs and line digraphs*. Springer, 2021.

[8] N.L. Biggs, E.K. Lloyd, and R.J. Wilson. *Graph Theory, 1736-1936*. Oxford University Press, 1986.

[9] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications*. Macmillan London, (290), 1976.

[10] C. Bueger, T. Liebetrau, J. Franken, et al. Security threats to undersea communications cables and infrastructure–consequences for the eu. *Report for SEDE Committee of the European Parliament, PE702*, 557:11–56, 2022.

[11] Anthony H Dekker and Bernard D Colbert. Network robustness and graph topology. In *ACSC*, volume 4, pages 359–368, 2004.

[12] L. Euler. Solutio problematis ad geometriam situs pertinentis. *Commentarii academiae scientiarum Petropolitanae*, pages 128–140, 1741.

[13] R. Frucht and F. Harary. On the corona of two graphs. *A.E.Q. Math*, (4):322–325, 1970.

[14] P.A. Fuhrmann. *A Polynomial Approach to Linear Algebra*. Springer Science & Business Media, 2011.

[15] G. Gajender and H. Sharma. Hurwitz polynomial. *International journal of innovative research in technology*, 1(7):340–342, 2014.

[16] J.A. Gallian. A dynamic survey of graph labeling. In *The Electronic Journal of Combinatorics*. Citeseer, 1998.

[17] A. Ganz, M. Camellini, E. Hine, C. Novelli, H. Roberts, and L. Floridi. Submarine cables and the risks to digital sovereignty. *Minds and Machines*, 34(3):31, 2024.

[18] R.L. Graham and N.J. Alexander Sloane. On additive bases and harmonious graphs. *SIAM Journal on Algebraic Discrete Methods*, 1(4):382–404, 1980.

[19] F. Harary. *Graph Theory.* CRC Press, 1969.

[20] M. Hasan and Major S. Goraya. Fault tolerance in cloud computing environment: A systematic survey. *Computers in Industry, Elsevier*, 99:156–172, 2018.

[21] T.W. Haynes. *Domination in Graphs: Volume 2: Advanced Topics.* Routledge, 2017.

[22] T.W. Haynes, S. Hedetniemi, and P. Slater. *Fundamentals of Domination in Graphs.* CRC press, 2013.

[23] M.A. Henning. Bounds on domination parameters in graphs: A brief survey. *Discussiones Mathematicae Graph Theory,Uniwersytet Zielonogórski. Wydział Matematyki, Informatyki i Ekonometrii*, 42(3):665–708, 2022.

[24] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43(4):439–561, 2006.

[25] H. Hosoya. On some counting polynomials in chemistry. *Discrete applied mathematics*, 19(1-3):239–257, 1988.

[26] A. Joshi. *Wiener Index and Wiener Polynomial for Graphs*. PhD Thesis., 2009.

[27] P.B. Joshi and M. Joseph. P-energy of generalized petersen graphs. *Communications in Combinatorics and Optimization, Azarbaijan Shahid Madani University*, 7(1):1–16, 2022.

[28] E. S. Kumar and H.B. Walikar. On the splitting graph of a graph. *The Karnataka University Journal*, pages 1980–1981, 1980.

[29] E.S. Kumar and L. Pushpa Latha. Pons domination number of a graph. *Indian J. pure appl. Math*, 24(4):225–229, 1993.

[30] E.S. Kumar and L. Pushpa Latha. Set domination in graphs. *Journal of Graph Theory, Wiley Online Library*, 18(5):489–495, 1994.

[31] P. Kumari and P. Kaur. A survey of fault tolerance in cloud computing. *Journal of King Saud University-Computer and Information Sciences, Elsevier*, 33(10):1159–1176, 2021.

[32] M. Lepović and I. Gutman. Some spectral properties of starlike trees. *Bulletin (Académie serbe des sciences et des arts. Classe des sciences mathématiques et naturelles. Sciences mathématiques), JSTOR*, pages 107–113, 2001.

[33] Chung-Shou Liao and Der-Tsai Lee. Power domination problem in graphs. In *Computing and Combinatorics: 11th Annual International Conference, COCOON 2005 Kunming, China, August 16–19, 2005 Proceedings 11*, pages 818–828. Springer, 2005.

[34] M.S. Matis. The protection of undersea cables: A global security threat. 2012.

[35] P.M. Merlin and A. Segall. A failsafe distributed routing protocol. *IEEE Transactions on Communications*, 27(9):1280–1287, 1979.

[36] J. Nieminen. Two bounds for the domination number of a graph. *IMA Journal of Applied Mathematics, Oxford University Press*, 14(2):183–187, 1974.

[37] J. Nieminen and M. Peltola. The minumum geodetic communication overlap graphs. *Applied mathematics letters*, 11(2):99–102, 1998.

[38] Y. M. Omar and P. Plapper. A survey of information entropy metrics for complex networks. *Entropy*, 22(12):1417, 2020.

[39] S.V. Pemmaraju and I.A. Pirwani. Energy conservation via domatic partitions. In *Proceedings of the 7th ACM International Symposium on Mobile Ad hoc Networking and Computing*, pages : 143–154, 2006.

[40] A.U. Rehman, R.L. Aguiar, and J.P. Barraca. Fault-tolerance in the scope of cloud computing. *IEEE Access*, (10):63422–63441, 2022.

[41] G. Sahathevan and P. Panneerselvam. Needed, a framework to protect undersea cable. 2024.

[42] B. Şahin. New network entropy: The domination entropy of graphs. *Information Processing Letters, Elsevier*, 174:106195, 2022.

[43] M Shikhi. *A study on common neighbor polynomial of graphs.* PhD thesis, Ph. D. Thesis, Calicut University, 2019.

[44] P Sini. On completely homogeneous c-spaces. *Heliyon*, 6(12), 2020.

[45] N. Trinajstic. *Chemical graph theory*. CRC press, 2018.

[46] N.K. Vishnoi. Zeros of polynomials and their applications to theory: a primer. In *FOCS 2013 Workshop on Zeros of Polynomials and their Applications to Theory*, pages 1–18, 2013.

[47] H. Wiener. Structural determination of paraffin boiling points. *Journal of the American chemical society*, 69(1):17–20, 1947.

[48] Y.N. Yeh and I. Gutman. On the sum of all distances in composite graphs. *Discrete Mathematics*, 135(1-3):359–365, 1994.

[49] J. Yu, Q. Zhang, D. Yu, C. Chen, and G. Wang. Domatic partition in homogeneous wireless sensor networks. *Journal of Network and Computer Applications, Elsevier*, (37):186–193, 2014.

[50] Y. Zhao. Integer polynomials. *MOP 2007 Black Group*, 69(1):17–20, 1947.

APPENDIX I

List of Publications

(i) Priya K. and Anil Kumar V., ***CC-Domination in graphs***, Palestine Journal of Mathematics, 12(3), 2023, 532-541.
$https://pjm.ppu.edu/sites/default/files/papers/PJM_12\%283\%29_2023_532_to_541.pdf$

(ii) Priya K. and Anil Kumar V., ***The number of cc-dominating sets of some graphs***, South East Asian Journal of Mathematics and Mathematical Sciences, 19(2), 2023, 211-222.
$http://dx.doi.org/10.56827/SEAJMMS.2023.1902.15$

(iii) Priya K. and Anil Kumar V., ***On the roots and stability of vertex connectivity polynomial of graphs***, Ratio Mathematica, 42, 2022, 145-155.
$http://dx.doi.org/10.23755/rm.v42i0.731$

(iv) Priya K. and Anil Kumar V., ***On the vertex connectivity index of***

graphs, Chinese Journal of Mathematical Sciences, 1(1), 2021, 49-59.
https://romanpub.com/resources/cjms%20v1-1-paper5tex.pdf

(v) Priya K. and Anil Kumar V., ***On the vertex connectivity polynomial of graphs***, Advances and Applications in Discrete Mathematics, 26(2), 2021, 133-147.
*http://dx.doi.org/*10.17654/*DM*026020133

APPENDIX II

Paper Presentations

(i) The research paper entitled ***"CC-Domination Polynomial of Graphs"*** was presented in the International Conference on Analysis and Applied Mathematics 2022 (ICAAM-2022) jointly organized by Ayya Nadar Janaki Ammal College(Autonomous) & Ramco Institute of Technology, Tamil Nadu.

(ii) The research paper entitled ***"On the Vertex Connectivity Index of Graphs"*** was presented in the International Conference on Graphs, Networks and Combinatorics 2023 (ICGNC-2023) organized by the Department of Mathematics, Ramanujan College, University of Delhi.

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