A STUDY ON MAGIC LABELING OF GRAPHS USING NON-ABELIAN GROUPS

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CERTIFICATE

I hereby certify that the thesis entitled " A STUDY ON MAGIC LABELING OF GRAPHS USING NON-ABELIAN GROUPS " is a bonafide record of the original research work carried out by **Smt. Anusha C.**, under my guidance for the award of Degree of Ph.D. in Mathematics from the University of Calicut and that this work or any part of it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

I hereby declare that the work presented in the thesis entitled **"A Study on Magic Labeling of Graphs Using Non-abelian Groups"** is based on the original work done by me under the guidance of **Dr. Anil Kumar V., Retd. Senior Professor**, Department of Mathematics, University of Calicut and has not been included in any other thesis submitted previously for the award of any degree. The contents of the thesis are undergone plagiarism check using **iThenticate** software at C.H.M.K. Library, University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.

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ABSTRACT

Graph labeling is an emerging area in graph theory research. Graph labeling is a mathematical technique that assigns labels to vertices, edges, or both, subject to specific conditions. One of the most interesting areas of graph labeling is the study of magic labeling. A finite connected simple graph G is said to be a magic graph if there exist real numbers, the edge labels of G with the following properties.

(1) Different edges have different labels

(2) The sum of the label values assigned to all edges, which are in incidence to the certain vertex, is the same for all vertices of graph G.

Motivated by this definition, the area of group magic labeling of graphs has been developed.

The magic labeling of graphs using abelian groups is a well-established research area in graph theory. This involves labeling the edges (or vertices) of a connected simple graph G with the non-identity elements of a finite abelian group A in such a way that the sum of the labels incident to each vertex (or edge) is constant for all vertices (or edges) of G.

This thesis attempts to generalize the magic labeling of graphs using finite abelian groups to any finite group (abelian or non-abelian) A. Since the group operation of a non-abelian group is not commutative, investigating group magic labeling of graphs using a finite non-abelian group is an interesting research topic. When defining group magic labeling for graphs with finite groups, it is crucial to ensure consistency with the existing definition of A-magic labeling, where A is a finite abelian group, as established by S. M. Lee, Doob, and others. If the group A is non-abelian, then from the definition of A-magic labeling, it follows that the sum of the labels of edges or the sum of the labels of adjacent vertices incident to a particular vertex or edge may change according to the order in which we sum. To address this, we impose an additional ordering in the magic labeling.

This thesis introduces the concept of A-magic labeling of graphs with a finite non-abelian group A. Specifically, the non-abelian groups S_3, D_4, and Q_8 are considered, and necessary and sufficient conditions are determined for several well-known graphs to be S_3-magic, D_4-magic, and Q_8-magic. This thesis also discusses the idea of induced S_3-magic labeling of graphs and extends the concept of A-barycentric magic labeling of graphs with abelian groups to any finite group by defining the same for non-abelian groups. Furthermore, this study introduces a new magic labeling concept called conjugate A-magic labeling of graphs and investigates the conjugate S_3-magic labeling of some well-known graphs. Additionally, a study of neighborhood magic labeling of graphs using the finite non-abelian group A is also included in this thesis.

Keywords: Non-abelian group, A-magic labeling, S_3-magic graphs, D_4-magic graphs, Q_8-magic graphs, induced S_3-magic labeling, conjugate S_3-magic, S_3-barycentric magic labeling,neighborhood S_3-magic labeling.

Contents

List of Symbols

G	Simple, connected and undirected graph
$E(G)$	Edge set of G
$V(G)$	Vertex set of G
S_3	Permutation group on 3 symbols
D_4	Dihedral group
Q_8	Quaternion group
W_n	Wheel graph
H_n	Helm graph
F_n	Fan graph
$B(n,k)$	n-gon book of k pages
SF_n	Sunflower graph
$W(2,n)$	Web graph
CB_n	Comb graph
$O(L_n)$	Open ladder graph
Sun_n	Sun graph
$CBSun_{p,q}$	Consecutive broken sun graph
$S_{n,n-3}$	Shell graph of width n
B_n	Bistar graph $B_{n,n}$
P_n	Path on n vertices
C_n	Cycle on n vertices
K_n	Complete graph
$K_{1,n}$	Star graph
$K_{m,n}$	Complete bipartite graph
$\mathbb{S}(G)$	Subdivision graph of G
L_n	Ladder graph

F_m	Friendship graph
$M(G)$	Middle graph of G
G_n	Gear graph
T_n	Triangular snake graph
$A(T_n)$	Alternate triangular snake graph
$D(T_n)$	Double triangular snake graph
$G_1 \circ G_2$	Lexicographic product of G_1 and G_2
$G_1 \times G_2$	Cartesian product of G_1 and G_2
$\mathcal{I}^{\mathrm{m}}\ (S_3)$	The set of all induced S_3-magic graphs
$K_m^{(n)}$	Windmill graph
Fl^n	Flower graph
Fl_n	Flag graph
ω_a	The class of all Q_8-magic graphs with magic constant a, where $a \in \mathbb{B} = \{\pm i, \pm j, \pm k\}$
ω_1	The class of all Q_8 -magic graphs with magic constant 1
ω_{-1}	the class of all Q_8 -magic graphs with magic constant -1
$\omega_{1,-1}$	$\omega_1 \cap \omega_{-1}$
$\omega_{a,1}$	$\omega_a \cap \omega_1, a \in \mathbb{B}$
$\omega_{a,-1}$	$\omega_a \cap \omega_{-1}, a \in \mathbb{B}$
Ω	$\omega_a \cap \omega_1 \cap \omega_{-1}$, for all $a \in \mathbb{B}$
$S(G)$	Splitting graph of the graph G
$\mathcal{BS}_{\rho_0}$	the class of graphs that are S_3-barycentric magic with constant ρ_0
$\mathcal{BS}_{\rho}$	the class of graphs that are S_3-barycentric magic with constant belongs to the set $\{\rho_1, \rho_2\}$
$\mathcal{BS}_{\mu}$	the class of graphs that are S_3-barycentric magic with constant belongs to the set $\{\mu_1, \mu_2, \mu_3\}$
$B_t(n,k)$	(n,k)-banana tree

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Introduction

Graph theory is a dynamic and thriving field of study, boasting a rich collection of elegant and powerful theorems with diverse and broad applications. Graphs are a versatile tool for solving problems in various fields, such as computer science, biology, engineering, genetics, social science, network analysis, and many others.

Background and Motivation for study

Graph labeling is a mathematical technique that assigns labels to vertices, edges, or both, subject to specific conditions. Various restrictions are imposed on the labeling to satisfy particular conditions, often driven by practical applications and inherent mathematical interest. The concept of labeling was first introduced by Sedláček, then formally developed by Kotzig and Rosa [1]. Graph labeling encompasses several types, including:

- Graceful labeling
- Radio labeling
- Magic labeling
- Harmonic labeling
- Cordial labeling
- Group distance magic labeling
- Antimagic labeling, etc

This field has grown into a diverse and extensive area of graph theory research, with numerous applications in:

- Coding theory
- X-ray technology
- Radar systems
- Communication networks

- Radio astronomy

For a comprehensive overview, refer to Gallian's dynamic survey, "A Dynamic Survey of Graph Labeling" [2], covering approximately 200 graph labeling techniques from over 2000 papers since the 1960s.

One of the most fascinating areas of graph labeling study is magic labeling, drawing inspiration from the concept of Magic Squares. For generations, magic squares have captivated both mathematicians and laypeople. A magic square is a square grid filled with distinct integers, arranged such that the sum of numbers in every row, column, and major diagonal (and sometimes other diagonals) is identical. The oldest recorded magic square is the Lo Shu Square, originating in ancient China approximately $4,000$ years ago. Legend has it that during a devastating flood, the people sought the river gods' intervention. In response, a mystical turtle emerged, bearing a pattern on its back. This pattern consisted of a 3×3 grid with numbers $1 - 9$, where the sum of each row, column, and diagonal was consistently 15. This arrangement is now recognized as the 3×3 magic square. Here is the renowned Lo Shu Square with a constant sum of 15:

4	9	2
3	5	7
8	5	6

Magic squares continued to appear in texts and artworks from numerous cultures, including the Mayans, the Hausa people of Africa, and Renaissance Europe. Due to the historical interest in magic squares, in the early 1960s, Sedláček explored applying 'magic' concepts to mathematics. Sedláček [3] first introduced the concept of a magic graph in his 1963 report at the Smolenice symposium. According to Sedláček, a finite, connected, simple graph G is considered a magic graph if:

1. Different edges have distinct labels (real numbers).

2. The sum of label values assigned to edges incident to any vertex is constant across all vertices of G [3].

Group magic labeling of graphs is a fascinating topic in magic labeling of graphs. For any abelian group A with identity element 0, written additively, any mapping $l : E(G) \to A \setminus \{0\}$ is called a labeling. A graph G is said to be A-magic [4] if there exists a labeling $l : V(G) \to A \setminus \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \to A$ defined by $l^+(v) = \Sigma\{l(uv)|uv \in E(G)\}$

is a constant map. In general, a graph G may admit more than one labeling to become an A-magic graph. Several authors studied about A-magic labeling of graphs including V_4-magic graphs [4–8], $\mathbb{Z}$-magic graphs [9–11], $\mathbb{Z}_k$-magic graphs [12] etc.

Our research aims to address the question of whether graphs can be magically labeled using non-abelian groups. This thesis explores the existence of graphs admitting A-magic labeling for non-abelian groups A. We investigate the potential of various magic labeling schemes using finite non-abelian groups, focusing on three specific cases:

1. The symmetric group S_3
2. The dihedral group D_4
3. The quaternion group Q_8

Organization of the thesis

The research work presented in the thesis is organized and structured in the form of nine chapters, which are briefly described as follows:

In **chapter 1**, basic definitions and terminologies in graph theory and group theory which are needed in the ensuing chapters, are provided.

Chapter 2 explores the concept of A-magic labeling of graphs, focusing on the smallest non-abelian group, S_3, the symmetric group on 3 symbols. We investigated the S_3-magic labeling of some well-known graphs. The first section is devoted to A-magic labeling of graphs, where A is an additive abelian group due to S.M. Lee, F. Saba, Ebrahim Salehi, and Hugo Sun. In the second section, we introduced the A-magic labeling of graphs using a non-abelian group A. In the third section, we investigated S_3-magic graphs. And also, derived necessary and sufficient conditions for some general graphs that admit S_3-magic labeling. We have shown that any regular graph is S_3-magic. Also, if the degrees of the vertices of a graph G are either all even or all odd, then it is S_3-magic. The third section deals with the A-magic property of the graph obtained from the product of two A-magic graphs and also proves that the cartesian product and the lexicographic product of two A-magic labelings of graphs are A-magic, where A-is any finite non-abelian group. The fourth section investigates the S_3-magic labeling of some graph products on cycles and paths.

Chapter 3 deals with the D_4-magic labeling of graphs, where D_4 is the dihedral group of order 8. The first section of the chapter includes a brief description of the dihedral group D_4. The second section discusses the D_4-magic labeling of graphs. We proved that if A is a non-abelian group having an element of order 2 and if G is a graph whose degree of the vertices is either all even or all odd, then G is A-magic. Also, the second section discusses the D_4-magic labeling of the cycle-related graphs, namely wheel W_n, shell $S_{n,n-3}$, helm H_n, web graph $W(2,n)$, n-gon book of k pages $B(n,k)$ and path-generated graphs, namely splitting graph of the path graph, middle graph of the path P_n, triangular snake T_n, alternate triangular snake, and double triangular snake.

In **chapter 4**, we discuss the Q_8-magic labeling of graphs, where Q_8 is the quaternion group of order 8. In the first section, we include the definition of the quaternion group Q_8 and define the Q_8-magic labeling of graphs. Also, we provide an example of a Q_8-magic labeling. The second section of the chapter deals with the Q_8-magic labeling of some graphs and their subdivision graphs, and also classifies them according to the magic constants.

Chapter 5 introduces the concept of induced A-magic labeling of graphs when A is a finite non-abelian group. The first section of the chapter gives an introduction to induced A-magic labeling, and the second section deals with the induced A-magic labeling of graphs when A is a finite non-abelian group. Also, we consider the particular non-abelian group symmetric group S_3 and investigate the induced S_3-magic labeling of graphs. The third section of the chapter discusses a necessary and sufficient condition for some cycle-related graphs that admit induced S_3-magic labeling. The graphs considered in the third section are cycle C_n, wheel, helm, fan, gear, sunflower, flag, sun graph, broken sun graph $CBSun_{p,q}$, web, flower, friendship graph, and n-gon book of k pages.

Chapter 6 introduces a new concept, conjugate A-magic labeling of graphs, where A is a finite non-abelian group. The chapter consists of two main sections:

1. Definition and introduction to conjugate A-magic labeling.
2. Investigation of necessary and sufficient conditions for well-known graphs to admit conjugate S_3-magic labeling.

Additionally, we establish the following key result: If G is a conjugate S_3-magic graph with a vertex of degree 2, then the conjugate S_3-magic constant cannot belong to the set $\{\mu_1, \mu_2, \mu_3\}$.

Chapter 7 explores the concept of A-barycentric magic labeling of graphs, with a focus on S_3-barycentric magic labeling. We investigate whether various graphs admit S_3-barycentric magic labeling, including:

- Cycle C_n
- Star $K_{1,n}$
- Wheel W_n
- Helm H_n
- Shell $S_{n,n-3}$
- n-gon book of k-pages
- Flag Fl_n
- Complete graph K_n
- Sun and gear graph G_n

We classify these graphs into three categories:

1. $\mathcal{BS}\rho_0$: Graphs that are S_3-barycentric magic with constant ρ_0.
2. $\mathcal{BS}\rho$: Graphs that are S_3-barycentric magic with constants in $\{\rho_1, \rho_2\}$.
3. $\mathcal{BS}\mu$: Graphs that are S_3-barycentric magic with constants in $\{\mu_1, \mu_2, \mu_3\}$.

Chapter 8 introduces the concept of neighborhood A-magic labeling, where A is a finite non-abelian group. The chapter consists of three sections:

- Introduction to group distance magic labeling.
- Neighborhood S_3-magic labeling of well-known graphs.
- Neighborhood S_3-magic labeling of cycle-related graphs.

In **chapter 8**, we introduce the notion of neighborhood A-magic labeling, where A is a finite non-abelian group. The first section of this chapter gives an introduction about the group distance magic labeling and the second and third sections deal with the neighborhood S_3-magic labeling of some well-known graphs and some cycle-related graphs respectively.

Chapter 9 presents a concise summary of the research findings and suggests avenues for future exploration. The thesis is supplemented by the following:

- Compilation of presented and published papers
- An extensive bibliography
- A comprehensive index

located at the end of the document.

Chapter 1

Preliminaries

This chapter provides a quick revision of the preliminary topics in graph theory and group theory that will be used in subsequent chapters.

1.1 Basic Definitions

1.1.1 Basic Definitions from Graph Theory

Definition 1. *[13] A graph G is an ordered pair (V, E), where V is a non-empty set and E is a symmetric and irreflexive relation on V. The set V is called the vertex set and the elements of V are called vertices. The set E is called the edge set and the elements of E are called edges. An edge of the form $e = (u, v)$ is usually denoted by uv. If $e = uv$ is an edge, then we say that e is said to join u and v. The vertex u is called the initial vertex and u is called the terminal vertex of e. Moreover, we say that u and v are the neighbors of the edge e.*

Pictorically a graph $G = (V, E)$ is representing the vertices as points and edges by line segments in a plane.

Definition 2. *[14] A vertex of a graph G which is not the end of any edge is called isolated. The set of all neighbors of a fixed vertex v of G is called the neighborhood set of v and is denoted by $N(v)$. That is,*

$$N(v) = \{vu : u \in V\}$$

Definition 3. *[14] An edge e of a graph G is said to be incident with the vertex v if v is an end vertex of e. In this case, we also say that v is incident with e. Two edges e and f are incident with a common vertex v are said to be adjacent.*

Definition 4. *[15] In a graph G, the number of elements in a vertex set is called the order of G and is denoted by $|V(G)|$ and the number of elements in* *the edge set of G is called the size of G and is denoted by $|E(G)|$. A graph is finite if its vertex set and edge set are finite.*

Throughout this thesis, we consider only finite graphs until it is mentioned in other ways.

Definition 5. *[15] The degree $d_G(v)$ or $d(v)$ or $deg(v)$ of a vertex v in a graph G is the number of edges of G incident with v, each loop counting as two edges.*

The minimum degree among the vertices of G is denoted by $mindegG$ or $\delta(G)$, and the maximum degree among the vertices of G is denoted as max $degG$ or $\Delta(G)$. If all the vertices have the same degree, ie. if $\delta(G) = \Delta(G) = r$, say, then the graph G is called regular graph of degree r. Such a graph is also called r-regular graph.

Definition 6. *[13] If the degree of a vertex in G is zero, then the vertex is called an isolated vertex and if the degree of a vertex in G is one, then it is called a pendant vertex or a leaf or an end vertex of G.*

Definition 7. *[13] A walk of length k is a sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$ of vertices and edges such that $e_i = v_{i-1}v_i$ for all i.*

A trail is a walk with no repeated edge.

A path is a walk with no repeated vertex. This walk is also denoted by $v_0v_1 \ldots v_{n-1}v_n$ and called a v_0, v_n walk. If $v_0 = v_n$, then the is a closed and open otherwise.

A closed path is called cycle. The length of the walk is the number of edges occurring in it. C_n denotes the cycle with n vertices and hence length of C_n is n. P_n is a path with n vertices and length $n-1$.

Definition 8. *[13] A graph G is connected if each pair of vertices in G belongs to a path. Otherwise, G is disconnected.*

Definition 9. *[15] A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph. Up to isomorphism, there is just one complete graph on n vertices; it is denoted by K_n.*

Definition 10. *[13] A graph G is bipartite if $V(G)$ is the union of two disjoint(possibly empty) independent sets called partite sets of G.*

Definition 11. *[13] A complete bipartite graph or biclique is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have sizes m and n the complete bipartite graph is denoted by $K_{m,n}$.*

Definition 12. *[13] A graph is Eulerian if it has a closed trail containing all edges. We call a closed trail a circuit when we do not specify the first vertex but keep the list in cyclic order. An Eulerian circuit or Eulerian trail in a graph is a circuit or trail containing all the edges.*

Definition 13. *[13] A graph with no cycle is acyclic. A tree is a connected acyclic graph. A leaf(or pendant vertex) is a vertex of degree 1.*
A star is a tree consisting of one vertex adjacent to all the others. The n-vertex star is a biclique $K_{1,n-1}$.

Definition 14. *[15] A wheel W_n of order $n+1$ is a graph obtained from a cycle C_n by adding a new vertex (which is known as hub) and edges joining it to all the vertices of the cycle; the new edges are called the spokes of the wheel. Equivalently, $W_n = K_1 + C_n$.*

1.1.2 Basic Definitions from Group Theory

Definition 15. *[16] A binary operation on a set S is a function mapping $*$ from $S \times S$ into S. Usually, the image of $(a,b) \in S \times S$ under $*$ is denoted by $a * b$. That is, $*(a,b) = a * b$.*

Definition 16. *[16] A binary operation on $*$ a set S is commutative if(and only if) $a * b = b * a$ for all $a, b \in S$.*

Definition 17. *[16] A binary operation $*$ on a set S is associative if $(a*b)*c = a*(b*c)$ for all $a, b, c \in S$.*

Definition 18. *[16] A set S together with a binary operation $*$ is called a binary algebraic structure or simply binary structure, denoted by $< S, * >$.*

Definition 19. *[16] A group $< G, * >$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied:*

(i) The operation $*$ is associative.

(ii) There exists an element $e \in G$ such that $e * x = x = x * e$ for all $x \in G$. (The element e is called the identity element of the binary operation on $*$ on G.)

(iii For each $a \in G$, there is an element $a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$ for all $a \in G$. (The element a^{-1} is called the inverse of the element a.)

Definition 20. *[16] A group G is abelian if its binary operation $*$ is commutative. Otherwise it is said to be non-abelian.(i.e., there exists at least one pair of elements a and b of G such that $a * b \neq b * a$.)*

Definition 21. *[16] If G is a group, then the order $|G|$ of G is the number of elements in G.*

Example 22. *[16] The set $\mathbf{M}_{m\times n}$ $(\mathbb{R})$ of all $m \times n$ matrices under matrix addition is a group. The $m \times n$ matrix with all entries 0 is the identity matrix. This group is abelian.*

Example 23. *[16] The set $\mathbf{M}_n$ $(\mathbb{R})$ of all $n \times n$ matrices under matrix multiplication is not a group. The $n \times n$ matrix with all entries 0 has no inverse.*

Example 24. *[16] The subset $\mathbf{S}$ of $\mathbf{M}_n$ $(\mathbb{R})$ consisting of all invertible $n \times n$ matrices under matrix multiplication is a group and this group is not commutative. So it is an example of a non-abelian group.*

Definition 25. *[16] A permutation of a set A is a function $\phi : A \to A$ that is both one one and onto.*

We observe that the collection of all permutation of a non empty set A forms a group under permutation multiplication(or function composition).

Definition 26. *[16] Let A be the finite set $\{1, 2, 3, \ldots, n\}$. The group of all permutations of A is the symmetric group on n letters, and is denoted by S_n.*

Definition 27. *[16] Let $X = \{1, 2, 3\}$. The group of permutations of X with the the composition of functions as a binary operation is a non-abelian group. It is called the symmetric group on 3 letters and denoted by S_3.*

The elements of S_3 are listed below.

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$
$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$
$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	ρ_0	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	ρ_0

The multiplication table for the elements S_3 is shown in the above Table. There is a natural correspondence between the elements of S_3 and the ways in which two copies of an equilateral triangle with vertices 1,2, and 3 can be places, one covering the other with vertices on top of vertices. For this reason S_3 is also the group D_3 of symmetries of an equilateral triangle [16].

Definition 28. *[16] The nth dihedral group D_n is the group of symmetries of the regular n-gon. D_n are non abelian for $n > 2$.*

Example 29. *[16] The group of symmetries of a square is the dihedral group D_4. The elements of D_4 are given by*

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$
$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$
$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix},$$
$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

Now the group table for the group D_4 is given in the following table below.

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_1	δ_2	μ_2	μ_1
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_2	δ_1	μ_1	μ_2
μ_1	μ_1	δ_2	μ_2	δ_1	ρ_0	ρ_2	ρ_3	ρ_1
μ_2	μ_2	δ_1	μ_1	δ_2	ρ_2	ρ_0	ρ_1	ρ_3
δ_1	δ_1	μ_1	δ_2	μ_2	ρ_1	ρ_3	ρ_0	ρ_2
δ_2	δ_2	μ_2	δ_1	μ_1	ρ_3	ρ_1	ρ_2	ρ_0

Definition 30. *[16] Let G be a group with identity element e. Then the order of an element a in G is the smallest positive integer m such that $a^m = e$ and it is denoted by $o(a)$.*

Definition 31. *[16] If a subset H of a group G is closed under the binary operation of G and if H with the induced operation from G is itself a group, then H is a subgroup of G.*

Definition 32. *[16] Let G be a group and $a \in G$. Then the subgroup $\{a^n | n \in \mathbb{Z}\}$ of G, is called the cyclic subgroup generated by a, and denotes by $< a >$. An element a of a group G and is a generator for G if $< a >= G$. A group G is cyclic if there is some element a in G that generates G.*

Definition 33. *[16] Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Q_8 is a non-abelian group under quaternion multiplication. This group is generated by j and j, where $i^4 = 1, j^2 = i^2$ and $ji = i^3 j$.*

Chapter 2

S_3-Magic Labeling of Graphs

In this chapter, we introduced the concept of A-magic labeling of graphs, when A is a non-abelian group. The first section of this chapter gives an introduction about the A-magic labeling of graphs when A is an abelian group and then extends this concept to A-magic labeling of graphs when A is non-abelian. The second section of the chapter gives a necessary and sufficient condition for some general graphs which admits S_3-magic labeling, where S_3 is the group of permutation on 3 symbols. In the third section, we investigate the A-magic property of the graph obtained from the product of two A-magic graphs. The fourth section of this chapter deals with the the S_3-magic labeling of some graph products on cycles and paths.

2.1 A-Magic Labeling of Graphs

For any abelian group A, written additively, any mapping $\ell : E(G) \to A \setminus \{0\}$ is called a labeling. Given a labeling on the edge set of G, one can introduce a vertex set labeling $\ell^+ : V(G) \to A$ as follows:

$$\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$$

A graph G is said to be A-magic [4] if there is a labeling $\ell : E(G) \to A \setminus \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; that is, $\ell^+(v) = a$ for some fixed $a \in A$. The original concept of A-magic graph was introduced by Sedláček [17]. According to him, a graph G is A magic if there exists an edge labeling on G such that

(i) distinct edges have distinct nonnegative labels; and

(ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

It is natural to ask whether the definition of A-magic graphs can be extended to non-abelian groups. In this chapter, we address this question and then investigate graphs that are S_3-magic, where S_3 is the group of permutations of $X = \{1, 2, 3\}$ with the composition of functions as a binary operation.

2.2 A-Magic Labeling of Graphs Using Non-abelian Group A

Definition 34. *[18]*

Let $G = (V(G), E(G))$ be a finite (p, q) graph and A be a finite non-abelian group with identity element 1. Let $f : E(G) \to N_q = \{1, 2, \ldots, q\}$ and let $g : E(G) \to A \setminus \{1\}$ be two edge labelings of G such that f is bijective. Define an edge labeling $\ell : E(G) \longrightarrow N_q \times A \setminus \{1\}$ by

$$l(e) := (f(e), g(e)),\ e \in E(G).$$

Define a relation $\leq$ on the range of ℓ by:

$$(f(e), g(e)) \leq (f(e^{'}), g(e')) \ \text{ if and only if } \ f(e) \leq f(e').$$

Then obviously the relation $\leq$ is a partial order on the range of ℓ.
Let $\{(f(e_1), g(e_1)), (f(e_2), g(e_2)), \ldots, (f(e_k), g(e_k))\}$ be a chain in the range of ℓ. We define the product of the elements of this chain follows:

$$\prod_{i=1}^{k}(f(e_i), g(e_i)) := ((((g(e_1) * g(e_2)) * g(e_3)) * g(e_4)) * \ldots) * g(e_k).$$

Let $u \in V$ and let $N^*(u)$ be the set of all edges incident with u. Note that the range of $\ell|_{N^*(u)}$ is a chain, say $(f(e_1), g(e_1)) \leq (f(e_2), g(e_2)) \leq \cdots \leq (f(e_n), g(e_n))$. We define

$$\ell^*(u) = \prod_{i=1}^{n}(f(e_i), g(e_i)). \tag{2.2.1}$$

If $\ell^*(u)$ is a constant, say a for all $u \in V(G)$, we say that the graph G is A-magic. The map ℓ^* is called an A-magic labeling of G and the corresponding

[1] The sections 2.1, 2.2, and 2.3 of this chapter have been published in the South East Asian Journal of Mathematics and Mathematical Sciences, Volume 18, Number 3, pages 317-328, (2022).

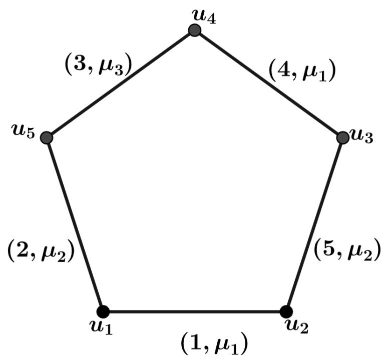

Figure 2.1: S_3-magic labeling of C_5

constant a is called the magic constant.

Observe that when A is an abelian group our definition coincides with that by Doob [10, 11] and Sin Min Lee et al [4].

Example 35. *Consider the cycle graph C_5 with vertex set $V(C_5) = \{u_1, u_2, u_3, u_4, u_5\}$, and the permutation group S_3. Note the group S_3 is a non abelian group [16] of order 6 and its elements are given by*

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$
$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$
$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Define $f : E(C_5) \longrightarrow N_q = \{1, 2, 3, 4, 5\}$ as $f(u_1u_2) = 1, f(u_1u_5) = 2, f(u_4u_5) = 3, f(u_3u_4) = 4, f(u_2u_3) = 5$ and $g : E(C_4) \to S_3 \setminus \{\rho_0\}$ as

$$g(e) = \begin{cases} \mu_1, & \text{if } e = u_1u_2, u_3u_4, \\ \mu_2, & \text{if } e = u_1u_5, u_2u_3, \\ \mu_3, & \text{if } e = u_4u_5. \end{cases}$$

Thus $\ell^*(u_1) = (1, \mu_1)(2, \mu_2) = \mu_1\mu_2 = \rho_1$. Similarly, $\ell^*(u_2) = \mu_1\mu_2 = \rho_1$, $\ell^*(u_3) = \mu_1\mu_2 = \rho_1$, $\ell^*(u_4) = \mu_3\mu_1 = \rho_1$ and $\ell^*(u_5) = \mu_2\mu_3 = \rho_1$. Thus C_5 is S_3-magic with magic constant ρ_1.

2.3 S_3-Magic Labeling of Graphs

In this section, we consider the symmetric group S_3 and investigate graphs that are S_3-magic. We also find a necessary and sufficient conditions for some graphs to be S_3-magic.

Theorem 36. *Any regular graph is S_3-magic.*

Proof. Let $G = (V(G), E(G))$ be a regular graph with $|E(G)| = q$. Let $g : E(G) \to S_3 \setminus \{\rho_0\}$ be any constant map and let $f : E(G) \to N_q = \{1, 2, \ldots, q\}$ be any bijective map. Then obviously, ℓ^* is a constant map. This completes the proof of the theorem. □

Corollary 37. *For any $n \geq 3$, the cycle graph C_n is S_3-magic.*

Corollary 38. *For any $n \geq 2$, the complete graph K_n is S_3-magic.*

Theorem 39. *If the degrees of the vertices of a graph G are either all even or odd, then it is S_3-magic.*

Proof. Let G be a (p, q) graph. We consider two cases:

Case(i): Assume that all the vertices of G are of even degree. Define a map $g : E(G) \to S_3 \setminus \{\rho_0\}$ by $g(e) = \mu_1$, $\forall e \in E(G)$ and let $f : E(G) \to N_q$ be any bijective map. Then obviously, G is S_3-magic with $\ell^*(u) = \rho_0$, for all $u \in V(G)$.

Case(ii): Assume that all vertices are of odd degree. The proof is exactly similar to case (i) and the magic constant is μ_1.

□

Corollary 40. *All Eulerian graphs are S_3-magic.*

Definition 41. *[19] The graph obtained by joining a single pendant edge to each vertex of a cycle is called a sun graph.*

Corollary 42. *A sun graph is S_3-magic.*

Proof. Since all the vertices of sun graph has odd degree (1 or 3), the proof follows from Theorem 39. □

Theorem 43. *For any $n \geq 3$, the path of order n, is not S_3-magic.*

Proof. Let $P_n = (u_1, u_2, \ldots, u_n)$ be a path of order n. Assume to the contrary that P_n admits a S_3-magic labeling. This implies that there exist two maps f and g such that $\ell^*(u_1) = \ell^*(u_2) = \cdots = \ell^*(u_n) = a$, for some $a \in S_3 \setminus \{\rho_0\}$. Since u_1 and u_n are vertices of degree 1, $g(u_1u_2) = g(u_{n-1}u_n) = a$. Let $g(u_2u_3) = b$, $b \in S_3 \setminus \{\rho_0\}$ and let $f(u_1u_2) = m_1, f(u_2u_3) = m_2$ for some $m_1, m_2 \in N_{n-1}$. Now $\ell^*(u_2) = \begin{cases} (m_1, a)(m_2, b), & \text{if} \quad m_1 < m_2, \\ (m_2, b)(m_1, a), & \text{if} \quad m_2 < m_1. \end{cases}$
This implies that $\ell^*(u_2) = ab$, if $m_1 < m_2$ and $\ell^*(u_2) = ba$, if $m_2 < m_1$. This implies that either $a = \rho_0$ or $b = \rho_0$, which is a contradiction. Hence the path P_n is not S_3-magic. □

Definition 44. *[20] Comb graph is a graph obtained by joining a single pendant edge to each vertex of a path P_n.*

Theorem 45. *Comb graphs are not S_3-magic.*

Proof. Let the vertices of P_n be $u_1, u_2, \ldots, u_n$ and the end vertex of each pendent edge at u_i be u_{n+i}. Suppose to the contrary that comb graph G is S_3-magic. Then by the definition, there exist functions $f : E(G) \to N_{2n-1}$ and $g : E(G) \to S_3 \setminus \{\rho_0\}$ such that $l^*(u_1) = l^*(u_2) = \cdots = l^*(u_{2n}) = a$, for some $a \in S_3 \setminus \{\rho_0\}$. Since each $u_{n+i}, 1 \le i \le n$ are of degree 1, it follows that $g(u_iu_{n+i}) = a, 1 \le i \le n$. This implies that there exists $b \in S_3 \setminus \{\rho_0\}$ such that $g(u_1u_2) = b$ and $l^*(u_1) = a * b$ or $l^*(u_1) = b * a$ according to the value of $f(u_1u_2)$ and $f(u_1u_{n+1})$. Since $l^*(u_1) = a$, it follows that $ab = a$ or $ba = a$ which implies either $a = \rho_0$ or $b = \rho_0$. This contradiction shows that G is not S_3-magic. □

Definition 46. *[21] A splitting graph $S(G)$ of a graph G is that graph obtained from G by adding to G a new vertex z' for each vertex z of G and joining z' to the neighbors of z in G.*

Theorem 47. *Splitting graph of a path P_n, where $n \ge 3$ is S_3-magic.*

Proof. Let P_n be a path of order n, where $n \ge 3$. Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n. Then $S(P_n)$ has $2n$ vertices and $3n-3$ edges. Let u_{n+i} be the vertex corresponding to the i^{th} vertex in $S(P_n)$. Observe that there are two pendant edges in $S(P_n)$, one with end points u_2 and u_{n+1} and the other with end points u_{n-1} and u_{2n}. Here we consider 2 cases.

Case (i): $n = 3$.

Define $f : E(S(P_3)) \to N_6$ as $f(u_1u_2) = 1, \quad f(u_3u_5) = 2, \quad f(u_2u_3) =$

3, $f(u_1u_5) = 4$, $f(u_2u_4) = 5$, $f(u_2u_6) = 6$ and $g : E(S(P_3)) \to S_3 \setminus \{\rho_0\}$ as

$$g(u_1u_2) = g(u_3u_5) = \rho_1,\ g(u_2u_4) = g(u_2u_6) = \mu_1,$$
$$g(u_1u_5) = g(u_2u_3) = \mu_2.$$

Note that $\ell^*(u) = \mu_1, \forall u \in V(S(P_3))$. Hence, the graph $S(P_3)$ is S_3-magic.

Case(ii): $n > 3$.

Define $f : E(S(P_n)) \to N_{3n-3}$ as

$$f(u_1u_2) = 1, f(u_2u_{n+1}) = 2n, f(u_{n-1}u_n) = n,$$
$$f(u_iu_{n+i+1}) = n + i, \quad 1 \leq i \leq n-2, f(u_iu_{n+(i-1)}) = i-1, \quad 3 \leq i \leq n,$$
$$f(u_iu_{i+1}) = 2n + (i-1), \quad 2 \leq i \leq n-2, f(u_{n-1}u_{2n}) = 2n-1.$$

Now define $g : E(S(P_n)) \to S_3 \setminus \{\rho_0\}$ as

$$g(u_1u_2) = \rho_1,\ g(u_2u_{n+1}) = \mu_1 = g(u_{n-1}u_{2n}),$$
$$g(u_iu_{i+1}) = \mu_1, \quad 2 \leq i \leq n-2,\ g(u_iu_{n+(i+1)}) = \mu_2, 1 \leq i \leq n-2,$$
$$g(u_{n-1}u_n) = \mu_2,\ g(u_iu_{n+(i-1)}) = \rho_1,\ 3 \leq i \leq n.$$

Obviously, $S(P_n)$ is S_3-magic with magic constant μ_1.

This completes the proof of the theorem. □

Theorem 48. *The star graph $K_{1,n}$ is S_3-magic if and only if either n is odd or $n \equiv 1(mod\ 3)$.*

Proof. Let $G = K_{1,n}$. First, assume that n is odd. Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ by $g(e) = \mu_1$, $\forall e \in E(G)$. Let $f : E(G) \to N_n = \{1, 2, \ldots, n\}$ be any bijection. Obviously, $\ell^*(u) = \mu_1$, $\forall u \in V(G)$. Similarly, we can prove that if $n \equiv 1(\text{mod } 3)$ then $K_{1,n}$ is S_3-magic.

Conversely, assume that $K_{1,n}$ is S_3-magic. Thus, each pendant edge should be labeled by the same element of S_3 under the map g. Hence, $g : E(G) \to S_3 \setminus \{\rho_0\}$ must be a constant map. Let $u_1, u_2, \ldots, u_n$ be the vertices of $K_{1,n}$ having degree 1 and let v be the vertex of $K_{1,n}$ having degree n. Let $f : E(G) \to \{1, 2, 3, \ldots, n\}$ be a bijection which make $K_{1,n}$ S_3-magic. By our assumption $\ell^*(u_i) = a$, for some $a \in S_3 \setminus \{\rho_0\}$, $i = 1, 2, \ldots, n$. Thus $\ell^*(v) = \ell^*(u_i) = a$.

This implies that $\underbrace{aa\cdots a}_{n\,\text{times}} = a$. Since the maximum order of an element in S_3 is 3 this implies that $n \equiv 1 \pmod 3$ or n is odd. Hence the proof. $\square$

Theorem 49. *For $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is S_3-magic.*

Proof. Let G be the graph $K_{m,n}$. Here we consider four cases.

Case (i): m and n have the same parity.

We can define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as $g(e) = \mu_1, \forall e \in E(G)$ and $f : E(G) \to \{1, 2, \ldots, m+n\}$ be any bijection. Then obviously $\ell^*(u)$ is either ρ_0 or $\mu_1, \forall u \in V(G)$.

Case(ii): $m \equiv 0(\text{mod } 2)$ and $n \equiv 0(\text{mod } 3)$.

In this case, $m = 2k$ for some k and $n = 3l$ for some l. Let $U := \{u_1, u_2, \ldots, u_{2k}\}$ and $V := \{v_1, v_2, \ldots, v_{3l}\}$ be the two partite sets of $K_{m,n}$. Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ by

$$g(u_iv_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases}$$

Now define $f : E(G) \to \{1, 2, \ldots, m+n\}$ by
$f(u_iv_j) = (i-1)m + j,\ 1 \leq i \leq 2k,\ 1 \leq j \leq 3l$. Obviously, $\ell^*(u) = \rho_0,\ \forall u \in V(G)$.

Case (iii): $m \equiv 0(\text{mod } 2)$, $n \equiv 2(\text{mod } 3)$ and n odd.

Note that, in this case $n = 5 + (k-1)6,\ k \in \mathbb{N}$. Let $U = \{u_1, u_2, \ldots, u_{2l}\}$ and $V = \{v_1, v_2, \ldots, v_n\}$, where $2l = m$ be the two partite sets of $K_{m,n}$. Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ by:

$$g(u_iv_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd and } j \leq n-2, \\ \rho_2, & \text{if } i \text{ is even and } j \leq n-2, \\ \mu_1, & \text{if } j = n-1, n. \end{cases}$$

Now define $f : E(G) \to N_{m+n} = \{1, 2, \ldots, m+n\}$ by
$f(u_iv_j) = (i-1)m + j,\ i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.
Then $\ell^*(u) = \rho_0,\ \forall u \in V(G)$.

Case(iv): $m \equiv 0(\text{mod } 2)$, $n \equiv 1(\text{mod } 3)$ and n is odd.

Here the number n is of the form $7 + (k-1)6$, where $k \in \mathbb{N}$. Define

$g : E(G) \to S_3 \setminus \{\rho_0\}$ by

$$g(u_iv_j) = \begin{cases} \rho_1, & \text{if } i \text{ and } j \text{ odd}, j \neq 3, j \leq 6, \\ \rho_2, & \text{if } i \text{ is odd and } j = 3, \\ \mu_1, & \text{if } j \text{ is even}, \\ \mu_1, & \text{if } j \geq 6, \\ \rho_2, & \text{if } i \text{ is even } j \text{ is odd}, j \neq 3, j \leq 6, \\ \rho_1, & \text{if } i \text{ is even and } j = 3. \end{cases}$$

Now define the map $f : E(G) \to N_{m+n} = \{1, 2, \ldots, m+n\}$ by $f(u_iv_j) = (i-1)m + j$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then $\ell^*(u) = \rho_0, \forall u \in V(G)$.

This completes the proof. □

Theorem 50. *If $n \geq 3$, the wheel W_n is S_3-magic.*

Proof. Let G be the wheel W_n and let the vertices of C_n be $u_1, u_2, \ldots, u_n$ and the vertex of K_1 be k. Here we consider two cases:

Case(i): n is odd.

In this case, define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as follows:
Label each spokes by μ_1 and all the outer edges by μ_2 and define $f : E(G) \to N_{2n} = \{1, 2, \ldots, 2n\}$ as:

$$f(ku_i) = i, \; i = 1, 2, \ldots, n, \; f(u_iu_{i+1}) = n + i, \; i < n, \; f(u_nu_1) = 2n.$$

Obviously, $\ell^*(e) = \mu_1$, for all $e \in E(W_n)$.

Case(ii): n is even.

Here we define $g : E(G) \to S_3 \setminus \{\rho_0\}$ by labeling each spokes by μ_1 and all the outer edges by μ_2 and ρ_2 alternatively such that

$$g(u_iu_{i+1}) = \begin{cases} \mu_2, & \text{if } i \text{ is odd}, \\ \rho_2, & \text{if } i \text{ is even}. \end{cases} \quad , \; g(u_nu_1) = \rho_2.$$

Now for $i = 1, 2, \ldots, n$, define $f : E(G) \to N_{2n}$ as:

$$f(ku_i) = i, \; f(u_1u_n) = 2n, \; f(u_iu_{i+1}) = \begin{cases} \frac{(i+1)}{2} + n, & \text{if } i \text{ is odd}, \\ \frac{i}{2} + \frac{3n}{2}, & \text{if } i \text{ is even}. \end{cases}$$

Hence the wheel W_n becomes S_3-magic with magic constant ρ_0.

□

Definition 51. *[22] A shell $S_{n,n-3}$ of width n is a graph obtained by taking $n-3$ concurrent chords in a cycle C_n of n vertices. The vertex at which all chords are concurrent is called apex. The two vertices adjacent to the apex have degree 2, apex has degree $n-1$ and all other vertices have degree 3.*

Theorem 52. *Shell graphs $S_{n,n-3}$ are S_3-magic.*

Proof. Let G be the shell graph $S_{n,n-3}$ and denote the vertices of $S_{n,n-3}$ by $u_1, u_2, \ldots, u_n$. There are n vertices and $2n-3$ edges in $S_{n,n-3}$. Without loss of generality, let the *apex* be u_1. Here we consider two cases:

Case(i): n is even.

We define $f : E(G) \to N_{2n-3}$ as follows:

$$f(u_1u_2) = 1, f(u_nu_1) = \frac{n}{2} + 1, f(u_{n-1}u_n) = 2n-3,$$

$$f(u_iu_{i+1}) = \begin{cases} \frac{i}{2}+1, & \text{if } i \text{ is even and } 2 \le i \le n-2, \\ \frac{n+i+1}{2}, & \text{if } i \text{ is odd and } 3 \le i \le n-3. \end{cases}$$

$$f(u_1u_{n-1}) = n \text{ and } f(u_1u_i) = n + (i-2) \text{ where } i \ne n-1, 2.$$

and now define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as
$g(u_1u_2) = \rho_1,\ g(u_1u_n) = \rho_2,\ g(u_{n-1}u_n) = \mu_1,\ g(u_1u_{n-1}) = \mu_2,$
$g(u_1u_i) = \mu_1,$ where $i \ne 2, n-1, n.$

$$g(u_iu_{i+1}) = \begin{cases} \mu_2, & \text{if } i \text{ is odd and } 3 \le i \le n-3, \\ \mu_3, & \text{if } i \text{ is even and } 2 \le i \le n-1. \end{cases}$$

Under these maps, shell graphs $S_{n,n-3}$ with even number of vertices are S_3-magic with magic constant μ_2.

Case(ii): n is odd.

Define $f(u_iu_{i+1}) = \begin{cases} \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{n+i}{2}, & \text{if } i \text{ is odd.} \end{cases}$,$f(u_1u_n) = n$ and $f(u_1u_i) = n + (i-2)$, where $i \ne 2, n$.

Now define $g(u_1u_2) = g(u_nu_1) = g(u_2u_3) = g(u_{n-1}u_n) = \rho_1,\ g(u_1u_i) = \mu_1$, where $i \ne 2, n$ and $g(u_iu_{i+1}) = \begin{cases} \mu_3, & \text{if } i \text{ is odd,} \\ \rho_1, & \text{if } i \text{ is even.} \end{cases}$

Thus the shell graph $S_{n,n-3}$ with odd number of vertices becomes S_3-magic with magic constant ρ_2.

Hence the proof. □

Definition 53. *[4] When k copies of C_n share a common edge it will form the n-gon book of k pages and is denoted by $B(n,k)$.*

Theorem 54. *For any $n \geq 3$ and $k \geq 1$, the $n-gon\,book$ of k pages are S_3-magic.*

Proof. Here we consider two cases:

Case(i): k is odd.

In this case, degree of all the vertices of $B(n,k)$ will be even. Define $g(e) = \mu_1,\ \forall e \in E(B(n,k))$ and f as any bijection from $E(G)$ to $\{1,\ 2, \ldots,\ k(n-1)+1\}$. Then the graph $B(n,k)$ becomes S_3-magic with magic constant ρ_0.

Case(ii): k is even.

We denote the common edge of $B(n,k)$ by c. Now, define the labeling $g : E(B(n,k)) \to S_3 \setminus \{\rho_0\}$ as follows:

Let $g(c) = \rho_1$ also label the outer edges of the first page by μ_1 and all other edges by μ_3. Denote the edges in the first page by $c,\ a_1,\ a_2, \ldots,\ a_{n-1}$. Now define $f(c) = 1$ and $f(a_i) = i+1$ and map other edges to the set $\{n+1, \ldots, k(n-1)+1\}$ such that $f(e_i) \neq f(e_j),\ \ e_i, e_j \in E(B(n,k))$. Then obviously, $\ell^*(v) = \rho_0,\ \forall v \in V(B(n,k))$.

This completes the proof of the theorem. □

We observe that the cycle C_{2n} with a pendant edge is non-magic when we label the edges with the nonzero elements of an abelian group [4]. But we can show that the above said graph is S_3-magic for all $n > 2$.

Theorem 55. *The cycle graph $C_n, n > 2$ with a pendant edge is S_3-magic.*

Proof. Let us denote the vertices of C_n by $u_1, u_2, \ldots, u_n$. Without loss of generality, assume that the pendant edge e is on the vertex u_1 and let its other end vertex be u_{n+1}.

Case(i): n is odd.

Define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as

$$g(u_iu_{i+1}) = \begin{cases} \mu_1, & \text{if } \ i \ \text{ is odd and } \ i < n, \\ \mu_3, & \text{if } \ i \ \text{ is even and } \ i < n. \end{cases},$$

$$g(u_nu_1) = \mu_1 \ \text{ and } g(u_1u_{n+1}) = \rho_2.$$

Now define

$$f(u_iu_{i+1}) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd and } i < n, \\ \frac{n+1}{2} + \frac{i}{2}, & \text{if } i \text{ is even and } i < n. \end{cases},$$

$$f(u_nu_1) = \frac{n+1}{2}, \; f(u_1u_{n+1}) = n+1.$$

Hence the graph is S_3-magic with magic constant ρ_2.

Case(ii): n is even.

In this case, define

$$g(u_iu_{i+1}) = \begin{cases} \mu_3, & \text{if } i \text{ is odd and } i < n, i \neq 1, \\ \mu_2, & \text{if } i \text{ is even and } i \neq n. \end{cases}$$

$$g(u_1u_2) = \mu_1 = g(u_nu_1) \text{ and } g(u_1u_{n+1}) = \rho_1.$$

Moreover, define f as:

$$f(u_1u_2) = 1, f(u_nu_1) = n, f(u_1u_{n+1}) = n+1,$$

$$f(u_iu_{i+1}) = \begin{cases} \frac{i}{2} + 1, & \text{if } i \text{ even and } i < n, \\ \frac{n}{2} + \frac{i-1}{2}, & \text{if } i \text{ is odd } i \neq 1 \text{ and } i < n. \end{cases}$$

Hence the magic constant is ρ_1.

This completes the proof of the theorem. □

2.4 Product of A-Magic Labeling of Graphs

In this section, we examine the A-magic property of the graph obtained from the product of two A-magic graphs.

Definition 56. *[23] The cartesian product of two simple graphs H and K is the graph $G = H \times K$ with $V(G) = V(H) \times V(K)$ in which vertices (h, k) and (h', k') are adjacent if and only if either*

1. *$h = h'$ and k, k' are adjacent in K.*

2. *$k = k'$ and h, h' are adjacent in H.*

Definition 57. *[24] Let H and K be two graphs the lexicographic product $G = H \circ K$ is a graph with vertex set $V(H) \times V(K)$. Two vertices (h, k) and (h', k') are adjacent if*

1. *h is adjacent to h' in H or*

2. *$h = h'$ and k is adjacent to k' in K*

Example 58. *The Cartesian product $P_3 \times P_3$ is S_3-magic(see Figure 2.2). Observe that P_3 is not S_3-magic.*

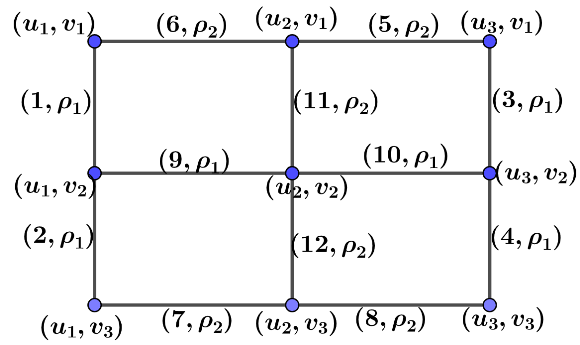

Figure 2.2: S_3-magic labeling of $P_3 \times P_3$.

Example 59. *Consider the graph P_4 and the null graph N_2 of order 2. The lexicographic product $P_4 \circ N_2$ is S_3-magic(see Figure 2.3) even though both the graphs P_4 and N_2 are not S_3-magic.*

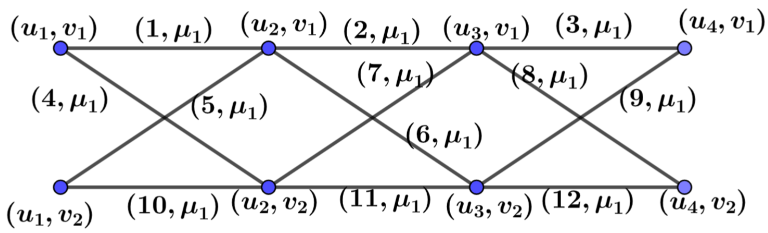

Figure 2.3: S_3-magic labeling of $P_4 \circ N_2$.

2.5 A-Magic Labeling of Cartesian and Lexicographic Products of Two A-Magic Graphs

In this section, we investigate whether the product of two A-magic graphs is A-magic or not.

Let A be a non-abelian group with identity element 1 and let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two A-magic graphs with magic constants γ_1 and γ_2 respectively, where $\gamma_1, \gamma_2 \in A$.

To illustrate the following theorems, we will use the S_3-magic graphs in the Figure 2.4.

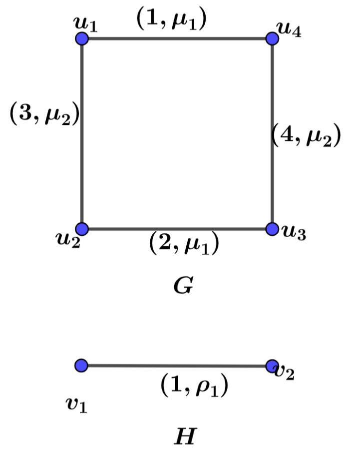

Figure 2.4: S_3-magic labeling of $G = C_4$ and $H = P_2$.

Theorem 60. *Let $(A, *)$ be a non-abelian group with identity element 1. If $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two A-magic graphs, then the Cartesian product $G \times H$ is A-magic.*

Proof. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two A-magic graphs with magic constants γ_1 and γ_2 respectively, where $\gamma_1, \gamma_2 \in A$. Denote the vertices of G by $g_1, g_2, \ldots, g_n$ and the vertices of H by $h_1, h_2, \ldots, h_n$. Now, let $f_1 : E(G) \to N_{|E(G)|}$, $g_1 : E(G) \to A \setminus \{1\}$ be two maps which determine an A-magic labeling, say ℓ_1^* for G and let $f_2 : E(G) \to N_{|E(H)|}$, $g_2 : E(H) \to A \setminus \{1\}$ be two maps which determine an A-magic labeling, say ℓ_2^* for H. Now define $F : E(G \times H) \to N_{|E(G)||E(H)|}$ and $G : E(G \times H) \to A \setminus \{1\}$ as follows:
For $1 \leq i, j \leq n$ and $1 \leq k, l \leq m$,

$$F((g_i, h_k)(g_j, h_k)) = f_1(g_i g_j) + (k-1)|E(G)|, \text{ if } g_i \text{ adjacent to } g_j \text{ and}$$

$$F((g_i, h_k)(g_i, h_l)) = m|E(G)| + f_2(h_k h_l) + (i-1)|E(H)|, \text{ if } h_k \text{ adjacent to } h_l$$

$$G((g_i, h_l)(g_j, h_k)) = \begin{cases} g_1(g_i g_j), & \text{if } h_l = h_k, \\ g_2(h_l h_k), & \text{if } g_i = g_j. \end{cases}$$

Then the induced vertex labeling of every vertex (g, h) is

$$\prod_{(g', h')} (F((g, h)(g', h'), G((g, h)(g', h')) =$$

$$\prod_{h'} (F((g, h)(g', h'), G((g, h)(g', h')) \prod_{g'} (F((g, h)(g', h'), G((g, h)(g', h'))$$

$= \prod_{gg' \in N^*(g)}((f_1(gg'), g_1(gg')) * \prod_{hh' \in N^*(h)}(f_2(hh'), g_2(hh'))$

$= \gamma_1 * \gamma_2.$ □

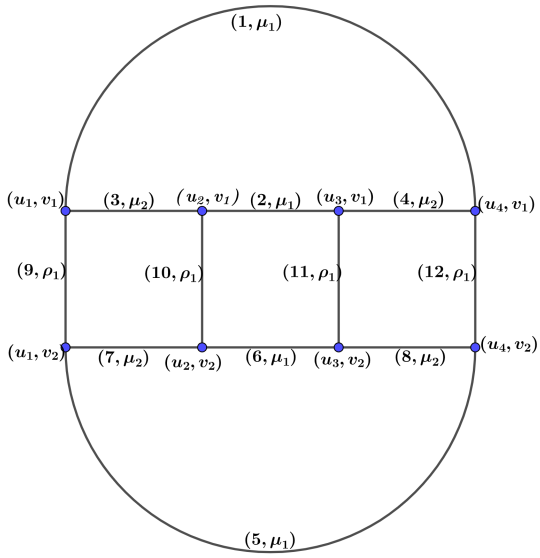

Figure 2.5: S_3-magic labeling of $C_4 \times H$.

Theorem 61. *Let $(A, *)$ be a non-abelian group with identity element 1. If $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two A-magic graphs, then the lexicographic product $G \circ H$ is A-magic.*

Proof. Define G, H, $f_1, f_2, g_1, g_2, \gamma_1, \gamma_2$ as in the above theorem. Also denote the vertices of G and H as in Theorem 60. Now define the map $F : E(G \circ H) \to N_{n|E(H)|+m|E(G)|+(m-1)|E(G)|}$ and $G : E(G \circ H) \to A$ be defined as follows:
For $1 \leq i, j \leq n,\ 1 \leq l, k \leq m$,

$$F((g_i, h_l)(g_i, h_k)) = f_2(h_l, h_k) + (i-1)|E(G)|$$
$$F((g_i, h_k)(g_j, h_k)) = F_1(g_i g_j) + (k-1)|E(G)| + n|E(H)|$$

if $i < j$, then

$$F((g_i, h_l)(g_j, h_{k+1})) = n|E(H)| + m|E(G)| + kf_1(g_i g_j)$$

if $i > j$, then

$$F((g_i, h_l)(g_j, h_{k+1})) = n|E(H)| + m|E(G)| + (m-1)|E(G)| + kf_1(g_i g_j)$$
$$= n|E(H)| + (2m-1)|E(G)| + kf_1(g_i g_j) \text{ and}$$

$$G((g_i, h_l)(g_j, h_k)) = \begin{cases} g_2(h_l h_k), & \text{if } i = j, \\ g_1(g_i g_j), & \text{otherwise.} \end{cases}$$

Here, we have if $g \neq g'$, $F((g,h)(g,h')) < F((g,h)(g',h'))$. Now the induced vertex labeling of every vertex (g,h) is

$$\prod_{g',h'}(F((g,h)(g',h')), G((g,h)(g',h'))) =$$
$$\prod_{\substack{g',h' \\ g=g'}}(F((g,h)(g',h')), G((g,h)(g',h'))) \prod_{\substack{g',h' \\ g\neq g'}}(F((g,h)(g',h')), G((g,h)(g',h')))$$
$$= \prod_{hh' \in N^*(h)}(f_2(hh'), g_2(hh')) * (\prod_{gg' \in N^*(g)}((f_1(gg'), g_1(gg')))^{|E(H)|}$$
$$= \gamma_2 * \gamma_1^{|E(H)|}.$$

□

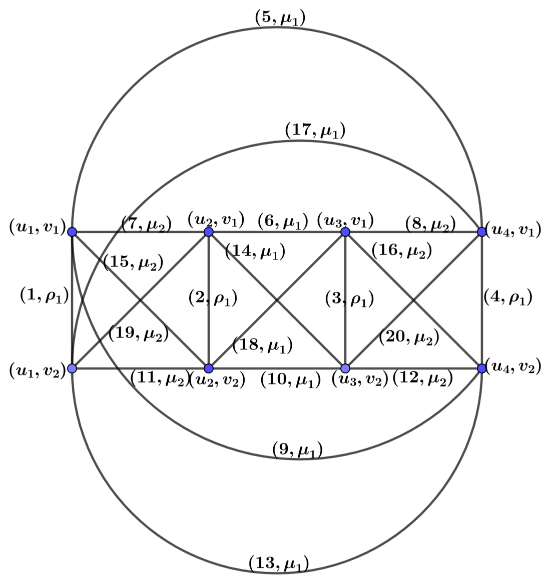

Figure 2.6: S_3-magic labeling of $G \circ H$.

2.6 S_3-Magic labeling of Cartesian Products on Cycles and Paths

This section discusses the S_3-magic labeling of the Cartesian products on cycles and paths.

Theorem 62. *The Cartesian product of two paths $P_n \times P_m$, $m, n \geq 2$ is S_3-magic.*

Proof. Let the vertices of the paths P_n and P_m be $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_m$, respectively, where u_i is adjacent to u_{i+1} and v_j is adjacent to v_{j+1} , $1 \leq i \leq n-1$, $1 \leq j \leq m-1$.There are mn vertices and $m(n-1) + n(m-1)$ edges in $P_n \times P_m$. Here we consider the following 4 cases.

Case(i): Let $n = 2$ and $m = 2$.

In this case $P_n \times P_m$ is the cycle C_4. Take f as any bijection from $E(P_2 \times P_2)$ to N_4 and define g by labeling the edges alternatively by ρ_1 and ρ_2. Then clearly $\ell^*(x) = \rho_0, \forall x \in V(P_2 \times P_2)$.

Case(ii): $n = 2$ and $m > 2$.

Let $f : E(P_n \times P_m) \to N_{m(n-1)+n(m-1)}$ be any bijection and let $g : E(P_n \times P_m) \to S_3 \setminus \{\rho_0\}$ be defined by $g((u_1, v_j)\ (u_2, v_j)) = \begin{cases} \rho_1, & \text{if } j = 1, m, \\ \rho_2, & \text{otherwise.} \end{cases}$

, $g((u_1, v_j)\ (u_1, v_{j+1})) = g((u_2, v_j)\ (u_2, v_{j+1})) = \rho_2,\ 1 \leq j \leq m-1$. The above maps f and g will determine a S_3-magic labeling of $P_2 \times P_m, m > 2$.

Case(iii): $n > 2$ and $m = 2$.

Take f as above and define g by $g((u_i, v_1)\ (u_i, v_2)) = \begin{cases} \rho_1, & \text{if } i = 1, n, \\ \rho_2, & \text{otherwise.} \end{cases}$

$1 \leq i \leq n$,

$g((u_i, v_1)\ (u_{i+1}, v_1)) = g((u_i, v_2)\ (u_{i+1}, v_2)) = \rho_2, 1 \leq i \leq n-1$. Clearly $\ell^*(x) = \rho_0, \forall x \in V(P_n \times P_2)$.

Case(iv): $n > 2$ and $m > 2$.

Here, take f as in case (ii). Now let g be the function defined by

$$g((u_i, v_j)(u_i, v_{j+1})) = \begin{cases} \rho_1, & \text{if } i = 1, n, \\ \rho_2, & \text{otherwise.} \end{cases}$$

$$g((u_i, v_j)(u_{i+1}, v_j)) = \begin{cases} \rho_2, & \text{if } j = 1, m, \\ \rho_1, & \text{otherwise.} \end{cases}$$

Clearly $\ell^*(x) = \rho_0, \forall x \in V(P_n \times P_m)$.

From the above four cases we have $P_n \times P_m$ is S_3-magic with magic constant ρ_0. □

Theorem 63. *The Cartesian product of two cycles $C_n \times C_m, m, n > 1$ is S_3-magic.*

Proof. Since the cycle graph $C_n, n > 1$ is S_3-magic by Theorem 60 $C_n \times C_m$, is S_3-magic.

□

Theorem 64. *The Cartesian product of a path P_n and a cycle C_m is S_3-magic, $m, n > 1$.*

Proof. Let P_n be a path with n vertices and C_m be a cycle with n vertices. Let $G = P_n \times C_m$. We consider the following cases:

Case(i): $n = 2$ and $m = 2$.

In this case, $P_n \times C_m$ is the cycle graph C_4. Then by Case(i) of Theorem 62 it is S_3-magic with constant ρ_0.

Case(ii): $n = 2$ and $m > 2$.

In this case, the degree of all vertices of $P_n \times C_n$ is 3. So we can define a S_3-magic labeling with constant ρ_0 by taking g as the constant map $g(e) = \rho_1, \forall e \in E(G)$ and f as any bijective map from $E(G)$ to N_{3m}.

Case(iii): $n > 2$ and $m = 2$.

In this case, G is same as the graph considered in Case(iii) of Theorem 62. Hence it is S_3-magic with constant ρ_0.

Case(iv): $n > 2$ and m is odd.

Here we take f as any bijection from $E(G) \to N_{2mn-m}$ and define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as

$$g((u_i, v_j)(u_{i+1}, v_j)) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases},$$

$$g((u_i, v_j)(u_i, v_{j+1})) = \begin{cases} \rho_1, & \text{if } i = 1, \\ \mu_1, & \text{if } 2 \leq i \leq n-1, \\ \rho_1, & \text{if } i = n \text{ and } n \text{ is even,} \\ \rho_2, & \text{if } i = n \text{ and } n \text{ is odd.} \end{cases},$$

$$g((u_i, v_1)(u_i, v_m)) = \begin{cases} \rho_1, & \text{if } i = 2, \\ \mu_1, & \text{if } 2 \leq i \leq n-1, \\ \rho_1, & \text{if } i = n \text{ and } i \text{ is even,} \\ \rho_2, & \text{if } i = n \text{ and } i \text{ is odd.} \end{cases}$$

Then clearly $\ell^*(x) = \rho_0, \forall x \in V(G)$.

Case(v): $n > 2$ and $m > 2$.

In this case, let f as above and define g as follows:

$$g((u_i, v_j)(u_{i+1}, v_j)) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even}, 1 \leq i \leq n, 1 \leq j \leq m \end{cases},$$

$$g((u_i, v_j)(u_i, v_{j+1})) = \begin{cases} \rho_1, & \text{if } i = 1, 1 \leq j \leq m, \\ \rho_1, & \text{if } i + j \text{ is odd, } 1 < i \leq n-1, 1 \leq j \leq m, \\ \rho_2, & \text{if } i + j \text{ is even}, 1 < i \leq n-1, 1 \leq j \leq m, \\ \rho_1, & \text{if } i = n \text{ and } n \text{ is even,} \\ \rho_2, & \text{if } i = n \text{ and } n \text{ is odd.} \end{cases}$$

$$g((u_i, v_1)(u_i, v_m)) = \begin{cases} \rho_1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \text{ and } i = n \text{ and } n \text{ is even,} \\ \rho_2, & \text{if } i \text{ is even, } 1 \leq i \leq n-1 \text{ and } i = n \text{ and } n \text{ is odd.} \end{cases}$$

Hence $\ell^*(x) = \rho_0, \forall x \in V(G)$.

This completes the proof of the theorem. □

Corollary 65. *The Cartesian product of C_m and P_n is S_3-magic.*

Chapter 3

D_4-Magic Graphs

In this chapter, we discuss the D_4-magic labeling of graphs, where D_4 denotes the dihedral group of order 8. The first section of this chapter gives an introduction to D_4 magic labeling, and the second section discusses the D_4 magic labeling of some general graphs. The third and fourth sections of this chapter deal with a necessary and sufficient condition for some cycle-generated graphs and path-generated graphs (respectively), which admits D_4-magic labeling.

3.1 Introduction

Consider the set $X = \{1, 2, 3, 4\}$ with 4 elements. A permutation of X is a function from X to itself that is both one one and on to. The permutations of X with the composition of functions as a binary operation is a non-abelian group, called the symmetric group S_4. Now consider the collection of all permutations corresponding to the ways that two copies of a square with vertices $1, 2, 3$ and 4 can be placed one covering the other with vertices on the top of vertices. This collection form a non-abelian subgroup of S_4, called the dihedral group D_4 [16]. The elements of D_4 are given by

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix},$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

[1]The sections 3.2, 3.3 and 3.4 of this chapter have been published in the journal Ratio Mathematica, Volume 42, 167-181, (2022).

In this chapter, we consider the non-abelian group D_4 and investigate graphs that are D_4-magic.

3.2 D_4-magic labeling of graphs

Recall the definition of A-magic labeling of graphs in Chapter 1(see Definition 34). In this section, we take $A = D_4$ and discuss a necessary and sufficient condition for some well known graphs that admit D_4-magic labeling. Consider the following example.

Example 66. *Consider the cycle graph $C_4 = (uv, vw, wx, xu)$ and the permutation group D_4.*

Define $f : E(G) \to N_q = \{1, 2, 3, 4\}$ as $f(uv) = 1, f(wx) = 2, f(vw) = 3, f(xu) = 4$ and $g : E(G) \to D_4 \setminus \{\rho_0\}$ as $g(uv) = g(wx) = \rho_1,\ g(vw) = g(xu) = \delta_1$. Thus

$$\ell^*(u) = (1, \rho_1)(4, \delta_1) = \rho_1\delta_1 = \mu_2,$$

$\ell^*(v) = (1, \rho_1)(3, \delta_1) = \rho_1\delta_1 = \mu_2$. Similarly, $\ell^*(w) = \mu_2$ and $\ell^*(x) = \mu_2$. Thus C_3 is D_4-magic with magic constant μ_2.

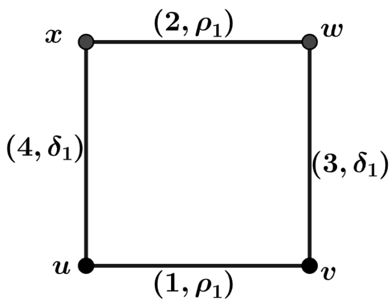

Figure 3.1: D_4-magic labeling of C_4.

Theorem 67. *Let A be a non-abelian group having an element of order 2 and let G be a graph. If either the degree of the vertices of G are all even or odd. Then G is A-magic.*

Proof. Let G be a (p, q) graph and A be a non-abelian group having an element of order 2. Let $a \in G$ is of order 2. Let $g : E(G) \to A \setminus \{1\}$ be the constant map $g(e) = a,\ \forall e \in E(G)$ and let f be any bijection from $E(G) \to N_q$. First, assume that all the vertices of G are of even degree then $l^*(u) = 1,\ \forall u \in V(G)$. Similarly, if all the vertices of G are of odd degree then $l^*(u) = a,\ \forall u \in V(G)$. Hence the proof. ∎

Corollary 68. *All Eulerian graphs are D_4-magic.*

Theorem 69. *Any regular graph is D_4-magic.*

Proof. Let $G = (V(G), E(G))$ be a regular graph with $E(G) = q$. Let $f : E(G) \to N_q$ be any bijection and g be any constant map from $E(G) \to D_4 \setminus \{\rho_0\}$. Obviously, f and g will determine a D_4-magic labeling of G. This completes the proof of the theorem. □

Corollary 70. *For any $n \geq 3$, the cycle graph C_n is D_4-magic.*

Corollary 71. *For any $n \geq 2$, the complete graph K_n is D_4-magic.*

Corollary 72. *The Petersen graph is D_4-magic.*

Theorem 73. *The star graph $K_{1,n}, n \geq 2$ is D_4-magic if and only if n is odd.*

Proof. Let $G = K_{1,n}$. Suppose that n is odd. Let $f : E(G) \to N_{n+1}$ be a bijection. Define $g : E(G) \to D_4 \setminus \{\rho_0\}$ by $g(e) = \mu_1$. Clearly it is D_4-magic with magic constant μ_1.
Conversely, suppose $K_{1,n}$ is D_4-magic with magic constant, say 'a'. So every pendent edge of $K_{1,n}$ should be mapped to a under g. Let u be the vertex of $K_{1,n}$ with degree n. Then

$$\ell^*(u) = \underbrace{aa \cdots a}_{n\,\text{times}} = a.$$

This implies that $a^{n-1} = \rho_0$. If n is odd, the equation $a^{n-1} = \rho_0$ has five solutions in D_4 viz.$\mu_1, \mu_2, \delta_1, \delta_2$ and ρ_2. On the other hand, if n is even there are no element in D_4 such that $a^{n-1} = \rho_0$. This completes the proof. □

Definition 74. *[25] A bistar graph B_n is the graph obtained by connecting the apex vertices of two copies of star $K_{1,n}$ by a bridge.*

Theorem 75. *The bistar graph B_n, $n > 1$ is D_4-magic when $n \not\equiv 1(mod\ 4)$.*

Proof. First, observe that there are $2n$ pendant edges and one bridge in B_n. Here we consider the following cases:

Case (i): n is even ($n \equiv 2(\text{mod } 4)$ or $n \equiv 0(\text{mod } 4)$).

If n is even, define $g : E(B_n) \to D_4 \setminus \{\rho_0\}$ by $g(e) = \mu_1$, $\forall e \in E(B_n)$. Let f be any bijective map from $E(B_n) \to N_{2n+1}$. Obviously, B_n is D_4-magic with magic constant μ_1.

Case (ii): $n \equiv 3 (\text{mod } 4)$.

In this case, we define $g : E(B_n) \to D_4 \setminus \{\rho_0\}$ by

$$g(e) = \begin{cases} \rho_1, & \text{if } e \text{ is a pendant edge,} \\ \rho_2, & \text{if } e \text{ is the bridge.} \end{cases}$$

Let f be any bijective map from $E(B_n)$ to N_{2n+1}. Obviously B_n is D_4-magic with the magic constant ρ_1.

Case (iii): $n \equiv 1 (\text{mod } 4)$.

Suppose that $n \equiv 1 (\text{mod } 4)$. Let k_1 and k_2 be the apex vertices of the bistar graph. Assume that B_n is D_4-magic with magic constant μ_1. Therefore, $g(e) = \mu_1$, for all pendant edges e. Assume that $g(k_1k_2) = a$, where $a \in D_4 \setminus \{\rho_0\}$. Without loss of generality, assume that $f(k_1k_2) > f(b)$, $\forall b \in E(G)$, where b denotes the pendant edge with one end point k_1. Then

$$\ell^*(k_1) = \underbrace{\mu_1\mu_1 \ldots \mu_1}_{(n\ times)} a = \mu_1.$$

The above equation tells us that $a = \rho_0$, which is a contradiction. This contradiction shows that B_n is not D_4-magic with magic constant μ_1. In a similar manner, we can prove that B_n is not D_4-magic with magic constants μ_2, ρ_1, ρ_2, ρ_3, δ_1 or δ_2. Thus the bistar graph B_n is not D_4-magic when $n \equiv 1 (\text{mod } 4)$. This completes the proof of the theorem.

□

Theorem 76. *The complete bipartite graph $K_{n,m}$ is D_4-magic, $m, n > 1$.*

Proof. Let $G = K_{n,m}$. Suppose $U = \{u_1, u_2, \ldots, u_n\}$, and $V = \{v_1, v_2, \ldots, v_m\}$. be the two partite sets of $K_{n,m}$. If m and n are both even or odd then the theorem is obvious by taking any constant map $g : E(G) \to \{\rho_2, \mu_1, \mu_2, \delta_1, \delta_2\}$.

Case (i): $n \equiv 0 (\text{mod } 2)$ and $m \equiv 1 (\text{mod } 4)$.

Let $U = \{u_1, u_2, \ldots, u_{2l}\}$ and $V = \{v_1, v_2, \ldots, v_{4k+1}\}$. Define

$$g(u_iv_1) = \mu_1,$$
$$g(u_iv_2) = \mu_2,$$
$$g(u_iv_j) = \rho_2, \ \forall\ 2 < j \leq m,$$
$$f(u_iv_j) = (i-1)m + j, \ 1 \leq i \leq n, \ 1 \leq j \leq m.$$

The maps f and g will determine a D_4-magic labeling for $K_{m,n}$ with magic constant ρ_0.

Case (ii): $n \equiv 0 (\text{mod } 2)$ and $m \equiv 3 (\text{mod } 4)$.
Define g as follows:

$$g(u_i v_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd } 1 \leq j < m, \\ \rho_3, & \text{if } i \text{ is even } 1 \leq j < m, \end{cases} \quad \text{and}$$
$$g(u_i v_m) = \rho_2, \ \forall i, \ 1 \leq i \leq n,$$

and let f be any bijection from $E(G)$ to $\{1, 2, \ldots, mn\}$. Clearly f and g will determine a D_4-magic labeling of $K_{m,n}$ with magic constant ρ_0.

This completes the proof of the theorem. □

3.3 Cycle Generated Graphs

In this section, we consider certain graphs which are constructed from cycles.

Theorem 77. *If $n \geq 3$, the wheel W_n is D_4-magic.*

Proof. Let the vertices of C_n be $u_1, u_2, \ldots, u_n$ such that $u_i u_{i+1} \in E(C_n), \ i = 1, 2, \ldots, n$ and $u_{n+1} = u_1$. Denote the vertex of K_1 by k. Now consider the following cases:

Case (i): n is odd.
If n is odd then every vertex of W_n is of odd degree. Thus we can take $g : E(W_n) \to D_4 \backslash \{\rho_0\}$ as any constant map from $E(W_n)$ to $\{\rho_2, \mu_1, \mu_2, \delta_1, \delta_2\}$. Since g is constant we can take f as any bijection from $E(W_n)$ to N_{2n}. Clearly this f and g will constitute a D_4-magic labeling for W_n.

Case (ii): n is even.
Suppose n is even define $f : E(W_n) \to N_{2n}$ as

$$f(ku_i) = i, \ i = 1, 2, \ldots, n,$$
$$f(u_i u_{i+1}) = n + i, \ 1 \leq i \leq n - 1,$$
$$f(u_1 u_n) = 2n.$$

Now we can define $g : E(W_n) \to D_4 \setminus \{\rho_0\}$ by labeling each spokes by μ_1 and all the outer edges by μ_2 and ρ_2 alternatively. Then W_n becomes D_4-magic with magic constant ρ_0.

This completes the proof of the theorem. □

Definition 78. *[26] The helm H_n is a graph obtained from a wheel W_n by attaching a pendant edge at each vertex of the n cycle.*

Theorem 79. *The helm graph H_n is D_4-magic.*

Proof. Let $\{k, u_i, v_i : i = 1, 2, \ldots, n\}$ be the vertex set of H_n, where k be the central vertex, $u_1, u_2, \ldots, u_n$ are the vertices of the cycle, $v_1, v_2, \ldots, v_n$ are the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$. The edge set of H_n is $E(H_n) = \{u_iu_{i+1}, ku_i, u_iv_i : i = 1, 2, \ldots, n,\ u_{n+1} = u_1\}$. Now consider the following two cases:

Case(i): n is odd.

Suppose that n is odd. Define f and g as follows: Let $g : E(G) \to D_4 \setminus \{\rho_0\}$ be defined as $g(ku_i) = \rho_2,\ 1 \leq i \leq n,\ g(u_ju_{j+1}) = \rho_1,\ 1 \leq j \leq n-1,\ g(u_1u_n) = \rho_1,\ g(u_kv_k) = \rho_2,\ 1 \leq k \leq n$. Now let $f : E(G) \to N_{2n+1}$ be any bijection. In this case, we can easily prove that f and g will determine a D_4-magic labeling for H_n, where n is odd.

Case(ii): n is even.

Let f be defined as above and define $g : E(G) \to D_4 \setminus \{\rho_0\}$ by

$$g(u_iv_i) = \rho_2,\ 1 \leq i \leq n,\ g(u_1u_n) = \rho_1,$$

$$g(ku_j) = \begin{cases} \rho_2, & \text{if } i \leq j \leq n-2, \\ \rho_1, & \text{if } j = n-1, n. \end{cases},$$

$$g(u_ku_{k+1}) = \begin{cases} \rho_1, & \text{if } 1 \leq k \leq n-2, \\ \rho_2, & \text{if } k = n-1. \end{cases}$$

It follows that $l^*(u) = \rho_2, \forall u \in V(G)$. Hence H_n is D_4-magic when n is even.

This completes the proof of the theorem. □

Definition 80. *[2] The web graph $W(2, n)$ is a graph obtained joining the pendant points of a helm to form a cycle and adding a single pendant edge to each vertex of this outer graph.*

Theorem 81. *The web graph $W(2, n), n \geq 3$ is D_4-magic.*

Proof. Let $\{k, u_i, v_i, w_i : i = 1, 2, 3, \ldots, n\}$ be the vertex set of $W(2, n)$, where k be the central vertex, $u_1, u_2, u_3, \ldots, u_n$ are the vertices of inner cycle, $v_1, v_2, v_3, \ldots, v_n$ are the vertices of outer cycle and $w_1, w_2, w_3, \ldots, w_n$ are the pendant vertices adjacent to $v_1, v_2, v_3, \ldots, v_n$ of $W(2,n)$. Let $E(W(2,n)) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_i, v_iw_i, ku_i : i = 1, 2, \ldots, n \text{ and } u_{n+1} = u_1, v_{n+1} = v_1\}$. We define a D_4-magic labeling for $W(2,n)$ with magic constant ρ_2 as follows:

Case (i): n is odd.

Let $f : E(G) \to N_{3n+1}$ be any bijection.

Define $g : E(G) \to D_4 \setminus \{\rho_0\}$ as

$$\begin{aligned} g(ku_i) &= \rho_2 = g(u_iv_i) = g(v_iw_i),\ 1 \le i \le n, \\ g(u_iu_{i+1}) &= \rho_1 = g(v_iv_{i+1}),\ 1 \le i \le n-1, \\ g(u_1u_n) &= \rho_1 = g(v_1v_n). \end{aligned}$$

Case (ii): n is even.

Let $f : E(G) \to N_{3n+1}$ be any bijection. Define $g : E(G) \to D_4 \setminus \{\rho_0\}$ as $g(ku_i) = \rho_2,\ 1 \le i < n-1,\ g(ku_n) = g(ku_{n-1}) = \rho_1,\ \ g(v_iv_{i+1}) = \rho_1 = g(u_iu_{i+1}) = \rho_1,\ 1 \le i < n-1,\ g(v_iw_i) = \rho_2 = g(u_iv_i),\ 1 \le i \le n,\ g(v_1v_n) = g(v_{n-1}v_n) = \rho_1,\ g(u_{n-1}u_n) = \rho_2,\ g(u_1u_n) = \rho_1$.

This completes the proof of the theorem. □

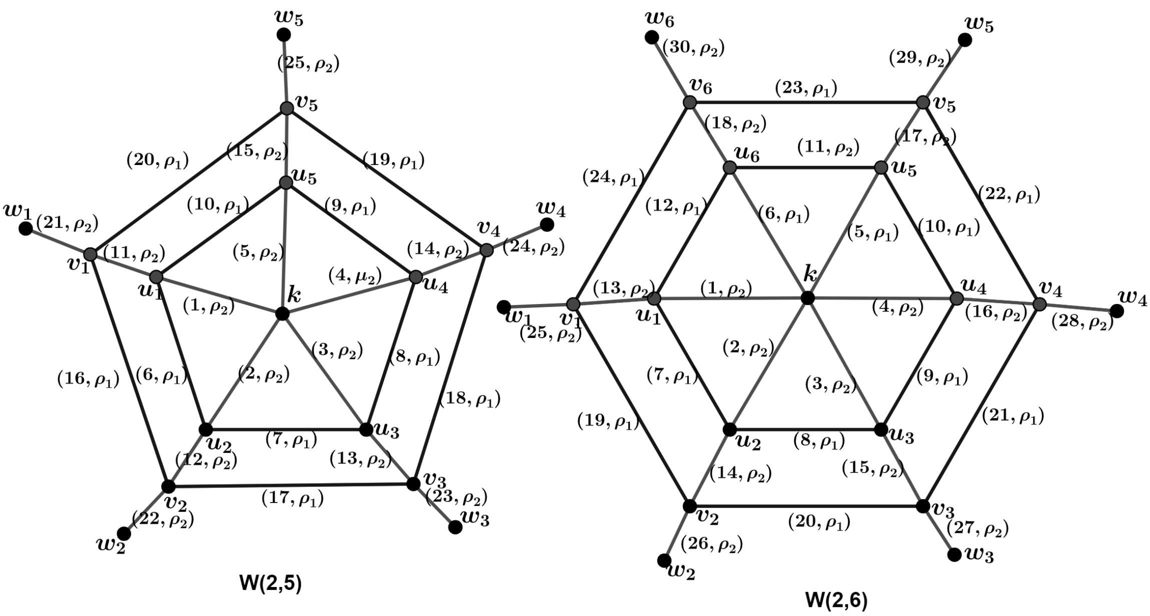

Figure 3.2: D_4-magic labeling of $W(2,5)$ and $W(2,6)$.

The Figure 3.2 represents a D_4 magic labeling of the web graphs $W(2,5)$ and $W(2,6)$.

Theorem 82. *Shell graphs $S_{n,n-3}$ are D_4-magic.*

Proof. Let us denote the vertices of the shell graph $S_{n,n-3}$ by $u_1, u_2, \ldots, u_n$ such that u_i is adjacent to u_{i+1}, where $i = 1, 2, \ldots, n$ and $u_{n+1} = u_1$. Without loss of generality, let the apex be u_1. Now consider the following cases:

Case (i): n is even.

We will define the map $f : E(S_{n,n-3}) \to N_{2n-3}$ as

$$\begin{aligned} f(u_i u_{i+1}) &= i,\ 1 \le i \le n-1, \\ f(u_n u_1) &= n, \\ f(u_1 u_j) &= n + (j-2),\ 3 \le j \le n-1 \end{aligned}$$

and we define $g : E(S_{n,n-3}) \to D_4 \setminus \{\rho_0\}$ as

$$\begin{aligned} g(u_1 u_2) = g(u_n u_1) &= \rho_2, \\ g(u_1 u_i) &= \mu_1,\ i \ne 2, n, \\ g(u_i u_{i+1}) &= \mu_2,\ 2 \le i \le n-1. \end{aligned}$$

Clearly, f and g define a D_4-magic labeling with magic constant μ_1.

Case (ii): n is odd.

Define f as

$$\begin{aligned} f(u_i u_{i+1}) &= i,\ 1 \le i \le n-1, \\ f(u_1 u_n) &= n, \\ f(u_1 u_j) &= n + (j-2),\ j \ne 2, n-1 \end{aligned}$$

and define g as

$$\begin{aligned} g(u_1 u_2) = g(u_1 u_n) &= \rho_2, \\ g(u_1 u_j) &= \mu_1,\ 3 \le j \le n-1, \\ g(u_i u_{i+1}) &= \begin{cases} \rho_2, & \text{if } i \text{ is even}, \\ \mu_2, & \text{if } i \text{ is odd}, 1 < i \le n-1. \end{cases} \end{aligned}$$

Obviously, the functions f and g define a D_4-magic labeling of $S_{n,n-3}$ with magic constant ρ_0.

This completes the proof of the theorem. □

A D_4-magic labeling of the shell graphs $S_{8,5}$ and $S_{7,3}$ are shown in the Figure 3.3.

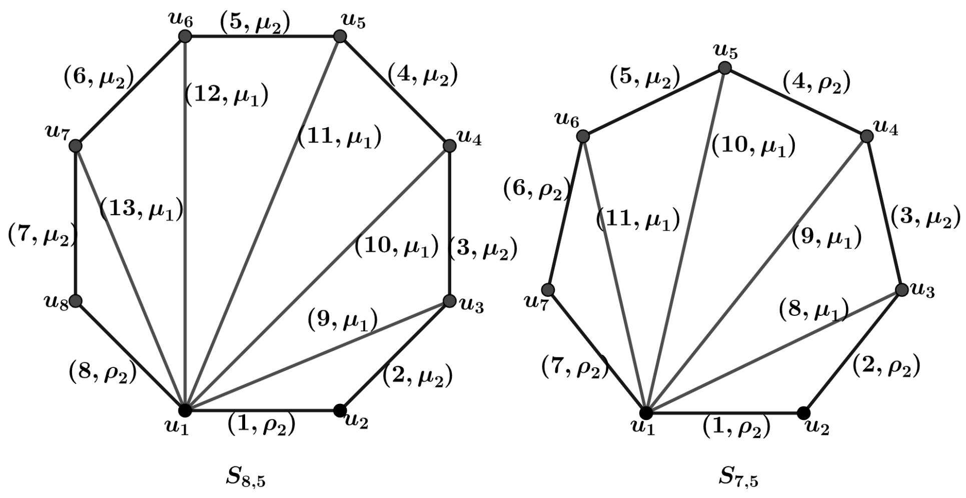

Figure 3.3: D_4-magic labeling of $S_{8,5}$ and $S_{7,4}$.

Theorem 83. *The graph n-gon book of k pages $B(n, k)$ is D_4-magic.*

Proof. Let G be the graph $B(n, k)$. Denote the vertices of common edge by k_1 and k_n and the edges of i^{th} page other than k_1 and k_n by $u_{i2}, u_{i3}, \ldots, u_{in-1}$ such that u_{i2} is adjacent to k_1 and u_{in-1} adjacent to k_n and u_{ij} adjacent to u_{ij+1} for all $2 \leq j < n - 1$. Consider the following cases:

Case (i): k is even.

Define $g : E(G) \to D_4 \setminus \{\rho_0\}$ as

$$
\begin{aligned}
g(k_1 k_n) &= \rho_2, \\
g(u_{1j} u_{1j+1}) &= \mu_1, \quad 2 \leq j \leq n - 2, \\
g(u_{1n-1} k_n) &= \mu_1 = g(k_1 u_{12}), \\
g(u_{ij} u_{ij+1}) &= \mu_2, \ \forall i \geq 2, \ 2 \ \leq j \leq n - 1, \\
g(k_1 u_{l2}) &= g(u_{ln-1} k_n) = \mu_2, \ 2 \leq l \leq k.
\end{aligned}
$$

Now define f as

$$
\begin{aligned}
f(k_1 k_n) &= 1, \ f(k_1 u_{12}) = 2, f(u_{1n-1} k_n) = n, \\
f(u_{1j} u_{1j+1}) &= j + 1, \ \forall \ 2 \leq j \leq n - 2, \\
f(k_1 u_{i2}) &= n + (i - 2)(n - 1) + 1, \ i \geq 2, \\
f(u_{ij} u_{ij+1}) &= n + (i - 2)(n - 1) + j, \ 2 \leq j \leq n - 2, \ 2 \leq i \leq k, \\
f(u_{i(n-1)} k_n) &= n + (i - 2)(n - 1) + (n - 1), \ 2 \leq i \leq k.
\end{aligned}
$$

The functions f and g determine a D_4-magic labeling with magic constant ρ_0.

Case (ii): k is odd.

Here define g as $g(e) = \rho_2$, $\forall e \in E(G)$ then g together with any bijection $f : E(G) \to N_{kn-1}$ will define a D_4-magic labeling of $B(n, k)$ with magic constant ρ_0.

This completes the proof of the theorem. □

Note that, for any $n \geq 3$ the path graph of order n is not D_4-magic. We observe that the cycle C_{2n} with a pendant edge is non-magic when we label the edges with the nonzero elements of an abelian group [4] and it is S_3-magic for all $n > 2$ [18]. Now we investigate whether the cycle graph C_n with a pendant edge is D_4-magic or not.

Theorem 84. *The cycle graph C_n with a pendant edge (Fl_n) is D_4-magic if and only if n is odd and $n > 3$.*

Proof. Let us denote the vertices of C_n by $u_1, u_2, \ldots, u_n$. Let one of the end vertex of the pendant edge is u_1 and denote the other vertex as u_p. Suppose that n is odd. Now define the maps f and g as follows:

$$f(u_1u_2) = 1, f(u_1u_n) = 2, f(u_1u_p) = n + 1,$$

$$f(u_iu_{i+1}) = \begin{cases} 2 + \frac{i}{2}, & \text{if } i \text{ is even and } 2 < i < n, \\ 1 + \frac{n+i}{2}, & \text{if } i \text{ is odd and } 2 < i < n, \end{cases} ,$$

$$g(u_iu_{i+1}) = \begin{cases} \mu_2, & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\ \rho_2, & \text{if } i \text{ is even and } 2 \leq i \leq n-1, \end{cases} \quad \text{and } g(u_1u_{n+1}) = \mu_1.$$

Clearly, the maps will determine a D_4-magic labeling of Fl_n with magic constant μ_1.

Now, suppose that n is even and Fl_n is D_4-magic with magic constant $a \in D_4$. Since Fl_n has a pendant vertex $a \neq \{\rho_0\}$. For our convenience let us call the sets $\{\rho_0, \rho_1, \rho_2, \rho_3\}$, $\{\mu_1, \mu_2\}, \{\delta_1, \delta_2\}$ as ρ-set, μ-set and δ-set respectively. Consider the following cases:

Case (i): $a = \rho_1$.

Suppose that the graph Fl_n is D_4 magic with magic constant ρ_1. Then $g(u_1u_p) = \rho_1$. If $g(u_1u_2) \in \rho$-set. From our assumption $\lambda^*(u_2) = \rho_1$ implies $g(u_1u_2) \neq \rho_0, \rho_1$. Without loss of generality, let $g(u_1u_2) = \rho_2$ then $\lambda^*(u_3) = \rho_1$ implies $g(u_2u_3) = \rho_3$. Proceeding like this, we obtain

$g(u_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is odd, } 1 \leq i \leq n, \\ \rho_3, & \text{if } i \text{ is even, } 1 \leq i \leq n, \end{cases}$, where $i+1$ is taken over modulo n. This mapping implies $\lambda^*(u_1) = \rho_1 * \rho_2 * \rho_3 = \rho_2$, which is a contradiction. So $a \neq \rho_1$.

If $g(u_1u_2) \in \mu$-set then $\lambda^*(u_i) = \rho_1$, $i \leq i \leq n$, implies

$$g(u_iu_{i+1}) \in \begin{cases} \mu\text{-set, if } i \text{ is odd and } 1 \leq i \leq n, \\ \delta\text{-set, if } i \text{ is even and } 1 \leq i \leq n. \end{cases}$$

Then $\lambda^*(u_1) = \prod g(u_1v)$ where v is adjacent to u_1. There are 24 possible product for $\lambda^*(u_1)$ which are listed below,

$\mu_1\rho_1\delta_1$	$\mu_1\rho_1\delta_2$	$\delta_1\rho_1\mu_1$	$\delta_2\rho_1\mu_1$
$\mu_2\rho_1\delta_1$	$\mu_2\rho_1\delta_2$	$\delta_1\rho_1\mu_2$	$\delta_2\rho_1\mu_2$
$\rho_1\delta_1\mu_1$	$\rho_1\delta_2\mu_1$	$\rho_1\delta_2\mu_1$	$\rho_1\delta_2\mu_1$
$\rho_1\delta_1\mu_2$	$\rho_1\delta_2\mu_2$	$\rho_1\delta_2\mu_2$	$\rho_1\delta_2\mu_2$
$\delta_1\mu_1\rho_1$	$\delta_2\mu_1\rho_1$	$\mu_1\delta_1\rho_1$	$\mu_1\delta_2\rho_1$
$\delta_1\mu_2\rho_1$	$\delta_2\mu_2\rho_1$	$\mu_2\delta_1\rho_1$	$\mu_2\delta_2\rho_1$

None of the above product yield the value ρ_1. So such a magic labeling is not possible for Fl_n. Hence Fl_n does not has a D_4-magic labeling with magic constant ρ_1, when n is even. Similarly we can prove that $a \neq \rho_3$.

Case(ii): $a = \rho_2$.

Suppose that $a = \rho_2$ and $g(u_1u_2) \in \rho$-set then $g(u_1u_2)$ is either ρ_1 or ρ_3 and $g(u_1u_p) = \rho_2$. Without loss of generality, $g(u_1u_2) = \rho_1$. Since $\lambda^*(u_i) = \rho_2$, $\forall i$ we have $g(u_iu_{i+1}) = \rho_1, 1 \leq i \leq n$ and then $\lambda^*(u_1) = \rho_2\rho_1^2 = \rho_0$, which is a contradiction. So $a \neq \{\rho_2\}$ when $g(u_1u_2) \in \rho$-set. Now suppose that $g(u_1u_2) \in \mu$-set or δ-set. Observe that $\mu_1\mu_2 = \mu_2\mu_1 = \rho_2$ and $\delta_1\delta_2 = \delta_2\delta_1 = \rho_2$. If we take $g(u_1u_2) = \mu_1$ then

$$g(u_iu_{i+1}) = \begin{cases} \mu_1, & \text{if } i \text{ is odd,} \\ \mu_2, & \text{if } i \text{ is even.} \end{cases}$$

But this imply that $\lambda^*(u_1) = \rho_2 * \mu_1 * \mu_2 = \rho_0$. Which is a contradiction. Similarly, we can show that $g(u_1u_2) \neq \mu_2, \delta_1, \delta_2$. Hence $a \neq \rho_2$.

Case(iii): $a \in \mu$-set or δ-set.

If $a \in \mu$-set then $g(u_1u_p) \in \mu$-set. If possible let $g(u_1u_2) \in \rho$-set. Since

$\lambda^*(u_i) \in \mu$-set for all $2 \leq i \leq n$ we get,

for $1 \leq i \leq n$, $g(u_iu_{i+1}) \in \begin{cases} \rho\text{-set, if } i \text{ is odd,} \\ \mu\text{-set or } \delta\text{-set, if } i \text{ is even.} \end{cases}$

This labeling implies $\lambda^*(u_1)$ is a product of an element from ρ-set, an element from μ-set and an element from δ-set(or μ-set) but this product always belongs to ρ-set. Which is a contradiction to our assumption. In a similar manner, we can show that $g(u_1u_2) \notin \mu$-set and δ-set. Hence $a \notin \mu$-set. Similarly, we can prove that $a \notin \delta$-set.

All the above cases illustrate that there does not exist a D_4-magic labeling for the graph Fl_n when n is even. □

3.4 Path Generated Graphs

In this section, we consider some graphs that are constructed from the path graph. We start with the splitting graph [21] of the path graph.

Theorem 85. *Splitting graph of the path graph $P_n, n \geq 3$ is D_4-magic.*

Proof. Let P_n be a path graph of order n, where $n \geq 3$. Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n, where $u_iu_{i+1} \in E(P_n), i = 1, 2, \ldots, n-1$. There are $2n$ vertices and $3n - 3$ edges in the splitting graph of the path, $S(P_n)$. Let u_{n+i} be the vertex corresponding to the i^{th} vertex in $S(P_n)$. Observe that, there are two pendant edges in $S(P_n)$, one with end points u_2 and u_{n+1} and the other with end points u_{n-1} and u_{2n}.

Case (i): $n = 3$.

In this case, define $f : E(S(P_3)) \to N_6$ as
$f(u_1u_2) = 1,\ f(u_2u_3) = 3,\ f(u_3u_5) = 2,\ f(u_1u_5) = 4,\ f(u_2u_4) = 5,\ f(u_2u_6) = 6$. Now define $g : E(G) \to D_4 \setminus \{\rho_0\}$ as
$g(u_1u_2) = g(u_3u_5) = \rho_1,\ g(u_2u_3) = g(u_1u_5) = \delta_2,\ g(u_2u_4) = g(u_2u_6) = \mu_1$.

Case (ii): $n > 3$.

In this case, define f and g as follows:

$$\begin{aligned}
f(u_iu_{i+1}) &= i,\ 1 \leq i \leq n-1,\\
f(u_iu_{n+(i-1)}) &= n+(i-2),\ 2 \leq i \leq n,\\
f(u_iu_{n+(i+1)}) &= (2n-2)+i,\ 1 \leq i \leq n-1 \text{ and}\\
g(u_1u_2) &= \rho_2,\ g(u_{n-1}u_n) = \mu_2,\\
g(u_2u_{n+1}) &= g(u_{n-1}u_{2n}) = \mu_1,\\
g(u_iu_{i+1}) &= \mu_1,\ 2 \leq i < n-1,\\
g(u_iu_{n+(i-1)}) &= \rho_2,\ 3 \leq i \leq n,\\
g(u_iu_{n+(i+1)}) &= \mu_2,\ 1 \leq i \leq n-2.
\end{aligned}$$

In all the above cases, we can prove that the functions f and g defines a D_4-magic labeling of $S(P_n)$ with magic constant μ_1.

This completes the proof of the theorem. □

The Figure 3.4 represents a D_4-magic labeling of the splitting graph of the path P_5.

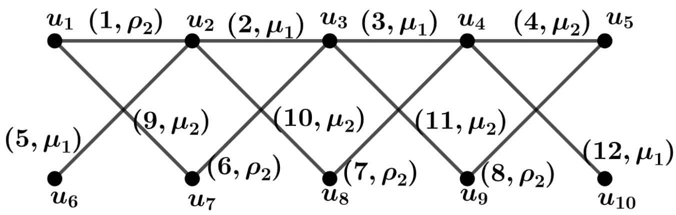

Figure 3.4: D_4-magic labeling of $S(P_5)$.

Definition 86. *[27] The middle graph of a connected graph G denoted by $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if*

(i) They are adjacent edges of G or

(ii) One is a vertex of G and the other is an edge incident with it.

Theorem 87. *Middle graph of the path graph P_n is D_4-magic.*

Proof. Let $M(P_n)$ be the middle graph of the path P_n. Denote the vertices of P_n by $u_1, u_2, \ldots, u_n$ and edges by $e_1, e_2, \ldots, e_{n-1}$ where e_i incident with u_i and u_{i+1}. There are $2n-1$ vertices and $3n-4$ edges in $M(P_n)$. Consider the following cases:

Case(i): $n = 3$.

Define $f : E(M(P_3)) \to N_{3n-4}$ as $f(e_1u_1) = 1,\ f(e_1u_2) = 2,\ f(e_1e_2) = 3,\ f(e_2u_2) = 4$ and $f(e_2u_3) = 5$ and define $g : E(M(P_3)) \to D_4 \setminus \{\rho_0\}$ as $g(e_1u_1) = \rho_2 = g(e_2u_3),\ \ g(e_1u_2) = \rho_1 = g(e_2u_2),\ \ g(e_1e_2) = \rho_3$. Then the middle graph of the path P_3 is D_4-magic with magic constant ρ_2.

Case(ii): $n > 3$.

Define $f : E(M(P_n)) \to N_{3n-4}$ as follows:
For $1 \leq i \leq n-1,\ 1 \leq j \leq n,\ f(e_iu_j) = 2(i-1) + j$ and $f(e_ie_{i+1}) = 3i$.
Now define $g : E(M(P_n)) \to D_4 \setminus \{\rho_0\}$ by

$$
\begin{aligned}
g(e_1u_1) &= \rho_2 = g(e_{n-1}u_n),\\
g(e_1u_2) &= \mu_2,\ g(e_2u_2) = \mu_1,\ g(e_2e_3) = \rho_3,\\
g(e_2u_3) &= \rho_1 = g(e_3u_3),\\
g(e_ie_{i+1}) &= \mu_2,\ \text{where } i \neq 2 \text{ and } 1 \leq i \leq n-1,\\
g(e_iu_j) &= \begin{cases} \mu_1, & \text{if } \ j = i+1 \text{ and } 3 \leq i < n-1,\\ \mu_2, & \text{if } \ i = j \text{ and } 4 \leq i \leq n-1. \end{cases}
\end{aligned}
$$

The above functions f and g will define a D_4-magic labeling of $M(P_n)$ with magic constant ρ_2.

This completes the proof of the theorem. □

Definition 88. *[28] A triangular snake T_n is obtained from the path P_n by replacing each edge of the path by a triangle C_3.*

Theorem 89. *The triangular snake T_n is D_4-magic.*

Proof. Note that every vertex of T_n has even degree . So the proof is indisputable from Theorem 67. □

Definition 90. *[28] The alternate triangular snake $A(T_n)$ is obtained from the path $u_1, u_2, \ldots, u_n$ by joining u_iu_{i+1}(alternatively) to a new vertex v_i.*

Theorem 91. *The alternate triangular graph $A(T_n)$ is D_4-magic.*

Proof. Let us denote the vertices of the path P_n be $u_1, u_2, \ldots, u_n$ and the vertex which join u_i and u_{i+1} be denoted by v_i. Now consider the following cases:

Case (i): n is even and triangle starts from u_1.

Suppose that n is even and the triangle starts from the first vertex u_i, then there are $n + \frac{n}{2}$ vertices and $2n - 1$ edges.

Suppose $n = 2$ then $A(T_n)$ is C_3 itself. So there is nothing to prove.
Suppose $n = 4$ then take f be any bijection from $E(A(T_n))$ to N_7 and define $g : E(A(T_n)) \to D_4 \setminus \{\rho_0\}$ by
$g(u_1v_1) = g(u_4v_3) = \rho_1,\ g(u_2u_3) = \rho_2,\ g(u_2v_1) = g(u_3v_3) = g(u_1u_2) = g(u_3u_4) = \rho_3$. Then $A(T_4)$ becomes D_4-magic with magic constant ρ_0.
Suppose $n > 4$, then let $f : E(A(T_n)) \to N_{2n-1}$ be any bijection and define $g : E(A(T_n)) \to D_4 \setminus \{\rho_0\}$ as

$$g(u_1u_2) = g(u_{n-1}u_n) = g(u_2v_1) = g(u_{n-1}v_{n-1}) = \rho_3,$$
$$g(u_1v_1) = g(u_nv_{n-1}) = \rho_1.\text{ For } 2 \leq i < n-1,\ \text{define}$$
$$g(u_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is even,} \\ \mu_2, & \text{if } i \text{ is odd.} \end{cases}$$
$$g(u_{2k+1}v_{2k+1}) = g(u_{2k+2}v_{2k+1}) = \mu_1, k = 1, 2, \ldots, \frac{n-4}{2}.$$

Obviously the functions f and g will constitute a D_4-magic labeling for $A(T_n)$ with $l^*(u) = \rho_0, \forall u \in V(A(T_n))$.

Case (ii): n is even and the triangle starts from the second vertex u_2.
We can define a magic labeling for $A(T_n)$, where n is even and cycle starts from u_2 as follows:
Let f be any bijection as above and define g as

$$g(u_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is odd,} \\ \rho_3, & \text{if } i \text{ is even,} \end{cases} \quad , 1 \leq i \leq n-1,$$
$$g(u_{2k}v_{2k}) = g(u_{2k+1}v_{2k}) = \rho_1,\ k = 1, 2, 3, \ldots, \frac{n-2}{2}.$$

Clearly l^* is a constant map, i.e., $l^*(u) = \rho_2,\ \forall u \in V(A(T_n))$.

Case (iii): n is odd and the triangle starts from the first vertex.
Suppose $n = 3$ and the triangle starts from the first vertex u_1.
Let $f : E(A(T_n)) \to N_4$ be any bijection.
Now define $g : E(A(T_n)) \to D_4 \setminus \{\rho_0\}$ by
$g(u_1u_2) = g(u_2v_1) = \delta_2,\ g(u_1v_1) = \delta_1,\ g(u_2u_3) = \rho_2$. Using these maps we can show that the graph is D_4-magic with magic constant ρ_2.
When n is odd and $n > 3$, there are $n + \frac{(n-1)}{2}$ vertices and $2(n-1)$ edges in $A(T_n)$. Suppose that $n > 3$, n is odd and the triangle of $A(T_n)$ starts from the first vertex u_1. Here we take f as any bijection and

$g : E(A(T_n)) \to D_4 \setminus \{\rho_0\}$ be defined as follows:

$$g(u_1u_2) = g(u_2v_1) = \delta_2,\ g(u_1v_1) = \delta_1,$$

$$g(u_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is even,} \\ \rho_3, & \text{if } i \text{ is odd,} \end{cases}, 1 < i < n.$$

$$g(u_iv_i) = g(u_{i+1}v_i) = \rho_1, 1 < i < n \text{ and } i \text{ is odd.}$$

Then clearly $l^*(u) = \rho_2,\ \forall u \in V(A(T_n))$.

Case (iv): n is odd and triangle starts from the second vertex.

$A(T_n)$ with n odd and the triangle starts from the first vertex is just the mirror image of the $A(T_n)$ in Case (iii). So we can define f and g similarly as in Case(iii) and obtain a D_4-magic labeling for $A(T_n)$ with magic constant ρ_2.

This completes the proof of the theorem. □

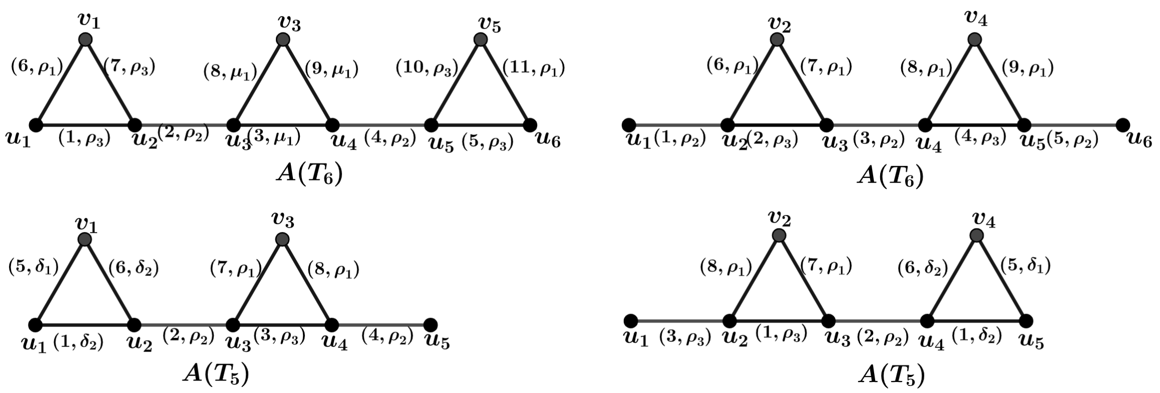

Figure 3.5: D_4-magic labeling of $A(T_6)$ and $A(T_5)$.

The Figure 3.5 represents a D_4 magic labeling of the graphs $A(T_6)$ and $A(T_5)$.

Definition 92. *[28] A double triangular snake $D(T_n)$ consists of two triangular snakes that have a common path P_n.*

Theorem 93. *The double triangular graph is D_4-magic.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n and let $v_1, v_2, \ldots, v_{n-1}$, $w_1, w_2, \ldots, w_{n-1}$ be the remaining vertices of $D(T_n)$ such that the vertex v_i is adjacent to u_i and u_{i+1}, where $1 \leq i < n$. Similarly the vertex w_i is adjacent to u_i and u_{i+1}. Without loss of generality, let $v_1, v_2, \ldots, v_{n-1}$ and $w_1, w_2, \ldots, w_{n-1}$

be the vertices of upper triangles and lower triangles respectively.
Now we define a D_4-magic labeling for $D(T_n)$ as follows:
Let $f : E(D(T_n)) \rightarrow N_{5(n-1)}$ be any bijection and let $g : E(D(T_n)) \rightarrow D_4 \setminus \{\rho_0\}$ be defined by $g(u_iu_{i+1}) = \rho_2,\ 1 \leq i < n,\ g(u_iv_i) = g(u_{i+1}v_i) = \rho_1,$ and $g(u_iw_i) = g(u_{i+1}w_i) = \rho_3,\ 1 \leq i < n$. Thus we can see that $l^*(u) = \rho_2,\ \forall u \in V(D(T_n))$. This completes the proof. $\square$

Chapter 4

Q_8-Magic Labeling of Graphs

In this chapter, we discuss the Q_8-magic labeling of graphs. The first section of this chapter gives an introduction about Q_8-magic labeling and the second section of this chapter deals with the Q_8-magic labeling of some graphs and its subdivision graphs. Also, this chapter classifies these graphs according to the magic constant.

4.1 Introduction

Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the Quaternion group with identity element 1. A graph $G = (V(G), E(G))$ with p vertices and q edges is said to be Q_8-magic if there exist two maps $f : E(G) \to N_q$ and $g : E(G) \to Q_8 \setminus \{1\}$ such that the map f is bijective and the map $\ell^*(v) : V(G) \to Q_8$ defined by $\ell^*(u) = \prod_{e \in N^*(u)}((f(e), g(e))$ is a constant map, where $N^*(u)$ is the set of all edges incident with u. The map ℓ^* is called a Q_8-magic labeling of G. Recall the definition of A-magic labeling of graphs where A is a non-abelian group [29]. In this chapter, we consider the Quaternion group [16] $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, which is a non-abelian group of order 8. The Cayley table for Q_8 is given by

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

[1]This chapter has been published in the journal Advances and Applications of Discrete Mathematics, Volume 34, 67-85, (2022).

4.2 Q_8-Magic Labeling

Definition 94. *A graph G is Q_8-magic if there exist two maps $f : E(G) \to N_q$ and $g : E(G) \to Q_8 \setminus \{1\}$ as in the definition 34 such that the map $\ell^* : V(G) \to Q_8$ defined by $\ell^*(u) = \prod_{e \in N^*(u)}(f(e), g(e))$ is a constant map, where $N^*(u)$ is the set of all edges incident with u.*

Example 95. *Consider the complete graph K_4. Let u_1, u_2, u_3, u_4 be the vertices of K_4. Define $f : E(K_4) \to N_6$ and $g : E(K_4) \to Q_8 \setminus \{1\}$ as follows:*

$$f(u_1u_2) = 1,\ f(u_2u_3) = 3,\ f(u_3u_4) = 2,$$
$$f(u_1u_4) = 4,\ f(u_1u_3) = 5,\ f(u_2u_4) = 6,$$
$$g(u_1u_2) = g(u_3u_4) = i,\ g(u_2u_3) = g(u_1u_4) = j,\ g(u_1u_3) = g(u_2u_4) = k.$$

$$\text{Now } \ell^*(u_1) = \prod_{e \in N^*(u_1)} (f(e), g(e))$$
$$= (f(u_1u_2), g(u_1u_2)) * (f(u_1u_4), g(u_1u_4)) * (f(u_1u_3), g(u_1u_3))$$
$$= (1, i) * (4, j) * (5, k) = ijk = -1.$$

Similarly, $\ell^*(u_2) = \ell^*(u_3) = \ell^*(u_4) = -1$. Therefore, K_4 is Q_8-magic with magic constant -1.

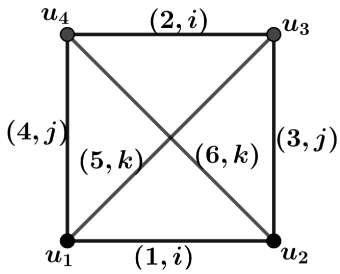

Figure 4.1: Q_8-magic labeling of K_4.

Observe that the cycle C_{2n} with a pendant edge is non-magic when we label the edges with the nonzero elements of an abelian group. Now we consider the graph C_n with a pendant edge and investigate whether it admits a Q_8-magic labeling or not.

Theorem 96. *The cycle graph C_n with a pendant edge is Q_8-magic if and only if n is odd.*

Proof. Let Fl_n be the cycle graph C_n with a pendant edge. Let us denote the vertices of C_n by $u_1, u_2, \ldots, u_n$. Let one of the end vertex of the pendant edge is u_1 and denote the other vertex as u_p. Suppose that n is odd. Now define the maps f and g as follows:

$$f(u_m u_{m+1}) = \begin{cases} \frac{m+1}{2}, & \text{if } m \text{ is odd, } 1 \leq m \leq n-2, \\ \frac{n+m-1}{2}, & \text{if } m \text{ is even, } 1 \leq m \leq n-1. \end{cases}, f(u_n u_1) = n,$$

$f(u_1u_p) = n+1$ and $g(u_1u_p) = i$, $g(u_nu_1) = -j$, $g(u_mu_{m+1}) = \begin{cases} j, & \text{if } m \text{ is odd,} \\ k, & \text{if } m \text{ is even.} \end{cases}$,

where $1 \leq m \leq n-1$,

Clearly the above maps will determine a Q_8-magic labeling for Fl_n with magic constant i.

Suppose that Fl_n is Q_8-magic with magic constant $a, a \in Q_8$. Since Fl_n has a pendant vertex $a \neq 1$. Suppose that $a = -1$, then $g(u_1u_p) = -1$. By our assumption $\lambda^*(u_m) = -1, 2 \leq m \leq n$, so we must have $g(u_mu_{m+1}) = b, b \in \{\pm i, \pm j, \pm k\}$. Then $\lambda^*(u_1) = g(u_1u_p) * g(u_1u_n) * g(u_1u_2) = b^2 * -1 = 1$, which is a contradiction. Hence $a \neq -1$.

Now suppose that n is even and Fl_n is Q_8-magic with magic constant $a, a \in \{\pm i, \pm j, \pm k\}$. Without loss of generality, let $f(u_1u_p < f(u_1u_2) < f(u_2u_3) < f(u_1u_n)$. Suppose that $g(u_1u_2) = b$ and $g(u_2u_3) = c$, where $a = b * c,\ b, c \in Q_8 \setminus \{1\}$. Then $\lambda^*(u_m) = a,\ 2 \leq i \leq m$ implies $g(u_mu_{m+1}) = \begin{cases} \pm b, & \text{if } m \text{ is odd}, 3 \leq i \leq n, \\ \pm c, & \text{if } m \text{ is even}, 3 \leq i \leq n. \end{cases}$ Then $\lambda^*(u_1) = a * b * \pm c = \pm 1$, which is a contradiction. Hence $a, a \notin \{\pm i, \pm j, \pm k\}$ when n is even. So Fl_n does not have a Q_8-magic labeling when n is even. This completes the proof of the theorem. □

Theorem 97. *The wheel graph W_n is Q_8-magic for all $n \geq 3$.*

Proof. Let us $V(W_n) = \{u_1, u_2, \dots, u_n, k\}$ and $E(W_n) = \{u_mu_{m+1}, ku_m, 1 \leq m \leq n, m+1 \text{ is taken modulo } n\}$. Consider the following two cases:

Case(i): n is even.

Let $f : E(W_n) \to N_{2n}$ be any bijective function and let $g :\to Q_8 \setminus \{1\} \to$ be defined by $g(e) = \begin{cases} i, & \text{if } e = u_mu_{m+1}, 1 \leq m \leq n, \\ & \quad m+1 \text{ is taken over modulo } n, \\ -1, & \text{otherwise.} \end{cases}$

Case(ii): n is odd.

Let f as above and g be the constant function $g(e) = -1, \forall e \in E(W_n)$.

Then in both cases, we can easily verify that f and g determine a Q_8-magic labeling for W_n. Hence the proof. □

The subdivision graph [30] of a graph G is denoted by $\mathbb{S}(G)$ and is obtained by inserting an additional vertex to each edge of G. Now we investigate Q_8-magic labeling of some graphs and its subdivision graphs. Also we classify them into the following 7 categories:

$$
\begin{aligned}
\omega_a &:= \text{ the class of all } Q_8\text{-magic graphs with magic constant } a, \\
&\quad \text{ where } a \in \mathbb{B} = \{\pm i, \pm j, \pm k\} \\
\omega_1 &:= \text{ the class of all } Q_8\text{-magic graphs with magic constant } 1. \\
\omega_{-1} &:= \text{ the class of all } Q_8\text{-magic graphs with magic constant } -1. \\
\omega_{1,-1} &:= \omega_1 \cap \omega_{-1} \\
\omega_{a,1} &:= \omega_a \cap \omega_1, a \in \mathbb{B} \\
\omega_{a,-1} &:= \omega_a \cap \omega_{-1}, a \in \mathbb{B} \\
\Omega &:= \omega_a \cap \omega_1 \cap \omega_{-1}, \text{ for all } a \in \mathbb{B}
\end{aligned}
$$

4.3 Q_8-magic labeling of some graphs and its subdivision graphs

Theorem 98. *If a graph G has a pendant edge then $G \notin \omega_1$.*

Proof. Let G be a graph having a pendant edge e with end vertices u_1 and u_2. Without loss of generality, assume that degree of u_1 is one. Suppose $G \in \omega_1$ then $\ell^*(u_1) = 1 = \prod(f(u_1u_2), g(u_1u_2)) = g(u_1u_2)$, which is a contradiction to the definition 94. Hence $G \notin \omega_1$. □

Theorem 99. *The star graph $K_{1,n}, n \geq 1$ is Q_8-magic if and only if n is odd.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of $K_{1,n}$ having degree 1 and let v be the vertex of $K_{1,n}$ having degree n. Suppose that n is odd. Let f be a bijective map from $E(K_{1,n})$ to N_n. Define $g : E(K_{1,n}) \to Q_8 \setminus \{1\}$ as $g(e) = -1, \ \forall e \in E(G)$. Then $\ell^*(u) = -1, \ \forall u \in V(G)$. Hence $K_{1,n}$ is Q_8-magic.
Conversely, suppose that $K_{1,n}$ is Q_8-magic with the magic constant 'a', where $a \in Q_8$. There are n pendant edges and each pendant edge should be mapped to a under the map g. Then $\ell^*(v) = a$ (by our assumption). i.e., $\underbrace{aa\cdots a}_{n\,\text{times}} = a$. Which implies either n is odd or $n \equiv 1(\text{mod } 4)$. If n is odd then we can take a as -1. If $n \equiv 1(\text{mod } 4)$ then we can take a as $\pm i$, $\pm j$ or $\pm k$. But we observe that $n \equiv 1(\text{mod } 4)$ also implies n is odd. This completes the proof of the theorem. □

Corollary 100. *The star graph* $K_{1,n} \in \omega_{a,-1}$ *if and only if* $n \equiv 1(\ mod\ 4)$.

Corollary 101. $K_{1,n} \notin \Omega$, *for all* $n \geq 1$.

Theorem 102. *Subdivision graph of* $K_{1,n}$, $\mathbb{S}(K_{1,n}) \notin \omega_a$, *for all* $a \in \mathbb{B}$ *and for all* $n \geq 1$.

Proof. Let $u_1, u_2, \ldots, u_n$ be the pendant vertices of $K_{1,n}$ and let v be the vertex having degree n. Now let $u_1^{'}, u_2^{'}, \ldots, u_n^{'}$ be the vertices of $\mathbb{S}(K_{1,n})$ corresponding to the edges $vu_1, vu_2, \ldots, vu_n$ of $K_{1,n}$. Suppose that $\mathbb{S}(K_{1,n}) \in \omega_a$, for some $a \in \mathbb{B}$. Then all pendant edge should be mapped to 'a' in the map g. In particular $g(u_1u_1^{'}) = a$. Note that

$$\begin{aligned} \ell^*(u_1^{'}) &= (f(vu_1^{'}), g(vu_1^{'}))(f(u_1u_1^{'}), g(u_1u_1^{'})) \\ &= g(vu_1^{'}) * g(u_1u_1^{'}) \text{ or } g(u_1u_1^{'}) * g(vu_1^{'}) \end{aligned}$$

This implies that $\ell^*(u_1^{'}) = g(vu_1^{'}) * a$ or $a * g(vu_1^{'})$.

Now $\ell^*(u_1^{'}) = a$ implies $g(vu_1^{'}) = 1$, which is a contradiction. This completes the proof of the theorem.

□

Theorem 103. *For all* $n \geq 1$, $\mathbb{S}(K_{1,n}) \notin \omega_{1,-1}$.

Proof. Since $\mathbb{S}(K_{1,n})$ has pendant edges by Theorem 98 $\mathbb{S}(K_{1,n}) \notin \omega_1$. We can prove $\mathbb{S}(K_{1,n}) \notin \omega_{-1}$ as in the above theorem. Hence the proof. □

Corollary 104. $\mathbb{S}(K_{1,n}) \notin \Omega$, *for all* $n \geq 1$.

Theorem 105. *For* $n \geq 3$, $C_n \in \omega_a$, *for all* $a \in \mathbb{B}$ *if and only if* n *is even.*

Proof. Let us denote the vertices of C_n by $u_1, u_2, \ldots, u_n, u_{n+1}$, where $u_{n+1} = u_1$. Suppose that n is even. Let $f : E(C_n) \to N_n$ and $g : E(C_n) \to Q_8 \setminus \{1\}$ be defined as follows:

$$f(u_pu_{p+1}) = \begin{cases} \frac{p+1}{2}, & \text{if } p \text{ is odd,} \\ \frac{n+p}{2}, & \text{if } p \text{ is even.} \end{cases}$$

$$g(u_pu_{p+1}) = \begin{cases} i, & \text{if } p \text{ is odd,} \\ j, & \text{if } p \text{ is even.} \end{cases} \quad , 1 \leq p \leq n.$$

Then clearly $\ell^*(u) = k$. Thus, $C_n \in \omega_k$. Similarly, we can prove that $C_n \in \omega_a$, for all $a \in \mathbb{B}$.

Suppose n is odd. Then if $C_n \in \omega_a$, where $a \in \mathbb{B}$. Let $a, b, c \in Q_8$ such that $bc = a$ also $b, c \neq 1$. Without loss of generality, let $f(u_1u_2) < f(u_2u_3)$. Let $g(u_1u_2) = b,\ g(u_2u_3) = c$ then $g(u_3u_4) = \pm b,\ g(u_4u_5) = \pm c,\ g(u_5u_6) = \pm b, \ldots, g(u_{n-1}u_n) = \pm c$. But $g(u_{n-1}u_n) = \pm c$ implies $g(u_nu_1) = \pm b$. Then we have $\ell^*(u_1) = g(u_nu_1) * g(u_1u_2)$ or $g(u_1u_2) * g(u_nu_1)$. Both cases imply that $\ell^*(u_1) = \pm 1$. Which is a contradiction to our assumption that the magic constant is a. Hence $C_n \notin \omega_a, \forall a \in \mathbb{B}$ when n is odd. This completes the proof of the theorem. □

Theorem 106. *For $n \geq 3,\ C_n \in \omega_{1,-1}$.*

Proof. Define $g : E(C_n) \to Q_8 \setminus \{1\}$ as the constant map $g(e) = -1, \forall e \in E(G)$ and let and $f : E(C_n) \to N_n$ be a bijective map. Then we obtain a Q_8-magic labeling of C_n with magic constant 1. Similarly, if we define g as the map $g(e) = i,\ \forall e \in E(C_n)$ and f as above we get $\ell^*(u) = -1,\ \forall u \in V(C_n)$. This completes the proof of the theorem. □

Theorem 107. *For $n \geq 3,\ \mathbb{S}(C_n) \in \Omega$.*

Proof. Observe that $\mathbb{S}(C_n) = C_{2n}$. So the theorem is indisputable from Theorem 105 and Theorem 106. □

Definition 108. *[31] A ladder graph L_n is defined by $L_n = P_n \times K_2$, where P_n is a path with n vertices, $\times$ denote the cartesian product and K_2 is a complete graph with two vertices.*

Theorem 109. *For $n \geq 2$ the ladder $L_n \in \Omega$.*

Proof. Consider the ladder graph L_n with vertex set $V = \{u_p, v_p : 1 \leq p \leq n\}$ and edge set $E = \{u_pu_{p+1}, v_pv_{p+1} : 1 \leq p \leq n-1\} \cup \{u_pv_p : 1 \leq p \leq n\}$.

First, we prove that $L_n \in \omega_1$. For this, let f be any bijection from $E(L_n)$ to N_{3n-2} and $g : E(L_n) \to Q_8 \setminus \{1\}$ be defined as follows:

$$\text{For } 1 \leq p \leq n,\ g(u_pv_p) = \begin{cases} i, \text{ if } p = 1, n, \\ -1, \text{ otherwise.} \end{cases} \quad \text{and}$$

$$g(u_pu_{p+1}) = g(v_pv_{p+1}) = -i, \text{ where } 1 \leq p \leq n-1.$$

Clearly, $\ell^*(u) = 1,\ \forall u \in V(L_n)$. Therefore $L_n \in \omega_1$.

Next we show that $L_n \in \omega_{-1}$. Take f as above and for $1 \leq p \leq n-1$ define g as follows:

$$g(u_pu_{p+1}) = g(v_pv_{p+1}) = \begin{cases} i, \text{ if } p \text{ is odd,} \\ -i, \text{ if } p \text{ is even.} \end{cases},$$

$$g(u_m v_m) = \begin{cases} i, \text{ if } m = 1, \\ -1, \text{ if } 1 < m \leq n-1, \\ -i, \text{ if } m = n \text{ and } n \text{ is odd}, \\ i, \text{ if } m = n \text{ and } n \text{ is even}. \end{cases}$$

Using the above f and g we can determine a Q_8-magic labeling of L_n with magic constant -1. Hence $L_n \in \omega_{-1}$.

Finally, we show that $L_n \in \omega_k$. To show this, define f and g as follows: $f(u_1v_1) = 1$, $f(u_pu_{p+1}) = p+1$, $f(v_pv_{p+1}) = n+p$, where $1 \leq p \leq n-1$ and $f(u_mv_m) = 2n-1+(m-1)$, $1 < m \leq n$. For $1 \leq p \leq n-1$ and for $1 \leq m \leq n$, define

$$g(u_pu_{p+1}) = g(v_pv_{p+1}) = \begin{cases} j, & \text{if } p \equiv 1 (\text{mod } 4), \\ i, & \text{if } p \equiv 2 (\text{mod } 4), \\ -j, & \text{if } p \equiv 3 (\text{mod } 4), \\ -i, & \text{if } p \equiv 0 (\text{mod } 4). \end{cases},$$

$$g(u_m v_m) = \begin{cases} i, & \text{if } m = 1, \\ -1, & \text{if } 1 < m < n-1, \\ j, & \text{if } m = n \text{ and } n \equiv 3 (\text{mod } 4), \\ -i, & \text{if } m = n \text{ and } n \equiv 2 (\text{mod } 4), \\ i, & \text{if } m = n \text{ and } n \equiv 0 (\text{mod } 4), \\ -j, & \text{if } m = n \text{ and } n \equiv 1 (\text{mod } 4). \end{cases}$$

Note that the above maps f and g will determine a Q_8-magic labeling of L_n with magic constant k. Similarly we can prove that $L_n \in \omega_a, \forall a \in \mathbb{B}$.
Hence $L_n \in \Omega$. This completes the proof of the theorem. □

Theorem 110. *The subdivision graph of L_n, $\mathbb{S}(L_n) \in \omega_{1,-1}$.*

Proof. Consider the ladder L_n with vertex set $V = \{u_p, v_p : 1 \leq p \leq n\}$ and edge set $E = \{u_pu_{p+1}, v_pv_{p+1} : 1 \leq p \leq n-1\} \cup \{u_pv_p : 1 \leq p \leq n\}$. Let $u_{p'}, v_{p'}, w_p$ be the vertices of $S(L_n)$ corresponding to the edges $u_pu_{p+1}, v_pv_{p+1}, u_pv_p$ respectively. Define $f : S(L_n) \to N_{6n-4}$ as follows: for $1 \leq p \leq n-1$,

$f(u_pu_{p'}) = 1+3(p-1),\ f(u_{p'}u_{p+1}) = 2+3(p-1),\ f(v_pv_{p'}) = (3n-2)+3(p-1),\ f(v_{p'}v_{p+1}) = 3(n+(p-1)),\ 1 \leq p \leq n-1,\ f(u_1w_1) = 6n-5,\ f(w_1v_1) = 6n-4$ and for $1 \leq m \leq n,\ f(u_mw_m) = 3(m-1),\ m \neq 1, f(w_mv_m) = (3n-1)+3(m-2),\ m \geq 2$. For $1 \leq p \leq n-1$ and for $1 \leq m \leq n$, define $g : E(G) \to Q_8 \setminus \{1\}$ as follows:

$$g(u_pu_{p'}) = \begin{cases} -i, & \text{if } p \text{ is odd,} \\ -k, & \text{if } p \text{ is even.} \end{cases}, g(v_pv_{p'}) = \begin{cases} i, & \text{if } p \text{ is odd,} \\ k, & \text{if } p \text{ is even.} \end{cases}$$

$$g(u_{p'}u_{p+1}) = \begin{cases} i, & \text{if } p \text{ is odd,} \\ k, & \text{if } p \text{ is even.} \end{cases}, g(v_{p'}v_{p+1}) = \begin{cases} -i, & \text{if } p \text{ is even,} \\ -k, & \text{if } p \text{ is even.} \end{cases}$$

$$g(u_mw_m) = \begin{cases} i, & \text{if } m = 1, \\ j, & \text{if } m \text{ is even and } m < n, \\ -j, & \text{if } m \text{ is odd and } m < n, \\ -k, & \text{if } m = n \text{ and } n \text{ is odd,} \\ -i, & \text{if } m = n \text{ and } n \text{ is even.} \end{cases}$$

$$g(w_mv_m) = \begin{cases} -i, & \text{if } m = 1, \\ -j, & \text{if } m \text{ is even and } m < n, \\ j, & \text{if } m \text{ is odd and } m < n, \\ k, & \text{if } m = n \text{ and } n \text{ is odd,} \\ i, & \text{if } m = n \text{ and } n \text{ is even.} \end{cases}$$

By defining f and g as above we get a Q_8-magic labeling of $\mathbb{S}(L_n)$ with magic constant 1. Therefore $\mathbb{S}(L_n) \in \omega_1$.

Now define g as

$$g(u_pu_{p'}) = g(u_{p'}u_{p+1}) = g(v_pv_{p'}) = g(v_{p'}v_{p+1}) = \begin{cases} i, & \text{if } p \text{ is odd, } 1 \leq p \leq n-1, \\ k, & \text{if } p \text{ is even, } 1 \leq p \leq n-1. \end{cases}$$

and for $1 \leq m \leq n$, $g(u_mw_m) = g(w_mv_m) = \begin{cases} i, & \text{if } m = 1, \\ j, & \text{if } m \text{ is even and } 1 < m \leq n-1, \\ -j, & \text{if } \text{ is odd and } 1 < m \leq n-1, \\ i, & \text{if } m = n \text{ and } n \text{ is even,} \\ k, & \text{if } m = n \text{ and } n \text{ is odd.} \end{cases}$

Also define $f : S(L_n) \to N_{6n-4}$ by for $1 \leq p \leq n-1,\ f(u_pu_{p'}) = 1+3(p-1),\ f(u_{p'}u_{p+1}) = 2+3(p-1),\ f(v_pv_{p'}) = (3n-2)+3(p-1),\ f(v_{p'}v_{p+1}) = (3n-1)+3(p-1),\ f(u_1w_1) = 6n-5,\ f(w_1v_1) = 6n-4$ and for $1 \leq m \leq n$,

$f(u_m w_m) = 3(m-1)$, $m \neq 1, f(w_m v_m) = 3(n+(m-2))$, $m \geq 2$. Then g and f will determine a Q_8-magic labeling of $S(L_n)$ with magic constant -1. That is $\mathbb{S}(L_n) \in \omega_{-1}$. Hence $\mathbb{S}(L_n) \in \omega_{1,-1}$. This completes the proof of the theorem. □

Theorem 111. *When n is even, $\mathbb{S}(L_n) \in \omega_a$ for all $a \in \mathbb{B}$.*

Proof. Suppose that n is even. Let V and E be the vertex set and edge set of $\mathbb{S}(L_n)$ and denote the vertices and edges as in Theorem 110. Consider the following cases:

Case (i): $n = 2$.

Suppose that $n = 2$ and $a \in \mathbb{B}$. Let f be any bijective map from $E(S(L_2))$ to N_8 and g be the map defined as follows: label the edges of $\mathbb{S}(L_2)$ alternatively by -1 and $-a$. Clearly $\ell^*(u) = a, \forall u \in V(\mathbb{S}(L_2))$. Thus $\mathbb{S}(L_2) \in \omega_a$, $\forall a \in \mathbb{B}$.

Case (ii): $n > 2$ and n is even.

First we show that, when n is even $\mathbb{S}(L_n) \in \omega_k$. In this case, we take $f : E(\mathbb{S}(L_n)) \rightarrow N_{6n-4}$ to be the map defined by

For $1 \leq p \leq n-1$, $f(u_p u_{p'}) = 1 + 2(p-1)$, $f(u_{p'} u_{p+1}) = 2 + 2(p-1)$, $f(v_p v_{p'}) = (2n-1) + 2(p-1)$, $f(v_{p'} v_{p+1}) = 2n + 2(p-1)$ and for $1 \leq m \leq n, f(u_m w_m) = 4n - 4 + m, f(w_m v_m) = 5n - 4 + m$. If $n \equiv 2 (\bmod\ 4)$ then define $f(u_n w_n) = 4n-4, f(w_n v_n) = 5n-4, f(v_{n-1'} v_n) = 6n-4$ and label other edges as above.

For $1 \leq p \leq n-1$ and $1 \leq m \leq n$, define $g : E(\mathbb{S}(L_n)) \rightarrow Q_8 \setminus \{1\}$ as follows:

$$g(u_pu_{p'}) = \begin{cases} -k, & \text{if } p = 1, \\ j, & \text{otherwise.} \end{cases}, g(u_p^{'}u_{p+1}) = \begin{cases} -1, & \text{if } p = 1, \\ -i, & \text{otherwise.} \end{cases}$$

$$, g(v_pv_{p'}) = \begin{cases} -1, & \text{if } p = 1, \\ i, & \text{if } p \equiv 1 (\text{mod } 4), p \neq 1, \\ j, & \text{if } p \equiv 2 (\text{mod } 4), \\ -i, & \text{if } p \equiv 3 (\text{mod } 4), \\ -j, & \text{if } p \equiv 0 (\text{mod } 4). \end{cases},$$

$$g(v_{p'}v_{p+1}) = \begin{cases} -k, & \text{if } p = 1, \\ j, & \text{if } p \equiv 1 (\text{mod } 4), p \neq 1, \\ -i, & \text{if } p \equiv 2 (\text{mod } 4), \\ -j, & \text{if } p \equiv 3 (\text{mod } 4), \\ i, & \text{if } p \equiv 0 (\text{mod } 4). \end{cases}$$

$$g(u_mw_m) = \begin{cases} i, & \text{if } m = 2, \\ -j, & \text{if } p = n, \\ -1, & \text{otherwise.} \end{cases}, g(w_mv_m) = \begin{cases} j, & \text{if } m = 2, \\ i, & \text{if } p = n, \\ -k, & \text{otherwise.} \end{cases}$$

Obviously, $\ell^*(u) = k, \forall u \in V(\mathbb{S}(L_n))$. Thus $\mathbb{S}(L_n) \in \omega_k$ when n is even. Similarly, we can prove that $S(L_n) \in \omega_a, \forall a \in \mathbb{B}$.

□

Corollary 112. *When n is even, $\mathbb{S}(L_n) \in \Omega$.*

Theorem 113. $\mathbb{S}(L_3) \notin \omega_k$.

Proof. Without loss of generality, suppose that $f(u_1w_1) < f(w_1v_1)$.
The possible map g for $S(L_3)$ to get the magic constant k is shown in Figure 4.2. Observe that in all the possible labelings we cannot find a suitable image for the edge $v_2^{'}v_3$ in g. Hence $S(L_3) \notin \omega_k$. □

Corollary 114. $\mathbb{S}(L_3) \notin \omega_a$*, for all* $a \in \mathbb{B}$.

Corollary 115. $\mathbb{S}(L_n) \notin \Omega$ *for all* n.

Definition 116. *[2] The friendship graph or the Dutch windmill graph denoted by $F_m(or D_3^{(m)} or C_3^t)$ is the graph obtained by taking m copies of C_3 with one vertex in common.*

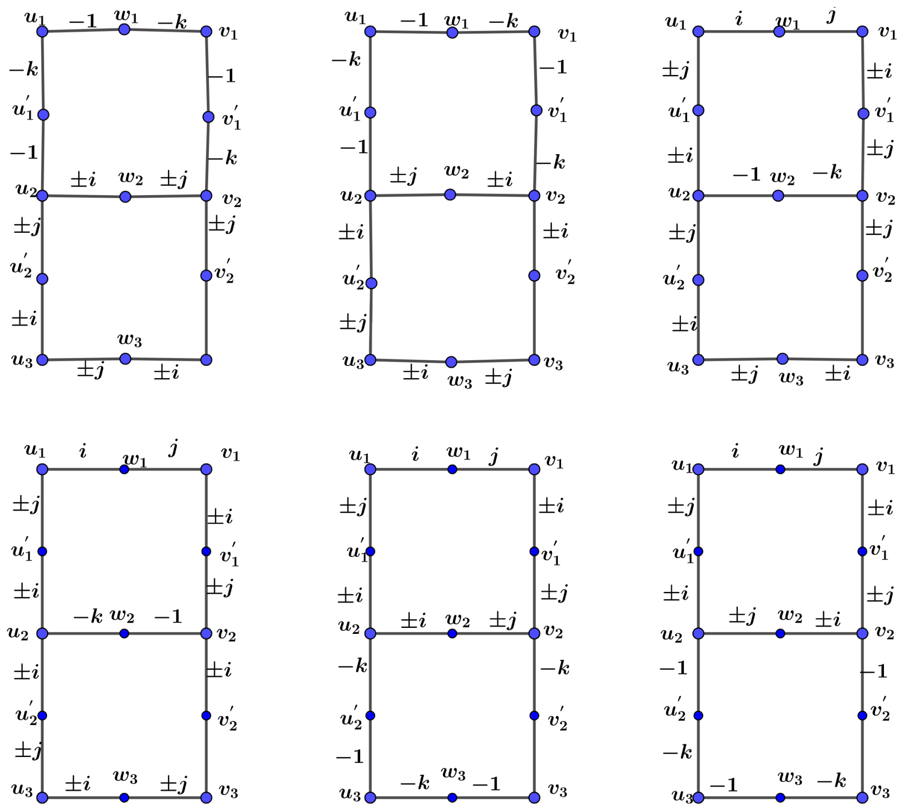

Figure 4.2: possible labelings for $\mathbb{S}(L_3)$ in the map g

Theorem 117. *The friendship graph $F_m \notin \omega_a$, for all $a \in \mathbb{B}$.*

Proof. Let F_m be the friendship graph with vertex set $\{a_p, b_p, w : 1 \leq p \leq m\}$ and edge set $\{a_p b_p, wa_p, wb_p\}$. Suppose that $F_m \in \omega_a$, for some $a \in \mathbb{B}$. Let $a = b * c$ where $b, c \in Q_8 \setminus \{1, -1\}$ also $a = (-1) * (-a)$. We have $\ell^*(a_p) = \ell^*(b_p) = a$. If we take $g(wa_p) = \pm b($ or $\pm c)$ then $g(a_p b_p) = \pm c($or $\pm b)$ and $g(wb_p) = \pm b($ or $\pm c)$ also if we take $g(wa_p) = -1($ or $-a)$ then $g(a_p b_p) = -a($ or -1) and $g(wb_p) = -1($ or $-a)$. So $\ell^*(w)$ consists of a product of even number of b's(b and $-b$), even number of c's(c and $-c$), even number of -1 and even number of a's(a and $-a$). Which implies $\ell^*(w) = \pm 1$. Which is a contradiction. Hence the proof of the theorem. □

Corollary 118. *The friendship graph $F_m \notin \Omega$.*

Theorem 119. *For all $m \geq 2$, $\mathbb{S}(F_m) \in \omega_{1,-1}$.*

Proof. Let F_m be the friendship graph with vertex set $\{a_p, b_p, w : 1 \leq p \leq m\}$ and edge set $\{a_p b_p, wa_p, wb_p\}$. Now let a_p', b_p', w_p' be the vertices in $\mathbb{S}(F_m)$ corresponding to the edges $wa_p, wb_p, a_p b_p$ of F_m respectively. Suppose we define

g as

$$g(wa_p') = g(a_pw_p') = g(b_pb_p') = i,$$
$$g(a_p'a_p) = g(w_p'b_p) = g(wb_p') = -i, \ 1 \leq p \leq m.$$

and let f be any bijection from $E(\mathbb{S}(F_m))$ to N_{6m}. Clearly $\ell^*(u) = 1, \ \forall u \in V(G)$. Thus, $\mathbb{S}(F_m) \in \omega_1$.

Suppose that m is odd. If we define g as the constant map $g(e) = i, \ \forall e \in E(\mathbb{S}(F_m))$ and f be a bijection from $E(\mathbb{S}(F_m))$ to N_{6m} then $\mathbb{S}(F_m) \in \omega_{-1}$. Now suppose that m is even. Here we consider two cases:

Case (i): m is an odd multiple of 2.

For $1 \leq p \leq m$, define f and g as follows:

$$f(wa_p') = p, \ f(wb_p') = m + p, \ f(a_p'a_p) = 2m + p,$$
$$f(a_pw_p') = 3m + p, \ f(w_p'b_p) = 4m + p, \ f(b_pb_p') = 5m + p$$

and $g(wa_p') = g(wb_p') = g(a_p'a_p) = g(a_pw_p') = g(w_p'b_p) = g(b_pb_p') = \begin{cases} i, & \text{if } p \text{ is odd}, \\ j, & \text{if } p \text{ is even}. \end{cases}$

Clearly the above maps f and g will determine a Q_8-magic labeling of $\mathbb{S}(F_m)$ with magic constant -1.

Case (ii): m is an even multiple of 2.

In this case, we take f as a bijection from $\mathbb{S}(F_m)$ to N_{6m} which map $wa_1{}'$ to 1, wa_2' to 2 , wa_3' to 3, wb_1' to 4, wb_2' to 5, and wb_3' to 6. For $1 \leq p \leq m$, let g be the function defined by

$$g(wa_p') = g(wb_p') = g(a_p'a_p) = g(a_pw_p') = g(w_p'b_p) = g(b_pb_p') = \begin{cases} j, & \text{if } p = 2, \\ k, & \text{if } p = 3, \\ i, & \text{otherwise.} \end{cases}$$

Obviously $\ell^*(u) = -1, \ \forall u \in V(\mathbb{S}(F_m))$.

In both cases, $\mathbb{S}(F_m) \in \omega_{-1}$. Hence $\mathbb{S}(F_m) \in \omega_{1,-1}$. □

Theorem 120. *$\mathbb{S}(F_m) \in \Omega$ if and only if m is odd.*

Proof. Let F_m be the friendship graph with vertex set $\{a_p, b_p, w : 1 \leq p \leq m\}$ and edge set $\{a_pb_p, wa_p, wb_p\}$. Now let a_p', b_p', w_p' be the vertices in $\mathbb{S}(F_m)$ corresponding to the edges wa_p, wb_p, a_pb_p of F_m. Suppose that m is odd. First we show that $\mathbb{S}(F_m) \in \omega_i$. Let f be the function defined by

$$
\begin{aligned}
f(wa_p^{'}) &= p,\ f(a_p^{'}a_p) = m + (p-1)5 + 1,\\
f(a_pw_p^{'}) &= m + (p-1)5 + 2, f(w_p^{'}b_p) = m + (p-1)5 + 3,\\
f(b_pb_p^{'}) &= m + (p-1)5 + 4, f(wb_p^{'}) = m + (p-1)5 + 5,\ 1 \leq p \leq m.
\end{aligned}
$$

Now define $g : E(\mathbb{S}(F_m)) \to Q_8 \setminus \{1\}$ by

$$
\begin{aligned}
g(wa_p^{'}) &= g(b_pb_p^{'}) = j, g(a_pw_p^{'}) = -j,\\
g(a_p^{'}a_p) &= g(wb_p^{'}) = k, g(w_p^{'}b_p) = -k,\ \text{where } 1 \leq p \leq m.
\end{aligned}
$$

Clearly $\ell^*(u) = i,\ \forall u \in V(\mathbb{S}(F_m))$. Similarly, we can show that $\mathbb{S}(F_m) \in \omega_a, \forall a \in \mathbb{B}$. From Theorem 119, we have $\mathbb{S}(F_m) \in \omega_{1,-1}$. Thus $\mathbb{S}(F_m) \in \Omega$.

Suppose that m is even. If possible, assume that $\mathbb{S}(F_m) \in \omega_a$, for some $a \in \mathbb{B}$. Let $a = bc$ where $b, c \in Q_8 \setminus \{1, -1\}$ also we have $a = (-1)(-a)$. We have $\ell^*(u) = a, \forall u \in V(\mathbb{S}(F_m))$. If we let $g(wa_p^{'}) = \pm b$ then $g(a_p^{'}a_p) = \pm c$, $g(a_pw_p^{'}) = \pm b$, $g(w_p^{'}b_p) = \pm c$, $g(b_pb_p^{'}) = \pm b$ and $g(wb_p^{'}) = \pm c$ (if we take $g(wa_p^{'}) = \pm c$ then replace $\pm b$ by $\pm c$ and vice versa). If we take $g(wa_p^{'}) = -1$ then $g(a_p^{'}a_p) = -a$, $g(a_pw_p^{'}) = -1$, $g(w_p^{'}b_p) = -a$, $g(b_pb_p^{'}) = -1$ and $g(wb_p^{'}) = -a$. Thus $\ell^*(w)$ is the product determined by a combination of $\pm b$, $\pm c$, -1 and $-a$. In the product $\ell^*(w)$ the number of b's(either b or $-b$) is equal to the number of c's(either c or $-c$) and the number of -1's equal to the number of $-a$'s. But for any function f the product is either $+1$ or -1. Which is a contradiction to our assumption that $\ell^*(u) = a,\ \forall u \in V(\mathbb{S}(F_m))$. Hence $\mathbb{S}(F_m) \notin \omega_a, \forall a \in \mathbb{B}$. This completes the proof of the theorem. □

Chapter 5

Induced S_3-Magic Labeling of Graphs

In this chapter, we define the induced A-magic labeling of graphs, for any non-abelian group A and we investigate the induced S_3-magic labeling of some graphs. The first section of this chapter gives an introduction to induced A-magic labeling and the second section deals with the induced group magic labeling of graphs using non abelian groups. The third section of this chapter discusses a necessary and sufficient condition for some cycle-related graphs that admit induced S_3-magic labeling. The fourth and fifth sections deal with the induced S_3-magic labeling of some star-related and path-related graphs respectively.

5.1 Introduction

Let $G = (V(G), E(G))$ be the graph with vertex set $V(G)$ and edge set $E(G)$ and $(A, +)$ be an abelian group with identity element 0. Suppose $f : V(G) \to A$ be a vertex labeling and $f^* : E(G) \to A$ denote the induced edge labeling of f defined by $f^*(uv) = f(u) + f(v)$ for all $u, v \in E(G)$. Then f^* again induces a vertex labeling $f^{**} : V(G) \to A$ defined by $f^{**}(u) = \Sigma f^*(uv)$, where the summation is taken over all the vertices v which are adjacent to u. A graph $G = (V(G), E(G))$ is said to be an induced A-magic graph and it is denoted by IAMG or simply IMG if there exists a non-zero vertex labeling $f : V(G) \to A$ such that $f \equiv f^{**}$. The function f, so obtained is called an induced A-magic labeling of G(See [7]).

In this chapter, we define the induced A-magic labeling of graphs, for any non-abelian group A and we investigate the induced S_3-magic labeling of some graphs, where S_3 is the symmetric group of order 6.

5.2 Induced Group-Magic Labeling of Graphs Using Non-abelian Groups

We extend the above definition to a non-abelian group as follows.

Definition 121. *Let $G = (V(G), E(G))$ be a finite graph with p vertices and q edges, and let $(A, *)$ be a finite non-abelian group with identity element 1. Let $f : V(G) \to N_p = \{1, 2, \ldots, p\}$ and $g : V(G) \to A$ be two vertex labelings of G such that f is bijective. We denote the induced edge labeling of f by $f^* : E(G) \to N_{2p}$ and the induced edge labeling of g by $g^* : E(G) \to A$, where f^* and g^* are defined as follows:*

$$f^*(uv) = f(u) + f(v), \text{ and}$$

$$g^*(uv) = \begin{cases} g(u) * g(v), & \text{if } f(u) < f(v), \\ g(v) * g(u), & \text{if } f(u) > f(v), \end{cases}$$

for all $uv \in E(G)$.

Define an edge labeling $\mathbb{T} : E(G) \to N_{2p} \times A$ by $\mathbb{T}(e) = (f^*(e), g^*(e))$. Define a relation $\preceq$ on the range of $\mathbb{T}$ by

$$(f^*(e), g^*(e)) \preceq (f^*(e^{'}), g^*(e^{'})) \text{ if and only if } f^*(e) \leq f^*(e^{'}).$$

Let $\{(f^*(e_1), g^*(e_1)), (f^*(e_2), g^*(e_2)), \ldots, (f^*(e_m), g^*(e_m))\}$ be a chain in the range of $\mathbb{T}$. We define the product of the elements of this chain as follows:

$$\prod_{i=1}^{m}(f^*(e_i), g^*(e_i)) = ((((g^*(e_1) * g^*(e_2)) * g^*(e_3)) * \ldots) * g^*(e_m)). \tag{5.2.1}$$

Let $u \in V(G)$ and let $N^*(u)$ be the set of all edges incident with u. Consider the restriction of the function $\mathbb{T}$ on $N^*(u)$. Then the range of $\mathbb{T}|_{N^*(u)}$ is a chain, say $(f^*(e_1), g^*(e_1)) \preceq (f^*(e_2), g^*(e_2)) \preceq \cdots \preceq (f^*(e_n), g^*(e_n))$.
We define $g^{**}(u) = \prod_{i=1}^{n}(f^*(e_i), g^*(e_i))$.
A graph G is said to be an induced A-magic graph and it is denoted by IAMG if there exists a non-zero vertex labeling(i.e $g(u) \neq 1, \forall u \in V(G)$) $g : V(G) \to A$ such that $g \equiv g^{**}$. Then the function g is called an induced A-magic labeling.

Observe that definition 121 coincides with the definition of IAMG by Libeeshkumar and Anil Kumar [7], when A is an abelian group.

Example 122. *Consider the non-abelian group symmetric group S_3 of order 6. Observe that S_3 is the group of permutations of a 3 element set, say $\{1, 2, 3\}$. The elements of S_3 are denoted by $\rho_0, \rho_2, \rho_3, \mu_1, \mu_2, \mu_3$(see [16]). Let G be the cycle graph C_3. Denote the vertices of G by $u_1, u_2,$ and u_3. Define f and g as follows: $f(u_1) = 1,\ f(u_2) = 2,\ f(u_3) = 3,\ g(u_1) = \mu_1,\ g(u_2) = \mu_2,\ g(u_3) = \rho_1$. Then $f^*(u_1u_2) = 3, f^*(u_2u_3) = 5, f^*(u_1u_3) = 4$ and $g^*(u_1u_2) = \rho_1,\ g^*(u_2u_3) =$*

$\mu_3,\ g^*(u_1u_3) = \mu_2$.
Hence

$$\begin{aligned} g^{**}(u_1) &= (f^*(u_1u_2), g^*(u_1u_2))(f^*(u_1u_3), g^*(u_1u_3)) = g^*(u_1u_2) * g^*(u_1u_3) \\ &= \rho_1\mu_2 = \mu_1, \\ g^{**}(u_2) &= (f^*(u_2u_3), g^*(u_2u_3))(f^*(u_1u_2), g^*(u_1u_2)) = g^*(u_1u_2) * g^*(u_2u_3) \\ &= \rho_1\mu_3 = \mu_2, \\ g^{**}(u_3) &= (f^*(u_2u_3), g^*(u_2u_3))(f^*(u_1u_3), g^*(u_1u_3)) = g^*(u_1u_3) * g^*(u_2u_3) \\ &= \mu_2\mu_3 = \rho_1. \end{aligned}$$

Clearly $g^{**} \equiv g$. *Hence the graph* C_3 *is an induced* S_3*-magic graph.*

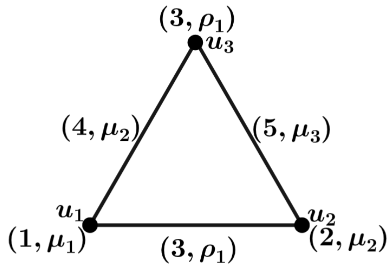

Figure 5.1: Induced S_3-magic labeling of C_3.

Define the following categories of sets as follows:
$\mathcal{I}^{\text{m}}\ (S_3)$:= the set of all induced S_3-magic graphs and
$\mathcal{I}^{\text{m}}_{\lambda}\ (S_3)$:= set of all induced S_3-magic labeling with the induced magic labeling g satisfying $g(V(G)) = \{\lambda\}$, for some $\lambda \in S_3 \setminus \{\rho_0\}$.
In this chapter, we discuss the induced S_3-magic labeling of some graphs that fall into the above categories of sets.

Theorem 123. *Let* G *be an induced* S_3*-magic graph determined by the functions* f *and* g *as in the definition 121. If* g *is a constant map then the degree of each vertex of* G *gives a remainder of 2 when divided by* 3.

Proof. Let G be an induced S_3-magic graph with $g(u) = a$, where $a \in S_3 \setminus \{\rho_0\}$. Then $g^*(uv) = a^2, \forall uv \in E(G)$. Since $g = g^{**} = a$, a is either ρ_1 or ρ_2. Without loss of generality, let $g(u) = \rho_1, \forall u \in V(G)$. Now $g^{**}(u) = ((\rho_1)^2)^{\deg(u)} = \rho_2^{\deg(u)}$. Thus $g(u) = g^{**}(u)$ if $\deg(u) \equiv 2 (\text{mod } 3)$. $\square$

5.3 Cycle Related Graphs

Theorem 124. *The cycle graph* C_n *belongs to* $\mathcal{I}^m\ (S_3)$, $n > 2$.

Proof. Denote the vertices of C_n by $u_1, u_2, \ldots, u_n$. Let f be any bijective function from $V(C_n)$ to $\{1, 2, \ldots, n\}$. If we take $g(v) = \rho_1, \forall v \in V(C_n)$. Then $g^*(u_i u_{i+1}) = \rho_2, 1 \le i \le n$. Thus $g^{**}(u_i) = g^*(u_i u_{i+1}) * g^*(u_i u_{i-1}) = \rho_2 * \rho_2 = \rho_1$. Hence $g^{**} \equiv g$. This completes the proof. □

From the proof of the above theorem we have the following corollary.

Corollary 125. *The cycle graph $C_n, n > 2 \in \mathcal{I}^m{}_\lambda\ (S_3)$, for some $\lambda \in S_3 \backslash \{\rho_0\}$.*

Theorem 126. *C_n has a non-constant induced S_3-magic labeling if and only if n is a multiple of 3.*

Proof. Suppose that n is a multiple of 3. Denote the vertices of C_n by $u_1, u_2, \ldots,$ u_n. Now, let f be any bijective function from $V(C_n)$ to N_n and define $g : V(C_n) \to S_3$ as

$$g(u_i) = \begin{cases} \rho_0, & \text{if } i \equiv 1(\text{mod } 3), \\ \rho_1, & \text{if } i \equiv 2(\text{mod } 3), \\ \rho_2, & \text{if } i \equiv 0(\text{mod } 3). \end{cases}$$

Then

$$g^*(u_i u_{i+1}) = \begin{cases} \rho_1, & \text{if } i \equiv 1(\text{mod } 3), \\ \rho_0, & \text{if } i \equiv 2(\text{mod } 3), \\ \rho_2, & \text{if } i \equiv 0(\text{mod } 3). \end{cases}$$

Hence $g^{**}(u_i) = g^*(u_i u_{i+1}) * g^*(u_i u_{i-1}) = g(u_i)$. Thus $g^{**} = g$.

Suppose that n is not a multiple of 3 and there exists a non-constant induced S_3-magic labeling for C_n. Consider the following 3 cases:

Case (i): Suppose that $g(V(C_n)) \subseteq \{\rho_0, \rho_1, \rho_2\}$.

Since g is non-constant, at least 1 pair of adjacent vertices should be labeled by different elements in $\{\rho_0, \rho_1, \rho_2\}$ under g. Without loss of generality, let $g(u_1) = \rho_1$ and $g(u_2) = \rho_2$. Then $g^*(u_1 u_2) = \rho_0$, since $g(u_2) = \rho_2$ and $g \equiv g^{**}$, we get $g(u_3) = \rho_0$. Since $g(u_3) = g^{**}(u_3)$, we have $g(u_4) = \rho_1$. Proceeding like this, we obtain g as

$$g(u_i) = \begin{cases} \rho_1, & \text{if } i \equiv 1(\text{mod } 3), \\ \rho_2, & \text{if } i \equiv 2(\text{mod } 3), \\ \rho_0, & \text{if } i \equiv 0(\text{mod } 3). \end{cases}$$

This type of induced magic labeling is possible only if n is a multiple of 3. Hence $g(V(C_n)) \nsubseteq \{\rho_0, \rho_1, \rho_2\}$.

Case(ii): At least one $g(u_i) \in \{\mu_1, \mu_2, \mu_3\}$.

Case (i) implies that at least one $g(u_i)$ should belongs to $\{\mu_1, \mu_2, \mu_3\}$. Without loss of generality, let $g(u_1) \in \{\mu_1, \mu_2, \mu_3\}$. Since g is an IAMG for C_n, we have either $g(u_2) \in \{\rho_0, \rho_1, \rho_2\}$ and $g(u_n) \in \{\mu_1, \mu_2, \mu_3\}$ or $g(u_2) \in \{\mu_1, \mu_2, \mu_3\}$ and $g(u_n) \in \{\rho_0, \rho_1, \rho_2\}$. Without loss of generality, take $g(u_2) \in \{\rho_0, \rho_1, \rho_2\}$ and $g(u_n) \in \{\mu_1, \mu_2, \mu_3\}$. Since $g(u_2) \in \{\rho_0, \rho_1, \rho_2\}$, we must have $g(u_3) \in \{\mu_1, \mu_2, \mu_3\}$ and $g(u_4) \in \{\mu_1, \mu_2, \mu_3\}$, $g(u_5) \in \{\rho_0, \rho_1, \rho_2\}$. Proceeding like this, we obtain

$$g(u_i) \in \begin{cases} \{\rho_0, \rho_1, \rho_2\}, & \text{if } i \equiv 2 (\text{mod } 3), \\ \{\mu_1, \mu_2, \mu_3\}, & \text{otherwise.} \end{cases}$$

Such an induced magic labeling is possible only when n is a multiple of 3.

Case(iii): $g(V(G)) \subseteq \{\mu_1, \mu_2, \mu_3\}$.

Since $g(u_i) \in \{\mu_1, \mu_2, \mu_3\}$ we have $g^*(u_i u_{i+1}) \in \{\rho_0, \rho_1, \rho_2\}$, $\forall i$. Then $g^{**}(u_i) \in \{\rho_0, \rho_1, \rho_2\}$. Hence $g \neq g^{**}$.

This completes the proof of the theorem. □

Theorem 127. *The wheel graph W_n is an induced S_3-magic graph if and only if n is even.*

Proof. Let the vertex set and edge set of W_n be $\{u_i, k : 1 \leq i \leq n\}$, $\{u_i u_{i+1}, k u_i : 1 \leq i \leq n,$ and $i+1$ is taken modulo $n\}$ respectively. For our convenience, let us call the set $\{\rho_0, \rho_1, \rho_2\}$ as the ρ-set and the set $\{\mu_1, \mu_2, \mu_3\}$ as μ-set. Suppose that n is even. Then, any bijective map f from $V(W_n)$ to N_{n+1} together with the map $g(k) = \rho_0$, $g(u_i) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even,} \end{cases}$ will determine an induced S_3-magic labeling of W_n.

Conversely, suppose that $W_n \in \mathcal{I}^{\mathrm{m}}(S_3)$. Now consider the following three cases:

Case(i): $g(k) \in \mu$-set.

If all the $g(u_i) \in \mu$-set, then $g^*(u_i u_{i+1}), g^*(k u_i) \in \rho$-set. Since $g^*(k) = \prod_{i=1}^{n}(f^*(k u_i), g^*(k u_i))$, we get $g^{**}(k) \in \rho$-set. Which is a contradiction to the assumption that $g(k) = g^{**}(k)$. Hence at least one $g(u_i) \in \rho$-set. Without loss of generality, let $g(u_1) \in \rho$-set then $g^*(u_1 k) \in \mu$-set. Consider $g^{**}(u_1) = \prod_{v \in N^*(u_1)}(f^*(u_1 v), g^*(u_1 v))$. There are 6 possible product for $g^{**}(u_1)$(varies according to f). Since $g^{**}(u_1) = g(u_1)$ either

(a) $g^*(u_1u_2) \in \mu$-set and $g^*(u_1u_n) \in \rho$-set or

(b) $g^*(u_1u_2) \in \rho$-set and $g^*(u_1u_n) \in \mu$-set.

(a) implies $g(u_i) \in \begin{cases} \mu\text{-set, if } i = 2+4m \text{ or } 3+4m, m = 0,1,2,\dots, \\ \rho\text{-set, if } i = 1, 4+4m, 5+4m. \end{cases}$

This labeling is possible only when n is an even number(a multiple of 4). Similarly, we can show that the case(b) also implies n is an even number. But then $g^{**}(k)$ will be the product of even number of elements from μ-set and even number of elements from ρ-set. So $g^{**}(k) \in \rho$-set. Which is a contradiction.

Case(ii): $g(k) \in \rho$-set.

Suppose there exist a vertex u_i such that $g(u_i) \in \mu$-set. Without loss of generality, let $g(u_1) \in \mu$-set. Then $g^*(ku_1) \in \mu$-set. Since $g^{**}(u_1) = g(u_1) \in \mu$-set. Then either $g(u_2), g(u_n) \in \mu$-set or $g(u_2), g(u_n) \in \rho$-set. If $g(u_2), g(u_n) \in \rho$-set then this implies $g(u_3) \in \mu$-set, $g(u_4) \in \rho$-set. Proceeding like this, we get $g(u_i) \in \begin{cases} \mu\text{-set, if } i \text{ is odd,} \\ \rho\text{-set, if } i \text{ is even.} \end{cases}$ and this labeling is possible only if n is even. Similarly, if $g(u_2), g(u_n) \in \mu$-set then $g^{**} = g$ implies $g(u_i) \in \mu$-set, for all i. Then we have $g^*(ku_i) \in \mu$-set and $g^{**}(k) = \prod_{i=1}^{n}(f^*(ku_i), g^*(ku_i)) \in \rho$-set implies n is even.
Suppose that $g(u_i) \in \rho$-set, for all i. We consider the following sub-cases:

Subcase(1): $g(k) = \rho_0$.

In this case, at least one $g(u_i) \neq \rho_0$. Without loss of generality, let $g(u_1) = \rho_1$ then either

(a) $g(u_2) = \rho_0$ and $g(u_n) = \rho_1$ or

(b) $g(u_2) = \rho_1$ and $g(u_n) = \rho_0$.

(c) $g(u_2), g(u_n) = \rho_2$.

(a) implies, for $m = 0,1,2,\dots, g(u_i) = \begin{cases} \rho_0, \text{ if } i = 2+3m, \\ \rho_1, \text{ if } i = 6m, 6m+1, \\ \rho_2, \text{ if } i = 6m+3, 6m+4. \end{cases}$

But this labeling is possible only when n is an even multiple of 3. i.e., n should be an even number. The similar argument holds for the case (b). If we consider the case (c), then we get $g(u_i) = \begin{cases} \rho_1, \text{ if } i \text{ is odd,} \\ \rho_2, \text{ if } i \text{ is even.} \end{cases}$

Then $g^{**}(k) = \rho_1^{l_1} * \rho_2^{l_2}$, where $l_1 + l_2 = n$. Then $g^{**}(k) = g(k)$ implies $l_1 = l_2$. i.e., n is even.

Subcase(2): $g(k) = \rho_1$.

If g is the constant map $g(v) = \rho_1, \forall v \in V(W_n)$. Then $g^{**}(u_i) = \rho_2 * \rho_2 * \rho_2 = \rho_0, \forall i$. Then $g \not\equiv g^{**}$. So g must be a nonconstant map. Without loss of generality, let $g(u_1) \neq \rho_1$.

Suppose that $g(u_1) = \rho_0$ then this implies $g(u_i) = \begin{cases} \rho_1, & \text{if } i \text{ is even,} \\ \rho_0, & \text{if } i \text{ is odd.} \end{cases}$

or $g(u_i) = \begin{cases} \rho_0, & \text{if } i = 1 + 6m, 2 + 6m, m = 0, 1, 2, \ldots, \\ \rho_2, & \text{if } i = 3l, l = 1, 2, \ldots, \\ \rho_1, & \text{otherwise.} \end{cases}$

But this labeling is possible when n is an even number(a multiple of 6). If $g(u_1) = \rho_2$, then $g^{**}(u_1) = \rho_2$ implies $g(u_2) = g(u_n) = \rho_2$. Similarly, $g^{**}(u_2) = \rho_2$ implies $g(u_3) = \rho_2$. Proceeding like this, we obtain $g(u_i) = \rho_2, \forall i$. But then $g^{**}(k) = \prod_{i=1}^{n} g^*(ku_i) = \rho_0$. Hence $g^{**} \neq g$.

Subcase (3): $g(k) = \rho_2$.

In this case, we can prove that there does not exist an induced S_3-magic labeling when n is odd as in sub case(2).

All the above cases and sub-cases lead to the conclusion that the wheel graph W_n is an induced S_3-magic graph when n is even.

□

Corollary 128. *The wheel graph $W_n \notin \mathcal{I}_\lambda^m(S_3)$, for any $\lambda \in S_3 \setminus \{\rho_0\}$ and $n \geq 3$.*

Proof. The proof directly follows from Theorem 123 and Theorem 127. □

Theorem 129. *The helm graph H_n is an induced S_3-magic graph if and only if n is odd or $n \equiv 1(mod\ 3)$.*

Proof. Let us denote the vertices of H_n by $u_1, u_2, \ldots, u_n, k, v_1, v_2, \ldots, v_n$ and let the edge set of H_n be $\{u_iv_i, u_iu_{i+1}, ku_i : 1 \leq i \leq n\}$. Suppose that there exists an induced S_3-magic labeling for H_n. Since $v_i's$ are pendant vertices $g^{**}(v_i) = g(v_i)$ implies $g(u_i) = \rho_0, 1 \leq i \leq n$. Hence $g^*(u_iu_{i+1}) = \rho_0, \forall i$. Suppose that, for some m, $1 \leq m \leq n$, $g(v_m) = a, a \in S_3$ then $g^*(v_mu_m) = a, g^*(u_mu_{m+1}) = g^*(u_mu_{m-1}) = \rho_0$. If $g^{**}(u_m) = g(u_m)$ implies $g^*(u_mv_m) = g^*(u_mk)^{-1}.i.e., g(k) = a^{-1}$. Then $g(v_i) = a, \forall i$. Now $g^*(u_ik) = a^{-1}$ implies

$g^{**}(k) = \underbrace{a^{-1}a^{-1}\cdots a^{-1}}_{n\,\text{times}} = a^{-1}$. This is possible when either n is odd or $n \equiv 1(\text{mod } 3)$.

Conversely, suppose that n is odd then the labeling $g(v) = \begin{cases} \rho_0, & \text{if } v = u_i, \\ \mu_1, & \text{otherwise.} \end{cases}$ together with any bijection f from $V(H_n)$ to N_{n+1} will give an induced S_3-magic labeling for H_n. Similarly, If $n \equiv 1(\text{mod } 3)$ then the map $g(v) = \begin{cases} \rho_0, & \text{if } v = u_i, \\ \rho_1, & \text{if } v = v_i, \\ .\rho_2, & \text{if } u = k. \end{cases}$ will determine an induced S_3-magic labeling. □

Corollary 130. *The helm graph H_n does not belong to $\mathcal{I}^m_\lambda(S_3)$, for any n.*

Proof. It is clear that, since v_i's are pendant verttices $g(u_i) = \rho_0$, for all $1 \leq i \leq n$. Hence the corollary. □

Definition 131. *[32] A fan graph, denoted by F_n, is defined as $P_n + K_1$, where P_n is a path on n vertices. We can identify a fan graph, F_n with a shell graph of width $n+1$. Where a shell graph $S_{n,n-3}$ of width n is a graph obtained by taking $n-3$ concurrent chords in a cycle C_n of n vertices.*

Theorem 132. *The fan graph F_n is an induced S_3-magic graph if and only if n is even.*

Proof. Let F_n be the fan graph with vertex set $\{k, u_i : i = 1, 2, \ldots, n\}$, and edge set $\{u_iu_{i+1}, ku_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}$. Suppose that n is even. Define $f : V(F_n) \to N_{n+1}$ be the map $f(u_i) = i, 1 \leq i \leq n$ and $f(k) = n+1$. Now define $g : V(F_n) \to S_3$ as follows:

$$g(u) = \begin{cases} \rho_0, & \text{if } u = k, \\ \mu_1, & \text{if } u = u_i, 1 \leq i \leq n. \end{cases}$$

Clearly, $g(u) = g^{**}(u), \forall u \in V(F_n)$. Hence $F_n \in \mathcal{I}^{\text{m}}(S_3)$, when n is even. Conversely, suppose that $F_n \in \mathcal{I}^{\text{m}}(S_3)$. That is, there exist two functions f and g such that $g^{**} = g$. Now we will show that n must be even. Consider the following cases:

Case(i): $g(k) = \rho_0,\ g(u_1) = \rho_0$.

This case $g^{**}(u_1) = \rho_0 = g^*(ku_1) * g^*(u_1u_2)$ implies $g(u_2) = \rho_0$. Which in turn implies $g(u_3) = \rho_0$. Proceeding like this, we obtain $g(u) = \rho_0$, for all $u \in V(F_n)$. So this case is not possible.

Case (ii): $g(k) = \rho_0,\ g(u_1) = \rho_1$.

If $g(k) = \rho_0, g(u_1) = \rho_1$ then $g^{**}(u_1) = \rho_1 =$ implies

$$g(u_i) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases}$$

But $g^{**}(k) = \rho_0 = g(k)$ implies that n must be an even number.

Case(iii): $g(k) = \rho_0,\ g(u_1) = \rho_2$.

This case is exactly similar to Case(ii).

Case (iv): $g(k) = \rho_1,\ g(u_1) = \rho_1$.

Suppose that $g(k) = \rho_1, g(u_1) = \rho_1$ then $g^*(ku_1) = \rho_1^2 = \rho_2$. Then $g^{**}(u_1) = g(u_1)$ implies $g(u_2) = \rho_1$. Similarly, applying $g^{**}(u_i) = g(u_i)$, for $1 \leq i \leq n$, leads

$$g(u_i) = \begin{cases} \rho_1, & \text{if } i \equiv 1 (\text{mod } 6) \text{ or } i \equiv 2 (\text{mod } 6), \\ \rho_2, & \text{if } i \equiv 3 (\text{mod } 6) \text{ or } i \equiv 0 (\text{mod } 6), \\ \rho_0, & \text{if } i \equiv 4 (\text{mod } 6) \text{ or } i \equiv 5 (\text{mod } 6). \end{cases}$$

We can easily see that this labeling g satisfies $g = g^{**}$ only if $n \equiv 2(\text{mod } 6)$, which is an even number.

Case(v): $g(k) = \rho_1, g(u_1) = \rho_2$.

Suppose $g(k) = \rho_1, g(u_1) = \rho_2$ then as in the case(iv) we get

$$g(u_i) = \begin{cases} \rho_2, & \text{if } i \equiv 1 (\text{mod } 6) \text{ or } i \equiv 4 (\text{mod } 6), \\ \rho_0, & \text{if } i \equiv 2 (\text{mod } 6) \text{ or } i \equiv 3 (\text{mod } 6), \\ \rho_1, & \text{if } i \equiv 5 (\text{mod } 6) \text{ or } i \equiv 0 (\text{mod } 6). \end{cases}$$

and this g satisfy $g = g^{**}$ only when $n \equiv 2(\text{mod } 6)$. So n is an even number.

Case(vi): $g(k) = \rho_1, g(u_1) = \rho_0$.

If $g(k) = \rho_1, g(u_1) = \rho_0$ then applying $g^{**}(u_i) = g(u_i)$, implies that

$$g(u_i) = \begin{cases} \rho_0, & \text{if } i \equiv 1 (\text{mod } 6) \text{ or } i \equiv 0 (\text{mod } 6), \\ \rho_2, & \text{if } i \equiv 2 (\text{mod } 6) \text{ or } i \equiv 5 (\text{mod } 6), \\ \rho_1, & \text{if } i \equiv 3 (\text{mod } 6) \text{ or } i \equiv 4 (\text{mod } 6). \end{cases}$$

and this g satisfy $g = g^{**}$ only when $n \equiv 2(\text{mod } 6)$ as in the above two cases. Hence n is an even number and $n \equiv 2(\text{mod } 3)$.

Similarly, we can prove that when n is odd there does not exist an induced S_3-magic labeling f and g for F_n with $g(k) = \rho_2$ and $g(u_1) \in \{\rho_0, \rho_1, \rho_2\}$.

Case(vii): $g(k) \in \rho$-set, $g(u_1) \in \mu$-set.

Suppose there exist an induced S_3-magic labeling g such that $g(k) \in \rho$-set, $g(u_1) \in \mu$-set

Then $g^*(ku_1) \in \mu$-set. Now, $g^{**}(u_1) = g^*(ku_1) * g^*(u_1u_2) \in \mu$-set implies $g(u_2) \in \mu$-set. Similarly, $g(u_2) \in \mu$-set implies $g(u_3) \in \mu$-set. Proceeding like this, we get $g(u_i) \in \mu$-set for all i. Then $g^{**}(k)$ is a product of n elements from the μ-set. But $g^{**}(k) = g(k) \in \rho$-set implies n must be an even number.

Case(viii): $g(k) \in \mu$-set, $g(u_1) \in \mu$-set.

Suppose that $g(k) \in \mu$-set, $g(u_1) \in \mu$-set then $g(u_i) = g^{**}(u_i)$ implies that

$$g(u_i) = \begin{cases} \mu\text{-set}, \text{ if } i \equiv 1(\text{mod } 8) \text{ or } i \equiv 4(\text{mod } 8), i \equiv 5(\text{mod } 8), \\ \qquad\qquad i \equiv 0(\text{mod } 8), \\ \rho\text{-set}, \text{ if } i \equiv 2(\text{mod } 8) \text{ or } i \equiv 3(\text{mod } 8), i \equiv 6(\text{mod } 8), \\ \qquad\qquad i \equiv 7(\text{mod } 8), \end{cases}$$

But this map g is an induced S_3-magic labeling if n is of the form $2 + 4k, k = 0, 1, 2, \ldots$. Which means n is not an odd number.

Case(ix): $g(k) \in \mu$-set, $g(u_1) \in \rho$-set.

This case is similar to Case (viii).

From all the above cases we observe that when n is an odd number, there does not exist an induced S_3-magic labeling for F_n. This completes the proof of the theorem. □

Theorem 133. *The fan graph does not belong to $\mathcal{I}^m_\lambda(S_3)$ for any $\lambda \in S_3 \setminus \{\rho_0\}$ and $n \geq 3$.*

Proof. Suppose there exist a g such that $g(u) = \lambda = g^{**}(u), \forall u \in V(F_n)$ and some $\lambda \in S_3 \setminus \{\rho_0\}$. Denote the vertices of F_n as in the above theorem. Then $g(u_i) = \lambda, \forall i, 1 \leq i \leq n$ and $g^{**}(u_1) = \lambda = g^*(ku_1) * g^*(u_1u_2) = \lambda^2 * \lambda^2$. Which implies $\lambda^4 = \lambda$. Hence, the order of λ is three. So $\lambda = \rho_1$ or ρ_2. Now

$g^{**}(u_2) = g^*(u_2u_1) * g^*(u_2u_3) * g^*(ku_2) = \lambda^6 = \rho_0$. Which is a contradiction. Hence $F_n \notin \mathcal{I}_\lambda^{\mathrm{m}}(S_3)$, for any $\lambda \in S_3 \setminus \{\rho_0\}$ and $n \geq 4$.

□

Definition 134. *[32] A gear graph is a graph G_n obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the n-cycle.*

Theorem 135. *The gear graph G_n is an induced S_3-magic graph if and only if n is even.*

Proof. Let $V(G_n) = \{u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_n, k\}$ and $E(G_n) = \{ku_i, u_iw_i, w_iu_{i+1} : 1 \leq i \leq n$ and $i+1$ is taken modulo $n\}$. Suppose that n is even.

Then define f and g as follows: $f(u_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \frac{3n}{2} + \frac{i}{2}, & \text{if } i \text{ is even.} \end{cases}$

$f(k) = 2n+1$, $f(w_i) = \frac{n}{2} + i$ and $g(u_i) = \mu_1, g(w_i) = \rho_1, 1 \leq i \leq n, g(k) = \rho_0$. Clearly, the maps f and g determine an induced S_3-magic labeling.

Conversely, suppose that the gear graph G_n is an induced S_3-magic graph. Consider the following cases:

Case(i): $g(k) \in \rho$-set and at least one $g(u_i) \in \rho$-set.

Suppose that there exist a u_i such that $g(u_i) \in \rho$-set. Without loss of generality, let $g(u_1) \in \rho$-set. Since $g(u_1) = g^{**}(u_1)$ either $g(w_1), g(w_n) \in \mu$-set or $g(w_1), g(w_n) \in \rho$-set.

If $g(w_1), g(w_n) \in \mu$-set then $g(u_i) = g^{**}(u_i)$ implies

$$g(u_i) \in \begin{cases} \rho\text{-set}, & \text{if } i \text{ is odd,} \\ \mu\text{-set}, & \text{if } \text{ is even.} \end{cases}$$

and $g(w_i) \in \mu$-set. This type of labeling is possible only when n is even. If $g(w_1), g(w_n) \in \rho$-set. Then we obtain $g(u_i)$ and $g(w_i)$ belongs to the ρ-set, for all $1 \leq i \leq n$. Now consider the following subcases.

Subcase(a): Let $g(k) = \rho_0$.

Then there exist at least one vertex u_i with $g(u_i) \neq \rho_0$. Let $g(u_1) \neq \rho_0$. So $g(u_1) = \rho_1$ or ρ_2. If $g(u_1) = \rho_1$ then $g(w_1) = \rho_1$ and $g(w_n) = \rho_0$ or $g(w_1) = \rho_0$ and $g(w_n) = \rho_1$ or $g(w_1) = g(w_n) = \rho_2$. Suppose $g(w_1) = \rho_1$ and $g(w_n) = \rho_0$ then map $g(u_i)$ and $g(w_i)$ will be as follows:

$$g(u_i) = \begin{cases} \rho_1, & \text{if } i = 1+4m, 2+4m, m = 0,1,2,3,\dots, \\ \rho_2, & \text{if } i = 3+4m, 4+4m, m = 0,1,2,3,\dots, \end{cases},$$

$$g(w_i) = \begin{cases} \rho_1, & \text{if } i = 1+4l, l = 0,1,2,\dots, \\ \rho_0, & \text{if } i \text{ is even}, \\ \rho_2, & \text{if } i = 3+4l, l = 0,1,2,3,\dots \end{cases}$$

Then $g^{**}(k) = \prod g^*(ku_i) = (\rho_1)^h(\rho_2)^j = \rho_0$, where h,k are two non negative integers such that $h+j=n$. But the definition of g implies that $h=j$. So n must be even. Similarly, If $g(w_1) = \rho_0$ and $g(w_n) = \rho_1$ then we arrive at a conclusion that n must be even. If $g(w_1) = g(w_n) = \rho_2$. Then the map g will be

$$g(u_i) = \begin{cases} \rho_0, & \text{if } i \text{ is even}, \\ \rho_1, & \text{if } i = 1+4l, l = 0,1,2,3,\dots, \\ \rho_2, & \text{if } i = 3+4l, l = 0,1,2,3,\dots, \end{cases} \quad \text{and}$$

$$g(w_i) = \begin{cases} \rho_2, & \text{if } i = 1+4l, 4+4l, l = 0,1,2,\dots, \\ \rho_1, & \text{if } i = 2+4l, 3+4l, l = 0,1,2,\dots \end{cases}$$

So it is clear that such a map is possible when n is a multiple of 4. i.e., n must be even.

Subcase (b): $g(k) = \rho_1$ and $g(u_i), g(w_i) \in \rho$-set.

Suppose there exist a vertex u_i such that $g(u_i) = \rho_0$. Without loss of generality, let $g(u_1) = \rho_0$. Since $g(u_1) = g^{**}(u_1)$ there are three possibilities for $g(w_1)$ and $g(w_n)$.

(1) $g(w_1) = g(w_n) = \rho_1$ or

(2) $g(w_1) = \rho_0$ and $g(w_n) = \rho_2$ or

(3) $g(w_1) = \rho_2$ and $g(w_n) = \rho_0$.

If we consider (1), $g(u_i) = g^{**}(u_i)$ implies that

$$g(u_i) = \begin{cases} \rho_0, & \text{if } i = 1+4l, l = 0,1,2,\dots, \\ \rho_2, & \text{if } i \text{ is even}, \\ \rho_1, & \text{if } i = 3+4l, l = 0,1,2,\dots \end{cases} \quad \text{and}$$

$$g(w_i) = \begin{cases} \rho_1, & \text{if } i = 1+4l, 4+4l, l = 0,1,2,\dots, \\ \rho_0, & \text{if } i = 2+4l, 3+4l, l = 0,1,2,\dots \end{cases}$$

Then such a mapping is possible when n is an even multiple of 4.

So n must be an even number. Similarly (2) implies

$$g(u_i) = \begin{cases} \rho_0, & \text{if } i = 1+4l, 2+4l, l = 0,1,2,\dots, \\ \rho_1, & \text{if } i = 3+4l, 4+4l. \end{cases} \quad \text{and}$$

$$g(w_i) = \begin{cases} \rho_0, & \text{if } i = 1+4l, l = 0,1,2,\dots, \\ \rho_2, & \text{if } i \text{ is even}, \\ \rho_1, & \text{if } i = 3+4l, l = 0,1,2,\dots. \end{cases}$$

this g also leads to the same conclusion as in (1). Case (3) is exactly similar to (2). Similarly, we can prove that n is even if there exist a $g(u_i)$ such that $g(u_i) = \rho_1$ or ρ_2 and $g(k) = \rho_1$.

Subcase (c): $g(k) = \rho_2$ and $g(u_i), g(w_i) \in \rho$-set.
This case is similar to subcase (b).

Case(ii): $g(k) \in \rho$-set and at least one $g(u_i) \in \mu$-set.
Suppose $g(u_1) \in \mu$-set. Then either

(1) $g(w_1), g(w_n) \in \rho$-set or

(2) $g(w_1), g(w_n) \in \mu$-set.

First case together the condition $g(u_i) = g^{**}(u_i)$ implies that $g(u_i) \in \mu$-set and $g(w_i) \in \rho$-set, for all $1 \le i \le n$. Then $g^{**}(k) = \prod_{i=1}^{n} g(ku_i)$ is a product consisting of elements from μ-set. But our assumption $g^{**}(k) = g(k) \in \rho$-set implies n should be an even number.
Similarly, if we consider (2), we obtain $g(u_i) \in \begin{cases} \mu\text{-set, if } i \text{ is odd,} \\ \rho\text{-set, if } i \text{ is even.} \end{cases}$ and $g(w_i) \in \mu-$set, this kind of labeling is possible when n is even.

Case(iii): $g(k) \in \mu$-set and at least one $g(u_i) \in \rho$-set.
If possible, let $g(u_1) \in \rho$-set. Then $g^*(u_1k) \in \mu$-set. Since $g^{**}(u_1) = g(u_1)$ either

(1) $g(w_n) \in \mu$-set and $g(w_1) \in \rho$-set or

(2) $g(w_n) \in \rho$-set and $g(w_1) \in \mu$-set.

First case together with the assumption $g^{**}(u_i) = g(u_i)$ and $g^{**}(w_i) =$

$g(w_i)$ implies

$$g(u_i) \in \begin{cases} \rho\text{-set, if } i = 1+4m, 2+4m, m = 0, 1, 2, \ldots, \\ \mu\text{-set, if } i = 3+4m, 4+4m. \end{cases} \quad \text{and}$$

$$g(w_i) \in \begin{cases} \rho\text{-set, if } i \text{ is odd,} \\ \mu\text{-set, if } i \text{ is even.} \end{cases}$$

Hence n must be a multiple of 4. That is n is even. Similarly, case (2) also implies n is an even number.

Case(iv): $g(k) \in \mu$-set and at least one $g(u_i) \in \mu$-set.
Without loss of generality, let $g(u_1) \in \mu$-set. Then, proceeding as in case (iii), we find the possible values of $g(u_i)$ and $g(w_i)$ with the assumption $g^{**} = g$. This leads to the conclusion that n should be an even number.

All the above cases show that n is an even number. This completes the proof of the theorem. □

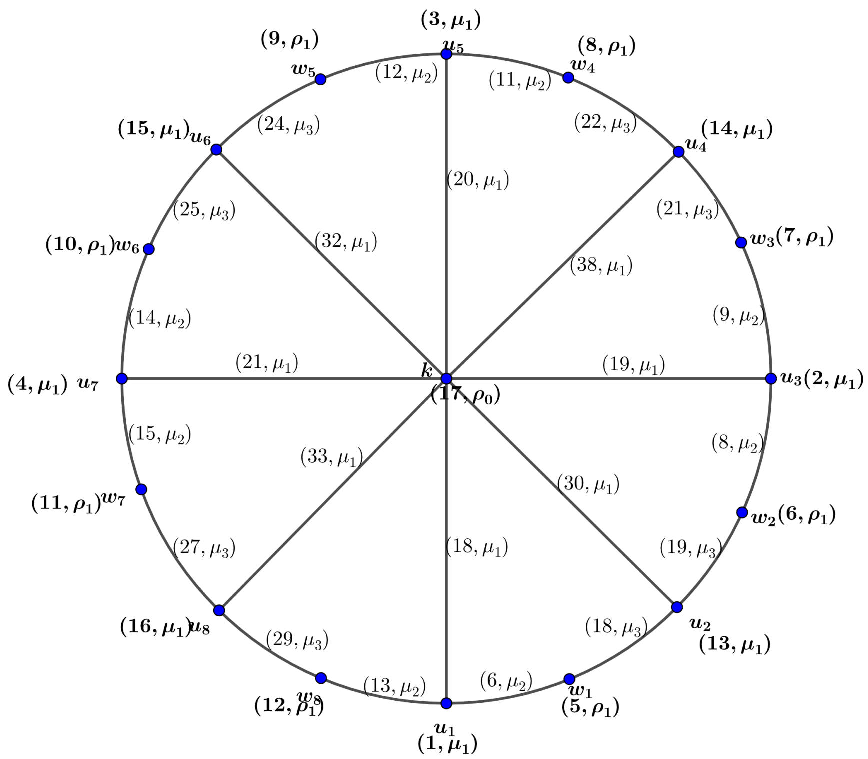

Figure 5.2: Induced S_3-magic labeling of gear graph G_8.

The Figure 5.2 represents a gear graph G_8 with an induced S_3-magic labeling.

Theorem 136. *The gear graph $G_n \notin \mathcal{I}^{m}_{\lambda}(S_3)$, for any n.*

Proof. From the above theorem, gear graph $G_n \in \mathcal{I}^{\mathrm{m}}(S_3)$ when n is even. Suppose that n is even and $G_n \in \mathcal{I}^{\mathrm{m}}_{\lambda}(S_3)$, for some $\lambda \in S_3 \setminus \{\rho_0\}$. Now $g(u) = a$, for some $a \in S_3 \setminus \{\rho_0\}$. Then $g^*(u_i w_i) = g^*(u_i w_{i-1}) = g(ku_i) = a^2$. But then $g^{**}(u_i) = (a^2)^3 = \rho_0 \neq g(u_i)$. Hence the proof. □

Definition 137. *[33] A sunflower graph is denoted by SF_n and is obtained by taking a wheel with the central vertex v_0 and the n-cycle $v_1, v_2, \ldots, v_n$ and additional vertices $w_1, w_2, \ldots, w_n$, where w_i is joined by edges to the vertices v_i and v_{i+1}, where $i+1$ is taken modulo n.*

Theorem 138. *The sunflower graph $SF_n \in \mathcal{I}^{m}(S_3)$, for all n.*

Proof. Let the vertex set of SF_n be the set $\{u_i, v_i, k\}$ and the edge set is $\{u_i u_{i+1}, ku_i, u_i v_i, u_{i+1} v_i | 1 \leq i \leq n, i+1 \text{ is taken modulo } n\}$. Consider the following three cases:

Case(i): $n \equiv 0 (\text{mod } 3)$.

Here we define f as any bijection from $V(G)$ to N_{2n+1} and define g as follows:

$$g(k) = \rho_0,\ g(u_i) = \begin{cases} \rho_0, & \text{if } i \equiv 1 (\text{mod } 3), \\ \mu_1, & \text{otherwise.} \end{cases}$$

$$g(w_i) = \begin{cases} \rho_0, & \text{if } i \equiv 2 (\text{mod } 3), \\ \mu_1, & \text{otherwise.} \end{cases}$$

Clearly, g defines an induced S_3-magic labeling.

Case(ii): $n \equiv 1 (\text{mod } 3)$.

Here also define f as above and if n is odd , let g be the map

$$g(u_i) = \begin{cases} \rho_0, & \text{if } i = 1, \\ \mu_1, & \text{otherwise.} \end{cases},\ g(w_i) = \begin{cases} \mu_1, & \text{if } i = 1, n, \\ \rho_0, & \text{otherwise .} \end{cases},\ g(k) = \rho_0.$$

If n is even, then define g as $g(u) = \begin{cases} \mu_1, & \text{if } u = u_i, i = 1, 2, \ldots, n, \\ \rho_0, & \text{otherwise .} \end{cases}$

Here we can see that $g = g^{**}$.

Case(iii): $n \equiv 2 (\text{mod } 3)$.

Here define f as above and g be defined by the constant map $g(u) = \rho_1$, for all $u \in V(G)$. Then $g^{**}(u) = \rho_1$.

This completes the proof of the theorem. □

Theorem 139. *Sunflower graph* $SF_n \in \mathcal{I}_\lambda^m(S_3)$ *if and only if and only if* $n \equiv 2(mod\ 3)$.

Proof. If $n \equiv 2(\text{mod } 3)$. Then case(iii) of Theorem 138 shows $SF_n \in \mathcal{I}_\lambda^{\mathrm{m}}(S_3)$. Suppose that $SF_n \in \mathcal{I}_\lambda^{\mathrm{m}}(S_3)$. Suppose $g(u) = s, s \in S_3 \setminus \{\rho_0\}$. If $s \in \{\mu_1, \mu_2, \mu_3\}$ then $g^*(ku_i) = \rho_0, \forall i = 1, 2, \ldots, n$. Then $g^{**}(k) = \prod_{i=1}^n (ku_i) = \rho_o$. Then $g^{**} \neq g$. Hence $s \in \{\rho_1, \rho_2\}$. Without loss of generality, let $s = \rho_1$. Then $g^*(ku_i) = g^*(u_i u_{i+1}) = g^*(u_i v_i) = g^*(v_i u_{i+1}) = \rho_1 * \rho_1 = \rho_2$. Then $g^{**}(u_i) = \rho_2^5 = \rho_1, g^{**}(v_i) = \rho_2^2 = \rho_1$ and $g^{**}(k) = \prod_{i=1}^n (ku_i) = \rho_2^n$. Hence $g = g^{**}$ implies that $n \equiv 2(\text{mod } 3)$. This completes the proof. □

Definition 140. *[32] A flag graph is denoted by* Fl_n *and is obtained by joining one vertex of* C_n *to an extra vertex called the root.*

Theorem 141. *For* $n \geq 3$, *the flag graph* $Fl_n \in \mathcal{I}^m(S_3)$ *if and only if* n *is a multiple of 3.*

Proof. Let $u_1, u_2, \ldots, u_n, k$ be the vertex set of Fl_n where u_i's, $1 \leq i \leq n$ are the vertices corresponding to the cycle graph C_n and k is the root vertex adjacent to the vertex u_1, Suppose that n is a multiple of 3. Then define $g : V(Fl_n) \to N_{n+1}$ as follows: $g(u_i) = \begin{cases} \rho_0, & \text{if } n \equiv 1(\text{mod } 3), \\ \rho_1, & \text{if } n \equiv 2(\text{ mod } 3), \\ \rho_2, & \text{if } n \equiv 0(\text{ mod } 3), \end{cases}$ and $g(k) = \rho_0$. Clearly, this g together with any bijective map f from $V(G)$ to N_{n+1} will determine an induced S_3-magic labeling for Fl_n.

Conversely, suppose that the flag graph Fl_n is induced S_3-magic. Since k is a pendant vertex $g^{**}(k) = g^*(ku_1) = g(k)$ implies $g(u_1) = \rho_0$. Now let $g(k) = a, a \in S_3$. If $g(u_2) = b, b \in S_3 \setminus \{\rho_0\}$ then $g(u_2) = g^{**}(u_2)$ implies $g(u_3) = b^{-1}$. Proceeding like this, we obtain $g(u_i) = \begin{cases} \rho_0, & \text{if } i \equiv 1(\text{mod } 3), \\ b, & \text{if } i \equiv 2(\text{mod } 3), \\ b^{-1}, & \text{if } i \equiv 0(\text{mod } 3). \end{cases}$ Then $g^{**}(u_1) = g(u_1) = \rho_0 = b * a * g(u_n)$. If $n \equiv 1(\text{mod } 3)$ then $g(u_n) = \rho_0, g(u_{n-1}) = b^{-1}$. Hence $g^{**}(u_n) = g^*(u_n u_{n-1}) * g^*(u_1 u_n) = b^{-1}$. So $g(u_n) \neq g^{**}(u_n)$. Similarly, we can show that $n \not\equiv 2(\text{mod } 3)$. Also, we can show that this type of induced magic labeling is possible when n is a multiple of 3 and $g(k) = \rho_0$. Hence the proof. □

Corollary 142. *The flag graph* $Fl_n \notin \mathcal{I}_\lambda^m(S_3)$ *for any* $\lambda \in S_3 \setminus \{\rho_0\}$ *and* $n \geq 3$.

Proof. Since Fl_n has a pendant edge, the value of g of one end vertex of that edge will be ρ_0. □

Definition 143. *[34] The sun graph on $n = 2p$ vertices, denoted by Sun_n is the graph obtained by appending a pendant vertex to each vertex of a p-cycle. A broken sun graph is a connected unicyclic subgraph of a sun graph. We denote by BS(p,q) the set of broken suns with $n = p + q$ vertices and with a p-cycle. For $p > 2$ and $0 < q < p$, a consecutive broken sun graph, denoted by $CBSun_{p,q}$ is the graph belonging to BS(p,q) such that the subgraph induced by the vertices of degree 2 is a path on $p - q$ vertices.*

Theorem 144. *The sun graph Sun_n is not an induced A-magic for any group (abelian / non-abelian) A.*

Proof. Let the vertex set of Sun_n be $\{u_i, v_i : 1 \leq i \leq n\}$, where u_i's are the vertices of the corresponding cycle C_n and v_i is the pendant vertices attached to u_i. Suppose that Sun_n is an induced A-magic graph. Since v_i's are pendant vertices we have $g(u_i) = e$, where 'e' is the identity element of A. Thus $g^{**}(u_i) = (f^*(u_iv_i), g^*(u_iv_i)) * (f^*(u_iu_{i+1}), g^*(u_iu_{i+1})) * (f^*(u_iu_{i-1}), g^*(u_iu_{i-1}))$ $= g(u_i) * g(v_i) * g(u_i) * g(u_{i+1}) * g(u_i) * g(u_{i-1}) = g(v_i)$. Then $g^{**} = g$ implies $g(v_i) = e$. So $g(u) = e, \forall u \in V(\text{Sun}_n)$. Which is a contradiction. Hence the proof. ☐

Theorem 145. *The graph $CBSun_{p,q} \in \mathcal{I}^m(S_3)$ if and only if $p - q \equiv 2(mod\ 3)$.*

Proof. Consider the graph $\text{CBSun}_{p,q}$ with vertex set $\{u_i, v_j : 1 \leq i \leq p, 1 \leq j \leq q\}$, where v_j is the pendant vertex adjacent to u_j. Suppose that, $p - q \equiv 2(\text{mod } 3)$. Let f be any bijection from $V(G)$ to N_{p+q}. Define the map g as follows:

$$g(u_i) = \begin{cases} \rho_0, & \text{if } i = 1 \leq i \leq q, q + 3k, k = 0, 1, 2, \ldots, \\ \rho_2, & \text{if } i = q + 1 + 3k, k = 0, 1, 2, \ldots, \\ \rho_1, & \text{if } i = q + 2 + 3k, k = 0, 1, 2, \ldots, \end{cases} \quad \text{and}$$

$$g(v_j) = \begin{cases} \rho_2, & \text{if } j = 1, \\ \rho_1, & \text{if } j = q, \\ \rho_0, & \text{if } 2 \leq j \leq q - 1. \end{cases}$$

We can easily prove that g is an induced S_3-magic labeling.

Now, suppose $p - q \not\equiv 2(\text{mod } 3)$. If possible, let g be an induced S_3-magic labeling for $\text{CBsun}_{p,q}$. Since v_i's are pendant vertices we have $g(u_i) = \rho_0, 1 \leq i \leq q$. If $g(v_1) = \rho_0$ then $g(u_1) = g^{**}(u_1)$ implies $g(u_p) = \rho_0$. Then we can see that $g(u) = \rho_0, \forall u \in V(\text{CBSun}_{p,q})$. So $g(v_1) \neq \rho_0$. Similarly, we can show

that $g(v_q) \neq \rho_0$. Let $g(v_q) = b$, for some $b \in S_3 \setminus \{\rho_0\}$. Then our assumption
$g = g^{**}$ implies $g(u_i) = \begin{cases} b^{-1}, \text{ if } i = (q+1)+3m, m = 0,1,2,\dots, \\ b, \text{ if } i = (q+2)+3m, m = 0,1,2,\dots, \\ \rho_0, \text{ if } i = q+3k, k = 1,2,\dots. \end{cases}$
If $p-q \equiv 0(\text{mod } 3)$. Then $u_p = u_{q+(p-q)} = u_{q+3k}$, for some k. Thus, $g(u_p) = \rho_0$, $g(u_{p-1}) = b$ and $g^*(u_pu_1) = \rho_0$, $g^*(u_pu_{p-1}) = b$. Then $g^{**}(u_p) = b$. Then $g^{**}(u_p) \neq g(u_p)$. Which is a contradiction. Similarly, if $p-q \equiv 1(\text{mod } 3)$ we get $g(u_p) = b^{-1}$ and $g^{**}(u_p) = (b^{-1})^2$. Which implies $b = \rho_0$. So $p-q \not\equiv 1(\text{mod } 3)$. Hence $\text{CBSun}_{p,q}$ is an induced S_3-magic if and only if $p-q \equiv 2(\text{mod } 3)$. □

Corollary 146. *The graph $CBSun_{p,q} \notin \mathcal{I}_\lambda^m(S_3)$, for any $\lambda \in S_3 \setminus \{\rho_0\}$ and any $n > 2$.*

Proof. The proof directly follows from the above theorem. □

Theorem 147. *The web graph $W(2,n)$ is an induced S_3-magic when $n \equiv 0(mod\ 3)$ or $n \equiv 2(mod\ 3)$.*

Proof. Let the vertex set of $W(2,n)$ be $\{k, u_i, v_i, w_i : 1 \leq i \leq n\}$ and the edge set be $\{u_iu_{i+1}, u_ik, u_iv_i, v_iv_{i+1}, v_iw_i, 1 \leq i \leq n, \text{i+1 is taken modulo } n\}$. Now, let f be any bijective map from $V(W(2,n))$ to N_{3n+1} and define $g : V(W(2,n) \to S_3$ as follows:

Case(i): $n \equiv 0(\text{mod } 3)$.

For $1 \leq i \leq n$,

define $g(u) = \begin{cases} \rho_0, \text{ if } u = k, u_i, v_j, w_i, i \equiv 1(\text{mod } 3), 1 \leq j \leq n \\ \mu_1, \text{ if } u = w_i, u_i, \text{ where } i \equiv 2(\text{mod } 3) \text{ or} \\ \qquad i \equiv 0(\text{mod } 3). \end{cases}$

Case(ii): $n \equiv 2(\text{mod } 3)$.

For $1 \leq j \leq n$, let $g(u) = \begin{cases} \rho_0, \text{ if } u = v_j, \\ \rho_1, \text{ if } u = w_j, \\ \rho_2, \text{ otherwise }. \end{cases}$

Thus, in each case one can easily verify that the vertex labeling f and g will determine an induced S_3-magic labeling for the graph $W(2,n)$.

□

Corollary 148. *The web graph $W(2,n)$ does not belong to $\mathcal{I}_\lambda^m(S_3)$, for any n.*

Proof. Since $W(2,n)$ has pendant vertices $W(2,n) \notin \mathcal{I}_\lambda^m(S_3)$. □

Definition 149. *[35] A flower graph Fl^n is a graph obtained from a helm by joining each pendant vertex to the central vertex of the helm.*

Theorem 150. *The flower graph $Fl^n \in \mathcal{I}^m(S_3)$ for all n.*

Proof. Let the vertex set of Fl^n be $V(Fl^n) = \{k, u_i, w_i : i = 1, 2, 3, \ldots, n\}$, where k is the central vertex, $u_1, u_2, \ldots, u_n$ are the vertices of the corresponding cycle and $w_1, w_2, \ldots, w_n$ are the vertices adjacent to the central vertex k. Here we take f as any bijection from $V(Fl^n)$ to N_{2n+1}.

Case(i): n is even

Here, we define $g : V(Fl^n) \to S_3$ as

$$g(u) = \begin{cases} \rho_0, & \text{if } u = k, \\ \mu_1, & \text{otherwise.} \end{cases}$$

Case(ii): n is odd

In this case, define $g : V(Fl^n) \to S_3$ as

$$g(u) = \begin{cases} \rho_0, & \text{if } u = u_i, \text{ for } i = 1, 2, \ldots, n, \\ \mu_1, & \text{if } u = k, u = w_i, \text{for } i = 1, 2, \ldots, n. \end{cases}$$

Then in both cases, we can verify that the maps f and g determine an induced S_3-magic labeling of the graph Fl^n. This completes the proof. □

Corollary 151. *The flower graph $Fl^n \notin \mathcal{I}^m_\lambda(S_3)$, for any $\lambda \in S_3 \backslash \{\rho_0\}$ and $n > 2$.*

Proof. The proof directly follows from the Theorem 123. □

Theorem 152. *The friendship graph or Dutch 3-windmill graph C_3^t is an induced S_3-magic graph for all t.*

Proof. Let $V(C_3^t) = \{u, v_i, w_i\}$, where u is the common vertex, u_i and w_i are other the vertices corresponding to the i^{th} copy of C_3 other than the common vertex u. Define f as any bijection from $V(C_3^t)$ to N_{2t+1} and define g as

$$g(v) = \begin{cases} \rho_0, & \text{if } v = u, \\ \mu_1, & \text{otherwise.} \end{cases}$$

Clearly, f and g determine an induced S_3-magic labeling of the friendship graph. □

Corollary 153. *The friendship graph $C_3^t \in \mathcal{I}^m_\lambda(S_3)$ if and only if $t \equiv 1(mod\ 3$.*

Proof. By Theorem 123 a graph $G \in \mathcal{I}^{\mathrm{m}}_{\lambda}(S_3)$ if and only if the degree of each vertex of G gives a remainder 2 when divided by 3. In C_3^t all vertices except the central vertex are of degree 2 and the degree of the central vertex is $2t$. So $C_n^t \in \mathcal{I}^{\mathrm{m}}_{\lambda}(S_3)$ if and only if $t \equiv 1(\text{mod } 3)$. □

Theorem 154. *The graph $C_n^{(t)} \in \mathcal{I}^m(S_3)$, for $t \equiv 1(mod\ 3)$ or $n \equiv 0(mod\ 3)$.*

Proof. Let us denote the vertex set of $C_n^{(t)}$ be $V(C_n^{(t)}) = \{u_{ij} : i = 1, 2, \ldots, t, j = 1, 2, \ldots, n\}$, where the vertices $\{u_{i1}, u_{i2}, \ldots, u_{in}\}$ be the vertex set of i^{th} copy of C_n and the vertices $u_{i1}, 1 \leq i \leq t$ are identified with the vertex u.

Suppose that $t \equiv 1(\text{mod } 3)$. Then we define g as the constant map $g(u) = \rho_1$, for all $u \in V(C_n^{(t)})$ and let f as any bijective map from $V(C_n^{(t)})$ to N_{tn}. Then one can easily verify that f and g define an induced S_3-magic labeling. Now suppose that $n \equiv 0(\text{mod } 3)$ and let $i = 1, 2, \ldots, t$. Here also we take f as any bijective map from $V(C_n^{(t)})$ to N_{tn}.

Case (i): t is even.

In this case, define $g : V(C_n^{(t)}) \to S_3$ as

$$g(u) = \begin{cases} \mu_1, & \text{if } v = u_{ij}, \text{ where } j \equiv 0, 2(\text{mod } 3), \\ \rho_0, & \text{if } v = u, u_{ij}, \text{ where } j \equiv 1(\text{mod } 3). \end{cases}$$

Case (ii): t is odd.

In this case, define $g : V(C_n^{(t)}) \to S_3$ as

$$g(u) = \begin{cases} \rho_0, & \text{if } v = u_{ij}, \text{ where } j \equiv 2(\text{mod } 3), \\ \mu_1, & \text{if } v = u, u_{ij}, \text{ where } j \equiv 0, 1(\text{mod } 3). \end{cases}$$

It is easy to verify that f and g in both cases define an induced S_3-magic labeling of $C_n^{(t)}$, when $n \equiv 0(\text{mod } 3)$.

This completes the proof of the theorem. □

Corollary 155. *The graph $C_n^t \in \mathcal{I}^m_{\lambda}(S_3)$ if and only if $t \equiv 1(mod\ 3$.*

Proof. The proof directly follows from the Corollary 153. □

Theorem 156. *The n-gon book graph $B(n, k) \in \mathcal{I}^m(S_3)$ when $k \equiv 1(mod\ )3$ or k odd and $n \equiv 0(mod\ 3)$.*

Proof. Let $\{u_{i1}, u_{i2}, \ldots, u_{i,n}\}$ denote the vertex set of the i^{th} page of $B(n, k)$, where $i = 1, 2, \ldots, k$. Also the vertices $u_{i1}, 1 \leq i \leq k$ are identified with the vertex k_1 and the vertices $u_{in}, 1 \leq i \leq k$ are identified with the vertex k_2.

Suppose that $k \equiv 1(\text{mod } 3)$. Let f be any bijection from $V(B(n, k))$ to $N_{k(n-2)+2}$ and let g be the constant function $g(v) = \rho_1, v \in V(B(n, k))$. Clearly f and g defines an induced S_3-magic labeling with constant ρ_2.

Suppose that $n \equiv 0 (\text{mod } 3)$ and k is odd. Here, define f as above and let

$$g(u) = \begin{cases} \mu_1, & \text{if } u = k_1, k_2 \text{ or } u = u_{ij}, \text{ where } j \equiv 1, 0 (\text{mod } 3), \\ \rho_0, & \text{if } u = u_{ij}, \text{ where } j \equiv 2 (\text{mod } 3). \end{cases}$$

Thus one can easily verify that the above f and g define an induced S_3-magic labeling of $B(n, k)$. □

Corollary 157. *The n-gon book of k pages $B(n, k) \in \mathcal{I}^m_\lambda(S_3)$ if and only if $k \equiv 1(mod\ 3)$.*

5.4 Star Related graphs

In this section, we discuss the induced S_3-magic labeling of some star related graphs.

Theorem 158. *The star graph $K_{1,n}$ belongs to $\mathcal{I}^m(S_3)$, for all $n \geq 2$.*

Proof. Let G be the star graph $K_{1,n}$ of order $n+1$. Denote the pendant vertices of G by $u_1, u_2, \ldots, u_n$ and the vertex having degree n by k. Now consider the following cases:

Case(i): $n = 2$.

In this case, let $g(k) = \rho_0$, $g(u_1) = \rho_1$, $g(u_2) = \rho_2$ and let f be any bijection from $V(G)$ to N_{n+1}. Clearly $g \equiv g^{**}$.

Case (ii): n is even.

In this case, let f as above and define g by $g(u_i) = \mu_1$ and $g(k) = \rho_0$. This g will determine an induced S_3-magic labeling of G.

Case (iii): $n \equiv 0(\text{mod } 3)$ and n is odd.

Here also define f as in Case(i) and define g as $g(u_i) = \rho_1$ and $g(k) = \rho_0$. Then $g \equiv g^{**}$.

Case(iv): $n \equiv 1(\text{mod } 3)$ and n is odd.

Let $f(u_i) = i, 1 \leq i \leq n$ and $f(k) = n + 1$.

Define $g(k) = \rho_0$ and $g(u_i) = \begin{cases} \mu_1, & \text{if } i = 1, \\ \rho_1, & \text{if } i = n - 1, \\ \mu_2, & \text{otherwise.} \end{cases}$

Case(v): $n \equiv 2(\text{mod } 3)$ and n is odd.

Define f as in the case(iv) and define g as $g(u_i) = \begin{cases} \mu_1, & \text{if } i = 1, 2, \\ \rho_1, & \text{otherwise.} \end{cases}$

and $g(k) = \rho_0$. Clearly $g \equiv g^{**}$.

This completes the proof. □

Corollary 159. *The star graph $K_{1,n}$ does not belong to $\mathcal{I}^m_\lambda(S_3)$, for any $\lambda \in S_3 \setminus \{\rho_0\}$.*

Definition 160. *[36] The bistar $B_{m,n}$ is the graph obtained by joining the central or apex vertex of $K_{1,m}$ and $K_{1,n}$ by an edge.*

Theorem 161. *The bistar graph $B_{m,n}$ belongs to $\mathcal{I}^m(S_3)$, for all $m, n > 1$.*

Proof. Let $B_{m,n}$ be a bistar graph with vertex set $\{u_i, v_j, k_1, k_2 : 1 \leq i \leq m, 1 \leq j \leq n\}$ and the edge set $\{k_1u_i, k_2v_j, k_1k_2 : 1 \leq i \leq m, 1 \leq j \leq n\}$. Consider the following cases:

Case(i): Both m and n are even.

Here we take f as any bijection from $V(B_{m,n}) \to N_{m+n+2}$ and g be the map

$$g(u) = \begin{cases} \rho_0, & \text{if } u = k_1, k_2, \\ \mu_1, & \text{otherwise.} \end{cases}$$

Clearly f and g defines a conjugate S_3-magic labeling of $B_{m,n}$.

Case(ii): m is even , $n \equiv 0(\text{mod } 3)$ and n is odd.

In this case, define f as above and g be defined as

$$g(u) = \begin{cases} \mu_1, & \text{if } u = u_i, 1 \leq i \leq m, \\ \rho_0, & \text{if } u = k_1, k_2, \\ \rho_1, & \text{if } u = v_j, 1 \leq j \leq m, \end{cases}$$

Case(iii): m is even, $n \equiv 1(\text{mod } 3)$ and n is odd.

Let f be the function, $f(u_i) = i, 1 \leq i \leq m, f(k_1) = m + 1, f(k_2) = m + 2, f(v_j) = m + 2 + j$ and define g as

$$g(u) = \begin{cases} \mu_1, & \text{if } u = v_1, u_i, 1 \leq i \leq m, \\ \rho_0, & \text{if } u = k_1, k_2, \\ \rho_1, & \text{if } u = v_{n-1}, \\ \mu_2, & \text{if } u = v_n, v_l, 2 \leq l \leq n - 2. \end{cases}$$

Case(iv): m is even, $n \equiv 2(\text{mod } 3)$ and n is odd.

Let f be as in the Case(i) and let g be defined as

$$g(u) = \begin{cases} \mu_1, & \text{if } u = u_i, 1 \leq i \leq m, \\ \rho_0, & \text{if } u = k_1, k_2, \\ \rho_1, & \text{if } u = v_j, 1 \leq j \leq n-1, \\ \rho_2, & \text{if } u = v_n. \end{cases}$$

Clearly all the cases imply that the bistar graph $B_{m,n} \in \mathcal{I}^{\mathrm{m}}(S_3)$.

□

Corollary 162. *Bistar graph $B_{m,n}$ does not belong to $\mathcal{I}_{\lambda}^{m}(S_3)$, for all m,n.*

Proof. Since $B_{m,n}$ has pendant vertices u_i, v_j,$g(k_1) = g(k_2) = \rho_0$. Hence $g(V(B_{m,n})) \neq \{\lambda\}$, for any $\lambda \in S_3 \setminus \{\rho_0\}$. □

Theorem 163. *The complete graph $K_n \in \mathcal{I}^m(S_3)$ if and only if n is odd or $n \equiv 0(mod\ 3)$.*

Proof. Let the vertices of K_n be denoted by $u_1, u_2, \ldots, u_n$. Suppose that n is odd or $n \equiv 0(\text{mod } 3)$. Then define f be any bijection from $V(G)$ to N_n. If n is odd, define g as: For $1 \leq i \leq n$, $g(u_i) = \begin{cases} \rho_0, & \text{if } i = 1, \\ \mu_1, & \text{otherwise.} \end{cases}$
If $n \equiv 0(\text{mod } 3)$, define g as the constant map $g(u) = \rho_1, \forall u \in V(K_n)$. Clearly we get $g(u) = g^{**}(u), \forall u \in V(G)$.

Conversely, suppose that K_n is an induced S_3-magic graph. Now we will show that n is odd or $n \equiv 0(\text{mod } 3)$. Consider the following cases:

Case(i): $g(u) \in \{\rho_0, \rho_1, \rho_2\}$, for all $u \in V(G)$.

Since all $g(u_i) \in \{\rho_0, \rho_1, \rho_2\}$, the product $g^{**}(u_i) = \prod_{e_j \in N^*(u_i)} (f^*(e_j), g^*(e_j))$ is commutative. Hence assumption $g(u_i) = g^{**}(u_i)$ leads to the following equations:

$$\begin{aligned} g(u_1) &= g(u_2) * g(u_3) * g(u_n) * \cdots * (g(u_1))^{n-1} \\ g(u_2) &= g(u_1) * g(u_3) * g(u_n) * \cdots * (g(u_2))^{n-1} \\ &\vdots \\ g(u_n) &= g(u_1) * g(u_2) * g(u_3) * \cdots * g(u_{n-1}) * (g(u_n))^{n-1} \end{aligned}$$

Which implies

$$\begin{aligned}\rho_0 &= g(u_2) * g(u_3) * g(u_n) * \cdots * (g(u_1))^{n-2} \\ &= g(u_1) * g(u_3) * g(u_n) * \cdots * (g(u_2))^{n-2} \\ &\vdots \\ &= g(u_1) * g(u_2) * g(u_3) * \cdots * g(u_{n-1}) * (g(u_n))^{n-2}.\end{aligned}$$

Thus we get $g(u_i)^{(n-3)} = \rho_0, \forall u_i \in V(K_n)$. Which implies either $n \equiv 0(\text{mod } 3)$ or $g(u_i) = \rho_0, \forall u_i \in V(K_n)$. Hence $g \equiv \rho_0$ if $n \not\equiv 0(\text{mod } 3)$. Which implies $g(V(K_n)) \not\subseteq \{\rho_0, \rho_1, \rho_2\}$ when $n \not\equiv 0(\text{mod } 3)$.

Case(ii): At least one $g(u_i) \in \{\mu_1, \mu_2, \mu_3\}$.
First case implies that if $n \not\equiv 0(\text{mod } 3)$, $g(u_i) \in \{\mu_1, \mu_2, \mu_3\}$, for some i, $1 \leq i \leq n$. Let us call the set $\{\mu_1, \mu_2, \mu_3\}$ as μ-set and the set $\{\rho_0, \rho_1, \rho_2\}$ as the ρ-set for our convenience. If $g(u_i) \in \mu$-set for all i, we get $g^*(u_iu_j) \in \rho$-set. Then $g^{**}(u) \in \rho$-set, for all $u \in V(K_n)$. Then $g \neq g^{**}$. Without loss of generality, let $g(u_1) \in \mu$-set. Then there exist a vertex say, u_2 such that $u_2 \in \rho$-set. Thus $g^*(u_1u_2) \in \mu$-set. Then $g^{**}(u_2) = g(u_2)$ implies there exist a vertex say u_3 such that $u_3 \in \mu$-set. So K_n has at least 3 vertices. If there exists a vertex $u_4 \neq u_1, u_2, u_3$ such that $u_4 \in \mu$-set. Then $g^{**}(u_2) = g(u_2)$ implies there must exist a vertex say, u_5 with $g(u_5) \in \mu$-set. Similarly, if $g(u_4) \in \rho$-set then $g^{**}(u_1) = g(u_1)$ implies there must exist a vertex say, u_5 such that $g(u_5) \in \rho$-set. Similarly, if we examine the possibility of one more vertex on the complete graph we find that there is one more point besides that point. So this kind of induced magic labeling is possible only when n is of the form $3 + 2k, k = 0, 1, 2, \ldots$, which is always an odd number.

Case(i) and (ii) shows that $K_n \in \mathcal{I}^{\text{m}}(S_3)$ when $n \equiv 0(\text{mod } 3)$ or n is odd. This completes the proof of the theorem.

□

Corollary 164. *The complete graph $K_n \in \mathcal{I}^m_\lambda(S_3)$ if and only if $n \equiv 0(\text{mod } 3)$.*

Definition 165. *[2] The (n,k)-banana tree, $Bt(n,k)$ is the graph obtained by starting with n number of k-stars$(K_{1,k-1})$ and connecting one end vertex from each to a new vertex.*

Theorem 166. *The (n,k)-banana tree $Bt(n,k) \in \mathcal{I}^m(S_3)$, for all n and k.*

Proof. Let the vertices of j^{th} copy of k-star be u_{ji}, $1 \leq i \leq k, 1 \leq j \leq n$ where u_{j1} is the center of the k-stars, u_{jk} is the vertex which is connected by the new vertex and the vertex connecting each k-star be denoted by v. Now we can define an induced S_3-magic labeling of $Bt(n,k)$ as follows:
Let f be any bijective map from $V(Bt(n,k))$ to N_{k+1}. Consider the following two cases:

Case(i): k is even.

In this case, define $g(u) = \begin{cases} \rho_0, & \text{if } u = u_{jk}, u_{j1}, v, 1 \leq j \leq n, \\ \mu_1, & \text{otherwise.} \end{cases}$

Case(ii): k is odd.

Here we define g as $g(u) = \begin{cases} \rho_0, & \text{if } u = u_{jk}, u_{j1}, v, u_{j2}, 1 \leq j \leq n, \\ \rho_1, & \text{if } u = u_{ji}, \text{ where } i \text{ is odd and } 2 < i < k, \\ \rho_2, & \text{if } u = u_{ji}, \text{ where } i \text{ is even and } 2 < i < k \end{cases}$

Both cases show that $g^{**}(u) = g(u), \forall u \in V(Bt(m,n))$.

This completes the proof of the theorem. □

Corollary 167. *The (n,k)-banana tree $Bt(n,k) \notin \mathcal{I}^m_\lambda(S_3)$, for any n and k.*

Proof. Since $Bt(n,k)$ has pendant vertices, the corollary follows. □

Definition 168. *[2] Let $< K_{1,n} : m >$ denote the graph obtained by taking 'm' disjoint copies of $K_{1,n}$ and joining a new vertex to the centers of m copies of $K_{1,n}$.*

Theorem 169. *The graph $< K_{1,n} : m >$ is an induced S_3-magic graph for all $n \geq 2$, $m \geq 1$.*

Proof. Let the pendant vertices of i^{th} copy of $K_{1,n}$ be denoted by u_{ij}, $1 \leq i \leq m, 1 \leq j \leq n$ and the center of i^{th} copy of $K_{1,n}$ be k_i. Denote the vertex connecting the centers of m star graphs $K_{1,n}$ in $< K_{1,n} : m >$ be K. We can show that the graph $< K_{1,n} : m > \in \mathcal{I}^{\text{m}}(S_3)$ by defining f and g as follows: Let f be any bijective map from $V(< K_{1,n} : m >) \to N_{m(n+1)+1}$ and define $g : V(< K_{1,n} : m >) \to S_3$ as follows:

Case(i): n is odd.

Let $g(u) = \begin{cases} \rho_0, & \text{if } u = u_{i1}, K, k_i, 1 \leq i \leq m, \\ \mu_1, & \text{otherwise.} \end{cases}$

Case(ii): n is even.

Let $g(u) = \begin{cases} \rho_0, & \text{if } u = K, k_i, 1 \leq i \leq m, \\ \mu_1, & \text{otherwise.} \end{cases}$

One can easily show that the vertex labeling g in the above two cases defines an induced S_3-magic labeling. □

Corollary 170. *The graph* $< K_{1,n} : m > \notin \mathcal{I}_\lambda^m(S_3)$ *for any* n *and* m.

Proof. Since $< K_{1,n} : m >$ has pendant vertices it does not belong to $\mathcal{I}_\lambda^{\mathrm{m}}(S_3)$. □

Definition 171. *[37] The windmill graph* $K_m^{(n)}$ *is the graph consisting of* n *copies of the complete graph* K_m *with a vertex in common.*

Theorem 172. *The graph* $K_m^{(n)} \in \mathcal{I}^m(S_3)$, *for all* m *and* n.

Proof. Let the set of vertices of the i^{th} copy of K_m in $K_m^{(n)}$ be $\{k, u_{i2}, u_{i3}, \ldots u_{im}\}$, where $i = 1, 2, \ldots, n$ and k is the common vertex. Consider the following three cases:

Case(i): m is odd or m is even and n is even.

Define f as any bijection from $V(K_m^{(n)})$ to $N_{n(m-1)+1}$ and let g from $V(K_m^{(n)})$ to S_3 as $g(u) = \begin{cases} \rho_0, & \text{if } u = k, \\ \mu_1, & \text{otherwise.} \end{cases}$

Case(ii): m is even and n is odd.

Define f as above and define g as $g(u) = \begin{cases} \rho_0, & \text{if } u = k, u_{1j}, 2 \leq j \leq m, \\ \mu_1, & \text{otherwise.} \end{cases}$

One can easily prove that $g(u) = g^{**}(u), \forall u \in V(K_m^{(n)})$. □

Corollary 173. *The windmill graph* $K_m^{(n)}$ *belongs to* $\mathcal{I}_\lambda^m(S_3)$ *when* $m \equiv 0(mod\ 3)$ *and* $n \equiv 1(mod\ 3)$.

Proof. Let us denote the vertices of $K_m^{(n)}$ as in the above theorem. Now suppose there exist a map $g : V(K_m^{(n)}) \to S_3 \setminus \{\rho_0\}$ such that $g(u) = \lambda, \forall u \in V(K_m^{(n)})$. Clearly $\lambda \in \{\rho_1, \rho_2\}$. Without loss of generality, let $\lambda = \rho_1$. Then ,for $1 \leq i \leq n, 2 \leq j \leq m$ $g^{**}(u_{ij}) = g(u_{ij})$ implies $(\rho_1^2)^{m-1} = \rho_1$. i,e., $(\rho_2)^{m-1} = \rho_1$. This implies $m \equiv 0(\text{mod } 3)$. Now $g^{**}(k) = g(k)$ implies $((\rho_1^2)^{m-1})^n = \rho_1$. i.e $\rho_1^n = \rho_1$. Which implies $n \equiv 1(\text{mod } 3)$. This completes the proof of the corollary. □

5.5 Path Related Graphs

Observe that, when A is an abelian group, the path graph P_n is an induced A-magic graph if and only if n is a multiple of 3 [7]. Similarly, if A is non-abelian we can prove that the path P_n is induced A magic if and only if n is a multiple of 3, using definition 121. So in general, we have the following result.

Theorem 174. *For any finite group A, the path P_n is induced A-magic if and only if n is a multiple of 3.*

Theorem 175. *The comb graph $CB_n \notin \mathcal{I}^m(S_3)$ for any n.*

Let CB_n denote the comb graph with vertex set $\{u_i, v_i : i = 1, 2, 3, \ldots, n\}$ and edge set $\{u_iu_{i+1}, u_jv_j : i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, n\}$. If possible, there exist two functions f and g as in the definition 121. Since v_i's are pendant vertices adjacent to u_i, we get $g(u_i) = \rho_0, \forall i$. Now $g^{**}(u_i) = g(u_i)$ implies $g^{**}(u_i) = g(v_i) = g(u_i) = \rho_0$. Thus $g(u) = \rho_0, \forall u \in V(CB_n)$. Hence $CB_n \notin \mathcal{I}^{\mathrm{m}}(S_3)$ for any n.

Corollary 176. *The comb graph $CB_n \notin \mathcal{I}^m_\lambda(S_3)$ for any n.*

Theorem 177. *The triangular snake graph T_n is an induced S_3-magic graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of path P_n and denote v_i be the vertex joining u_i and u_{i+1}. Define $f : V(T_n) \to N_{2n-1}$ be any bijection and define $g : V(T_n) \to S_3$ as

$$g(u) = \begin{cases} \mu_1, & \text{if } u = u_1, u_n, v_1, v_{n-1}, \\ \rho_0, & \text{otherwise.} \end{cases}$$

Clearly, the maps defined above will determine an induced S_3-magic labeling for T_n. □

Corollary 178. *The triangular snake graph $T_n \notin \mathcal{I}^m_\lambda(S_3)$ for any n.*

Proof. Let us denote the vertices of T_n as in the above theorem. Suppose that $T_n \in \mathcal{I}^{\mathrm{m}}_\lambda(S_3)$ then there exist a constant map $g(u) = a, \forall u \in V(T_n)$, $a \in S_3 \backslash \{\rho_0\}$ such that $g = g^{**}$. In particular, we have $a = g^{**}(u_2) = (a^2)^4 = a^8$. *i.e.*, $a = a^2$. Which implies $a = \rho_0$, a contradiction. Hence the proof. □

Theorem 179. *The double triangular graph $D(T_n)$ is an induced S_3-magic graph if and only if $n \equiv 0 (mod\ 3)$.*

Proof. Let $\{u_i, v_j, w_j : i = 1, 2, \ldots, n, j = 1, 2, \ldots, n-1\}$ be the vertex set of $D(T_n)$ where u_i's are the vertices of the common path P_n and v_i, w_i are the vertices adjacent to u_i and u_{i+1}.

Suppose that $n \equiv 0 (\text{mod } 3)$. Then define f as any bijective map from $V(D(T_n)) \to N_{3n-2}$ and let $g : V(D(T_n)) \to S_3$ be defined as

$$g(u) = \begin{cases} \rho_0, & \text{if } u = u_i \text{ with } i \equiv 2(\text{mod } 3), \\ & \quad \text{or } u = v_{3k}, w_{3k}, k = 1, 2, \ldots, \\ \mu_1, & \text{otherwise.} \end{cases}$$

Clearly, we can show that $g(u) = g^{**}(u), \forall u \in V(D(T_n))$.

Conversely, suppose that $D(T_n) \in \mathcal{I}^{\text{m}}(S_3)$. We know that $g(V(D(T_n))) \nsubseteq \{\mu_1, \mu_2, \mu_3\}$. Now consider the following cases:

Case(i): $g(v_1) \in \{\rho_0, \rho_1, \rho_2\}$.

Here we consider the following sub cases:

Subcase(a): $g(v_1) = \rho_1$ and $g(u_1), g(u_2) \in \{\rho_0, \rho_1, \rho_2\}$.

From our assumption $g(v_1) = g^{**}(v_1)$. Then

1. $g(u_1) = g(u_2) = \rho_1$ or
2. $g(u_1) = \rho_0$ and $g(u_2) = \rho_2$ or
3. $g(u_1) = \rho_2$ and $g(u_2) = \rho_0$.

If $g(u_1) = g(u_2) = \rho_1$ then $g(u_1) = \rho_1 = g^{**}(u_1) = (g(u_1))^3 * g(v_1) * g(w_1) * g(u_2) = (\rho_1)^3 * \rho_1 * g(w_1) * \rho_1 = \rho_2 * g(w_1)$ implies $g(w_1) = \rho_2$. Then $g^{**}(w_1) = g(u_1) * g(u_2) * g(w_1)^2 = \rho_1 * \rho_1 * (\rho_2)^2 = \rho_0 \neq g(w_1)$. So $g(u_1) = g(u_2) \neq \rho_1$. Similarly, if we consider (2), we will get $g(w_1) = \rho_0$ and $g^{**}(w_1) = \rho_2$. Thus (2) is not possible. In a similar manner, we can show that (3) is also not possible. Hence $g(v_1) \neq \rho_1$ when $g(u_1), g(u_2) \in \{\rho_0, \rho_1, \rho_2\}$.

Subcase(b): $g(v_1) = \rho_2$ and $g(u_1), g(u_2) \in \{\rho_0, \rho_1, \rho_2\}$.

This case is similar to subcase(a) and we can prove that $g(v_1) \neq \rho_2$, when $g(u_1), g(u_2) \in \{\rho_0, \rho_1, \rho_2\}$.

Subcase(c): $g(v_1) = \rho_0$ and $g(u_1), g(u_2) \in \{\rho_0, \rho_1, \rho_2\}$.

If $g(v_1) = \rho_0$ then either

1. $g(u_1) = \rho_1$ and $g(u_2) = \rho_2$.

2. $g(u_1) = \rho_2$ and $g(u_2) = \rho_1$ or
3. $g(u_1) = \rho_0 = g(u_2)$.

If we consider the case (1), $g(u_1) = g^{**}(u_1)$ implies $g(w_1) = \rho_2$. Then we get $g^{**}(w_1) = \rho_1$. Which is a contradiction. So (1) is not possible. Similarly, we can show that (2) is also not possible. Now consider (3). $g = g^{**}$ implies that $g(w_1) = \rho_0$. To get a non zero labeling (i.e., $g(u) \neq \rho_0, \forall u \in V(D(T_n))$) there exists one v_i or w_i such that $g(v_i) \neq \rho_0$ or $g(w_i) \neq \rho_0$. Without loss of generality, let $g(v_2) = \rho_1$ or ρ_2. Again without loss of generality, let $g(v_2) = \rho_1$. Then $g(u_2) = g^{**}(u_2) = \rho_0$ implies

(a) $g(u_3) = g(w_2) = \rho_1$

(b) $g(u_3) = \rho_2$ and $g(w_2) = \rho_0$ or

(b) $g(u_3) = \rho_0$ and $g(w_2) = \rho_2$

Then (a) implies $g^{**}(w_2) = \rho_1 * \rho_2 = \rho_0 \neq g(w_2)$. (b) implies $g^{**}(w_2) = \rho_0 * \rho_2 = \rho_2 \neq g(w_2)$ and (c) implies $g^{**}(w_2) = \rho_2 * \rho_2 = \rho_1 \neq g(w_2)$. So (3) is also not possible.

Subcase(d): $g(v_1) \in \rho$-set $g(u_1), g(u_2) \in \{\mu_1, \mu_2, \mu_3\}$.
Suppose $g(u_1), g(u_2) \in \mu$-set. Then $g(u_1) = g^{**}(u_1)$ implies $g(w_1) \in \mu$-set. Then we can see that $g^{**}(w_1) = g^*(u_1w_1) * g^*(u_2w_1) \in \rho$-set. Hence $g(w_1) \neq g^{**}(w_1)$. So $g(v_1) \notin \rho$-set.
All the above subcases show that there does not exist an induced S_3-magic labeling of $D(T_n)$ with $g(v_1) \in \{\rho_0, \rho_1, \rho_2\}$.

Case(ii): $g(v_1) \in \mu$-set.
This case implies that either

1. $g(u_1) \in \rho$-set and $g(u_2) \in \mu$-set or
2. $g(u_1) \in \mu$-set and $g(u_2) \in \rho$-set.

Consider the case (1). Then $g(u_1) = g^{**}(u_1)$ implies that $g(w_1)$ should belongs to ρ-set. Which in turn implies that $g^{**}(w_1) \in \mu$-set. Hence $g(w_1) \neq g^{**}(w_1)$. So g does not provide an induced S_3-magic labeling for $D(T_n)$.

Now consider the case (2). Then by the definition of induced S_3-magic labeling, we get $g(w_1) \in \mu$-set. If $g(v_2) \in \rho$-set then $g(u_2) = g^{**}(u_2)$ implies either $g(u_3) \in \mu$-set and $g(w_2) \in \rho$-set or $g(u_3) \in \rho$-set and

$g(w_2) \in \mu$-set. But both of these values lead to the contradiction that $g^{**}(w_2) \neq g(w_2)$. Hence $g(v_2) \in \{\mu_1, \mu_2, \mu_3\}$.

Now, $g(v_2) \in \mu$-set implies $g(u_3) \in \mu$-set. Which again implies $g(w_2) \in \mu$-set. So when $n = 3$, there we can find an induced S_3-magic labeling like this.

Suppose $n > 3$ and $g(v_3) \in \mu$-set then $g(u_4) \in \rho$-set(since $g^{**}(v_3) = g(v_3)$). Then $g^{**}(u_3) = g(u_3)$ implies $g(w_3) \in \rho$-set. But then we get $g^{**}(w_3) \in \mu$-set, which is a contradiction. Hence $g(v_3) \in \rho$-set. Also $g(v_3) \in \rho$-set implies $g(u_4) \in \mu$-set and $g(w_3) \in \rho$-set.

If $g(v_4) \in \rho$-set then $g(u_5) \in \mu$-set and $g^{**}(u_4) = g(u_4)$ implies $g(w_4) \in \mu$-set. But then we get $g^{**}(w_4) \in \rho$-set, a contradiction. So $g(v_4) \in \mu$-set. Hence $g(u_5) \in \rho$-set and $g(w_4) \in \mu$-set. Proceeding like this, we get For $1 \leq i \leq n, 1 \leq j \leq n-1$,

$$g(v_j), g(w_j) \in \begin{cases} \rho\text{-set, if } i = 3k, k = 1, 2, \ldots, \\ \mu\text{-set, otherwise.} \end{cases} \quad \text{and}$$

$$g(u_i) \in \begin{cases} \rho\text{-set, if } i = 2(\text{mod } 3), \\ \mu\text{-set, otherwise.} \end{cases}$$

One can easily prove that this kind of induced S_3-magic labeling is possible only when n is a multiple of 3. This completes the proof of the theorem.

□

Corollary 180. *The double triangular graph $D(T_n)$ does not belong to $\mathcal{I}_\lambda^m(S_3)$ for any n.*

Proof. The above theorem shows that the $g(V(G))$ can not be singleton. □

Theorem 181. *The alternate triangular graph $A(T_n)$ is induced S_3-magic for all $n > 4$.*

Proof. Denote the vertices of the path P_n by $u_1, u_2, \ldots, u_n$ and the vertex which join u_i and u_{i+1} be denoted by v_i. Now consider the following cases:

Case(i): n is odd.

Suppose n is odd, then there exists one pendant vertex in $A(T_n)$. Without loss of generality, let u_1 be the pendant vertex in $A(T_n)$. Define f :

$V(A(T_n)) \to N_{n+\lfloor \frac{n}{2} \rfloor}$ be any bijective map and define $g : V(A(T_n)) \to S_3$ as follows:

$$g(u) = \begin{cases} \rho_0, & \text{if } u = u_i, i = 1+4k, 2+4k, k = 0,1,2,\dots, \\ \mu_1, & \text{otherwise.} \end{cases}$$

We can easily show that $g(u) = g^{**}(u), \forall u \in V(A(T_n))$.

Case(ii): n is even and the triangle starts from the first vertex u_1.

In this case, define f as above and define g as:

For $k = 0,1,2,\dots, g(u) = \begin{cases} \rho_1, & \text{if } u = u_{1+4k}, u_{2+4k}, v_{1+4k}, \\ \rho_2, & \text{if } u = u_{3+4k}, u_{4+4k}, v_{3+4k}. \end{cases}$

Clearly $g = g^{**}$.

Case(iii): n is even and the triangle starts from the second vertex u_2.

Here define f as above and define g as

$$g(u) = \begin{cases} \rho_0, & \text{if } u = u_1, u_2, u_{n-1}, u_n, v_j, 4 \le j \le n-4, \\ \mu_1, & \text{if } v = u_i, 3 \le i \le n-2, v_2, v_{n-2}. \end{cases}$$

Then the above f and g will determine an induced S_3-magic labeling of $A(T_n)$.

□

Corollary 182. *The alternate triangular graph $A(T_4)$ is not an induced S_3-magic when the triangle starts from the second vertex u_2.*

Proof. Since u_1 and u_4 are pendant vertices we must have $g(u_2) = g(u_3) = \rho_0$. If $g(v_2) = a, a \in S_3 \setminus \{\rho_0\}$. Then $g^{**}(v_2) = a^2$. Suppose $g^{**} = g$ then $a^2 = a$ implies $a = \rho_0$. Which is a contradiction. Hence the proof of the corollary.

□

Corollary 183. *The graph $A(T_n) \notin \mathcal{I}^{m}_{\lambda}(S_3)$ for any n and any $\lambda \in S_3 \setminus \{\rho_0\}$.*

Proof. If $A(T_n)$ has pendant vertices then clearly $A(T_n) \notin \mathcal{I}^{m}_{\lambda}(S_3)$. Suppose that there is no pendant vertex in $A(T_n)$. Suppose there exists a $g : V(A(T_n)) \to S_3$ such that $g(u) = \lambda, \forall u \in V(A(T_n)), \lambda \in S_3 \setminus \{\rho_0\}$. Then $g^{**}(u_2) = (\lambda^2)^3 = \lambda^6 = \rho_0$. Which is a contradiction. Hence the corollary. □

Theorem 184. *The ladder graph L_n is an induced S_3-magic graph if and only if either n is even or n is a multiple of* 3.

Proof. Let the vertex set of L_n be $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E(G) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_i : 1 \leq i \leq n\}$ where $i+1$ is taken over modulo n. Now consider the following cases:

Case(i): n is a multiple of 3.

Let f be any bijective map from $V(L_n)$ to N_{2n} and let $g : V(G) \to S_3$ be defined as:

$$\text{For } 1 \leq i \leq n,\ g(u_i) = \begin{cases} \rho_1, & \text{if } i \equiv 1 (\text{mod } 3), \\ \rho_0, & \text{if } i \equiv 2 (\text{mod } 3), \\ \rho_2, & \text{if } 1 \equiv 0 (\text{mod } 3). \end{cases} \quad \text{and}$$

$$g(v_i) = \begin{cases} \rho_2, & \text{if } i \equiv 1 (\text{mod } 3), \\ \rho_0, & \text{if } i \equiv 2 (\text{mod } 3), \\ \rho_1, & \text{if } 1 \equiv 0 (\text{mod } 3). \end{cases}$$

We can see that $g(u) = g^{**}(u), \forall u \in V(L_n)$.

Case(ii): n is even.

In this case, let f as in the above case and let g be the map

$$g(u_i) = g(v_i) = \begin{cases} \rho_1, & \text{if } i = 1+4k, 2+4k, k = 0, 1, 2, \ldots, \\ \rho_2, & \text{if } i = 3+3k, 4+4k, k = 0, 1, 2, \ldots \end{cases}$$

The above f and g will determine an induced S_3-magic labeling of L_n.

Conversely, suppose that L_n is induced S_3-magic. So there exist two functions f and g as in the definition 121. Now, we will show that n is either an even number or a multiple of 3. Consider the following cases:

Case(1): $g(u_1) \in \rho$-set and $g(v_1) \in \mu$-set.

Suppose $g(u_1) \in \rho$-set and $g(v_1) \in \mu$-set. Then $g^{**}(u) = g(u), \forall u \in V(L_n)$ implies that

$$g(u_i) \in \begin{cases} \rho\text{-set, if } i \equiv 1 (\text{mod } 6) \text{ or } i \equiv 0 (\text{mod } 6), \\ \mu\text{-set, otherwise.} \end{cases}$$

and

$$g(v_i) \in \begin{cases} \rho\text{-set, if } i \equiv 3 (\text{mod } 6) \text{ or } i \equiv 4 (\text{mod } 6), \\ \mu\text{-set, otherwise.} \end{cases}$$

Such an induced magic labeling is possible only when n is a multiple of 6. Similarly, we can show that if $g(u_1) \in \mu$-set and $g(v_1) \in \rho$-set then n is a multiple of 6.

Case(2): $g(u_1), g(v_1) \in \mu$-set.
In this case, $g(u) = g^{**}(u), \forall u \in V(L_n)$ implies

$$g(u_i), g(v_i) \in \begin{cases} \mu\text{-set, if } i \equiv 1 (\text{mod } 3) \text{ or } i \equiv 0 (\text{mod } 3), \\ \rho\text{-set, if } i \equiv 2 (\text{mod } 3). \end{cases}$$

Which implies n should be a multiple of 3.

Case(3): $g(u_1), g(v_1) \in \rho$-set. Consider the following subcases:

Subcase(a) : $g(u_1) = g(v_1) = \rho_1$.
Suppose that $g(u_1) = g(v_1) = \rho_1$ then $g(u_1) = g(v_1) = \rho_1 = g^{**}(u_1) = g^{**}(v_1)$ implies $g(u_2) = g(v_2) = \rho_1$ and $g(u_2) = g(v_2) = g^{**}(u_2) = g^{**}(v_2)$ implies $g(u_3) = g(v_3) = \rho_2$. Proceeding like this, we get

$$g(u_i) = g(v_i) = \begin{cases} \rho_1, & \text{if } i = 1 + 4k, 2 + 4k, k = 0, 1, 2, \ldots, \\ \rho_2, & \text{if } i = 3 + 4k, 4 + 4k, k = 0, 1, 2, \ldots \end{cases}$$

It is clear that this g determine an induced S_3-magic labeling for L_n if n is even. We get a similar observation if $g(u_1) = g(v_1) = \rho_2$.

Subcase (b) : $g(u_1) = \rho_1$ and $g(v_1) = \rho_2$.
In this case applying $g(u_i) = g^{**}(u_i), g(v_i) = g^{**}(v_i)$ starting from $i = 1$ we get

$$g(u_i) = \begin{cases} \rho_1, & \text{if } i \equiv 1 (\text{mod } 3), \\ \rho_0, & \text{if } i \equiv 2 (\text{mod } 3), \\ \rho_2, & \text{if } i \equiv 0 (\text{mod } 3). \end{cases} \quad \text{and}$$

$$g(v_i) = \begin{cases} \rho_2, & \text{if } i \equiv 1 (\text{mod } 3), \\ \rho_0, & \text{if } i \equiv 2 (\text{mod } 3), \\ \rho_1, & \text{if } i \equiv 0 (\text{mod } 3). \end{cases}$$

It is clear that this g will determine an induced S_3-magic labeling for L_n only when n is a multiple of 3. A similar observation is obtained if $g(u_1) = \rho_2$ and $g(v_1) = \rho_1$.

Subcase(c): $g(u_1) = \rho_1$ and $g(v_1) = \rho_0$.

As in the above cases $g = g^{**}$ leads

$$g(u_i) = \begin{cases} \rho_1, & \text{if } i \equiv 1(\text{mod } 12), i \equiv 8(\text{mod } 12), i \equiv 10(\text{mod } 12) \\ & \quad \text{or } \equiv 11(\text{mod } 12), \\ \rho_2, & \text{if } i \equiv 2(\text{mod } 12), i \equiv 3(\text{mod } 12), i \equiv 5(\text{mod } 12) \\ & \quad \text{or } i \equiv 0(\text{mod } 12), \\ \rho_0, & \text{if } i \equiv 4(\text{mod } 12), i \equiv 6(\text{mod } 12), i \equiv 7(\text{mod } 12) \\ & \quad \text{or } i \equiv 9(\text{mod } 12). \end{cases}$$

and

$$g(v_i) = \begin{cases} \rho_0, & \text{if } i \equiv 1(\text{mod } 12), i \equiv 3(\text{mod } 12), i \equiv 10(\text{mod } 12) \\ & \quad \text{or } i \equiv 0(\text{mod } 12), \\ \rho_2, & \text{if } i \equiv 2(\text{mod } 12), i \equiv 4(\text{mod } 12), i \equiv 5(\text{mod } 12) \\ & \quad \text{or } i \equiv 7(\text{mod } 12), \\ \rho_1, & \text{if } i \equiv 6(\text{mod } 12), i \equiv 8(\text{mod } 12), i \equiv 9(\text{mod } 12) \\ & \quad \text{or } i \equiv 11(\text{mod } 12). \end{cases}$$

Such an induced S_3-magic labeling is possible when n is a multiple of 12. Similarly, we see that if $g(u_1) = \rho_0$ and $g(v_1) = \rho_1$, $g(u_1) = \rho_2$ and $g(v_1) = \rho_0$ or $g(u_1) = \rho_0$ and $g(v_1) = \rho_2$, then n must be a multiple of 12. Which in turn a multiple of 3.

From all the cases and sub cases we see that L_n is an induced S_3-magic graph if either n is an even number or n is a multiple of 3.

Hence the theorem. □

Definition 185. *[31] An open ladder graph $O(L_n), n \geq 2$ is obtained from two paths of length $n-1$ with $V(O(L_n)) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 2 \leq i \leq n-1\}$.*

Theorem 186. *For $n \geq 2$, the open ladder graph is an induced S_3-magic if and only if n is odd or $n \equiv 0(\text{mod } 3)$.*

Proof. Suppose that n is odd or $n \equiv 0(\text{mod } 3)$. Then define $f : V(O(L_n)) \to N_{2n}$ as for $1 \leq i \leq n, f(u_i) = i, f(v_i) = n + i$.
If n is odd then let $g : V(O(L_n)) \to S_3$ as

$$g(u_i) = g(v_i) = \begin{cases} \rho_0, & \text{if } i \text{ is even}, \\ \rho_1, & \text{if } i = 1 + 4k, k = 0, 1, 2, \ldots, \\ \rho_2, & \text{if } i = 3 + 4k, k = 0, 1, 2, \ldots \end{cases}$$

Suppose $n \equiv 0 (\text{mod } 3)$, define g by

$$g(u_i) = \begin{cases} \rho_0, & \text{if } i = 2 + 3k, k = 0, 1, 2, \ldots, \\ \mu_1, & \text{if } i \equiv 1(\text{mod } 9) \text{ or } 3(\text{mod } 9), \\ \mu_3, & \text{if } i \equiv 4(\text{mod } 9) \text{ or } 6(\text{mod } 9), \\ \mu_2, & \text{if } i \equiv 7(\text{mod } 9) \text{ or } 0(\text{mod } 9). \end{cases}$$

$$g(v_i) = \begin{cases} \rho_0, & \text{if } i = 2 + 3k, k = 0, 1, 2, \ldots, \\ \mu_2, & \text{if } i \equiv 1(\text{mod } 9) \text{ or } 3(\text{mod } 9), \\ \mu_1, & \text{if } i \equiv 4(\text{mod } 9) \text{ or } 6(\text{mod } 9), \\ \mu_3, & \text{if } i \equiv 7(\text{mod } 9) \text{ or } 0(\text{mod } 9). \end{cases}$$

We can easily prove that $g(u) = g^{**}(u), \forall u \in V(O(L_n))$.

Conversely, suppose that $O(L_n)$ is an induced S_3-magic graph. So there exist two functions f and g satisfying $g(u) = g^{**}(u), \forall u \in V(O(L_n))$. We know that $g(V(O(L_n))) \nsubseteq \{\mu_1, \mu_2, \mu_3\}$. Since u_1, v_1, u_n, v_n are pendant vertices we have $g(u_2) = g(v_2) = g(u_{n-1}) = g(v_{n-1}) = \rho_0$. Now consider the following cases:

Case (i): $g(V(O(L_n))) \subseteq \{\rho_0, \rho_1, \rho_2\}$.

Subcase (a): $g(u_1) = \rho_0 = g(v_1)$.

Clearly $g = g^{**}$ implies $g(u) = \rho_0, \forall u \in V(O(L_n))$.

Subcase (b): $g(u_1) = \rho_1 = g(v_1)$.

Suppose $g(u_1) = \rho_1 = g(v_1)$. Then $g^{**}(u_2) = g(u_2)$ and $g^{**}(v_2) = g(v_2)$ implies $g(u_i) = g(v_i) = \begin{cases} \rho_0, & \text{if } i \text{ is even}, \\ \rho_1, & \text{if } i = 1 + 4k, k = 0, 1, 2, 3, \ldots, \\ \rho_2, & \text{if } i = 3 + 4k, k = 0, 1, 2, 3, \ldots \end{cases}$

Since u_n and v_n are pendant vertices, $g(u_{n-1}) = g(v_{n-1}) = \rho_0$, $n - 1$ must be an even number for this induced S_3-magic labeling. Which implies n must be an odd integer. A similar observation is obtained when $g(u_1) = \rho_2 = g(v_1)$.

Subcase (c): $g(u_1) = \rho_1$, $g(v_1) = \rho_2$.

If $g(u_1) = \rho_1, g(v_1) = \rho_2$. Then $g^{**}(u) = g(u)$ implies,

$$\text{for } k = 0, 1, 2, \ldots, g(u) = \begin{cases} \rho_1, & \text{if } u = u_{1+3k}, v_{3+3k}, \\ \rho_2, & \text{if } u = u_{3+3k}, v_{1+3k}, \\ \rho_0, & \text{if } u = u_{2+3k}, v_{2+3k}. \end{cases}$$

Since u_n and v_n are pendant vertices, $g(u_{n-1}) = g(v_{n-1}) = \rho_0$, so we must have $n-1 = 2+3k$, for some k. i.e., n is a multiple of 3. We get a similar conclusion if $g(u_1) = \rho_2, g(v_1) = \rho_1$.

Subcase (d): $g(u_1) = \rho_0,\ g(v_1) = \rho_1$.

Here by assuming $g(u_2) = g^{**}(u_2)$ and $g(v_2) = g^{**}(v_2)$ leads that

$$g(u_i) = \begin{cases} \rho_0, & \text{if } i \equiv 1(\text{mod } 12), 2(\text{mod } 12), 3(\text{mod } 12), 8(\text{mod } 12), \\ \rho_2, & \text{if } i \equiv 5(\text{mod } 12), 6(\text{mod } 12), 7(\text{mod } 12), 0(\text{mod } 12), \\ \rho_1, & \text{if } i \equiv 4(\text{mod } 12), 9(\text{mod } 12), 10(\text{mod } 12), 11(\text{mod } 12). \end{cases}$$

and

$$g(v_i) = \begin{cases} \rho_0, & \text{if } i = 2(\text{mod } 12), 7(\text{mod } 12), 8(\text{mod } 12), 9(\text{mod } 12), \\ \rho_1, & \text{if } i \equiv 1(\text{mod } 12), 6(\text{mod } 12), 11(\text{mod } 12), 0(\text{mod } 12), \\ \rho_2, & \text{if } i \equiv 3(\text{mod } 12), 4(\text{mod } 12), 5(\text{mod } 12), 10(\text{mod } 12). \end{cases}$$

Then we obtain a similar observation. As in the above subcase $g(u_{n-1}) = g(v_{n-1}) = \rho_0$ implies either $n-1 \equiv 2(\text{mod } 12)$ or $n-1 \equiv 8(\text{mod } 12)$. Which implies $n \equiv 0(\text{mod } 3)$. A similar observation is obtained if we assume

(1) $g(u_1) = \rho_0$ and $g(v_1) = \rho_2$

(2) $g(u_1) = \rho_1$ and $g(v_1) = \rho_0$ or

(3) $g(u_1) = \rho_2$ and $g(v_1) = \rho_0$.

All the above subcases implies that if $g(V(O(L_n))) \subseteq \{\rho_0, \rho_1, \rho_2\}$ then either n is even or n is a multiple of 3.

Case(ii): $g(u_1) \in \rho$-set and $g(v_1) \in \mu$-set.

Since $g(u_2) = g(v_2) = \rho_0$ and $g = g^{**}$ then we must have $g(u_3) \in \rho$-set and $g(v_3) \in \mu$-set. Then $g^*(u_3v_3) \in \mu$-set. Which again implies that

$$g(u_i) \in \begin{cases} \mu\text{-set}, & \text{if } i \equiv 4(\text{mod } 6) \text{ or } i \equiv 0(\text{mod } 6), \\ \rho\text{-set}, & \text{otherwise }. \end{cases}$$

$$g(v_i) \in \begin{cases} \mu\text{-set}, & \text{if } i \equiv 1(\text{mod } 6) \text{ or } i \equiv 3(\text{mod } 6), \\ \rho\text{-set}, & \text{otherwise.} \end{cases}$$

Thus $g(u_{n-1}) = g(v_{n-1}) = \rho_0$ implies that either $n-1 \equiv 2(\text{mod } 6)$ or $n-1 \equiv 5(\text{mod } 6)$. Which shows that n must be a multiple of 3.

Case(iii): $g(u_1) \in \mu$-set and $g(v_1) \in \rho$-set.

This case is similar to case (ii) and we get n is a multiple of 3.

Case(iv): $g(u_1), g(v_1) \in \mu$-set.

By assuming $g(u) = g^{**}(u),\ \forall u \in V(O(L_n))$ and starting with $g(u_2) = g(v_2) = \rho_0$, we see that

$$g(u_i), g(v_i) \in \begin{cases} \mu\text{-set}, & \text{if } i \equiv 1(\text{mod } 3) \text{ or } i \equiv 0(\text{mod } 3), \\ \rho\text{-set} & \text{if } i \equiv 2(\text{mod } 3). \end{cases}$$

As in the above case, this type of induced S_3-magic labeling is possible only when n is a multiple of 3.

All the above cases and sub cases show that, the open ladder graph $O(L_n), n > 2$ is an induced S_3-magic if and only if n is odd or a multiple of 3. □

Corollary 187. *The graph $O(L_n) \notin \mathcal{I}^m_\lambda(S_3)$, for any $\lambda \in S_3 \setminus \{\rho_0\}$.*

Chapter 6

Conjugate S_3-Magic Labeling of Graphs

In this chapter, we introduce a new magic labeling of graphs using a non-abelian group namely, the conjugate A-magic labeling of graphs, and investigate conjugate S_3-magic labeling of some graphs.

6.1 Introduction

In this chapter, we introduce a new magic labeling of graphs using a non-abelian group namely, the conjugate A-magic labeling of graphs and investigate conjugate S_3-magic labeling of some graphs. Recall the definition of A-magic labeling of graphs, where A is non-abelian group [29].

Definition 188. *[29] Let $G = (V(G), E(G))$ be a finite graph with p vertices and q edges, and let $(A, *)$ be a finite non-abelian group with identity element 1. Let $f : E(G) \to N_q = \{1, 2, \ldots, q\}$ and let $g : E(G) \to A \setminus \{1\}$ be two edge labelings of G such that f is bijective. Define an edge labeling $\ell : E(G) \to N_q \times A \setminus \{1\}$ by*

$$l(e) := (f(e), g(e)),\, e \in E(G).$$

Define a relation $\leq$ on the range of ℓ by:

$$(f(e), g(e)) \leq (f(e^{'}), g(e')) \quad \text{if and only if} \quad f(e) \leq f(e').$$

Then obviously, the relation $\leq$ is a partial order on the range of ℓ.
Let $\{(f(e_1), g(e_1)), (f(e_2), g(e_2)), \ldots, (f(e_k), g(e_k))\}$ be a chain in the range of ℓ. We define the product of this chain as follows:

$$\prod_{i=1}^{k}(f(e_i), g(e_i)) := ((((g(e_1) * g(e_2)) * g(e_3)) * g(e_4)) * \ldots) * g(e_k).$$

Let $u \in V(G)$ and let $N^*(u)$ be the set of all edges incident with u. Consider the restriction of the function ℓ on $N(u)$, that is, $\ell|_{N^*(u)}$. Observe that the range

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of $\ell|_{N^*(u)}$ is a chain, say $(f(e_1), g(e_1)) \leq (f(e_2), g(e_2)) \leq \cdots \leq (f(e_n), g(e_n))$. We define

$$\ell^*(u) = \prod_{i=1}^{n} (f(e_i), g(e_i)). \tag{6.1.1}$$

If $\ell^*(u)$ is a constant, say a for all $u \in V(G)$, we say that the graph G is A-magic. The map ℓ^* is called an A-magic labeling of G and the corresponding constant a is called the magic constant.

In the above definition if the elements $g(e_2), g(e_3), \ldots, g(e_n)$ belong to the conjugacy class determined by $g(e_1)$, then we say that the graph G is conjugate A-magic. Formally, we have the following:

Definition 189. *Let $G = (V(G), E(G))$ be a graph with p vertices and q edges, and A be a finite non-abelian group of order n with identity 1. The graph G is said to be a conjugate A-magic graph if*

(i) for all $u \in V(G)$,

$$\ell^*(u) = \prod_{i=1}^{n} (f(e_i), g(e_i)) = \text{constant in } A (\text{see definition } 188).$$

(ii) the elements $g(e_2), g(e_3), \ldots, g(e_n)$ belong to the conjugacy class determined by $g(e_1)$.

Definition 190. *If the map g in the definition 189 is a constant map then the conjugate A-magic labeling is said to be constant conjugate A-magic labeling otherwise it is said to be non-constant conjugate A-magic labeling.*

6.2 Main Results

Consider the non-abelian group S_3 and investigate the graphs which are conjugate S_3-magic.

Theorem 191. *Let G be a conjugate S_3-magic graph. If G has a vertex of degree 2, then the conjugate S_3-magic constant does not belong to $\{\mu_1, \mu_2, \mu_3\}$.*

Proof. Let G be a conjugate S_3-magic graph with magic constant a. Let v be the vertex of G having degree 2. Let u_1 and u_2 be the vertices adjacent to v. Then $\ell^*(v) = a = g(u_1v) * g(vu_2)$ or $\ell^*(v) = g(vu_2) * g(u_1v)$. Since the product of any two elements(need not be distinct) from a conjugacy class always belongs to $\{\rho_0, \rho_1, \rho_2\}$, we have $a \notin \{\mu_1, \mu_2, \mu_3\}$. □

Corollary 192. *If G is a non-constant conjugate S_3-magic graph determined by the functions $f : E(G) \to N_q$ and $g : E(G) \to S_3 \setminus \{\rho_0\}$. Suppose G has a vertex of order 2 then the range of ℓ^* always belongs to the set $\{\rho_0, \rho_1, \rho_2\}$.*

6.3 Conjugate S_3-Magic Labeling of Some Well Known Graphs

Theorem 193. *There does not exist a non-constant conjugate S_3-magic labeling for the cycle graph C_3.*

Proof. Let the vertices of C_3 be denoted by u_1, u_2 and u_3. Suppose, on the contrary, that there exists a non-constant conjugate S_3-magic labeling for C_3. Without loss of generality, we take $g(u_1u_2) = \rho_1$ then $g(u_2u_3) = \rho_2$ or $g(u_1u_3) = \rho_2$, since g is non-constant. If $g(u_2u_3) = \rho_2$ then $\ell^*(u_2) = \rho_0$ then the magic constant must be ρ_0, so $\ell^*(u_3) = \rho_0 = g(u_1u_3) * g(u_2u_3)$ implies $g(u_1u_3) = \rho_1$ but then $\ell^*(u_1) = g(u_1u_2) * g(u_1u_3) = \rho_1 * \rho_1 = \rho_2$, which is a contradiction. Hence, we cannot label C_3 using ρ_1 and ρ_2 under the map g. Similarly, we can prove that there does not exist a non-constant mapping $g : E(G) \to \{\mu_1, \mu_2, \mu_3\}$ to make the cycle C_3 conjugate S_3-magic. This completes the proof of the theorem. □

Theorem 194. *If $n > 3$, there exists a non-constant S_3-magic labeling for cycle C_n.*

Proof. Let the vertices of C_n be denoted by $u_1, u_2, \ldots, u_n$. We consider the following two cases:

Case(i): n is even.

Suppose n is even. Then take f as any bijective map from $E(C_n)$ to N_n and define the map g as follows: label the adjacent edges of C_n by ρ_1 and ρ_2 alternatively. Clearly $\ell^*(u) = \rho_0, \forall u \in V(G)$.

Case(ii): n is odd and $n > 3$.

Suppose that n is odd and $n > 3$. we define a conjugate S_3-magic labeling of C_n with magic constant ρ_1. Define f and g as follows:

Subcase(a): n is odd and $n \equiv 2 (\text{mod } 3)$.

$$\text{Let } g(u_iu_{i+1}) = \begin{cases} \mu_1, & \text{if } i \equiv 1 (\text{mod } 3), \\ \mu_2, & \text{if } i \equiv 2 (\text{mod } 3), \\ \mu_3, & \text{if } i \equiv 0 (\text{mod } 3). \end{cases} \quad \text{and } f(u_iu_{i+1}) = i, \ 1 \le i \le n,$$

$i+1$ is taken modulo n.

Subcase(b): n is odd and $n \equiv 1 (\text{mod } 3)$.

$$\text{Let } g(u_iu_{i+1}) = \begin{cases} \mu_1, & \text{if } i \equiv 1(\text{mod } 3), 1 \leq i < n, \\ \mu_2, & \text{if } i \equiv 2(\text{mod } 3) \text{ and } i = n, \\ \mu_3, & \text{if } i \equiv 0(\text{mod } 3), 1 \leq i < n. \end{cases}$$

Now define $f(u_iu_{i+1}) = i,\ 1 \leq i \leq n-2,\ f(u_{n-1}u_n) = n,\ f(u_nu_1) = n-1$.

Subcase(c): n is odd and $n \equiv 0 (\text{mod } 3)$.

$$\text{Let } g(u_iu_{i+1}) = \begin{cases} \mu_1, & \text{if } i \equiv 1(\text{mod } 3), 1 \leq i \leq n-2, \\ \mu_2, & \text{if } i \equiv 2(\text{mod } 3), 1 \leq i \leq n-2 \text{ and } i = n, \\ \mu_3, & \text{if } i \equiv 0(\text{mod } 3), 1 \leq i \leq n-2 \text{ and } i = n-1. \end{cases}$$

Now define $f(u_iu_{i+1}) = i, 1 \leq i < n-2,\ f(u_{n-2}u_{n-1}) = n,\ f(u_{n-1}u_n) = n-1, f(u_nu_1) = n-2$.

This completes the proof of the theorem. □

Theorem 195. *There does not exist a non-constant conjugate S_3-magic labeling for the star graph, $K_{1,n}, n \geq 2$.*

Proof. Suppose that $K_{1,n}$ is conjugate S_3-magic with magic constant a. Since there are n pendant vertices in $K_{1,n}$, all pendant edges should be mapped to a under g. So g must be a constant map. We observe that, the star graph $K_{1,n}$ is S_3-magic if and only if either n is odd or $n \equiv 1(\text{mod } 3)$ [29]. Thus we have $K_{1,n}$ is S_3-magic if and only if it is conjugate S_3-magic. So $K_{1,n}$ is conjugate S_3-magic if and only if n is odd or $n \equiv 1(\text{mod } 3)$. □

Theorem 196. *The bistar graph B_n is conjugate S_3-magic except when n is odd and $n \equiv 1(\text{mod } 3)$.*

Proof. Let the end vertices of the bridge be k_1 and k_2. Label the pendant vertices of first star by $u_1, u_2, \ldots, u_n$ and the pendant vertices of the second star by $v_1, v_2, \ldots, v_n$.

Case(i): n is even.

Let $f : E(B_n) \rightarrow N_{2n+1}$ be any bijective map. Define $g : E(G) \rightarrow S_3\backslash\{\rho_0\}$ as $g(e) = \mu_1, \forall e \in E(B_n)$. Then clearly $\ell^*(u) = \mu_1, \forall u \in V(B_n)$.

Case(ii): n is odd and $n \equiv 0(\text{mod } 3)$.

In this case, let f as above and define g as $g(e) = \rho_1, \forall e \in E(B_n)$. Then $\ell^*(u) = \rho_1, \forall u \in V(G)$.

Case(iii): n is odd and $n \equiv 2 (\text{mod } 3)$.

In this case also let f as above and define $g(e) = \begin{cases} \rho_2, & \text{if } e = k_1k_2, \\ \rho_1, & \text{otherwise.} \end{cases}$

Clearly $\ell^*(u) = \rho_1, \forall u \in V(B_n)$.

Case(iv): n is odd and $n \equiv 1 (\text{mod } 3)$.

Suppose that, B_n is conjugate S_3-magic with magic constant 'a', $a \in S_3$. So each pendant edge should be mapped to a under the map g. Now let $g(k_1k_2) = b, b \in S_3 \setminus \{\rho_0\}$, then there are n possible values for $\ell^*(k_1)$. But in all the cases $\ell^*(k_1) = a$ implies $b = \rho_0$. Which is a contradiction. So B_n is not conjugate S_3-magic when n is odd and $n \equiv 1 (\text{mod } 3)$.

This completes the proof of the theorem. □

Theorem 197. *The cycle graph C_n with a pendant edge is not conjugate S_3-magic.*

Proof. Let G be the graph C_n with a pendant edge e. Denote the vertices of C_n by $u_1, u_2, \ldots, u_n$. Without loss of generality, let the one end vertex of the pendant edge e is at u_1 and let the other end vertex of e be denoted by u_{n+1}.

Suppose to the contrary that the graph G is conjugate S_3-magic with magic constant 'a', where $a \in S_3$. Clearly $a \neq \rho_0$. Let $g(u_iu_{i+1}) = a_i$, where $a_i \in S_3 \setminus \{\rho_0\}$. Suppose that the conjugate magic constant is ρ_1. Then $g(u_1u_{n+1}) = \rho_1$. But $\ell^*(u_1) = \rho_1$ implies that $g(u_1u_2) * g(u_nu_1) = \rho_0$ also $a_i \in \{\rho_1, \rho_2\}$. Without loss of generality, let $g(u_1u_2) = \rho_1$ and $g(u_nu_1) = \rho_2$. But then $\ell^*(u_2) = \rho_1$ implies $g(u_2u_3) = \rho_0$, which is a contradiction. Similarly, we can prove that the magic constant cannot be ρ_2. Now, suppose that $a = \mu_1$, then $g(u_1u_{n+1}) = \mu_1$. There are 6 possible products for $\ell^*(u_1)$. i.e., $a_1 * a_n * \mu_1 = \mu_1,\ a_n * a_1 * \mu_1 = \mu_1,\ \mu_1 * a_1 * a_n = \mu_1,\ \mu_1 * a_n * a_1 = \mu_1,\ a_1 * \mu_1 * a_n = \mu_1$ or $a_n * \mu_1 * a_1 = \mu_1$. But all these products lead to the same contradiction as above. Hence the proof. □

Theorem 198. *If $n \geq 3$, the wheel W_n is S_3-magic.*

Proof. Let G be the wheel W_n and let the vertices of C_n be $v_1, v_2, \ldots, v_n$ and the vertex of K_1 be k. Consider the following four cases, for all the following cases let f be any bijection from $E(W_n)$ to N_{2n}:

Case(i): n is odd.

In this case, define $g(e) = \mu_1, \forall e \in E(W_n)$. Clearly W_n is conjugate S_3-magic with magic constant μ_1.

Case(ii): n is even and $n \equiv 0 (\text{mod } 3)$.

Here, we define $g : E(W_n) \to S_3 \setminus \{\rho_0\}$ be the constant map $g(e) = \rho_1, \forall e \in E(W_n)$. Then W_n becomes conjugate S_3-magic with constant ρ_0.

Case(iii): n is even and $n \equiv 1 (\text{mod } 3)$.

Here we define f as above and let $g : E(W_n) \to S_3 \setminus \{\rho_0\}$ be defined as : for $1 \leq i \leq n$, $g(v_i v_{i+1}) = \begin{cases} \rho_1, & \text{if } i \text{ is odd}, \\ \rho_2, & \text{if } i \text{ is even}. \end{cases}$ and $g(kv_i) = \rho_1, i+1$ is taken modulo n.

Clearly, $\ell^*(u) = \rho_1, \forall u \in V(W_n)$. Hence, the theorem is valid in this case.

Case(iv): n is even and $n \equiv 2 (\text{mod } 3)$.

We define g as : $g(e) = \begin{cases} \rho_1, & \text{if } e = kv_i, 1 \leq i \leq n, \\ \rho_2, & \text{if } e = v_i v_{i+1}, 1 \leq i \leq n, n+1 = 1. \end{cases}$

In this case, W_n is conjugate S_3-magic with magic constant ρ_2.

This completes the proof of the theorem. □

Theorem 199. *The helm graph $H_n, n \geq 3$ is conjugate S_3-magic if and only if $n \not\equiv 0 (mod\ 3)$.*

Proof. Let H_n be the helm graph of order $2n+1$. Denote the vertices of C_n by $u_1, u_2, \ldots, u_n$, vertex of C_1 be k and denote the other vertices by $v_1, v_2, \ldots, v_n$ such that $u_i v_i$ is a pendant edge. Suppose that $n \not\equiv 0 (\text{mod } 3)$. Now consider the following cases:

Case(i): $n \equiv 1 (\text{mod } 3)$.

Let f be any bijective map from $E(H_n)$ to N_{3n} and let g be the constant map $g(e) = \rho_1, \forall e \in E(H_n)$. Clearly H_n becomes conjugate S_3-magic with constant ρ_1.

Case(ii): $n \equiv 2 (\text{mod } 3)$.

Here, we define f as above and let g be the map

$$g(e) = \begin{cases} \rho_1, & \text{if } e \text{ is a pendant edge}, \\ \rho_2, & \text{otherwise}. \end{cases}$$

Then the above f and g determine a conjugate S_3-magic labeling of H_n with constant ρ_1.

Suppose to the contrary that H_n is conjugate S_3-magic when $n \equiv 0 (\text{mod } 3)$. Let the magic constant be a, $a \in S_3$. Observe that $a \neq \rho_0$. Now if possible, let $a \in \{\rho_1, \rho_2\}$ then $g(e) \in \{\rho_1, \rho_2\}$ and f can be any bijection. Also $g(u_iv_i) = a, 1 \leq i \leq n$. Now

$$\ell^*(u_i) = g(u_iv_i) * g(u_iu_{i+1}) * g(u_{i-1}u_i) * g(u_ik). \tag{6.3.1}$$

Since $\ell^*(u_i) = a$ the equation 6.3.1 implies $g(u_iu_{i+1}) = g(u_{i-1}u_i) = g(u_ik) = b$, where $b \in \{\rho_1, \rho_2\}$. Thus $\ell^*(k) = \underbrace{b * b * \cdots * b}_{n \text{ times}} = \rho_0$. Which is a contradiction. Hence $a \notin \{\rho_1, \rho_2\}$. Now suppose that $a \in \{\mu_1, \mu_2, \mu_3\}$. Without loss of generality, let $a = \mu_1$ then $g(u_iv_i) = \mu_1, \forall 1 \leq i \leq n$. Let $g(u_1u_2) = p,\ g(u_1u_n) = q,\ g(u_1k) = r,$ where $p, q, r \in \{\mu_1, \mu_2, \mu_3\}$ then

$$\ell^*(u_1) = \prod_{e \in N^*(u_1)} (f(e), g(e)) = \mu_1. \tag{6.3.2}$$

Suppose $f(u_1u_2) < f(u_1u_n) < f(u_1k)$ then $\ell^*(u_1) = p * q * r * \mu_1$ or $\mu_1 * p * q * r$ or $p * \mu_1 * q * r$ or $p * q * \mu_1 * r$. If $p * q * r * \mu_1 = \mu_1$ or $\mu_1 * p * q * r* = \mu_1$ then $p * q * r = \rho_0$, which is not possible for any value of $p, q, r \in \{\mu_1, \mu_2, \mu_3\}$. Similarly, we cannot find $p, q, r \in \{\mu_1, \mu_2, \mu_3\}$ satisfying equation 6.3.2 for any possible values of $f(u_1u_2), f(u_1u_n)$ and $f(u_1k)$. Hence $a \notin \{\mu_1, \mu_2, \mu_3\}$. This completes the proof of the theorem.

□

Theorem 200. *The gear graph G_n is conjugate S_3-magic if and only if n is even.*

Proof. Let $G = G_n$. Denote the central vertex of G by k and the vertices of W_n by $u_1, u_2, \ldots, u_n$ and let $v_1, v_2, \ldots, v_n$ be the vertices such that v_i is adjacent to u_i and u_{i+1}. Suppose that n is even. Define f be any bijective map from $E(G)$ to N_{3n}. For $1 \leq i \leq n$, define $g : E(G) \to S_3 \setminus \{\rho_0\}$ as $g(ku_i) = g(u_iv_i) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases}$ and $g(v_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is odd,} \\ \rho_1, & \text{if } i \text{ is even.} \end{cases}$
Clearly, G is conjugate S_3-magic with magic constant ρ_0.

Conversely, suppose that n is odd and G is conjugate S_3-magic with magic constant a. Note that $a \notin \{\mu_1, \mu_2, \mu_3\}$. We have $\ell^*(v_1) = a$ and $g(u_1v_1)$ is conjugate to $g(v_1u_2)$. Observe that the product of two elements from a conjugacy class in S_3 always belongs to $\{\rho_0, \rho_1, \rho_2\}$. Suppose that $a = \rho_0$ and g is a map from $E(G)$ to $\{\rho_1, \rho_2\}$. Without loss of generality, let $g(u_1v_1) = \rho_1$ and $g(v_1u_2) = \rho_2$. Then $\ell^*(u_2) = \rho_0$ implies $g(ku_2) = g(u_2v_2) = \rho_2$. Then

$g(u_2v_2) = \rho_2$ implies $g(v_2u_3) = \rho_1$ and $g(ku_3) = \rho_1$. Proceeding like this, we obtain $g(ku_n) = g(u_nv_n) = \rho_1$ and $g(v_nu_1) = \rho_2$. Then $\ell^*(u_1) = \rho_0$ implies $g(ku_1) = \rho_0$. Which is a contradiction. So $g(e) \notin \{\rho_1, \rho_2\}$. Now suppose g is a map from $E(G)$ to $\{\mu_1, \mu_2, \mu_3\}$. Then $a = \rho_0$ implies $g(u_1v_1) = g(v_1u_2)$. Without loss of generality, let $g(u_1v_1) = g(v_1u_2) = \mu_1$. Now $\ell^*(u_2) = \rho_0$ implies $g(ku_2)*\mu_1*g(u_2v_2) = \rho_0$ or $g(ku_2)*g(u_2v_2)*\mu_1 = \rho_0$ or $g(u_2v_2)*\mu_1*g(ku_2) = \rho_0$ or $g(u_2v_2)*g(ku_2)*\mu_1 = \rho_0$ or $\mu_1*g(ku_2)*g(u_2v_2) = \rho_0$ or $\mu_1*g(u_2v_2)*g(ku_2) = \rho_0$. But we cannot find $g(u_2v_2), g(ku_2) \in \{\mu_1, \mu_2, \mu_3\}$ satisfying any of the above 6 equations.

Now, suppose that $a = \rho_1$ and $g(e) \in \{\rho_1, \rho_2\}$. So $\ell^*(v_1) = \rho_1$ implies $g(u_1v_1) = g(v_1u_2) = \rho_2$. Also $\ell^*(u_2) = \rho_1$ implies $g(u_2v_2) = g(ku_1) = \rho_1$ but then $\ell^*(v_2) = g(u_2v_2) * g(v_2u_3) = \rho_1$ implies $g(v_2u_3) = \rho_0$, which is a contradiction.

Suppose that $a = \rho_1$ with $g(e) \in \{\mu_1, \mu_2, \mu_3\}$. Without loss of generality, let $f(u_1v_1) < f(v_1u_2)$ and $g(u_1v_1) = \mu_1$ and $g(v_1u_2) = \mu_2$. There are 6 possible products for $\ell^*(u_2)$ as above depending on the function f. But all the 6 product leads to a contradiction as in the above cases. So $a \neq \rho_1$. Similarly, we can prove that $a \neq \rho_2$. Hence n can not be an odd number. This completes the proof of the theorem. □

Theorem 201. *The shell graph $S_{n,n-3}$ is conjugate S_3-magic for all $n > 4$ except when $n \neq 6$.*

Proof. Let $S_{n,n-3}$ be the shell graph and denote the vertices of $S_{n,n-3}$ by $u_1, u_2, \ldots, u_n$. Without loss of generality, let the *apex* be u_1. Consider the following four cases also for all the following cases let f be any bijection from $E(S_{n,n-3})$ to N_{2n-3} and let $u_{n+1} = u_1$. Let $1 \leq i \leq n$ and $3 \leq j \leq n-1$.

Case(i): $n \equiv 1 (\text{mod } 3), n \neq 4$.

In this case, define $g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i = 1, 2, n-1, n, \\ \rho_2, & \text{otherwise.} \end{cases}$ and

$$g(u_1u_j) = \begin{cases} \rho_2, & \text{if } j = 3, n-1, \\ \rho_1, & \text{otherwise.} \end{cases}$$

Then clearly $\ell^*(u) = \rho_2, \forall u \in V(S_{n,n-3})$.

Case(ii): $n \equiv 2 (\text{mod } 3)$.

$$\text{Here we define } g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i = 1, n, \\ \rho_2, & \text{if } 2 \leq i \leq n-1. \end{cases} \quad \text{and}$$

$$g(u_1v_j) = \rho_2, \; 3 \leq j \leq n-1.$$

Hence the above f and g will determine a conjugate S_3-magic labeling of $S_{n,n-3}$ with magic constant ρ_0.

Case(iii): $n \equiv 0 (\text{mod } 3)$ and n odd.

$$\text{In this case, define } g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i \text{ is even and } 4 \leq i \leq n-1, \\ & i = 1, 2, n-1, n, \\ \rho_2, & \text{if } i \text{ is odd and } 3 \leq i \leq n-2. \end{cases}$$

and $g(u_1u_j) = \rho_2$. Hence $\ell^*(u) = \rho_2, \forall u \in V(S_{n,n-3})$.

Case(iv): $n \equiv 0 (\text{mod } 3)$, n is even and $n \neq 6$.

Here we define

$$g(u_1u_2) = g(u_2u_3) = g(u_1u_n) = g(u_{n-1}u_n) = \rho_1,$$
$$g(u_3u_4) = g(u_4u_5) = g(u_5u_6) = g(u_{n-2}u_{n-1}) = g(u_{n-3}u_{n-2}) = \rho_2,$$
$$g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i \text{ is even and } 6 \leq i \leq n-4, \\ \rho_2, & \text{if } i \text{ is odd and } 7 \leq i \leq n-4. \end{cases} \quad \text{and}$$
$$g(u_1u_j) = \begin{cases} \rho_1, & \text{if } j = 4, 5, n-2, \\ \rho_2, & \text{otherwise.} \end{cases}$$

Thus $\ell^*(u) = \rho_2, \forall u \in V(S_{n,n-3})$.

This completes the proof of the theorem. □

Theorem 202. *The shell graphs $S_{4,1}$ and $S_{6,3}$ are not conjugate S_3-magic.*

Proof. Let the vertices of $S_{4,1}$ be u_1, u_2, u_3 and u_4. Let the apex be u_1. Suppose that $S_{4,1}$ is conjugate S_3-magic with magic constant a, $a \in S_3$. Since u_2 and u_4 have degree 2, by Theorem 191 $a \in \{\rho_0, \rho_1, \rho_2\}$. Consider the following cases.

Case (i): $a = \rho_0$ and $g(e) \in \{\rho_1, \rho_2\}$.

Without loss of generality, let $g(u_1u_2) = \rho_1$. $\ell^*(u_2) = \rho_0$ implies $g(u_2u_3) = \rho_2$. Similarly, $\ell^*(u_3) = \rho_0$ implies $g(u_1u_3) = g(u_3u_4) = \rho_2$. Then $\ell^*(u_4) = \rho_0$ implies $g(u_4u_1) = \rho_1$. Hence $\ell^*(u_1) = g(u_1u_2) * g(u_1u_3) * g(u_4u_1) = \rho_1 * \rho_2 * \rho_1 = \rho_1 \neq \rho_0$. Which is a contradiction.

Case(ii): $a = \rho_0$ and $g(e) \in \{\mu_1, \mu_2, \mu_3\}$.

In this case, without loss of generality, let $g(u_1u_2) = \mu_1$ then $\ell^*(u_2) = \rho_0$ implies $g(u_2u_3) = \mu_1$. Now $\ell^*(u_3) = \prod_{e \in N^*(u_3)}(f(e), g(e))$. We have $g(u_2u_3), g(u_1u_3) \in \{\mu_1, \mu_2, \mu_3\}$ but the product of any three elements in $\{\mu_1, \mu_2, \mu_3\}$(need not be distinct) does not yield the value ρ_0. Hence $\ell^*(u_3) \neq \rho_0$, which is a contradiction.

The above 2 cases show that $a \neq \rho_0$.

Case(iii): $a = \rho_1$ and $g(e) \in \{\rho_1, \rho_2\}$.

Since $a = \rho_1$, $\ell^*(u_2) = \rho_1$. So $g(u_1u_2) = g(u_2u_3) = \rho_2$. Similarly, $\ell^*(u_3) = \rho_1$ implies $g(u_1u_3) = g(u_3u_4) = \rho_1$. But $\ell^*(u_4) = \rho_1$ implies $g(u_4u_1) = \rho_0$, a contradiction.

Case(iv) $a = \rho_1$ and $g(e) \in \{\mu_1, \mu_2, \mu_3\}$.

In the graph $S_{4,1}$ the vertices u_1 and u_3 are of degree 3. Observe that $g(u_2u_3), g(u_1u_3),\ g(u_1u_4) \in \{\mu_1, \mu_2, \mu_3\}$ but the product of any three elements in $\{\mu_1, \mu_2, \mu_3\}$ (need not be distinct) does not yield the values ρ_0, ρ_1 and ρ_2. Hence the above case does not exist.

Case(v): $a = \rho_2$.

We can prove that, the magic constant a cannot be ρ_2 when $g(e) \in \{\rho_1, \rho_2\}$ or $g(e) \in \{\mu_1, \mu_2, \mu_3\}$. The proof is similar to the above cases (iii) and (iv). Hence there does not exist a conjugate S_3 magic labeling for $S_{4,1}$ with magic constant ρ_2.

All the above cases show that there does not exist a conjugate S_3-magic labeling for the shell graph $S_{4,1}$. Similarly, we can prove that $S_{6,1}$ is not conjugate S_3-magic. □

Theorem 203. *The fan graph F_n is conjugate S_3-magic whenever $n \neq 3, 5$.*

Proof. We have $F_n = P_n + K_1$. Let $V(F_n) = \{k, u_1, u_2, \ldots,\ u_n\}$, where $u_1, u_2, \ldots, u_n$ be the vertices corresponding to P_n and k be the vertex corresponding to K_1. Now consider the following four cases. For all the cases, take f be any bijection from $E(F_n)$ to N_{2n-1}.

Case(i): $n \equiv 0(\text{mod } 3)$ and $n > 3$.

In this case, define $g : E(F_n) \to \{\rho_1, \rho_2\}$ as:

$$\text{for } 1 \leq i \leq n-1, g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i = 1 \text{ and } i = n-1, \\ \rho_2, & \text{otherwise.} \end{cases}$$
$$g(ku_i) = \begin{cases} \rho_2, & \text{if } i = 2, i = n-1, \\ \rho_1, & \text{otherwise.} \end{cases}$$

Clearly, f and g will determine a conjugate S_3-magic labeling for F_n with magic constant ρ_2.

Case(ii): $n \equiv 1(\text{mod } 3)$.

In this case, let g be defined by

$$g(u_iu_{i+1}) = \rho_1, \text{ if } i = 1 \leq i \leq n-1 \text{ and}$$
$$g(ku_i) = \begin{cases} \rho_2, & \text{if } i = 1, n, \\ \rho_1, & \text{otherwise.} \end{cases}$$

Then the maps f and g will define a conjugate S_3-magic labeling for F_n with the magic constant ρ_0.

Case(iii): $n \equiv 2(\text{mod } 3)$ and n is even.

Here we define g as

$$g(u_iu_{i+1}) = \begin{cases} \rho_2, & \text{if } i \text{ is odd,} \\ \rho_1, & \text{if } i \text{ is even.} \end{cases} \quad \text{and } g(ku_i) = \begin{cases} \rho_2, & \text{if } i = 1, i = n, \\ \rho_1, & \text{otherwise.} \end{cases}$$

Clearly $\ell^*(u) = \rho_1, \forall u \in V(F_n)$.

Case(iv): $n \equiv 2(\text{mod } 3)$, n is odd and $n \neq 5$.

In this case also we define f as above and for $1 \leq i \leq n$ define g as follows:

$$g(u_1u_2) = g(u_{n-1}u_n) = g(u_{n-3}u_{n-2}) = \rho_2,$$
$$g(u_2u_3) = g(u_3u_4) = g(u_{n-4}u_{n-3}) = g(u_{n-2}u_{n-1}) = \rho_1,$$
$$g(u_iu_{i+1}) = \begin{cases} \rho_1, & \text{if } i \text{ is even and } 4 \leq i \leq n-4, \\ \rho_2, & \text{if } i \text{ is odd and } 4 \leq i \leq n-4. \end{cases} \quad \text{and}$$
$$g(ku_i) = \begin{cases} \rho_2, & \text{if } i = 1, n, 3, 4, n-4, \\ \rho_1, & \text{otherwise.} \end{cases}$$

By defining f and g as above we get a conjugate S_3 -magic labeling for

F_n with magic constant ρ_1.

□

Theorem 204. *The fan graphs F_3 and F_5 are not conjugate S_3-magic.*

Proof. Consider the fan graph F_3. Let the vertices of P_3 be denoted by u_1, u_2 and u_3 and the vertex of K_1 be denoted by k. Suppose on the contrary that F_3 is conjugate S_3-magic with magic constant a, where $a \in \{\rho_0, \rho_1, \rho_2\}$. Suppose that $a = \rho_0$. Without loss of generality, let $g(u_1u_2) = \rho_1$ then $g(u_1k) = \rho_2$ and $g(u_2k) = g(u_2u_3) = \rho_1$. Then $g(u_2u_3) = \rho_1$ implies $g(u_3k) = \rho_2$. But then $\ell^*(k) = \rho_2 * \rho_1 * \rho_2 = \rho_2$, which is a contradiction. Hence $a \neq \rho_0$. Suppose that $a = \rho_1$, then $\ell^*(u_1) = \ell^*(u_3) = \rho_1$ implies $g(u_1u_2) = \rho_2 = g(ku_1) = g(ku_3) = g(u_2u_3)$. But then $\ell^*(u_2) = g(u_1u_2) * g(u_2u_3) * g(u_2k) = \rho_1$ implies $g(ku_2) = \rho_0$, which is a contradiction. Hence $a \neq \rho_1$. Similarly, we can prove that $a \neq \rho_2$. Hence F_3 is not a conjugate S_3-magic graph. In a similar manner, we can prove that the fan graph F_5 is not conjugate S_3-magic. □

Theorem 205. *The complete bipartite graph $K_{m,n}$ is conjugate S_3-magic for $m, n > 1$.*

Proof. Let U and V be the two partite sets of $V(K_{m,n})$. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices in U and V respectively. If m and n are of same parity then the constant map $g(e) = \mu_1$, together with any bijection $f : E(K_{m,n}) \to N_{mn}$ will give a conjugate S_3-magic labeling. Now, without loss of generality, assume that m is an even number and n is an odd number. Now consider the following cases: For all the following cases, let f be any bijection from $E(K_{m,n})$ to N_{mn}.

Case(i): m is even and $n \equiv 0(\text{mod } 3)$.

In this case define $g : E(K_{mn}) \to S_3 \setminus \{\rho_0\}$ as follows: For $1 \leq i \leq m$ and $1 \leq j \leq n$ define $g(u_iv_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases}$

Case(ii): m is even and $n \equiv 1(\text{mod } 3)$.

For $1 \leq i \leq m$ and $1 \leq j \leq n-3$, define

$$g(u_iv_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd and } j \text{ is odd }, i \text{ is even and } j \text{ is even,} \\ \rho_2, & \text{if } \text{ if } i \text{ is odd and } j \text{ is even }, i \text{ is even and } j \text{ is odd} \end{cases}$$

and $g(u_iv_{n-2}) = g(u_iv_{n-1}) = g(u_iv_n) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases}$

Case(iii): m is even and $n \equiv 2 (\text{mod } 3)$.

For $1 \leq i \leq m$, let

$$g(u_iv_j) = \begin{cases} \rho_1, & \text{if } i \text{ is odd and } 1 \leq j \leq n-1; i \text{ is even and } j = n, \\ \rho_2, & \text{if } i \text{ is even and } 1 \leq j \leq n-1; i \text{ is odd and } j = n. \end{cases}$$

In all the above cases, we can prove that f and g will determine a conjugate S_3-magic labeling for $K_{m,n}$ with magic constant ρ_0. Hence the proof. □

Theorem 206. *The flag graph Fl_n is not conjugate S_3-magic for all $n > 2$.*

Proof. Let us denote the vertices of Fl_n by $u_1, u_2, \ldots, u_n, k$, where u_i is adjacent to u_{i+1}, $1 \leq i \leq n$, $i+1$ is taken over modulo n and u_1 is adjacent to k. Suppose that there exists a conjugate S_3-magic labeling for Fl_n with the magic constant $a \in S_3$. Since Fl_n has a pendant edge $a \neq \rho_0$. Also, since Fl_n has vertices of degree 2 by Theorem 191, $a \in \{\rho_1, \rho_2\}$. Without loss of generality, let $a = \rho_1$. Since u_1k is the pendant edge $g(u_1k) = \rho_1$. So $g(E(Fl_n)) \in \{\rho_1, \rho_2\}$. Since $g^*(u_1) = \rho_1$, we must have $g(u_1u_2) * g(u_1u_n) = \rho_0$. Without loss of generality, let $g(u_1u_2) = \rho_1$ and $g(u_1u_n) = \rho_2$. But $g^*(u_2) = \rho_1$ implies $g(u_2u_3) = \rho_0$. Which is a contradiction. Hence Fl_n does not have a conjugate S_3-magic labeling. □

Chapter 7

S_3-Barycentric Magic Labeling of Graphs

In this chapter, we define S_3-barycentric magic labeling of graphs and investigate graphs that are barycentric S_3-magic.

7.1 Introduction

Let A be an abelian group. A graph $G = (V; E)$ is said to be A-barycentric magic [38] if there exists a labeling $l : E(G) \to A \setminus \{0\}$of the edges of G by non-zero elements of A such that the induced vertex set labeling $\ell^+ : V(G) \to A$ satisfies:

(i) $\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$ is a constant map.

(ii) $\ell^+(v) = deg(v)\ell(u_v v),$ for all $v \in V(G)$, and for some vertex u_v adjacent to v.

In this chapter, we define the barycentric magic labeling of graphs using the non-abelian group. Instead of considering any non-abelian group, we particularly choose the smallest non-abelian group S_3 and define the S_3-barycentric magic labeling of graphs as follows.

7.2 S_3-Barycentric Magic Labeling of Graphs

Definition 207. *[18] Let $G = (V(G), E(G))$ be a finite graph with p vertices and q edges and let $(A, *)$ be a finite non-abelian group with identity element 1. Let $f : E(G) \to N_q = \{1, 2, \ldots, q\}$ and let $g : E(G) \to A \setminus \{1\}$ be two edge labelings of G such that f is bijective. Define an edge labeling $\ell : E(G) \to N_q \times A \setminus \{1\}$ by*

$$l(e) := (f(e), g(e)),\ e \in E(G).$$

Define a relation $\leq$ on the range of ℓ by:

$$(f(e), g(e)) \leq (f(e^{'}), g(e')) \ \ \textit{if and only if} \ \ f(e) \leq f(e').$$

Then obviously, the relation $\leq$ is a partial order on the range of ℓ.
Let $\{(f(e_1), g(e_1)), (f(e_2), g(e_2)), \ldots, (f(e_k), g(e_k))\}$ be a chain in the range of ℓ. We define the product of this chain as follows:

$$\prod_{i=1}^{k}(f(e_i), g(e_i)) := ((((g(e_1) * g(e_2)) * g(e_3)) * g(e_4)) * \ldots) * g(e_k).$$

Let $u \in V(G)$ and let $N^*(u)$ be the set of all edges incident with u. Consider the restriction of the function ℓ on $N(u)$, that is, $\ell|_{N^*(u)}$. Observe that the range of $\ell|_{N^*(u)}$ is a chain, say $(f(e_1), g(e_1)) \leq (f(e_2), g(e_2)) \leq \cdots \leq (f(e_n), g(e_n))$. We define

$$\ell^*(u) = \prod_{i=1}^{n}(f(e_i), g(e_i)). \tag{7.2.1}$$

If $\ell^*(u)$ is a constant, say a for all $u \in V(G)$, we say that the graph G is A-magic. The map ℓ^* is called an A-magic labeling of G and the corresponding constant a is called the magic constant.

Definition 208. *Let A be a group (abelian or non-abelian) with identity 1. A graph $G = (V, E)$ is said to be A-barycentric magic if there exist functions $f : E(G) \to N_{|V(G)|}$ and $g : E(G) \to A \setminus \{1\}$such that the labeling $\ell^* : V(G) \to A$ as in definition 207 satisfying:*

(i) $\ell^*(v) = \prod_{e \in N^*(v)}(f(e), g(e))$ is a constant map, where $N^*(v)$ denote the set of all edges incident with u.

(ii) $\ell^*(v) = (g(u_v v))^{deg(v)}$, for all $v \in V(G)$ and for some vertex u_v adjacent to v.

If A is an abelian group and $\ell^*(u) = a, \forall u \in V(G)$ then we say that G is a-sum magic graph [38]. Similarly, suppose A is a non-abelian group and G is A-barycentric magic graph with magic constant 'k' then G is said to be a k-barycentric magic graph.

Remark 209. *Let A be an abelian group and let G be a regular graph then the graph G is A-barycentric magic [38].*

Theorem 210. *[29] Any regular graph is S_3-magic.*

Note 211. *[38] Notice that if A is a finite(abelian) group of order n and $deg(v) \equiv 0(modn)$ for all $v \in V(G)$, the barycentric-magic graphs coincide with the zero-sum magic graph(here zero is the identity element of A).*

Lemma 212. *[38] For every abelian group A, P_2 is A-barycentric magic and $P_n, n \geq 3$, is not A-barycentric magic.*

Theorem 213. *[38] C_n is A-barycentric magic for every abelian group A.*

In this chapter, we consider the non-abelian group S_3, symmetric group of order 6 and investigate graphs that are S_3-barycentric magic and the graphs that belong to the following classes.

(i) $\mathcal{BS}_{\rho_0}$ = the class of graphs that are S_3-barycentric magic with constant ρ_0.

(ii) $\mathcal{BS}_{\rho}$ = the class of graphs that are S_3-barycentric magic with constant belongs to the set $\{\rho_1, \rho_2\}$.

(iii) $\mathcal{BS}_{\mu}$ = the class of graphs that are S_3-barycentric magic with constant belongs to the set $\{\mu_1, \mu_2, \mu_3\}$.

7.3 Main Results

Lemma 214. *Let A be a group(abelian or non-abelian) and let G be a regular graph then G is A-barycentric magic.*

Proof. The proof is indisputable from Remark 209 and Theorem 210. □

Theorem 215. *Let G be a graph such that all of its vertices are of odd degree then $G \in \mathcal{BS}_{\mu}$.*

Proof. Let $g : E(G) \to S_3 \setminus \{\rho_0\}$ be the constant map $g(e) = \mu_1, \forall e \in E(G)$. Then g together with any bijective map $f : E(G) \to N_q$ will determine a barycentric magic labeling with constant μ_1. This completes the proof of the theorem. □

Theorem 216. *If a graph G has a vertex of even degree, then $G \notin \mathcal{BS}_{\mu}$.*

Proof. Suppose that $G \in \mathcal{BS}_{\mu}$ with the magic constant a, where $a \in \{\mu_1, \mu_2, \mu_3\}$. Let v be a vertex having even degree k. Then we have $\ell^*(v) = g(vu_v)^k = a$, for some vertex u_v adjacent to v. This implies $a \in \{\rho_0, \rho_1, \rho_2\}$. Which is a contradiction. Hence the proof. □

From the proof of the above theorem we have the following corollaries.

Corollary 217. *If a graph G belongs to the class $\mathbb{BS}_{\mu}$ if and only if all the vertices of G are of odd degree.*

Corollary 218. *The cycle graph C_n is A-barycentric magic for every group A.*

Theorem 219. *If a graph G has a vertex whose degree is a multiple of 3 then $G \notin \mathbb{BS}_\rho$.*

Proof. Let G be a graph having a vertex v, whose degree is a multiple of 3, say m. Suppose that $G \in \mathbb{BS}_\rho$. Without loss of generality, let the barycentric constant be ρ_1. Then there exists a vertex u_v adjacent to v such that $\ell^*(v) = g(vu_v)^m = \rho_1$. This implies $g(vu_v) \in \{\rho_1, \rho_2\}$. But the order of ρ_1 and ρ_2 is 3. Hence $g(u_v v)^m = \rho_0$. Which is a contradiction to our assumption. This completes the proof of the theorem. □

Theorem 220. *If a graph G has a pendant edge then $G \notin \mathbb{BS}_{\rho_0}$.*

Proof. Suppose that the graph G has a pendant edge, say 'e'. Let u be the end vertex of e having degree 1. Then $\ell^*(u) = g(e)$. Since g is a function from $E(G)$ to $S_3 \setminus \{\rho_0\}$, $g(e) \neq \rho_0$. Hence $G \notin \mathbb{BS}_{\rho_0}$. □

Theorem 221. *The cycle graph C_n belongs to the classes $\mathcal{BS}_{\rho_0}$ and $\mathcal{BS}_\rho$.*

Proof. If we take f as any bijective map from $E(C_n)$ to N_n and if take $g : E(C_n) \to S_3 \setminus \{\rho_0\}$ be the constant map $g(e) = \mu_1, \forall e \in E(C_n)$ then clearly C_n is S_3-barycentric magic with constant ρ_0. If we take g as $g(e) = \rho_1$, or ρ_2 then $C_n \in \mathcal{BS}_\rho$. □

Theorem 222. *The cycle graph C_n does not belong to the class $\mathcal{BS}_\mu$, for any $n > 2$.*

Proof. Suppose that the graph $C_n \in \mathcal{BS}_\mu$. Then adjacent edges are labeled by the elements from $\{\rho_1, \rho_2\}$ and $\{\mu_1, \mu_2, \mu_3\}$ alternatively under the map g. Hence, n must be an even number; moreover, if v is a vertex of C_n, we cannot find a vertex u_v such that $g(vu_v)^2 \in \{\mu_1, \mu_2, \mu_3\}$. Hence $C_n \notin \mathcal{BS}_\mu$. This completes the proof of the theorem. □

Theorem 223. *The star graph $K_{1,n}$ is S_3-barycentric magic if and only if n is odd or $n \equiv 1(mod\ 3)$.*

Proof. The proof directly follows from Theorem 48 of Chapter 2. □

Theorem 224. *The star graph $K_{1,n} \notin \mathbb{BS}_{\rho_0}$.*

Proof. The proof directly follows from the Theorem 220. □

Theorem 225. *The star graph $K_{1,n} \in \mathbb{BS}_\rho$ if and only if $n \equiv 1(mod\ 3)$.*

Proof. Observe that, a star graph is S_3-magic with magic constant 'a', the function $g : E(K_{1,n}) \to S_3 \setminus \{\rho_0\}$ must be a constant function. Say, $g(e) = a, \forall a \in S_3 \setminus \{\rho_0\}$. Suppose that $K_{1,n} \in \mathbb{BS}_\rho$. Without loss of generality, let the magic constant be ρ_1. Then $g(e) = \rho_1, \forall e \in E(K_{1,n})$. Since $K_{1,n}$ has a vertex of degree n, $\rho_1^n = \rho_1$ implies $n \equiv 1(\text{mod } 3)$. Similarly, we can prove that if $n \equiv 1(\text{mod } 3)$ then $K_{1,n} \in \mathbb{BS}_{\rho_0}$. Hence the proof. □

Theorem 226. *The star graph $K_{1,n} \in \mathbb{BS}_\mu$ if and only if n is an odd number.*

Proof. The proof is exactly similar to the above theorem(Theorem 225). □

Theorem 227. *If $n \geq 3$, the wheel graph $W_n \in \mathcal{BS}_\mu$ if and only if n is odd.*

Proof. Let G be the wheel W_n and let the vertices of C_n be $\{u_1, u_2, \ldots, u_n\}$ and the vertex of K_1 be k. Then $deg(k) = n$ and $deg(u_i) = 3, i = 1, 2, \ldots, n$.

Suppose that n is an odd number. Then, we define f and g as in the Case(i) of the Theorem 50 (Chapter 2). Then $\ell^*(u) = \mu_1, \forall u \in V(G)$ also $g(u_1k)^{deg(k)} = \underbrace{\mu_1 * \mu_1 * \ldots * \mu_1}_{n\,\text{times}} = \mu_1 = \ell^*(k)$ and $(g(ku_i))^{deg(u_i)} = \mu_1^3 = \mu_1 = \ell^*(u_i)$. Thus W_n is S_3-barycentric magic when n is odd.

Conversely, suppose that $G \in \mathbb{BS}_\mu$. Then Theorem 216 implies that G can not have a vertex of even degree. Hence n must be odd. This completes the proof of the theorem. □

Theorem 228. *The wheel graph $W_n \notin \mathbb{BS}_\rho$, for any $n \geq 3$.*

Proof. Observe that W_n has n vertex of degree 3 and $a^3 \notin \{\rho_1, \rho_2\}$ for any $a \in S_3 \setminus \{\rho_0\}$. Hence the proof. □

Theorem 229. *The wheel graph $W_n \in \mathbb{BS}_{\rho_0}$ if and only if either $n \equiv 0(\text{mod } 3)$ or n is even.*

Proof. Let us denote the vertices of W_n as in the above theorem. Now consider the following cases:

Case(i): n is even.

In this case, define $f : E(W_n) \to N_{2n}$ as

$f(u_iu_{i+1}) = \begin{cases} n + \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \frac{3n}{2} + \frac{i}{2}, & \text{if } i \text{ is even.} \end{cases}$ and $f(ku_i) = i, 1 \leq i \leq n$. Define

$g : E(W_n) \to S_3 \setminus \{\rho_0\}$ as $g(u_iu_{i+1}) = \begin{cases} \mu_2, & \text{if } i \text{ is odd}, 1 \leq i \leq n, \\ \rho_2, & \text{if } i \text{ is even}, 1 \leq i \leq n. \end{cases}$ and $g(ku_i) = \mu_1$, for all i, $1 \leq i \leq n$. We can easily verify that f and g

determine a S_3-magic labeling of W_n with constant ρ_0 also $\rho_0 = \ell^*(k) = g(u_1k)^n$ and $\rho_0 = \ell^*(u_i) = g(u_iu_j)^3$, for some $i-1 \leq j \leq i+1$. Hence f and g determine a S_3-barycentric magic labeling of W_n.

Case(ii): $n \equiv 0(\text{mod } 3)$.

In this case, we can take f as any bijective map from $E(W_n) \to N_{2n}$ and $g : E(G) \to S_3 \setminus \{\rho_0\}$ as the constant map $g(e) = \rho_1$, for all $e \in E(W_n)$. Clearly, this f and g defines a S_3-barycentric magic with constant ρ_0.

Conversely, suppose that $W_n \in \mathbb{BS}_{\rho_0}$. Then $\ell^*(k) = \rho_0$ implies that $g(u_ik)^n = \rho_0$, for some $u_i \in V(W_n)$. Since $g(u_ik) \in S_3 \setminus \{\rho_0\}$ implies that n is even or $n \equiv 0(\text{mod } 3)$. This completes the proof of the theorem. □

Theorem 230. *The helm graph H_n does not belong to the classes $\mathbb{BS}_{\rho_0}$ and $\mathbb{BS}_{\mu}$, for all $n > 2$.*

Proof. Since the helm graph has n pendant edges, by Theorem 220 , $H_n \notin \mathbb{BS}_{\rho_0}$. Also the helm graph has n vertices of degree 4, so the Theorem 216 implies that $H_n \notin \mathbb{BS}_{\mu}$. □

Theorem 231. *The helm graph H_n belong to the class $\mathbb{BS}_{\rho}$ if and only if $n \not\equiv 0(mod\ 3)$.*

Proof. Let the vertex set of H_n be $V(H_n) = \{u_i, v_i, k : 1 \leq i \leq n\}$ and edge set of H_n be $E(H_n) : \{u_iu_{i+1}, u_iv_i, ku_i : 1 \leq i \leq n, i+1$ is taken over modulo $n\}$. Suppose $n \not\equiv 0(\text{mod } 3)$. Then either $n \equiv 1(\text{mod } 3)$ or $n \equiv 2(\text{mod } 3)$. Here we take f as any bijective map from $E(G)$to N_{3n}. If $n \equiv 1(\text{mod } 3)$. Let $g : E(G) \to S_3 \setminus \{\rho_0\}$ be the constant map $g(e) = \rho_1, \forall e \in E(H_n)$. Then we can see that f and g defines a S_3-barycentric magic labeling of H_n. If $n \equiv 2(\text{mod } 3)$. Then let $g(e) = \begin{cases} \rho_1, & \text{if } e = u_iv_i, \\ \rho_2, & \text{otherwise.} \end{cases}$ Clearly, for every $v \in V(H_n)$ there exist a $u_v \in V(H_n)$ such that $\ell^*(v) = \rho_1 = (g(u_vv))^{deg(v)}$. So $H_n \in \mathbb{BS}_{\rho}$. Observe that the vertex k is of degree n. If $n \equiv 0(\text{mod } 3)$, then by Theorem 219 $H_n \notin \mathbb{BS}_{\rho}$. This completes the proof of the theorem.

□

Theorem 232. *The shell graph $S_{n,n-3}$ is does not belong to the classes $\mathbb{BS}_{\rho}$ and $\mathbb{BS}_{\mu}$.*

Proof. Let us denote the vertices of the shell graph $S_{n,n-3}$ by $u_1, u_2, \ldots, u_n$ and let the vertex with degree $n-1$ be u_1.

Observe that, the shell graph $S_{n,n-3}$ has vertices of degree 3, by Theorem 219 $S_{n-3} \notin \mathbb{BS}_{\rho}$.

Since the shell graph has two vertices of degree 2, Theorem 216 implies that $S_{n,n-3} \notin \mathbb{BS}_{\mu}$. □

Theorem 233. *The shell graph $S_{n,n-3} \in \mathbb{BS}_{\rho_0}$ if and only if either $n \equiv 1(mod\ 3)$ or n is an odd number.*

Proof. Denote the vertices of $S_{n,n-3}$ as in the above theorem. Suppose that $n \equiv 1(\text{mod } 3)$ or n is an odd number. Let $f : E(S_{n,n-3}) \to N_{2n-3}$ be defined as

$$
\begin{aligned}
f(u_2u_3) &= 1,\\
f(u_1u_i) &= (2i-3), 3 \leq i \leq n-1,\\
f(u_iu_{i+1}) &= 2(i-2), 3 \leq i \leq n-1,\\
f(u_1u_2) &= 2n-4,\\
f(u_1u_n) &= 2n-3.
\end{aligned}
$$

Now, define $g : E(S_{n,n-3}) \to S_3 \setminus \{\rho_0\}$ as follows:

$$
\begin{aligned}
g(u_1u_2) &= g(u_2u_3 = \mu_1\\
g(u_iu_{i+1}) &= \begin{cases} \mu_1, & \text{if } i \text{ is even and } 3 \leq i \leq n-1,\\ \mu_2, & \text{if } i \text{ is odd and } 3 \leq i \leq n-1. \end{cases}\\
g(u_nu_1) &= \begin{cases} \mu_1, & \text{if } n \text{ is odd,}\\ \mu_2, & \text{if } n \text{ is even.} \end{cases}\\
g(u_1u_i) &= \begin{cases} \rho_2, & \text{if } i \text{ is odd }, 3 \leq i \leq n-1,\\ \rho_1, & \text{if } i \text{ is even }, 3 \leq i \leq n-1. \end{cases}
\end{aligned}
$$

The above maps f and g will define a S_3-barycentric magic labeling of $S_{n,n-3}$. Hence $S_{n,n-3} \in \mathbb{BS}_{\rho_0}$ when either $n \equiv 1(\text{mod } 3)$ or n is odd.

Conversely, suppose that $S_{n,n-3} \in \mathbb{BS}_{\rho_0}$. Since the degree of the vertex u_1 is $n-1$, there exists a vertex u_j adjacent to u_1 such that $g(u_1u_j)^{n-1} = \rho_0$. $g(u_1u_j) \in S_3 \setminus \{\rho_0\}$ implies that $n-1 \equiv 0(\text{mod } 3)$ or $n-1$ is even. This completes the proof of the theorem. □

The Figure 7.1 represents a shell graph with a S_3-barycentric magic labeling.

Theorem 234. *The n-gon book of k pages $B(n,k) \in \mathbb{BS}_{\rho_0}$ for all $n, k > 1$.*

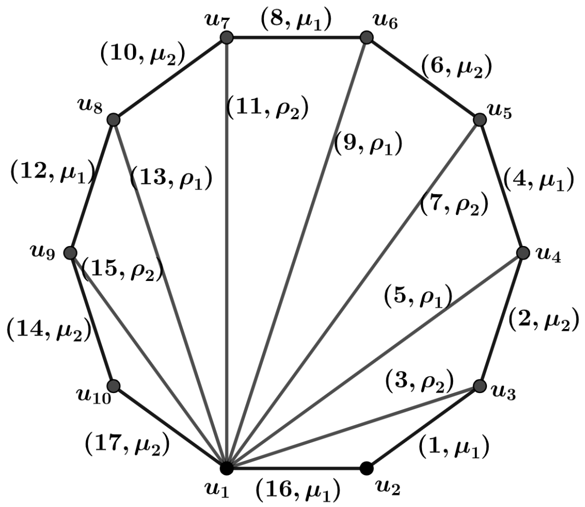

Figure 7.1: S_3-barycentric magic labeling of $S_{10,7}$

Proof. Let us denote the common edge of $B(n,k)$ be 'c' with end points k_1 and k_2 and denote the vertices in the the i^{th} page of $B(n,k)$ be $u_{i1}, u_{i2}, \ldots, u_{in},\ i = 1,2,3,\ldots,k$, where the vertices u_{i1} and u_{1n} identified with k_1 and k_2 respectively. Now consider the following 2 cases:

Case (i): k is an odd number.

If k is an odd number then all the vertices of $B(n,k)$ will be of even degree. So any bijective map$f : E(B(n,k)) \to N_{k(n-1)+1}$ together with the constant map $g(e) = \mu_1, \forall e \in E(B(n,k))$ will determine a S_3-barycentric magic with constant ρ_0.

Case (ii): k is an even number.

In this case, define $f(u_{ij}u_{ij+1}) = (n-1)(i-1)+j, f(k_1k_2) = k(n-1)+1, 1 \leq j \leq n-1, 1 \leq i \leq k$. Define $g : E(B(n,k)) \to S_3 \setminus \{\rho_0\}$ as follows:
For $1 \leq j \leq n-1$

$$g(u_{ij}u_{ij+1}) = \begin{cases} \mu_1, & \text{if } 1 \leq i \leq k-1, \\ \mu_2, & \text{if } i = k. \end{cases}$$

$,g(k_1k_2) = \rho_2$. Clearly, using this f and g one can easily prove that $B(n,k) \in \mathbb{BS}_{\rho_0}$.

□

Theorem 235. *The graph n-gon book of k pages $B(n,k) \in \mathbb{BS}_\rho$ if and only if $k \not\equiv 2(mod\ 3)$.*

Proof. Denote the vertices and edges of $B(n,k)$ as in the above theorem. Suppose that $k \not\equiv 2(\text{mod } 3)$. Then either $k \equiv 0(\text{mod } 3)$ or $k \equiv 1(\text{mod } 3)$. If

$k \equiv 1(\text{mod } 3)$ then we can take $f : E(B(n,k)) \to N_{k(n-1)+1}$ as any bijective function and take $g : E(B(n,k)) \to S_3 \setminus \{\rho_0\}$ as the constant map $g(e) = \rho_2, \forall e \in E(B(n,k))$. If $k \equiv 0(\text{mod } 3)$ then by taking f as above and g as $g(e) = \begin{cases} \rho_1, & \text{if } e = k_1k_2, \\ \rho_2, & \text{otherwise.} \end{cases}$ we can show that the graph $B(n,k) \in \mathbb{BS}_\rho$. Now let $k \equiv 2(\text{mod } 3)$, i.e., $k = 3m + 2$, for some $m \in \mathbb{N}$. Suppose that $B(n,k) \in \mathbb{BS}_\rho$. Without loss of generality, let ρ_1 be the S_3-barycentric magic constant. Then, all edges adjacent to a vertex of degree two should be mapped to ρ_2 under the map g. Then $\ell^*(k_1) = g(k_1k_2)*(\rho_2)^k = \rho_1$ implies $g(k_1k_2) = \rho_0$, which is a contradiction. Hence, we get $B(n,k) \notin \mathbb{BS}_\rho$, when $k \equiv 2(\text{mod } 3)$.

□

Theorem 236. *The graph $B(n,k) \notin \mathbb{BS}_\mu$, for any $n, k > 1$.*

Proof. Since $B(n,k)$ contains vertices of degree 2, by Theorem 216 $B(n,k) \notin \mathbb{BS}_\mu$. □

Theorem 237. *The flag graph Fl_n is not S_3-barycentric magic.*

Proof. Since the flag graph Fl_n has a pendant edge, by Theorem 220 $Fl_n \notin \mathbb{BS}_{\rho_0}$.

The flag graph has $n-1$ vertices of degree 2. So by Theorem 216, $Fl_n \notin \mathbb{BS}_\mu$.

The flag graph has a vertex of degree 3, hence by Theorem 219, $Fl_n \notin \mathbb{BS}_\rho$. Thus we can conclude that the flag graph Fl_n is not S_3-barycentric magic. □

Theorem 238. *The complete graph $K_n \in \mathbb{BS}_{\rho_0}$ if and only if either $n \equiv 1(\text{mod } 3)$ or n is an odd number.*

Proof. Suppose that $K_n \in \mathbb{BS}_{\rho_0}$. Let $v \in V(K_n)$ then there exist a $u_v \in V(K_n)$ such that $g(u_vv)^{n-1} = \rho_0$. If $g(u_vv) \in \{\rho_1, \rho_2\}$ then $n - 1 \equiv 0(\text{mod } 3)$, i.e., $n \equiv 1(\text{mod } 3)$. If $g(u_vv) \in \{\mu_1, \mu_2, \mu_3\}$, then $n-1$ is an even number. i.e., n is odd.

If $n \equiv 1(\text{mod } 3)$ then any bijective function $f : V(K_n) \to N_n$ together with the constant map $g(e) = \rho_1, \forall e \in E(K_n)$ will determine a S_3-barycentric magic labeling of K_n with constant ρ_0. Similarly, if n is odd then the above said f together with the constant map $g(e) = \mu_1, \forall e \in E(K_n)$ will define a S_3-barycentric magic labeling of K_n with constant ρ_0.

□

Theorem 239. *The complete graph $K_n \in \mathbb{BS}_\mu$ if and only if n is an even number.*

Proof. Observe that the degree of every vertex of K_n is $n-1$. So by Corollary 217, $K_n \in \mathbb{BS}_\mu$ if and only if n is even. □

Theorem 240. *The complete graph $K_n \in \mathbb{BS}_\rho$ if and only if $n \not\equiv 1(mod\ 3)$.*

Proof. Since the degree of each vertex of K_n is $n-1$, by Theorem 219, $K_n \in \mathbb{BS}_\rho$ if and only if $n-1 \not\equiv 0(\text{mod } 3)$. Hence the theorem. □

Theorem 241. *The sun graph Sun_n does not belong to the classes $\mathbb{BS}_{\rho_0}$ and $\mathbb{BS}_\rho$, and $Sun_n \in \mathbb{BS}_\mu$, for all $n > 2$.*

Proof. Observe that the sun graph Sun_n has n pendant edges so, by Theorem 220 $Sun_n \notin \mathbb{BS}_{\rho_0}$. Also, there are n vertices of degree 3, so by Theorem 219, $Sun_n \notin \mathbb{BS}_\rho$. Since all the vertices of Sun_n are of odd degree by Corollary 217, $Sun_n \in \mathbb{BS}_\mu$. □

Theorem 242. *For $n > 2$, the gear graph G_n belongs to the class $\mathbb{BS}_{\rho_0}$ if and only if either n is an even number or n is a multiple of 3.*

Proof. Let G_n, be the gear graph with vertex set $\{u_i, v_i, k : 1 \leq i \leq n\}$ and edge set $\{u_iv_i, v_iu_{i+1}, ku_i : 1 \leq i \leq n, i+1 \text{ is taken over modulo } n\}$. Suppose that $G_n \in \mathbb{BS}_{\rho_0}$. Then $\ell^*(k) = \rho_0 = g(ku_i)^n$, for some i. Since $g(u_ik) \in S_3 \setminus \{\rho_0\}$, either n is even or n is a multiple of 3. Now suppose n is even. Then define $f : E(G_n) \to N_{3n}$ and $g : E(G) \to S_3 \setminus \{\rho_0\}$ as follows:

$$\begin{aligned} f(u_iv_i) &= 2(i-1), 2 \leq i \leq n, \\ f(v_iu_{i+1}) &= (2i-1), 1 \leq i \leq n, \\ f(u_1v_1) &= 2n, \\ f(ku_i) &= 2n+i, 1 \leq i \leq n. \end{aligned}$$

For $1 \leq i, j \leq n$,

$$g(u_iv_j) = \begin{cases} \mu_2, & \text{if } j \text{ is odd}, \\ \rho_2, & \text{if } j \text{ is even and } i = j, \\ \rho_1, & \text{if } j \text{ is even and } i = j+1. \end{cases}$$
$$g(ku_i) = \mu_1.$$

Above functions f and g will determine a S_3-barycentric magic labeling of G_n with constant ρ_0. Suppose n is a multiple of 3. Then define f as above and let g as follows:

For $1 \leq i, j \leq n$, $g(u_i v_j) = \begin{cases} \mu_1, & \text{if } j \equiv 1 (\text{mod } 3), \\ \mu_2, & \text{if } j \equiv 2 (\text{mod } 3), \\ \mu_3, & \text{if } j \equiv 0 (\text{mod } 3), \end{cases}$ and $g(ku_i) = \rho_2$.

Using the above f and g, one can easily verify that $G_n \in \mathbb{BS}_{\rho_0}$. Hence the proof. □

Theorem 243. *The gear graph G_n does not belong to the classes $\mathbb{BS}_\rho$ and $\mathbb{BS}_\mu$.*

Proof. Since the gear graph has n vertices of degree 2 and n vertices of degree 3, by Theorem 219 and Theorem 216 $G_n \notin \mathbb{BS}_\rho$ and $G_n \notin \mathbb{BS}_\mu$. □

Chapter 8

Neighborhood S_3-Magic Labeling of Graphs

In this chapter, we introduced the notion of neighborhood A-magic labeling of graphs, where A is a finite non-abelian group. The first section of this chapter gives an introduction about the group distance magic labeling. The second section of this chapter deals with the neighborhood S_3-magic labeling of some known graphs. The third section of this chapter discusses the neighborhood S_3-magic labeling of some cycle related graphs.

8.1 Introduction

A group distance magic labeling or a Γ-distance magic labeling of a graph $G = (V(G), E(G))$ with $|V(G)| = n$ is an injection from $V(G)$ to an abelian group Γ of order n such that the weight of every vertex $x \in V(G)$ is equal to the same element $\mu \in \Gamma$, called the magic constant [39]. The concept of group distance magic labeling was introduced by D. Froncek [39].

If $u \in V(G)$, then open neighborhood of u is defined as $N(u) = \{v \in V(G) : uv \in E(G)\}$. A graph G is said to be a neighborhood magic graph if there exists a real-valued function $f : V(G) \to \mathbb{R}$ satisfying the condition $\sum_{v \in N(u)} f(v) = Q(f), \forall u \in V(G)$. The constant $Q(f)$ is called the neighborhood magic index of f and the function f is called neighborhood magic labeling. The term neighborhood magic labeling is put forward by B.D. Acharya et.al. [40]. In 2019, K.P. Vineesh and V. Anil Kumar [8] introduced the concept of neighborhood V_4-magic labeling of graphs, where V_4 is the Klein 4 group. A graph G is said to be neighborhood V_4-magic if there exists a labeling $f : V(G) \to V_4 \setminus \{0\}$ such that the sum

$$N_f^+(v) = \sum_{u \in N(v)} f(u)$$

[1]This chapter has been published in the Journal Advances and Applications of Discrete Mathematics, Volume 41, Number 2,(2024).

is a constant map [8].
Motivated by group distance magic labeling and neighborhood magic labeling, we introduce the notion of neighborhood A-magic labeling of graphs, where A is a non-abelian group. In this Chapter, we consider the group S_3 and investigate graphs that are neighborhood S_3-magic.

8.2 Neighborhood S_3-Magic Labeling of Graphs

Let $G = (V(G), E(G))$ be a finite graph with p vertices and q edges. That is, G is a finite (p, q) graph, and let A be a finite non-abelian group with identity element 1. Let $f : V(G) \to N_p = \{1, 2, \dots, p\}$ and let $g : V(G) \to A \setminus \{1\}$ be two edge labelings of G such that f is bijective. Define a vertex labeling labeling $\lambda : V(G) \to N_p \times A \setminus \{1\}$ by

$$\lambda(v) := (f(v), g(v)),\ v \in V(G).$$

Define a relation $\leq$ on the range of λ by:

$$(f(v), g(v)) \leq (f(v^{'}), g(v')) \text{ if and only if } \quad f(v) \leq f(v').$$

Then obviously, the relation $\leq$ is a partial order on the range of λ. Let $\{(f(v_1), g(v_1)), (f(v_2), g(v_2)), \dots, (f(v_k), g(v_k))\}$ be a chain in the range of λ. We define the product of this chain as follows:

$$\prod_{i=1}^{k}(f(v_i), g(v_i)) := ((((g(v_1) * g(v_2)) * g(v_3)) * g(v_4)) * \dots) * g(v_k)$$

Let $u \in V(G)$ and let $N(u)$ neighborhood of u. Note that the range of $\lambda|_{N(u)}$ is a chain, say $(f(v_1), g(v_1)) \leq (f(v_2), g(v_2)) \leq \cdots \leq (f(v_n), g(v_n))$. We define

$$\lambda^*(u) = \prod_{i=1}^{n}(f(v_i), g(v_i)). \tag{8.2.1}$$

Note that $\lambda^*(u) = \lambda|_{N(u)}$. We say that the graph G is neighborhood A-magic if there exist two functions f and g as above such that the map λ^* is a constant map. If this constant is γ where γ is an element in A, then we say that λ is a γ-neighborhood A-magic labeling of G and G is said to be a γ-neighborhood A-magic graph. The constant γ is called the neighborhood magic constant. For example, consider the cycle graph $C_4 = (u_1, u_2, u_3, u_4)$ and the permutation group S_3 with elements $\rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3$(See [16]). Define $f : V(G) \to N_p =$

$\{1, 2, 3, 4\}$ as $f(u_1) = 1,\ f(u_3) = 2, f(u_2) = 3,\ f(u_4) = 4$ and $g : V(G) \to S_3 \setminus \{\rho_0\}$ as $g(u_1) = g(u_3) = \mu_1,\ g(u_2) = g(u_4) = \mu_2$; as illustrated in Figure 8.1. Thus

$$\lambda^*(u_1) = (3, \mu_2)(4, \mu_2) = \mu_2\mu_2 = \rho_0 \text{ and}$$
$$\lambda^*(u_2) = (1, \mu_1)(2, \mu_1) = \mu_1\mu_1 = \rho_0.$$

Similarly, we get $\lambda^*(u_3) = \lambda^*(u_4) = \rho_0$. Thus C_4 is ρ_0-neighborhood S_3-magic with neighborhood magic constant ρ_0. Observe that, when A is abelian our definition coincides with that by Froncek [39] and Vineesh [8, 41].

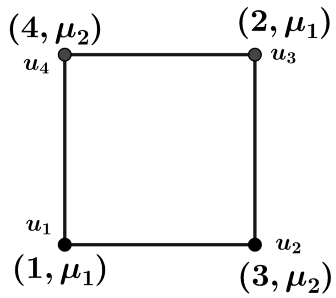

Figure 8.1: ρ_0-neighborhood S_3-magic labeling of C_4.

Theorem 244. *Let G be a graph and A be a non-abelian group having an element of order 2. If the degree of all the vertices of G is either all odd or all even then G is neighborhood A-magic.*

Proof. Let A be a non-abelian group having an element of order 2, say 'a' and let G be a (p, q) graph. Now, let f be a bijection from $V(G) \to N_p$ and let $g : V(G) \to A \setminus \{1\}$ be the constant map $g(v) = a,\ \forall v \in V(G)$. Suppose the cardinality of $N(u)$ is odd for all $u \in V(G)$ then $\lambda^*(u) = a, \forall u \in V(G)$. If the cardinality of $N(u)$ is even $\forall u \in V(G)$ then $\lambda^*(u) = 1,\ \forall u \in V(G)$. Hence the proof.

□

Corollary 245. *Any regular graph is neighborhood S_3-magic.*

Corollary 246. *For any $n \geq 3$, the cycle graph C_n is neighborhood S_3-magic.*

Corollary 247. *For any $n \geq 2$ the complete graph K_n is neighborhood S_3-magic.*

Corollary 248. *The Petersen graph is neighborhood S_3-magic.*

Theorem 249. *Any graph with a pendant edge is not 1-neighborhood A-magic graph, where 1 is the identity element of the nonabelian group A.*

Proof. Let G be a graph with a pendant edge, say 'e'. Let the end vertices of e be u_1 and u_2, such that u_2 is the pendant vertex of e. If G is 1-neighborhood A-magic graph then $\lambda^*(u) = 1, \ \forall u \in V(G)$. In particular, $\lambda^*(u_2) = 1$. Since u_1 is the only neighborhood of u_2, the value of $g(u_1)$ must be 1. This is a contradiction to the assumption that g is a map from $V(G) \to A \setminus \{1\}$. Hence the proof. □

Corollary 250. *The path $P_n, n \geq 2$ is not ρ_0-neighborhood S_3-magic.*

Theorem 251. *The path graph P_n is not neighborhood S_3-magic for $n \geq 4$.*

Proof. Suppose that $n \geq 4$. Denote the vertices of P_n by $u_1, u_2, \ldots, u_n$. Assume, on the contrary, that P_n is neighborhood S_3-magic. Let $g(u_1) = a, \ g(u_2) = b, \ g(u_3) = c$ and $g(u_4) = d$, where $a, b, c, d \in S_3 \setminus \{\rho_0\}$. Since $N(u_1) = u_2$, $\lambda^*(u_1) = b$. So the neighborhood magic constant is b. Then $\lambda^*(u_3) = bd$ or db according to the value of $f(u_2)$ and $f(u_4)$. Thus either $bd = b$ or $db = b$. This implies $d = \rho_0$. Which is a contradiction. Hence the path graph P_n is not neighborhood S_3-magic for $n \geq 4$. This completes the proof of the theorem. □

Theorem 252. *The complete bipartite graph $K_{m,n}, m, n > 1$ is neighborhood S_3-magic.*

Proof. Let $G = K_{m,n}$. Suppose $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the two partite sets of $K_{m,n}$. If m and n are both even or odd then the degree of all the vertices of G is either all odd or all even. Then the theorem is indisputable from Theorem 244. Without loss of generality, let m be an even number and n be an odd number. Now consider the following cases:

Case (i): m is even and $n \equiv 0 (\text{mod } 3)$.

Here we define $f : V(G) \to N_{mn}$ be any bijection and let g be the function defined as follows: For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$g(u_i) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even} \end{cases}$ and $g(v_j) = \rho_1$. By defining f and g as above we obtain a ρ_0-neighborhood S_3-magic labeling of G.

Case (ii): m is even and $n \equiv 1 (\text{mod } 3)$.

Define f as $f(u_i) = i, \ i = 1, 2, \ldots, m$ and $f(v_j) = m + j, \ j = 1, 2, \ldots, n$.

Now define g as follows:

$$\text{For } i = 1, 2, \ldots, m,\ g(u_i) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases} \quad \text{and}$$

$$\text{for } j = 1, 2, \ldots, n,\ g(v_j) = \begin{cases} \rho_1, & \text{if } 1 \leq j \leq n-4 \\ \mu_1, & \text{if } j \geq n-3. \end{cases}$$

Clearly $\lambda^*(u) = \rho_0, \forall u \in V(G)$.

Case (iii): m is even and $n \equiv 2 (\text{mod } 3)$.

Let f be defined as in Case(ii) and let g be defined by,

$$\text{For } i = 1, 2, \ldots, m,\ g(u_i) = \begin{cases} \rho_1, & \text{if } i \text{ is odd,} \\ \rho_2, & \text{if } i \text{ is even.} \end{cases} \quad \text{and}$$

$$\text{for } j = 1, 2, \ldots n,\ g(v_j) = \begin{cases} \rho_1, & \text{if } 1 \leq j \leq n-2, \\ \mu_1, & \text{if } j = n-1, n. \end{cases}$$

Obviously the above maps f and g define a ρ_o-neighborhood magic labeling for G.

This completes the proof of the theorem. □

Theorem 253. *The star graph $K_{1,n}, n > 2$ is neighborhood S_3-magic.*

Proof. Let $G = K_{1,n}$. Denote the pendant vertices of $K_{1,n}$ by $u_1, u_2, \ldots, u_n$ and the vertex of degree n by k. We will consider the following cases.

Case (i): $n \equiv 0 (\text{mod } 3)$.

Define $f : V(G) \to N_{n+1}$ by $f(u_i) = i,\ 1 \leq i \leq n$ and $f(k) = n+1$. Now define $g : V(G) \to S_3 \setminus \{\rho_0\}$ as $g(u_i) = \rho_1,\ 1 \leq i \leq n-1,\ g(u_n) = \mu_3,\ g(k) = \mu_1$. Then clearly $\lambda^*(u) = \mu_1,\ \forall u \in V(G)$.

Case (ii): $n \equiv 1 (\text{mod } 3)$.

Define f as above and g be defined by

$$g(u_i) = \begin{cases} \rho_1, & \text{if } 1 \leq i < n-2, \\ \rho_2, & \text{if } i = n-1, n-2, \\ \mu_3, & \text{if } i = n, \end{cases} \quad \text{and } g(k) = \mu_1.$$

Then $\lambda^*(u) = \mu_1, \forall u \in V(G)$.

Case (iii): $n \equiv 2 (\text{mod } 3)$.

Here also define f as in Case(i) and g be the map

$$g(u_i) = \begin{cases} \rho_1, & \text{if } 1 \leq i \leq n-2, \\ \rho_2, & \text{if } i = n-1, \\ \mu_3, & \text{if } i = n, \end{cases} \qquad \text{and } g(k) = \mu_1.$$

Then obviously, $\lambda^*(u) = \mu_1, \forall u \in V(G)$.

Hence $K_{1,n}, n > 2$ is neighborhood S_3-magic with the neighborhood magic constant μ_1. In particular, we can say that $K_{1,n}$ is a μ_1-neighborhood S_3-magic graph. □

Theorem 254. *Bistar graph B_n, $n \geq 2$ is neighborhood S_3-magic.*

Proof. There are $2n$ pendant vertices in B_n and let the end vertices of the bridge be represented by b_1 and b_2. Label the pendant vertices of the first star by $u_1, u_2, \ldots, u_n$ and the pendant vertices of the second star by $v_1, v_2, \ldots, v_n$. The neighborhood of every pendant vertex is either $\{b_1\}$ or $\{b_2\}$. Suppose B_n is a neighborhood S_3-magic graph with the neighborhood magic constant 'a' where $a \in S_3$. Then $\lambda^*(u) = g(b_1) = g(b_2)$. Hence $g(b_1) = g(b_2) = a$ and $a \neq \rho_0$. Consider the following cases:

Case(i): n is even, $n \geq 2$.

Suppose that n is even. Now let $g : V(B_n) \to S_3 \setminus \{\rho_0\}$ be the constant map $g(u) = \mu_1, \ \forall u \in V(B_n)$ and f be any bijective map from $V(B_n) \to N_{2n+2}$. Thus the maps f and g will determine a μ_1-neighborhood S_3-magic labeling for B_n.

Case (ii): n is odd and $n \equiv 0 (\text{mod } 3)$.

Define f as $f(u_i) = i, \ f(v_i) = n + i, \ 1 \leq i \leq n, \ f(b_1) = 2n + 1,$ and $f(b_2) = 2n + 2$ and let g be defined as follows:
$g(b_1) = g(b_2) = \mu_1$ and $g(v) = \rho_1, \forall$ pendant vertex $v \in V(B_n)$. Clearly, $\lambda^*(u) = \mu_1, \ \forall u \in V(B_n)$.

Case(iii): n is odd and $n \equiv 1 (\text{mod } 3)$.

Let f as in case(ii) and define $g : V(B_n) \to S_3 \setminus \{\rho_0\}$ as $g(b_1) = g(b_2) = \mu_1, \ g(u_i) = g(v_i) = \rho_1, \ 1 \leq i \leq n-2, \ g(u_{n-1}) = g(v_{n-1}) = \mu_1$ and $g(u_n) = g(v_n) = \mu_2$. Then $\lambda^*(u) = \mu_1, \ \forall u \in V(B_n)$.

Case (iv): n is odd and $n \equiv 2 (\text{mod } 3)$.

Let f be defined as in the case(ii) and let g be the function

$g(u_i) = g(v_i) = \begin{cases} \rho_1, & \text{if } 1 \leq i \leq n-2, \\ \mu_1, & \text{if } i = n, n-1. \end{cases}$ and $g(b_1) = g(b_2) = \mu_1$. Obviously the above f and g will determine a μ_1-neighborhood S_3-magic labeling of B_n.

This completes the proof of the theorem.

□

8.3 Cycle Generated Graphs

In this section, we consider certain graphs that are constructed from cycles.

Theorem 255. *The wheel graph W_n,$n \geq 3$, is neighborhood S_3-magic.*

Proof. Let W_n be a wheel of order $n+1$. Denote the vertices of C_n be $u_1, u_2, \ldots, u_n$ and the vertex of K_1 be k. Now consider the following cases:

Case(i): n is odd.

Suppose n is odd then the degree of every vertex is odd. Then by Theorem 244, W_n is neighborhood S_3-magic. If we take μ_1 as a in Theorem 244 then we get a μ_1-neighborhood S_3-magic labeling of W_n.

Case(ii): n is even and $n \equiv 0(\text{mod } 6)$.

In this case, we consider the following two subcases:

Subcase(a): $n \equiv 0(\text{mod } 6)$ and n is an odd multiple of 6.

Here we take $f : V(W_n) \to N_{n+1}$ as follows:

for $1 \leq i \leq n$, $f(u_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd}, \\ \frac{n+i}{2}, & \text{if } i \text{ is even}. \end{cases}$ and $f(k) = n+1$.

Now define $g : V(G) \to S_3 \setminus \{\rho_0\}$ as $g(k) = \rho_1$ and for $1 \leq i \leq n$,

$$g(u_i) = \begin{cases} \mu_1, & \text{if } i \text{ is odd}, \\ \mu_2, & \text{if } i \text{ is even}. \end{cases}$$

Clearly the above f and g determine a neighborhood S_3-magic labeling of W_n with neighborhood magic constant ρ_1.

Subcase(b): $n \equiv 0(\text{mod } 6)$ and n is an even multiple of 6.

For $1 \leq i \leq n$, define f as

$$f(u_i) = \begin{cases} \frac{(i+1)}{2}, \text{ if } i \text{ is odd and } 1 \leq i < \frac{n}{2}, \\ \frac{n}{2} + 1, \text{ if } i = \frac{n}{2} + 1, \\ \frac{n}{2} + 1 - k, \text{ if } i \text{ is odd and } i = \frac{n}{2} + (2k+1), k = 1, 2, \ldots, \frac{n-4}{4}, \\ \frac{n}{4} + 1, \text{ if } i = 2, \\ \frac{n}{2} + 1 + \frac{i-2}{2}, \text{ if } i \text{ is even and } i \neq 2. \end{cases}$$

and let $f(k) = n + 1$. Now let g be the function $g(k) = \rho_1$ and

$$g(u_i) = \begin{cases} \mu_1, \text{ if } i \text{ is odd and } i \equiv 1 (\text{mod } 6), 1 \leq i \leq n, \\ \mu_3, \text{ if } i \text{ is odd and } i \equiv 3 (\text{mod } 6), 1 \leq i < \frac{n}{2}, \\ \mu_2, \text{ if } i \text{ is odd and } i \equiv 5 (\text{mod } 6), 1 \leq i < \frac{n}{2}, \\ \mu_2, \text{ if } i \text{ is odd and } i \equiv 3 (\text{mod } 6), \frac{n}{2} \leq i < n, \\ \mu_3, \text{ if } i \text{ is odd and } i \equiv 5 (\text{mod } 6), \frac{n}{2} \leq i < n, \\ \rho_1, \text{ if } i \text{ is even}, 1 \leq i \leq n. \end{cases}$$

Obviously the functions f and g define a ρ_0-neighborhood S_3-magic labeling for W_n.

Case(iii): n is even and $n \equiv 2 (\text{mod } 6)$.

Define f as $f(u_i) = i, 1 \leq i \leq n$ and $f(k) = n + 1$. Let g be the function

$$g(k) = \rho_1 \text{ and for } 1 \leq i \leq n, g(u_i) = \begin{cases} \mu_1, \text{ if } i \text{ is odd}, \\ \mu_2, \text{ if } i \text{ is even}. \end{cases}$$

Then clearly $\lambda^*(u) = \rho_1, \forall u \in V(G)$.

Case(iv): n is even and $n \equiv 4 (\text{mod } 6)$.

Define f as in the case (iii) and g be $g(k) = \rho_2$ and for $1 \leq i \leq n$,

$$g(u_i) = \begin{cases} \mu_1, \text{ if } i \text{ is odd}, \\ \mu_2, \text{ if } i \text{ is even}. \end{cases}$$

Hence we get $\lambda^*(u) = \rho_2, \ \forall u \in V(G)$.

This completes the proof of the theorem. □

Theorem 256. *The helm graph $H_n, n \geq 3$ is neighborhood S_3-magic if and* *only if n is odd or $n \equiv 1 \pmod 3$.*

Proof. Let $\{k, u_i, v_i : i = 1, 2, \ldots, n\}$ be the set of vertices of H_n, where k be the central vertex, $u_1, u_2, \ldots, u_n$ are the vertices of the cycle, $v_1, v_2, \ldots, v_n$ are the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$. The edge set of H_n is $E(H_n) = \{u_iu_{i+1}, ku_i, u_iv_i : i = 1, 2, \ldots, n,\ u_{n+1} = u_1\}$.

Suppose that $H_n, n \geq 3$ is neighborhood S_3-magic with neighborhood magic constant a. So there exist two functions f and g satisfying the conditions for neighborhood S_3-magic labeling. Then, for every pendant vertex v_i, $\lambda^*(v_i) = g(u_i) = a$. So $g(u_i) = a,\ \forall i,\ 1 \leq i \leq n$. We have $N(k) = \{u_i : i = 1, 2, \ldots, n\}$. Hence $\lambda^*(k) = \underbrace{aa\cdots a}_{n\ times} = a$. Which is possible only when either n is odd or $n \equiv 1 (\mathrm{mod}\ 3)$. Since the order of an element in $S_3 \setminus \{\rho_0\}$ is either 2 or 3.

Conversely, assume that n is odd. Now define f by $f(k) = 1,\ f(u_i) = i + 1,\ f(v_i) = n + 1 + i,\ i = 1, 2, \ldots, n$ and let g be the function defined by $g(k) = \rho_1,\ g(u_i) = \mu_1,\ g(v_i) = \mu_2,\ i = 1, 2, \ldots, n$. Obviously, the above functions f and g determine a μ_1-neighborhood S_3-magic labeling of H_n. Next, suppose that $n \equiv 1 (\mathrm{mod}\ 3)$ is odd. Then define f be any bijective map from $V(H_n)$ to N_{2n+1} and g be the constant map $g(u) = \rho_1, \forall u \in V(H_n)$. Clearly $\lambda^*(u) = \rho_1,\ \forall u \in V(H_n)$. Hence the proof. □

Theorem 257. *The shell graph $S_{4,1}$ is neighborhood S_3-magic.*

Proof. Denote the vertices of $S_{4,1}$ by u_1, u_2, u_3, u_4. Without loss of generality, let the apex be u_1. Let f be any bijection from $V(S_{4,1})$ to N_4 and define $g : V(G) \to S_3 \setminus \{\rho_0\}$ as $g(u_1) = g(u_3) = \rho_1$ and $g(u_2) = g(u_4) = \rho_2$. Then the above maps f and g determine a ρ_2-neighborhood S_3-magic labeling of $S_{4,1}$. □

Theorem 258. *For $n \geq 5$, the shell graph $S_{n,n-3}$ is not neighborhood S_3-magic.*

Proof. Let $G = S_{n,n-3},\ n \geq 5$ and let us denote the vertices of the shell graph by $u_1, u_2, \ldots, u_n$. Without loss of generality, let the apex be u_1. Suppose that the graph G is neighborhood S_3-magic. So there exist two functions f and g such that the map λ^* is a constant function. Suppose $f(u_i) = m_i$ and $g(u_i) = a_i$, where $1 \leq i \leq n$ and $a_i \in S_3 \setminus \{\rho_0\}$. Without loss of generality, let

$f(u_2) < f(u_3) < f(u_4) < f(u_5)$. *i.e.*, $m_2 < m_3 < m_4 < m_5$. Then,

$$\lambda^*(u_1) = \prod_{i=2}^{n}(m_i, a_i)$$

$$\text{Similarly, } \lambda^*(u_2) = \begin{cases} a_1a_3, & \text{if } m_1 < m_3, \\ a_3a_1, & \text{if } m_1 > m_3. \end{cases}$$

$$\lambda^*(u_3) = \begin{cases} a_2a_4a_1, & \text{if } m_2 < m_4 < m_1, \\ a_2a_1a_4, & \text{if } m_2 < m_1 < m_4, \\ a_1a_2a_4, & \text{if } m_1 < m_2 < m_4. \end{cases}$$

$$\lambda^*(u_4) = \begin{cases} a_3a_5a_1, & \text{if } m_3 < m_5 < m_1, \\ a_1a_3a_5, & \text{if } m_1 < m_3 < m_5, \\ a_3a_1a_5, & \text{if } m_3 < m_1 < m_5. \end{cases}$$

If $\lambda^*(u_2) = a_1a_3$ then we must have $m_1 < m_3$. Hence $\lambda^*(u_4) = a_1a_3a_5$. Since λ^* is a constant map $a_1a_3 = a_1a_3a_5$ implies $a_5 = \rho_0$, which is a contradiction. If $\lambda^*(u_2) = a_3a_1$ then $m_3 < m_1$. Hence $\lambda^*(u_4) = a_3a_5a_1$ or $a_3a_1a_5$. Since λ^* is a constant map either $a_3a_1 = a_3a_5a_1$ or $a_3a_1 = a_3a_1a_5$. Both cases imply that $a_5 = \rho_0$, which is a contradiction to the assumption that $a_i \in S_3 \setminus \{\rho_0\}$. Hence G can not be a neighborhood S_3-magic graph. This completes the proof of the theorem. □

Chapter 9

Conclusions and Recommendations

A summary of the thesis is given in the first section of the chapter. The following section includes some guidelines for a researcher to explore more areas.

9.1 Summary of the Thesis

This thesis aims to generalize the concept of magic labeling of graphs from finite abelian groups to any finite non-abelian groups. While several authors have explored group magic labeling of graphs using abelian groups, the non-commutative nature of non-abelian groups presents an intriguing research opportunity.

When defining group magic labeling for graphs with finite groups, it is crucial to ensure consistency with the existing definition of A-magic labeling, where A is a finite abelian group, as established by S. M. Lee [4], Doob [10], and others.

For non-abelian groups, the definition of A-magic labeling requires additional consideration, as the sum of labels may depend on the order of summation. To address this, we introduce an ordering constraint in the magic labeling. This work primarily focuses on various types of magic labeling of graphs using non-abelian groups, exploring new avenues in graph theory research.

This thesis explores the concept of A-magic labeling of graphs, focusing on non-abelian groups. Instead of considering an arbitrary finite non-abelian group, we focus on the specific non-abelian groups S_3, D_4, and Q_8. We also determine a necessary and sufficient condition for several well-known graphs to be S_3-magic, D_4-magic, and Q_8-magic. We also developed the idea of induced S_3-magic labeling of graphs, and a necessary and sufficient condition for certain cycle, star, and path-related graphs that admit induced S_3-magic labeling was also discovered. We introduced a new magic labeling called conjugate A-magic labeling of graphs, and the conjugate S_3-magic labeling of some well-known graphs was also explored. We make a study on the A-barycentric magic labeling. Moreover, by defining the same for non-abelian groups, we extend the concept of A-barycentric magic labeling of graphs with abelian groups to any finite group. Finally, we defined neighborhood magic labeling of graphs using the finite non-abelian group A and evaluated the neighborhood magic labeling of various well-known graphs.

9.2 Recommendations

This thesis provides a foundation for further research in magic graph labeling. The following recommendations offer potential avenues for exploration:

- Study the necessary and sufficient conditions of S_3-magic labeling of some more graphs.
- Study the A-magic property of the graph obtained from the product of two A-magic graphs other than the cartesian product and lexicographic product.
- Examine the necessary and sufficient conditions of A-magic labeling of operation of two graphs.
- Examine the necessary and sufficient conditions of induced S_3-magic labeling of operation of two graphs.
- Study the induced D_4-magic labeling of some more well known graphs.
- Study the induced Q_8-magic labeling of some more graphs.
- Study the induced D_4-barycentric magic labeling of some more known graphs.
- Study the induced Q_8-barycentric magic labeling of some more known graphs.
- Generalize the concept S_3-magic labeling of graphs to S_n.
- Generalize the concept D_4 magic labeling of graphs to D_n.
- Study the necessary and sufficient conditions of Conjugate S_3-magic labeling of some more graphs and also try to extend it to the group S_n.
- Study the neighborhood S_3-magic labeling of some more graphs.
- Find some general results involving the different magic labeling discussed in the thesis.

- Make a comparative study on S_3-magic, D_4-magic and Q_8-magic labeling.

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Appendix I

List of Publications

[1] **C. Anusha** and **V. Anil Kumar**, "On The S_3-Magic Graphs", ***South East Asian Journal of Mathematics and Mathematical Sciences***, 2022, Volume 18, Number 3, Pages 317-328.
DOI: 10.56827/SEAJMMS.2022.1803.26**(Scopus-indexed)**

[2] **Anusha Chappokkil** and **Anil Kumar Vasu**,"D_4-Magic Graphs", ***Ratio Mathematica***, Volume 42, 2022, Pages 167-181.
DOI: 10.23755/rm.v41i0.738 **(UGC Care list)**

[3] **C. Anusha** and **V. Anil Kumar**,"Q_8 -Magic Labeling of Some Graphs and Its Subdivision Graphs", ***Advances and Applications of Discrete Mathematics***, Volume 34, 2022, Pages 67-85.
DOI: 10.17654/0974165822044**(Web of Science)**

[4] **C. Anusha** and **V. Anil Kumar,** "Conjugate S_3-Magic Graphs", ***South East Asian Journal of Mathematics and Mathematical Sciences***, 2023, Volume 19, Number 3, Pages 319-332.
DOI: 10.56827/SEAJMMS.2023.1903.25**(Scopus-indexed)**

[5] **C. Anusha** and **V. Anil Kumar**, "On Neighborhood S_3-Magic graphs", ***Advances and Applications in Discrete Mathematics,*** Volume 41, Number 2, 2024, Pages 135-148.
DOI: 10.17654/0974165824009(**Web of Science**).

[6] **C. Anusha** and **V. Anil Kumar,** "Induced S_3-Magic Labeling of Some Star and Path Related Graphs" (Communicated).

[7] **C. Anusha** and **V. Anil Kumar,** "On the product of A-magic labeling of Graphs" (Communicated).

[8] **C. Anusha** and **V. Anil Kumar,** "On Induced S_3-magic Labeling of Some Cycle Related Graphs" (Communicated).

[9] **C. Anusha** and **V. Anil Kumar,** " S_3-Barycentric Magic Labeling of Graphs" (Communicated).

Appendix II

List of paper presentations

[1] **"On S_3-magic Graphs",** in the ***International Conference on Graphs, Combinatorics and Optimization (ICGCO 2022)***, held during February 6–8, 2022 organized by Birla Institute of Technology and Science, Pilani, Dubai Campus.

[2] **"On D_4-magic Graphs",** in the ***International Conference of Graphs, Combinatorics ICGNC 2023*** organized by the Department of Mathematics, Ramanujan College, University of Delhi, during 10-12 January 2023.

[3] **"Induced S_3-magic labeling of some cycle related Graphs",** in the ***International Conference on Algebra & Discrete Mathematics (ICADM-2024)*** sponsored by the Directorate of Collegiate Education, DST(SERB) and co-sponsored by KSCSTE and organized by the Department of Mathematics, Government College Kattappana, with the academic support of the Institute of Mathematics Research and Training(IMRT) from 20^{th} to 22^{nd} February 2024.

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