ON THE STUDY OF FRAME MULTIRESOLUTION ANALYSIS IN THE SUPERSPACES $L^2(\mathbb{R})^n$

Thesis submitted to the
University of Calicut
in partial fulfillment of the requirements
for the award of the degree of
DOCTOR OF PHILOSOPHY
under Faculty of Science

BIJUMON RAMALAYATHIL

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALICUT
KERALA
INDIA

July 2009

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UNIVERSITY OF CALICUT
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Professor

Calicut University
$23^{\rm rd}$ July 2009

CERTIFICATE

Certified that the work presented in this thesis is a bonafide work done by Mr. Bijumon Ramalayathil under my guidance in the Department of Mathematics, University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree.

Dr. M. S. Balasubramani
Supervising Guide

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DECLARATION

I declare that the work presented in this thesis is based on the original work done by me under the guidance of Dr. M. S. Balasubramani, Professor, Department of Mathematics, University of Calicut and has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University/Institution.

University of Calicut

Bijumon Ramalayathil

23 July 2009

I

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Bijumon Ramalayathil

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Chapter 0

Introduction

Orthonormal Sets, Frames, Wavelets, MRA and Frame MRA ... Daubechies, Grossmann and Meyer [15] laid the foundations for the study of frames in Hilbert Spaces. Mallat [25] used these ideas and developed Wavelet Theory. The study of Wavelet Theory and its applications to Signal Processing and Image Processing are well described in the works of Niklas Grip [21] and James S. Walker [33]. Probably the main reason for the success of the wavelet theory was the introduction of the concept of Multiresolution Analysis (MRA), which provided the right frame work to construct orthogonal wavelet bases with good localization properties, which are very useful in physical applications. Similarly, the theory of Frames and Frame Multiresolution Analysis (FMRA) in $L^2(\mathbb{R})$ are highly developed subjects with origins in speech and image processing.

The frame concept for Hilbert Spaces generalizes the notion of an orthonormal basis in the sense that a frame $X = \{x_i : i \in I\}$ provides a stable representation for signals f by means of an expression $f = \sum\limits_i c_i(f)x_i$, but it is not necessarily an orthonormal or independent sequence. Further, and important for their applicability, the coefficients $c_i(f)$ in a frame expansion of a signal f are computable and depend continuously on f, and redundancies are allowed in the frame itself. This latter property can lead to more freedom when constructing frame elements $x_i \in X$ for specific types of expansions.

The notion of Frame Multiresolution Analysis in $L^2(\mathbb{R})$ developed and studied by J. J. Benedetto and S. Li [3], [2] and O. Christensen [9], combine the theories of frames and MRAs with the goal of attacking problems in signal processing. There has also been a general development on the construction of Tight Wavelet Frames by Ron and Shen [30], [28], [31] and [29]. In their work, Han and Larson [22] have shown that there is no MRA superwavelet in the superspace $L^2(\mathbb{R})^2$ analogous to MRA wavelets in $L^2(\mathbb{R})$. Stefan Bildea, Dorin Ervin Dutkay and Gabriel Picioroaga [5] have realized multiresolution constructions in the case of wavelets in the superspaces $L^2(\mathbb{R})^n$, n being a natural number, by making some slight modifications to the usual dilation and translation operators.

The main objective of our study is to look at Frame Multiresolution Analysis (FMRA) theory for superspaces $L^2(\mathbb{R})^n$ and study some consequences. This thesis is a result of our investigations along these lines. Our work on this study is presented in this thesis in five chapters. We hope that this study will simplify applications in data transmission, image and signal processing etc. In the study of wavelets, construction of examples plays an important role. In this work we have given emphasis for this aspect.

Chapter 1 begins with a quick review of some of the known fundamental definitions, results and examples needed for our study. For the sake of completeness we have added details to some of the important examples.

Working out the details have helped us in constructing new examples. We begin with the definitions and examples of wavelets and frame wavelets in 1.1. Starting with the description of Multiresolution Analysis for $L^2(\mathbb{R})$, MRA wavelets and MRA frame wavelets in $L^2(\mathbb{R})$ are discussed here. Wavelet Sets and Frame Sets are discussed in 1.2. Superwavelets and construction of superwavelets in $L^2(\mathbb{R})^2$ are discussed in 1.3. The important result of Han and Larson that superwavelet of length 2 exists is given in 1.3.6. 1.4 is a discussion on whether MRA wavelets (i.e., wavelets made out of an MRA) can be constructed in the superspace $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. As mentioned above, the fact that MRA superwavelets do not exist in the usual sense is stated in Theorem 1.4.2. However, Han and Larson have given a very simple definition of MRA superwavelet as: $(\eta_1, \eta_2) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is an MRA superwavelet if each η_1 and η_2 is an MRA *frame* wavelet in $L^2(\mathbb{R})$. Based on this idea an example of an MRA superwavelet is constructed in 1.4.5. Following the definition of MRA superwavelet given by Stefan Bildea *et al.* which involves modifications in the dilation and translation operators, superwavelets are constructed via MRA in 1.5.

As mentioned above, with a view to giving importance for the construction of examples, Chapter 2 is devoted for studying some important examples. In the first section construction of orthogonal scaling vectors associated with an MRA for the scale $N = 2$ is discussed. Using this idea superwavelets via MRA in $L^2(\mathbb{R})^2$ and $L^2(\mathbb{R})^6$ are constructed. In the final section of the chapter an example of superwavelet via MRA in $L^2(\mathbb{R})^3$ now using the scale $N = 3$ is constructed. These examples further help us in constructing Bessel maps.

Frames in the superspace $L^2(\mathbb{R})^n$ via the Bessel map is introduced in Chapter 3. We discuss about the frame theory in $L^2(\mathbb{R})^n$ giving various results. We discuss the relation between a countable collection and the associated Bessel map when the countable collection is a frame. 3.1.13 gives a characterization of frames in terms of adjoint of the Bessel map. In the second section of this chapter Frame Operators on $L^2(\mathbb{R})^n$ is introduced. That elements of $L^2(\mathbb{R})^n$ has a series representation in terms of the frame operator analogous to Fourier expansion of elements in a Hilbert space is given in 3.2.11.

Chapter 4 is a study about the frames generated by translations of functions in $L^2(\mathbb{R})^n$. In the first section of this chapter, we introduce translation and dilation operators on $L^2(\mathbb{R})^n$. In 4.2.1 of the second section we obtain a relation between the Bessel maps and the associated bounded $2\pi-$ periodic functions. 4.2.9 shows that if $\phi_1 \oplus \phi_2$ generates a frame for V_0, then the elements of V_0 and $\phi_1 \oplus \phi_2$ are closely related by their Fourier transforms.

Having discussed the ideas of superwavelets and frames in superspaces, we come to the study of Frame MRA in $L^2(\mathbb{R})^n$ in chapter 5. John J. Benedetto and Oliver M. Treiber [4] constructed multiresolution analysis frame (MRA frame) in the case of $L^2(\mathbb{R})$ and studied various results in this connection. We have used the ideas of John J. Benedetto, Oliver M. Treiber, D. Han and D. Larson [22] to develop FMRA theory in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. In 5.1.1, we give the definition of FMRA in the superspace $L^2(\mathbb{R})^n$. Important properties are given in 5.1.4 and 5.1.7. In 5.1.10 we give example of an FMRA generated by the given $\phi_1 \oplus \phi_2$. In the second section, Theorem 5.2.1 tells about a case for which there is no MRA frame wavelet in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. In 5.2.2 we see an example where the given $\phi_1 \oplus \phi_2$ cannot give a FMRA.

We conclude our thesis with an epilogue where we have listed some problems which can be investigated. For example, we can think of a result that ensures a

MRA frame wavelet $\psi_1 \oplus \cdots \oplus \psi_n$ in $L^2(\mathbb{R})^n$ corresponding to the FMRA scaling function $\phi_1 \oplus \cdots \oplus \phi_n$.

0.1 Preliminaries

The *Fourier transform* $\hat{f}$ of $f \in L^1(\mathbb{R})$ is defined as In what follows, $\mathbb{Z}$ is the set of integers; $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers and $\mathbb{T}$ is the unit circle in the complex plane. The points of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ in the complex plane form a multiplicative group known as the Circle group. When considered as a subset of $\mathbb{R}$, $\mathbb{T}$ is $[0, 2\pi)$. We sometimes identify $\mathbb{T}$ with $[-\pi, \pi)$ or with $[0, 1)$.

$L^2(\mathbb{R})$ is the Hilbert space of square-integrable functions on $\mathbb{R}$, i.e., $L^2(\mathbb{R})$ is the set of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ with norm

$$\|f\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |f|^2\right)^{1/2} < \infty.$$

The inner product $\langle \cdot , \cdot \rangle_{L^2(\mathbb{R})}$ for $L^2(\mathbb{R})$ is defined as

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f\bar{g}, \quad f, g \in L^2(\mathbb{R}).$$

$L^1(\mathbb{R})$ is the set of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ with norm

$$\|f\|_{L^1(\mathbb{R})} = \left(\int_{\mathbb{R}} |f|\right)^{1/2} < \infty.$$

Here we note that $L^1(\mathbb{R})$ is not an inner product space.

The *Fourier transform* $\hat{f}$ of $f \in L^1(\mathbb{R})$ is defined as

$$(1) \qquad \hat{f}(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\gamma} dt, \qquad \gamma \in \mathbb{R}$$

Example 0.1.1. If $f(t) = e^{-|t|}$, $t \in \mathbb{R}$, then $f \in L^1(\mathbb{R})$ and $\hat{f}(\gamma) = \frac{2}{1+\gamma^2}$.

Example 0.1.2. If

$$f(t) = \begin{cases} 1, & \text{for } |t| < 1 \\ 0, & \text{for } |t| \geq 1 \end{cases}$$

then $f \in L^1(\mathbb{R})$ and $\hat{f}(\gamma) = \frac{2\sin\omega}{\omega}$.

If $f \in L^1(\mathbb{R})$, by the Riemann-Lebesgue Lemma [8], $\hat{f}$ is a continuous function vanishing at $\pm\infty$ and $\hat{f} \in L^\infty(\mathbb{R})$ [19].

A corollary of Plancherel identity states that the Fourier transform extends from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to a unitary operator on $L^2(\mathbb{R})$ [20]. This extension is designated the Fourier transform $\hat{f}$ of $f \in L^2(\mathbb{R})$, and we use the notation of (1) for the $L^2(\mathbb{R})$ setting as well as for $L^1(\mathbb{R})$ [1]. The extended Fourier transform on $L^2(\mathbb{R})$ is sometimes denoted by $\mathcal{F}$, so that

$$\mathcal{F}(f) = \hat{f}.$$

The inverse Fourier transform operator is defined, both in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, as

$$\check{f}(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{it\gamma} dt.$$

$L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is the set of all tuples (f_1, f_2), where f_1 and $f_2 \in L^2(\mathbb{R})$.

$L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is also a Hilbert space under the inner product $\langle \cdot\,, \cdot \rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}$ defined by

$$\langle (f_1, f_2), (g_1, g_2) \rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = \langle f_1, g_1 \rangle_{L^2(\mathbb{R})} + \langle f_2, g_2 \rangle_{L^2(\mathbb{R})},$$

for (f_1, f_2), $(g_1, g_2) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

We can extend this for any natural number n: $L^2(\mathbb{R})^n$, the set of n tuples of elements in $L^2(\mathbb{R})$, is also a Hilbert space under the inner product

$$\langle f_1 \oplus \cdots \oplus f_n , g_1 \oplus \cdots \oplus g_n \rangle_{L^2(\mathbb{R})^n} = \sum_{j=1}^{n} \langle f_j, g_j \rangle_{L^2(\mathbb{R})},$$

for $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$. The Fourier transform on $L^2(\mathbb{R})^n$ is defined as:

$$(f_1, f_2, \ldots, f_n)\hat{} = \left(\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n\right),$$

for $(f_1, f_2, \ldots, f_n) \in L^2(\mathbb{R})^n$.

$l^2(\mathbb{Z})$ is the normed space of absolutely square-summable complex sequences. That is,

$$l^2(\mathbb{Z}) = \left\{ c = (c_k)_{k\in\mathbb{Z}} : \; c_k \in \mathbb{C} \text{ for } k \in \mathbb{Z} \text{ such that } \sum_{k\in\mathbb{Z}} |c_k|^2 < \infty \right\}.$$

and the associated norm is

$$\|c\|_{l^2(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |c_k|^2 \right)^{1/2}.$$

$l^2(\mathbb{Z})$ is a Hilbert space with the inner product

$$\langle c,\ d\rangle_{l^2(\mathbb{Z})} = \sum_{k\in\mathbb{Z}} c_k \bar{d}_k,$$

for $c,\ d \in l^2(\mathbb{Z})$.

A function F is 2π -periodic on $\mathbb{R}$ if $F(\gamma) = F(\gamma + 2\pi)$ for all $\gamma \in \mathbb{R}$. This does not necessarily mean that 2π is the largest period of F. From now on we view 2π periodic complex valued functions $f \in L^1[-\pi,\ \pi]$ or $L^2[-\pi,\ \pi]$ as having $\mathbb{T}$ as their domain, and denote them, respectively, by $L^1(\mathbb{T})$ or $L^2(\mathbb{T})$.

$L^2(\mathbb{T})$ is the space of 2π-periodic complex-valued Lebesgue square integrable functions F on $\mathbb{R}$ with norm

$$\|F\|_{L^2(\mathbb{T})} = \left(\int_{\mathbb{T}} |F|^2\right)^{1/2} = \left(\int_{-\pi}^{\pi} |F|^2\right)^{1/2} < \infty,$$

and inner product

$$\langle F,\ G\rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} F\bar{G}, \quad F,\ G \in L^2(\mathbb{T}).$$

If $c \in l^2(\mathbb{Z})$, $\hat{c} \in L^2(\mathbb{T})$ denotes its 2π-periodic Fourier transform

$$\widehat{c}(\gamma) = \sum_{k\in\mathbb{Z}} c_k e^{ik\gamma},$$

i.e., $\widehat{c}$ is the Fourier series associated with c and the Fourier coefficients are $c_k,\ k \in \mathbb{Z}$.

In the case of Hilbert spaces, convergence will be in H unless otherwise stated. In particular, if $H = L^2(\mathbb{R})$, or if we are dealing with Fourier series in $L^2(\mathbb{T})$, then convergence is in L^2-norm.

Translation by $y \in \mathbb{R}$ is denoted by τ_y, i.e., if $f : \mathbb{R} \to \mathbb{C}$ is a function, then $\tau_y f : \mathbb{R} \to \mathbb{C}$ is the function defined by

$$(\tau_y f)(t) = f(t - y).$$

If $y = k$ is an integer, then we use T in place of τ, and the translation by k is given by

$$T^k f(t) = f(t - k).$$

In particular,

$$Tf(t) = f(t - 1).$$

The translation operator is a unitary operator on $L^2(\mathbb{R})$.

The unitary dyadic dilation operator $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined as

$$(Uf)(t) = \sqrt{2} f(2t), \quad , \quad \text{for } f \in L^2(\mathbb{R})$$

Hence for all $k \in \mathbb{Z}$

$$(U^k f)(t) = 2^{k/2} f(2^k t).$$

In particular,

$$(U^{-1} f)(t) = \frac{1}{\sqrt{2}} f\left(\frac{t}{2}\right).$$

The function $e_\gamma : \mathbb{R} \to \mathbb{C}$ is defined by

$$e_\gamma(t) = e^{-it\gamma} \quad \forall \; t \in \mathbb{R}$$

and is 2π periodic.

With the above notations, we have for $f \in L^2(\mathbb{R})$

$$\hat{f}(\gamma) = \frac{1}{\sqrt{2\pi}} \int f e_\gamma.$$

Proposition 0.1.3. *For* $f \in L^2(\mathbb{R})$ *or* $L^1(\mathbb{R})$

$$\widehat{(\tau_y f)} = e_y \hat{f}.$$

and

$$\widehat{(Uf)} = U^{-1}\hat{f}.$$

Proof.

$$\begin{aligned}
\widehat{(\tau_y f)}(\gamma) &= \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} (\tau_y f)\,(t) e^{-it\gamma} dt \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} f(t-y) e^{-it\gamma} dt \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} f(u) e^{-i(u+y)\gamma} du \\
&= e^{-iy\gamma} \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} f(u) e^{-iu\gamma} du \\
&= e_y(\gamma) \hat{f}(\gamma).
\end{aligned}$$

$$\begin{aligned}
\widehat{(Uf)}(\gamma) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (Uf)(t) e^{-it\gamma} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{2} f(2t) e^{-it\gamma} dt \\
&= \frac{1}{\sqrt{2\pi}} \sqrt{2} \int_{\mathbb{R}} f(u) e^{-iu\frac{\gamma}{2}} \frac{1}{2} du \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-iu\frac{\gamma}{2}} du \\
&= \frac{1}{\sqrt{2}} \hat{f}(\tfrac{\gamma}{2}) \\
&= U^{-1} \hat{f}(\gamma).
\end{aligned}$$

□

The 2π *periodization operator* Per is defined for a function f on $\mathbb{R}$ as

$$\operatorname{Per}(f)(\cdot) = \sum_{k \in \mathbb{Z}} f(\cdot - 2\pi k)$$

Remark 0.1.4. $\operatorname{Per}(f)$ is a 2π-periodic function. If g is 2π-periodic and $gf \in L^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} gf = \int_{\mathbb{T}} g\operatorname{Per}(f).$$

Brackets are used to denote pullbacks; for example,

$$[f = 0] = \{t \in \operatorname{domain}(f) : \ f(t) = 0\} ;$$

$$[f \neq 0] = \{t \in \operatorname{domain}(f) : \ f(t) \neq 0\} ;$$

and so on.

The characteristic or indicator function of a set E is denoted by χ_E.

If $f : \mathbb{R} \to \mathbb{C}$ is a function, then $f(2\cdot)$ denotes the function $f(2x)$ and $f\left(\frac{\cdot}{2}\right)$ denotes the function $f\left(\frac{x}{2}\right)$.

Example 0.1.5. If $E \subseteq \mathbb{R}$ and $f = \chi_E$, then

$$f(2\cdot) = \chi_{\frac{1}{2}E}$$

and

$$f(\tfrac{\cdot}{2}) = \chi_{2E}.$$

If K_1 and K_2 are linear spaces and $\mathcal{L} : K_1 \to K_2$ is a linear operator, then $\mathcal{R}(\mathcal{L})$, the range of $\mathcal{L}$, is a subspace of K_2 and $\mathcal{N}(\mathcal{L})$, its kernel, is a subspace of K_1.

In the case of a separable Hilbert space H and a subset $X \subseteq H$, span $X \subseteq H$ is the set of finite linear combinations of elements from X; and $\overline{\text{span}}\ X$ is the closure of span X in H. If $X \subseteq H$, then

$$X^{\perp} = \{y \in H : \langle x,\ y\rangle_H = 0 \text{ for all } x \in X\}.$$

A linear operator $\mathcal{L} : H_1 \to H_2$, mapping the Hilbert space H_1 into the Hilbert space H_2, is a topological isomorphism if $\mathcal{L}$ is bijective and continuous. In this case, the open mapping theorem [24] ensures the continuity of $\mathcal{L}^{-1}$.

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Chapter 1

Wavelets and Superwavelets

This chapter is devoted to basic definitions and results on wavelets and frames that are necessary for our study.

1.1 Frames, Wavelets and MRA Wavelets

Definition 1.1.1. A *frame* in a Hilbert space H is a sequence $\{x_n\}$ of vectors in H with the property that there exist constants $A, B > 0$ such that for $f \in H$

$$A \|f\|^2 \leq \sum_n |\langle f,\ x_n \rangle|^2 \leq B \|f\|^2 .$$

The greatest possible such A is the *lower frame bound* and the least possible such B is the *upper frame bound.* If $A = B$, then the frame is called a *tight frame.* If $A = B = 1$, then the frame is called a *normalised tight frame.*

Definition 1.1.2. A function $\psi \in L^2(\mathbb{R})$ is a *wavelet* (resp. *frame wavelet*) if $\{U^j T^k \psi : j, k \in \mathbb{Z}\}$ is an orthonormal basis (resp. frame) for $L^2(\mathbb{R})$, where T and U are the translation and dilation unitary operators, respectively, on $L^2(\mathbb{R})$ defined by

$$(Tf)(t) = f(t-1) \tag{1.1}$$

and

$$(Uf)(t) = \sqrt{2}f(2t). \tag{1.2}$$

Definition 1.1.3. $\psi \in L^2(\mathbb{R})$ is a *tight frame wavelet* (resp. *normalized tight frame wavelet*) if $\{U^jT^k\psi : j, k \in \mathbb{Z}\}$ is a tight frame (resp. normalized tight frame).

Example 1.1.4. Consider the function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\psi(t) = \begin{cases} 1 \text{ if } t \in [0,\ \frac{1}{2}) \\ -1 \text{ if } t \in [\frac{1}{2},\ 1) \\ 0 \text{ otherwise} \end{cases}$$

Then it can be shown that $\{U^jT^k\psi : j,\ k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, so that ψ is a wavelet. This wavelet is called *Haar wavelet.*

Example 1.1.5. Let ψ be such that $\hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}}\chi_{[-2\pi,\ -\pi)\cup[\pi,\ 2\pi)}$. Then $\{U^jT^k\psi : j,\ k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, so that ψ is a wavelet.

Example 1.1.6. The function ψ whose Fourier transform satisfies

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\chi_{[-2\pi,\ -\pi)\cup[\pi,\ 2\pi)}$$

is a wavelet, called the *Shannon wavelet.*

Example 1.1.7. Let $\psi \in L^2(\mathbb{R})$ be such that $\text{supp}\,(\hat{\psi})$ is contained in

$\{s \in \mathbb{R} : \frac{1}{2} \leq |s| \leq 2\}$ and

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(2^j s)\right|^2 = \frac{1}{2\pi} \quad \text{for all } s \neq 0.$$

Then ψ is a normalised tight frame wavelet and these type of frame wavelets are called Frazier-Jawerth type frame wavelets.

Example 1.1.8. Consider the Mexican hat function

$$\psi(x) = \frac{2}{\sqrt{3}}\pi^{-\frac{1}{4}}(1-x)^2 e^{-\frac{1}{2}x^2},$$

which coincides with $-\dfrac{d^2}{dx^2}\left(e^{-\frac{1}{2}x^2}\right)$, when normalized in the space $L^2(\mathbb{R})$. Ingrid Daubechies [13] has reported frame bounds of 3.223 and 3.596 for the frame obtained by translations and dilations of the Mexican hat function.

An important concept in wavelet theory is the multiresolution analysis established by Mallat [25] which is used to derive wavelets. Such wavelets are called *MRA wavelets.* However, there exist non-MRA wavelets. e.g., Journe's wavelet [23] and wavelets constructed by Bownik., *et al.* [6]. But Papadakis [26] introduced the concept of generalised frame MRA (GFMRA) and announced that any wavelet in $L^2(\mathbb{R})$ is derived by GFMRA. A GFMRA may not give wavelets, e.g. [35].

Definition 1.1.9. A *Multiresolution Analysis (MRA)* for $L^2(\mathbb{R})$ consists of a sequence $\{V_j : j \in\mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying

1. $V_j \subset V_{j+1},\ j \in \mathbb{Z}$,
2. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
3. $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

4. $f \in V_j$ if and only if $Uf \in V_{j+1}$, $j \in \mathbb{Z}$,

5. There is a $\phi \in V_0$ such that $\{T^k\phi : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

The function ϕ given above is called an *orthogonal scaling function* for the mutliresolution analysis.

Example 1.1.10. Let V_j be the closed span of functions in $L^2(\mathbb{R})$ which are constant on intervals of the form $[2^{-j}k,\ 2^{-j}(k+1)]$, $k \in \mathbb{Z}$. Then $\{V_j : j \in \mathbb{Z}\}$ is an MRA and we can take the scaling function to be $\phi = \chi_{[-1,\,0]}$. As we shall see in Example 1.1.12, this MRA is related to the Haar wavelet.

It was shown by Hernandez and Weiss [23] that if ϕ is a scaling function for an MRA, then there is a 2π -periodic measurable function m such that

$$\hat{\phi}(2\xi) = m(\xi)\,\hat{\phi}(\xi)$$

for a.e. $\xi \in \mathbb{R}$. It is known that the function ψ given by

$$\hat{\psi}(\xi) = e^{\frac{i\xi}{2}}\overline{m\left(\frac{1}{2}\xi + \pi\right)}\hat{\phi}\left(\frac{1}{2}\xi\right)$$

is a wavelet, and moreover, it is known that every function of the form

$$\hat{\psi}(\xi) = e^{\frac{i\xi}{2}}k(\xi)\overline{m\left(\frac{1}{2}\xi + \pi\right)}\hat{\phi}\left(\frac{1}{2}\xi\right) \tag{1.3}$$

where k is any measurable unimodular 2π-periodic function, gives a wavelet. These are all contained in the difference space $W_0 = V_1 \ominus V_0$, and more importantly, every wavelet contained in W_0 has the form above.

Definition 1.1.11. A wavelet which has the form given in (1.3) for some MRA is called an *MRA wavelet*.

Example 1.1.12. The Haar wavelet seen in Example 1.1.4 is constructed (up to a translation) from the MRA generated by the scaling function $\phi = \chi_{[-1,\,0]}$ associated with the MRA described in Example 1.1.10.

Example 1.1.13. The Shannon wavelet seen in Example 1.1.6 is an MRA wavelet whose scaling vector ϕ is given by the Fourier transform $\hat{\phi}(\xi) = \chi_{[-\pi,\,\pi)}(\xi)$.

Remark 1.1.14. In practice, the wavelets ψ are constructed from the scaling function in such a way that

$$\{T^k\psi : k \in \mathbb{Z}\}$$

is an orthonormal basis for $W_0 := V_1 \ominus V_0$, the orthocomplement of V_0 in V_1.

Suppose ψ is a wavelet. For $j \in \mathbb{Z}$, let W_j be the subspace generated by $\{U^jT^l\psi : l \in \mathbb{Z}\}$ and let

$$V_j = \oplus_{k<j} W_k. \tag{1.4}$$

Then $\{V_j : j \in \mathbb{Z}\}$ satisfies (1) to (4) in the Definition 1.1.9 of MRA. If (5) in Definition 1.1.9 is also satisfied, then ψ is an *MRA wavelet.*

Definition 1.1.15. Let ψ be a normalized tight frame wavelet. For $j \in \mathbb{Z}$, let W_j be the subspace generated by $\{U^jT^l\psi : l \in \mathbb{Z}\}$ and V_j as in (1.4). Then $\{V_j : j \in \mathbb{Z}\}$ satisfies (1) to (4) in the Definition 1.1.9 of MRA. Also, if there is a function ϕ in V_0 such that $\{T^l\phi : l \in \mathbb{Z}\}$ is a normalized tight frame for V_0, then we call ψ an *MRA frame wavelet.*

1.2 Wavelet Sets and Frame Sets

Definition 1.2.1. A unitary system $\mathcal{U}$ is a subset of the set of all unitary operators acting on a separable Hilbert space H which contains the identity operator I.

Definition 1.2.2. A vector $x \in H$ is called a normalised tight frame vector (resp. frame vector with bounds A and B) for a unitary system $\mathcal{U}$ if $\mathcal{U}x = \{Ux : U \in \mathcal{U}\}$ forms a tight frame (resp. frame with bounds A and B) for span $(\mathcal{U}x)$. It is called a complete normalised tight frame vector (resp. complete frame vector with bounds A and B) when $\mathcal{U}x$ is a normalized tight frame (resp. frame with bounds A and B) for H.

Notation 1.2.3. Let $\mathcal{F}$ be the extended Fourier transform on $L^2(\mathbb{R})$ and U and T be defined as in (1.1) and (1.2). Then we use the following notations:

$$\hat{U} = \mathcal{F}U\mathcal{F}^{-1},$$

$$\hat{T} = \mathcal{F}T\mathcal{F}^{-1},$$

and

$$\mathcal{U}_{\hat{U},\hat{T}} = \left\{\hat{U}^n\hat{T}^m : n,\ m \in \mathbb{Z}\right\}.$$

Proposition 1.2.4. *For $f \in L^2(\mathbb{R})$*

$$\hat{T}f = e^{-i\cdot}f$$

and

$$\hat{U}f = U^{-1}f$$

Proof.

$$\begin{aligned}\left(\hat{T}f\right)(\xi) &= \left(\mathcal{F}T\mathcal{F}^{-1}f\right)(\xi)\\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}\left(T\mathcal{F}^{-1}f\right)(s)e^{-is\xi}ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\mathcal{F}^{-1} f\right)(s-1) e^{-is\xi} ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\mathcal{F}^{-1} f\right)(u) e^{-i(u+1)\xi} du \\
&= e^{-i\xi} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\mathcal{F}^{-1} f\right)(u) e^{-iu\xi} du \\
&= e^{-i\xi} \mathcal{F}\left(\mathcal{F}^{-1} f\right)(\xi) \\
&= e^{-i\xi} \left(\mathcal{F}\mathcal{F}^{-1}\right)(f)(\xi) \\
&= e^{-i\xi} f(\xi).
\end{aligned}
$$

$$
\begin{aligned}
\left(\hat{U} f\right)(\xi) &= \left(\mathcal{F} U \mathcal{F}^{-1} f\right)(\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(U \mathcal{F}^{-1} f\right)(s) e^{-is\xi} ds \\
&= \frac{1}{\sqrt{2\pi}} \sqrt{2} \int_{\mathbb{R}} \left(\mathcal{F}^{-1} f\right)(2s) e^{-is\xi} ds \\
&= \frac{1}{\sqrt{2\pi}} \sqrt{2} \int_{\mathbb{R}} \left(\mathcal{F}^{-1} f\right)(u) e^{-iu\frac{\xi}{2}} \frac{du}{2} \\
&= \frac{1}{\sqrt{2}} \mathcal{F}\left(\mathcal{F}^{-1} f\right)\left(\tfrac{\xi}{2}\right) \\
&= \frac{1}{\sqrt{2}} f\left(\tfrac{\xi}{2}\right) \\
&= U^{-1} f(\xi).
\end{aligned}
$$

□

Definition 1.2.5. A measurable subset E of $\mathbb{R}$ is called a *wavelet set* if $\frac{1}{\sqrt{2\pi}}\chi_E$, where χ_E is the characteristic function of E, is the Fourier transform of a wavelet in $L^2(\mathbb{R})$.

Example 1.2.6. [14] $\frac{1}{\sqrt{2\pi}}\chi_{[-2\pi,\ -\pi)\cup[\pi,\ 2\pi)}$ is the Fourier transform of the wavelet seen in Example 1.1.5, and hence $[-2\pi,\ -\pi) \cup [\pi,\ 2\pi)$ is a wavelet set.

Definition 1.2.7. A measurable subset E of $\mathbb{R}$ is called a *frame set* if $\frac{1}{\sqrt{2\pi}}\chi_E$ is a complete normalized tight frame vector for $\mathcal{U}_{\hat{U},\hat{T}}$. In other words, E is a frame set if $\frac{1}{\sqrt{2\pi}}\chi_E$ is the Fourier transform of a normalized tight frame wavelet.

Example 1.2.8. [22] $E = [-\pi, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \pi)$ is a frame set.

1.3 Superwavelets and Construction of Superwavelets in $L^2(\mathbb{R})^2$

Now we pass on to the super space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and study superwavelets in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Definition 1.3.1. [22] Suppose that η_1, η_2 are normalized tight frame wavelets. The ordered pair (η_1, η_2) is a *superwavelet* of length 2 if $\{U^kT^l\eta_1 \oplus U^kT^l\eta_2 : k, l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Definition 1.3.2. [22] Let G, E and F be measurable subsets of $\mathbb{R}$. G is 2-dilation congruent to F if there is a bijection $\tau : G \to F$ such that for any $s \in G$ there is $n \in \mathbb{Z}$ satisfying $\tau(s) = 2^n s$. G is $2\pi-$translation congruent to E if there is a bijection $\phi : G \to E$ such that $\phi(s) - s$ is an integral multiple of 2π for each $s \in G$.

Lemma 1.3.3. [12] *Let E and F be bounded measurable sets in $\mathbb{R}$ such that E contains a neighbourhood of 0, and F has nonempty interior and is bounded away from 0. Then there is a measurable set $G \subset \mathbb{R}$, which is 2-dilation congruent to F and $2\pi-$ translation congruent to E.*

Theorem 1.3.4. [22] *Let E be a measurable subset of $\mathbb{R}$. Then E is a frame set if and only if E is both $2\pi-$translation congruent to a subset F of $[0, 2\pi]$ and 2-dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$.*

Remark 1.3.5. $(\eta_1,\ \eta_2)$ is a superwavelet for $\mathcal{U}_{\hat{U},\hat{T}}$ means η_1 and η_2 are normalized tight frame wavelets and

$$\left\{\hat{U}^n\hat{T}^m\left(\eta_1\oplus\eta_2\right) : n,\ m\in\mathbb{Z}\right\} = \left\{\hat{U}^n\hat{T}^m\eta_1\oplus\hat{U}^n\hat{T}^m\eta_2 : n,\ m\in\mathbb{Z}\right\}$$

is an orthonormal basis for $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$.

Using the above results, we have the following examples:

Let $E_1=\left[-\pi,\ -\frac{1}{2}\pi\right)\cup\left[\frac{1}{2}\pi,\ \pi\right)$. Consider the mappings $\phi : E_1\to\left[\frac{1}{2}\pi,\ \frac{3}{2}\pi\right)$ and $\tau : E_1\to\left[-2\pi,\ -\pi\right)\cup\left[\pi,\ 2\pi\right)$ defined by:

$$\phi(s)=\begin{cases} s+2\pi, & s\in\left[-\pi,\ -\frac{1}{2}\pi\right)\\ s, & s\in\left[\frac{1}{2}\pi,\ \pi\right)\end{cases}$$

and

$$\tau(s)=2s,\quad s\in E_1$$

Then ϕ and τ are bijective mappings. Hence by Theorem 1.3.4, noting that $[-\pi,\ \pi]$ is 2π translation congruent to $[0,\ 2\pi]$, E_1 is a frame set.

Let $E=\left[-\frac{1}{2}\pi,\ \frac{1}{2}\pi\right)$ and $F=\left[-\pi,\ -\frac{1}{2}\pi\right)\cup\left[\frac{1}{2}\pi,\ \pi\right)$. Then by Lemma 1.3.3, there exists a measurable set G such that G is 2-dilation congruent to F and $2\pi-$translation congruent to E. Now G is 2-dilation congruent to F and F is 2-dilation congruent to $\left[-2\pi,\ -\pi\right)\cup\left[\pi,\ 2\pi\right)$. Hence G is 2-dilation congruent to $\left[-2\pi,\ -\pi\right)\cup\left[\pi,\ 2\pi\right)$. Similarly, G is 2π translation congruent to $\left[\frac{1}{2}\pi,\ \frac{3}{2}\pi\right)$. Hence, by Theorem 1.3.4, G is a frame set.

Claim:$(\eta_1,\ \eta_2)$ is a superwavelet, where $\eta_1=\frac{1}{\sqrt{2\pi}}\chi_{E_1}$ and $\eta_2=\frac{1}{\sqrt{2\pi}}\chi_G$.

We have, for every $j, l \in \mathbb{Z}, k \neq 0$

$$\begin{aligned}\left\langle \hat{U}^k\hat{T}^l\eta_1 \oplus \hat{U}^k\hat{T}^l\eta_2,\ \hat{T}^j\eta_1 \oplus \hat{T}^j\eta_2 \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} &= \left\langle \hat{U}^k\hat{T}^l\eta_1,\ \hat{T}^j\eta_1 \right\rangle_{L^2(\mathbb{R})} \\ &\quad + \left\langle \hat{U}^k\hat{T}^l\eta_2,\ \hat{T}^j\eta_2 \right\rangle_{L^2(\mathbb{R})} \\ &= 0,\end{aligned}$$

as $\left\langle \hat{U}^k\hat{T}^l\eta_i,\ \hat{T}^j\eta_i \right\rangle = 0 \quad (i = 1, 2)$.

Also

$$\left\langle \hat{T}^l\eta_1 \oplus \hat{T}^l\eta_2,\ \eta_1 \oplus \eta_2 \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = \begin{cases} 1 \text{ if } l = 0 \\ 0 \text{ if } l \neq 0 \end{cases}.$$

Hence $\left\{ \hat{U}^k\hat{T}^l\eta_1 \oplus \hat{U}^k\hat{T}^l\eta_2 : k, l \in \mathbb{Z} \right\}$ is an ONB for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Hence $(\eta_1,\ \eta_2)$ is a super-wavelet of length 2.

We can summarise the above discussion as:

Proposition 1.3.6. *There is a super-wavelet of length 2.*

Definition 1.3.7. If E and F are frame sets, we call (E, F) a strong complementary pair if $\left(\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_E\right),\ \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_F\right)\right)$ is a super-wavelet.

The following version of Theorem 1.3.4 is needed in the following example.

Theorem 1.3.8. *Let E be a measurable subset of $\mathbb{R}$. Then E is a frame set if and only if E is both 2π-translation congruent to a subset F of $[0, 2\pi]$ and 2-dilation generator for $\mathbb{R}$.*

Example 1.3.9. Let $E = [-\pi,\ -\frac{1}{2}\pi) \cup [\frac{1}{2}\pi,\ \pi)$. The argument before Proposition 1.3.6 gives us the existence of the strong complement frame set of E. Now we construct a concrete one. Consider a set of type $[a,\ \frac{\pi}{2}) \cup [2\pi,\ a + 2\pi)$. This set is a 2-dilation generator of a partition of $[0,\ \infty)$ if $\frac{1}{4}(a + 2\pi) = 2a$.

So we get $a = \frac{2\pi}{7}$. Thus $\left[\frac{2\pi}{7},\ \frac{\pi}{2}\right) \cup \left[2\pi,\ \frac{16\pi}{7}\right)$ is a 2-dilation generator of a partition of $[0,\ \infty)$. Symmetrically, $\left[-\frac{16\pi}{7},\ -2\pi\right) \cup \left[-\frac{\pi}{2},\ -\frac{2\pi}{7}\right)$ is a 2-dilation generator of a partition of $(-\infty, 0]$. Writing

$$A = \left[-\frac{16\pi}{7}, -2\pi\right), B = \left[-\frac{\pi}{2}, -\frac{2\pi}{7}\right),$$

$$C = \left[\frac{2\pi}{7}, \frac{\pi}{2}\right), D = \left[2\pi, \frac{16\pi}{7}\right),$$

and we let $L = A \cup B \cup C \cup D$. Then

$$(A + 2\pi) \cup B \cup C \cup (D - 2\pi) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{1.5}$$

Thus L is 2π translation congruent to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

L is a $2-$dilation generator of partition of $\mathbb{R}$ and since $[-\frac{\pi}{2}, \frac{\pi}{2})$ is 2π translation congruent to a subset of $[0, 2\pi)$, Theorem 1.3.8 shows that L is a frame set. Also

$$\tau(E) = [\tfrac{\pi}{2}, \tfrac{3}{2}\pi) \text{ and } \tau(L) = [0, \tfrac{1}{2}\pi) \cup [\tfrac{3}{2}\pi, 2\pi).$$

Hence

$$\tau(E) \cup \tau(L) = [0, 2\pi) \text{ and } \tau(E) \cap \tau(L) = \Phi.$$

Now we use the following result:

Proposition 1.3.10. *Let E and F be frame sets. Then (E, F) is a strong complementary pair if and only if both $\tau(E) \cup \tau(F) = [0, 2\pi)$ and $\tau(E) \cap \tau(F)$ has measure zero.*

By this proposition, the pair (E, L), where E and L are as obtained above, is a strong complementary pair. Thus L is a strong complementary frame set of E and $\left(\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_E\right), \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_L\right)\right)$ is a super-wavelet.

The above two crucial results 1.3.6 and 1.3.9 are the essence of what is done in [22].

We conclude this section by giving the definition of superwavelet having length n.

Definition 1.3.11. [22] Suppose that $\eta_1, \ldots, \eta_n$ are normalized tight frame wavelets. The n-tuple $(\eta_1, \ldots, \eta_n)$ is a *superwavelet* of length n if $\{U^kT^l\eta_1 \oplus \ldots \oplus U^kT^l\eta_n : k, l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})^n$.

It is shown in [22] that for any n, there is a superwavelet of length n. Just like in the $L^2(\mathbb{R})$ case, the superwavelet usually generates a normalized tight frame and to get an orthogonal basis, some extra conditions must be imposed on the initial low-pass filter from which the superwavelet is constructed such as the Cohen condition [10] or Lawton's condition [34], Theorem 3.9 in [5]. In [18] and [17], a certain affine structure on the space $L^2(\mathbb{R})^n$ was introduced which was shown to admit multiresolution wavelet bases.

1.4 MRA Superwavelets

We have seen in the previous section that $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ has a wavelet (superwavelet of length 2). In this section we discuss whether MRA wavelet (i.e., wavelet made out of an MRA) can be found in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

The following theorem tells us that there is no MRA superwavelet in the usual sense, i.e., as given in Definition 1.1.9.

Theorem 1.4.1. [22] *Let $\phi_1, \phi_2 \in L^2(\mathbb{R})$ and let V_0 be the closed subspace generated by $\{T^l\phi_1 \oplus T^l\phi_2 : l \in \mathbb{Z}\}$. Suppose $V_0 \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and $V_0 \subset (U \oplus U)V_0$ and that*

$$\{T^l\phi_1 \oplus T^l\phi_2 : l \in \mathbb{Z}\}$$

is an orthonormal basis for V_0. *Then*

$$\bigcup_{j\in\mathbb{Z}} \left(U^j \oplus U^j\right) V_0$$

is not dense in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Since MRA superwavelets cannot be defined similar to MRA wavelets, Han and Larson [22] have modified the definition of MRA superwavelet as follows:

Definition 1.4.2. A superwavelet $(\eta_1, \eta_2) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is an *MRA superwavelet* if each η_1 and η_2 are MRA *frame* wavelets.

Remark 1.4.3. In the case of MRA waveletes in $L^2(\mathbb{R})$, ψ is an MRA wavelet if $\{U^jT^k\psi : j,k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. However, $(\eta_1, \eta_2) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is an MRA superwavelet only when both η_1 and η_2 are MRA frame wavelets in $L^2(\mathbb{R})$.

Proposition 1.4.4. [11] *For a frame set* G, $\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_G\right)$ *is an MRA frame if and only if* $G^s := \bigcup_{j=1}^{\infty} 2^{-j}G$ *is* 2π*-translation congruent to a subset of* $[-\pi, \pi]$.

The following is an important result. In the proof given below we have explicitly constructed the function ϕ.

Proposition 1.4.5. [11] *There is an MRA superwavelet of length 2.*

Proof. Consider E and L as in Example 1.3.9.

Also, let

$$E^s = \bigcup_{j=1}^{\infty} 2^{-j}E, \ L^s = \bigcup_{j=1}^{\infty} 2^{-j}L.$$

Then $E^s = \left[-\frac{\pi}{2},\ 0\right) \cup \left(0,\ \frac{\pi}{2}\right]$ and

$$\begin{aligned} L^s &= \left[-\frac{8\pi}{7},\ -\pi\right) \cup \left[-\frac{4\pi}{7},\ -\frac{\pi}{2}\right) \cup \left[-\frac{2\pi}{7},\ 0\right) \\ &\cup \left[0,\ \frac{2\pi}{7}\right) \cup \left[\frac{\pi}{2},\ \frac{4\pi}{7}\right) \cup \left[\pi,\ \frac{8\pi}{7}\right) \end{aligned}$$

E^s is 2π-translation congruent to the set itself which is a subset of $[-\pi,\ \pi]$. The map ϕ defined by

$$\phi(s) = \begin{cases} s + 2\pi, & s \in \left[-\frac{8\pi}{7},\ -\pi\right) \\ s, & s \in \left[-\frac{4\pi}{7},\ -\frac{\pi}{2}\right) \cup \left[-\frac{2\pi}{7},\ 0\right) \cup \left[0,\ \frac{2\pi}{7}\right) \cup \left[\frac{\pi}{2},\ \frac{4\pi}{7}\right) \\ s - 2\pi, & s \in \left[\pi,\ \frac{8\pi}{7}\right) \end{cases}$$

shows that the set L^s is 2π-translation congruent to the set

$$\left[-\pi,\ -\frac{6\pi}{7}\right) \cup \left[-\frac{4\pi}{7},\ -\frac{\pi}{2}\right) \cup \left[-\frac{2\pi}{7},\ 0\right) \cup \left[0,\ \frac{2\pi}{7}\right) \cup \left[\frac{\pi}{2},\ \frac{4\pi}{7}\right) \cup \left[\frac{6\pi}{7},\ \pi\right),$$

which is a subset of $[-\pi,\ \pi]$. Hence by Proposition 1.4.4 both $\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_E\right)$ and $\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_L\right)$ are MRA frame wavelets. In Example 1.3.9 we have seen that

$$\left(\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_E\right),\ \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_L\right)\right)$$

is a superwavelet and hence by the discussion above and using the Definition 1.4.2, $\left(\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_E\right),\ \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_L\right)\right)$ is an MRA superwavelet. □

1.5 Superwavelets via MRA

In this section and the following sections we discuss the theory for the constructions of superwavelets via MRA giving examples [5]. For the discussion we need an abstract version of the situation existent on $L^2(\mathbb{R})$ and we replace $L^2(\mathbb{R})$

by an abstract Hilbert space H and the dilation and translation operators are replaced by two unitaries U and T satisfying the relation $UTU^{-1} = T^N$ for some integer $N \geq 2$ and the integer N is called the scale for U and T. The standard translation and dilation operators have scale 2.

We have seen in Theorem 1.4.2 that MRA superwavelets do not exist in the usual sense in $L^2(\mathbb{R})^2$. i.e., the technique of multiresolution analysis breaks down, when multiplexing is required, if one just amplifies the steps used in the construction of MRA wavelets. In [5] multiresolution constructions have been realized for multiple signals, provided some slight modifications are done to the usual dilation and translation operators.

Hereinafter, by *superwavelet via MRA* we mean a superwavelet made from a multiresolution analysis (MRA).

Notation 1.5.1. We often identify functions f on $\mathbb{T}$ with $2\pi-$ periodic functions on $\mathbb{R}$ or with functions on the interval $[-\pi,\ \pi)$. The identification is given by $f(z) \leftrightarrow f(\theta)$ where $z = e^{-i\theta}$.

Definition 1.5.2. [5] A *wavelet representation* is a triple $\tilde{\pi} := (H, U, \pi)$ where H is a Hilbert space, U is a unitary on H and π is a representation of $L^\infty(\mathbb{T})$ on H such that

$$U\pi(f)U^{-1} = \pi(f(z^N)), \quad f \in L^\infty(\mathbb{T})$$

(here, by $f\left(z^N\right)$ we mean the map $z \to f(z^N)$).

Definition 1.5.3. [5] A wavelet representation is called *normal* if for any sequence $(f_n)_{n\in\mathbb{N}}$ which converges pointwise a.e. to a function $f \in L^\infty(\mathbb{T})$ and such that $\|f_n\|_\infty \leq M,\ n \in \mathbb{N}$ for some $M > 0$, the sequence $\{\pi(f_n)\}$ converges to $\pi(f)$ in the strong operator topology.

Notation 1.5.4. U is called the dilation and $T := \pi(z)$ the translation of the wavelet representation, where z indicates the identity function on $\mathbb{T}$, $z \mapsto z$. We note that the identity function (denoted by z) on $\mathbb{T}$ is an element in $L^\infty(\mathbb{T})$.

Some of the examples in this chapter are outlined in [5]. For the sake of completeness, we give them with details.

Example 1.5.5. A classical example of a normal wavelet representation is the following: $H = L^2(\mathbb{R})$. We choose an integer $N \geq 2$.

$$U\xi(x) = \frac{1}{\sqrt{N}}\xi\left(\frac{x}{N}\right), \quad \xi \in L^2(\mathbb{R}),$$

and π is defined by its Fourier transform

$$\hat{\pi}(f)(\xi) = f\xi, \tag{1.6}$$

where $f \in L^\infty(\mathbb{T})$, $\xi \in L^2(\mathbb{R})$.

It is enough to check that for $\xi \in H$

$$\left(U\pi(f)U^{-1}(\xi)\right)\hat{}(\cdot) = \left(\pi(f(z^N))(\xi)\right)\hat{}(\cdot)$$

Now

$$\begin{aligned}
\left(U\pi(f)U^{-1}(\xi)\right)\hat{} &= U^{-1}\left(\left(\pi(f)U^{-1}(\xi)\right)\hat{}\right) \\
&= U^{-1}\left(\hat{\pi}(f)\left(U^{-1}(\xi)\right)\hat{}\right), \\
&\qquad \text{noting that } (Tf)\hat{} = \hat{T}\hat{f}, \text{ where } \hat{T}\hat{f} = e^{-i\cdot}\hat{f} \\
&= U^{-1}\left(\hat{\pi}(f)U(\hat{\xi})\right) \\
&= U^{-1}\left(fU(\hat{\xi})\right)
\end{aligned}$$

$$\begin{aligned} &= \sqrt{N} f(N\cdot) U(\hat{\xi})(N\cdot) \\ &= \sqrt{N} f(N\cdot) \frac{1}{\sqrt{N}} \hat{\xi}(\cdot) \\ &= f(N\cdot)\hat{\xi} \\ &= \left(\pi(f(z^N))(\xi)\right)\hat{}. \end{aligned}$$

In particular (1.6) gives

$$\hat{T}\xi(x) = e^{-ix}\xi\,(x)$$

so that

$$T\xi(x) = \xi\,(x-1),$$

for $\xi \in L^2(\mathbb{R})$, $x \in \mathbb{R}$.

We denote this normal wavelet representation by $\Re_0$.

Example 1.5.6. If $(H_i,\, U_i,\, \pi_i)$ are (normal) wavelet representations for $i \in \{1,\ldots,n\}$, then $(\oplus_{i=1}^n H_i, \oplus_{i=1}^n U_i, \oplus_{i=1}^n \pi_i)$ is a (normal) wavelet representation called the direct sum of the given wavelet representations.

Definition 1.5.7. For a given scale $N \geq 2$, an N *cycle* is a set $\{z_1,\ldots,z_p\}$ of distinct points in $\mathbb{T}$, such that $z_1^N = z_2, z_2^N = z_3, \ldots, z_p^N = z_1$. p is called the length of the cycle. $\{1\}$ is called the trivial cycle. $\{\omega,\, \omega^2\}$, where $\omega = e^{-\frac{2\pi i}{3}}$, is a 2-cycle having length 2.

Example 1.5.8. Let $\tilde{\pi} = (H,\, U, \pi)$ be a (normal) wavelet representation. Let $C := \{z_1,\ldots,z_p\}$ be a cycle and $\alpha_1,\ldots,\alpha_p \in \mathbb{T}$. Define

$$H_{C,\alpha} := \underbrace{H \oplus H \oplus \ldots \oplus H}_{p\,\text{times}}$$

and, for $f \in L^{\infty}(\mathbb{T})$, $\xi_1, \ldots, \xi_p \in H$.

$$U_{C,\alpha}(\xi_1, \ldots, \xi_p) := (\alpha_1 U\xi_2, \alpha_2 U\xi_3, \ldots, \alpha_{p-1} U\xi_p, \alpha_p U\xi_1),$$

$$\pi_{C,\alpha}(f)(\xi_1, \ldots, \xi_p) := (\pi(f(z_1 z))\xi_1, \pi(f(z_2 z))\xi_2, \ldots, \pi(f(z_p z))\xi_p).$$

Then $\tilde{\pi}_{C,\alpha} := (H_{C,\alpha}, U_{C,\alpha}, \pi_{C,\alpha})$ is a (normal) wavelet representation which we call the *cyclic amplification* of $\tilde{\pi}$ with cycle C and modulation α.
In particular,

$$T_{C,\alpha} := \pi_{C,\alpha}(z)$$

and

$$T := \pi(z),$$

where z is the identity map on $\mathbb{T}$. Since $(\pi \circ (z \to z_1 z))(\xi_1) = z_1 T\xi_1$, we have

$$T_{C,\alpha}(\xi_1, \ \ldots, \ \xi_p) = (z_1 T\xi_1, \ \ldots, \ z_p T\xi_p)$$

The cyclic amplification with the trivial cycle and $\alpha_1 = 1$ is the initial wavelet representation.

Notation 1.5.9. When $\alpha_1 = \ldots = \alpha_p = 1$ in Example 1.5.8, we use the following notation:

$$\tilde{\pi}_C := \tilde{\pi}_{C,\alpha}.$$

Example 1.5.10 (Special case of Example 1.5.8). If C is a cycle of length p and $\alpha_1, \ldots, \alpha_p$ are in $\mathbb{T}$, we denote by

$$\mathfrak{R}_{C,\alpha} = (L^2(\mathbb{R})_{C,\alpha}, U_{C,\alpha}, \pi_{C,\alpha})$$

the cyclic amplification $(\mathfrak{R}_0)_{C,\alpha}$ of the main representation $\mathfrak{R}_0$. When all α_i are

1, we use the notation

$$\Re_C = (L^2(\mathbb{R})_C, U_C, \pi_C).$$

Example 1.5.11. Another important wavelet representation is the direct sum of wavelet representations associated to several N cycles. That is, if $C_1, \ldots, C_n$ are distinct cycles and $\alpha_1, \ldots, \alpha_n$ are some finite sets of numbers in $\mathbb{T}$, then let $C := C_1 \cup \ldots \cup C_n$, and define

$$\Re_{C,\alpha} := \Re_{C_1,\alpha_1} \oplus \ldots \oplus \Re_{C_n,\alpha_n},$$

and is called the wavelet representation associated to the cycles $C_1, \ldots, C_n$ and the numbers $\alpha_1, \ldots, \alpha_n$.

Definition 1.5.12. [5] Let (H, U, π) be a wavelet representation. A sequence $(V_n)_{n\in\mathbb{Z}}$ of closed subspaces of H with the properties

i. $V_n \subset V_{n+1}$;

ii. $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = H$;

iii. $\overline{\bigcap_{n\in\mathbb{Z}} V_n} = \{0\}$;

iv. $U(V_n) = V_{n-1}$;

v. There is a $\phi \in V_0$ such that $\{T^k\phi : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0, is called a *multiresolution analysis (MRA)* for the wavelet representation.

A vector ϕ as in (v) for which there exists an MRA such that (i) to (iv) hold is called *orthogonal scaling vector.*

Theorem 1.5.13. [5] *Let $\Re_{C,\alpha}$ be the wavelet representation in Example 1.5.11. Denote by $e^{-i\theta_{i,j}}$ the j-th point of the cycle C_i. Then $\varphi_C = \varphi_{C_1} \oplus \varphi_{C_2} \oplus \ldots \oplus \varphi_{C_n}$ is an orthogonal scaling vector for this wavelet representation if and only if the following conditions are satisfied:*

i. [Orthogonality] $\sum_{i=1}^{n}\sum_{k=1}^{p_i}\left(\mathrm{Per}\left(\left|\hat{\varphi}_{C_{i,k}}\right|^2\right)\right)(\xi-\theta_{i,k})=1$ for a.e. $\xi\in\mathbb{R}$;

ii. [Scaling equation] *There exists a function* $m_0\in L^\infty(\mathbb{T})$ *such that for a.e.* $\xi\in\mathbb{R}$ *and for all* $i\in\{1,\ \dots,\ n\}$:

$$\begin{aligned}
\alpha_{i,1}\sqrt{N}\hat{\varphi}_{C_{i,2}}(N\xi) &= m_0\left(\theta_{i,1}+\xi\right)\hat{\varphi}_{C_{i,1}}(\xi),\\
\alpha_{i,2}\sqrt{N}\hat{\varphi}_{C_{i,3}}(N\xi) &= m_0\left(\theta_{i,2}+\xi\right)\hat{\varphi}_{C_{i,2}}(\xi),\\
\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\\
\alpha_{i,p_i}\sqrt{N}\hat{\varphi}_{C_{i,1}}(N\xi) &= m_0\left(\theta_{i,p_i}+\xi\right)\hat{\varphi}_{C_{i,p_i}}(\xi);
\end{aligned}$$

iii. [Cyclicity] *For each* $i\in\{1,\ \dots,\ n\}$, $j\in\{1,\ \dots,\ p_i\}$, $\hat{\varphi}_{C_i}$ *does not vanish on any subset* E *of* $\mathbb{R}$ *invariant under dilations by* N^{p_i} *(i.e.,* $N^{p_i}E=E$*) of positive measure.*

Proposition 1.5.14. [5] *Choose a scale* $N\geq 2$. *Let* $\tilde{\pi}$ *be a normal wavelet representation having an orthogonal scaling function* ϕ *with non-degenerate filter* m_0. *Denote by* $(V_n)_{n\in\mathbb{Z}}$ *the associated MRA. Assume that there are given the "high-pass filters"* $m_1,\dots,m_{N-1}\in L^\infty(\mathbb{T})$ *satisfying*

$$\frac{1}{\sqrt{N}}\begin{bmatrix} m_0(z) & m_0(\rho z) & \cdots & m_0(\rho^{N-1}z)\\ m_1(z) & m_1(\rho z) & \cdots & m_1(\rho^{N-1}z)\\ \vdots & \vdots & \vdots & \vdots\\ m_{N-1}(z) & m_{N-1}(\rho z) & \cdots & m_{N-1}(\rho^{N-1}z)\end{bmatrix} \tag{1.7}$$

is unitary for a.e. $z\in\mathbb{T}$, $\left(\rho=e^{\frac{2\pi i}{N}}\right)$, *and define* $\psi_i\in H$ *by*

$$\psi_i=:U^{-1}\pi(m_i)\phi,\quad (i\in\{1,\ \dots,\ N-1\}). \tag{1.8}$$

Then

$$\left\{T^k\psi_i : k\in\mathbb{Z},\ i\in\{1,\ \dots,\ N-1\}\right\}\ \text{is an orthonormal basis for } V_1\ominus V_0$$

and

$$\{U^m T^n \psi_i : m,\ n \in \mathbb{Z},\ i \in \{1, \ldots, N-1\}\} \text{ is an orthonormal basis for } H.$$

Remark 1.5.15. Thus by starting with a scaling vector ϕ, we can work out $N-1$ wavelets, where N is the scale.

1

ON THE STUDY OF FRAME MULTIRESOLUTION ANALYSIS IN THE SUPERSPACES $L^2(\mathbb{R})^n$

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Chapter 2

Examples of Superwavelets via MRA

As we have mentioned in the introduction we are devoting a chapter for constructing some examples of superwavelets via MRA in superspaces.

2.1 Construction of Orthogonal Scaling Vector with Scale $N = 2$

We construct an orthogonal scaling function ϕ for the wavelet representation $(\Re_{C_1} \oplus \ldots \oplus \Re_{C_p})$, where C_1, C_2, ..., C_p are 2-cycles. Also, let $C = C_1 \cup C_2 \cup \ldots \cup C_p$. Let $x \in \mathbb{R}$:

1. x is called a *cycle point* if there is a $c \in C$ such that $x \equiv \theta \bmod 2\pi$, where $e^{-i\theta} = c$;

2. x is called a *supplement* if $x - \pi$ is a cycle point;

3. x is called a *main point* if it is a cycle point or a supplement;

4. x is called a *mid-point* if $x = \frac{a+b}{2}$ with a, b consecutive main points;

5. x is called a *cycle midpoint* if $x = \frac{a+b}{2}$ with a, b consecutive cycle points.

In the above, when we say 'consecutive', we refer to the order on the real line. For $z \in C$ with $z = e^{-i\theta_0}$ define for $\theta \in [-\pi,\ \pi]$,

$$\hat{\varphi}_z(\theta) = \chi_{\left[\frac{a(\theta_0)+\theta_0}{2},\ \frac{\theta_0+b(\theta_0)}{2}\right]}(\theta+\theta_0),$$

where $a(\theta_0)$, θ_0, $b(\theta_0)$ are consecutive cycle points. It can be verified that

$$\sum_{z=e^{-i\theta_0}\in C} \mathrm{Per}\, |\hat{\varphi}_z|^2\,(\theta-\theta_0) = 1, \quad \theta \in \mathbb{R}.$$

Hence by Theorem 1.5.13, ϕ defined by its Fourier transform

$$\hat{\phi} = \oplus_{z\in C}\hat{\varphi}_z = \oplus_{i=1}^{p} \oplus_{z\in C_i} \hat{\varphi}_z =: \oplus_{i=1}^{p}\hat{\varphi}_{C_i}$$

is a good candidate for an orthogonal scaling vector corresponding to C built out of orthogonal scaling vector corresponding to each C_i. Next, let us define the filter m_0:

$$m_0 = \sum_{\theta_0\in[-\pi,\ \pi],\ \text{cycle point}} \chi_{\left[\frac{c(\theta_0)+\theta_0}{2},\ \frac{\theta_0+d(\theta_0)}{2}\right]\cap[-\pi,\ \pi]},$$

where for the cycle point $\theta_0 \in [-\pi,\ \pi], c(\theta_0)$, θ_0, $d(\theta_0)$ are consecutive main points. We will now check the scaling equation for the above defined ϕ and filter m_0. It suffices to show that for two consecutive elements $z_0 = e^{-i\theta_0}$, $z_1 = e^{-i\theta_1}$ of a cycle C_i (i.e., $z_0^2 = z_1$, or equivalently $2\theta_0 \equiv \theta_1 \bmod 2\pi$) the following holds:

$$\hat{\varphi}_{z_1}(2\theta) = m_0\,(\theta+\theta_0)\,\hat{\varphi}_{z_0}(\theta), \quad \text{a.e. } \theta \in \mathbb{R}$$

or, equivalently

$$(2.1) \qquad \hat{\varphi}_{z_1}(2\theta - 2\theta_0) = m_0(\theta)\,\hat{\varphi}_{z_0}(\theta - \theta_0), \quad \text{a.e. } \theta \in \mathbb{R}.$$

Suppose

$$\hat{\varphi}_{z_0}(\theta - \theta_0) = \chi_{\left[\frac{\alpha+\theta_0}{2},\, \frac{\beta+\theta_0}{2}\right]}(\theta),$$

where, $\alpha<\theta_0 < \beta$ are consecutive cycle points. By the definition of m_0, it follows that there are consecutive main points $a<\theta_0 < b$ such that

$$(2.2) \qquad m_0(\theta)\hat{\varphi}_{z_0}(\theta - \theta_0) = \chi_{\left[\frac{a+\theta_0}{2},\, \frac{b+\theta_0}{2}\right]}(\theta).$$

Actually a is either α or the first supplement on the left of θ_0. Suppose

$$\hat{\varphi}_{z_1}(\theta - \theta_1) = \chi_{\left[\frac{a_1+\theta_1}{2},\, \frac{b_1+\theta_1}{2}\right]}(\theta),$$

with $a_1 < \theta_1 < b_1$ consecutive main points. Then, we obtain

$$\hat{\varphi}_{z_1}(\theta) = \chi_{\left[\frac{a_1-\theta_1}{2},\, \frac{b_1-\theta_1}{2}\right]}(\theta),$$

and

$$(2.3) \qquad \hat{\varphi}_{z_1}(2\theta - 2\theta_0) = \chi_{\left[\frac{a_1-\theta_1}{2},\, \frac{b_1-\theta_1}{2}\right]}(2\theta - 2\theta_0).$$

We note that, under the map $x \mapsto 2x$, the consecutive main points $a < \theta_0 < b$ are mapped into consecutive cycle points . Since $2\theta_0 \equiv \theta_1 \bmod 2\pi$, and because a translation by an integer multiple of 2π maps consecutive cycle points to consecutive cycle points , it follows in particular that there exists $k \in \mathbb{Z}$ such that

$$a_1 = 2a + 2k\pi, \quad \theta_1 = 2\theta_0 + 2k\pi, \quad \text{and } b_1 = 2b + 2k\pi.$$

We have

$$\frac{a_1 - \theta_1}{2} \leq 2\theta - 2\theta_0 \leq \frac{b_1 - \theta_1}{2}$$

if and only if

$$\frac{a + \theta_0}{2} \leq \theta \leq \frac{b + \theta_0}{2}.$$

Hence with (2.2) and (2.3), the relation (2.1) is obtained.

The cylicity condition is automatically satisfied because all $\hat{\varphi}_z$ contain a neighbourhood of 0.

Consequently by Theorem 1.5.13 ϕ is an orthogonal scaling vector with filter m_0.

2.2 Examples of Superwavelets via MRA with Scale $N = 2$

In this section we workout particular cases of the filters, orthogonal scaling vectors, and wavelets for some superspaces. Here we take the *scale* as $N = 2$. Since the low pass filter m_0 is just a characterstic function, the corresponding high-pass filter m_1 can be chosen as the characterstic function of the complement of the set that gives m_0, and (1.7) holds. Then the wavelet is defined as in (1.8).

Example 2.2.1. (*Superwavelet via MRA in* $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$) As a special case of Example 1.5.8, we consider the dilation and translation operators on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. First consider a fixed 2 cycle, i.e., a periodic orbit for the map $z \mapsto z^2$ on the unit circle. Let $C = \{z_1, z_2\}$ be this cycle, so that $z_1^2 = z_2$, $z_2^2 = z_1$. Let $\alpha_1 = \alpha_2 = 1$ and let the Hilbert space be

$$H_C = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}).$$

For $f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, the translation operator T_C on H_C is given by

$$T_C\,(f_1 \oplus f_2) = z_1 T f_1 \oplus z_2 T f_2, \tag{2.4}$$

and the dilation operator U_C on H_C is given by

$$U_C\,(f_1 \oplus f_2) = U f_2 \oplus U f_1, \tag{2.5}$$

where we take T and U as the translation and dilation operators on $L^2(\mathbb{R})$ defined by:

$$Tf(x) = f(x-1) \tag{2.6}$$

$$Uf(x) = \frac{1}{\sqrt{2}} f(\tfrac{x}{2}) \tag{2.7}$$

for $f \in L^2(\mathbb{R})$, $x \in \mathbb{R}$. For this T_C and U_C the scale is 2.

Consider the 2-cycle $C = \{z_1, z_2\}$, where $z_1 = e^{-\frac{2\pi i}{3}}$, $z_2 = e^{-\frac{4\pi i}{3}} = e^{-\frac{-2\pi i}{3}}$, so that $z_1^2 = z_2$ and $z_2^2 = z_1$.

The cycle points are $-\frac{8\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$.

The supplements are $-\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$.

The main points are

$$-\frac{8\pi}{3}, -\frac{5\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{11\pi}{3}.$$

Hence

$$\begin{aligned}
\hat{\varphi}_1 &= \chi_{\left[\frac{a(\theta_0)+\theta_0}{2},\, \frac{\theta_0+b(\theta_0)}{2}\right]}(\theta+\theta_0), \qquad \theta_0 = \frac{2\pi}{3} \\
&= \chi_{[0,\, \pi]}(\theta + \frac{2\pi}{3}) \\
&= \chi_{\left[-\frac{2\pi}{3},\, \frac{\pi}{3}\right]}(\theta)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\varphi}_2 &= \chi_{[-\pi,\, 0]}\left(\theta - \frac{2\pi}{3}\right) \\
&= \chi_{\left[-\frac{\pi}{3},\, \frac{2\pi}{3}\right]}(\theta).
\end{aligned}$$

Then the orthogonal scaling vector $(\varphi_1, \varphi_2) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}))$ is given by the Fourier transforms of φ_1 and φ_2 as given above.

The filter m_0 is determined as follows:

$$\begin{aligned}
m_0 &= \sqrt{2} \sum_{\theta_0 \in [-\pi,\, \pi]} \chi_{\left[\frac{c(\theta_0)+\theta_0}{2},\, \frac{\theta_0+d(\theta_0)}{2}\right] \cap [-\pi,\, \pi]} \\
&= \sqrt{2}\left(\chi_{\left[-\pi,\, -\frac{\pi}{2}\right]} + \chi_{\left[\frac{\pi}{2},\, \pi\right]}\right) \\
&= \sqrt{2}\chi_{\left[-\pi,\, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2},\, \pi\right]}
\end{aligned}$$

The high pass filter is

$$m_1 = \sqrt{2}\chi_{\left[-\frac{\pi}{2},\, \frac{\pi}{2}\right]}$$

Determination of the orthogonal wavelet (ψ_1, ψ_2): Using (1.8), we have

$$U_C(\psi_1,\, \psi_2) = \pi_C(m_1)(\varphi_1,\, \varphi_2)$$

$$(U\psi_2,\, U\psi_1) = \pi_C(m_1)(\varphi_1,\, \varphi_2)$$

Taking the Fourier transform, we have

$$(U\psi_2,\ U\psi_1)^{\wedge} = \left(\hat{\pi}\left(m_1\left(z_1 z\right)\right)\hat{\varphi}_1,\ \hat{\pi}\left(m_1\left(z_2 z\right)\right)\hat{\varphi}_2\right)$$

Hence

$$\begin{aligned}
U^{-1}\hat{\psi}_2 &= \hat{\pi}(m_1(z_1 z))\hat{\varphi}_1 \\
&= m_1(z_1 z)\hat{\varphi}_1 \\
&= \sqrt{2}\chi_{[-\frac{3\pi}{6},\, \frac{3\pi}{6}]+\left(-\frac{4\pi}{6}\right)}\chi_{[-\frac{4\pi}{6},\, \frac{2\pi}{6}]} \\
&= \sqrt{2}\chi_{[-\frac{7\pi}{6},\, -\frac{\pi}{6}]\cap[-\frac{4\pi}{6},\, \frac{2\pi}{6}]} \\
&= \sqrt{2}\chi_{[-\frac{4\pi}{6},\, -\frac{\pi}{6}]}
\end{aligned}$$

$$\begin{aligned}
U^{-1}\hat{\psi}_1 &= \hat{\pi}(m_1(z_2 z))\hat{\varphi}_2 \\
&= m_1(z_2 z)\hat{\varphi}_2 \\
&= \sqrt{2}\chi_{[-\frac{\pi}{2},\, \frac{\pi}{2}]+\left(\frac{4\pi}{6}\right)}\chi_{[-\frac{2\pi}{6},\, \frac{4\pi}{6}]} \\
&= \sqrt{2}\chi_{[\frac{\pi}{6},\, \frac{7\pi}{6}]\cap[-\frac{2\pi}{6},\, \frac{4\pi}{6}]} \\
&= \sqrt{2}\chi_{[\frac{\pi}{6},\, \frac{4\pi}{6}]}.
\end{aligned}$$

Hence the orthogonal wavelet $(\psi_1,\ \psi_2)$ is given by

$$\hat{\psi}_1 = \chi_{[\frac{\pi}{3},\, \frac{4\pi}{3}]} \quad \text{and} \quad \hat{\psi}_2 = \chi_{[-\frac{4\pi}{3},\, -\frac{\pi}{3}]}.$$

Example 2.2.2. (*Superwavelet via MRA in* $L^2(\mathbb{R})^6$)

In this example, instead of considering a single cycle, we consider two 2-cycles and construct a superwavelet via MRA in $L^2(\mathbb{R})^6$.

$$C_1 = \left\{ z_1 = e^{-\frac{2\pi i}{5}},\ z_2 = e^{-\frac{4\pi i}{5}},\ z_3 = e^{-\frac{-2\pi i}{5}},\ z_4 = e^{-\frac{-4\pi i}{5}} \right\},$$

and

$$C_2 = \left\{ z_5 = e^{-\frac{2\pi i}{3}},\ z_6 = e^{-\frac{-2\pi i}{3}} \right\}.$$

The translation and dilation operators are defined as follows:

For $(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6) \in L^2(\mathbb{R})^6$

$$T_C\,(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6) = (z_1 T f_1,\ z_2 T f_2,\ z_3 T f_3,\ z_4 T f_4,\ z_5 T f_5,\ z_6 T f_6)$$

and

$$\begin{aligned} U_C\,(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6) &= \left(U_{C_1}\,(f_1,\ f_2,\ f_3,\ f_4),\ U_{C_2}\,(f_5,\ f_6)\right) \\ &= \left(U f_2,\ U f_3,\ U f_4,\ U f_1,\ U f_6,\ U f_5\right). \end{aligned}$$

Here again scale for T_C and U_C is 2.

The cycle points are:

$$-\frac{14\pi}{5},\ -\frac{8\pi}{3},\ -\frac{12\pi}{5},\ -\frac{8\pi}{5},\ -\frac{4\pi}{3},\ -\frac{6\pi}{5},\ -\frac{4\pi}{5},\ -\frac{2\pi}{3},\ -\frac{2\pi}{5},$$

$$\frac{2\pi}{5},\ \frac{2\pi}{3},\ \frac{4\pi}{5},\ \frac{6\pi}{5},\ \frac{4\pi}{3},\ \frac{8\pi}{5},\ \frac{12\pi}{5},\ ,\ \frac{8\pi}{3},\ \frac{14\pi}{5}.$$

The supplements are:

$$-\frac{9\pi}{5},\ -\frac{5\pi}{3},\ -\frac{7\pi}{3},\ -\frac{3\pi}{5},\ -\frac{\pi}{3},\ -\frac{\pi}{5},\ \frac{\pi}{5},\ \frac{\pi}{3},\ \frac{3\pi}{5},$$

$$\frac{7\pi}{5},\ \frac{5\pi}{3},\ \frac{9\pi}{5},\ \frac{11\pi}{5},\ \frac{7\pi}{3},\ \frac{13\pi}{5},\ \frac{17\pi}{5},\ \frac{11\pi}{3},\ \frac{19\pi}{5}.$$

The main points are:

$$-\frac{14\pi}{5},\ -\frac{8\pi}{3},\ -\frac{12\pi}{5},\ -\frac{9\pi}{5},\ -\frac{5\pi}{3},\ -\frac{7\pi}{3},\ -\frac{8\pi}{5},\ -\frac{4\pi}{3},\ -\frac{6\pi}{5},$$

$$-\frac{4\pi}{5},\ -\frac{2\pi}{3},\ -\frac{3\pi}{5},\ -\frac{2\pi}{5},\ -\frac{\pi}{3},\ -\frac{\pi}{5},\ \frac{\pi}{5},\ \frac{\pi}{3},\ \frac{2\pi}{5},$$

$$\frac{3\pi}{5},\ \frac{2\pi}{3},\ \frac{4\pi}{5},\ \frac{6\pi}{5},\ \frac{4\pi}{3},\ \frac{7\pi}{5},\ \frac{8\pi}{5},\ \frac{5\pi}{3},\ \frac{9\pi}{5},$$

$$\frac{11\pi}{5},\ \frac{7\pi}{3},\ \frac{12\pi}{5},\ \frac{13\pi}{5},\ \frac{8\pi}{3},\ \frac{14\pi}{5},\ \frac{17\pi}{5},\ \frac{11\pi}{3},\ \frac{19\pi}{5}.$$

Hence the orthogonal scaling function for the wavelet representation $\mathcal{R}_{C_1} \oplus \mathcal{R}_{C_2}$ is $\varphi = ((\varphi_1^1,\ \varphi_2^1,\ \varphi_3^1,\ \varphi_4^1),\ (\varphi_1^2,\ \varphi_2^2))$, where $\varphi_1^1,\ \varphi_2^1,\ \varphi_3^1,\ \varphi_4^1, \varphi_1^2$, and φ_2^2 are given by their Fourier transforms as below:

$$\hat{\varphi}_1^1(\theta) = \chi_{\left[-\frac{6\pi}{15},\ \frac{2\pi}{15}\right]}(\theta),\ \ \hat{\varphi}_2^1(\theta) = \chi_{\left[-\frac{\pi}{15},\ \frac{3\pi}{15}\right]}(\theta),\ \ \hat{\varphi}_3^1(\theta) = \chi_{\left[-\frac{2\pi}{15},\ \frac{6\pi}{15}\right]}(\theta),$$

$$\hat{\varphi}_4^1(\theta) = \chi_{\left[-\frac{3\pi}{15},\ \frac{\pi}{15}\right]}(\theta),\ \ \hat{\varphi}_1^2(\theta) = \chi_{\left[-\frac{2\pi}{15},\ \frac{\pi}{15}\right]}(\theta),\ \text{ and }\ \hat{\varphi}_2^2(\theta) = \chi_{\left[-\frac{\pi}{15},\ \frac{2\pi}{15}\right]}(\theta).$$

The filer m_0 is given by

$$m_0 = \sqrt{2}\chi_E,$$

where

$$E = \left[-\pi,\ -\tfrac{19\pi}{30}\right] \cup \left[-\tfrac{15\pi}{30},\ -\tfrac{11\pi}{30}\right] \cup \left[\tfrac{11\pi}{30},\ \tfrac{15\pi}{30}\right] \cup \left[\tfrac{19\pi}{30},\ \pi\right].$$

The high pass filter is

$$m_1 \;=\; \sqrt{2}\chi_{E_1},$$

where

$$E_1 = \left[-\tfrac{19\pi}{30},\ -\tfrac{15\pi}{30}\right] \cup \left[-\tfrac{11\pi}{30},\ \tfrac{11\pi}{30}\right] \cup \left[\tfrac{15\pi}{30},\ \tfrac{19\pi}{30}\right].$$

Using (1.8), and taking Fourier transforms, we obtain

$$U^{-1}\hat{\psi}_1^1 = 0,\ \ U^{-1}\hat{\psi}_2^1 = \sqrt{2}\chi_{\left[-\frac{12\pi}{30},\ -\frac{\pi}{30}\right]\cup\left[\frac{3\pi}{30},\ \frac{4\pi}{30}\right]},\ \ U^{-1}\hat{\psi}_3^1 = 0,$$

$$U^{-1}\hat{\psi}_4^1 = \sqrt{2}\chi_{\left[-\frac{4\pi}{30},\,-\frac{3\pi}{30}\right]\cup\left[\frac{\pi}{30},\,\frac{12\pi}{30}\right]},\ U^{-1}\hat{\psi}_1^2 = \sqrt{2}\chi_{\left[\frac{\pi}{30},\,\frac{4\pi}{30}\right]} \text{ and}$$

$$U^{-1}\hat{\psi}_2^2 = \sqrt{2}\chi_{\left[-\frac{4\pi}{30},\,-\frac{\pi}{30}\right]}.$$

Hence the MRA wavelet is $((\psi_1^1,\ \psi_2^1,\ \psi_3^1,\ \psi_4^1),\ (\psi_1^2,\ \psi_2^2))$, where $\psi_1^1,\ \psi_2^1,\ \psi_3^1,\ \psi_4^1, \psi_1^2,$ and ψ_2^2 are given by their Fourier transforms as below:

$$\hat{\psi}_1^1 = 0,\ \ \hat{\psi}_2^1 = \chi_{\left[-\frac{12\pi}{15},\,-\frac{\pi}{15}\right]\cup\left[\frac{3\pi}{15},\,\frac{4\pi}{15}\right]},\ \ \hat{\psi}_3^1 = 0,$$

$$\hat{\psi}_4^1 = \chi_{\left[-\frac{4\pi}{15},\,-\frac{3\pi}{15}\right]\cup\left[\frac{\pi}{15},\,\frac{12\pi}{15}\right]},\ \ \hat{\psi}_1^2 = \chi_{\left[\frac{\pi}{15},\,\frac{4\pi}{15}\right]} \text{ and } \hat{\psi}_2^2 = \chi_{\left[-\frac{4\pi}{15},\,-\frac{\pi}{15}\right]}.$$

2.3 Example of Superwavelets via MRA with Scale $N = 3$

Consider the 3-cycle $C = \{z_1,\ z_2,\ z_3\}$. For $z \in C$ with $z = e^{-i\theta_0}$ define for $\theta \in [-\pi,\ \pi]$,

$$\hat{\varphi}_z(\theta) = \chi_{\left[\frac{a(\theta_0)+\theta_0}{3},\ \frac{\theta_0+b(\theta_0)}{3}\right]}(\theta + \theta_0),$$

where $a(\theta_0)$, θ_0, $b(\theta_0)$ are consecutive cycle points. Hence by Theorem 1.5.13, ϕ defined by its Fourier transform

$$\hat{\phi} = (\hat{\varphi}_{z_1},\ \hat{\varphi}_{z_2},\ \hat{\varphi}_{z_3}) = (\hat{\varphi}_1,\ \hat{\varphi}_2,\ \hat{\varphi}_3)$$

is an orthogonal scaling vector corresponding to C. Next, the filter m_0 is given by

$$m_0 = \sqrt{3} \sum_{\theta_0 \in [-\pi,\ \pi],\ \text{cycle point}} \chi_{\left[\frac{c(\theta_0)+\theta_0}{3},\ \frac{\theta_0+d(\theta_0)}{3}\right]\cap[-\pi,\ \pi]},$$

where for the cycle point $\theta_0 \in [-\pi,\ \pi], c(\theta_0),\ \theta_0,\ d(\theta_0)$ are consecutive main points. Then the high pass filters m_1 and m_2 are such that

$$\frac{1}{\sqrt{3}}\begin{bmatrix} m_0(z) & m_0(\rho z) & m_0(\rho^2 z) \\ m_1(z) & m_1(\rho z) & m_1(\rho^2 z) \\ m_2(z) & m_2(\rho z) & m_2(\rho^2 z) \end{bmatrix}$$

is unitary for a.e. $z \in \mathbb{T}$ and $(\rho = e^{\frac{2\pi i}{3}})$. We define ψ_1 and ψ_2 in $L^2(\mathbb{R})^3$ as follows:

$$\psi_1 := U^{-1}\pi(m_1)\phi$$

and

$$\psi_2 := U^{-1}\pi(m_2)\phi$$

Then $\{U_C^m T_C^n \psi_i : i = 1, 2, m,\ n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})^3$. We conclude this chapter by giving an example.

Example 2.3.1. (*Superwavelet via MRA in* $L^2(\mathbb{R})^3$)
Now we take $z_1 = e^{-\frac{2\pi i}{26}}$, $z_2 = e^{-\frac{6\pi i}{26}}$ and $z_3 = e^{-\frac{18\pi i}{26}}$. Define, for $f_1 \oplus f_2 \oplus f_3 \in L^2(\mathbb{R})^3$,

$$T_C\left(f_1 \oplus f_2 \oplus f_3\right) = z_1 T f_1 \oplus z_2 T f_2 \oplus z_3 T f_3$$

and

$$U_C\left(f_1 \oplus f_2 \oplus f_3\right) = U f_2 \oplus U f_3 \oplus U f_1,$$

where, for $f \in L^2(\mathbb{R})$,

$$Tf(x) = f(x-1)$$

and

$$Uf(x) = \frac{1}{\sqrt{3}} f\left(\frac{x}{3}\right).$$

This T_C and U_C have scale $N = 3$ and $C = \{z_1,\ z_2,\ z_3\}$ is a 3-cycle.

The cycle points are:

$$-\frac{50\pi}{26},\ -\frac{46\pi}{26},\ -\frac{34\pi}{26},\ \frac{2\pi}{26},\ \frac{6\pi}{26},\ \frac{18\pi}{26},\ \frac{54\pi}{26},\ \frac{58\pi}{26},\ \frac{70\pi}{26}.$$

The supplements are:

$$-\frac{24\pi}{26},\ -\frac{20\pi}{26},\ -\frac{8\pi}{26},\ \frac{28\pi}{26},\ \frac{32\pi}{26},\ \frac{44\pi}{26},\ \frac{80\pi}{26},\ \frac{84\pi}{26},\ \frac{96\pi}{26}.$$

The main points are:

$$-\frac{50\pi}{26},\ -\frac{46\pi}{26},\ -\frac{34\pi}{26},\ -\frac{24\pi}{26},\ -\frac{20\pi}{26},\ -\frac{8\pi}{26},\ \frac{2\pi}{26},\ \frac{6\pi}{26},\ \frac{18\pi}{26},$$

$$\frac{28\pi}{26},\ \frac{32\pi}{26},\ \frac{44\pi}{26},\ \frac{54\pi}{26},\ \frac{58\pi}{26},\ \frac{70\pi}{26},\ \frac{80\pi}{26},\ \frac{84\pi}{26},\ \frac{96\pi}{26}.$$

Then

$$\hat{\varphi}_1 = \chi_{[-\frac{38\pi}{78},\, \frac{2\pi}{78}]},\quad \hat{\varphi}_2 = \chi_{[-\frac{10\pi}{78},\, \frac{6\pi}{78}]},\quad \hat{\varphi}_3 = \chi_{[0,\, \frac{48\pi}{78}]}$$

The filter m_0 is given by

$$m_0 = \sqrt{3}\chi_{[-\frac{6\pi}{78},\, \frac{46\pi}{78}]}.$$

Now we can choose

$$m_1 = \sqrt{3}\chi_{[-\pi,\, -\frac{58\pi}{78}]\cup[\frac{46\pi}{78},\, \pi]}$$

and

$$m_2 = \sqrt{3}\chi_{[-\frac{58\pi}{78},\, -\frac{6\pi}{78}]}.$$

The wavelets $\psi_1 = (\psi_{1,1},\ \psi_{1,2},\ \psi_{1,3})$ and $\psi_2 = (\psi_{2,1},\ \psi_{2,2},\ \psi_{2,3})$ are determined as follows:

$$U_C\psi_1 = U_C\left(\psi_{1,1},\ \psi_{1,2},\ \psi_{1,3}\right) = \pi_C(m_1)\left(\varphi_1,\ \varphi_2,\ \varphi_3\right)$$

Taking Fourier transform, we have

$$\left(U^{-1}\hat{\psi}_{1,2},\ U^{-1}\hat{\psi}_{1,3},\ U^{-1}\hat{\psi}_{1,1}\right) = \left(\hat{\pi}\left(m_1\left(z_1 z\right)\right)\hat{\varphi}_1,\ \hat{\pi}\left(m_1\left(z_2 z\right)\right)\hat{\varphi}_1,\ \hat{\pi}\left(m_1\left(z_3 z\right)\right)\hat{\varphi}_1\right)$$

$$\begin{aligned} U^{-1}\hat{\psi}_{1,2} &= m_1\left(z_1 z\right)\hat{\varphi}_1 \\ &= \sqrt{3}\chi_{\left\{[-\pi,\,-\frac{58\pi}{78}]\cup[\frac{46\pi}{78},\,\pi]\right\}+\left(-\frac{6\pi}{78}\right)\chi_{[-\frac{38\pi}{78},\,\frac{2\pi}{78}]}} \\ &= \sqrt{3}\chi_{[\frac{40\pi}{78},\,\frac{2\pi}{78}]} \end{aligned}$$

Similarly, we obtain

$$\hat{\psi}_{1,3} = 0, \quad \text{and } \hat{\psi}_{1,1} = \chi_{[0,\,\frac{24\pi}{26}]}.$$

Also,

$$\hat{\psi}_{2,1} = 0, \quad \hat{\psi}_{2,2} = \chi_{[-\frac{38\pi}{26},\,-\frac{12\pi}{26}]}, \quad \hat{\psi}_{2,3} = 0.$$

Hence $\{U_C^m T_C^n \psi_i : i = 1, 2, m,\ n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})^3$.

Remark 2.3.2. Thus for a given scale $N \geq 2$ and a superspace we can find orthogonal scaling vector ϕ and can work out $N - 1$ wavelets.

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ON THE STUDY OF FRAME MULTIRESOLUTION ANALYSIS IN THE SUPERSPACES $L^2(\mathbb{R})^n$

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Chapter 3

Frames in $L^2(\mathbb{R})^n$

In this chapter we describe frames in the superspace $L^2(\mathbb{R})^n$ via the Bessel map. We discuss about the frame theory in $L^2(\mathbb{R})^n$ giving various results. We discuss the relation between a countable collection and the associated Bessel map when the countable collection is a frame. We see that the adjoint of the Bessel map has more importance in the study.

3.1 Bessel Map and Frames

Definition 3.1.1. A countable collection $X = \{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})^n$ if there exist constants $A, B > 0$ such that

$$\begin{aligned} A \|f_1 \oplus \cdots \oplus f_n\|^2_{L^2(\mathbb{R})^n} &\leq \sum_{i\in\mathbb{Z}} |\langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle|^2 \\ &\leq B \|f_1 \oplus \cdots \oplus f_n\|^2_{L^2(\mathbb{R})^n}, \end{aligned} \tag{3.1}$$

for every $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$.

The greatest possible such A is the *lower frame bound* and the least possible such B is the *upper frame bound.* If $A = B$, then the frame is called a *tight frame.* If $A = B = 1$, the frame is called *normalized tight frame.*

Definition 3.1.2. A frame $\{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\}$ is called *minimal* or *exact* if

$$\forall\, i \in \mathbb{Z} \qquad x_{i,1} \oplus \cdots \oplus x_{i,n} \notin \overline{\operatorname{span}}\,\{x_{j,1} \oplus \cdots \oplus x_{j,n} : j \in \mathbb{Z} \setminus \{i\}\}\,.$$

A frame is *redundant* or *non exact* if it is not minimal.

Definition 3.1.3. Let $X = \{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\}$ be a countable system in the separable Hilbert space $L^2(\mathbb{R})^n$. If for $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$, the collection $\{\langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle : i \in \mathbb{Z}\} \in l^2(\mathbb{Z})$, then the correspondence

$$f_1 \oplus \cdots \oplus f_n \mapsto \{\langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle : i \in \mathbb{Z}\}$$

denoted by $\mathcal{L}$, is called the *Bessel map* associated with X.

In the case where the Bessel map $\mathcal{L}$ exists, then it is bounded by the uniform boundedness principle.

The following Proposition says that whenever the collection is a frame, then the Bessel map exists and also is a bounded invertible map.

Proposition 3.1.4. *Let* $X = \{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\}$ *be a frame for the Hilbert space* $L^2(\mathbb{R})^n$*. Then the map*

$$\mathcal{L} : L^2(\mathbb{R})^n \to l^2(\mathbb{Z})$$

defined by

$$f_1 \oplus \cdots \oplus f_n \mapsto \{\langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle : i \in \mathbb{Z}\}$$

is the Bessel map associated with X and is a bounded invertible map onto $\mathcal{R}(\mathcal{L})$.

Proof. X is a frame implies there exist constants $A, B > 0$ such that (3.1) holds. Hence

$$\|\mathcal{L}(f_1 \oplus \cdots \oplus f_n)\|^2_{l^2(\mathbb{Z})} = \sum_{i\in\mathbb{Z}} |\langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle|^2 < \infty,$$

and $\mathcal{L}$ is the Bessel map associated with X and we obtain

$$A\,\|f_1 \oplus \cdots \oplus f_n\|^2_{L^2(\mathbb{R})^n} \leq \|\mathcal{L}(f_1 \oplus \cdots \oplus f_n)\|^2_{l^2(\mathbb{Z})} \leq B\,\|f_1 \oplus \cdots \oplus f_n\|^2_{L^2(\mathbb{R})^n}\,.$$

Hence $\mathcal{L}$ is bounded and one-to-one. Hence $\mathcal{L}$ is a bounded invertible map onto $\mathcal{R}(\mathcal{L})$. □

Example 3.1.5. In Example 2.2.1, we have seen that a wavelet on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}))$ is $(\psi_1,\ \psi_2)$ given by

$$\hat{\psi}_1 = \chi_{[\frac{\pi}{3},\, \frac{4\pi}{3}]} \quad \text{and} \quad \hat{\psi}_2 = \chi_{[-\frac{4\pi}{3},\, -\frac{\pi}{3}]}.$$

Explicitly, $(\psi_1,\ \psi_2)$ is given by

$$\psi_1(\gamma) = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{i\frac{4\pi}{3}\gamma} - e^{i\frac{\pi}{3}\gamma}}{i\gamma}\right)$$

and

$$\psi_2(\gamma) = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-i\frac{\pi}{3}\gamma} - e^{-i\frac{4\pi}{3}\gamma}}{i\gamma}\right).$$

Hence $\{U_C^j T_C^k \psi_1 \oplus \psi_2 : \ j, \ k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and hence is a frame for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Hence by Proposition 3.1.4, the map

$$\mathcal{L} : L^2(\mathbb{R}) \oplus L^2(\mathbb{R})) \to l^2(\mathbb{Z})$$

defined by

$$f_1 \oplus f_2 \mapsto \left\{\left\langle f_1 \oplus f_2, \ U_C^j T_C^k \psi_1 \oplus \psi_2 \right\rangle : j, k \in \mathbb{Z}\right\}$$

is a Bessel map. Moreover, here $A = B = 1$, so that $\|\mathcal{L}\,(f_1 \oplus f_2)\| = \|f_1 \oplus f_2\|$ for $f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}))$.

Example 3.1.6. In Example 2.2.2, we have seen that a wavelet on $L^2(\mathbb{R})^6$ is $((\psi_1^1, \ \psi_2^1, \ \psi_3^1, \ \psi_4^1), \ (\psi_1^2, \ \psi_2^2))$, where $\psi_1^1, \ \psi_2^1, \ \psi_3^1, \ \psi_4^1, \psi_1^2,$ and ψ_2^2 given by their Fourier transforms as below:

$$\hat{\psi}_1^1 = 0, \ \ \hat{\psi}_2^1 = \chi_{\left[-\frac{12\pi}{15}, -\frac{\pi}{15}\right] \cup \left[\frac{3\pi}{15}, \frac{4\pi}{15}\right]}, \ \ \hat{\psi}_3^1 = 0,$$

$$\hat{\psi}_4^1 = \chi_{\left[-\frac{4\pi}{15}, -\frac{3\pi}{15}\right] \cup \left[\frac{\pi}{15}, \frac{12\pi}{15}\right]}, \ \ \hat{\psi}_1^2 = \chi_{\left[\frac{\pi}{15}, \frac{4\pi}{15}\right]} \ \text{and} \ \hat{\psi}_2^2 = \chi_{\left[-\frac{4\pi}{15}, -\frac{\pi}{15}\right]}.$$

Hence $\{U_C^j T_C^k \psi_1 \oplus \psi_2 \oplus \psi_3 \oplus \psi_4 \oplus \psi_5 \oplus \psi_6 : \ j, \ k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})^6$ and hence is a frame for $L^2(\mathbb{R})^6$. Hence by Proposition 3.1.4, the map

$$\mathcal{L} : L^2(\mathbb{R})^6 \to l^2(\mathbb{Z})$$

defined by

$$f_1 \oplus \cdots \oplus f_6 \mapsto \left\{\left\langle f_1 \oplus \cdots \oplus f_6, \ U_C^j T_C^k \psi_1 \oplus \cdots \oplus \psi_6 \right\rangle : j, k \in \mathbb{Z}\right\}$$

is a Bessel map. Moreover, here $A = B = 1$, so that $\|\mathcal{L}\,(f_1 \oplus \cdots \oplus f_6)\| = \|f_1 \oplus \cdots \oplus f_6\|$ for $f_1 \oplus \cdots \oplus f_6 \in L^2(\mathbb{R})^6$.

In view of 3.1.4 we make the following definition.

Definition 3.1.7. Let $\mathcal{L}$ be the Bessel map associated with a countable collection X. Then X is a *frame associated with* $\mathcal{L}$ if the map $\mathcal{L}$ is a bounded invertible operator from $L^2(\mathbb{R})^n$ onto a closed subspace of $l^2(\mathbb{Z})$. i.e., if there exist constants A and B such that for every $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$,

$$A \|f_1 \oplus \cdots \oplus f_n\|^2_{L^2(\mathbb{R})^n} \leq \|\mathcal{L}(f_1 \oplus \cdots \oplus f_n)\|^2_{l^2(\mathbb{Z})} \leq B \|f_1 \oplus \cdots \oplus f_n\|^2_{L^2(\mathbb{R})^n}$$

Definition 3.1.8. Let A be a bounded linear map from a Hilbert space H_1 to H_2. The *adjoint* of A is the bounded linear map B from H_2 to H_1 satisfying

$$\langle A(x),\ y\rangle_{H_2} = \langle x,\ B(y)\rangle_{H_1}$$

for all $x \in H_1$ and $y \in H_2$. The adjoint of A is unique and is denoted by A^*.

Proposition 3.1.9. *Let* $X = \{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\} \subseteq L^2(\mathbb{R})^n$ *be a countable system with a well-defined Bessel map* $\mathcal{L} : L^2(\mathbb{R})^n \to l^2(\mathbb{Z})$. *The adjoint of* $\mathcal{L}$ *is given by*

$$\mathcal{L}^* : l^2(\mathbb{Z}) \to L^2(\mathbb{R})^n.$$

$$c \mapsto \sum_{i\in\mathbb{Z}} c_i\, x_{i,1} \oplus \cdots \oplus x_{i,n}.$$

Proof. Fix $n \in \mathbb{Z}$. Consider a sequence of the form

$$c = (\ldots, 0, c_{-n}, \ldots, c_1, \ldots, c_n, 0, \ldots) \in l^2(\mathbb{Z}).$$

Then

$$\langle y_1 \oplus \cdots \oplus y_n,\ \mathcal{L}^*(c)\rangle_{L^2(\mathbb{R})^n} \;=\; \langle \mathcal{L}(y_1 \oplus \cdots \oplus y_n),\ c\rangle_{l^2(\mathbb{Z})}$$

$$\begin{aligned}
&= \langle \{\langle y_1 \oplus \cdots \oplus y_n, x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle : i \in \mathbb{Z}\}, c\rangle_{l^2(\mathbb{Z})} \\
&= \sum_{i=-n}^{n} \langle y_1 \oplus \cdots \oplus y_n, x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle \overline{c_i} \\
&= \sum_{i=-n}^{n} \langle y_1 \oplus \cdots \oplus y_n, c_i x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle \\
&= \left\langle y_1 \oplus \cdots \oplus y_n, \sum_{i=-n}^{n} c_i x_{i,1} \oplus \cdots \oplus x_{i,n} \right\rangle.
\end{aligned}$$

Hence,

$$\mathcal{L}^*(c) = \sum_{i=-n}^{n} c_i x_{i,1} \oplus \cdots \oplus x_{i,n}.$$

Now take any $c = (\ldots, c_{-n}, \ldots, c_1, \ldots, c_n, \ldots) \in l^2(\mathbb{Z})$. Then c can be considered as the limit of the sequence (d^n), where $d^n = (\ldots, 0, c_{-n}, \ldots, c_1, \ldots, c_n, 0, \ldots)$. Then, by the discussion above, we have $\mathcal{L}^*(d^n) = \sum\limits_{i=-n}^{n} c_i x_{i,1} \oplus \cdots \oplus x_{i,n}$.

Claim: $\{\mathcal{L}^*((d^n))\}$ is a Cauchy sequence in $L^2(\mathbb{R})^n$. Let $n > m$.

$$\begin{aligned}
\|\mathcal{L}^*((d^n)) - \mathcal{L}^*((d^m))\|^2_{L^2(\mathbb{R})^n} &= \|\mathcal{L}^*((d^n) - (d^m))\|^2_{L^2(\mathbb{R})^n} \\
&\le \|\mathcal{L}^*\|^2 \|(d^n) - (d^m)\|^2
\end{aligned}$$

Since (d^n) is a convergent sequence, it is Cauchy and from the above, we have $\mathcal{L}^*((d^n))$ is also a Cauchy sequence in the complete space $L^2(\mathbb{R})^n$ and hence is convergent.

Now

$$\lim_{n\to\infty} \mathcal{L}^*(d^n) = \mathcal{L}^*(\lim_{n\to\infty} d^n) = \mathcal{L}^*(c),$$

Hence

$$\mathcal{L}^*(c) = \lim_{n\to\infty} \sum_{i=-n}^{n} c_i x_{i,1} \oplus \quad \cdots \quad \oplus x_{i,n} = \sum_{i=-\infty}^{\infty} c_i x_{i,1} \oplus \quad \cdots \quad \oplus x_{i,n}.$$

□

We use the following Theorem in the proof of Proposition 3.1.11.

Theorem 3.1.10. [32] *Suppose H_1 and H_2 are Hilbert spaces, and T is a bounded linear transformation from H_1 to H_2. Then*

$$(R(T))^{\perp} = N(T^*)$$

and

$$(R(T^*))^{\perp} = N(T).$$

Proposition 3.1.11. *Let $X = \{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\} \subseteq L^2(\mathbb{R})^n$ be a countable system with a well-defined Bessel map $\mathcal{L} : L^2(\mathbb{R})^n \to l^2(\mathbb{Z})$ and adjoint map $\mathcal{L}^* : l^2(\mathbb{Z}) \to L^2(\mathbb{R})^n$. Then*

i. $(R(\mathcal{L}))^{\perp} = N(\mathcal{L}^*)$,

ii. $\overline{R(\mathcal{L})} = (N(\mathcal{L}^*))^{\perp}$, and

iii. $(R(\mathcal{L}^*))^{\perp} = N(\mathcal{L})$.

Proof. It remains to prove (ii), as the other two follows immediately from Theorem 3.1.10 because $\mathcal{L}$ is a bounded linear transformation from $L^2(\mathbb{R})^n$

to $l^2(\mathbb{Z})$. Now

$$\begin{aligned}
x \in \overline{R(\mathcal{L})} &\Leftrightarrow \text{there exists a sequence } (x_n) \text{ in } R(\mathcal{L}) \text{ converging to } x \\
&\Leftrightarrow \langle (x_n),\ y\rangle = 0\ \forall y \in N(\mathcal{L}^*) \\
&\Leftrightarrow \lim_{n\to\infty} \langle (x_n),\ y\rangle = 0\ \forall y \in N(\mathcal{L}^*) \\
&\Leftrightarrow \left\langle \lim_{n\to\infty} (x_n),\ y\right\rangle = 0\ \forall y \in N(\mathcal{L}^*) \\
&\Leftrightarrow \langle x,\ y\rangle = 0\ \forall y \in N(\mathcal{L}^*) \\
&\Leftrightarrow x \in (N(\mathcal{L}^*))^{\perp}
\end{aligned}$$

and this completes the proof. □

We need the following fundamental result in the proof of Proposition 3.1.13.

Proposition 3.1.12. *Let H_1 and H_2 be Hilbert spaces and T be a bounded linear transformation from H_1 to H_2. Then*

$$\|T\| = \|T^*\|.$$

If T is invertible, then T^* is also invertible and

$$(T^*)^{-1} = (T^{-1})^*.$$

Proof. For $x \in H_2$, $y \in H_1$, we have

$$\left|\langle T^*(x),\ y\rangle_{H_1}\right| = \left|\langle x,\ T(y)\rangle_{H_2}\right| = \left|\langle T(y),\ x\rangle_{H_2}\right|.$$

Thus

$$\begin{aligned}\|T^*\| &= \sup\left\{\left|\langle T^*(x),\ y\rangle_{H_1}\right| : x \in H_2,\ y \in H_1,\ \|x\| \leq 1,\ \|y\| \leq 1\right\}\\ &= \sup\left\{\left|\langle T(y),\ x\rangle_{H_2}\right| : x \in H_2,\ y \in H_1,\ \|x\| \leq 1,\ \|y\| \leq 1\right\}\\ &= \|T\|.\end{aligned}$$

If T is invertible, then there is $U : H_2 \to H_1$ such that

$$TU = I_{H_2} \text{ and } UT = I_{H_1}.$$

Hence

$$U^*T^* = I^*_{H_2} = I_{H_1} \tag{3.2}$$

and

$$T^*U^* = I^*_{H_1} = I_{H_2}. \tag{3.3}$$

From (3.2) and (3.3), T^* is invertible and

$$(T^*)^{-1} = U^* = \left(T^{-1}\right)^*.$$

□

Proposition 3.1.13. *Let* $X = \{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\} \subseteq L^2(\mathbb{R})^n$ *be a countable system with a well-defined Bessel map* $\mathcal{L} : L^2(\mathbb{R})^n \to l^2(\mathbb{Z})$. *Assume* $\overline{\operatorname{span}}\,\{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\} = L^2(\mathbb{R})^n$. *Then for every* $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$,

$$
\begin{aligned}
A \|f_1 \oplus \cdots \oplus f_n\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} &\leq \|\mathcal{L}(f_1 \oplus \cdots \oplus f_n)\|^2_{l^2(\mathbb{Z})} \\
&\leq B \|f_1 \oplus \cdots \oplus f_n\|^2_{L^2(\mathbb{R})^n} \qquad (3.4)
\end{aligned}
$$

if and only if for every $c \in (\mathcal{N}(\mathcal{L}^*))^\perp$,

$$A \|c\|^2_{l^2(\mathbb{Z})} \leq \|\mathcal{L}^*(c)\|^2_{L^2(\mathbb{R})^n} \leq B \|c\|^2_{l^2(\mathbb{Z})}. \qquad (3.5)$$

i.e., X *is a frame if and only if* $\mathcal{L}^*$ *is bounded and invertible.*

Proof. Let X be a frame. i.e., assume (3.4). Then $\mathcal{L}$ is a bounded and one-to-one map. Hence $\mathcal{L}$ is a bounded invertible map onto $\mathcal{R}(\mathcal{L})$. In this case $\mathcal{R}(\mathcal{L})$ is closed (by Definition 3.1.7) so that $\mathcal{R}(\mathcal{L}) = (\mathcal{N}(\mathcal{L}^*))^\perp$, by Proposition 3.1.11. Thus the adjoint map of $\mathcal{L} : L^2(\mathbb{R})^n \to \mathcal{R}(\mathcal{L})$ is $\mathcal{L}^*_{|(\mathcal{N}(\mathcal{L}^*))^\perp}$, and this latter map is invertible with bounds equal to those of $\mathcal{L}$, by Proposition 3.1.12. Hence the assertion (3.5) is obtained.

Conversely, assume (3.5). Then $\mathcal{L}^*_{|(\mathcal{N}(\mathcal{L}^*))^\perp}$ is a bounded one-to-one map from $\mathcal{R}(\mathcal{L})$ onto $L^2(\mathbb{R})^n$ and hence is a bounded invertible map onto $L^2(\mathbb{R})^n$. Thus the adjoint map of $\mathcal{L}^*_{|(\mathcal{N}(\mathcal{L}^*))^\perp}$ is $\mathcal{L}_{|L^2(\mathbb{R})^n} = \mathcal{L}$ itself and this latter map is also, by Proposition 3.1.12, invertible with bounds equal to those of $\mathcal{L}^*_{|(\mathcal{N}(\mathcal{L}^*))^\perp}$. Hence the assertion (3.4) is obtained. □

3.2 Frame Operators

Proposition 3.2.1. *If* $\{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\}$ *is a frame for* $L^2(\mathbb{R})^n$, *the series* $\sum_{i\in\mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle\, x_{i,1} \oplus \cdots \oplus x_{i,n}$ *is convergent.*

Proof. We consider the n−th partial sum

$$s_n = \sum_{i=-n}^{n} \langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle x_{i,1} \oplus \cdots \oplus x_{i,n}$$

For $m \geq n$, noting that

$$|\langle s_m - s_n, y_1 \oplus \cdots \oplus y_n \rangle| \leq \|s_m - s_n\| \, \|y_1 \oplus \cdots \oplus y_n\|$$

and

$$|\langle s_m - s_n, y_1 \oplus \cdots \oplus y_n \rangle| = \|s_m - s_n\|, \text{ if we let } y_1 \oplus \cdots \oplus y_n = \frac{s_m - s_n}{\|s_m - s_n\|},$$

it follows that

$$\begin{aligned}
&\|s_m - s_n\|^2 \\
&= \sup_{\|y_1 \oplus \cdots \oplus y_n\|=1} \left| \langle s_m - s_n, y_1 \oplus \cdots \oplus y_n \rangle_{L^2(\mathbb{R})^n} \right|^2 \\
&= \sup \left| \sum_{i \in J_{m,n}} \langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle \langle x_{i,1} \oplus \cdots \oplus x_{i,n}, y_1 \oplus \cdots \oplus y_n \rangle \right|^2, \\
&\qquad \text{where } J_{m,n} = \{-m, \ldots, -n-1, n+1, \ldots, m\} \\
&\leq \sup_{\|y_1 \oplus \cdots \oplus y_n\|=1} \sum_{i \in J_{m,n}} |\langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle|^2 \\
&\qquad \times \sup_{\|y_1 \oplus \cdots \oplus y_n\|=1} \sum_{i \in J_{m,n}} |\langle x_{i,1} \oplus \cdots \oplus x_{i,n}, y_1 \oplus \cdots \oplus y_n \rangle|^2, \\
&\qquad \text{by Holder's inequality} \\
&\leq \sup_{\|y_1 \oplus \cdots \oplus y_n\|=1} \sum_{i \in J_{m,n}} |\langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle|^2 \\
&\qquad \times B \sup_{\|y_1 \oplus \cdots \oplus y_n\|=1} \|y_1 \oplus \cdots \oplus y_n\|^2, \text{ by the definition of the frame}
\end{aligned}$$

$$
\begin{aligned}
&\le B \sum_{i \in J_{m,n}} \left| \langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle_{L^2(\mathbb{R})^n} \right|^2 \\
&\to 0 \text{ as } m,\ n \to \infty,
\end{aligned}
$$

since $\sum_{i \in \mathbb{N}} |\langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle|^2$ converges by the frame inequality. Hence (s_n) is a Cauchy sequence in the Hilbert space and therefore (s_n) converges to a vector $\mathcal{S}(f_1 \oplus \cdots \oplus f_n)$ of $L^2(\mathbb{R})^n$. □

Definition 3.2.2. Let $X = \{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\} \subseteq L^2(\mathbb{R})^n$ be a frame with Bessel map $\mathcal{L}$. The *frame operator* $\mathcal{S}$ is defined as

$$
f_1 \oplus \cdots \oplus f_n \mapsto \sum_{i \in \mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle\, x_{i,1} \oplus \cdots \oplus x_{i,n}.
$$

Remark 3.2.3. By the Proposition 3.2.1, $\mathcal{S}$ is a well-defined linear operator on $L^2(\mathbb{R})^n$.

Theorem 3.2.4. *The frame operator is a continuous operator on $L^2(\mathbb{R})^n$.*

Proof. To see that $\mathcal{S}$ is bounded, consider

$$
\|\mathcal{S}(f_1 \oplus \cdots \oplus f_n)\|^2 = \sup_{\|y_1 \oplus \cdots \oplus y_n\| = 1} \left| \langle \mathcal{S}(f_1 \oplus \cdots \oplus f_n),\ y_1 \oplus \cdots \oplus y_n \rangle_{L^2(\mathbb{R})^n} \right|^2
$$

$$
\begin{aligned}
&= \sup \left| \sum_{i \in \mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle \langle x_{i,1} \oplus \cdots \oplus x_{i,n},\ y_1 \oplus \cdots \oplus y_n \rangle \right|^2 \\
&\le \sup_{\|y_1 \oplus \cdots \oplus y_n\| = 1} \sum_{i \in \mathbb{Z}} |\langle f_1 \oplus \cdots \oplus f_n,\ x_i \oplus x_i' \rangle|^2 \sum_{i \in \mathbb{Z}} |\langle x_i \oplus x_i',\ y_1 \oplus \cdots \oplus y_n \rangle|^2 \\
&\le \sup_{\|y_1 \oplus \cdots \oplus y_n\| = 1} B \|f_1 \oplus \cdots \oplus f_n\|^2 B \|y_1 \oplus \cdots \oplus y_n\|^2 \\
&\le B^2 \|f_1 \oplus \cdots \oplus f_n\|^2
\end{aligned}
$$

Consequently $\mathcal{S}$ is bounded, and $\|\mathcal{S}\| \le B$. □

Theorem 3.2.5. *The frame operator $\mathcal{S}$ is a positive operator and*

$$AI \le \mathcal{S} \le BI.$$

Proof. For $f_1 \oplus \cdots \oplus f_n$ and $g_1 \oplus \cdots \oplus g_n \in L^2(\mathbb{R})^n$, using the linearity and continuity of the inner product,we have

$$\begin{aligned}
&\langle \mathcal{S}(f_1 \oplus \cdots \oplus f_n),\ g_1 \oplus \cdots \oplus g_n\rangle \\
&= \left\langle \sum_{i\in\mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle\, x_{i,1} \oplus \cdots \oplus x_{i,n},\ g_1 \oplus \cdots \oplus g_n \right\rangle \\
&= \sum_{i\in\mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle\, \langle x_{i,1} \oplus \cdots \oplus x_{i,n},\ g_1 \oplus \cdots \oplus g_n\rangle \\
&= \sum_{i\in\mathbb{Z}} \left\langle f_1 \oplus \cdots \oplus f_n,\ \overline{\langle x_{i,1} \oplus \cdots \oplus x_{i,n},\ g_1 \oplus \cdots \oplus g_n\rangle} x_{i,1} \oplus \cdots \oplus x_{i,n} \right\rangle \\
&= \sum_{i\in\mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n,\ \langle g_1 \oplus \cdots \oplus g_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle\, x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle \\
&= \left\langle f_1 \oplus \cdots \oplus f_n,\ \sum_{i\in\mathbb{Z}} \langle g_1 \oplus \cdots \oplus g_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle\, x_{i,1} \oplus \cdots \oplus x_{i,n} \right\rangle \\
&= \langle f_1 \oplus \cdots \oplus f_n,\ \mathcal{S}(g_1 \oplus \cdots \oplus g_n)\rangle
\end{aligned}$$

Hence $\mathcal{S}$ is self adjoint. Also,

$$\begin{aligned}
&\langle \mathcal{S}(f_1 \oplus \cdots \oplus f_n),\ f_1 \oplus \cdots \oplus f_n\rangle \\
&= \left\langle \sum_{i\in\mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle\, x_{i,1} \oplus \cdots \oplus x_{i,n},\ f_1 \oplus \cdots \oplus f_n \right\rangle \\
&= \sum_{i\in\mathbb{Z}} |\langle f_1 \oplus \cdots \oplus f_n,\ x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle|^2 \\
&\ge 0
\end{aligned}$$

so $\mathcal{S} \ge 0$, and hence we can rewrite the frame inequality for $f_1 \oplus \cdots \oplus f_n \in$

$L^2(\mathbb{R})^n$ as

$$\begin{aligned} & A\langle f_1 \oplus \cdots \oplus f_n, f_1 \oplus \cdots \oplus f_n\rangle \\ \leq \ & \langle \mathcal{S}(f_1 \oplus \cdots \oplus f_n), f_1 \oplus \cdots \oplus f_n\rangle \\ = \ & \sum_{i\in\mathbb{Z}} |\langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle|^2 \\ \leq \ & B\langle f_1 \oplus \cdots \oplus f_n, f_1 \oplus \cdots \oplus f_n\rangle \end{aligned}$$

or

$$AI \leq \mathcal{S} \leq BI.$$

□

Lemma 3.2.6. *Let X be a Banach space and let T be a bounded linear operator from X to itself. Suppose that $\|T\| < 1$. Then $I - T$ is invertible, $(I-T)^{-1}$ is bounded, and $\|(I-T)^{-1}\| < \frac{1}{1-\|T\|}$.*

Proposition 3.2.7. *Let H be a Hilbert space. Let T and $U : H \to H$ are positive maps, then*

$$T \leq U \Rightarrow \|T\| \leq \|U\|.$$

Proof. T and U are positive maps implies T and U are self adjoint. Hence

$$\begin{aligned} \|T\| &= \sup\{|\langle T(x), x\rangle| : x \in H, \|x\| = 1\} \\ &\leq \sup\{|\langle U(x), x\rangle| : x \in H, \|x\| = 1\}, \\ &\quad \text{as } T \leq U \text{ and } 0 \leq \langle T(x), x\rangle \leq \langle U(x), x\rangle \\ &= \|U\|. \end{aligned}$$

□

Theorem 3.2.8. *$\mathcal{S}^{-1}$ exists and is positive on $L^2(\mathbb{R})^n$.*

Proof. Since $AI \leq \mathcal{S} \leq BI$ by Theorem 3.2.5, we have

$$I - B^{-1}\mathcal{S} \geq 0$$

and

$$I - B^{-1}\mathcal{S} \leq I - AB^{-1}I.$$

Here $I - B^{-1}\mathcal{S}$ and $I - AB^{-1}I$ are both positive operators, and hence by Proposition 3.2.7,

$$\left\|I - B^{-1}\mathcal{S}\right\| \leq \left\|I - AB^{-1}I\right\| = 1 - AB^{-1}\left\|I\right\| = 1 - AB^{-1} < 1.$$

Hence, by Lemma 3.2.6, $[B^{-1}\mathcal{S}]^{-1} = (I - (I - B^{-1}\mathcal{S}))^{-1}$exists on all of $L^2(\mathbb{R})^n$ and is a bounded operator. The same is thus true for $\mathcal{S}^{-1}$. By 3.2.5 , $\mathcal{S}$ is self adjoint. Hence for $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$, we have

$$\begin{aligned}
\langle \mathcal{S}^{-1}f_1 \oplus \cdots \oplus f_n,\ f_1 \oplus \cdots \oplus f_n \rangle &= \langle \mathcal{S}^{-1}f_1 \oplus \cdots \oplus f_n,\ \mathcal{S}\left(\mathcal{S}^{-1}f_1 \oplus \cdots \oplus f_n\right)\rangle \\
&= \langle \mathcal{S}\left(\mathcal{S}^{-1}f_1 \oplus \cdots \oplus f_n\right),\ \mathcal{S}^{-1}f_1 \oplus \cdots \oplus f_n \rangle \\
&\geq A\left\|\mathcal{S}^{-1}f_1 \oplus \quad \cdots \quad \oplus f_n\right\|^2 \\
&\geq 0.
\end{aligned}$$

$\therefore$ $\mathcal{S}^{-1}$ is positive.

□

Theorem 3.2.9. *For the frame operator $\mathcal{S}$ with frame bounds A and B, $\mathcal{S}^{-1}$ is a frame operator with frame bounds B^{-1} and A^{-1}, and*

$$B^{-1}I \leq \mathcal{S}^{-1} \leq A^{-1}I.$$

Proof. Since $\mathcal{S}$ commutes with $\mathcal{S}^{-1}$ and I and

$$AI \leq \mathcal{S} \Leftrightarrow 0 \leq \mathcal{S} - AI,$$

it follows that

$$0 \leq (\mathcal{S} - AI)\,\mathcal{S}^{-1} \ \Leftrightarrow\ 0 \leq I - A\mathcal{S}^{-1}, \quad \text{or} \quad \mathcal{S}^{-1} \leq A^{-1}I.$$

Similarly, since $\mathcal{S} \leq BI$, it follows that $B^{-1}I \leq \mathcal{S}^{-1}$. Hence

$$B^{-1}I \leq \mathcal{S}^{-1} \leq A^{-1}I.$$

□

Corollary 3.2.10. *Let $\mathcal{S}$ be a frame operator on $L^2(\mathbb{R})^n$. Then $\mathcal{S}^{-1}$ is bounded.*

Proof. By Theorem 3.2.9, we have

$$\mathcal{S}^{-1} \leq A^{-1}I,$$

where A is the lower frame bound associated with $\mathcal{S}$. Also, by Theorem 3.2.8, $\mathcal{S}^{-1}$ is positive. Hence, by Proposition 3.2.7,

$$\|\mathcal{S}^{-1}\| \leq A^{-1},$$

which implies that $\mathcal{S}^{-1}$ is bounded. □

Theorem 3.2.11. *Let $X = \{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\} \subseteq L^2(\mathbb{R})^n$ be a frame with frame operator $\mathcal{S}$. Then for every $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$,*

$$\begin{aligned} f_1 \oplus \cdots \oplus f_n \\ = \sum_{i \in \mathbb{Z}} \left\langle f_1 \oplus \cdots \oplus f_n,\ \mathcal{S}^{-1}(x_{i,1} \oplus \cdots \oplus x_{i,n}) \right\rangle x_{i,1} \oplus \cdots \oplus x_{i,n} \end{aligned} \tag{3.6}$$

and

$$(3.7)\quad f_1 \oplus \cdots \oplus f_n = \sum_{i \in \mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle S^{-1}(x_{i,1} \oplus \cdots \oplus x_{i,n}).$$

In particular, if X is a tight frame with frame bound A, then for every $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$,

$$(3.8)\quad f_1 \oplus \cdots \oplus f_n = A^{-1} \sum_{i \in \mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle x_{i,1} \oplus \cdots \oplus x_{i,n} .$$

Proof. Because of Proposition 3.2.8, Theorems 3.2.4 and 3.2.5 and Corollary 3.2.10 and the fact that $\mathcal{S} = \mathcal{S}^*$, the decompositions (3.6) and (3.7) are consequences of the following calculations:

$$\begin{aligned} & f_1 \oplus \cdots \oplus f_n \\ = \; & \mathcal{S}(\mathcal{S}^{-1} f_1 \oplus \cdots \oplus f_n) \\ = \; & \sum_{i \in \mathbb{Z}} \big\langle \mathcal{S}^{-1} f_1 \oplus \cdots \oplus f_n,\; x_{i,1} \oplus \cdots \oplus x_{i,n} \big\rangle x_{i,1} \oplus \cdots \oplus x_{i,n} \\ = \; & \sum_{i \in \mathbb{Z}} \big\langle f_1 \oplus \cdots \oplus f_n,\; \mathcal{S}^{-1}(x_{i,1} \oplus \cdots \oplus x_{i,n}) \big\rangle x_{i,1} \oplus \cdots \oplus x_{i,n} \end{aligned}$$

and

$$\begin{aligned} f_1 \oplus \cdots \oplus f_n &= \mathcal{S}^{-1}\mathcal{S}(f_1 \oplus \cdots \oplus f_n) \\ &= \mathcal{S}^{-1} \sum_{i \in \mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n,\; x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle x_{i,1} \oplus \cdots \oplus x_{i,n} \\ &= \sum_{i \in \mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n,\; x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle \mathcal{S}^{-1}(x_{i,1} \oplus \cdots \oplus x_{i,n}) \end{aligned}$$

In the case X is a tight frame with frame bound A, $A^{-1/2}\mathcal{L}$ is an isometry. Thus, $A^{-1}\mathcal{S} = (A^{-1/2}\mathcal{L})^*(A^{-1/2}\mathcal{L})$ is the identity operator on $L^2(\mathbb{R})^n$, whence (3.8) follows. ☐

Proposition 3.2.12. *For the frame operator* $\mathcal{S}$, $(\mathcal{S}^{-1})^* = \mathcal{S}^{-1}$.

Proof. By Theorem 3.2.5, $\mathcal{S}$ is self adjoint. Also, by Proposition 3.1.12, since $\mathcal{S}$ is bounded and invertible, we have

$$\left(\mathcal{S}^{-1}\right)^* = \left(\mathcal{S}^*\right)^{-1} = \mathcal{S}^{-1}.$$

☐

Proposition 3.2.13. *Let* $X = \{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\} \subseteq L^2(\mathbb{R})^n$ *be a frame with Bessel map* $\mathcal{L}$ *and frame operator* $\mathcal{S}$. *The maps*

$$\mathcal{L}\mathcal{S}^{-1} : L^2(\mathbb{R})^n \to R(\mathcal{L})$$

$$f_1 \oplus \cdots \oplus f_n \mapsto \left\{\left\langle f_1 \oplus \cdots \oplus f_n, \mathcal{S}^{-1}(x_{i,1} \oplus \cdots \oplus x_{i,n}) \right\rangle : i \in \mathbb{Z}\right\},$$

and the restriction of $\mathcal{L}^*$ *to* $R(\mathcal{L})$, *viz.,*

$$\begin{aligned}\mathcal{L}^*_| &: R(\mathcal{L}) \to L^2(\mathbb{R})^n \\ c &\mapsto \sum_{i\in\mathbb{Z}} c_i x_{i,1} \oplus \quad \cdots \quad \oplus x_{i,n}\end{aligned}$$

are continuous maps and are inverse to each other. i.e.,

$$(\mathcal{L}^*_|)^{-1} = \mathcal{L}\mathcal{S}^{-1}.$$

Proof. By Propositions 3.1.4 and 3.1.12, $\mathcal{L}$ and $\mathcal{L}^*$ are continuous maps. By Corollary 3.2.10, $\mathcal{S}^{-1}$ is also continuous. Hence, being the composition of continuous maps, $\mathcal{L}\mathcal{S}^{-1}$ is also continuous.

Take

$\{f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in \mathbb{Z}\} \in R(\mathcal{L})$. Then

$$\begin{aligned}
&\mathcal{L}S^{-1}\mathcal{L}_{|}^{*}(\{\langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle : i \in \mathbb{Z}\}) \\
&= \mathcal{L}S^{-1}\left(\sum_{i\in\mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle x_{i,1} \oplus \cdots \oplus x_{i,n}\right) \\
&= \mathcal{L}\left(\sum_{i\in\mathbb{Z}} \langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle \mathrm{S}^{-1}(x_{i,1} \oplus \cdots \oplus x_{i,n})\right) \\
&= \mathcal{L}(f_1 \oplus \cdots \oplus f_n) \\
&= \{\langle f_1 \oplus \cdots \oplus f_n, x_{i,1} \oplus \cdots \oplus x_{i,n}\rangle : i \in \mathbb{Z}\}
\end{aligned}$$

Hence

$$\mathcal{L}S^{-1}\mathcal{L}_{|}^{*} = I_{l^2(\mathbb{Z})}.$$

Also

$$\mathcal{L}_{|}^{*}\mathcal{L}S^{-1} = SS^{-1} = I_{L^2(\mathbb{R})^n}.$$

Hence $\mathcal{L}_{|}^{*}$ and $\mathcal{L}S^{-1}$ are inverse to each other. i.e.,

$$(\mathcal{L}_{|}^{*})^{-1} = \mathcal{L}S^{-1}$$

□

Remark 3.2.14. Hence, minimality of a frame (Definition 3.1.2) is equivalent to

$$N(\mathcal{L}^*) = (R(\mathcal{L}))^{\perp} = \{0\}\,, \text{the zero sequence,}$$

because $N(\mathcal{L}^*) \neq \{0\}$ if and only if there is a non-zero sequence c such that $\sum_{i\in\mathbb{Z}} c_i x_{i,1} \oplus \cdots \oplus x_{i,n} = 0 \oplus \cdots \oplus 0$. This means that some $x_{i,1} \oplus \cdots \oplus x_{i,n}$ can be expressed as a linear combination of other $x_{j,1} \oplus \cdots \oplus x_{j,n}$, means that the frame is not minimal.

Remark 3.2.15. When X is not minimal $N(\mathcal{L}^*)$ is a non-zero subspace of $l^2(\mathbb{Z})$;

so that given a signal $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$ there are different coefficients $c, d \in l^2(\mathbb{Z})$ for which $\mathcal{L}^*(c) = \mathcal{L}^*(d) = f_1 \oplus \cdots \oplus f_n$.

Removal of a vector from a frame leaves either a frame or an incomplete set [7]. An orthonormal basis is obviously a normalized tight frame. The next result tells us that normalized tight frame of unit vectors form an orthonormal basis.

Proposition 3.2.16. *Let $X = \{x_{i,1} \oplus \cdots \oplus x_{i,n} : i \in I\} \subseteq L^2(\mathbb{R})^n$ be a tight frame with frame bound 1 and*

$$\|x_{i,1} \oplus \cdots \oplus x_{i,n}\| = 1 \ \forall\ i,$$

then X is an orthonormal basis (ONB).

Proof. By the definition of frame bound, for $j \in \mathbb{Z}$, we have

$$\sum_{i \in \mathbb{Z}} |\langle x_{j,1} \oplus \cdots \oplus x_{j,n}, x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle|^2 = 1 \tag{3.9}$$

The terms in the series on the LHS of (3.9) is a sum of non-negative numbers of which one term $\|x_{j,1} \oplus \cdots \oplus x_{j,n}\|^2 = 1$; hence all other terms are 0. That is

$$\langle x_{j,1} \oplus \cdots \oplus x_{j,n}, x_{i,1} \oplus \cdots \oplus x_{i,n} \rangle = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

□

ON THE STUDY OF FRAME MULTIRESOLUTION ANALYSIS IN THE SUPERSPACES $L^2(\mathbb{R})^n$

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Chapter 4

Frames Generated by Translation of a Function in $L^2(\mathbb{R})^n$

As mentioned in the introduction, our aim is to define frame MRA in $L^2(\mathbb{R})^n$. We first study the situation in $L^2(\mathbb{R})^2$ and these can be extended naturally to $L^2(\mathbb{R})^n$.

4.1 Translation and Dilation Operators on $L^2(\mathbb{R})^n$

Considering the Hilbert space $L^2(\mathbb{R})^n$ and taking the N cycle $C := \{z_1, \dots, z_n\}$ and $\alpha_1 = \ldots = \alpha_n = 1$ in Example 1.5.8, we have the translation operator given by

$$T_C\,(f_1 \oplus \cdots \oplus f_n) = z_1 T f_1 \oplus \cdots \oplus z_n T f_n$$

and the dilation operator given by

$$U_C\,(f_1 \oplus \cdots \oplus f_n) = U f_2 \oplus \cdots \oplus U f_n \oplus U f_1,$$

where we take T and U as the translation and dilation operators on $L^2(\mathbb{R})$ given by (1.1) and (1.2).

We begin with two simple basic theorems which will be helpful in proving results in this chapter.

Theorem 4.1.1. *Let* $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$. *Then*

$$\|f_1 \oplus \cdots \oplus f_n\|^2_{L^2(\mathbb{R})^n} = \sum_{j=1}^{n} \|f_j\|^2_{L^2(\mathbb{R})}$$

Proof. The inner product on $L^2(\mathbb{R})^n$ is defined by

$$\langle f_1 \oplus \cdots \oplus f_n\, ,\, g_1 \oplus \quad \cdots \quad \oplus g_n \rangle_{L^2(\mathbb{R})^n} = \sum_{j=1}^{n} \langle f_j,\ g_j \rangle_{L^2(\mathbb{R})}$$

Thus, we have

$$\begin{aligned} \|f_1 \oplus \cdots \oplus f_n\|^2_{L^2(\mathbb{R})^n} &= \langle f_1 \oplus \cdots \oplus f_n\, ,\, f_1 \oplus \cdots \oplus f_n \rangle_{L^2(\mathbb{R})^n} \\ &= \sum_{j=1}^{n} \langle f_j,\ f_j \rangle_{L^2(\mathbb{R})} \\ &= \sum_{j=1}^{n} \|f_j\|^2_{L^2(\mathbb{R})} \end{aligned}$$

□

Theorem 4.1.2. *For* $f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ *and* T_C *and* U_C *as in (2.4) and (2.5), we have*

$$(T_C^k(f_1 \oplus f_2))\widehat{\ } = e_k(z_1^k f_1 \oplus z_2^k f_2)\widehat{\ }$$

and

$$(U_C(f_1 \oplus f_2))\widehat{\ } = U^{-1}\widehat{f_2} \oplus U^{-1}\widehat{f_1}$$

Proof.

$$\begin{aligned}
(T_C^k(f_1 \oplus f_2))^\wedge &= (z_1^k T^k f_1 \oplus z_2^k T^k f_2)^\wedge \\
&= \widehat{z_1^k T^k f_1} \oplus \widehat{z_2^k T^k f_2}, \text{ as } \widehat{g_1 \oplus g_2} = \hat{g}_1 \oplus \hat{g}_2 \\
&= z_1^k \widehat{T^k f_1} \oplus z_2^k \widehat{T^k f_2}, \\
&\qquad \text{by the linearity property of Fourier transform} \\
&= z_1^k e_k \widehat{f_1} \oplus z_2^k e_k \widehat{f_2}, \quad \text{as } \widehat{T^k f} = e_k \hat{f} \\
&= e_k(\widehat{z_1^k f_1} \oplus \widehat{z_2^k f_2}) \\
&= e_k(z_1^k f_1 \oplus z_2^k f_2)^\wedge
\end{aligned}$$

Also,

$$\begin{aligned}
(U_C(f_1 \oplus f_2))^\wedge &= (Uf_2 \oplus Uf_1)^\wedge \\
&= \widehat{(Uf_2)} \oplus \widehat{(Uf_1)} \\
&= U^{-1}\widehat{f_2} \oplus U^{-1}\widehat{f_1}, \quad \text{as} \widehat{Uf} = U^{-1}\hat{f} \ for\ f \in L^2(\mathbb{R})
\end{aligned}$$

□

4.2 Frames Generated by Translation of a Function in $L^2(\mathbb{R})^n$

In the following theorem we associate two bounded 2π periodic function with a given family X in $L^2(\mathbb{R})^2$ and prove that whenever these associated 2π periodic functions are bounded, then the Bessel map $\mathcal{L}$ corresponding to X is also bounded. The converse is also proved.

Theorem 4.2.1. *Let* $X = \left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\} \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ *and define*

$\Phi_1(\gamma) = \sum\limits_{k\in\mathbb{Z}} \left|\widehat{\phi_1}(\gamma+2\pi k)\right|^2$ and $\Phi_2(\gamma) = \sum\limits_{k\in\mathbb{Z}} \left|\widehat{\phi_2}(\gamma+2\pi k)\right|^2$. Assume that the Bessel map $\mathcal{L}$ associated with X exists. If $\Phi_1 \le A$ and $\Phi_2 \le B$ a.e. then $\|\mathcal{L}\| \le (A+B)^{\frac{1}{2}}$. Conversely, $\|\mathcal{L}\| \le C^{\frac{1}{2}}$ implies $\Phi_1 \le C$ and $\Phi_2 \le C$ a.e.

Proof. By 3.1.12, we have $\|\mathcal{L}\| = \|\mathcal{L}^*\|$. Let c be a finitely generated sequence . Then

$$\begin{aligned}
\|\mathcal{L}^*(c)\|^2 &= \|\widehat{\mathcal{L}^*(c)}\|^2, \text{as Fourier transform is unitary} \\
&= \left\|\left(\widehat{\sum_{k\in\mathbb{Z}} c_k T_C^k \phi_1 \oplus \phi_2}\right)\right\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\
&= \left\|\left(\widehat{\sum_{k\in\mathbb{Z}} c_k z_1^k T^k \phi_1 \oplus z_2^k T^k \phi_2}\right)\right\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\
&= \left\|\left(\widehat{\sum_{k\in\mathbb{Z}} c_k z_1^k T^k \phi_1}\right)\right\|^2_{L^2(\mathbb{R})} + \left\|\left(\widehat{\sum_{k\in\mathbb{Z}} c_k z_2^k T^k \phi_2}\right)\right\|^2_{L^2(\mathbb{R})}, \\
&\qquad\qquad \text{using Theorem 4.1.1} \\
&= \left\|\sum_{k\in\mathbb{Z}} c_k z_1^k e_k \hat{\phi}_1\right\|^2_{L^2(\mathbb{R})} + \left\|\sum_{k\in\mathbb{Z}} c_k z_2^k e_k \hat{\phi}_2\right\|^2_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} \left|\sum_{k\in\mathbb{Z}} c_k z_1^k e_k \widehat{\phi_1}\right|^2 + \int_{\mathbb{R}} \left|\sum_{k\in\mathbb{Z}} c_k z_2^k e_k \widehat{\phi_2}\right|^2 \\
&= \int_{\mathbb{R}} \left|\sum_{k\in\mathbb{Z}} c_k z_1^k e_k(\gamma) \hat{\phi}_1(\gamma)\right|^2 d\gamma + \int_{\mathbb{R}} \left|\sum_{k\in\mathbb{Z}} c_k z_2^k e_k(\gamma) \hat{\phi}_2(\gamma)\right|^2 d\gamma \\
&= \int_{\mathbb{R}} \left|\sum_{k\in\mathbb{Z}} c_k z_1^k e_k(\gamma)\right|^2 \left|\hat{\phi}_1(\gamma)\right|^2 d\gamma + \int_{\mathbb{R}} \left|\sum_{k\in\mathbb{Z}} c_k z_2^k e_k(\gamma)\right|^2 \left|\hat{\phi}_2(\gamma)\right|^2 d\gamma \\
&= \sum_{l\in\mathbb{Z}} \int_{\mathbb{T}} \left|\sum_{k\in\mathbb{Z}} c_k z_1^k e_k(\gamma+2\pi l)\right|^2 \left|\hat{\phi}_1(\gamma+2\pi l)\right|^2 d\gamma \\
&\qquad + \sum_{l\in\mathbb{Z}} \int_{\mathbb{T}} \left|\sum_{k\in\mathbb{Z}} c_k z_2^k e_k(\gamma+2\pi l)\right|^2 \left|\hat{\phi}_2(\gamma+2\pi l)\right|^2 d\gamma
\end{aligned}$$

$$= \int_{\mathbb{T}} \left|\sum_{k\in\mathbb{Z}} c_k z_1^k e_k(\gamma)\right|^2 \sum_{l\in\mathbb{Z}} \left|\hat{\phi}_1(\gamma+2\pi l)\right|^2 d\gamma$$

$$+ \int_{\mathbb{T}} \left|\sum_{k\in\mathbb{Z}} c_k z_2^k e_k(\gamma)\right|^2 \sum_{l\in\mathbb{Z}} \left|\hat{\phi}_2(\gamma+2\pi l)\right|^2 d\gamma,$$

as e_k is $2\pi -$ periodic.

$$= \int_{\mathbb{T}} \left|\sum_{k\in\mathbb{Z}} c_k z_1^k e_k\right|^2 \Phi_1 + \int_{\mathbb{T}} \left|\sum_{k\in\mathbb{Z}} c_k z_2^k e_k\right|^2 \Phi_2$$

We note that $\sum_{k\in\mathbb{Z}} c_k z_1^k e_k$ is the Fourier transform of the sequence $(c_k z_1^k) \in l^2(\mathbb{Z})$ and $\sum_{k\in\mathbb{Z}} c_k z_2^k e_k$ is the Fourier transform of the sequence $(c_k z_2^k) \in l^2(\mathbb{Z})$. Hence, by the Parseval-Plancherel theorem for $\mathbb{T}$ [27]

$$\left\langle \left(c_k z_1^k\right), \left(c_k z_1^k\right)\right\rangle_{l^2(\mathbb{Z})} = \left\langle \left(c_k z_1^k\right)\hat{}, \left(c_k z_1^k\right)\hat{}\right\rangle_{L^2(\mathbb{T})},$$

we have

$$\left(\|(c_k)\|^2_{l^2(\mathbb{Z})} =\right) \left\|\left(c_k z_1^k\right)\right\|^2_{l^2(\mathbb{Z})} = \int_{\mathbb{T}} \left|\sum_{k\in\mathbb{Z}} c_k z_1^k e_k\right|^2$$

and, similarly,

$$\left(\|(c_k)\|^2_{l^2(\mathbb{Z})} =\right) \left\|\left(c_k z_2^k\right)\right\|^2_{l^2(\mathbb{Z})} = \int_{\mathbb{T}} \left|\sum_{k\in\mathbb{Z}} c_k z_2^k e_k\right|^2.$$

Thus, if $\Phi_1 \le A$ and $\Phi_2 \le B$ a.e. on $\mathbb{T}$, then

$$\begin{aligned} \|\mathcal{L}^*\|^2 &= \sup\left\{\|\mathcal{L}^*(c)\|^2 : c = (c_k) \in l^2(\mathbb{Z}),\ \|c\| \le 1\right\} \\ &< A\sup\left\{\left\|\left(c_k z_1^k\right)\right\|^2 : \|(c_k)\| \le 1\right\} + B\sup\left\{\left\|\left(c_k z_2^k\right)\right\|^2 : \|(c_k)\| \le 1\right\} \\ &< A\sup\left\{\|(c_k)\|^2 : \|(c_k)\| \le 1\right\} + B\sup\left\{\|(c_k)\|^2 : \|(c_k)\| \le 1\right\} \\ &< A + B \end{aligned}$$

so that

$$\|\mathcal{L}\| \leq (A+B)^{\frac{1}{2}}.$$

For the converse, consider for $\delta > 0$ the set $\Gamma = \left[\Phi_1 \geq C + \frac{\delta}{2}\right] \cup \left[\Phi_2 \geq C + \frac{\delta}{2}\right]$. Now, for any measurable set $\Gamma \subseteq \mathbb{T}$, there exists a sequence $\{p_n\}$ of trigonometric polynomials with $\|p_n\|^2_{L^2(\mathbb{T})} \leq |\Gamma|$ such that $\{p_n\}$ converges to $\mathbf{1}_\Gamma$ (the characteristic function of the set Γ) except on a set of arbitrarily small measure. Thus, if the measure $|\Gamma|$ of Γ was strictly greater than 0, there would be a finitely supported sequence c with $\|c\|^2_{l^2(\mathbb{Z})} \leq |\Gamma|$ such that

$$\|\mathcal{L}^*(c)\|^2 \;>\; |\Gamma|\left(C+\frac{\delta}{2}\right)$$

implies

$$\|\mathcal{L}^*(\frac{c}{|\Gamma|})\|^2 \;>\; |\Gamma|\left(C+\frac{\delta}{2}\right)$$

Hence

$$\|\mathcal{L}^*\|^2 = \sup\left\{\|\mathcal{L}^*(c)\|^2 : \|c\|_{l^2(\mathbb{Z})} \leq 1\right\} > C.$$

□

We can naturally extend the above result to $L^2(\mathbb{R})^n$ as follows:

Proposition 4.2.2 (General version of Proposition 4.2.1). *Let*

$$X = \left\{T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z}\right\} \subset L^2(\mathbb{R})^n$$

and define $\Phi_i(\gamma) = \sum_{k\in\mathbb{Z}} \left|\widehat{\phi_i}(\gamma + 2\pi k)\right|^2, i = 1,\ldots,n.$ *Denote the Bessel map associated with* X *by* $\mathcal{L}$. *If* $\Phi_i \leq A_i < \infty$, $i = 1,\ldots,n$ *a.e., then* $\|\mathcal{L}\| \leq (A_1 + \ldots + A_n)^{\frac{1}{2}}$. *Conversely,* $\|\mathcal{L}\| \leq C^{\frac{1}{2}}$ *implies* $\Phi_i \leq C$, $i = 1,\ldots,n$ *a.e.*

The next theorem deals with a situation in which the given collection becomes a frame.

Theorem 4.2.3. *Let* $X = \left\{T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\} \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ *and assume* $\Phi_1(\gamma) = \sum\limits_{k\in\mathbb{Z}} \left|\widehat{\phi_1}(\gamma + 2\pi k)\right|^2$ *and* $\Phi_2(\gamma) = \sum\limits_{k\in\mathbb{Z}} \left|\widehat{\phi_2}(\gamma + 2\pi k)\right|^2 \in L^\infty(\mathbb{T})$. *Then* X *has a well defined Bessel map* $\mathcal{L}$ *and*

$$N(\mathcal{L}^*) = \{c = (c_k) \in l^2(\mathbb{Z}) : \widehat{(c_k\omega^k)} = 0 \text{ on } [\Phi_1 > 0] \text{ and } \widehat{(c_k\omega^{2k})} = 0 \text{ on } [\Phi_2 > 0]\}.$$

Further, taking

$$A_1 = \inf\{a : |[\Phi_1 \le a] \cap [\Phi_1 > 0]| > 0\},$$

$$A_2 = \inf\{a : |[\Phi_2 \le a] \cap [\Phi_2 > 0]| > 0\},$$

$$\text{esssup}\Phi_1 = B_1 < \infty, \quad \text{and } \ \text{esssup}\Phi_2 = B_2 < \infty,$$

X *is a frame for* $V_0 = \overline{span}\left\{T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ *with lower frame bound greater than or equal to* $A_1 + A_2 > 0$ *and upper frame bound less than or equal to* $B_1 + B_2 < \infty$. *This frame is called the* frame generated *by* $\phi_1 \oplus \phi_2$

Proof. Φ_1 and Φ_2 are essentially bounded implies that there are $C_1 > 0$ and $C_2 > 0$ such that

$$|\Phi_1(x)| < C_1 \text{ and } |\Phi_2(x)| < C_2 \text{ a.e.}$$

Taking

$$\frac{C}{2} = \max\{C_1, C_2\}.$$

we have

$$\Phi_1 \le \frac{C}{2} \text{ and } \Phi_2 \le \frac{C}{2} \text{ a.e.}$$

Hence by Proposition 4.2.1, we have

$$\|\mathcal{L}\| < C^{1/2}.$$

Hence $\mathcal{L}$ takes values in $l^2(\mathbb{Z})$ and hence $\mathcal{L}$ is well-defined.

By the definition of V_0, $\left\{T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is complete in V_0.

Let $c \in l^2(\mathbb{Z})$ be arbitrary. Then

$$\begin{aligned}
\|\mathcal{L}^*(c)\|^2 &= \left\| \left(\sum_{k\in\mathbb{Z}} c_k T_C^k \phi_1 \oplus \phi_2 \right) \right\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\
&= \left\| \left(\sum_{k\in\mathbb{Z}} c_k \widehat{z_1^k T^k \phi_1 \oplus z_2^k} T^k \phi_2 \right) \right\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\
&= \left\| \left(\sum_{k\in\mathbb{Z}} \widehat{c_k z_1^k T^k} \phi_1 \right) \right\|^2_{L^2(\mathbb{R})} + \left\| \left(\sum_{k\in\mathbb{Z}} \widehat{c_k z_2^k T^k} \phi_2 \right) \right\|^2_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} \left| \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \widehat{\phi_1} \right|^2 + \int_{\mathbb{R}} \left| \sum_{k\in\mathbb{Z}} c_k z_2^k e_k \widehat{\phi_2} \right|^2
\end{aligned}$$

$$= \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \right|^2 \Phi_1 + \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_2^k e_k \right|^2 \Phi_2$$

Hence

$N(\mathcal{L}^*) = \left\{ c = (c_k) \in l^2(\mathbb{Z}) : \widehat{(c_k z_1^k)} = 0 \text{ on } [\Phi_1 > 0] \text{ and } \widehat{(c_k z_2^k)} = 0 \text{ on } [\Phi_2 > 0] \right\}.$

Now for the values A_1, A_2, B_1, B_2 chosen above,

$$A_1 \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \right|^2 \leq \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \right|^2 \Phi_1 \leq B_1 \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \right|^2$$

and

$$A_2 \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_2^k e_k \right|^2 \leq \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \right|^2 \Phi_2 \leq B_2 \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_2^k e_k \right|^2$$

implies

$$
\begin{aligned}
A_1 \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \right|^2 &+ A_2 \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_2^k e_k \right|^2 \\
&\leq \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \right|^2 \Phi_1 + \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k \omega^k e_k \right|^2 \Phi_2 \\
&\leq B_1 \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \right|^2 + B_2 \int_{\mathbb{T}} \left| \sum_{k\in\mathbb{Z}} c_k z_2^k e_k \right|^2
\end{aligned}
$$

Hence

$$(A_1 + A_2) \|c\|^2_{l^2(\mathbb{Z})} \leq \|\mathcal{L}^*(c)\|^2 \leq (B_1 + B_2) \|c\|^2_{l^2(\mathbb{Z})}$$

Consequently, by Proposition 3.1.13, X is a frame with lower frame bound greater than or equal to $A_1 + A_2$ and upper frame bound less than or equal to $B_1 + B_2$. □

Corollary 4.2.4. *For X, V_0, A_1, A_2, B_1 and B_2 as in Theorem 4.2.3, if $A_1+A_2 = B_1 + B_2$, then X is a tight frame for V_0 with bound $A_1 + A_2$.*

Corollary 4.2.5. *Take X, V_0, A_1, A_2, B_1 and B_2 as in Theorem 4.2.3 such that $A_1 = A_2 = B_1 = B_2 = A$. Then X is a tight frame for V_0 with bound $2A$.*

Theorem 4.2.6 (General version of Theorem 4.2.3). *Let*

$$X = \left\{T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z}\right\} \subset L^2(\mathbb{R})^n$$

and assume $\Phi_i(\gamma) = \sum_{k\in\mathbb{Z}} \left|\widehat{\phi}_i(\gamma + 2\pi k)\right|^2$, $i = 1, \cdots, n$. Then X has a well defined Bessel map $\mathcal{L}$ and

$$N(\mathcal{L}^*) = \{c = (c_k) \in l^2(\mathbb{Z}) : \widehat{(c_k z_i^k)} = 0 \text{ on } [\Phi_i > 0], \quad i = 1, \cdots, n.\}$$

Take $A_i = \inf\{a : |[\Phi_i \leq a] \cap [\Phi_i > 0]| > 0\}$ *and* $\operatorname{esssup}\Phi_i = B_i < \infty, i = 1, \cdots, n$ *. Then* X *is a frame for* $V_0 = \overline{span}\{T_C^k\phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z}\}$ *with lower frame bound greater than or equal to* $A_1 + \cdots + A_n > 0$ *and upper frame bound less than or equal to* $B_1 + \cdots + B_n$.

Corollary 4.2.7. *For* X, V_0, A_i *and* B_i *as in Theorem 4.2.6, if* $A_1 + \cdots + A_n = B_1 + \cdots + B_n$, *then* X *is a tight frame for* V_0 *with bound* $A_1 + \cdots + A_n$.

Corollary 4.2.8. *For* X, V_0, A_i *and* B_i *as in Theorem 4.2.3 such that* $A_1 = \cdots = A_n = B_1 = B_2 = A$, *then* X *is a tight frame for* V_0 *with bound* nA.

In the next result we will see that if $\phi_1 \oplus \phi_2$ generates a frame for V_0, then elements of V_0 and $\phi_1 \oplus \phi_2$ are closely related by their Fourier transforms.

Proposition 4.2.9. *Suppose* $\{T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\} \subseteq L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ *is a frame for its closed span* V_0. *Then*

$$f_1 \oplus f_2 \in V_0 \Rightarrow \hat{f_1} = F_1\hat{\phi_1} \text{ and } \hat{f_2} = F_2\hat{\phi_2},$$

for some F_1, $F_2 \in L^2(\mathbb{T})$ *depending on* $f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. *In particular, for such an* $f_1 \oplus f_2$, $\widehat{f_1 \oplus f_2} = (0,\ 0)$ *almost every where on the set* $[\widehat{\phi_1} = 0] \cap [\widehat{\phi_2} = 0]$. *Conversely, if* $\hat{f_1} = F_1\hat{\phi_1}$ and $\hat{f_2} = F_2\hat{\phi_2}$ *with* $F_1 = (c_k z_1^k)^\wedge$ *and* $F_2 = (c_k z_2^{2k})^\wedge$ *for some sequence* $(c_k) \in l^2(\mathbb{Z})$, *then* $f_1 \oplus f_2 \in V_0$.

Proof. Since $\{T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\}$ is a frame for its closed span V_0, $f_1 \oplus f_2 \in V_0$ implies

$$f_1 \oplus f_2 = \sum_{k \in \mathbb{Z}} c_k T_C^k \phi_1 \oplus \phi_2,$$

for some sequence $c = (c_k) \in l^2(\mathbb{Z})$. That is,

$$f_1 \oplus f_2 = \sum_{k \in \mathbb{Z}} c_k z_1^k T^k \phi_1 \oplus z_2^k T^k \phi_2.$$

Taking the Fourier transform of this equation gives

$$\widehat{f_1 \oplus f_2} = \left(\sum_{k\in\mathbb{Z}} c_k z_1^k e_k \hat{\phi}_1, \sum_{k\in\mathbb{Z}} c_k z_2^k e_k \hat{\phi}_2 \right)$$

implies

$$\hat{f}_1 = F_1 \hat{\phi}_1 \text{ and } \hat{f}_2 = F_2 \hat{\phi}_2,$$

where $F_1 = \sum_{k\in\mathbb{Z}} c_k z_1^k e_k$ and $F_2 = \sum_{k\in\mathbb{Z}} c_k z_2^k e_k$. Here note that F_1 is the Fourier transform of the sequence $(c_k z_1^k)$ and F_2 that of $(c_k z_2^k)$. The fact that F_1 and $F_2 \in L^2(\mathbb{T})$ follows from Parseval's theorem.

Conversely,

$$\hat{f}_1 = \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \hat{\phi}_1 \text{ and } \hat{f}_2 = \sum_{k\in\mathbb{Z}} c_k z_2^k e_k \hat{\phi}_2$$

implies

$$\hat{f}_1 = \sum_{k\in\mathbb{Z}} c_k z_1^k \left(T^k \phi_1\right)^{\wedge} \text{and } \hat{f}_2 = \sum_{k\in\mathbb{Z}} c_k z_2^k \left(T^k \phi_2\right)^{\wedge}$$

implies

$$f_1 = \sum_{k\in\mathbb{Z}} c_k z_1^k T^k \phi_1 \text{ and } f_2 = \sum_{k\in\mathbb{Z}} c_k z_2^k T^k \phi_2$$

implies

$$f_1 \oplus f_2 = \sum_{k\in\mathbb{Z}} c_k T_C^k \phi_1 \oplus \phi_2$$

implies

$$f_1 \oplus f_2 \in V_0.$$

□

Proposition 4.2.10 (General version of Theorem 4.2.9). *Suppose*

$$\left\{T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z}\right\} \subseteq L^2(\mathbb{R})^n$$

is a frame for its closed span V_0. *Then*

$$f_1 \oplus \cdots \oplus f_n \in V_0 \Rightarrow \hat{f}_i = F_i\hat{\phi}_i, i = 1, \cdots, n$$

for some $F_1, \cdots, F_n \in L^2(\mathbb{T})$ *depending on* $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. *Conversely, if* $\hat{f}_i = F_i\hat{\phi}_i$ *with* $F_i = \left(c_k z_i^k\right)\widehat{}$, $i = 1, \ldots, n$ *for some sequence* $(c_k) \in l^2(\mathbb{Z})$, *then* $f_1 \oplus \cdots \oplus f_2 \in V_0$.

ON THE STUDY OF FRAME MULTIRESOLUTION ANALYSIS IN THE SUPERSPACES $L^2(\mathbb{R})^n$

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Chapter 5

The Frame MRA Approach in the Superspace $L^2(\mathbb{R})^n$

Multiresolution analysis frame (MRA frame) in the case of $L^2(\mathbb{R})$ have been discussed in [4]. In this chapter we introduce the MRA frame theory in $L^2(\mathbb{R})^n$. In this chapter also, we first state and prove results in $L^2(\mathbb{R})^2$ and then we naturally extend those results to $L^2(\mathbb{R})^n$, where proof is omitted as it is exactly similar to those in $L^2(\mathbb{R})^2$. In 5.1.4, we see that if the set of translations of $\phi_1 \oplus \phi_2$ is a frame for V_0, then the set of dilation and translation of $\phi_1 \oplus \phi_2$ is a frame for V_j with same frame bounds. In 5.1.10 we give example of an FMRA generated by the given $\phi_1 \oplus \phi_2$. Theorem 5.2.1 tells about a case for which there is no frame wavelet in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

We first recall the Definition of FMRA in $L^2(\mathbb{R})$, noting T and U as in (1.1) and (1.2):

Definition: A *frame MRA (FMRA)* $\{V_j,\ \phi\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ is an increasing sequence of closed subspaces $V_j \subset L^2(\mathbb{R})$ and an element $\phi \in V_0$ for which the following hold:

1. $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$,
2. $f \in V_j \Leftrightarrow Uf \in V_{j+1} \quad \forall j \in \mathbb{Z}$
3. $f \in V_0 \Leftrightarrow T^k f \in V_0 \quad \forall k \in \mathbb{Z}$,
4. $\{T^k \phi : k \in \mathbb{Z}\}$ is a frame for the subspace V_0 .

5.1 The Frame MRA Approach in $L^2(\mathbb{R})^n$

Definition 5.1.1. A *frame MRA (FMRA)* $\{V_j, \phi_1 \oplus \cdots \oplus \phi_n\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})^n$ is an increasing sequence of closed subspaces $V_j \subset L^2(\mathbb{R})^n$ and an element $\phi_1 \oplus \cdots \oplus \phi_n \in V_0$ for which the following hold:

1. $\overline{\bigcup_j V_j} = L^2(\mathbb{R})^n$,
2. $f_1 \oplus \cdots \oplus f_n \in V_j \Leftrightarrow U_C f_1 \oplus \cdots \oplus f_n \in V_{j+1} \quad \forall j \in \mathbb{Z}$
3. $f_1 \oplus \cdots \oplus f_n \in V_0 \Leftrightarrow T_C^k f_1 \oplus \cdots \oplus f_n \in V_0 \quad \forall k \in \mathbb{Z}$,
4. $\{T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z}\}$ is a frame for the subspace V_0 .

An FMRA is called *redundant* or *exact* according to the nature of the frame in $\{T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z}\}$. (Ref. Definition 3.1.2).

Remark 5.1.2. The crucial difference in defining FMRA in the superspace is that we take T_C instead of T and U_C instead of U. We also observe that C can vary. Thus, in fact, for various choices of C we get the different FMRAs.

Remark 5.1.3. The condition (2) in the Definition 5.1.1 is equivalent to saying $U_C V_j = V_{j+1}$.

Proposition 5.1.4. *Let* $\{V_j, \phi_1 \oplus \phi_2\}$ *be an FMRA of* $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. *If* $\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\}$ *is a frame for* V_0, *then the system* $\{U_C^j T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\}$ *is a frame for* V_j *with the same frame bounds.*

Proof. $\{T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\}$ is a frame for V_0 implies that there exist lower frame bound, say, $A > 0$ and upper frame bound , say, $B < \infty$ such that for all $f_1 \oplus f_2 \in V_0$

$$A\,\|f_1 \oplus f_2\|^2 \leq \sum_{k\in\mathbb{Z}} \left|\langle f_1 \oplus f_2,\ T_C^k\phi_1 \oplus \phi_2\rangle\right|^2 \leq B\,\|f_1 \oplus f_2\|^2 \tag{5.1}$$

Also,

$$\{T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\} \subseteq V_0 \Leftrightarrow \{U_C^j T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\} \subseteq V_j,$$

by (2) in the Definition 5.1.1.

$$\begin{aligned}
\langle U_C f_1 \oplus f_2,\ U_C f_1 \oplus f_2\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} &= \langle U f_2 \oplus U f_1,\ U f_2 \oplus U f_1\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\\
&= \langle U f_2,\ U f_2\rangle_{L^2(\mathbb{R})} + \langle U f_1,\ U f_1\rangle_{L^2(\mathbb{R})}\\
&= \langle f_2,\ f_2\rangle_{L^2(\mathbb{R})} + \langle f_1,\ f_1\rangle_{L^2(\mathbb{R})},\\
&\qquad \text{as } U \text{ is unitary operator on } L^2(\mathbb{R})\\
&= \langle f_1,\ f_1\rangle_{L^2(\mathbb{R})} + \langle f_2,\ f_2\rangle_{L^2(\mathbb{R})}\\
&= \langle f_1 \oplus f_2,\ f_1 \oplus f_2\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}.
\end{aligned}$$

Hence U_C is a unitary operator on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, and hence

$$U_C U_C^* = I = U_C^* U_C,$$

which implies that

$$U_C^* = U_C^{-1}.$$

In general,

$$\left(U_C^j\right)^* = U_C^{-j} \quad \text{for } j \in \mathbb{Z}.$$

Hence for $f_1 \oplus f_2 \in V_j$,

$$(5.2)\quad \sum_{k\in\mathbb{Z}} \left|\left\langle f_1 \oplus f_2,\ U_C^j T_C^k \phi_1 \oplus \phi_2 \right\rangle\right|^2 = \sum_{k\in\mathbb{Z}} \left|\left\langle U_C^{-j} f_1 \oplus f_2,\ T_C^k \phi_1 \oplus \phi_2 \right\rangle\right|^2.$$

Since $U_C^{-j} f_1 \oplus f_2 \in V_0$ and $\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for V_0, we have

$$A \left\|U_C^{-j} f_1 \oplus f_2\right\|^2 \leq \sum_{k\in\mathbb{Z}} \left|\left\langle U_C^{-j} f_1 \oplus f_2,\ T_C^k \phi_1 \oplus \phi_2 \right\rangle\right|^2 \leq B \left\|U_C^{-j} f_1 \oplus f_2\right\|^2.$$

But,

$$\begin{aligned} \left\|U_C^{-j} f_1 \oplus f_2\right\|^2 &= \left\langle U_C^{-j} f_1 \oplus f_2,\ U_C^{-j} f_1 \oplus f_2 \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\ &= \left\langle \left(U_C^{-j}\right)^* U_C^{-j} f_1 \oplus f_2,\ f_1 \oplus f_2 \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\ &= \left\langle I f_1 \oplus f_2,\ f_1 \oplus f_2 \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}, \\ &\quad \text{where } I \text{ is the identity operator on } L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \\ &= \left\langle f_1 \oplus f_2,\ f_1 \oplus f_2 \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\ &= \left\|f_1 \oplus f_2\right\|^2. \end{aligned}$$

Now using (5.2), we have

$$A \left\|f_1 \oplus f_2\right\|^2 \leq \sum_{k\in\mathbb{Z}} \left|\left\langle f_1 \oplus f_2,\ U_C^j T_C^k \phi_1 \oplus \phi_2 \right\rangle\right|^2 \leq B \left\|f_1 \oplus f_2\right\|^2,$$

Hence $\left\{U_C^j T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for V_j with same frame bounds as the frame $\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ of V_0. ☐

In a natural way, we can extend the preceding result to get the following one:

Proposition 5.1.5. *Let $\{V_j,\ \phi_1 \oplus \cdots \oplus \phi_n\}_{j\in\mathbb{Z}}$ be an FMRA of $L^2(\mathbb{R})^n$. If $\left\{T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z}\right\}$ is a frame for V_0, then the system $\left\{U_C^j T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z}\right\}$ is a frame for V_j with the same frame bounds.*

Given an FMRA of $L^2(\mathbb{R})^n$, define W_0 to be the orthogonal complement of V_0 in V_1 (i.e., $W_0 = V_1 \ominus V_0$), and set for $j \in \mathbb{Z}$

$$W_j = \left\{ f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n : \; U_C^{-j} f_1 \oplus \cdots \oplus f_n \in W_0 \right\}.$$

Analogous to an MRA, there are two fundamental problems associated with FMRAs: first, finding FMRAs, and, second, given an FMRA, constructing a *frame wavelet* $\psi_1 \oplus \cdots \oplus \psi_n$ such that $\left\{ T_C^k \psi_1 \oplus \cdots \oplus \psi_n : k \in \mathbb{Z} \right\}$ is a frame for W_0, with given frame bounds. Then, as in the standard MRA setting,

$$L^2(\mathbb{R})^n = \operatorname{clos}\left(\oplus_j^{\perp} W_j \right)$$

as an orthogonal direct sum, and

$$\left\{ U_C^j T_C^k \psi_1 \oplus \cdots \oplus \psi_n : j, \; k \in \mathbb{Z} \right\}$$

will be a frame for $L^2(\mathbb{R})^n$ with given frame bounds.

Theorem 5.1.6. *Suppose* $\{V_j, \; \phi_1 \oplus \phi_2\}$ *be an FMRA of* $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. *Then there are 2π-periodic functions* H_0' and $H_0'' \in L^2(\mathbb{T})$ *such that*

$$\hat{\phi}_1(2\cdot) = H_0' \hat{\phi}_2 \tag{5.3}$$

and

$$\hat{\phi}_2(2\cdot) = H_0'' \hat{\phi}_1. \tag{5.4}$$

Also

$$\Phi_1 = \left| H_0'\left(\tfrac{\cdot}{2}\right) \right|^2 \Phi_2\left(\tfrac{\cdot}{2}\right) + \left| H_0'\left(\tfrac{\cdot}{2} + \pi\right) \right|^2 \Phi_2\left(\tfrac{\cdot}{2} + \pi\right).$$

and

$$\Phi_2 = \left|H_0''(\tfrac{\cdot}{2})\right|^2 \Phi_1(\tfrac{\cdot}{2}) + \left|H_0''(\tfrac{\cdot}{2} + \pi)\right|^2 \Phi_1(\tfrac{\cdot}{2} + \pi),$$

where Φ_1 *and* Φ_2 *are given by*

$$\Phi_1(\gamma) = \sum_{k\in\mathbb{Z}} \left|\widehat{\phi}_1(\gamma + 2\pi k)\right|^2 \text{ and } \Phi_2(\gamma) = \sum_{k\in\mathbb{Z}} \left|\widehat{\phi}_2(\gamma + 2\pi k)\right|^2.$$

Proof. V_1 is closed and invariant under translations. So $V_0 \subseteq V_1$ if and only if $\phi_1 \oplus \phi_2 \in V_1$: For

$$V_0 \subseteq V_1 \Rightarrow \phi_1 \oplus \phi_2 \in V_1$$

and

$$\phi_1 \oplus \phi_2 \in V_1 \Rightarrow \text{integer translates of } \phi_1 \oplus \phi_2 \in V_1 \Rightarrow V_0 \subseteq V_1.$$

As $\left\{U_C T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for V_1, we have

$$\phi_1 \oplus \phi_2 = \sum_{k\in\mathbb{Z}} c_k U_C T_C^k \phi_1 \oplus \phi_2$$

implies

$$\phi_1 \oplus \phi_2 = \sum_{k\in\mathbb{Z}} c_k U_C z_1^k T^k \phi_1 \oplus z_2^k T^k \phi_2$$

implies

$$\phi_1 \oplus \phi_2 = \sum_{k\in\mathbb{Z}} c_k U z_2^k T^k \phi_2 \oplus U z_1^k T^k \phi_1$$

Taking the Fourier transforms on both sides, we obtain

$$\widehat{\phi_1 \oplus \phi_2} = \left(\sum_{k\in\mathbb{Z}} c_k U^{-1}\left(\widehat{z_2^k T^k \phi_2}\right), \sum_{k\in Z} c_k U^{-1}\left(\widehat{z_1^k T^k \phi_1}\right)\right),$$

$$\text{as}(U\phi)\hat{} = U^{-1}\hat{\phi}, \text{ for } \phi \in L^2(\mathbb{R})$$

implies

$$\widehat{\phi}_1 = \sum_{k\in\mathbb{Z}} c_k U^{-1}\big(\widehat{z_2^k T^k \phi_2}\big),$$

and

$$\widehat{\phi}_2 = \sum_{k\in\mathbb{Z}} c_k U^{-1}\big(\widehat{z_1^k T^k \phi_1}\big),$$

implies

$$U\widehat{\phi}_1 = \sum_{k\in Z} c_k\big(\widehat{z_2^k T^k \phi_2}\big),$$

and

$$U\widehat{\phi}_2 = \sum_{k\in Z} c_k\big(\widehat{z_1^k T^k \phi_1}\big),$$

implies

$$U\hat{\phi}_1 = \sum_{k\in\mathbb{Z}} c_k z_2^k e_k \hat{\phi}_2 \quad \text{and} \quad U\hat{\phi}_2 = \sum_{k\in\mathbb{Z}} c_k z_1^k e_k \hat{\phi}_1$$

implies

$$\hat{\phi}_1(2\cdot) = H_0'\hat{\phi}_2$$

and

$$\hat{\phi}_2(2\cdot) = H_0''\hat{\phi}_1,$$

where $H_0' = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} c_k z_2^k e_k$ and $H_0'' = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} c_k z_1^k e_k$ and both belong to $L^2(\mathbb{T})$. Moreover, we may choose $H_0' = 0$ on $[\Phi_2 = 0]$, where $\Phi_2(\gamma) = \sum_{k\in\mathbb{Z}} \left|\widehat{\phi_2}\,(\gamma + 2\pi k)\right|^2$ and $H_0'' = 0$ on $[\Phi_1 = 0]$, where $\Phi_1(\gamma) = \sum_{k\in\mathbb{Z}} \left|\widehat{\phi_1}\,(\gamma + 2\pi k)\right|^2$.

Now we periodize the square modulus of $\hat{\phi}_1(\cdot) = H_0'\hat{\phi}_2(\frac{\cdot}{2})$:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \left|\hat{\phi}_1(\cdot + 2\pi k)\right|^2 &= \sum_{k\in\mathbb{Z}} \left|H_0'(\tfrac{\cdot+2\pi k}{2})\right|^2 \left|\hat{\phi}_2(\tfrac{\cdot+2\pi k}{2})\right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|H_0'\left(\frac{\cdot}{2} + \pi k\right)\right|^2 \left|\hat{\phi}_2\left(\frac{\cdot}{2} + \pi k\right)\right|^2
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k\in 2\mathbb{Z}} \left|H_0'\left(\frac{\cdot}{2}+\pi k\right)\right|^2 \left|\hat{\phi}_2\left(\frac{\cdot}{2}+\pi k\right)\right|^2 \\
&\quad + \sum_{k\in 2\mathbb{Z}+\{1\}} \left|H_0'\left(\frac{\cdot}{2}+2\pi k\right)\right|^2 \left|\hat{\phi}_2\left(\frac{\cdot}{2}+2\pi k\right)\right|^2 \ddagger \\
&= \sum_{k\in\mathbb{Z}} \left|H_0'(\tfrac{\cdot}{2}+2\pi k)\right|^2 \left|\hat{\phi}_2(\tfrac{\cdot}{2}+2\pi k)\right|^2 \\
&\quad + \sum_{k\in\mathbb{Z}} \left|H_0'(\tfrac{\cdot}{2}+\pi+2\pi k)\right|^2 \left|\hat{\phi}_2(\tfrac{\cdot}{2}+\pi+2\pi k)\right|^2 \\
&= \left|H_0'(\tfrac{\cdot}{2})\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\phi}_2(\tfrac{\cdot}{2}+2\pi k)\right|^2 \\
&\quad + \left|H_0'\left(\frac{\cdot}{2}+\pi\right)\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\phi}_2\left(\frac{\cdot}{2}+\pi+2\pi k\right)\right|^2 \dagger
\end{aligned}$$

That is

$$\Phi_1 = \left|H_0'\left(\frac{\cdot}{2}\right)\right|^2 \Phi_2\left(\frac{\cdot}{2}\right) + \left|H_0'\left(\frac{\cdot}{2}+\pi\right)\right|^2 \Phi_2\left(\frac{\cdot}{2}+\pi\right).$$

Similarly, we obtain

$$\Phi_2 = \left|H_0''(\tfrac{\cdot}{2})\right|^2 \Phi_1(\tfrac{\cdot}{2}) + \left|H_0''(\tfrac{\cdot}{2}+\pi)\right|^2 \Phi_1(\tfrac{\cdot}{2}+\pi).$$

‡ Rearrangement of abosulutely summable series doesn't effect the sum.

† As

$$\begin{aligned}
\left|H_0'(\tfrac{\cdot}{2}+2\pi k)\right|^2 &= \left|\sum_k c_k\omega^{2k} e_k(\tfrac{\cdot}{2}+2\pi k)\right|^2 \\
&= \left|\sum_k c_k\omega^{2k} e_k(\tfrac{\cdot}{2})\right|^2, \natural \\
&= \left|H_0'(\tfrac{\cdot}{2})\right|^2.
\end{aligned}$$

♮ since e_k is 2π – periodic as $e_k(t+2\pi) = e^{-i(t+2\pi 1)k} = e^{-itk} = e_k(t)$.

Similarly,

$$\left|H_0'(\tfrac{\cdot}{2}+\pi+2\pi k)\right|^2 = \left|H_0'(\tfrac{\cdot}{2}+\pi)\right|^2.$$

$\square$

Proposition 5.1.7 (General version of Proposition 5.1.6). *Suppose*

$$\{V_j,\ \phi_1\oplus\cdots\oplus\phi_n\}$$

be an FMRA of $L^2(\mathbb{R})^n$. *Then there are* 2π*-periodic functions* $H_{0,\,1},\ \ldots\ ,\ H_{0,\,n}\in L^2(\mathbb{T})$ *such that*

$$\hat{\phi}_1(2\cdot) = H_{0,\,1}\hat{\phi}_2 \tag{5.5}$$

$$\hat{\phi}_2(2\cdot) = H_{0,\,2}\hat{\phi}_3 \tag{5.6}$$

$$\vdots$$

$$\hat{\phi}_{n-1}(2\cdot) = H_{0,\,n-1}\hat{\phi}_n \tag{5.7}$$

and

$$\hat{\phi}_n(2\cdot) = H_{0,\,n}\hat{\phi}_1 \tag{5.8}$$

Proposition 5.1.8. *Let* $\{V_j,\ \phi_1\oplus\phi_2\}$ *be an FMRA of* $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ *and let* H_0' and H_0'' *be the functions defined in* (5.3) *and* (5.4). *Define*

$$\Phi_1(\gamma) = \sum_{k\in\mathbb{Z}}\left|\widehat{\phi_1}\,(\gamma+2\pi k)\right|^2$$

and

$$\Phi_2(\gamma) = \sum_{k\in\mathbb{Z}}\left|\widehat{\phi_2}\,(\gamma+2\pi k)\right|^2$$

Given any 2π-periodic functions H_1', $H_1'' \in L^2(\mathbb{T})$, define $\psi_1 \oplus \psi_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ by means of the equations

$$\hat{\psi}_1(2\cdot) = H_1'\hat{\phi}_2 \tag{5.9}$$

and

$$\hat{\psi}_2(2\cdot) = H_1''\hat{\phi}_1. \tag{5.10}$$

Define W_0 as the orthogonal complement of V_0 in V_1. Then $\overline{H_0'}H_1'\Phi_2 + \tau_\pi\left(\overline{H_0'}H_1'\Phi_2\right) = 0$ and $\overline{H_0''}H_1''\Phi_1 + \tau_\pi\left(\overline{H_0''}H_1''\Phi_1\right) = 0 \Rightarrow \psi_1 \oplus \psi_2 \in W_0$.

Conversely, $\left\langle \psi_1,\ z_1^k T^k \phi_1 \right\rangle_{L^2(\mathbb{R})} \geq 0$ and $\left\langle \psi_2,\ z_2^k T^k \phi_2 \right\rangle_{L^2(\mathbb{R})} \geq 0$ and $\psi_1 \oplus \psi_2 \in W_0$ implies that $\overline{H_0'}H_1'\Phi_2 + \tau_\pi\left(\overline{H_0'}H_1'\Phi_2\right) = 0$ and $\overline{H_0''}H_1''\Phi_1 + \tau_\pi\left(\overline{H_0''}H_1''\Phi_1\right) = 0$.

Proof.

$$\begin{aligned}
\psi_1 \oplus \psi_2 \in V_1 &\Leftrightarrow U_C^{-1}\psi_1 \oplus \psi_2 \in V_0 \\
&\Leftrightarrow U^{-1}\psi_2 \oplus U^{-1}\psi_1 \in V_0 \\
&\Leftrightarrow (\widehat{U^{-1}\psi_2}) = F_1\hat{\phi}_1 \text{ and } (\widehat{U^{-1}\psi_1}) = F_2\hat{\phi}_2, \\
&\qquad \text{for some } F_1,\ F_2 \in L^2(\mathbb{T}) \text{ by Proposition 4.2.9} \\
&\Leftrightarrow (U^{-1})^{-1}\hat{\psi}_2 = F_1\hat{\phi}_1 \text{ and } (U^{-1})^{-1}\hat{\psi}_1 = F_2\hat{\phi}_2 \\
&\Leftrightarrow U\hat{\psi}_2 = F_1\hat{\phi}_1 \text{ and } U\hat{\psi}_1 = F_2\hat{\phi}_2 \\
&\Leftrightarrow \hat{\psi}_2(2\cdot) = \frac{1}{\sqrt{2}}F_1\hat{\phi}_1 \text{ and } \hat{\psi}_1(2\cdot) = \frac{1}{\sqrt{2}}F_2\hat{\phi}_2.
\end{aligned}$$

By the assumption H_1' and $H_1'' \in L^2(\mathbb{T})$. Hence from the discussion just above and by the definition of $\psi_1 \oplus \psi_2$ in (5.9) and (5.10), we have $\psi_1 \oplus \psi_2 \in V_1$.

Moreover, $\forall k \in \mathbb{Z}$

$$H_1'\overline{H_0'}\Phi_2 + \tau_\pi H_1'\overline{H_0'}\Phi_2 = 0 \text{ and } H_1''\overline{H_0''}\Phi_1 + \tau_\pi H_1''\overline{H_0''}\Phi_1 = 0$$

implies, by the uniqueness theorem of the Fourier series, for all $k \in \mathbb{Z}$

$$\int_{[0,\,\pi)} (H_1'\overline{H_0'}\Phi_2 + \tau_\pi H_1'\overline{H_0'}\Phi_2)e_{-2k} = 0 \text{ and } \int_{[0,\,\pi)} \left(H_1''\overline{H_0''}\Phi_1 + \tau_\pi H_1''\overline{H_0''}\Phi_1\right) e_{-2k} = 0$$

implies

$$\int_{\mathbb{R}} H_1'\overline{H_0'}\left|\hat{\phi}_2\right|^2 e_{-2k} = 0 \text{ and } \int_{\mathbb{R}} H_1''\overline{H_0''}\left|\hat{\phi}_1\right|^2 e_{-2k} = 0$$

implies for all $k \in \mathbb{Z}$

$$\int_{\mathbb{R}} H_1'\hat{\phi}_2\overline{z_1^k e_{2k}\hat{\phi}_2 H_0'} = 0$$

and

$$\int_{\mathbb{R}} H_1''\hat{\phi}_1\overline{z_2^k e_{2k}\hat{\phi}_1 H_0''} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle H_1'(\cdot)\hat{\phi}_2(\cdot),\ z_1^k e_{2k} H_0'(\cdot)\hat{\phi}_2(\cdot)\right\rangle_{L^2(\mathbb{R})} = 0$$

and

$$\left\langle H_1''(\cdot)\hat{\phi}_1(\cdot),\ z_2^k e_{2k} H_0''(\cdot)\hat{\phi}_1(\cdot)\right\rangle_{L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle H_1'(\tfrac{\cdot}{2})\hat{\phi}_2(\tfrac{\cdot}{2}),\ z_1^k e_k H_0'(\tfrac{\cdot}{2})\hat{\phi}_2(\tfrac{\cdot}{2})\right\rangle_{L^2(\mathbb{R})} = 0$$

and

$$\left\langle H_1''(\tfrac{\cdot}{2})\hat{\phi}_1(\tfrac{\cdot}{2}),\ z_2^k e_k H_0''(\tfrac{\cdot}{2})\hat{\phi}_1(\tfrac{\cdot}{2})\right\rangle_{L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle \hat{\psi}_1,\ z_1^k e_k \hat{\phi}_1 \right\rangle_{L^2(\mathbb{R})} = 0$$

and

$$\left\langle \hat{\psi}_2,\ z_2^k e_k \hat{\phi}_2 \right\rangle_{L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle \psi_1,\ z_1^k T^k \phi_1 \right\rangle_{L^2(\mathbb{R})} = 0 \text{ and } \left\langle \psi_2,\ z_2^k T^k \phi_2 \right\rangle_{L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle \psi_1,\ z_1^k T^k \phi_1 \right\rangle_{L^2(\mathbb{R})} + \left\langle \psi_2,\ z_2^k T^k \phi_2 \right\rangle_{L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle \psi_1 \oplus \psi_2,\ z_1^k T^k \phi_1 \oplus z_2^k T^k \phi_2 \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle \psi_1 \oplus \psi_2,\ T_C^k \phi_1 \oplus \phi_2 \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} = 0$$

implies

$$\psi_1 \oplus \psi_2 \in V_0^{\perp}.$$

Conversely, assume

$$\left\langle \psi_1,\ z_1^k T^k \phi_1 \right\rangle_{L^2(\mathbb{R})} \geq 0, \quad \left\langle \psi_2,\ z_2^k T^k \phi_2 \right\rangle_{L^2(\mathbb{R})} \geq 0 \text{ and } \psi_1 \oplus \psi_2 \in W_0.$$

Then for all $k \in \mathbb{Z}$

$$\left\langle \psi_1 \oplus \psi_2,\ T_C^k \phi_1 \oplus \phi_2 \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle \psi_1 \oplus \psi_2,\ z_1^k T^k \phi_1 \oplus z_2^k T^k \phi_2 \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle \psi_1,\ z_1^k T^k \phi_1 \right\rangle_{L^2(\mathbb{R})} + \left\langle \psi_2,\ z_2^k T^k \phi_2 \right\rangle_{L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle \psi_1,\ z_1^k T^k \phi_1 \right\rangle_{L^2(\mathbb{R})} = 0 \text{ and } \left\langle \psi_2,\ z_2^k T^k \phi_2 \right\rangle_{L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle \hat{\psi}_1,\ z_1^k e_k \hat{\phi}_1 \right\rangle_{L^2(\mathbb{R})} = 0 \text{ and } \left\langle \hat{\psi}_2,\ z_2^k e_k \hat{\phi}_2 \right\rangle_{L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle H_1'(\tfrac{\cdot}{2})\hat{\phi}_2(\tfrac{\cdot}{2}),\ z_1^k e_k H_0'(\tfrac{\cdot}{2})\hat{\phi}_2(\tfrac{\cdot}{2}) \right\rangle_{L^2(\mathbb{R})} = 0$$

and

$$\left\langle H_1''(\tfrac{\cdot}{2})\hat{\phi}_1(\tfrac{\cdot}{2}),\ z_2^k e_k H_0''(\tfrac{\cdot}{2})\hat{\phi}_1(\tfrac{\cdot}{2}) \right\rangle_{L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\left\langle H_1'(\cdot)\hat{\phi}_2(\cdot),\ z_1^k e_{2k} H_0'(\cdot)\hat{\phi}_2(\cdot) \right\rangle_{L^2(\mathbb{R})} = 0$$

and

$$\left\langle H_1''(\cdot)\hat{\phi}_1(\cdot),\ z_2^k e_{2k} H_0''(\cdot)\hat{\phi}_1(\cdot)\right\rangle_{L^2(\mathbb{R})} = 0$$

implies for all $k \in \mathbb{Z}$

$$\int_{\mathbb{R}} H_1' \hat{\phi}_2 \overline{z_1^k e_{2k} \hat{\phi}_2 H_0'} = 0 \text{ and } \int_{R} H_1'' \hat{\phi}_1 \overline{z_2^k e_{2k} \hat{\phi}_1 H_0''} = 0$$

implies for all $k \in \mathbb{Z}$

$$\int_{\mathbb{R}} H_1' \overline{H_0'} \left|\hat{\phi}_2\right|^2 e_{-2k} = 0 \text{ and } \int_{\mathbb{R}} H_1'' \overline{H_0''} \left|\hat{\phi}_1\right|^2 e_{-2k} = 0$$

implies for all $k \in \mathbb{Z}$

$$\int_{[0,\,\pi)} \left(H_1' \overline{H_0'} \Phi_2 + \tau_\pi H_1' \overline{H_0'} \Phi_2\right) e_{-2k} = 0$$

and

$$\int_{[0,\,\pi)} \left(H_1'' \overline{H_0''} \Phi_1 + \tau_\pi H_1'' \overline{H_0''} \Phi_1\right) e_{-2k} = 0$$

implies, by the uniqueness theorem of the Fourier series, that for all $k \in \mathbb{Z}$

$$H_1' \overline{H_0'} \Phi_2 + \tau_\pi H_1' \overline{H_0'} \Phi_2 = 0 \text{ and } H_1'' \overline{H_0''} \Phi_1 + \tau_\pi H_1'' \overline{H_0''} \Phi_1 = 0.$$

□

Proposition 5.1.9 (General version of Proposition 5.1.8). *Let*

$$\{V_j,\ \phi_1 \oplus \cdots \oplus \phi_n\}$$

be an FMRA of $L^2(\mathbb{R})^n$ and let $H_{0,\,1}, \ \ldots\ , H_{0,\,n}$ be the functions defined in (5.5) *to* (5.8). *Given any* 2π*-periodic functions* $H_{1,\,1}, \ldots, H_{1,\,n}$, *define* $\psi_1 \oplus \cdots \oplus \psi_n \in L^2(\mathbb{R})^n$ *by means of the equations*

$$\hat{\psi}_1(2\cdot) = H_{1,\,1}\hat{\phi}_2 \tag{5.11}$$

$$\hat{\psi}_2(2\cdot) = H_{1,\,2}\hat{\phi}_3 \tag{5.12}$$

$$\vdots$$

$$\hat{\psi}_{n-1}(2\cdot) = H_{1,\,n-1}\hat{\phi}_n \tag{5.13}$$

and

$$\hat{\psi}_n(2\cdot) = H_{1,\,n}\hat{\phi}_1. \tag{5.14}$$

Define W_0 *as the orthogonal complement of* V_0 *in* V_1. *Then* $\overline{H_{0,\,1}}H_{1,\,1}\Phi_2 + \tau_\pi\left(\overline{H_{0,\,1}}H_{1,\,1}\Phi_2\right) = 0$, $\overline{H_{0,\,2}}H_{1,\,2}\Phi_3 + \tau_\pi\left(\overline{H_{0,\,2}}H_{1,\,2}\Phi_3\right) = 0, \cdots,$ *and* $\overline{H_{0,\,n}}H_{1,\,n}\Phi_1 + \tau_\pi\left(\overline{H_{0,\,n}}H_{1,\,n}\Phi_1\right) = 0 \Rightarrow \psi_1 \oplus \cdots \oplus \psi_n \in W_0$.

Conversely, $\langle \psi_1,\ z_1^k T^k \phi_1 \rangle_{L^2(\mathbb{R})} \geq 0, \cdots$, *and* $\langle \psi_n,\ z_n^k T^k \phi_n \rangle_{L^2(\mathbb{R})} \geq 0$ *and* $\psi_1 \oplus \cdots \oplus \psi_n \in W_0$ *implies that* $\overline{H_{0,\,1}}H_{1,\,1}\Phi_2 + \tau_\pi\left(\overline{H_{0,\,1}}H_{1,\,1}\Phi_2\right) = 0$, $\overline{H_{0,\,2}}H_{1,\,2}\Phi_3 + \tau_\pi\left(\overline{H_{0,\,2}}H_{1,\,2}\Phi_3\right) = 0, \cdots$, and $\overline{H_{0,\,n}}H_{1,\,n}\Phi_1 + \tau_\pi\left(\overline{H_{0,\,n}}H_{1,\,n}\Phi_1\right) = 0$.

Example 5.1.10. Consider $\phi_1 \oplus \phi_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ defined by

$$\hat{\phi}_1 = \hat{\phi}_2 = e^{i3\theta}_{|[-\pi,\,\pi)}.$$

Then $\Phi_1 = \Phi_2 = \chi_{[-\pi,\ \pi)}$ on $[-\pi,\ \pi)$. Hence by Corollary 4.2.4, the system $\{T_C^k\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\}$ is a tight frame with frame bound 2 for its closed span V_0 and

$$V_0 = \left\{ f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) : \operatorname{supp} \hat{f}_1 \subseteq [-\pi,\ \pi) \text{ and } \operatorname{supp} \hat{f}_2 \subseteq [-\pi,\ \pi) \right\}.$$

We take $H_0' = H_0'' = e^{i30}_{|[-\frac{\pi}{2},\frac{\pi}{2})}$. By means of proof of the Proposition 5.1.6, we first prove $\phi_1 \oplus \phi_2 \in V_1$:

We first note that $\{z_1,\ z_2\}$, where $z_1 = \omega$ and $z_2 = \omega^2$ with $\omega^3 = 1$, is a 2-cycle. In this example, $\hat{\phi}_1(2\cdot) = H_0'\hat{\phi}_2$ and $\hat{\phi}_2(2\cdot) = H_0''\hat{\phi}_1$ becomes $\hat{\phi}_1(2\cdot) = H_0'\hat{\phi}_1$ as $\hat{\phi}_1 = \hat{\phi}_2$ and $H_0' = H_0''$. This implies $\hat{\phi}_1(\cdot) = H_0'\left(\frac{\cdot}{2}\right)\hat{\phi}_1\left(\frac{\cdot}{2}\right)$ implies $\hat{\phi}_1(\cdot) = H_0'\left(\frac{\cdot}{2}\right)\sqrt{2}U^{-1}\hat{\phi}_1(\cdot)$ implies $\hat{\phi}_1(\cdot) = h_0'(\cdot)\,\widehat{U\phi_1}(\cdot)$, where $h_0' = H_0'\left(\frac{\cdot}{2}\right)\sqrt{2}$ implies

$$\hat{\phi}_1 \oplus \hat{\phi}_2 = \sum_{k\in\mathbb{Z}} c_k \omega^k e_k \widehat{U\phi_1} \oplus \omega^{2k} e_k \widehat{U\phi_2}$$

implies

$$\hat{\phi}_1 \oplus \hat{\phi}_2 = \sum_{k\in\mathbb{Z}} c_k \widehat{T_C^k U\phi_1 \oplus U\phi_2}$$

implies

$$\phi_1 \oplus \phi_2 = \sum_{k\in\mathbb{Z}} c_k T_C^k U\phi_1 \oplus U\phi_2 \in V_1$$

as $U_C T_C^0 \phi_1 \oplus \phi_2 = U_C \phi_1 \oplus \phi_2 = U\phi_2 \oplus U\phi_1 \in V_1$ and V_1 is closed and invariant under translations. Hence $\phi_1 \oplus \phi_2 \in V_1$, so that $V_0 \subseteq V_1$, and $\{V_j,\ \phi_1 \oplus \phi_2\}$ forms a FMRA.

Setting $H_1' = H_1'' = e^{i3\theta}_{|[-\pi,\ -\frac{\pi}{2}) \cup [\frac{\pi}{2},\ \pi)}$ and defining $\psi_1 \oplus \psi_2$ by means of the equations (5.9) and (5.10), we get $\sum_{k\in\mathbb{Z}} \left|\widehat{\psi_1}(\cdot + k)\right|^2 = \sum_{k\in\mathbb{Z}} \left|\widehat{\psi_2}(\cdot + k)\right|^2 = \chi_{|[-\pi,-\frac{\pi}{2}) \cup [\frac{\pi}{2},\pi)}$, so that by Theorem 4.2.3, $\{T_C^k\psi_1 \oplus \psi_2 : k \in \mathbb{Z}\}$ is a tight frame with frame bound 2 for the closure of its span. Let

$$W_0 = \overline{\operatorname{span}}\left\{T_C^k\psi_1 \oplus \psi_2 : k \in \mathbb{Z}\right\}.$$

If, also, $W_0 = V_1 \ominus V_0$, then $\left\{U_C^j T_C^k \psi_1 \oplus \psi_2 : j,\ k \in \mathbb{Z}\right\}$ will be a frame for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, so that $\psi_1 \oplus \psi_2$ is a frame wavelet for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

5.2 On Non-Existence of FMRA Frame Wavelets

The following result tells about a case for which there is no frame wavelet in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ associated with the given FMRA.

Theorem 5.2.1. *Suppose* $\{V_j,\ \phi_1 \oplus \phi_2\}$ *be an FMRA of* $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ *and let* H_0' and H_0'' *be the functions defined in* (5.1) *and* (5.4). *Set*

$$\Gamma = \{\gamma \in [-\pi,\ \pi) : \Phi_1(2\gamma) = \Phi_2(2\gamma) = 0,\ \Phi_1(\gamma) > 0,\ \Phi_2(\gamma) > 0,$$

$$\Phi_1(\gamma + \pi) > 0,\ \Phi_2(\gamma + \pi) > 0\}.$$

If $|\Gamma| > 0$, *there is no* $\psi_1 \oplus \psi_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ *for which* $\left\{T_C^k\psi_1 \oplus \psi_2 :\ k \in \mathbb{Z}\right\}$ *is a frame for the orthogonal complement* W_0 *of* V_0 *in* V_1.

Proof. Suppose that $|\Gamma| > 0$ and set $\Gamma_+ = \Gamma \cap [0,\ \pi)$ and $\Gamma_- = \Gamma \cap [-\pi,\ 0)$. By the definition of Γ, neither Γ_+ nor Γ_- has measure 0. Now define the functions $F_1,\ F_2,\ F_1',\ F_2' \in L^\infty([-\pi,\ \pi))$ by the properties: $F_1 = F_2 = a \neq 0$ on $\Gamma_+, F_1 = F_2 = 0$ on $[-\pi,\ \pi)\backslash\Gamma, F_1 = F_2 = b \neq 0$ on Γ_- and $F_1' = F_2' = 1$ on $\Gamma_+, F_1' = F_2' = 0$ on $[-\pi,\ \pi)\backslash\Gamma, F_1' = 1$ and $F_2' = -1$ on Γ_-. Next, we define functions $f_i,\ f_i'$ by the dilation equations

$$\widehat{f}_i(2\cdot) = F_i\hat{\phi}_1 \qquad i = 1,\ 2 \tag{5.15}$$

and

$$\widehat{f_i'}(2\cdot) = F_i'\hat{\phi}_2 \qquad i = 1,\ 2. \tag{5.16}$$

We note that f_i and f_i', $i = 1,\ 2$ are not identically zero. In fact, if $f_i = 0$, then from (5.15), we see that

$$|F_i|^2\,\Phi_1 = 0$$

on $[-\pi,\ \pi)$, which contradicts the hypothesis that $|F_i|^2$ and Φ_1 are positive on Γ. Similarly, $f_i' = 0$ also leads to contradiction. Also note by (5.3), that

$$\Phi_1 = \left|H_0'\left(\tfrac{\cdot}{2}\right)\right|^2 \Phi_2\left(\tfrac{\cdot}{2}\right) + \left|H_0'\left(\tfrac{\cdot}{2}+\pi\right)\right|^2 \Phi_2\left(\tfrac{\cdot}{2}+\pi\right). \tag{5.17}$$

Hence, we have

$$\Phi_1(2\gamma) = |H_0'(\gamma)|^2\,\Phi_2(\gamma) + |H_0'(\gamma+\pi)|^2\,\Phi_2(\gamma+\pi).$$

Hence, if $\gamma \in \Gamma$, then

$$0 = |H_0'(\gamma)|^2\,\Phi_2(\gamma) + |H_0'(\gamma+\pi)|^2\,\Phi_2(\gamma+\pi). \tag{5.18}$$

As $\Phi_2(\gamma) > 0$ and $\Phi_2(\gamma+\pi) > 0$ for $\gamma \in \Gamma$, (5.18) leads us to conclude that

$$H_0'(\gamma) = H_0'(\gamma+\pi) = 0 \text{ for } \gamma \in \Gamma$$

That is

$$H_0' = \tau_\pi H_0' = 0 \text{ on } \Gamma. \tag{5.19}$$

Similarly,

$$H_0'' = \tau_\pi H_0'' = 0 \text{ on } \Gamma. \tag{5.20}$$

Since $H_0' = H_0'' = 0$ on Γ and $F_1 = F_2 = F_1' = F_2' = 0$ on $[-\pi, \pi)\backslash\Gamma$. for $k \in \mathbb{Z}$, we have

$$\int_{\mathbb{T}} \left(F_i' \, \overline{H_0'}\Phi_2\right) e_{-2k} = 0 \tag{5.21}$$

and

$$\int_{\mathbb{T}} \left(F_i \, \overline{H_0''}\Phi_1\right) e_{-2k} = 0 \tag{5.22}$$

By Parseval-Plancherrel Theorem and using (5.15), we have

$$\begin{aligned} \langle f_i, T^k\phi_2\rangle_{L^2(\mathbb{R})} &= \left\langle \hat{f}_i, e_k\hat{\phi}_2\right\rangle_{L^2(\mathbb{R})} \\ &= \left\langle F_i\left(\tfrac{\cdot}{2}\right)\hat{\phi}_1\left(\tfrac{\cdot}{2}\right), H_0''\left(\tfrac{\cdot}{2}\right)\hat{\phi}_1\left(\tfrac{\cdot}{2}\right)e_k\left(\tfrac{\cdot}{2}\right)\right\rangle_{L^2(\mathbb{R})}, \end{aligned} \tag{5.23}$$

as H_0'' is 2π periodic.

L.H.S. of (5.22) is the R.H.S. of (5.23) and so, for $k \in \mathbb{Z}$

$$\langle f_i, T^k\phi_2\rangle_{L^2(\mathbb{R})} = 0, \quad i = 1, 2. \tag{5.24}$$

Similarly, for $k \in \mathbb{Z}$

$$\langle f_i', T^k\phi_1\rangle_{L^2(\mathbb{R})} = 0, \quad i = 1, 2. \tag{5.25}$$

Hence, we have

$$\begin{aligned}\left\langle f_i' \oplus f_i,\ T_C^k\phi_1 \oplus \phi_2\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} &= \left\langle f_i' \oplus f_i,\ z_1^k T^k\phi_1 \oplus z_2^k T^k\phi_2\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\ &= \left\langle f_i',\ z_1^k T^k\phi_1\right\rangle_{L^2(\mathbb{R})} + \left\langle f_i,\ z_2^k T^k\phi_2\right\rangle_{L^2(\mathbb{R})} \\ &= \overline{z_1^k}\left\langle f_i',\ T^k\phi_1\right\rangle_{L^2(\mathbb{R})} + \overline{z_2^k}\left\langle f_i,\ T^k\phi_2\right\rangle_{L^2(\mathbb{R})} \\ &= 0.\end{aligned}$$

Hence $f_i' \oplus f_i \in W_0, i = 1, 2$.

Now assume there is a function $\psi_1 \oplus \psi_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ for which $\left\{T_C^k\psi_1 \oplus \psi_2 :\ k \in \mathbb{Z}\right\}$ is a frame for the orthogonal complement W_0 of V_0 in V_1. Combining Proposition 4.2.9 with both this assumption and the hypothesis that $\left\{T_C^k\phi_1 \oplus \phi_2 :\ k \in \mathbb{Z}\right\}$ is a frame for V_0, we claim the existence of 1-periodic functions $H_1,\ H_1' \in L^2(\mathbb{T})$ and $D_i,\ D_i' \in L^2(\mathbb{T}),\ \ i = 1,\ 2$ for which

$$\hat{f}_i' = D_i\hat{\psi}_1 = D_iH_1'(\tfrac{\cdot}{2})\hat{\phi}_2(\tfrac{\cdot}{2})$$

and

$$\hat{f}_i = D_i'\hat{\psi}_2 = D_i'H_1''(\tfrac{\cdot}{2})\hat{\phi}_1(\tfrac{\cdot}{2})$$

where $F_i'(\frac{\cdot}{2}) = D_iH_1'(\frac{\cdot}{2})$ and $F_i(\frac{\cdot}{2}) = D_i'H_1''(\frac{\cdot}{2})$ by (5.15) and (5.16).

Proof of the claim: $f_i' \oplus f_i \in W_0$ implies there exists $D_i,\ D_i' \in L^2(\mathbb{T})$ such that $\hat{f}_i' = D_i\hat{\psi}_1$ and $\hat{f}_i = D_i'\hat{\psi}_2$.

Also,

$$\begin{aligned}\psi_1 \oplus \psi_2 \in W_0 &\Rightarrow U_C^{-1}(\psi_1 \oplus \psi_2) \in V_0 \\ &\Rightarrow U^{-1}\psi_2 \oplus U^{-1}\psi_1 \in V_0\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad (\widehat{U^{-1}\psi_2}) = G_1''\hat{\phi}_1 \text{ and } (\widehat{U^{-1}\psi_1}) = G_1'\hat{\phi}_2, \\
&\qquad\quad \text{for some } G_1',\ G_1'' \in L^2(\mathbb{T}), \text{ by Proposition 4.2.9} \\
&\Rightarrow \quad U\hat{\psi}_2 = G_1''\hat{\phi}_1 \text{ and } U\hat{\psi}_1 = G_1'\hat{\phi}_2 \\
&\Rightarrow \quad \hat{\psi}_2 = U^{-1}\left(G_1''\hat{\phi}_1\right) \text{ and } \hat{\psi}_1 = U^{-1}\left(G_1'\hat{\phi}_2\right) \\
&\Rightarrow \quad \hat{\psi}_2 = \frac{1}{\sqrt{2}}G_1''(\tfrac{\cdot}{2})\hat{\phi}_1(\tfrac{\cdot}{2}) \text{ and } \hat{\psi}_1 = \frac{1}{\sqrt{2}}G_1'(\tfrac{\cdot}{2})\hat{\phi}_2(\tfrac{\cdot}{2})
\end{aligned}$$

Hence the claim is proved.

If $\frac{\gamma}{2} \in \Gamma$, then

$$H_1'(\tfrac{\gamma}{2} + \pi) = H_1'(\tfrac{\gamma}{2}). \tag{5.26}$$

To prove this, note by the definition of Γ and F_1' that

$$F_1'(\tfrac{\gamma}{2} + \pi) = F_1'(\tfrac{\gamma}{2}) = 1.$$

$$D_1(\gamma + 2\pi)H_1'(\tfrac{\gamma}{2} + \pi) = F_1'(\tfrac{\gamma}{2} + \pi) = 1$$

and

$$D_1(\gamma)H_1'(\tfrac{\gamma}{2}) = F_1'(\tfrac{\gamma}{2}) = 1 \tag{5.27}$$

implies

$$D_1(\gamma)\left(H_1'(\tfrac{\gamma}{2} + \pi) - H_1'(\tfrac{\gamma}{2})\right) = 0,$$

since D_1 is 2π-periodic. Since by (5.27), $D_1(\gamma) \neq 0$, from the above we obtain (5.26). Similarly, if we replace F_1' by F_2', and if $\frac{\gamma}{2} \in \Gamma$, then two cases arise:

Case 1: $F_2'(\frac{\gamma}{2}) = 1, \qquad F_2'(\frac{\gamma}{2} + \pi) = -1:$

Then

$$D_2(\gamma+2\pi)H_1'(\tfrac{\gamma}{2}+\pi)=F_2'(\tfrac{\gamma}{2}+\pi)=-1$$

and

$$D_2(\gamma)H_1'(\tfrac{\gamma}{2})=F_2'(\tfrac{\gamma}{2})=1.$$

This implies that

$$D_2(\gamma)\left(H_1'(\tfrac{\gamma}{2}+\pi)+H_1'(\tfrac{\gamma}{2})\right)=0,$$

since D_2 is 2π-periodic. Hence

$$H_1'(\tfrac{\gamma}{2}+\pi)=-H_1'(\tfrac{\gamma}{2}).$$

Case 2: $F_2'(\frac{\gamma}{2})=-1, \qquad F_2'(\frac{\gamma}{2}+\pi)=1:$

Then

$$D_2(\gamma+2\pi)H_1'(\tfrac{\gamma}{2}+\pi)=F_2'(\tfrac{\gamma}{2}+\pi)=1$$

and

$$D_2(\gamma)H_1'(\tfrac{\gamma}{2})=F_2'(\tfrac{\gamma}{2})=-1.$$

This implies that

$$D_2(\gamma)\left(H_1'(\tfrac{\gamma}{2}+\pi)+H_1'(\tfrac{\gamma}{2})\right)=0,$$

since D_2 is $2\ \pi$-periodic. Hence

$$H_1'(\tfrac{\gamma}{2}+\pi)=-H_1'(\tfrac{\gamma}{2}).$$

i.e., for any $\frac{\gamma}{2}\in\Gamma$, we have

$$H_1'(\tfrac{\gamma}{2}+\pi)=-H_1'(\tfrac{\gamma}{2}) \tag{5.28}$$

Combining (5.26) and (5.28), we get

$$H_1'(\gamma) = 0 \qquad \forall\ \gamma \in \Gamma$$

and this contradicts (5.27). Hence the theorem. □

Example 5.2.2. Consider $\phi_1 \oplus \phi_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ defined by

$$\hat{\phi}_1 = \hat{\phi}_2 = \chi_{[-\pi,\ \pi)}.$$

Then $\Phi_1 = \Phi_2 = \chi_{[-\pi,\ \pi)}$ on $[-\pi,\ \pi)$. Hence, by Corollary 4.2.4, the system $\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\}$ is a tight frame with frame bound 2 for its closed span V_0 and

$$V_0 = \left\{ f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) : \operatorname{supp} \hat{f}_1 \subseteq [-\pi,\ \pi) \text{ and } \operatorname{supp} \hat{f}_2 \subseteq [-\pi,\ \pi) \right\}.$$

But the above $\phi_1 \oplus \phi_2$ cannot give an FMRA: If so, by Theorem 5.1.6, there exists H_0', $H_0'' \in L^2(\mathbb{T})$ such that (5.3) and (5.4) hold. For the given $\phi_1 \oplus \phi_2$, we have $H_0' = H_0'' = \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}$. But there is no 2-cycle $\{z_1,\ z_2\}$ and no sequence $c = (c_k)$ such that $\left(c_k z_1^k\right)\hat{} = \left(c_k z_2^k\right)\hat{} = \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}$.

ON THE STUDY OF FRAME MULTIRESOLUTION ANALYSIS IN THE SUPERSPACES $L^2(\mathbb{R})^n$

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Epilogue

We have discussed FMRA in $L^2(\mathbb{R})^n$, where n is a natural number. It was shown in [5] that for any n there is a superwavelet of length n, which assures the existence of frame wavelets in $L^2(\mathbb{R})^n$. In the Example 5.1.10, we have seen an FMRA in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. One can search for a result that ensures a frame wavelet $\psi_1 \oplus \cdots \oplus \psi_n$ in $L^2(\mathbb{R})^n$ corresponding to the FMRA scaling function $\phi_1 \oplus \cdots \oplus \phi_n$. Theorem 5.2.1 tells about a case where there is no frame wavelet in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Whether there is frame wavelet in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ in the case $|\Gamma| > 0$, where Γ as in Theorem 5.2.1, is to be studied. Examples of frame wavelets in the superspaces obtained through FMRA are to be constructed.

We note that the following result in [16] gives a sufficient condition for the existence of an MRA in $L^2(\mathbb{R})$: *Let* $\phi \in L^2(\mathbb{R})$, *and* $V_0 = \overline{\text{span}}\{\tau_k\phi : k \in \mathbb{Z}\}$ *and* $V_j = \{U^j f : f \in V_0\}$ *for each* $j \in \mathbb{Z}$. *Suppose* $V_0 \subseteq V_1$. *If* $|\hat{\phi}| > 0$ *almost everywhere on some neighbourhood of 0, then* $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$ *and* $\bigcap_j V_j = \{0\}$. A study for obtaining sufficient conditions on $\phi_1 \oplus \cdots \oplus \phi_n$ in $L^2(\mathbb{R})^n$ that gives an MRA in the superspaces can be carried out.

The following Proposition in [4] gives us conditions which ensures the existence of frame wavelet via FMRA: *Let* $\{V_j,\ \phi\}$ *be an FMRA of* $L^2(\mathbb{R})$, *and*

H_0 and H_1 are 2π periodic functions in $L^1(\mathbb{T})$ that relates with ϕ and ψ as below.

$$\hat{\phi}(2\cdot) = H_0\hat{\phi}$$

and

$$\hat{\psi}(2\cdot) = H_1\hat{\phi}$$

Assume $\psi \in W_0$, where W_0 is the orthogonal complement of V_0 in V_1, and $P(|\hat{\psi}|^2) \in L^\infty(\mathbb{T})$. If there are $G_0, G_1 \in L^\infty(\mathbb{T})$ such that

$$H_0G_0\Phi + H_1G_1\Phi = \Phi$$

and

$$\tau_\pi\left(H_0\Phi\right)G_0 + \tau_\pi\left(H_1\Phi\right)G_1 = 0$$

then $\{\tau_k\psi : \ k \in \mathbb{Z}\}$ is a frame for W_0. Analogous results can be investigated in the superspaces $L^2(\mathbb{R})^n$.

In 5.1.2, we have observed that the FMRA depend on the choice of the cycle C. One can look at the relation between these FMRAs.

Finally, whether superwavelets can be made through MRA in $L^2(\mathbb{R})^\infty$ can be studied. In this connection, we feel that the study of FMRA in $L^2(\mathbb{R})^\infty$ and frame wavelets through FMRAs will be interesting.

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ON THE STUDY OF FRAME MULTIRESOLUTION ANALYSIS IN THE SUPERSPACES $L^2(\mathbb{R})^n$

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