D 104383 **(Pages : 3)** **Name.......................................**

Reg. No..................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT 2C 11—MULTIVARIABLE CALCULUS AND GEOMETRY

(2022 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Verify whether the operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $A(x, y) = (x + y, x + y)$ is one to one.

2. Let $f(x, y) = \dfrac{xy}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Find $D_1 f$.

3. Give a parametrization of the circles, $x^2 + y^2 = 1$.

4. Find the curvature of the curve $\gamma(t) = (\cos^3 t, \sin^3 t)$.

5. Show that $\sigma(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$ is a parametrization of the unit sphere in $\mathbb{R}^3$.

6. Verify whether $S = \{(x, y, x + y) \in \mathbb{R}^3\}$ is a smooth surface.

7. Show that the second fundamental form of a plane is zero.

8. Find the normal curvature of a curve on the sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.

(8 × 1 = 8 marks)

Turn over

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Let A, B be linear operators on $\mathbb{R}^n$. Show that $\| A + B \| \leq \| A \| + \| B \|$.

10. Let X be a complete metric space and $\phi : X \to X$ be a contraction. Show that fixed points of ϕ are unique.

11. Show that if a square matrix A has two equal columns then det A = 0.

12. Let $\gamma(t)$ be a regular curve with arc length s. Show that s is a smooth function of t.

13. Find the torsion of the circle $\gamma(\theta) = (\cos\theta, \sin\theta)$.

14. Verify whether $S = \{(x, y, z) : x^2 + y^2 = z^2, z > 0\}$ is a surface in $\mathbb{R}^3$.

15. Let $f(x, y, z) = x^2 + y^2 - z^2$. Find ∇f and verify the smoothness of the surface given by f.

16. Describe the image of the Gauss map G_s where S is a plane.

17. Let k_g be the geodesic curvature of a normal section of a surface. Show that $k_g = 0$.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. (a) Define the derivative $f'(x)$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point x.

    (b) Let E be an open set in $\mathbb{R}^n$. Let f map E to $\mathbb{R}^m$ and g map $f(E)$ to $\mathbb{R}^k$. Show that if f is differentiable at $x_0 \in E$ and g is differentiable at $f(x_0)$ then $F(x) = g(f(x))$ is differentiable at x_0.

19. (a) Define unit speed parametrization of a curve.

    (b) Show that a parametrized curve γ has a unit speed parametrization if and only if it is regular.

20. (a) Define regular surface patch.

    (b) Let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch and $\Phi : U_1 \to U$ be a bijective smooth map with smooth inverse. Show that $\overline{\sigma} = \sigma \circ \Phi : U_1 \to \mathbb{R}^3$ is a regular surface patch.

21. (a) Define surface of revolution.

    (b) Give an example of a surface of revolution specifying the profile curve and surface patches.

22. (a) Define Gaussian curvature and mean curvature of a surface.

    (b) Deduce the formulas

$$K = \frac{LN - M^2}{EG - F^2} \text{ and } H = \frac{LG - 2MF + NE}{2\left(EG - F^2\right)}$$

where K is the Gaussian curvature and H is the mean curvature and L, M, N, E, F, G are usual notations.

(3 × 8 = 24 marks)

D 104378 **(Pages : 4)** **Name....................................**

Reg. No................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT2C10—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

*Answer **all** the questions.*
Each question carries 2 marks.

1. State the implicit function theorem.
2. Find parametrisation of the level curve $y^2 - x^2 = 1$.
3. Calculate the Gauss map $\mathcal{G}$ of the paraboloid S with equation $z = x^2 + y^2$. What is the image of $\mathcal{G}$.
4. Compute the curvature of the curve $\gamma(t) = (\cos^3 t, \sin^3 t)$.
5. Show that any open disc in the xy-plane is a surface.
6. Calculate the first fundamental form of the surface $\sigma(u, v) = (u - v, u + v, u^2 + v^2)$.
7. What are quadrics ? Give an example of a quadric which is not a surface.
8. Compute the second fundamental form of the elliptic paraboloid $\sigma(u, v) = \{u, v, u^2 + v^2)$.

(8 × 2 = 16 marks)

Part B

*Answer any **four** questions.*

Each question carries 4 marks.

9. Prove that a linear operator A on a finite dimensional vector space X is one - to -one if and only if the range of A is all of X.

Turn over

10. Prove that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

11 Prove that the total signed curvature of a closed plane curve is an integral multiple of 2π.

12. Compute κ, τ, t, n and b for the curve, $\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$ and verify that the Frenet-Serret equations are satisfied.

13. What is meant by an oriented surface ? Show that *Möbius band* is not orientable.

14. Show that $(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$, where $a > b > 0$ are constants is a smooth surface.

(4 × 4 = 16 marks)

Part C

Answer A **or** *B of the following questions.*
Each question carries 12 marks.

Unit I

15. (A) (a) State and prove the inverse function theorem.

(b) Suppose E is an open set in $\mathbb{R}^n$, f maps. E into $\mathbb{R}^m$, f is differentiable at $x_0 \in E$, g maps an open set containing f (E) into $\mathbb{R}^k$ and g is differentiable at $f(x_0)$. Prove that $F : E \to \mathbb{R}^k$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0)) f'(x_0)$.

(B) (a) State and prove the Contraction Principle.

(b) Suppose f maps an open set $E \subset R^n$ into R^m. Prove that $f \in \mathscr{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.

Unit II

16. (A) (a) Let γ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature so that the torsion τ of γ is defined. Then, prove that the image of γ is contained in a plane if and only if τ is zero at every point of the curve.

(b) Prove that a parametrized curve has a unit-speed reparametrization if and only if it is regular.

(B) (a) Let γ be a unit-speed curve in R^3 with constant curvature and zero torsion. Prove that γ is a parametrization of (part of) a circle.

(b) A plane curve is given by $\gamma(\theta) = (r \cos \theta, r \sin \theta)$, where r is a smooth function of θ. Under what conditions is γ regular ? Find all functions (r, θ) for which γ is unit speed. Show that if γ is unit speed, the image of γ is a circle ; what is its radius ?

Unit III

17. (A) (a) Let $\sigma : U \rightarrow \mathbb{R}^3$ be a patch of a surface S containing a point $p \in S$, and let (u, v) be co-ordinates in U. Prove that the tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v.

(b) Show that $\sigma(u, v) = (\text{sech}\,u \cos v, \text{sech}\,u \sin v, \tanh u)$ is a regular surface patch for S^2. Show that meridians and parallels on S^2 correspond under σ to perpendicular straight lines in the plane.

(B) (a) Show that the first fundamental form of the generalized cone

$a(u, v) = (1 + v)\gamma(u) - v\mathbf{p}$ with vertex $\mathbf{p}$ does not depend on the curve γ.

(b) Prove that $\|\sigma_u \times \sigma_v\| = \left(EG - F^2\right)^{1/2}$, where $E = \|\sigma_u\|^2$, $G = \|\sigma_v\|^2$ and $F = \sigma_u \cdot \sigma_v$.

Turn over

UNIT IV

18. (A) (a) Let S be a surface of which every point is an umblic Prove that S is an open subset of a plane or a sphere.

(b) What is the effect on the second fundamental form of a surface of applying an isometry of R^3 ? Or a dilation ?

(B) (a) Prove that the principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point.

(b) Compute the principal curvatures of $\sigma\ (u, v) = (\cosh u \cos v, \cosh u \sin v, u)$.

(4 × 12 = 48 marks)

D 104376 **(Pages : 2)** **Name.......................................**

Reg. No...................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 mark.

1. Find the general solution of $y'' - x\,f(x)\,y' + f(x)\,y = 0$.

2. Locate and classify the singular points of the differential equation

$$x^2\left(x^2-1\right)^2 y'' + x\left(x-1\right)y' + 2y = 0$$

on the x-axis.

3. Determine the nature of the point $x = \infty$ for $x^2\,y'' + xy' + \left(x^2 - p^2\right)y = 0$.

4. Define Gamma function and show that $\Gamma(n+1) = n!$ for any $n \geq 0$.

5. Describe the phase portrait of the system $\frac{dx}{dt} = -x, \frac{dy}{dt} = -y$.

6. State and prove the necessary condition for the quadratic form $ax^2 + bxy + cy^2$ to be positive definite.

(6 × 2 = 12 marks)

Turn over

Part B

Answer any **five** *questions.*
Each question carries 4 marks.

7. Consider the two functions $f(x) = x^2$ and $g(x) = x^2 |x|$ on the interval $[-1, 1]$. Find the Wronskian $W(f, g)$ Whether f and g are linearly dependent or independent ? justify.

8. Find a series solution for the equation $y'' + y' - xy = 0, y(0) = 1, y'(0) = 0$.

9. Find the general solution of the differential equation $(x^2 - x - 6) y'' + (5 + 3x) y' + y = 0$ at the singular point $x = 3$.

10. Find the curve of fixed length L that joins the points $(0, 0)$ and $(1, 0)$, lies above the x-axis, and encloses the maximum area between itself and the x-axis.

11. Find the first three terms of the Legendre series of $f(x) = e^x$.

12. Find the stationary function of $\int_0^4 (xy' - (y')^2) \, dx$, which is determined by the boundary conditions $y(0) = 0$ and $y(4) = 3$.

13. Determine the nature and the stability properties of system $\frac{dx}{dt} = 4x - 2y$, $\frac{dy}{dt} = 5x + 2y$ at the critical point $(0, 0)$.

14. Verify that $(0, 0)$ is a simple critical point of

$$\frac{dx}{dt} = x + y - 2xy, \quad \frac{dy}{dt} = -2x + y + 3y^2.$$

Also find nature and stability properties.

(5 × 4 = 20 marks)

Part C

Answer **either** *A* **or** *B of each of the following four questions.*
Each question carries 16 marks.

15. (A) (a) Let $f(x, y)$ be a continuous function that satisfies a Lipschitz condition $|f(x, y_1) - f(x, y_2)| \leq \kappa |y_1 - y_2|$ on a strip defined by $a \leq x \leq b$ and $-\infty < y < c\infty$. Show that if (x_0, y_0) is any point of the strip, then the initial value problem $y' = f(x, y), y(x_0) = y_0$ has one and only solution $y = y(x)$ on $a \leq x \leq b$.

(b) If n is a positive integer, find two linearly independent solutions of

$$xy'' - (x+n)y' + ny = 0.$$

Also find the general solution for the cases $n = 1$ and 2.

(B) (a) Show that $\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$ by solving the equation $y' = 1 + y^2$; $y(0) = 0$ in two ways.

(b) Find two independent Frobenius series solutions of the equation

$$x^2y'' - x^2y' + (x^2 - 2)y = 0.$$

16. (A) (a) Derive Rodrigues formula.

(b) Show that the Bessel function are orthogonal with respect respect to the weight function x on the interval $0 \leq x \leq 1$.

(B) (a) Derive Euler's differential equation for an extremal.

(b) Let $f(x)$ be a function defined on the interval $-1 \leq x \leq 1$. Determine the polynomial $p(n)$ of degree $\leq n$ which minimize the value of the integral $\mathrm{I} = \int_{-1}^{1} [f(x) - p(x)]^2 \, dx$.

17. (A) (a) Find the general solution of the system :

$$\frac{dx}{dt} = 7x + 6y,\ \frac{dy}{dt} = 2x + 6y.$$

(b) If $a_1 b_2 - a_2 b_1 \neq 0$, show that the system $\frac{dx}{dt} = a_1 x + b_1 y + c_1, \frac{dy}{dt} = -a_2 x + b_2 y + c_2$. has a single isolated critical point (x_0, y_0).

(B) (a) Show that $(0, 0)$ is an asymptotically stable critical point for the system :

$$\frac{dx}{dt} = -3x^3 - y, \frac{dy}{dt} = x^5 - 2y^3.$$

(b) Consider the equation of motion for the damped vibration of a pendulum is $\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{g}{a}\sin(x) = 0$, where a is the length of the pendulum, m is a mass and c is a constant. Find the critical point and its nature, also give physical interpretation.

(3 × 16 = 48 marks)

D 104381 **(Pages : 4)** **Name.......................................**

Reg. No..................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT2C09—ORDINARY DIFFERENTIAL EQUATIONS

(2022 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Consider the equation $y'' + a_1 y' + a_2 y = 0$, where a_1, a_2 are constants. Let ϕ_1, ϕ_2 be two solutions of the equation satisfying $\phi_1(x_0) = 1, \phi_1'(x_0) = 0, \phi_2(x_2) = 0, \phi_2'(x_0) = 1$. If ϕ is any solution of the equation satisfying $\phi(x_0) = \alpha,\ \phi'(x_0) = \beta,$ then show that $\phi(x) = \alpha\, \phi_1(x) + \beta\, \phi_2(x)$.
2. State the initial value problem for n^{th}–order linear equation with constant co-efficients.
3. Let $p(x)$ be any polynomial of degree n, and let $p(x) = c_0 P_0(x) + c_1 P_1(x) + ... + c_n P_n(x)$, where $P_n(x)$ denote the n^{th} degree Legendre polynomial and $c_0, c_1, ..., c_n$ are constants. Show that $c_k = \dfrac{2k+1}{2}\int_{-1}^{1} P(x) P_k(x)$ for k = 0, 1,...., n.
4. Compute the indicial polynomial and its roots of the equation $x^2 y'' + (\sin x)\, y' + (\cos x)\, y = 0$.
5. Find all real-valued solutions of the equation $y' = \dfrac{e^{x-y}}{1+e^x}$.
6. Show that the function f given by $f(x, y) = x^2 \,|\, y \,|$ satisfies a Lipschitz condition on $R : |\, x \,| \le 1, |\, y \,| \le 1$.

Turn over

7. Describe the phase portrait of the system $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2$.

8. Show that a function of the form $ax^3 + bx^2y + cxy^2 + dy^3$ cannot be either of positive type or of negative type.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Let ϕ_1, ϕ_2 be two solutions of the equation $y'' + a_1 y' + a_2 y = 0$, where a_1, a_2 are constants, on an interval I containing x_0. Show that ϕ_1, ϕ_2 are linearly independent on I iff $W(\phi_1, \phi_2)(x_0) \neq 0$.

10. Verify that the function $\phi_1(x) = x, (0 < x < 1)$ satisfies the equation $(1 - x^2)y'' - 2xy' + 2y = 0$, and find a second independent solution.

11. Show that the co-efficient of x^n in the n^{th} degree Legendre polynomial $P_n(x)$ is $\frac{(2n)!}{2^n \cdot (n!)^2}$.

12. Find the solution ϕ of the equation $(1 + x^2)y'' + y = 0$ of the form $\phi(x) = \sum_{x=0}^{\infty} c_k x^k$, which satisfies $\phi(0) = 1, \phi'(0) = 0$.

13. Show that infinity is not a regular singular point for the Bessel equation $x^2y'' + xy' + (x^2 - 2^2)y = 0$. Find all other singular points and classify them.

14. Show that the equation $(2y e^{2x} + 2x \cos y)dx + (e^{2x} - x^2 \sin y)dy = 0$ is exact and hence solve it.

15. Let y be a non-trivial solution of the equation $y'' + qy = 0$ on the closed interval $[a, b]$, where q is a positive function. Show that y has at most a finite number of zeros in the interval $[a, b]$.

16. Determine the nature and stability properties of the critical point (0. 0) for the system :

$$\frac{dx}{dt} = 4x - 2y, \quad \frac{dy}{dt} = 5x + 2y.$$

17. Show that (0, 0) is an asymptotically stable critical point of the system :

$$\frac{dx}{dt} = -2x + xy^3, \frac{dy}{dt} = x^2y^2 - y^3$$

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. (a) Find all solutions of the equation $y'' + y = 2 \sin x \sin 2x$.

(b) Show that $\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$.

19. (a) Suppose that $\phi_1, \phi_2, ..., \phi_n$ are n-linearly independent solutions of the equation $y^{(n)} + a_1(x) y^{(n-1)} + ... + a_n(x) y = 0$ on an interval I. Show that $W(\phi_1, \phi_2, ..., \phi_n) x \neq 0$ for all x in I.

(b) Obtain two linearly independent solutions which are valid near $x = 0$ for the equation $x^2y'' + 5xy' + (3 - x^3) y = 0$.

20. (a) Obtain a solution $J_\alpha(x)$, the Bessel function of order α of the first kind, for the Bessel equation $x^2y'' + xy' + (x^2 - \alpha^2) y = 0$.

(b) Find the normal form of Bessel's equation $x^2y'' + xy' + (x^2 - p^2) y = 0$ and show that every non-trivial solution has infinitely many positive zeros.

21. (a) Consider the initial value problem $y' = 1 + y^2, y(0) = 0$.

(i) using separation of variables, find the solution ϕ.

(ii) show that all the successive approximations $\phi_0, \phi_1, \phi_2...$ exist for all real x.

(iii) show that $\phi_k(x) \to \phi(x)$ for each x satisfying $|x| \leq \frac{1}{2}$.

Turn over

(b) Verify that (0, 0) is a simple critical point for the system :

$$\frac{dx}{dt} = -x - y - 3x^2y, \quad \frac{dy}{dt} = -2x - 4y + y\sin x.$$

22. (a) Consider the non-linear system : $\frac{dx}{dt} = y\left(x^2 + 1\right), \quad \frac{dy}{dt} = -x\left(x^2 + 1\right).$

(i) Find the critical points.

(ii) Find the differential equation of the paths.

(iii) Solve this differential equation to find the paths.

(b) Show that every non-trivial solution of the equation $y'' + \left(\sin^2 x + 1\right) y = 0$ has an infinite number of positive zeros.

(3 × 8 = 24 marks)

D 104380 **(Pages : 4)** **Name.................................**

Reg. No..............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT2C08—REAL ANALYSIS—II

(2022 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries one mark.

1. Let P and Q be two partitions of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that

$$U(P \cup Q, f) \leq U(Q, f)$$

where U (Q, f) denotes the upper Darboux sum associated with Q and f.

2. Prove that every countable subset of $\mathbb{R}$ is of measure 0.

3. Let (X, $\mathcal{A}$, μ) be a measure space and A, B $\in \mathcal{A}$. Prove that A $\subseteq$ B implies μ (A) $\leq$ μ (B).

4. Let X be a measurable space. If f and g are real measurable functions on X, then prove that fg is also measurable.

5. Suppose $f : X \to [0, \infty)$ is integrable, where (X, $\mathcal{A}$, μ) is a measure space. Prove that

$$\mu\left(\left\{x \in X : f(x) \geq n\right\}\right) \to 0 \text{ as } n \to \infty.$$

6. Suppose f and g are complex measurable functions such that $f = 0$ a.e. and $g = 0$ a.e. Prove that $f + g = 0$ a.e.

7. Define an absolutely continuous function.

Turn over

8. Suppose that $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be measure spaces. Let $E = A \times B$, with $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$, be a measurable rectangle. Prove that the y-section of the set E is :

$$E^y = \begin{cases} A & \text{if } y \in B \\ \varnothing & \text{if } y \notin B \end{cases}$$

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 marks.

9. Prove that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if $L(f) = U(f)$.

10. Let $\mathcal{M}$ denote the set of all Lebesgue measurable sets. If $A_1, A_2 \in \mathcal{M}$, then prove that $A_1 \cup A_2 \in \mathcal{M}$.

11. Suppose $(X, \mathcal{A})$ is a measurable space and $X_0 \in \mathcal{A}$. Prove that $\mathcal{A}_0 = \{A \cap X_0 : A \in \mathcal{A})$ is a σ-algebra on X_0 and it is the largest σ-algebra on X_0 contained in $\mathcal{A}$.

12. Let $(X, \mathcal{A}, \mu)$ be a measure space and $A_n \in \mathcal{A}$ be such that $A_n \supseteq \mathcal{A}_{n+1}$ for all $n \in \mathbb{N}$ and $\mu(A_k) < \infty$ for some $k \in \mathbb{N}$. Prove that

$$\mu(A_n) \to \mu\left(\bigcap_{i=1}^{\infty} A_i\right) \text{ as } n \to \infty.$$

13. Let (f_n) be a sequence of real or extended real valued or complex measurable functions on a measure space $(X, \mathcal{A}, \mu)$. If $f_n \to f$ pointwise on X, then prove that f is measurable.

14. Let $\{B_1, ..., B_k\}$ be a disjoint family of measurable sets and let $\beta_1, ..., \beta_k$ be non-negative real numbers (not necessarily distinct). Let $\psi := \sum_{i=1}^{k} \beta_i \chi_{B_i}$. Prove that

$$\int_x \Psi d\mu = \sum_{i=1}^{k} \beta_i \mu(B_i).$$

15. Let (f_n) be a sequence of extended real valued non-negative measurable functions defined on a measure space $(X, \mathcal{A}, \mu)$ such that $f_n \leq f_{n+1}$ *a.e.* for every $n \in \mathbb{N}$. Prove that (f_n) converges a.e. to a measurable function f and

$$\lim_{n \to \infty} \int_X f_n = \int_X f.$$

16. Suppose $\varphi : [a, b] \to \mathbb{K}$ be a function of bounded variation. Prove that φ is differentiable a.e. and $\varphi' \in \mathcal{L}[a, b]$.

17. Suppose that $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ are measure spaces and $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Prove that, for every $(x, y) \in X_1 \times X_2$, the x-section E_x is in $\mathcal{A}_2$.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 marks.

18. (a) If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then prove that for every $\in > 0$, there exists a partition P of $[a, b]$ such that

$$\left| S(P, T, f) - \int_a^b f(x)\, dx \right| < \in$$

for every tag set T of P.

(b) Prove that the Cantor set is uncountable but the measure of Cantor set is 0.

19. (a) Prove that every G_δ-set is Lebesgue measurable.

(b) Prove that a subset E of $\mathbb{R}$ is Lebesgue measurable if and only if

$$m^*(I) \geq m^*(I \cap E) + m^*(I \cap E^c)$$

for every open interval I.

Turn over

20. (a) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and (f_n) be a sequence of extended real valued measurable functions on X which converges a.e. to a measurable function f. Prove that for every $\in > 0$, there exists $E \in \mathcal{A}$ such that $\mu \{X \setminus E\} < \in$ and $f_n \to f$ uniformly on E.

(b) Prove that the Lebesgue measure on the Borel σ-algebra $\mathcal{B}$ is not complete.

21. (a) Let φ and ψ be non-negative simple measurable functions on a measure space $(X, \mathcal{A}, \mu)$ Prove, that

$$\int_X (\varphi + \psi)\, d\mu = \int_X \varphi d\mu + \int_X \psi d\mu.$$

(b) Let J be an interval and let $f \in \mathcal{L}(J)$. For any given $a \in J$, if $\int_a^x f\, dm = 0$ for every $x \in J$, then prove that $f = 0$ a.e. on J.

22. Let $f \in \mathcal{L}[a, b]$ and $g : [a, b] \to K$ be defined by :

$$g(x) = \int_a^x f\, dm \; ; x \in [a, b].$$

Prove that g is differentiable a.e., $g' \in \mathcal{L}\ [a, b]$ and $g' = f$ a.e.

(3 × 8 = 24 marks)

D 104382 **(Pages : 3)** **Name.......................................**

Reg. No..................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT2C10—TOPOLOGY

(2022 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *the questions.*
Each question carries 1 *mark.*

1. Show that if $n > 1$, there is a bijective correspondence of $A_1 \times \ldots \times A_n$ with $(A_1 \times \ldots \times A_{n-1}) \times A_n$.
2. Show that interior of a set is an open set in the topology.
3. Define a metric on a set X.
4. If X is a topological space and $f, g : X \to \mathcal{R}$ are continuous functions, then prove that $f + g$ is also continuous.
5. Prove that the set of real numbers R with standard topology is not compact.
6. Is the set of real numbers with standard topology Hausdorff ? Justify your answer.
7. Prove that every normal space is regular.
8. Prove that regular space is- Hausdorff.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Prove that subset of a countable set is countable.
10. Define basis for a topology. Give an example of a basis' for the standard topology on the set of real numbers.
11. If A is a subspace of X and B is a subspace of Y, then prove that the product topology on A × B is the same as the topology A × B inherits as a subspace of X × Y.

Turn over

12. Let Y be a subspace of X. Then prove that a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

13. Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathscr{R}$ by the equation $\bar{d}(x,y) = \min\{\alpha(x,y),1\}$. Then prove that $\bar{d}$ is a metric that induces the same topology as d on X.

14. Let $f : X \to Y$. If the function f is continuous then prove that for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. Also prove that the converse holds if X is metrizable.

15. If the sets C and D form a separation of X, and if Y is a connected subspace of X, then prove that Y lies entirely within either C or D.

16. Prove that the interval (0,1] is not compact in the set of real numbers with standard topology.

17. Prove that every compact Hausdorff space is normal.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. (a) Prove that a finite product of countable sets is countable.

    (b) Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element $\mathcal{C}$ of $\mathcal{C}$ such that $x \in \mathcal{C} \subset X$. Then prove that $\mathcal{C}$ is a basis for the topology on X.

19. (a) Let A be a subset of a topological space X. Let A' and $\bar{A}$ denote the set of all limit points of A and the closure of A respectively, then prove that $\bar{A} = A \cup A'$.

    (b) Prove that in a Hausdorff space, a sequence of points of X converges to at most one point of X.

20. (a) Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space X to the metric space Y. If $\{f_n\}$ converges uniformly to f, then prove that f is continuous.

    (b) Let R denotes the set of real numbers with standard topology. Then proves that an uncountable product of R with itself is not metrizable.

21. (a) Prove, that the union of a collection of connected subspaces of *X* that have a point in common is connected.

    (b) Prove that a finite Cartesian product of connected spaces is connected.

22. (a) Let Y be a subspace of X. Then prove that Y is compact if and only if every covering of Y by open sets in X contains a finite sub collection covering Y.

    (b) Prove that every metrizable space is normal.

(3 × 8 = 24 marks)

D 104377 **(Pages : 4)** **Name..................................**

Reg. No...............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT2C09—TOPOLOGY

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*

Each question carries 2 *marks.*

1. Let X = {a, b, c, d, e}. Determine whether or not T = {X, (ϕ, {a, b, c} , {a, b, d} , {a, b, c, d}} is a topology on X.
2. Define the second axiom of countability. Give an example for a second countable space.
3. Define what is homeomorphism and show that the closed interval A = [a, b] is home-omorphic to the closed unit interval I = [0, 1].
4. Define accumulation point of a set. Give an example of a space in which no point is an accumulation point of any set.
5. Prove that every closed, surjective map is a quotient map.
6. Prove that every metric space is a T_1 space.
7. Distinguish between T_3 and T_4 spaces.
8. Show that regularity is a hereditary property.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Prove that metrisability is a hereditary property.

10. Prove that a subset of a topological space is open if and only if it is a neighbourhood of each of its points.

11. Prove that every quotient space of a discrete space is discrete.

12. Let (X, T) be a topological space and $A \subset X$. Prove that A is a compact subset of X if and only if the subspace (A, T/A) is compact.

13. Prove that the projection functions are open.

14. Prove that every completely regular space is regular.

(4 × 4 = 16 marks)

Part C

Answer **either** A **or** B *part of the following questions.*
Each question carries 12 *marks.*

15. (A) (a) Given any family $\mathscr{S}$ of subsets of X, Prove that there is a unique topology T on X having $\mathscr{S}$ as sub-base. Further, prove that every member of T can be expressed as the union of sets each of which can be expressed as the intersection of finitely many members of $\mathscr{S}$.

(b) Let X be a set, T a topology on X and $\mathscr{S}$ be a family of subsets of X. Prove that $\mathscr{S}$ is a sub-base for T if and only if $\mathscr{S}$ generates T.

(B) (a) Let $\mathscr{B}$ be a base for a toplogy T on X and let $Y \subset X$. Let $\mathscr{B}/Y = \{B \cap Y : B \in \mathscr{B}\}$. Prove that $\mathscr{B}/Y$ is a base for the topology T/Y on Y.

(b) Let (X, d) be a metric space. Then show that :

(i) The empty set ϕ and the entire set X are open ;

(ii) The union of any family of open sets is open ; and

(iii) The intersection of any finite number of open sets is open.

16. (A) (a) Let $\{(Y_i, T_i) : i \in I\}$ be an indexed family of topological spaces, X any set and $\{f_i : i \in I\}$ an indexed collection of functions such that for each $i \in I, f_i$ is a function from X to Y_i. Prove that there exists a unique smallest topology T on X which makes each f_i continuous.

(b) Prove that every closed and bounded interval is compact.

(B) (a) Let X, Y be topological spaces, $x \in X$ and $f : X \to Y$ a function. Suppose X is first countable at x. Prove that f is continuous, at x if and only if for every sequence $\{x_n\}$ which converges to x in X, the sequence $\{f(x_n)\}$ converges to f (x) in Y.

(b) What is meant by a component of a space ? State and prove any two properties of components of a space.

17. (A) (a) Le A, B be subsets of a topological space (X, T). Prove the following statements :

(1) A is closed in X if and only if $\bar{A} = A$.

(2) $\overline{A \cup B} = \bar{A} \cup \bar{B}$, where $\bar{A}$ denotes the closure of A.

(b) Prove that there is a one - to - one correspondence between the set of topologies on a set and the set of all nearness relations on that set.

(B) (a) Prove that compositions of continuous functions are continuous.

(b) Let T be the topology on $\mathbb{N}$ which consists of ϕ and all subsets of $\mathbb{N}$ of the form $E_n = \{n, n+1, n+2, ...\}$, where $n \in \mathbb{N}$. Determine :

(1) The closed subsets of ($\mathbb{N}$, T).

(2) The closure of the sets {7, 24, 47, 85} and {3, 6, 9, 12,....}.

(3) Those subsets of $\mathbb{N}$ which are dense in $\mathbb{N}$.

18. (A) (a) Show that for a topological space (X, T) the following are equivalent :

(i) The space X is a T_1 space.

(ii) For any $x \in X$ the singleton set $\{x\}$ is closed.

(iii) Every finite subset of X is closed.

(iv) The topology T is stronger than the cofinite topology on X.

(b) Prove that for any sets Y, I and J, $(Y^I)^J = Y^{I \times J}$ upto a bijection.

(B) (a) Prove that the axioms T_0, T_1, T_2, T_3 and T_4 form a hierarchy of progressively stronger conditions.

(b) Let $X = \Pi X_i, i \in I$, each X_i being a topological space. Suppose $\{x_n\}$ is a sequence in X and that $x \in X$. Then prove that $\{x_n\}$ converges to x in X if and only if for each $i \in I$, the sequence $\{\pi_i(x_n)\}$ converges to $\pi_i(x)$ in X_i, where $\pi_i' s$ are projection functions.

(4 × 12 = 48 marks)

D 104379 **(Pages : 3)** **Name.......................................**

Reg. No..................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT 2C 07—ALGEBRA—II

(2022 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Show that every ideal in the ring $\mathbb{Z}$ of integers is a principal ideal.

2. Show that 5 is a product of two irreducibles in $\mathbb{Z}[i]$.

3. Find the irreducible polynomial for $\sqrt{2}+\sqrt{3}$ in $\mathbb{Q}[x]$.

4. Find the degree of the extension $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{2}\right)$ over $\mathbb{Q}$.

5. Let E be an extension of degree 3 of the field $\mathbb{Z}_2$. Find a polynomial $f(x) \in \mathbb{Z}_2[x]$ such that E is the splitting field of $f(x)$.

6. Find the order of the Galois group $G(\mathbb{Q}(\alpha)/Q)$ where α is the real cube root of 2.

7. Let $s_1 = y_1 + y_2$ and $s_2 = y_1 y_2$ be elementary symmetric functions in $F(y_1 y_2)$ for a field F. Show that $F(y_1 y_2)$ is a splitting field over $F(s_1, s_2)$.

8. Find the Galois group of the 8th cyclotomic extension of $\mathbb{Q}$.

(8 × 1 = 8 marks)

Turn over

Part B

Answer any **six** *questions.*
Each question carries 3 marks.

9. Let D be a Euclidean domain with Euclidean norm v. Show that if $u \in \mathrm{D}$ is a unit in D then $v(u) = v(1)$.

10. Show that the ring $\mathbb{Z}[i]$ of Gaussian integers is an integral domain.

11. Let p be an odd prime. Show that if $p = a^2 + b^2$ for integers a, b then $p \equiv 1 \pmod 4$.

12. Let E be an extension of a filed F and let $\alpha \in \mathrm{E}$. Let $p(x)$ be an irreducible polynomial in $\mathrm{F}[x]$ such that $p(\alpha) = 0$. Show that if $f(x) \in \mathrm{F}[x]$ and $f(\alpha) = 0$ then $p(x)$ divides $f(x)$.

13. Prove that trisection of an angle is an impossible geometric construction.

14 Let E be an extension of degree 3 over the field $\mathbb{Z}_5$. Find a polynomial $p(x) \in \mathbb{Z}_5[x]$ such that every element of E is zero of $p(x)$.

15. Let E be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. . Find the degree $[\mathrm{E} : \mathbb{Q}]$.

16. Let K be the splitting field of $(x^2 - 2)(x^2 - 3)$ over $\mathbb{Q}$. Find the Galois group $\mathrm{G}(\mathrm{K}/\mathbb{Q})$.

17. Verify whether a regular 60-gon is constructible.

(6 × 3 = 18 marks)

Part B

Answer any **three** *questions.*
Each question carries 8 marks.

18. (a) Define multiplicative norm on an integral domain and give an example.

(b) Let N be a multiplicative norm on an integral domain D such that $|\mathrm{N}(u)| = 1$ for every unit $u \in \mathrm{D}$. Show that if $\pi \in \mathrm{D}$ is such that $\mathrm{N}(\pi)$ is a prime number then π is irreducible in D.

19. Let $p(x)$ be an irreducible polynomial in $F[x]$ where F is a field. Show that :

(a) $E = F[x]/p(x)$ is an extension of F.

(b) $p(x)$ has a zero in E.

(c) If degree of $p(x)$ is n then $[E:F] = n$.

20. (a) Let E, K be finite extensions of a field F with F < E < K. Show that $[K:F][K:E][E:F]$.

(b) Let E be a finite extension of F of degree n and $a \in E$. Show that degree (α, F) divides n.

21. (a) Let K be an extension of F and σ be an automorphism of K such that $\sigma(a) = a$ for all $a \in F$.

Prove that for every $\alpha \in K$, $irr(\alpha ; F) = irr(\sigma(\alpha) ; F)$.

(b) Let $f(x) \in \mathbb{R}[x]$ be such that $f(a + ib) = 0$ where $a, b \in \mathbb{R}$. Show that $f(a - ib) = 0$.

22. (a) Define the n^{th} cyclotomic polynomial.

(b) Describe the 4^{th} cyclotomic polynomial over the rationals.

(c) Show that for a prime p, the Galois group of the p^{th} cyclotomic polynomial over $\mathbb{Q}$ is isomorphic to the multiplicative groups of non-zero elements of the field $\mathbb{Z}_p$.

(3 × 8 = 24 marks)