D 22060 **(Pages : 4)** **Name..**

Reg. No.....................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2011

(CCSS)

Mathematics

MAT 3E 02—OPERATIONS RESEARCH

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Show that a positive semidefinite quadratic form is a convex function ?
2. Write the general linear programming problem with *m* constraints and *n* variables. Express it also in matrix form.
3. Prove that the dual of the dual is the primal in L.P. Problems.
4. What is Caterer problem ? Express it in the standard transportation form.
5. Define the following terminologies.
   (a) Slack variable. (b) Surplus variable.
   (c) Feasible solution. (d) Basic solution.
6. Prove that $f(X) = \|X\|, X \in E_n$ is a convex function.
7. Give all the three conditions any one of which helps of terminate branching in Branch and bound method.
8. Obtain Kuhn-Tuker conditions for the problem

   Maximize x^4

   subject to $-1/2 \le x \le 1$.

(8 × 4 = 32 marks)

Part B

Answer **either** *part A or Part B in each questions.*
Each question carries 16 *marks.*

9. A. (a) Let $f(x)$ be a convex differentiable function defined in a convex domain $K \le E_n$. Prove that for $X_0 \in K$, $f(x_0)$ is a global minimum if and only if

   $$(X - X_0)' \nabla f(x_0) \ge 0 \,\forall\, X \text{ in } K.$$

Turn over

(b) Solve the following problem using simplex method.

Maximize $5x_1 - 3x_2 + 4x_3$

subject to $x_1 - x_2 \leq 1$

$-3x_1 + 2x_2 + 2x_3 \leq 1$

$4x_1 - x_3 = 1$

$x_2 \geq 0, \quad x_3 \geq 0,$ x_1 unrestricted in sign.

(8 + 8 = 16 marks)

B. (a) Define convex functions. Prove that sum of two convex functions is again a convex function.

(b) Solve the following problem using simplex method

Maximize $5x_1 + 3x_2 + x_3$

subject to $2x_1 + x_2 + x_3 = 3$

$-x_1 + 2x_3 = 4$

$x_1, x_2, x_3 \geq 0$

(8 + 8 = 16 marks)

10. A. (a) Solve by dual simplex method

Minimize $x_1 + 2x_2 + 3x_3 + 4x_4$

subject to $x_1 + 2x_2 + 2x_3 + 3x_4 \geq 30$

$2x_1 + x_2 + 3x_3 + 2x_4 \geq 20$

$x_1, x_2, x_3, x_4 \geq 0.$

(b) Solve the following quadratic programming problem.

Minimize $f(x) = -x_1 - x_2 - x_3 + \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$

subject to $g_1(x) = x_1 + x_2 + x_3 - 1 \leq 0$

$g_2(x) = 4x_1 + 2x_2 - 7/3 \leq 0$

$x_1, x_2, x_3, \geq 0.$

(8 + 8 = 16 marks)

B. (a) Solve the transportation problem for minimum cost with the cost coefficients, demands and supplies as given in the following table. Obtain three optimal solutions.

	D_1	D_2	D_3	D_4	
O_1	1	2	–2	3	70
O_2	2	4	0	1	38
O_3	1	2	–2	5	32
	40	28	30	42	

(b) Solve by the method of quadratic programming.

Minimize $-6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$

subject to $x_1 + x_2 \le 2$

$x_1, x_2 \ge 0.$

(8 + 8 = 16 marks)

11. A. (a) Solve the integer programming problem by the cutting plane method

Maximize $x_1 + x_2$

subject to $7x_1 - 6x_2 \le 5$

$6x_1 + 3x_2 \ge 7$

$-3x_1 + 8x_2 \le 6$

x_1, x_2 non-negative integers.

(b) Use Kuhn-Tucker conditions to find the minima and maxima of $(x_1 - 4)^2 + (x_2 - 3)^2$ subject to $36(x_1 - 2)^2 + (x_2 - 3)^2 < 9.$

(8 + 8 = 16 marks)

B. (a) Solve by branch and bound method

Maximize $11x_1 + 22x_2$

subject to $4x_1 + 7x_2 + x_3 = 13$

x_1, x_2, x_3 non-negative integers

(b) Solve the following problem using simplex method, taking the entering variable at each iteration to be the non-basic variable with the most negative relative cost at that stage.

Minimize $x_1 - 2x_2 - 3x_3$

subject to $-2x_1 + 4x_3 \leq 12$

$-4x_1 + 8x_2 + 3x_3 \leq 10$

$3x_1 + 2x_2 - x_3 \leq 7$

$x_1, x_2, x_3 \geq 0.$

(8 + 8 = 16 marks)

[3 × 16 = 48 marks]

D 22059 (Pages : 2) Name.....................................

Reg. No...................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2011

(CCSS)

Mathematics

MAT 3 E 01—ADVANCED TOPOLOGY

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Prove that the topologist's sine curve

$$X = \left\{\left(x, \sin\frac{1}{x}\right) \in R^2 : 0 < x \leq 1\right\} \cup \left\{(0, y) \in R^2 : -1 \leq y \leq 1\right\}$$ is not path connected.

2. Let X be a topological space with the property that for every closed subset A of X, every continuous real valued function on A has a continuous extension to X. Prove that X is normal.
3. Prove that arbitrary product of T_2 spaces is T_2.
4. Show that second countability is preserved under continuous open functions.
5. Let X be a the topological product of a family of spaces $\{X_i : i \in I\}$. Prove that a net $S : D \to X$ converges to a point $x \in X$ iff for each $i \in I$, the net π_i oS converges to $\pi_i(x)$ in X_i.
6. Prove that a topological space is compact iff every family of closed subsets of it, which has the finite intersection property has non-empty intersection.
7. Prove that every compact metric space is complete.
8. Let (X, *d*) be a metric space and $(\hat{X}, e)$ its completion. Prove that for any complete metric space (*y*, *f*), any uniformly continuous function $g : (X, d) \to (y, f)$ can be extended uniquely to a uniformly continuous function from $(\hat{X}, e)$ to (*y*, *f*).

(8 × 4 = 32 marks)

Part B

Answer either Part A or Part B in each question.
Each question carries 12 *marks.*

9. A (a) Prove that every regular Lindel off space is normal.

(b) Suppose $\mathcal{D}$ is a decomposition of a space X each of whose members is compact and suppose the projection $p : X \to \mathcal{D}$ is closed. Prove that the quotient space $\mathcal{D}$ is Hausdorff whenever X is Hausdorff.

Turn over

B State and prove Urysohn's lemma

10. A (a) Prove that every quotient of a locally connected space is locally connected.

(b) Prove that subset C is a path component of a space X iff C is a maximal subset of C with respect to the property of being path-connected.

B (a) Prove that product of spaces is connected iff each co-ordinate space is connected.

(b) Let $\mathcal{S}$ be a sub-base for a topological space X. Prove that X is completely regular iff for each $V \in \mathcal{S}$ and for each $x \in V$, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $Y \in V$.

11. A (a) Let $(D, \geq)$ be a directed set and let E be an eventual subset of D. Prove that E with the restriction of $\geq$ is a directed set. Also prove that a net $S : D \to X$ where X is a topological space converges to x in X iff the restriction $S|E : E \to X$ converges to x in X.

(b) Let X, Y be topological spaces, $x \in X$ and $F : X \to Y$ a function. Prove that f is continuous at x iff whenever a filter $\mathcal{F}$ converges to x in X, the image filter $f \neq (\mathcal{F})$ converges to $f(x)$ in Y.

B (a) Prove that a topological space is Hausdorff iff limits of all nets in it are unique.

(b) Let $\mathcal{F}$ be a filter in a space X and let S be a the associated net in X. Let $x \in X$. Prove that x is a cluster point of the filter $\mathcal{F}$ iff it is a cluster point of the net S.

12. A (a) Let A be a subset of a metric space (X, d) such that A is complete with respect to the metric induced on it. Prove that A is closed in X.

(b) Prove that an open subspace of a metrically topologically complete space is metrically topologically complete.

B (a) Prove that every contraction of a complete metric space into itself has a unique fixed point.

(b) Let (X, d) be a complete metric space and $T : X \to X$ a map such that for some positive integer m, T^m is a contraction. Prove that T has a unique fixed point.

(4 × 12 = 48 marks)

D 22058 **(Pages : 3)**

Name.......................................

Reg. No....................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION NOVEMBER 2011

(CCSS)

Mathematics

MAT 3C 12—PARTIAL DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Show that $(x-a)^2+(y-b)^2+z^2=1$ is a complete integral of $z^2(1+p^2+q^2)=1$. By taking $b=2a$, show that the envelope of the sub-family is $(y-2x)^2+5z^2=5$. Show further that the singular integrals are $z=\pm 1$.
2. Find the general integral of $yzp+xzq=x+y$.
3. Find the complete integral of $p+q=pq$.
4. Solve the initial value problem for the quasi-linear equation $zz_x+z_y=1$ with the initial conditions $x=s, y=s, z=\frac{1}{2}.s$ for $0\le s\le 1$.
5. Derive D'Alembert's solution which describes the vibrations of an infinite string.
6. Reduce the equation $u_{xx}-4x^2u_{yy}=\frac{1}{x}\cdot u_x$ into its canonical form.
7. State Neumann problem and show that its solution is unique up to the addition of a constant.
8. Solve the Dirichlet exterior problem for a circle.
9. Show that the solution $u(x,t)$ of the differential equation

   $u_t-ku_{xx}=F(x,t), 0<x<l,\ t>0$ satisfying the initial condition $u(x,0)=f(x), 0\le x\le l$, and the boundary conditions $u(o,t)=u(l,t)=0, t\ge 0$, is unique.
10. Transform the problem $\frac{d^2y}{dx^2}+y=x, y(0)=0, y'(1)=0$ to a Fredholm integral equation.
11. Show that the characteristic numbers of a Fredholm equation with a real symmetric Kernel are all real.
12. Determine the resolvent Kernel associated with $K(x,\xi)=\cos(x+\xi)$ in $(0,2\pi)$, in the form of a power series in λ.

(12 × 4 = 48 marks)

Turn over

Part B

Answer **either** *part A* **or** *part B of each question.*
Each question carries 8 *marks.*

13. (A) (a) Determine the partial differential equation satisfied by all surfaces of revolution of the form $z = F(r)$, $r = (x^2 + y^2)^{1/2}$. (2 marks)

(b) Show that the general solution of the quasi-linear equation

$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$, where P, Q and R are given continuously differential functions of x, y and z is $F(u, v) = 0$, where F is an arbitrary function of u and v and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are the solution of

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)}.$$ (6 marks)

(B) (a) Show that the Ptaffian differential equation $yzdx + xzdy + xydz = 0$ is integrable and find the corresponding integral. (3 marks)

(b) Show that the equations $p^2 + q^2 - 1 = 0$ and $(p^2 + q^2)x = pz$ are compatible and find one–parameter family of common solutions. (5 marks)

14. (A) (a) Find the integral surface passing through the parabola $x = 0, z^2 = 4y$ of the differential equation $(p^2 + q^2)x = pz$. (5 marks)

(b) Write short notes on :

(i) Domain of dependence.

(ii) Range of influence. (3 marks)

(B) Determine the two solutions of the equation $pq = 1$ passing through the straight line $x_0 = 2s, y_0 = 2s, z_0 = 5s$. (8 marks)

15. (A) (a) State the Dirichlet problem. (2 marks)

(b) Show that the solution for the Dirichlet problem for a circle of radius a is given by the Poisson integral formula. (6 marks)

(B) (a) State the necessary assumptions needed for the heat conduction problem. (2 marks)

(b) Solve :

$$u_t = u_{xx},\ 0 < x < l,\ t > 0$$
$$u(0, t) = u(l, t) = 0,$$
$$u(x, 0) = x(l - x),\ 0 \le x \le l.$$

(6 marks)

16. (A) (a) If $y''(x) = F(x)$, and y satisfies the end conditions $y(0) = 0$ and $y(1) = 0$, show that

$$y(x) = \int_0^x (x-\xi)F(\xi)d\xi - x\int_0^1 (1-\xi)F(\xi)d\xi .$$ **(3 marks)**

(b) Solve the integral equation by iterative method :

$$y(x) = \lambda \int_0^1 (x+\xi)y(\xi)d\xi + 1 .$$ **(5 marks)**

(B) Show that any solution of the equation

$y(x) = \lambda \int_0^1 (1-3x\xi)y(\xi)d\xi + F(x)$ can be expressed as the sum of F(x) and some linear combination of the characteristic functions. **(8 marks)**

[4 × 8 = 32 marks]

D 22057 (Pages : 2) Name..................................

Reg. No..................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2011

(CCSS)

Mathematics

MAT 3C 11—FUNCTIONAL ANALYSIS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Show that if X is a separable metric space and $Y \subset X$, then Y is separable is the induced metric.
2. Show that the set of all simple functions is dense in $L^a(E)$, where E is a measurable subset of R.
3. Show that the three norms $\| \ \|_1$, $\| \ \|_2$ and $\| \ \|_\infty$ defined on R^n are equivalent.
4. State and prove Holder's inequality.
5. Let X and Y be normed spaces, $F \in BL(X,Y)$ and $Z \subset Z(F)$ be a closed subspace of X. Show that the map $\tilde{F} : X/Z \to Y$ given by $\tilde{F}(x+Z) = F(x), x \in X$, is well defined, $\tilde{F} \in BL(X/Z, Y)$ and $\|\tilde{F}\| = \|F\|$.
6. Show that every finite dimensional normed linearspace is complete.
7. Let X be a normed space over *K*, and f be a nonzero linear functional on X. Show that if E is an open subset of X, then $f(E)$ is an open subset of K.
8. Show that the linear space C_{00} cannot be a Banach space in any norm.
9. With normal notations, show that if T is a metric space, then $C_0(T)$ is a Banach space.
10. Let X be a normed space over *K* and E be a subset of X. Show that E is bounded in X iff $f(E)$ is bounded in *K* for every $f \in X'$.
11. Let X, Y and Z be normed spaces. Show that if $F : X \to Y$ is continuous and $G : Y \to Z$ is closed, then $GoF : X \to Z$ is closed.
12. Show that if a linear map on a normed space X is open then it is surjective.

(12 × 4 = 48 marks)

Turn over

Part B

Answer **either** *part A* **or** *part B of each questions.*
Each question carries 8 *marks.*

13. A. (a) Show that for $1 \le p \le \infty$, the metric space l^p is complete.

(b) Show that the property of completeness of a metric may not be shared by an equivalent metric.

B. (a) Show that the metric space $L^\infty([a,b])$ is not separable.

(b) Define n^{th} Dirichlet Kernel $D_n(t)$ and evaluate $\int_{-\pi}^{\pi} D_n(t)\,dt$. Further, show that $|D_n|$ is not well-behaved as $n \to \infty$.

14. A. (a) Let X be a normed space. Show that every closed and bounded subset of X is compact iff X is finite dimensional.

(b) Show that every bijective linear map on a finite dimensional normed space is a homeomorphism.

B. (a) Let X and Y be normed spaces and $F: X \to Y$ be a linear map. Show that F is continuous at 0 iff $\|F(x)\| \le \alpha \|x\|$ for all $x \in X$ and some $\alpha > 0$.

(b) Show that a linear map on a linear space X may be continuous with respect to some norm on X, but discontinuous with respect to another norm on X.

15. A. (a) State and prove Hahn Banach separation theorem.

(b) Show that there exist $f_1, f_2, \ldots, f_m$ in X' such that $f_j(a_i) = \delta_{ij}$, $1 \le i, j \le m$, for any linearly independent set $\{a_1, a_2, \ldots, a_m\}$ in a normed linear space X.

B. (a) Let X be a normed space and Y be a closed subspace of X. Show that X is a Banach space iff Y and X/Y are Banach spaces in the induced norm and the quotient norm respectively.

(b) Show that a Banach space cannot have a denumerable basis.

16. A. (a) Let X and Y be Banach spaces and $F: X \to Y$ be a closed linear map. Show that F is continuous.

(b) Show that a linear functional on a normed space X is continuous iff it is closed.

B. (a) State and prove open mapping theorem.

(b) Show by an example that the open mapping theorem may not hold if the normed spaces X and Y are not Banach spaces.

($4 \times 8 = 32$ marks)

D 22056 (Pages : 3) Name.......................................

Reg. No...................................

THIRD SEMESTER M.Sc. (Mathematics) DEGREE EXAMINATION NOVEMBER 2011

(CCSS)

MAT 3C 10—COMPLEX ANALYSIS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Let S be the Riemann sphere. For the points $\frac{1}{2}$, i, $2i$ of $\mathbb{C}$ what are the corresponding points of S ? Which subsets of S correspond to

   (a) real axis and

   (b) the circle $\{z : |z| = 2\}$?

2. Find the radius of convergence of each of the following power series.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$.

   (b) $\sum_{R=0}^{\infty} k^w z^w, k \in \mathbb{Z}, k \neq 0$.

3. Discuss the mapping properties of $\cos z$.

4. Let r be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{r\}$. Show that for all integers

   $n \geq 2, \int_r (z-a)^{-n}\, dz = 0$.

5. Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $u(z) \geq 0$ for all z in $\mathbb{C}$. Prove that u is a constant.

6. Let f and g be two analytic functions on a region G such that $f(z)\, g(z) = 0$ for all z in G. Prove that $f \equiv 0$ or $g \equiv 0$.

7. Let G be an open convex set. Show that every closed rectifiable curve r in G is homotopic to zero.

8. If G is a simply connected region and if f is an analytic function on G, prove that f has a primitive in G.

9. Let G be a region and $z_0 \in G$. Suppose f is analytic in G $\{z_0\}$, with a pole at z_0. Show that there exists a positive integer m and an analytic function g on G such that $f(z) = (z - z_0)^{-m} g(z)$ for all z in G.

Turn over

10. Let $f(z)=(1+z^4)^{-1}$. Find the poles of f and obtain the residues at the poles.

11. If f is a meromorphic function in a region G, show that the poles of f does not have a limit point in G.

12. If f is a nonconstant analytic function on a bounded open set G and if f is continuous on G, prove that either f has a zero in G or $|f|$ assumes its minimum value on ∂G.

(12 × 4 = 48 marks)

Part B

Answer **either** *Part A* **or** *Part B in each question.*
Each question carries 8 *marks.*

13. A. (a) Show that a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely for each z in B(a ; R) where $R^{-1} = \limsup |a_n|^{1/n}$. (2 marks)

(b) If f is a branch of the logarithm in a region G, show that f is analytic in G and its derivative is $1/z$. (4 marks)

(c) Show that a Möbius transformation preserves cross ratio. (2 marks)

B. (a) Show that a power series is analytic in the disc of convergence. (3 marks)

(b) Let D be an open disk and $u : D \to \mathbb{R}$ be a harmonic function. Show that u has a harmonic conjugate. (3 marks)

(c) Show that every Möbius transformation is a composition of translations, dilations and the inversion. (2 marks)

14. A. (a) Let $r : [a, b] \to ¢$ be piece wise smooth and $f : [a, b] \to ¢$ be continuous. Prove that $\int_a^b f dr = \int_a^b f'(t)\, r'(t)\, dt$. (2 marks)

(b) Suppose f is analytic in B(a ; R). Show that $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ for $|z-a|<$ R where $a_n = \frac{1}{n!} f^{(n)}(a)$. (3 marks)

(c) State and prove Morera's Theorem. (3 marks)

B. (a) If $p(z)$ is a non constant polynomial, show that there is a complex number a with $p(a) = 0$. (2 marks)

(b) Let f be analytic in B(0 ; 1). Suppose $|f(z)| \le 1$ for $|z| < 1$. Show that $|f'(0)| \le 1$. (2 marks)

(c) State and prove the first version of Cauchy's Integral Formula. (4 marks)

15. A. (a) State and prove the second version of Cauchy's Theorem. (5 marks)

(b) Give the Laurent series expansion of $f(z)=\frac{1}{z(z-1)(z-2)}$ in the following annuli

(i) ann (0 ; 0, 1)

(ii) ann (0 ; 1, 2)

(iii) (0 ; 2, ∞). (3 marks)

B. (a) State and prove the Open Mapping Theorem. (4 marks)

(b) State and prove the theorem on Laurent series development in an annulus. (4 marks)

16. A. (a) State and prove the Residue Theorem. (3 marks)

(b) State Rouche's Theorem and deduce the fundamental theorem of algebra from it. (2 marks)

(c) State and prove the Maximum Modulus Theorem (second version). (3 marks)

B. (a) Show that for $a>1$, $\int_0^{\pi}\frac{d\theta}{a+\cos\theta}=\frac{\pi}{\sqrt{a^2-1}}$. (3 marks)

(b) State and prove the Argument Principle. (3 marks)

(c) Is the Maximum Modulus Theorem true for unbounded regions ? Justify. (2 marks)

[4 × 8 = 32 marks]

D 10234

(Pages : 2)

Name.....................................

Reg. No.................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2010

(CCSS)

Mathematics

MAT 3E 01—ADVANCED TOPOLOGY

Time : Three Hours

Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Give an example of a space in which every compact subset is closed but not Hausdorff. Justify your claim.
2. Prove that the real line with semi-open interval topology is normal.
3. Prove that a path connected space is connected. Give an example of a connected space which is not path connected.
4. Give an example of a Tychnoff space which is not normal.
5. Prove that a subnet of a convergent net is again convergent.
6. Prove that in a compact, first countable space, every sequence has a convergent subsequence.
7. Prove that a closed subset of a complete metric space is complete with respect to the induced metric.
8. Prove that every contraction is a weak contraction. Is the Converse true ? Justify your answer.

($8 \times 4 = 32$ marks)

Part B

Answer A **or** *B of each question.*
Each question carries 12 *marks.*

9. A State and prove Tietze characterization of normality.

   B (a) Show that every regular, Lindeloff space is normal.

   (b) Let X be a topological space and (Y, d) be a metric space. Let $\{f_n\}$ be a sequence of functions from X toY which converges uniformly to the function $f : X \rightarrow Y$. Show that if each f_n is continuous, then f is continuous.

10. A (a) Let X be a topological space. Show that the following are equivalent :

    (i) X is locally connected.

    (ii) Components of open subsets of X are open in X.

    (iii) X has a base consisting of connected subsets.

    (iv) For every $n \in X$ and every neighbourhood N of x there exist a connected open neighbourhood M of x such that $M \subset N$.

Turn over

(b) Show that a product space is connected if and only if each co-ordinate space is connected.

B Show that metrizability is a countably productive property.

11. A (a) Let $S : D \to X$ be a net in a topological space and let $x \in X$. Show that x is a cluster point of S if and only if there exists a subnet of S which converges to x in X.

(b) Show that a topological space is Hausdorff if and only if no filter can converge to more than one point in it.

B Show that for a topological space X, the following statements are equivalent :

(i) X is compact.

(ii) Every net in X has a cluster point in X.

(iii) Every net in X has a convergent subnet in X.

12. A (a) Show that every compact metric space is complete.

(b) Show that a topological space is metrically topologically complete if and only if it is an absolute G_δ.

B (a) Prove that every complete metric space with no isolated points is uncountable.

(b) Show that every metric space can be isometrically embedded as a dense subspace of a complete metric space.

(4 × 12 = 48 marks)

P 0233 (Pages : 2) Name.....................................

Reg. No.................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2010

(CCSS)

Mathematics

MAT 3C 12—PARTIAL DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Find the general integral of $y^2p - xyq = x(z - 2y)$.
2. Show that the equations $p^2 + q^2 = 1$ and $(p^2 + q^2)x = pz$ are compatible and find the one-parameter family of common solutions.
3. Find the complete integral of the equation $zpq - p - q = 0$.
4. Solve the Cauchy problem for $2z_x + yz_y = z$, when the initial data curve C is given by $x_0 = s, y_0 = s^2, z_0 = s, 1 \le s \le 2$.
5. Find the characteristic strips of the equation $xp + yq - pq = 0$.
6. Show that $y = y(x, t)$, the transverse displacement of a string from the mean position (x-axis) at time t at the point x, satisfies the equation $y_{xx} = \frac{1}{c^2} y_{tt}$ where c is a constant.
7. Suppose that $u(x, y)$ is harmonic in a bounded domain D and continuous in $\bar{D} = D \cup B$, where B is the boundary of D. Show that u attains its maximum and minimum on the boundary B.
8. State the Neumann problem and show that the solution of the Neumann problem is unique up to the addition of a constant.
9. Solve the following problem :—

$$\nabla^2 u = u_{xx} + u_{yy} = 0, 0 < x < a, 0 < y < b$$

with the boundary conditions $u(x, 0) = f(x), 0 \le x \le a, u(x, b) = u(0, y) = u(a, y) = 0$.

10. If $y''(x) = F(x)$, and y satisfies the end conditions $y(0) = 0$ and $y(1) = 0$, show that $y(x) = \int_0^x (x - \xi) F(\xi) d\xi - x \int_0^1 (1 - \xi) F(\xi) d\xi$.
11. Show that the characteristic values of λ for the equation $y(x) = \lambda \int_0^{2\pi} \sin(x + \xi) y(\xi) d\xi$ are $\lambda_1 = \frac{1}{\pi}$ and $\lambda_2 = -\frac{1}{\pi}$, with corresponding characteristic function $y_1(x) = \sin x + \cos x$ and $y_2(x) = \sin x - \cos x$.
12. Determine the resolvent Kernel associated with $k(x, \xi) = \cos(x + \xi)$ in $(0, 2\pi)$, in the form of a power series in λ.

(12 × 4 = 48 marks)

Turn over

Part B

Answer A **or** *B of each question.*
Each question carries 8 marks.

13. A (a) Obtain the partial differential equation satisfied by all surfaces of revolution of the form $z = F(r)$, $r = (x^2 + y^2)^{1/2}$, where F is an arbitrary function having continuous partial derivatives.

(b) Find the complete integral of $(p^2 + q^2)y = qz$.

B (a) Show that $(x - a)^2 + (y - b)^2 + z^2 = 1$ in a complete integral of $z^2(1 + p^2 + q^2) = 1$. Find a particular solution corresponding to $b = 2a$. Show that $z = \pm 1$ are the singular integrals.

(b) Show that a necessary and sufficient condition that the Pfaffian differential equation $\vec{X}.d\vec{r} = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0$ be integrable is that $\vec{X}$. curl $\vec{X} = 0$.

14. A (a) Define the Monge cone at a point (x_0, y_0, z_0) characterised by the differential equation $f(x, y, z, p, q) = 0$ and find the Monge cone at (0, 0, 0) for the differential equation $p^2 + q^2 = 1$.

(b) Reduce the following equation into its canonical form and solve it $x^2 u_{xx} - y^2 u_{yy} = 0$.

B (a) Find an integral surface of $p^2 x + qy - z = 0$ containing the initial line $y = 1$, $x + z = 0$.

(b) Derive the D'Alembert's solution which describes the vibration of an infinite string.

15. A (a) State the Dirichlet problem and show that the solution of the Dirichlet problem if it exists is unique.

(b) Show that the solution for the Dirichlet problem for a circle of radius a is given by the Poisson integral formula.

B (a) State and prove Harnack's theorem.

(b) Solve :

$$u_t = u_{xx},\ 0 < x < l,\ t > 0\ ;\ u\ (0, t) = u\ (l, t) = 0\ ;\ u\ (x, 0) = x\ (l - x),\ 0 \le x \le l.$$

16. A (a) Transform the problem $\frac{d^2y}{dx^2} + xy = 1$, $y\ (0) = y\ (1) = 0$ to the integral equation $y(x) = \int_0^1 G(x, \xi)\ \xi y(\xi)\ d\xi - \frac{1}{2}x(1 - x)$ where $G(x, \xi) = x(1 - \xi)$ when $x < \xi$ and $G(x, \xi) = \xi(1 - x)$ when $x > \xi$.

(b) Solve the Fredhom integration equation by iterative method :

$$y(x) = 1 + \lambda \int_0^1 (1 - 3x\xi)\ y(\xi)\ d\xi$$

B (a) Show that the characteristic numbers of a Fredholm integral equation with a real symmetric Kernel are all real.

(b) Obtain an approximate solution of the integral equation $y(x) = \int_0^1 \sin(x\xi) y(\xi)\ d\xi + x^2$, by replacing sin $(x\xi)$ by the first two terms of its power series $\sin(x\xi) = (x\xi) - \frac{(x\xi)^3}{3!} + \ldots$

(4 × 8 = 32 marks)

D10232 (Pages : 2) Name..............................

Reg. No..............................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2010

(CCSS)

Mathematics

MAT 3C 11—FUNCTIONAL ANALYSIS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Show that if X is a separable metric space and $Y \subseteq X$, then Y is separable in the induced metric.
2. Let E be a measurable subset of R. Show that the set of all simple functions is dense in $L^{\infty}(E)$.
3. Define norm on a vector space X over K and show that it is uniformly continuous.
4. Let Y be a subspace of a normed space X. Show that $Y^{\circ} \neq \phi$ iff Y = X.
5. Show that $S = \{x \in l^2 : \|x\|_2 \leq 1\}$ is closed, convex and bounded, but not compact.
6. Show that the functional f on c defined by $f(x) = \lim_{n \to \infty} x(n)$, $x \in c$, is continuous and $\|f\| = 1$.
7. Let Y be a subspace of a normed space X and $a \in X$ but $a \notin \bar{Y}$. Show that there is some $f \in X'$ such that $f/_{y} = 0$, $f(a) = \text{dist}\,(a, \bar{Y})$ and $\|f\| = 1$.
8. Show that the linear space c_{00} cannot be a Banach space in any norm.
9. Show that C $([a, b])$ is a proper dense subspace of $L^p([a, b])$; $1 \leq p < \infty$.
10. Let X be a normed space and E be a subset of X. Show that E is bounded in X iff f (E) is bounded in K for every $f \in X'$.
11. Show that every continuous map on a metric space is closed. Is the converse true ? Justify your answer.
12. Let Y be a finite dimensional subspace of a normed space X. Show that there is a continuous projection P defined on X such that R (P) = Y.

($12 \times 4 = 48$ marks)

Part B

Answer either Part A **or** *Part B of each question.*
Each question carries 8 *marks.*

13. A (a) Show that every sequence in a compact metric space has a convergent subsequence.

(b) State dominated convergence theorem and deduce bounded convergence theorem from it.

(c) State and prove Holder's inequality for measurable functions on a measurable subset of R.

Turn over

11. Suppose f has a simple pole at $z = a$ and let g be analytic in an open set containing a. Prove that $\text{Res}\,(fg\,;a) = g(a)\,\text{Res}\,(f\,;a)$.

12. Let f and g be analytic in $\bar{B}\,(0\,;R)$ with $|f(z)| = |g(z)|$ on $|z| = R$. If neither f nor g vanishes in B (0 ; R), then show that there exists a constant λ, with $|\lambda| = 1$, such that $f = \lambda g$.

(12 × 4 = 48 marks)

Part B

13. A (a) Which subsets of the Riemann sphere S correspond to the real and imaginary axes of $\mathbb{C}$.

(b) Let f and g be analytic on open sets G and Ω respectively. If $f(G) \subset \Omega$, prove that gof is analytic on G and for all z in G, $(gof)'(z) = g'(f(z)) \cdot f'(z)$.

(c) Show that Mobius transformations preserve cross ratios.

(2 + 3 + 3 = 8 marks)

B (a) If f is analytic on a region G and if $f' \equiv 0$ on G, prove that f is constant on G.

(b) Let f be a branch of log z on a region G. Show that the totality of all branches of log z on G are the functions $f(z) + 2\pi ki$, $k \in Z$.

(c) Using Orientation Principle, prove that there exists an analytic function $f : G \to \mathbb{C}$, where $G = \{z : \text{Re}\, z > 0\}$ such that $f(G) = \{z : |z| < 1\}$.

(2 + 3 + 3 = 8 marks)

14. A (a) Let G be an open set and let f be an analytic function on G. Prove that f is infinitely differentiable on G.

(b) Let f be an analytic function on a region G. Prove that the following statements are equivalent :—

(i) $f \equiv 0$.

(ii) There is a point a in G such that $f^{(n)}(a) = 0$ for each $n \geq 0$.

(iii) $\{z \in G : f(z) = 0\}$ has a limit point in G.

(4 + 4 = 8 marks)

B (a) Let f be analytic in B (a ; R). Show that f has a power series expansion in B (a ; R).

(b) Prove that a bounded entire function is a constant.

(c) State and prove the Maximum Modulus theorem.

(2 + 3 + 3 = 8 marks)

15. A (a) Let f be a function analytic on a simply connected region G. Prove that f has a primitive in G.

(b) State and prove the Open Mapping Theorem.

(c) Suppose f has an isolated singularity at z_0. Prove that z_0 is a removable singularity of f if and only if $\lim_{z \to z_0} (z - z_0)\, f(z) = 0$.

(2 + 3 + 3 = 8 marks)

B (a) Let r be a closed rectifiable curve in $\mathbb{C}$. Prove that $n(r ; a) = n(r, b)$ whenever a, b belong to the same component of $\mathbb{C} - \{r\}$. Also show that $n(r ; a) = 0$ whenever a belongs to the unbounded component of $\mathbb{C} - \{r\}$.

(b) State and prove the theorem on Laurent series development in an annulus.

(4 + 4 = 8 marks)

A (a) Prove that $\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2$.

(b) State and prove Rouche's theorem.

(c) Is the Maximum Modulus Principle true for unbounded regions. Justify.

(3 + 3 + 2 = 8 marks)

B (a) State and prove the Argument Principle.

(b) State and prove the Residue theorem.

(c) Show that if f is analytic on a region G and if $|f|$ attains a maximum on G, then f is a constant on G.

(3 + 3 + 2 = 8 marks)

[4 × 8 = 32 marks]