24075

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Name.................................

Reg. No.............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, FEBRUARY 2003

Mathematics

Paper I—ALGEBRA—I

(Effective from 2001 admissions)

Time : Three Hours — Maximum : [illegible]0 Marks

Part A *is* **compulsory.**

Answer any **four** *questions from Part B without omitting any unit.*

Part A

1. (a) Show that $\mathbb{Z}$ has no composition series.

(b) Show that no group of order 20 is simple.

(c) Find all solutions of the congruence

$$2x \equiv 6 \pmod 4.$$

(d) Show that the polynomial [illegible]

[illegible]

Part B

Unit I

2. (a) Show that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.

(8 marks)

(b) Find the order of (8, 4, 10) in the group $\mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$. (2 marks)

(c) Let G be a finite abelian group of order n. Show that for every divisor m of n the group G has a subgroup of order m.

(6 marks)

3. (a) Suppose that a group G has a composition series. Prove that for any proper normal subgroup N of G there is a composition series of G containing N.

(10 marks)

(b) Define a solvable group. Show that S_3 is solvable while S_5 is not solvable. (6 marks)

4. (a) Let X be a G-set and let $x \in X$. Show that $|Gx| = (G : G_x)$. (12 marks)

(b) Find the number of orbits in {1, 2, 3, 4, 5, 6, 7, 8} under the cyclic subgroup ⟨(1, 3, 5, 6)⟩ of S_8.

(4 marks)

Turn over

UNIT II

5. (a) Let G be a group and let H, N be subgroups of G with N normal in G. Prove that :

   (i) $H \vee N = HN = NH$.

   (ii) If H is normal is G then HN is normal in G.

(8 marks)

(b) State and prove the second isomorphism theorem for groups. (8 marks)

6. (a) Let G be a finite group and let the prime p be a divisor of G. Prove that the number of Sylow p-subgroups of G is congruent to 1 modulo p.

(12 marks)

(b) Show that any two Sylow 2-subgroups of S_3 are conjugate. (4 marks)

7. (a) Show that any integral domain D can be enlarged to a field F such that every element of F can be expressed as a quotient of two elements of D.

(12 marks)

(b) Prove that if a field F contains $\mathbb{Z}$ then it contains a subfield that is isomorphic to $\mathbb{Q}$.

(4 marks)

UNIT III

8. (a) Let F be a field and $\alpha \in F$. Show that the evaluation at α $\phi_\alpha : F[x] \to F$ defined by

$$\phi_\alpha\left(a_0 + a_1x + \cdots + a_nx^n\right) = a_0 + a_1\alpha + \cdots + a_n\alpha^n$$

is a homomorphism of F [x] into F.

(10 marks)

(b) Show that a non-zero polynomial $f(x) \in F[x]$ of degree n can have at most n zeros in a field F.

(6 marks)

9. (a) Let $\phi : R \to R'$ be a ring homomorphism with kernel N. Show that $\phi(R)$ is a ring and the map $\mu : R/N \to \phi(R)$ given by

$$\mu[x + N] = \phi(x)$$

is an isomorphism of rings.

(12 marks)

(b) Find a subring of the ring $\mathbb{Z} \times \mathbb{Z}$ that is not an ideal of $\mathbb{Z} \times \mathbb{Z}$. (4 marks)

10. (a) Let R be a commutative ring with unity. Show that M is a maximal ideal of R if and only if R/M is a field.

(12 marks)

(b) Let R be a finite commutative ring with unity. Show that every prime ideal in $\mathbb{R}$ is maximal.

(4 marks)

M.Sc. (PREVIOUS) DEGREE EXAMINATION, MAY 2003

Mathematics

Paper II—LINEAR ALGEBRA

ne : Three Hours | Maximum : 120 Marks

Answer **all** *questions from Part A.*
Answer any **four** *questions from Part B without omitting any unit.*
Each question in Part A carries 4 marks.
Each question in Part B carries 24 marks.

Part A

1. Verify whether {(1, 2, 0), (1, 3, 0), (1, 1, 0)} is linearly independent in $\mathbb{R}^3$ over $\mathbb{R}$.
2. Find the characteristic roots of the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Give all inequivalent 3 × 3 Jordan matrices with characteristic value 2.
4. Find the matrix of the transformation $T(x_1, x_2, x_3) = (x_1 + x_2, x_2, x_3)$ over the basis {(1, 1, 0), (0, 1, 1), (1, 0, 1)}.
5. Let M be a finite abelian group. Show that M is not a free $\mathbb{Z}$-module.
6. Show that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}) \cong \mathbb{Q}$.

Part B

UNIT I

7. (a) Let V be the space of all 2 × 2 matrices of the form $\begin{pmatrix} x & y \\ z & 0 \end{pmatrix}$ where $x, y, z \in \mathbb{R}$. Show that with the usual operations V is a vector space over $\mathbb{R}$.

   (b) Find a finite set which spans the vector space V given above.

   (c) Show that the set $\left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$ is a subspace of the space V given in (*a*).

8. (a) Let V be a finite dimensional space, $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\mathcal{B}' = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be ordered basis of V. Show that there exists an invertible matrix P such that

   $[\alpha]_{\mathcal{B}} = P\,[\alpha]_{\mathcal{B}'}$ for each $\alpha \in V$.

   Here $[\alpha]_{\mathcal{B}}$ is the vector representation of $\alpha \in V$ w.r.t the basis $\mathcal{B}$.

   (b) Let $V = \mathbb{R}^3$. $\mathcal{B}$ = {(1, 1, 0), (0, 1, 1), (1, 0, 1)} and $\mathcal{B}'$ = {(1, 1, 1), (0, 1, 0), (0, 0, 1)}. Find the matrix P such that

   $[\alpha]_{\mathcal{B}} = P\,[\alpha]_{\mathcal{B}'}$ for all $\alpha \in V$.

Turn over

9. (a) Let $T: V \to W$ be a linear transformation of a vector space V to a vector space W. Show that T is non-singular if and only if T maps each linearly independent set in V onto linearly independent set in W.

(b) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$. Show that T is non-singular.

UNIT II

10. (a) Let T be a linear operator on a finite dimensional vector space V. Let $c_1, c_2, \ldots, c_k$ be district characteristic values of T and $w_1, w_2, \ldots, w_k$ be the associated characteristic spaces. Show that T is diagonalizable if and only if $\dim W_1 + \dim W_2 + \ldots + \dim W_k = \dim V$.

(b) Verify whether the matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

is diagonalizable.

11. (a) Let T be a linear operator on an n-dimensional space V. Show that the minimal polynomial of T divides the characteristic polynomial of T.

(b) Find the characteristic polynomial and minimal polynomial of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

12. (a) Define Companion matrix of a monic polynomial.

(b) Show that if A is the companion matrix of a monic polynomial $p(x)$ then the characteristic polynomial of A and minimal polynomial of A are both $p(x)$.

(c) Find the Jordan form of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

UNIT III

13. (a) Let M be an abelian group and R = End (M) be the ring of all endomorphisms of M. Show that with the usual operations M is a unitary R-module.

(b) Define direct sum of submodules ; and direct summond of a submodule.

(c) Show that for the $\mathbb{Z}$-module $\mathbb{Z}$ the subgroup generated by 5 is a submodule. Is it a direct summond. Justify your answer.

14. (a) Let M be a module and N be a submodule. Describe the quotient module M/N.

(b) Let M be a module and N a submodule of M. Show that there is a one-to-one correspondence between the set of all submodules of M/N and the set of all submodules of M containing N.

(c) If R is a commutative ring, show that $\lambda a: R \to R$ defined by $x \mapsto ax$ is an R-module homomorphism.

15. (a) Show that every finitely generated torsion free module over a PID is free.

(b) Show that the module $(\mathbb{Q}, +)$ over $\mathbb{Z}$ is torsion free and that it is not free.