D 51891

(Pages : 3)

Name..................................

Reg. No..............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2009

Mathematics

Paper V—DISCRETE MATHEMATICS

(2003 Admissions)

Time : Three Hours | Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

1. (a) Prove that every distributive lattice is modular. Is the converse true ? Justify your answer.

   (b) Prove that, for any graph $G, K(G) \leq \lambda(G) \leq \delta(G)$. Draw a graph with $K = 1$, $\lambda = 2$ and $\delta = 3$.

   (c) Define : Eulerian graph, Hamiltonian graph. Give an example of a graph which is Eulerian but not Hamiltonian. Justify your answer.

   (d) If the language $L = \{a\ w\ a : w\ \varepsilon\ \{a, b\}^*\}$ regular. Justify your answer.

(4 × 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

UNIT I

2. Let $(L, \leq)$ be a lattice. Prove or disprove the following :—

   (a) For any $a, b, c\ \varepsilon\ L, b \leq C \Rightarrow a * b \leq a * c$.

   (b) Every element of a lattice has a unique complement.

   (c) Every lattice with 0 and 1 is complemented.

   (d) Is there a nondistributive lattice on S elements ? Justify your answer.

3. (a) Write the disjunctive normal form of $f(x, y, z) = (x + y')z' + yx'(y + x'z)$.

   (b) Is it true that every distributive lattice is a chain ? Justify your answer.

   (c) Draw the Hasse diagram of D (45) – the lattice of all the divisors of 45. Is it a chain ? Justify your answer.

   (d) Prove that in a distributive lattice, every element has a unique complement.

4. (a) Prove that every finite Boolean algebra is isomorphic to the power set Boolean algebra of the net of all its atoms.

   (b) Draw the Hasse diagram of the Boolean algebra of the power set of $X = \{a, b, c\}$.

Turn over

Unit II

5. (a) **State Havel-Hakimi Theorem. Is the sequence (6, 6, 6, 6, 4, 3, 3, 0) graphical ? Justify your answer.**

(2 + 6 = 8 marks)

(b) **Does there exist a self complementary graph of order 58 ? Justify your answer.** (4 marks)

(c) **Prove that if G is disconnected, then its complement is connected.** (4 marks)

6. (a) Derive the Euler's formula for a connected plane graph. (6 marks)

(b) Does there exist a polyhedral graph with exactly seven edges ? Justify your answer.

(4 marks)

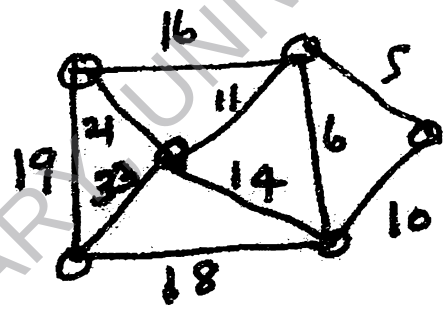
(c) Draw two non isomorphic graphs on 6 vertices, 9 edges and having the same degree sequence.

(6 marks)

7. (a) Show that a graph G is 2-chromatic if and only if G is bipartite. (4 marks)

(b) Prove that a nontrivial tree has at least two vertices of degree ore. (4 marks)

(c) Using Kruskal's algorithm find a MS T for G



(8 marks)

Unit III

8. (a) Explain with one example each the terms, Language, grammars and automata. (6 marks)

(b) Find the grammar that generates $L = \{a^n b^{n+1}, n \geq 0\}$. (4 marks)

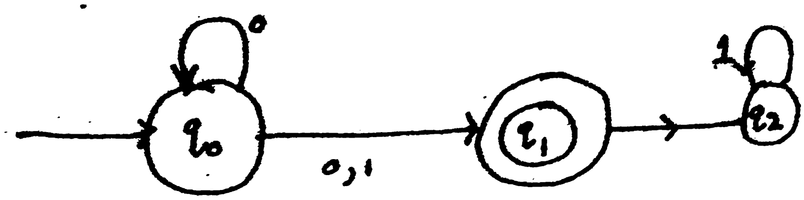
(c) Explain the terms "deterministic finite acceptor" and "regular language". (6 marks)

9. (a) What is "nondeterministic finite acceptor" ? Illustrate with an example. (6 marks)

(b) Explain the "language accepted by an nfa M'. Illustrate. (6 marks)

(c) Define "equivalence of two finite acceptors. (4 marks)

10. (a) Convert the *n.f a* in the figure below to an equivalent *d.f a*.



(8 marks)

(b) Find a regular expression for the language $L = W \in \{0, 1\}^*$: *w* has no pair of consecutive zeros.

(8 marks)

[4 × 16 = 64 marks]

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FIRST SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2009

Mathematics

Paper IV—ORDINARY DIFFERENTIAL EQUATION AND CALCULUS OF VARIATIONS

(2003 admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

I. (a) If $y(x)$ is a non-trivial solution of $\frac{d^2y}{dx^2} + q(x)\,y = 0$, show that $y(x)$ has an infinite number of positive zeros if $q(x) > \frac{k}{x^2}$ for some $k > \frac{1}{4}$ and only a finite number if $q(x) < \frac{1}{4x^2}$.

(b) Show that in Least Square's approximation the minimizing polynomial is the sum of terms of the Legendre series.

(c) Find the general solution of the system $\frac{dx}{dt} = x, \frac{dy}{dt} = y$. Show that any second order equation obtained from the system is not equivalent to this system in the sense that it has solutions that are not part of any solution of the system.

(d) In polar co-ordinates the length of a curve from a point P to a point Q in a plane is $\int_P^2 ds = \int_P^2 \sqrt{dr^2 + r^2\,d\theta^2}$. Find the polar equations of a straight line by minimizing this integral taking θ as the independent variable.

($4 \times 4 = 16$ marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

II. (a) Verify that the equation $y'' + y' - xy = 0$ has a three term recursion formula and obtain a series solution $y(x)$ satisfying $y(0) = 0,\ y'(0) = 1$.

(b) Define regular singular joint of an equation $y'' + P(x)y' + Q(x)\,y = 0$.

Assuming a power series solution obtain its indicial equation.

Turn over

III. (a) Transform the equation $\left(1-x^2\right) y'' - xy^1 + p^2 y = 0$ into a hypergeometric equation and find the general solution near $x = 1$.

(b) Show that $F^1(a, b, c, x) = \frac{ab}{c} F(a+1, b+1, c+1, x.)$

IV. (a) Obtain the Rodrigue's formula and use it to obtain the first three Legendre's polynomials.

(b) Discuss the Least squares approximation of a function.

UNIT II

V. (a) Show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

(b) State Bessel expansion theorem. Also find the Bessel series of the function $f(x) = 1$.

VI. (a) If two solutions of the linear system

$$\begin{cases} \frac{dx}{dt} = a_1(t)\, x + b_1(t)\, y \\ \frac{dy}{dt} = a_2(t)\, x + b_2(t)\, y \end{cases}$$

has a Wronskian that does not vanish at a point in an interval $[a, b]$, show that the general solution is given by a linear combination of them.

(b) Solve the systems $\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = y \end{cases}$

by differentiating and eliminating one of the variables.

VII. (a) Show that the critical point (0, 0) of a linear system with constant coefficients is state if and only if both roots of the auxiliary equation have non-positive real parts and asympathially stable if and only if negative real parts.

(b) Find a Liapunov function and analyse the stability of $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + k\,x = 0$.

UNIT III

VIII. (a) State and prove Sturm separation theorem.

(b) Show that the eigen functions of the Sturm-Liouville boundary value problem on an interval $[a, b]$ are orthogonal.

IX. (a) Let $f(x, y)$ and $\frac{dt}{dy}$ be continous functions of x and y on a closed rectangle R with sides parallel to the axes. If (x_0, y_0) is an interior point of R show that the sequence $\{y_n(x)\}$ defined by $y_o(x) = y_0, y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\,dt$ converges.

(b) Solve the initial value problem $\begin{cases} \frac{dy}{dx} = z \; y(0) = 1 \\ \frac{dz}{dx} = \in y \; z(0) = 0 \end{cases}$ by Picards' method and compare with the exact solution.

X. (a) Obtain Euler's differential equation for minimization of $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$.

(b) Using Lagrange multipliers find the point on the plane $ax + by + cz = d$ that is nearest to the origin.

(4 × 16 = 64 marks)

D 51889 (Pages : 2) Name..................................

Reg. No..............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2009

Mathematics

Paper III—REAL ANALYSIS—I

(2003 admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 marks.

I. (a) Give an example of an open cover of the segment (0, 1) which has no finite subcover. Justify your answer.

(b) Let f be a continuous real function on a metric space X. Let Z (f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that Z (f) is closed.

(c) If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x and show that $f^{(3)}(0)$ does not exist.

(d) Suppose $f \geq 0$, f is continuous on $[a, b]$ and $\int_a^b f(x)\, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

(4 × 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 marks.

Unit I

II. (a) Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = 1 \quad \text{if } p \neq q$$
$$= 0 \quad \text{if } p = q.$$

Prove that this is a metric. Which subsets of the resulting metric space are open ? Which are closed ? Which are compact ? Justify your answers.

(b) Show that a mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

III. (a) If a set E in R^k is such that every infinite subset of E has a limit point in E then show that E is closed and bounded.

(b) Define uniformly continuous mappings on a metric space. Let E be a bounded non-compact set in R. Show that there exists a continuous function on E which is not uniformly continuous.

IV. (a) Prove that in a metric space, any infinite subset of a compact set has a limit point in the compact set.

(b) Let f be a real uniformly continuous function on the bounded set E in R. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Unit II

V. (a) State and prove the chain rule for differentiation.

(b) If $f \in \mathbb{R}(\alpha)$ on $[a, b]$ then prove that if $f \in \mathbb{R}(\alpha)$ and $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| \, d\alpha$.

VI. (a) If f is continuous on $[a, b]$ then prove that $f \in \mathbb{R}(\alpha)$ on $[a, b]$.

(b) Suppose f is a bounded real function on $[a, b]$ and $f^2 \in \mathbb{R}$ on $[a, b]$, Does it follow that $f \in \mathbb{R}$? Does the answer change if we assume that $f^3 \in \mathbb{R}$? Justify your conclusions.

VII. (a) If $C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$, where $C_0, C_1, \ldots, C_n$ are real constants, prove that the equation $C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$ has atleast one real root between 0 and 1.

(b) Suppose $f \in \mathbb{R}(\alpha)$ on $[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a, b]$. Then prove that $h \in \mathbb{R}(\alpha)$ on $[a, b]$.

Unit III

VIII. (a) Define a curve r in R^k and its length $\wedge(r)$. When we say the curve is rectifiable ? If r' is continuous on $[a, b]$ then prove that r is rectifiable and $\wedge(r) = \int_a^b |r'(t)| \, dt$.

(b) Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

IX. (a) If $\{f_n\}$ is a sequence of continuous functions on E and if $\{f_n\}$ converges to f uniformly on E then prove that f is continuous on E. Is the converse true ? Justify.

(b) Suppose A is a self-adjoint algebra of complex continuous functions on a compact set K which separates points on K and vanishes at no point of K. Show that the uniform closure of A consists of all complex continuous functions on K.

X. (a) Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$. Is the converse true ? Justify.

(b) Let α be monotonically increasing on $[a, b]$ and $f_n \in \mathbb{R}(\alpha)$ on $[a, b]$ for $n = 1, 2, \ldots$. If $\{f_n\}$ converges uniformly to f on $[a, b]$ then prove that $f \in \mathbb{R}(\alpha)$ on $[a, b]$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$.

(4 × 16 = 64 marks)

D 51888 (Pages : 2) Name..

Reg. No......................................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2009

Mathematics

Paper II—LINEAR ALGEBRA

(2003 admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 16 *marks.*

I. Answer *all* questions in this part.

(a) Let $W = \{(x_1, x_2, x_3) \in R^3 \mid x_1 + x_2 + x_3 = 0\}$. Show that W is a subspace of R^3 and exhibit a basis for W.

(b) Describe the linear operator $T : R^2 \to R^2$ for which T(1,0) = (1,1), T(0,1) = (1,1). Obtain the range and null space of T.

(c) Let T be the linear operator on R^2 defined by $T(x, y) = (-y, x)$. Show that the only subspaces of R^2 invariant under T are R^2 and the zero space.

(d) Let V be an n-dimensioned vector space over a field F. Find the characteristic polynomials of (i) the identity operator on V and (ii) the zero-operator on V.

($4 \times 4 = 16$ marks)

Part B

Answer **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

II. (a) Show that any *two* bases of a finite dimensional vector space contain the same number of elements.

(b) Show that the vectors :

$\alpha_1 = (1,1,0), \alpha_2 = (2,1,1), \alpha_3 = (1,3,2)$ form a basis of $\mathbb{R}^3$. Express each of the standard basis vectors as a combination of $\alpha_1, \alpha_2, \alpha_3$.

III. (a) Let A be on $n \times n$ matrix over a field F. Suppose that the row vectors of A form a linearly independent set in F^n. Prove that A is invertible.

(b) Let V be the vector space of all 2×2 matrices over a field F. Prove that V has dimension 4 by exhibiting a basis which has four elements.

IV. (a) Let V and let W be vector spaces over a field F and let T be a linear transformation from V into W. If V is finite dimensional, prove that

rank (T) + nullity (T) = dim V.

(b) Let $T : V \to W$ be a linear transformation. Show that T is non-singular if and only if T carries each linearly independent subset of V into a linearly independent subset of W.

UNIT II

V. (a) Let V be a finite dimensional vector space over a field F and letter $\mathcal{B} = \{\beta_1, \ldots\ldots, \beta_k\}$ be a basis of V. Describe the dual basis V* of $\mathcal{B}^*$.

(b) Let $\mathcal{B} = \{\beta_1, \ldots\ldots, \beta_k\}$ be a basis of $\mathcal{B}$ and let $\mathcal{B}^* = \{f_1, \ldots f_k\}$ be the dual basis. Show that each α in V, $\sum_{i=1}^{k} f_i(\alpha)\, \beta_i = \alpha$.

VI. (a) Let V be a finite dimensional vector space over a field F, show that there exists a natural isomorphism of V onto its second dual V**.

(b) Let V and W be vector spaces over a field F and let $T : V \to W$ be a linear transformation. Define T^t, the transpose of T. Show that the null space of T^t is the annihilator of the range of T.

VII. (a) Let A be the 3×3 matrix given by $A = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Find the characteristic minimal polynomials for A.

(b) Let T be a linear operator on a finite dimensional vector space V. Prove that the characteristic and normal polynomials for T have the same roots except for multiplicities.

UNIT III

VIII. (a) Definer the projection operator. If E is a projection of a vector space V, prove that $V = R \oplus N$, where R is the range of E and N is the null space of E.

(b) Let $E_i, \ldots . E_k$ be projections of a vector space V such that $E_iE_j = 0$ for $i \neq j$ and $T = E_i + \ldots + E_k$. If W_i is the range of E_i for $i = 1, \ldots, k$, prove that $V = W_1 \oplus \ldots \oplus W_k$.

IX. (a) Let V be a finite dimensional vector space and α be any vector in V. Define :

(i) the T-cyclic subgroup generated by α.

(ii) the T-annihilation of α.

(b) Let α be a non-zero vector in a finite dimensional space V and let p_α be the T-annihilator of α. Prove that the degree of p_α is equal to the dimension of the cyclic subspace Z(α, T).

X. (a) Show that every finite dimensioned inner product space has an orthonormal basis.

(b) If S is any subset of an inner product space V, prove that $S^\perp$, the orthogonal complement of S, is a subspace of V.

($4 \times 16 = 64$ marks)

D 51887 (Pages : 2) Name..............................

Reg. No..............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2009

Mathematics

Paper I—ALGEBRA—I

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A (Compulsory)

Answer **all** *questions.*

I. (a) Describe all symmetries of a line segment in R^2.

(b) Show that $Z_2 \times Z_3$ is a cyclic group.

(c) Show that B^n with operations of word addition is a group.

(d) Classify the group $Z_4 \times Z_2 / \{(a) \times Z_2\}$ according to the fundamental theorem of finitely generated abelian groups.

(4 × 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*

Unit I

II. (a) Let H be a sub-group of a group G. Then show that the left coset multiplication is well defined by the equation

$(aH) \cdot (bH) = (ab) H$

if and only if left cosets and right cosets coincide so that $aH = Ha$ for all $a \in G$.

(b) Find the order of the factor group $Z_n \times Z_2 / (\langle 2 \rangle \times \langle 2 \rangle)$.

III. (a) Compute the factor group of $Z_4 \times Z_6 / \langle (2, 3) \rangle$.

(b) Show that M is a maximal normal subgroup of G if and only if G/M is simple.

(c) Show that the commutator subgroup C of S_3 contains A_3.

IV. (a) Show that any two compositions (principal) series of a group G are isomorphic.

(b) Find the isomorphic refinement of the two series :

$\{0\} < \langle 18 \rangle < \langle 3 \rangle < Z_{12}$ and

$\{0\} < \langle 24 \rangle < \langle 12 \rangle < Z_{12}$.

(c) Find all composition series of $Z_5 \times Z_5$.

Turn over

UNIT II

V. (a) If G has a composition (principal) series, and if N is a proper normal subgroup of G then show that there exists a composition (principal) series containing N.

(b) Find all composition series of

$Z_5 \times Z_5$.

VI. (a) Let G be a finite group and r be a finite G_s set. If r is the member of orbits in X under G, then show that

$$r|G| = \sum_{g \in G} |X_g|, \text{ where}$$

$$X_g = \{x \in X \,/\, gx = x\}$$

(b) Find the number of orbits in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ under the cyclic subgroup $\langle(4, 3, 5, 6)\rangle$ of S_8.

VII. (a) Let H and K be normal subgroups of a group G with $K \leq H$. Then show that

$$G/H \simeq (G)/K \left(H/K\right).$$

(b) Let $G = Z_{24}$, $H = \langle 4 \rangle$ and $K = \langle 8 \rangle$. Compute the isomorphism described in the proof of (a).

UNIT III

VIII. (a) Give a presentation of S_3 involving three generators.

(b) Give a presentation of Z_4 involving one generator, involving two generators, involving three generators.

IX. (a) Show that the set $R[x]$ of all polynomials in an indeterminate x with coefficients in a ring R is a ring under polynomial addition and multiplication. Show that if R is a commutative ring so is $R[x]$.

(b) Find the sum and product of the polynomials

$$f(x) = 2x^2 + 3x + 4 \text{ and}$$

$$g(x) = 3x^2 + 2x + 3 \text{ in } Z_6[x].$$

X. (a) Let H be a subring of a ring R. Show that multiplication of additive cosets of H is well defined by the equation

$$(a + H)(b + H) = a, b + H \text{ if and only if a } b \in H \text{ and } hb \in H \text{ for all } a, b \in R \text{ and } h \in H.$$

(b) Describe all ring homomorphisms of $Z \times Z$ into Z.

(4 × 16 = 64 marks)

D 1391 (Pages : 3)

Name..............................

Reg. No..............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2009

Mathematics

MAT 1 C 05—NUMBER THEORY

(CCSS)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

I. 1 Find all integers n such that $\varphi(n) = 6$.

2 Prove that $\prod_{t/n} t = n^{d(n)/2}$ $(n > 1)$.

3 Prove that $[x] + [x + ½] = [2x]$.

4 Define Chebyshev's functions $\psi(x)$ and $\theta(x)$.

5 Assuming the prime number theorem prove :

If $0 < a < b$, there exist an x_0 such that for $x \geq x_0$. there is at least one prime between ax and bx.

6 For a positive integer n, prove that :

$$2^n \leq \binom{2n}{n} < 4^n.$$

7 For an odd prim p, prove that :

$$(n/p) \equiv n^{(p-1)/2} \pmod{p}.$$

8 If P and Q are odd positive integers, prove that $(n / P)(n / Q) = (n / PQ)$.

9 Compute the Legendre symbols (13 / 59).

10 In the 27-letter alphabet (with blank = 26), use the affine encephering transformation with key $a = 13$, $b = 9$ to encephar the message "HELP ME".

Turn over

11 Find the inverse of the matrix $\begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix}$ mod 29.

12 Explain how to send a signature using RSA Crypto system.

(12 × 4 = 48 marks)

Part B

Answer A or B *of each question.*
Each question carries 8 *marks.*

II. A (a) Let g be a multiplecative function. Prove that f is multiplicative if and only if $f * g$ is multiplicative.

(b) Is the following statement true or false ? If m/n, then $\varphi(m)/\varphi(n)$. Justify your answer.

B (a) State and prove Euler's summation formula.

(b) Prove that for $x \geq 2$, $\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x)$ p prime.

III. A (a) State and prove Abel's identity.

(b) Prove that the n^{th} prime p_n statistic the inequality $P_n < 12\left(n \log n + n \log \frac{12}{e}\right)$.

B (a) Prove that the following relations are logically equivalent :

(i) $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1$

(ii) $\lim_{x \to \infty} \frac{\theta(\kappa)}{x} = 1$

(b) Prove that $\lim_{x \to \infty} \left(\frac{M(x)}{x} - \frac{H(x)}{x \log x}\right) = 0.$

IV. A (a) Let p be an odd prime. Prove that every reduced residue system mod p contains exactly $(p-1)/2$ quadratic residues.

(b) State and prove Gauss' lemma used in the proof of the Quadratic reciprocity law.

B (a) State and prove the reciprocity law for Jacobi symbols.

(b) Let p be a prime ≡ 3 (mod 4). Prove that

$$\sum_{r=1}^{p+} r^2 (r \mid p) = p \sum_{r=1}^{p+} r (r \mid p).$$

IV. A (a) Write a brief note on digraph transformations.

(b) Suppose it is known that the adversary is using an encepharing matrix A in the 26-letter, alphabet. The message "WKNCCHSSJH" is intercepted and it is known that the first word is "GIVE". Find the decepharing matrix A^{-1} and read the message.

B (a) Explain the RSA cryptosystem, illustrating with an example.

(b) Explain the advantages and disadvantages of public key Crypto system as compared to classical Crypto systems.

($4 \times 8 = 32$ marks)

D 1389 (Pages : 3) Name..................................

Reg. No..................................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2009

Mathamatics

MAT 1 C. 03—LINEAR ALGEBRA

(CCSS Scheme)

Time : Three Hours Maximum : 80 Marks

Answer **all** *questions under Part A. This part contains 12 questions of 4 marks each. In Part B each question has two parts A and B. Answer either A or B of each question. Each question carries* 8 *marks.*

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

I. 1 Let V be a vector space. Show that the intersection of any collection of subspaces of V is a subspace of V.

2 Let F be the field of reals. Verify whether the set $\{w_1, w_2, w_3\}$ is linearly dependent in F^3 where $w_1 = (3, 1, 0)$, $w_2 = (-3, 0, 1)$, $w_3 = (0, 2, 2)$.

3 Let $B = \{(1,1,0), (1,0,1), (0,1,1)\}$ and $B' = \{(1,1,1), (1,2,1), (1,1,0)\}$ be ordered bases of $\mathbb{R}^3$. Find the co-ordinates of the point $(1, 2, 3) \in \mathbb{R}^3$ w.r.t. B and w.r.t. B'.

4 Let $T : V \to W$ be a linear transformation between vector spaces V and W such that T is an invertible map. Show that T^{-1} is linear.

5 Let $T : V \to W$ be a linear transformation such that $T(\alpha) = 0$ implies $\alpha = 0$. Show that T is one - to - one.

6 Let V be the space of all $n \otimes n$ matrices over a field F. Let $f : V \to F$ be defined by $A = (A_{ij}) \mapsto A_{11} + A_{22} + \dots + A_{nn}$. Show that f is a linear functional.

7 Define characteristic value of a linear operator T. Find one characteristic value of the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ define by $T(x_1, x_2) = (x_1, x_2)$.

8 Verify whether the matrix

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

is diagonalizable.

Turn over

9 Find the minimal polynomial of the matrix :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

10 Let $V = \mathbb{R}^3$, $W_1 = \text{span}\{(1,0,1),(2,0,2)\}$ and $W_2 = \text{span}\ \{(1,2,3),(1,2,0)\}$. Verify whether W_1 and W_2 are independent subspaces.

11 Let $V = W_1 \oplus W_2$. Describe a projection E_1 with range W_1 and a projection E_2 with range W_2 such that $E_1 E_2 = 0$.

12 State generalized Cayley-Hamilton theorem.

Part B

Answer Part A or Part B of each question.
Each question carries 8 marks.

II. A (a) Let V be a vector space spanned by a finite set of m vectors. Show that the number of elements in any linearly independent subset of V is less than or equal to m.

(b) Show that in a finite dimensional vector space, any two bases have the same number of elements.

B (a) Let P be an invertible $n \times n$ matrix over a field F. Let V be an n-dimensional vector space over F and B be an ordered basis of V. Show that there exists a unique ordered basis B' of V such that :

$$[\alpha]_B = P[\alpha]_{B'} \text{ for every } \alpha \in V.$$

(b) Let $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and B be the standard ordered basis of F^2 where F is a field. Find the basis B' of F^2 such that $[\alpha]_B = P[\alpha]_{B'}$ for every $\alpha \in F^2$.

III. A (a) Let V be an n-dimensional vector space over a field F and W be an m-dimensional space over F. Let $L(V, W)$ be the space of all linear transformations from V to W.

1 Describe a basis for $L(V, W)$.

2 Prove that $\dim L(V, W) = mn$.

B (a) Let V be a vector space of dimension n over a field F and $\{v_1, v_2, ..., v_n\}$ be a basis of V.

1 Describe the dual basis $\{v_1*, v_2*, ..., v_n*\}$.

2 Show that $\{v_1*, v_2*, ..., v_n*\}$ is a basis of V*.

IV. A (a) Define characteristic polynomial of a matrix. Show that similar matrices have same characteristic polynomial.

(b) Show that if A is diagonalizable then the characteristic polyomial in A is the form $(x - c_1)^{r_1} (x - c_2)^{r_2} ... (x - c_k)^{r_k}$ where $c_1, c_2, ..., c_k$ are in the field F where A is a matrix over F.

B. Define minimal polynomial of an $n \times n$ matrix ; A. Show that the characteristic polynomial of A and the minimal polynomial of A have the same roots.

V. A Define independence of subspaces of a vector space.

(b) Let $W_1, W_2, ..., W_k$ be subspaces of a vector space V. Let $W = W_1 + W_2 + ... + W_k$. Prove that $W_1, W_2, ..., W_k$ are independent if and only if for each j with $2 \leq j \leq \kappa$,

$$W_j \; n = (W_1 + ... + W_{j-1}) = \{0\}.$$

B (a) Define T-annihilator of a vector α, where T is a linear operator.

Let α be a non-zero vector and p_α be the T-annihilator of α. Then prove that

(i) Degree $p_\alpha = \dim 6\, Z(\alpha, T)$ where Z (α, T) is the T-cyclic subspace generated by α.

(ii) If degree $p_\alpha = k$ then $\{\alpha, T\alpha, ..., T_\alpha^{k-1}\}$ is a basis for Z (α, T).

D 1387 **(Pages : 3)** Name..............................

Reg. No..............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2009

Mathematics

MAT 1C. 01—ALGEBRA—I

(CCSS)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

I. 1 Find the order of the element (3, 6, 12, 16) in $Z_4 \times Z_{12} \times Z_{20} \times Z_{24}$.

2 Let H and K be subgroups of a group G. Give an example showing that we may have $H \simeq K$ while G/H is not isomorphic to G/K.

3 Prove or disprove : Every infinite abelian group has composition series.

4 Find the number of orbits in {1, 2, 3, 4, 5, 6, 7, 8} under the cyclic subgroup $< (1, 3, 5, 6) >$ of s_8.

5 Illustrate the first Isomorphism theorem using a nontrivial homomorphism from Z_{12} to Z_3.

6 Let K and L be normal subgroups of a group G with $K \vee L = G$ and $K \cap L = \{e\}$. Show that $G/K \simeq L$ and $G/L \simeq K$.

7 Show that no group of order 255 is simple.

8 Write the class equation for D_4 ; the group of symmetries of a square.

9 Describe the field of quotients of the integral subdomain $\{m + ni : m, n \in z\}$ of *c*.

10 Find all zeros of $x^5 + 3x^3 + x^2 + 2x \in Z_5[x]$ in Z_5.

11 Demonstrate that $x^3 + 3x^2 - 8$ is irreducible over Q.

12 Prove that if F is a field, every proper non-trivial prime ideal of F [*x*] is maximal.

(12 × 4 = 48 marks)

Part B

Answer A *or* B *of each question.*
Each question carries 8 *marks.*

II. A (a) Show that the group $Z_m \times Z_n$ is isomorphic to Z_{mn} iff m and n are relatively prime.

(5 marks)

(b) Show that the finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.

(3 marks)

B (a) Give isomorphic refinements of two series.

$$\{0\} < 60\,Z < 20\,Z < Z$$
and $\{0\} < 245\,Z < 49\,Z < Z.$

(3 marks)

(b) Show that if G has a composition series and N is a proper normal subgroup of G, then there exists a compoition series containing N.

(5 marks)

III. A Let X be a G-set. Show that $G_x = \{g \in G \mid gx = x\}$ is a subgroup of G for each $x \in X$ and that $|Gx| = (G : G_x)$.

(3 + 5 = 8 marks)

B (a) Let H and N be subgroups of a group G. Show that if N is normal in G, then $H \cap N$ is normal in H.

(2 marks)

(b) Let H be a subgroup of a group G and let N be a normal subgroup of G. Show that $HN/N \simeq H/H \cap N$.

(6 marks)

IV. A (a) Let G be a finite group and let a prime p divide $|G|$. Show that G has a subgroup of order p.

(5 marks)

(b) Show that D_4 is solvable. (3 marks)

B (a) Show that for a prime number p, every group of order p^2 is abelian. (5 marks)

(b) Show that any two fields of quotient of an integral domain D are isomorphic.

(3 marks)

A (a) State and prove Division Algorithm for F $[x]$, where F is a field. (5 marks)

(b) Factorize $x^4 + 4$ into linear factors in $Z_5[x]$. (3 marks)

B (a) Show that if F is a field, then every ideal in F $[x]$ is principal. (5 marks)

(b) Is $Q[x]/\langle x^2 - 6x + 6 \rangle$ a field ? Justify your answer. (3 marks)