D 91956 **(Pages : 3)** **Name..................................**

Reg. No...............................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2015

(CCSS)

Mathematics

MAT 3E 02—OPERATIONS RESEARCH

Time : Three Hours **Maximum : 80 Marks**

Part A

Answer **all** *the questions.*
Each question carries 4 marks.

1. Prove that the linear function $f(x) = CX,\ X \in E_n$ is both convex and concave.

2. Define the following terminologies in LP problems.

    (i) Basic Solution.

    (ii) Feasible Solution.

3. Write the dual of the LP problem :

$$\begin{aligned}
&\text{Minimize } x_1 - 3x_2 - 2x_3 \\
&\text{Subject to } 2x_1 - 4x_2 \geq 12 \\
&\quad 3x_1 - x_2 + 2x_3 \leq 7 \\
&\quad -4x_1 + 3x_2 + 8x_3 = 10 \\
&\quad x_1 \geq 0, x_2 \geq 0. \\
&\quad x_3 \text{ unrestricted in sign.}
\end{aligned}$$

4. Prove that if the K^{th} constraint of the primal is an equality, then the dual variable y_k is unrestricted in sign.

5. How would you test for optimality of a given basic feasible solution of a transportation problem ?

6. What is degeneracy in transport ation problems ?

7. Explain whether an integer programming problem can be solved by rounding off the corresponding simplex solution.

8. Write the Kuhn-Tucker conditions for the problem

$$\begin{aligned}
&\text{Minimize } f = (x_1 - 2)^2 + x_2^2 \\
&\text{Subject to } x_1^2 + x_2 - 1 \leq 0 \\
&\quad x_1, x_2 \geq 0.
\end{aligned}$$

(8 × 4 = 32 marks)

Turn over

Part B

Answer **either** *A or B of each question.*
Each question carries 16 *marks.*

9. A (a) Let $f(x)$ be a convex differentiable function defined in a convex domain $K \le E_n$. Prove that $f(X_0), X_0 \in K$ is a global minimum iff $(X - X_0)^1 \nabla f(X_0) \ge 0$ for all X in K.

(b) In the problem :

$$\begin{aligned} \text{Maximize } & 20x_1 + x_2 + 10x_3 \\ \text{Subject to } & x_1 + 4x_2 - x_3 \le 20 \\ & x_1 + x_2 \le 10 \\ & 3x_1 + 5x_2 - 3x_3 \le 50 \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

In which direction is the solution space unbounded ? Without any computations what can be said about the optimal solution to the problem ?

B (a) Prove that a basic feasible solution of an LP problem is a vertex of the convex set of feasible solutions.

(b) Use the simplex method to verify that the following problem has no finite optimal solution.

$$\begin{aligned} \text{Maximize } & 2x_1 + x_2 \\ \text{Subject to } & x_1 - x_2 - x_3 \le 1 \\ & x_1 - 2x_2 + x_3 \le 2 \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

From the simplex table construct a feasible solution with value of the objective function greater than 2000.

0 A (a) Explain briefly the dual simplex method to solve an LP problem.

(b) Solve by dual simplex method :

$$\begin{aligned} \text{Minimize } & x_1 + 3x_2 + 2x_3 \\ \text{Subject to } & 4x_1 - 5x_2 + 7x_3 \le 8 \\ & 2x_1 - 4x_2 + 2x_3 \ge 2 \\ & x_1 - 3x_2 + 2x_3 \le 2 \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

B (a) Prove that a transportation problem has a triangular basis.

(b) Solve the following transportation problem for minimum cost with the cost coefficients, demands and supplies as given in the following table :

	D_1	D_2	D_3	
O_1	4	5	2	30
O_2	4	1	3	40
O_3	3	6	2	20
O_4	2	3	7	60
	40	50	60	

A (a) Decribe the cutting plane method to solve an integer linear programming problem.

(b) Solve using branch and bound method :

Maximize $3x_1 + 4x_2$

Subject to $2x_1 + 4x_2 \leq 13$

$-2x_1 + x_2 \leq 2$

$2x_1 + 2x_2 \geq 1$

$6x_1 - 4x_2 \leq 15$

$x_1, x_2 \geq 0$

x_1, x_2 integers.

B (a) Briefly describe the Wolfe's algorithm for solving a quadratic programming problem.

(b) Minimize $f = (x_1 + 1)(x_2 - 2)$ over the region $0 \leq x_1 \leq 2, 0 \leq x_2 \leq 1$ by writing the Kuhn-Tucker conditions and obtaining the saddle point.

(3 × 16 = 48 marks)

D 91955 (Pages : 2) Name..................................

Reg. No..............................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2015

(CCSS)

Mathematics

MAT 3E 01—ADVANCED TOPOLOGY

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

I. 1 State a condition under which a Hausdorff space will become a T_3 space. Prove this result.

2 Prove that any continuous real-valued function on a closed subset of a normal space can be extended continuously to the whole space.

3 Give an example of a space that is connected but not locally connected.

4 Define large box in topological spaces. Prove that intersection of a family of large boxes is a large box.

5 Prove that second countability is preserved under continuous open functions.

6 Prove that a subnet of a convergent net is convergent.

7 Show that in a Hausdorff space no filter can converge to more than one point in it.

8 Define nowhere dense subsets. Write an example of a nowhere dense subset of the set of real numbers with usual topology.

(8 × 4 = 32 marks)

Part B

Answer **either** *A* **or** *B of each question.*
Each question carries 12 *marks.*

II. (A) (i) Prove that every regular Lindelöff space is normal.

(ii) Prove that the unit circle in $\mathbb{R}^2$ is compact.

(B) (i) If a space X has the property that for any two mutually disjoint closed subsets A, B of it, there exists a continuous function $f : X \rightarrow [0.1]$ taking the value 0 at all points on A and value 1 at all points on B, then prove that X is normal.

(ii) Prove that T_4 spaces are Tychonoff spaces.

Turn over

III. (A) (i) Prove that every open subset of the real line in the usual topology can be expressed as the mutually disjoint union of open intervals.

(ii) Give an example of a Tychonoff space which is not normal.

(B) Prove that metrisability is a countably productive property.

IV. (A) (i) Prove that a topological space is Hausdorff if and only if limits of all nets in it are unique.

(ii) Prove that a topological space is compact if and only if every family of closed subsets of it, which has the finite intersection property has a non-empty intersection.

(B) (i) Prove that an open subspace of a locally connected space is locally connected.

(ii) Let S be a family of subsets of a set X. Then prove that there exists a filter on X having S as a subbase if and only if S has the finite intersection property.

V. (A) Prove that a topological space is metrically topologically complete if and only if it is an absolute G_δ.

(B) Prove that a metric space X is complete if and only if it contains a dense subset D such that every Cauchy sequence in D has a limit point in X.

($4 \times 12 = 48$ marks)

D 91954 (Pages : 3) Name.................................

Reg. No...............................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2015

(CCSS)

Mathematics

MAT 3C 12—PARTIAL DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Obtain the first order partial differential equation satisfied by the surfaces $F(u, v) = 0$, where $u = u\ (x, y, z(x, y))$ and $v = v\ (x, y, z(x, y))$ are known functions of x, y and z and F is an arbitrary function of u and v, having first order derivatives with respect to u and v.

2. Show that $(x-a)^2 + (y-b)^2 + z^2 = 1$ is a complete integral of $z^2\left(1 + p^2 + q^2\right) = 1$. Show further that $z = \pm 1$ are the singular integrals.

3. Find the complete integral of the equation $zpq - p - q = 0$.

4. Find the integral surface of the differential equation :

$$z\left(xz_x - yz_y\right) = y^2 - x^2$$

passing through $(2s, s, s)$.

5. Reduce the equation :

$$4u_{xx} - 4u_{xy} + 5\,u_{yy} = 0$$

into canonical form.

6. Find the characteristic strips of the equation $xp + yq - pq = 0$ where the initial data curve is $z = x/2,\ y = 0$.

7. Suppose that $u(x, y)$ is harmonic in a bounded domain D and is continuous on $\bar{D} = D \cup B$, where B in the boundary of D. Show that u attains its minimum on B.

Turn over

8. Solve the Dirichlet problem for the upper half plane.

9. Solve :

$$u_t = ku_{xx}, -\infty < x < \infty, t > 0$$

$$u(x, 0) = f(x), -\infty < x < \infty.$$

10. Transform the problem :

$$\frac{d^2y}{dx^2} + xy = 1, y(0) = y(1) = 0$$

to the integral equation

$$y(x) = \int_0^1 G(x, \xi)\, \xi y(\xi) \quad d\xi - \frac{1}{2} x(1-x),$$

where $G(x, \xi) = \begin{cases} x(1-\xi) & \text{when } x < \xi \\ \xi(1-x) & \text{when } x > \xi \end{cases}$.

11. Show that the Kernel $k(x, \xi) = 1 + 3x\xi$ has a double characteristic number associated with $(-1, 1)$, with two independent characteristic functions.

12. Determine the iterated kernel $k_2(x, \xi) = |x - \xi|$ in $(0, 1)$.

(12 × 4 = 48 marks)

Part B

Answer **either** *part (A)* **or** *part (B) of each question.*
Each question carries 8 *marks.*

13. (A) (i) Find the general integral of :

$$\left(x^2 + y^2\right) p + 2xyq = (x + y) z.$$

(ii) Show that the equations :

$xp - yq - x = 0$ and $x^2p + q - xz = 0$ are compatible on the domain D where D does not contain the points (x, y) such that $xy + 1 = 0$ and find the one parameter family of common solutions on D.

(B) (i) Show that the Pfaffian differential equation $yzdz + \left(x^2y - xz\right)dy + \left(x^2z - xy\right)dz = 0$ is integrable and find the corresponding integral.

(ii) Write the equation

$$zpq = p^2\left(p^2 + xq\right) + q^2\left(q^2 + yp\right)$$

into Clairant form and solve it.

14. (A) Find a solution of $\left(p^2 + q^2\right)x = pz$ passing through the parabola $x = 0, z^2 = 4y$.

(B) Describe the classification of second order semi-linear partial differential equation.

15. (A) (i) State the Neumann problem and obtain the necessary condition for the Neumann problem.

(ii) Solve the Newmann problem for a circle.

(B) Solve the Dirichlet interior problem for a circle and show that the solution is unique.

16. (A) (i) Explain the iterative method for solving integral equations of the second kind.

(ii) Solve by iterative method :

$$y(x) = \sin x + \lambda \int_0^{2\pi} \cos(x + \xi)\, y(\xi)\, d\xi.$$

(B) (i) Define separable kernel and give an example of a separable kernel.

(ii) Determine the characteristic values of λ and the corresponding characteristic functions of the equation :

$$y(x) = \lambda \int_0^{2\pi} \sin(x + \xi)\, y(\xi)\, d\xi.$$

(4 × 8 = 32 marks)

D 91953

(Pages : 3)

Name.......................................

Reg. No...................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2015

(CCSS)

Mathematics

MAT 3C 11—FUNCTIONAL ANALYSIS

Time : Three Hours

Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Let $a_j, b_j \varepsilon K, j = 1, 2, 3, \ldots\ldots, n$ and let $1 \leq p < \infty$. Show that :

$$\left(\sum_{j=1}^{n} |a_j + b_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |b_j|^p\right)^{1/p}.$$

2. Show that the metric space $L^{\infty}([a, b])$ is not separable.

3. Let $\| \ \|_1, \| \ \|_2, \ldots, \| \ \|_m$ be norms on a linear space X. Let $r_1, r_2, \ldots, r_m$ be positive real numbers. For $x \varepsilon X$, let $\| x \| = \max\{r_1 \| x \|_1, r_2 \| x \|_2, \ldots, r_m \| x \|_m\}$. Show that $\| \ \|$ is a norm on X, and $\| x_n - x \| \to 0$ iff $\| x_n - x \|_j \to 0$ for each $j = 1, 2, \ldots, m$.

4. State and prove Riesz Lemma.

5. Show that every linear map from a finite dimensional normed space to a normed space is continuous.

6. Show that the linear functional f on c defined by $f(x) = \lim_{j \to \infty} x(s)$, $x \varepsilon C$, is continuous and $\| f \| = 1$.

7. Let Y be a subspace of a normed space X and $a \varepsilon X$ but $a \notin \bar{Y}$. Show that there is some $f \varepsilon X'$ such that $f / Y = 0$, $f(a) = \text{dist}(a, \bar{Y})$ and $\| f \| = 1$.

8. Show that a normed space X is a Banach space iff every absolutely summable series of elements in X is summable in X.

9. Show that the linear space c_{00} cannot be a Banach space in any norm.

Turn over

10. Let $1 \leq p \leq \infty$ and $X = c_{00}$ with the norm $\| \ \|_p$. For $n = 1, 2, \ldots$, let $f_n(x) = nx(n)$, $x \in X$. Show that $f_n(x) \to 0$ for every $x \in X$, but $\| f_n \| \to \infty$.

11. Let X and Y be normed spaces. Show that if $F : X \to Y$ is continuous and $G : X \to Y$ is closed, then $F + G : X \to Y$ is closed.

12. Let Y be a finite dimensional subspace of a normed space X. Show that there is a continuous projection P defined on X such that R (P) = Y.

(12 × 4 = 48 marks)

Part B

Answer **either** *(A)* **or** *(B) of each question.*
Each question carries 8 *marks.*

13. (A) (a) Show that for $1 \leq p \leq \infty$, the metric space l^p is complete.

(b) Show that the property of completeness of a metric may not be shared by an equivalent metric.

(B) (a) Let X be a metric space. Show that if X is complete and totally bounded then X is compact.

(b) Let m be a measurable subset of R. Show that if $m(E) < \infty$ and $1 \leq p < \infty$, then the set of all bounded continuous functions on E is dense in $L^p(E)$.

14. (A) (a) Show that every finite dimensional subspace of a normed space X is closed in X.

(b) Give an infinite dimensional subspace of a normed space X which is not closed in X.

(B) (a) Show that a linear functional f on a normed space X is continuous iff Z (f) is closed in X.

(b) Show that a linear map on a linear space X may be continuous with respect to some norm on X, but discontinuous with respect to another norm on X.

15. (A) (a) State and prove Hahn Banach separation theorem.

(b) Let X be a normed space over K and let $0 \neq a \in X$. Show that there is some $f \in X'$ such that $f(a) = \| a \|$ and $\| f \| = 1$.

(B) (a) Let X and Y be normed space and $X \neq \{ 0 \}$. Show that BL (X, Y) is a Banach space in the operator norm iff Y is a Banach space.

(b) Show that for a normed space Y, BL (X, Y) = { 0 } iff Y = { 0 }.

16. (A) (a) State and prove uniform boundedness principle.

(b) Interpret uniform boundedness principle geometrically.

(B) (a) Show that if a linear map from a normed space X to a normed space Y is open, then it is surjective.

(b) Let X and Y be Banach spaces and $F : X \rightarrow Y$ be a linear map which is closed and surjective. Show that F is continuous and open.

($4 \times 8 = 32$ marks)

D 91952 (Pages : 3) Name..................................

Reg. No..............................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2015

(CCSS)

Mathematics

MAT 3C 10—COMPLEX ANALYSIS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Let S be the Riemann sphere. For the points i, $1 + i$, $2 + 3i$ in $\mathbb{C}$, give the corresponding points in S. Also obtain the subsets in S that correspond to the real and imaginary axes in $\mathbb{C}$.

2. Give the principal branch of $\sqrt{1-z}$.

3. Show that a Möbius transformation has ∞ as its only fixed point if and only if it is a translation.

4. Let $r(t) = 2e^{it}$, $0 \le t \le 2\pi$. Evaluate $\int_r \left(z^2 - 1\right)^{-1} dz$.

5. Let f and g be analytic functions on a region G such that $f(z)\,g(z) = 0$ for all z in G. Then prove that either $f \equiv 0$ or $g \equiv 0$.

6. Let f be an entire function. Suppose there exists constants M and R > 0 and an integer $n \ge 1$ such that $|f(z)| \le M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\le n$.

7. Let G be an open convex set in C. If r is any closed rectifiable curve in $\mathbb{C}$, prove that $r \sim 0$.

8. If f is analytic in a simply connected region G, prove that f has a primitive in G.

9. Let $f(z) = \dfrac{1}{z(z-1)(z-2)}$. Obtain the Laurent series expansion of f in each of the following regions :—

   (i) $\{z : 0 < |z| < 1\}$; (ii) $\{z : 1 < |z| < 2\}$; (iii) $\{z : |z| > 2\}$.

10. Suppose f has a pole of order m at $z = a$. If $g(z) = (z-a)^m f(z)$, prove that Res $(f ; a) = \dfrac{1}{(m-1)!}$ $g^{(m-1)}(a)$.

Turn over

11. Suppose f is analytic in the closed disk $\{z:|z|\leq 1\}$. If $|f(z)|<1$ for $|z|=1$, show that there exists a unique z with $|z|<$ such that $f(z)=z$.

12. Let f and g be analytic in the closed disk $\{z:|z|\leq R\}$ with $|f(z)|=|g(z)|$ on $|z|=R$. If f and g have no zeros in this disk, prove that $f=\lambda g$, for some constant λ with $|\lambda|=1$.

(12 × 4 = 48 marks)

Part B

Answer **either** *(A)* **or** *(B) of each question.*
Each question carries 8 *marks.*

13. (A) (a) Find the radius of convergence of the following series :—

(i) $\sum_{n=0}^{\infty} z^{n!}$.

(ii) $\sum_{n=0}^{\infty} a^n z^n, \begin{matrix} a\in\mathbb{C} \\ a\neq 0 \end{matrix}$.

(2 marks)

(b) Let $f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R>0$. Prove that f is analytic in $B(a\,;R)$.

(3 marks)

(c) Prove that a Möbius transformation takes circles into circles. (3 marks)

(B) (a) State and prove the Chain Rule for analytic functions. (3 marks)

(b) Prove that every harmonic function on $\mathbb{C}$ has a harmonic conjugate. (3 marks)

(c) Let Γ be a circle with centre a and radius R. If the points z and z^* are symmetric with respect to Γ, prove that $(z^*-a)(\bar{z}-\bar{a})=R^2$. (2 marks)

14. (A) (a) Prove that $\int_0^{2\pi} \frac{e^{is}}{e^{is}-z}\,ds=2\pi$, if $|z|<1$. (2 marks)

(b) Let f be analytic in $B(a\,;R)$. Prove that for all z in this disk, $f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n$, where $a_n=\frac{f^{(n)}(a)}{n!}$.

(3 marks)

(c) Let f be analytic in a region G. If there exists a in G such that $f^{(n)}(a)=0$ for all $n\geq 0$, prove that $f\equiv 0$ in G.

(3 marks)

(B) (a) Let f be analytic in the disk B $(a\,;\,R)$. If γ is a closed rectifiable curve in B $(a\,;\,R)$, prove that

$$\int_r f = 0.$$

(2 marks)

(b) State and prove Cauchy's Integral Formula (First version). (4 marks)

(c) Let r be a closed rectifiable curve in $\mathbb{C}$. If $a \notin \{r\}$, prove that $h(r, a)$ is an integer.

(2 marks)

(A) (a) State and prove the Open Mapping Theorem. (4 marks)

(b) State and prove Goursat's Theorem. (4 marks)

(B) (a) State and prove the theorem on Laurent Series Development in an annulus. (4 marks)

(b) Suppose f has an isolated singularity at $z = a$. Show that this is a removable singularity of f if and only f

$$\lim_{z \to a} (z-a) f(z) = 0.$$

(2 marks)

(c) If f is a meromorphic function on an open set G, show that the poles of f cannot have a limit point in G.

(2 marks)

(A) (a) Let $a > 1$. Prove that

$$\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}.$$

(4 marks)

(b) State and prove Rouche's Theorem. (4 marks)

(B) (a) State and prove the Argument Principle. (3 marks)

(b) State and prove the Residue Theorem. (3 marks)

(c) If f is a non-constant analytic function on a bounded open set G and continuous on $\bar{G}$, prove that f has a zero in G or $|f|$ assumes its minimum value on ∂G.

(2 marks)

[4 × 8 = 32 marks]