Ph.D. THESIS **MATHEMATICS**

A STUDY ON COMPLEMENT DEGREE AND VERTEX CUT POLYNOMIALS OF GRAPHS

Thesis submitted to the University of Calicut for the

award of the degree of Doctor of Philosophy

in Mathematics under the Faculty of Science

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ACKNOWLEDGEMENT

I thank God, the Almighty, for all his benevolence and mercy upon me to do this research work in a successful manner.

Next I like to express my deep sense of gratitude to my research supervisor, Dr.Anil Kumar V, Retired Senior Professor, Department of Mathematics, University of Calicut, for his invaluable guidance provided during the entire period of my research work. He gave me complete freedom in my research work and his friendly approaches helped make the research period peaceful and stress-free. His timely suggestions and constant inspiration are the main factors behind the materialization of the thesis.

I sincerely acknowledge Dr.Preethi Kuttipulakkal, Head and Associate Professor, Department of Mathematics, University of Calicut for the affectionate support she has given me. I am also thankful to the faculty members of the department Dr.Raji Pilakkat (Retired Professor), Dr.Sini P (Assistant Professor), Dr.Mubeena T (Assistant Professor) and Dr.Prasad T(Assistant Professor) for their timely support. I extend my gratitude to the non-teaching staff and the librarian of the department for their support during the entire period of research work.

I remember with gratitude Dr.Rajendran V, the former HoD, Department of Mathematics, PSMO College, Tirurangadi for the kind support.

I acknowledge my gratitude to the Council of Scientific and Industrial Research (CSIR) for providing financial support.

I express my heartful thanks to my research friends Darsana C, Anusha C, Priya K, Ajeesh TT, Saleel Mohamed K, Naheeda Farhath CP, Archana S, Angela sunny, Ameena PP, Abhinand M, Nithya S, Fazil K, Noushida PP, Saira Kurian, Rafia Yoosuf and Saji NR for their mental support and cheerful companionship.

I take this opportunity to place on record my sincere gratitude to my parents, brothers and sisters for their unfailing encouragement and support during the research period.

I would like to thank my better half Mr.Mufeed KP and my loving kids Fadi Mufeed KP and Faiza Rashi KP for their heartfelt support.

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ABSTRACT

Graph theory is one of the flourished branches of mathematics that studies the properties of graphs, which are mathematical objects that represent pairwise relationships between objects. The beauty of graph theory lies in its wide scope of applications in fields ranging from network theory, chemistry and operational research to architecture and linguistics.

Among various branches of graph theory, graph polynomials are one of the well-studied concepts, as they are used to unveil the structural properties of graphs. Also, the graph polynomials are used for the characterization of graphs. Generally speaking, a graph polynomial is a polynomial assigned to a graph whose coefficients are indicators of some graph-theoretic parameters.

In this thesis, we introduce two new graph polynomials named the complement degree polynomial and the vertex cut polynomial of graphs. We derive these two polynomials of some well-known graphs and graph operations.Then we investigate stability, real roots, and the location of roots of complement degree polynomial. Moreover, we define equivalent classes of graphs of these two polynomials. Finally, we discuss the complement degree polynomial of some chemical graphs.

Key Words: graph polynomial, complement of a graph, degree of a vertex, vertex connectivity, roots of the polynomial, stability of a polynomial.

സംഗ്രഹം

ഗ്രാഫ് സിദ്ധാന്തം ഗ്രാഫുകളുടെ സവിശേഷതകളെ കുറിച്ച് പഠിക്കുന്ന ഗണിതശാസത്രത്തിന്റെ അഭിവൃദ്ധി പ്രാപിച്ച ശാഖകളിലൊന്നാണ്. അവ രണ്ട് ഒബ്ജെക്ടുകൾ തമ്മിലുള്ള ജോഡി ബന്ധങ്ങളെ പ്രതിനിധീകരിക്കുന്നു. നെറ്റ് വർക്ക് സിദ്ധാന്തത്തിലും, രസതന്ത്ര, പ്രവർആന ഗവേഷണം മുതൽ വാസ്തുവിദ്യയും ഭാഷാശാസ്ത്രവും വരെയുള്ള മേഖലകളിലെ സാധ്യതകളിലാണ് ഗ്രാഫ് സിദ്ധാന്തത്തിന്റെ ഭംഗി.

ഗ്രാഫ് സിദ്ധാന്തത്തിന് ഒരുപാട് ശാഖകൾ ഉണ്ട്. അതിൽ വളരെയധികം പഠന സാധ്യതയുള്ള ഒന്നാണ് ഗ്രാഫ് പോളിനോമിയൽ. അത് ഗ്രാഫുകളുടെ ഘടനാപരമായ സവിശേഷതകൾ പഠിക്കാൻ ഉപയോഗിക്കുന്നു. കൂടാതെ ഗ്രാഫ് പോളിനോമിയലുകൾ ഗ്രാഫുകളുടെ പ്രത്യേകതകൾ മനസ്സിലാക്കാൻ സഹായിക്കുന്നു. ഒരു ഗ്രാഫ് പോളിനോമൽ എന്നത് ഒരു ഗ്രാഫുമായി ബന്ധപ്പെട്ട പോളിനോമിയൽ ആണ്. അതിന്റെ ഗുണകങ്ങൾ ചില ഗ്രാഫ് തിയററ്റിക് പരാമീറ്ററുകളുടെ സൂചകങ്ങളാണ്.

ഈ തിസീസിൽ കോംപ്ലിമെന്റ് ഡിഗ്രി പോളിനോമിയലും വെർട്ടെക്സ് കട്ട് പോളിനോമിയലും എന്ന് പേരുള്ള രണ്ട് പുതിയ പോളിനോമിയലുകളെ കുറിച്ചുള്ള പഠനമാണ് ഉള്ളത്. വിവിധ ഗ്രാഫുകളുടെയും ഗ്രാഫ് ഓപ്പറേഷനുകളുടെയും ഈ രണ്ട് പോളിനോമിയലുകൾ കണ്ടുപിടിക്കുന്നു. ഒരേ പോളിനോമിയലുകളുള്ള ഗ്രാഫുകളെ ഒന്നിച്ച് ഒരു ഗണമാക്കി നിർവ്വചിക്കുന്നു. അവസാനമായി ചില കെമിക്കൽ ഗ്രാഫുകളുടെ കോംപ്ലിമെന്റ് ഡിഗ്രി പോളനോമിയലുകളും കണ്ടു പിടിക്കുന്നു.

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List of Symbols

G	A simple finite graph
$\cong$	is isomorphic to
$\ncong$	is non isomorphic to
$\deg v$	the degree of the vertex v
$\Delta(G)$	the maximum degree of G
$\delta(G)$	the minimum degree of G
$\overline{G}$	the complement of a simple graph G
$V(G)$	vertex set of G
$\|V\|$	cardinality of the set V
$E(G)$	edge set of G
$N(v)$	neighborhood of the vertex v
$\kappa(G)$	vertex connectivity of G
K_n	complete graph on n vertices
N_n	null graph
P	Peterson graph
W_n	wheel graph

$K_{m,n}$	complete bipartite graph
H_n	helm graph
G_n	gear graph
$B_{n,n}$	bistar graph
$L_{m,n}$	lollipop graph
$T_{m,n}$	tadpole graph
$W_n^{(m)}$	windmill graph
F_n	friendship graph
Sh_n	Shell graph
Sl_n	sunlet graph
S_n	sun graph
$F_{m,n}$	fan graph
$U_{m,n}$	umbrella graph
$FC_{m,n}$	firecracker graph
$Sh_{m,n}$	bow graph
$BF_{n,n}$	butterfly graph
Wb_n	web graph
Bk_n	book graph
$K_{n_1,n_2,\ldots,n_r}$	complete r-partite graph
Cr_n	crown graph
$C_n \bigodot P_m$	armed crown graph
$BT_{m,n}$	banana tree graph
Sf_n	sunflower graph
TS_n	triangular snake graph
Q_n	quadrilateral snake graph

$A(Q_n)$	alternate quadrilateral snake graph
$D(Q_n)$	double quadrilateral snake graph
$A(D(Q_n))$	alternate double quadrilateral snake graph
$A(TS_n)$	alternate triangular snake graph
$D(TS_n)$	double triangular snake graph
L_n	ladder graph
CL_n	circular ladder graph
ML_n	Mobius ladder graph
TL_n	triangular ladder graph
DL_n	diagonal ladder graph
SL_n	step ladder graph
DSL_n	double sided step ladder graph
B_n	bipartite cocktail party graph
CP_n	cocktail party graph CP_n
$G \circ H$	corona graph
$\mu(G)$	Mycielski graph
$Sh(G)$	shadow graph
$C_p \bigodot C_q{}^t$	chaplet graph
$M(G)$	middle graph
$S(G)$	splitting graph
$CS(G)$	cosplitting graph
$d(G)$	derivative of a graph G
$L(G)$	line graph
$T(G)$	total graph
$C(G)$	central graph

$G_1 \cup G_2$	union of graphs G_1 and G_2
$G_1 + G_2$	sum of graphs G_1 and G_2
$G_1 \vee G_2$	join of graphs G_1 and G_2
$G \circ_v H$	rooted product graph of G and H
$G \Box H$	Cartesian product of G and H
$CD(G, i)$	set of vertices of degree i in $\overline{G}$
$Cd_i(G)$	cardinality of $CD(G, i)$
$CD[G, x]$	complement degree polynomial of G
$cd(G)$	number of real *cd-roots* of a graph G
$\mathcal{CD}[G]$	CD- equivalent polynomials of G
C_nH_{2n+2}	alkanes
S	stand A of human insulin
$G(U) \sqcap G'$	generalized hierarchial product of G and G'
LP_{6n}	linear phenylene
D	dopamine
C	caffeine
R	ribonucleic acid (RNA)
CR	conditional Random Field
$V(G, i)$	family of vertex cuts with cardinality i
$d_v(G, i)$	cardinality of $V(G, i)$
$V[G; x]$	vertex cut polynomial of graph G
$VC[G]$	VC-equivalent polynomials of G

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Introduction

Graph theory is one of the most burgeoning branches of mathematics. The origin of graph theory started with the problem of the Königsberg bridge; in 1735, Euler studied and solved the problem of the Königsberg bridge [8]. A wide range of studies are ongoing in graph theory, and graph polynomials are one of the well-studied concepts in graph theory.

A graph polynomial is a polynomial assigned to a graph whose coefficients are indicators of some graph-theoretic parameters [4]. The first graph polynomial is the edge difference polynomial introduced by J.J. Sylvester in 1878 [9]. The study of graph polynomials, their stability, real roots, and location of roots in the complex plane is one of the main areas in graph theory.

In the present work, two new graph polynomials are introduced, namely the complement degree polynomial of graphs and the vertex cut polynomial of graphs. The complement degree polynomial of graphs relates to the degrees of the vertices in the complement graph. The vertex cut polynomial of graphs relates to vertex connectivity and vertex cuts of graphs.

Overview of the Thesis

In this thesis, we introduce two new graph polynomials: the complement degree polynomial of graphs and the vertex cut polynomial of graphs. The thesis comprises nine chapters, including an introductory chapter. In the introductory chapter, we discuss the motivation and importance of studying graph polynomials.

Chapter Overview

Chapter 1: Basic definitions and notations in graph theory are described in the first section. Definitions of some well-known graphs and graph operations are described in the second and third sections, respectively.

Chapter 2: Introduces a new graph polynomial called the complement degree polynomial of graphs. The definition and some general results are explained in section 1. The complement degree polynomial of some well-known graphs, graph operations, and some chemical graphs is identified in sections 2, 3, and 4, respectively.

Chapter 3: There are some details and results on stability and roots of polynomials are described in the first section and includes results on the stability of the complement degree polynomial of graphs in the second section. In section 3, we define *cd-roots* and real roots of complement degree polynomials of graphs. In section 4, it identifies the location of the roots of the complement degree polynomial of graphs.

Chapter 4: Focuses on equivalent classes of graphs. CD-equivalent classes of graphs are defined and studied in the first section, and we identify CD-equivalent graph classes in section 2.

Chapter 5: Introduces another new graph polynomial named the vertex cut polynomial of graphs. We define the vertex cut polynomial in section 1 and derive the vertex cut polynomial of some well-known graphs in section 2.

Chapter 6: Includes the vertex cut polynomial of some unary graph operations and binary graph operations in sections 1 and 2, respectively.

Chapter 7: VC-equivalent graphs and VC-equivalent classes of graphs are defined and studied in this chapter .

Chapter 8: Concludes and gives some directions for further research. Also, we include a bibliography, a list of publications in Appendix I, and a list of paper presentations in Appendix II.

Chapter 1

Preliminaries

This chapter includes basic terminology and notations of graphs and polynomials. We collect the basic concepts of graphs as in graph theory [1], written by Frank Harary. In the first section we collect basic terminology of graphs. The second and third sections include definitions of some well-known graphs and graph operations, respectively. The fourth section deals with basic definitions and results of polynomials.

1.1 Basic terminology of graphs

A **graph** $G = (V, E)$ consists of a finite non empty set V of p points called vertices together with a prescribed set E of q unordered pairs of distinct points of V. Each pair $e = uv$ of points in E is a line called edge of G [1].

- If $e = uv$, then we say that u and v are adjacent vertices, vertex u and e are incident with each other, as are v and e.

- If two distinct edges e_1 and e_2 are incident with a common vertex, then they are adjacent edges.

- The number of elements in V and in E are called order and the size of G respectively.

- Two or more edges having same end vertices are called multiple edges and an edge with identical end vertices is called a loop.

- A graph having finite number of vertices and edges is called a finite graph.

- a subgraph of G is a graph having all of its vertices and edges in G.

- Let $v \in V(G)$, then neighbourhood of v is defined as $N(v) = \{$u: u adjacent to v$\}$.

Two graphs G and H are **isomorphic** (written $G \cong H$) if there exist a one-to-one correspondence between their vertex sets which preserves adjacency. The **complement** $\overline{G}$ **of a graph** G also has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A **self complementary graph** is isomorphic with its complement [1].

The degree of a vertex $v \in V$, written $deg(v)$ is the number of edges in G which are incident with v.

- The minimum degree among the vertices of G is denoted $\delta(G)$ or min deg G while $\Delta(G) =$ max deg G is the largest such number.

- A pendant vertex (or leaf) is any vertex of degree 1 (that is, a vertex adjacent to exactly one other vertex).

- A graph having all the vertices with same degree is called a **regular graph**.

A **walk** is an alternating sequence $v_0, e_1, v_1, e_2, \ldots, v_{i-1}, e_i, v_i, \ldots, v_n$ of vertices and edges in which the vertices v_{i-1} and v_i are the end points of the edge e_i. A **path** is a walk having all the vertices distinct. A **trail** is a walk where all the edges are distinct. A closed trail in which all the vertices are distinct is called a **cycle** [1].

A graph is **connected** if every vertices are joined by a path. A maximal connected subgraph of G is called a connected component or simply a **component** of G. Thus a disconnected graph has at least two components [1].

A **vertex cut** of a non complete graph G is a set S of vertices of G such that $G - S$ is disconnected. A vertex cut of minimum cardinality in G is called a minimum vertex cut of G and this cardinality is called the **vertex connectivity** of G and is denoted by $\kappa(G)$. A vertex v in a graph G is called a **cut vertex** if deleting v from G increases the number of components of G [1].

1.2 Definitions of some well known graphs

The **complete graph** K_n has every pairs of its n vertices are adjacent. Thus K_n has $\binom{n}{2}$ edges and is regular of degree $n - 1$. A **null graph** N_n is a graph with n vertices in which there are no edges between its vertices. The **Petersen graph** is an undirected cubic graph with 10 vertices and 15 edges.

A **wheel graph** W_n is a graph formed by connecting a single vertex to all vertices of a cycle C_{n-1} [15]. A **complete bipartite graph** $K_{m,n}$ is a graph whose vertices can be partitioned into two subsets V and U with cardinality m and n respectively such that no edge has both end points in the same subset,

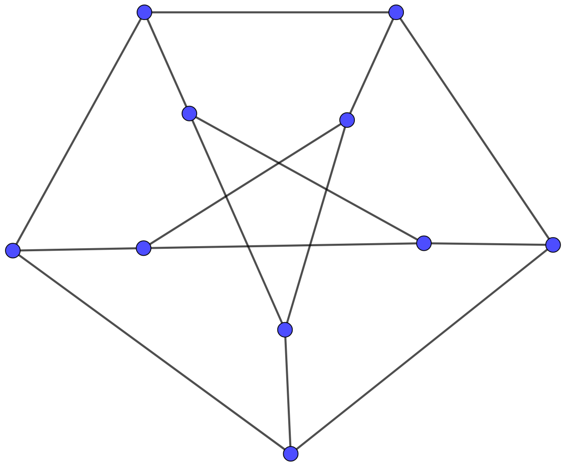

Figure 1.1: Petersen graph

and every possible edge that could connect vertices in different subsets is part of the graph [16].

A **star graph** is the complete biparite graph $K_{1,n}$ with one central vertex and n leaves. A **helm graph** denoted H_n is a graph obtained by attaching a single edge vertex to each vertex of the outer circuit of a wheel graph W_n [18]. A **gear graph**, denoted by G_n is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of a wheel graph W_n [17].

A **bistar graph** $B_{n,n}$ is the union of two star graphs $K_{1,n}$ with centers u and v together with a new edge uv. A **lollipop graph** $L_{m,n}$ is a special type of graph consisting of a complete graph on m vertices and a path graph on n vertices connected with a bridge [19]. A **tadpole graph** $T_{m,n}$ is a special type of graph consisting of a cycle graph on m vertices and path graph on n vertices connected with a bridge [20]. The **Pan graph** $T_{m,1}$ is the graph obtained by joining a cycle graph to a singleton graph with a bridge. The m-pan graph is therefore isomorphic with the $(m, 1)$- tadpole graph [22].

The **windmill graph** $W_n^{(m)}$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common [22]. The **friendship (or Dutch windmill or n-fan) graph** F_n is a graph joining n copies of the cycle graph C_3 with a common vertex. A **shell graph** Sh_n is defined as a cycle C_n with $n-3$ chords sharing a common end point called the apex [23].

The **sunlet graph** Sl_n is the graph on $2n$ vertices obtained by attaching n pendant edges to a cycle graph C_n [22]. The **sun graph** S_n is a graph on $2n$ vertices consisting of central complete graph with an outer ring of n vertices, each of which is joined to both end points of the closest outer edge of the central core [22]. A **fan graph** $F_{m,n}$ is defined as the join of two graphs $\overline{K_m} + P_n$ where $\overline{K_m}$ is the empty set on m vertices and P_n is the path graph on n vertices. The case $m = 1$ corresponds to the usual fan graph while $m = 2$ corresponds to the double fan graph, etc [22].

For any integer $m > 2$ and $n > 1$, an **umbrella graph** $U_{m,n}$ is the graph obtained by appending a path P_n to the central vertex of a fan graph $F_m = P_m + K_1$ [24]. A **firecracker graph** $FC_{m,n}$ is obtained by the concatenation of m $n-stars$ by linking one leaf from each [22]. A **bow graph** $Sh_{m,n}$ is defined to be a double shell with common apex in which each shell has any order [25].

A **butterfly graph** $BF_{n,n}$ is a bow graph along with exactly two pendant edges at the apex [25]. A **web graph** Wb_n, $n > 2$ is obtained by joining the pendant vertices of a helm graph H_n to form a cycle and then adding a single pendant edge to each vertex of this outer cycle [4]. The **book graph** Bk_n is defined as the graph Cartesian product $K_{1,n} \Box P_2$, where $K_{1,n}$ is a star graph and P_2 is a path graph on two vertices [22].

A complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ is a graph whose vertex set can be partitioned in to r non empty sets V_i, $i = 1, 2, \ldots, r$ such that every vertex in V_i is adjacent to every vertex in V_j for every $i \neq j$ and $i, j \in \{1, 2, \ldots, r\}$ [4]. A **crown graph** Cr_n on $2n$ vertices is graph with two sets of vertices $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ and with an edge from u_i to v_j whenever $i \neq j$ [26]. An **armed crown graph** $C_n \bigodot P_m$ is a graph obtained by attaching a path P_m to every vertex of the cycle C_n [4].

A **banana tree** graph $BT_{m,n}$ is a graph obtained by connecting one leaf of each of m copies of an n-star graph with a single root vertex that is distinct from all the stars [22]. The **sunflower graph** Sf_n is a graph on $2n + 1$ vertices consisting of central wheel graph with an outer ring of n vertices, each of which is joined to both end points of the outer edge of the wheel graph.

A **triangular snake graph** TS_n is the graph on n vertices with n odd defined by starting with the path graph P_{n-1} and adding edges $(2i - 1, 2i + 1)$ for $i = 1, 2, \ldots, n - 1$ [27]. A **quadrilateral snake graph** Q_n is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to new vertices u_i and w_i, respectively and joining the vertices u_i and w_i for $i = 1, 2, \ldots, n - 1$. That is every edge of path is replaced by a cycle C_4 [27]. An **alternate quadrilateral snake graph** $A(Q_n)$ is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} (alternatively) to new vertices u_i and u_{i+1}, respectively and joining the vertices u_i and u_{i+1} . The **double quadrilateral snake graph** $D(Q_n)$ consists of two quadrilateral snakes that have a common path [27]. An **alternate double quadrilateral snake graph** $A(D(Q_n))$ is obtained from two alternative quadrilateral snakes that have a common path. An **alternate triangular snake graph** $A(TS_n)$

is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i, v_{i+1} (alternatively) to a new vertex u_i. A **double triangular snake graph** $D(TS_n)$ consists of two triangular snakes that have a common path.

The **ladder graph** L_n is the Cartesian product of two path graphs P_n and P_2 [21]. The **circular ladder graph** CL_n is contractible by connecting the four 2-degree vertices in a straight way, or by the Cartesian product of a cycle of length $n \geq 3$ and an edge [21]. Connecting the four 2-degree vertices crosswise creates a cubic graph called a **Mobius ladder graph** ML_n [21]. The **triangular ladder graph** TL_n is a graph that has a set of vertices. $V(TL_n) = \{u_i, v_i : 1 \leq i \leq n\}$ where many vertices $|V| = 2n$ and set the $E(TL_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\}$ where many edges $|E| = 4n-3$ [21]. The **diagonal ladder graph** $DL_n(n \geq 2)$ is a graph obtained from a ladder graph by by adding the edges u_iv_{i+1} and $u_{i+1}v_i$ for $1 \leq i \leq n-1$ such graph has $2n$ vertices and $5n-4$ edges. Let P_n be a path of n vertices denoted by $(1,1), (1,2), \ldots, (1,n)$ and $n-1$ edges denoted by $e_1, e_2, \ldots, e_{n-1}$ where e_i is the edge joining the vertices $(1,i)$ and $(1,i+1)$ on each edge e_i, $i = 1, 2, \ldots, n-1$ we erect a ladder with $n-(i-1)$ steps including the edge e_i [21]. The graph obtained is called **step ladder graph** and is denoted by SL_n where n denote the number of vertices in the base. It has $\frac{n^2+3n-2}{2}$ vertices and $n(n+1)-2$ edges. Let P_{2n} be a path of length $2n-1$ with $2n$ vertices $(1,1), (1,2), \ldots, (1,2n)$ with $2n-1$ edges $e_1, e_2, \ldots, e_{2n-1}$ where e_i is the edge that joins the vertices $(1,i)$ and $(1,i+1)$ on each edge e_i for $i = 1, 2, \ldots, n$. We erect a ladder with $i+1$ steps that include the edge e_i and on each edge for $i = n+1, n+2, \ldots, 2n-1$ we erect a ladder with $2n+1-i$ steps including the edge e_i [21]. The **double sided step ladder graph** DSL_{2n} has vertices denoted by $(1,1),(1,2),\ldots,(1,2n),(21,(2,2),\ldots,(2,2n),(3,2),(3,3),\ldots,(3,2n-1),(4,3),(4,4),\ldots,(4,2n-2),\ldots,(n+1,n),(n+1,n+1)$. In the ordered pair (i,j) where i denotes the row number (counted from bottom to top) and j denotes the column number (from left to right) in which the vertex occurs. It has n^2+3n vertices and $2n^2+3n-1$ edges [21].

The **bipartite cocktail party graph** B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$ [4]. The **cocktail party graph** CP_n is a graph consisting of two rows of paired vertices in which all vertices except the paired ones are connected with straight lines; it is the complement of the ladder graph [22].

1.3 Graph Operations

The **corona graph** $G \circ H$ of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. **The Mycielski graph**, $\mu(G)$ of a graph G contains G itself as an isomorphic subgraph together with $n+1$ additional vertices; a vertex u_i corresponding to each vertex v_i of G and another w. Each u_i is connected by an edge to w and for each edge v_iv_j of G, $\mu(G)$ includes two additional edges v_iu_j and u_iv_j [4]. The **shadow graph** $Sh(G)$ of a graph G is obtained by taking two copies of G, say G_1 and G_2 and joining each vertex of G_1 to the neighbors of the corresponding vertex of G_2 [4].

Duplication of a vertex v of a graph G is the graph denoted by G' obtained by adding a vertex v' in G with $N(v) = N(v')$ [4]. A **chaplet graph** $C_p \bigodot {C_q}^t$

where $p, q.t \geq 3$ is obtained by taking one point union of t-copies of the cycle C_q and attaching the same to each vertex of the cycle C_p [4]. The **middle graph** $M(G)$ of a graph G is the graph in which the vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if either they are adjacent edges of G or one is vertex of G and the other is an edge incident with it.

For a graph G the **splitting graph** $S(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ [10]. The **cosplitting graph** $CS(G)$ is the graph obtained from G, by adding a new vertex w_i for each vertex v_i and joining w_i to all vertices which are not adjacent to v_i in G. The **derivative of a graph** G is a graph $d(G)$ obtained from G by deleting all the pendant vertices of G [4].

A graph $L(G)$ is said to be a **line graph** of an undirected simple graph G if the vertex set of $L(G)$ is in one-one correspondence with the edge set of G and two vertices of $L(G)$ are joined by an edge if and only if the correspondence edge of G are adjacent in G. **Total graph** denoted by $T(G)$ of a graph G is a graph in which the set of vertices and edge set of G and any two vertices in $T(G)$ are said to be adjacent if and only if their corresponding elements are either adjacent or incident in G. The **central graph** denoted by $C(G)$ of a graph G is the graph obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, the **union** $G_1 \cup G_2$ is defined to be $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, the **sum** $G_1 + G_2$ is defined as $G_1 \cup G_2$ together with all the lines joining points of V_1 to V_2. The **join** of two graphs G_1 and G_2 is a graph formed from disjoint copies of G_1 and

G_2 by connecting every vertex of G_2, it is denoted by $G_1 \vee G_2$.

The **rooted product graph** $G \circ_v H$ of a graph G and a rooted graph H with root vertex v is defined as the graph by taking one copy of G and $|V(G)|$ copies of H and identifying the i^{th} vertex of G with the root vertex v in the i^{th} copy of H for every $i \in \{1, 2, \ldots, |V(G)|\}$. The **Cartesian product** of two graphs G and H is the graph $G \Box H$ with vertex set $V(G) \times V(H)$ and the vertices (u, v) and (x, y) are adjacent if and only if $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$ [4].

Chapter 2

Complement Degree Polynomial of Graphs

In this chapter is divided into three sections. In the first section, we introduce the concept of the complement degree polynomial of a graph. Subsequent sections focus on deriving the complement degree polynomials of notable graphs, exploring the effects of various graph operations and some chemical graphs. Throughout this chapter, our attention is restricted to finite, undirected, and simple graphs

2.1 Complement degree polynomial of graphs

Definition 2.1.1. *Let $G = (V, E)$ be a graph, and let $CD(G, i)$ be the set of vertices of degree i in the complement graph $\overline{G}$ and let $Cd_i(G) = |CD(G, i)|$.*

Then the complement degree polynomial of G is defined as

$$CD[G,x] := \sum_{i=\delta(\overline{G})}^{\Delta(\overline{G})} Cd_i(G)x^i.$$

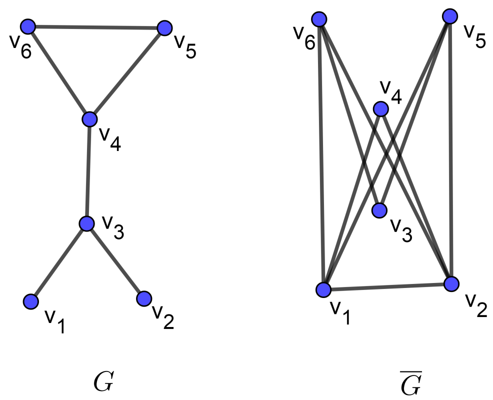

Figure 2.1: The graph G and $\overline{G}$

Example 2.1.2. *Consider the graph G and its complement graph $\overline{G}$ as shown in Figure 2.1. Obviously, $Cd_1(G) = 0 = Cd_5(G)$, $Cd_2(G) = 2$, $Cd_3(G) = 2$, and $Cd_4(G) = 2$. Hence, the complement degree polynomial of G is $CD[G,x] =$* $2x^2 + 2x^3 + 2x^4$.

Theorem 2.1.3. *If G is a graph of order n and v be a vertex of degree d, then degree of v in $\overline{G}$ is $n-1-d$.*

Proof. Note that ($deg(v)$ in G) + ($deg(v)$ in $\overline{G}$) $= n-1$. This implies that $deg(v)$ in $\overline{G}$= $n-1$ -($deg(v)$ in G) =$n-1-d$. Thus, $deg(v) = n-1-d$ in $\overline{G}$. This completes the proof. □

Theorem 2.1.4. *If G and H are two isomorphic graphs, then $CD[G,x] = CD[H,x]$.*

Proof. Let G and H be two isomorphic graphs. Then the complement graphs $\overline{G}$ and $\overline{H}$ are also isomorphic, and the degree sequence of $\overline{G}$ and $\overline{H}$ are the same. Therefore, the complement degree polynomial of G and H is the same. This completes the proof. □

Remark 2.1.5. *The converse of the above theorem need not be true. That is, the complement degree polynomial of two graphs G and H are the same, but G and H need not be isomorphic. For example, consider the graphs $G = P_5$ and $H = P_2 \cup C_3$. Note that the degree sequences of G and H are the same but $G \ncong H$.*

Theorem 2.1.6. *A graph G is a $r-$regular graph if and only if $CD[G, x] = nx^{n-r-1}$.*

Proof. Let G is a $r-$regular graph with n vertices. Note that, if G is a regular graph then $\overline{G}$ is also a regular graph, that is, $\overline{G}$ is $(n - r - 1)$ -regular graph. Therefore, $CD[G, x] = nx^{n-r-1}$. Conversely, suppose that $CD[G, x] = nx^{n-r-1}$. Since $\overline{G}$ is $(n - r - 1)$ -regular graph and $n - 1 - (n - r - 1) = r$, it follows that G is r-regular. This completes the proof. □

2.2 Complement degree polynomial of some graphs

Theorem 2.2.1. *For a path graph P_n, we have*

$$CD[P_n, x] = \begin{cases} 2x^{n-2}, & n = 2; \\ (n-2)x^{n-3} + 2x^{n-2}, & n \geq 3. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n. Assume that the vertices v_1 and v_n have degree 1 in P_n. Therefore, v_1 and v_n have degree $n-2$ in $\overline{P_n}$. Observe that $v_2, v_3, \ldots, v_{n-1}$ have the degree 2 in P_n and hence they have degree $n-3$ in $\overline{P_n}$. Therefore, $CD[P_n, x] = (n-2)x^{n-3} + 2x^{n-2}$. This completes the proof. □

Theorem 2.2.2. *For cycle graph C_n, Petersen graph P, and complete graph K_n, we have the following*

(1) if $n \geq 3$, then $CD[C_n, x] = nx^{n-3}$,

(2) $CD[P, x] = 10x^6$,

(3) if $n \geq 1$, then $CD[K_n, x] = n$.

Proof. Note that the cycle graph C_n, the Petersen graph P and the complete graph K_n are regular graphs. Then by Theorem 2.1.6 the result follows. □

Theorem 2.2.3. *For a wheel graph W_n, where $n \geq 4$, we have the following*

$$CD[W_n, x] = (n-1)x^{n-4} + 1.$$

Proof. Observe that the central vertex of the wheel graph is adjacent to all other vertices. Thus, the degree of the central vertices in $\overline{W_n}$ is 0. Note that the outer ring of the wheel graph is C_{n-1}. Since $CD[C_n, x] = nx^{n-3}$, it follows that $CD[C_{n-1}, x] = (n-1)x^{n-1-3} = (n-1)x^{n-4}$. Thus $CD[W_n, x] = (n-1)x^{n-4} + 1$. This completes the proof. □

Theorem 2.2.4. *For a complete bipartite graph $K_{m,n}$, where $m \geq 1$ and $n \geq 1$,* *we have the following*

$$CD[K_{m,n}, x] = mx^{m-1} + nx^{n-1}.$$

Proof. Let U and V be the bipartition of $V(K_{m,n})$ with cardinality m and n, respectively. Since no edge has both endpoints in the same subset and every possible edge that could connect vertices in different subsets of $V(K_{m,n})$, we have two components in $\overline{K_{m,n}}$, each component is complete graphs with m and n vertices. Thus, $CD[K_{m,n}, x] = mx^{m-1} + nx^{n-1}$. This completes the proof. □

Theorem 2.2.5. *For a helm graph H_n, where $n \geq 3$, we have the following*

$$CD[H_n, x] = x^n + nx^{2n-4} + nx^{2n-1}.$$

Proof. Let $v_0, v_1, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of H_n, where v_0 is the central vertex , $v_1, v_2, \ldots, v_n$ are the vertices adjacent to central vertex and $u_1, u_2, \ldots, u_n$ be the pendant vertices. Since v_0 is not adjacent to any vertices of $u_1, u_2, \ldots, u_n$ in H_n, we have v_0 adjacent to all vertices of $u_1, u_2, \ldots, u_n$ in $\overline{H_n}$. Therefore, degree of v_0 is n in $\overline{H_n}$. Note that any v_i adjacent to v_{i-1}, v_{i+1}, v_0 and u_i. Therefore, $deg(v_i) = 2n - 4$, $i = 1, 2, \ldots, n$. Similarly, the pendant vertex u_i is adjacent to the only vertex v_i. Thus, $deg(u_i) = 2n - 1$, $i = 1, 2, \ldots, n$. Thus $CD[H_n, x] = x^n + nx^{2n-4} + nx^{2n-1}$. This completes the proof. □

Theorem 2.2.6. *For a gear graph G_n, where $n \geq 3$, we have the following*

$$CD[G_n, x] = x^n + nx^{2n-3} + nx^{2n-2}.$$

Proof. Let $v_0, v_1, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of G_n (where v_0 is central vertex, $v_1, v_2, \ldots, v_n$ are vertices on the perimeter of a wheel graph and $u_1, u_2, \ldots, u_n$ are inserting vertices between each pair of adjacent vertices of $v_1, v_2, \ldots, v_n$). Note that v_0 adjacent to n vertices $v_1, v_2, \ldots, v_n$ in G_n. Thus v_0 adjacent to $u_1, u_2, \ldots, u_n$ in $\overline{G_n}$. Therefore, $deg(v_0) = n$ in $\overline{G_n}$. Since each v_i adjacent to u_i, u_{i+1} and v_0 in G_n, we have v_i adjacent to other $2n-3$ vertices in $\overline{G_n}$. Therefore, $deg(v_i) = 2n-3$, $i = 1, 2, \ldots, n$. Similarly, each u_i adjacent to v_{i-1} and v_i in G_n. Thus u_i is adjacent to the other $2n-2$ vertices in $\overline{G_n}$. Thus $CD[G_n, x] = x^n + nx^{2n-3} + nx^{2n-2}$. This completes the proof. □

Theorem 2.2.7. *For a bistar graph $B_{n,n}$, where $n \geq 1$, we have the following*

$$CD[B_{n,n}, x] = 2nx^{2n} + 2x^n.$$

Proof. Let $v, u, v_1, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of $B_{n,n}$ with pendant vertices $u_1, u_2, \ldots, u_n$ adjacent to u and the other pendant vertices $v_1, v_2, \ldots, v_n$ adjacent to v. Note that each u_i is adjacent to $2n$ vertices v, $v_1, \ldots, v_n$, $u_1, \ldots u_{i-1}$, $u_{i+1}, \ldots, u_n$ in $\overline{B_{n,n}}$. Therefore, $deg(u_i) = 2n$ in $\overline{B_{n,n}}$, $i = 1, 2, \ldots, n$. Similarly, $deg(v_i) = 2n, i = 1, 2, \ldots, n$. Since u is adjacent to $u_1, u_2, \ldots, u_n$ and v in $B_{n,n}$, u is adjacent to $v_1, v_2, \ldots, v_n$ in $\overline{B_{n,n}}$. Therefore $deg(u) = n$ in $\overline{B_{n,n}}$. Similarly, $deg(v) = n$ in $\overline{B_{n,n}}$. Thus $CD[B_{n,n}, x] = 2nx^{2n} + 2x^n$. This completes the proof. □

Theorem 2.2.8. *For a lollipop graph $L_{m,n}$, where $n \geq 1$ and $m \geq 1$, we have the following*

$$CD[L_{m,n}, x] = x^{m+n-2} + (n-1)x^{m+n-3} + (m-1)x^n + x^{n-1}.$$

Proof. Let $v_1, \ldots, v_n, u_1, u_2, \ldots, u_m$ be the vertices of $L_{m,n}$ with u_m adjacent to v_1. Since $u_1, u_2, \ldots, u_{m-1}$ adjacent to every vertex of the complete graph in $L_{m,n}$, they are adjacent to $v_1, v_2, \ldots, v_n$ in $\overline{L_{m,n}}$. Therefore, $deg(u_i) = n$ in $\overline{L_{m,n}}$, $i = 1, 2, \ldots, m-1$. The remaining vertex in the complete graph is u_m, u_m is adjacent to every vertex of a complete graph K_m and v_1 in $L_{m,n}$. Therefore, $deg(u_m) = m$ in $L_{m,n}$, this follows that $deg(u_m) = n-1$ in $\overline{L_{m,n}}$. Note that each v_i is adjacent to v_{i-1} and v_{i+1}, $i = 2, 3, \ldots, n-1$. Similarly, v_1 is adjacent to u_m and v_2. Therefore, $deg(v_i) = 2$ in $L_{m,n}$, $i = 1, 2, \ldots, n-1$: this implies that $deg(v_i) = m+n-3$ in $\overline{L_{m,n}}$, $i = 1, 2, \ldots, n-1$. Since v_n is adjacent to only v_{n-1}, therefore $deg(v_n) = 1$ in $L_{m,n}$. This implies $deg(v_n) = m+n-2$ in $\overline{L_{m,n}}$. Thus $CD[L_{m,n}, x] = x^{m+n-2} + (n-1)x^{m+n-3} + (m-1)x^n + x^{n-1}$. This completes the proof. □

Theorem 2.2.9. *For a tadpole graph $T_{m,n}$, where $m \geq 3$ and $n \geq 1$, we have the following*

$$CD[T_{m,n}, x] = x^{m+n-2} + (m+n-2)x^{m+n-3} + x^{m+n-4}.$$

Proof. Let $v_1, \ldots, v_n, u_1, u_2, \ldots, u_m$ be the vertices of $T_{m,n}$ with u_m adjacent to v_1 and v_n be the pendant vertex. Note that each vertex (other than u_m and v_n) is adjacent to another two vertices in $T_{m,n}$. Therefore, each vertices (other than u_m and v_n) adjacent to $m+n-3$ in $\overline{T_{m,n}}$. The vertex u_m is adjacent to two vertices in a cycle and on the vertices of a path in $T_{m,n}$. Thus u_m adjacent to other $m-3$ vertices of a cycle and $n-1$ vertices of the path in $\overline{T_{m,n}}$. Therefore, $deg(u_m) = m+n-4$ in $T_{m,n}$. Since v_n be the pendant vertex, it follows that $deg(v_n) = 1$ in $T_{m,n}$ implies $deg(v_n) = m+n-2$ in $\overline{T_{m,n}}$. Therefore, $CD[T_{m,n}, x] =$

$x^{m+n-2} + (m+n-2)x^{m+n-3} + x^{m+n-4}$. This completes the proof. □

Theorem 2.2.10. *For a windmill graph $W_n^{(m)}$, where $n \geq 2$ and $m \geq 1$, we have the following*

$$CD[W_n^{(m)}, x] = m(n-1)x^{(m-1)(n-1)} + 1.$$

Proof. The common vertex in the windmill graph is adjacent to every vertex of $W_n^{(m)}$. Thus this that common vertex is an isolated vertex in $\overline{W_n^{(m)}}$. If we delete the common vertex, we get m copies of the complete graph K_{n-1}. The vertices in each complete graph are adjacent to vertices of other $m-1$ complete graphs K_{n-1} in $W_n^{(m)}$. Thus the degree of vertices other than the common vertex is $(n-1)(m-1)$ in $\overline{W_n^{(m)}}$. Therefore, $CD[W_n^{(m)}, x] = m(n-1)x^{(m-1)(n-1)} + 1$. This completes the proof. □

Corollary 2.2.11. *For a friendship graph F_n, where $n \geq 3$, we have the following*

$$CD[F_n, x] = 2nx^{2(n-1)} + 1.$$

Theorem 2.2.12. *For a shell graph Sh_n, where $n \geq 4$, we have the following*

$$CD[Sh_n, x] = 2x^{n-3} + (n-3)x^{n-4} + 1.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of Sh_n with v_1 is the apex and $v_3, \ldots, v_{n-1}$ are the end points of chords with v_1. Note that the apex of the shell graph is adjacent to every vertex. Therefore, v_1 is an isolated vertex in $\overline{Sh_n}$. Also, we have $v_3, \ldots, v_{n-1}$ as the endpoints of chords in Sh_n. Thus the degree of any v_i, $i = 3, 4, \ldots, n-1$, is 3 in Sh_n. That is the degree of v_i, $i = 3, 4, \ldots, n-1$, is $n-4$ in $\overline{Sh_n}$. Since v_2 and v_n are part of C_n but not sharing with any chords, therefore, the degree of v_2 and v_n is 2 in Sh_n. Thus, the degree of v_2 and v_n is $n-3$ in $\overline{Sh_n}$. Therefore, $CD[Sh_n, x] = 2x^{n-3} + (n-3)x^{n-4} + 1$. This completes the proof. □

Theorem 2.2.13. *For a sunlet graph Sl_n, where $n \geq 3$, we have the following*

$$CD[Sl_n, x] = nx^{2n-2} + nx^{2n-4}.$$

Proof. Let $v_1, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of the sunlet graph Sl_n with pendant vertices $v_1, v_2, \ldots, v_n$. Note that each v_i is adjacent to u_i, $i = 1, 2, \ldots, n$. Any pendant vertex v_i, $i = 1, 2, \ldots, n$ adjacent to other pendant vertices and all vertices in the inner circle except u_i, $i = 1, 2, \ldots, n$ in $\overline{Sl_n}$. Then $deg(v_i) = 2n-2$ in $\overline{Sl_n}$, $i = 1, 2, \ldots, n$. Note that each u_i is adjacent to u_{i-1}, u_{i+1} and v_i in Sl_n. Then each u_i adjacent to all vertices except u_{i-1}, u_{i+1} and v_i in $\overline{Sl_n}$. That is $deg(u_i) = 2n-4$ in $\overline{Sl_n}$, $i = 1, 2, \ldots, n$. Thus $CD[Sl_n, x] = nx^{2n-2} + nx^{2n-4}$. This completes the proof. □

Theorem 2.2.14. *For a sungraph S_n, where $n \geq 3$, we have the following*

$$CD[S_n, x] = nx^{2n-3} + nx^{n-2}.$$

Proof. Let $v_1, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of the sun graph S_n with $u_1, u_2, \ldots, u_n$ be the vertices of the complete graph K_n and $v_1, \ldots, v_n$ be the vertices in the outer ring of S_n. The vertices in the outer ring are adjacent to only two vertices of the complete graph. That is degree of each v_i is 2 in S_n. Thus the degree of each v_i is $2n-3$ in $\overline{S_n}$. Note that the vertices $u_1, u_2, \ldots, u_n$ are part of a complete graph K_n, and each u_i is adjacent to only two vertices of the outer ring of S_n. That is the degree of each u_i is $2 + n - 1 = n + 1$ in S_n. Thus the degree of each u_i is $n - 2$ in $\overline{S_n}$, $i = 1, 2, \ldots, n$. Therefore, $CD[S_n, x] = nx^{2n-3} + nx^{n-2}$. This completes the proof. □

Theorem 2.2.15. *For a fan graph $F_{1,n}$, where $n \geq 3$, we have the following*

$$CD[F_{1,n}, x] = 2x^{n-2} + (n-2)x^{n-3} + 1.$$

Proof. The fan graph is the sum of two graphs $K_1 + P_n$. Note that the vertex of K_1 adjacent to all vertices of P_n. Therefore, K_1 is an isolated vertex of $\overline{F_{1,n}}$. If we remove K_1 in $F_{1,n}$, then we get the path graph. Also note that the degree of end vertices of the path graph is $n-2$ in $\overline{P_n}$ and the degree of remaining vertices of P_n is $n-3$ in $\overline{P_n}$ (by Theorem 2.2.1). Thus $CD[F_{1,n}, x] = 2x^{n-2} + (n-2)x^{n-3} + 1$. This completes the proof. □

Theorem 2.2.16. *For a double fan graph $F_{2,n}$, where $n \geq 3$, we have the following*

$$CD[F_{2,n}, x] = 2x^{n-2} + (n-2)x^{n-3} + 2x.$$

Proof. Observe that $\overline{F_{2,n}}$ graph is a disconnected graph with two components K_2 and $\overline{P_n}$. The degree of each vertex in K_2 is 1, and by Theorem 2.2.1, $CD[P_n, x] = 2x^{n-2} + (n-2)x^{n-3}$. Therefore, $CD[F_{2,n}, x] = 2x^{n-2} + (n-2)x^{n-3} + 2x$. This completes the proof. □

Theorem 2.2.17. *For a umbrella graph $U_{m,n}$, where $m \geq 3$ and $n \geq 2$, we have the following*

$$CD[U_{m,n}, x] = x^{m+n-1} + (n+1)x^{m+n-2} + (m-2)x^{m+n-3} + x^{n-1}.$$

Proof. Let $v_1, \ldots, v_m, w, u_1, u_2, \ldots, u_n$ be the vertices of umbrella graph $U_{m,n}$ (where $v_1, \ldots, v_m$ be the vertices of the path graph P_m of fan graph, w be the central vertex of fan graph, and $u_1, u_2, \ldots, u_n$ be the vertices of path graph P_n and u_1 adjacent to w). Since degree of vertices $v_2, v_3, \ldots, v_{m-1}$ is 3 in $U_{m,n}$, it follows that the degree of these $m-2$ vertices in $\overline{U_{m,n}}$ is $m+n-3$. The central vertex w is adjacent to m vertices $v_1, \ldots, v_m$ of the fan graph and vertex u_1 of the path graph. Thus $deg(w) = m+1$ in $U_{m,n}$. Therefore, $deg(w) = n-1$ in $\overline{U_{m,n}}$. Similarly, the degree of $n-1$ vertices $u_1, u_2, \ldots, u_{n-1}$ of the path graph is 2 in $U_{m,n}$; and the degree of these $n-1$ vertices is $m+n-2$ in $\overline{U_{m,n}}$. The degree of leaf vertex u_n of the umbrella graph is $m+n-1$ in $\overline{U_{m,n}}$. Thus $CD[U_{m,n}, x] = x^{m+n-1} + (n+1)x^{m+n-2} + (m-2)x^{m+n-3} + x^{n-1}$. This completes the proof. □

Theorem 2.2.18. *For firecracker graph $FC_{m,n}$, where $m \geq 3$ and $n \geq 2$, we have the following*

$$CD[FC_{m,n}, x] = m(n-2)x^{mn-2} + 2x^{mn-3} + (m-2)x^{mn-4} + mx^{n(m-1)}.$$

Proof. The degree of $m-2$ linking leaves is 3 in $FC_{m,n}$, but the degree of the first and last linking leaves is 2 in $FC_{m,n}$. The degree of these $m-2$ linking leaves is $mn-4$ and the degree of the other two linking leaves are $mn-3$ in $\overline{FC_{m,n}}$. The central vertices of each star graph are adjacent to $(m-1)n$ vertices of other star graphs in $\overline{FC_{m,n}}$. The degree of remaining $m(n-2)$ vertices (leaves of $FC_{m,,n}$) is $mn-2$ in $\overline{FC_{m,n}}$. Therefore, $CD[FC_{m,n}, x] = m(n-2)x^{mn-2} + 2x^{mn-3} + (m-2)x^{mn-4} + mx^{n(m-1)}$. This completes the proof. □

Theorem 2.2.19. *For a bow graph $Sh_{m,n}$, where $m \geq 3$ and $n \geq 2$, we have the* *following*

$$CD[Sh_{m,n}, x] = 4x^{m+n-2} + (m + n - 4)x^{m+n-3} + 1.$$

Proof. Let $v_1, \ldots, v_n, w, u_1, u_2, \ldots, u_m$ be the vertices of a bow graph (where w is the apex, $v_1, \ldots, v_n$ are the vertices of one shell graph, and $u_1, u_2, \ldots, u_m$ are the vertices of another shell graph). The apex of the bow graph is adjacent to all vertices of the bow graph; therefore, w is an isolated vertex in $\overline{Sh_{m,n}}$. Note that in each shell graph, $n-2$ vertices $v_2, v_3, \ldots, v_{n-1}$ and $m-2$ vertices $u_2, u_3, \ldots, u_{m-1}$ occur at the ends of the chords. Therefore, the degree of $m - 2 + n - 2$ vertices is 3 in $Sh_{m,n}$, then the degree of these $m + n - 4$ vertices is $m + n - 3$ in $\overline{Sh_{m,n}}$, it follows that the degree of the remaining 4 vertices is 2 in $Sh_{m,n}$; then the degree of these 4 vertices is $m + n - 2$ in $\overline{Sh_{m,n}}$. Thus $CD[Sh_{m,n}, x] = 4x^{m+n-2} + (m + n - 4)x^{m+n-3} + 1$. This completes the proof. □

Corollary 2.2.20. *For a butterfly graph $BF_{n,n}$, where $n \geq 2$, we have the following*

$$CD[BF_{n,n}, x] = 2x^{2n+1} + 4x^{2n} + (2n - 4)x^{2n-1} + 1.$$

Proof. The pendant vertices of the butterfly graph are adjacent to only the apex; therefore, these two pendant vertices are adjacent to each vertex of the shell graph, and these two pendant vertices are adjacent to each other in $\overline{BF_{n,n}}$. Thus the degree of these pendant vertices is $2n + 1$ in $\overline{BF_{n,n}}$. Also, the degree of other vertices except the apex is the same as the bow graph and the apex is adjacent to all vertices in $BF_{n,n}$. Note that the number of vertices of the butterfly graph is $2n+3$. Then, by Theorem 2.2.19, $CD[BF_{n,n}, x] = 2x^{2n+1}+4x^{2n}+(2n-4)x^{2n-1}+1$. This completes the proof. □

Theorem 2.2.21. *For a web graph Wb_n, where $n \geq 3$, we have the following*

$$CD[Wb_n, x] = nx^{3n-2} + nx^{3n-4} + nx^{3n-5}.$$

Proof. Let $v_1, \ldots, v_n, u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n$ be the vertices of web graph Wb_n (where $v_1, \ldots, v_n$ are the vertices of the inner cycle, $u_1, u_2, \ldots, u_n$ are the vertices of the outer cycle, and $w_1, w_2, \ldots, w_n$ are the pendant vertices of the web graph). Note that any v_i adjacent to v_{i-1}, v_{i+1}, and u_i, that is, the degree of v_i is 3 in Wb_n. Therefore, the degree of v_i, $i = 1, 2, \ldots, n$ is $3n - 4$ in $\overline{Wb_n}$. Since any u_i is adjacent to u_{i-1}, u_{i+1}, v_i and w_i, that is, the degree of u_i is 4 in Wb_n, it follows that the degree of u_i, $i = 1, 2, \ldots, n$ is $3n - 5$ in $\overline{Wb_n}$. Sine the vertices $w_1, w_2, \ldots, w_n$ are the pendant vertices; therefore the degree of $w_1, w_2, \ldots, w_n$ is $3n - 2$ in $\overline{Wb_n}$. Thus $CD[Wb_n, x] = nx^{3n-2} + nx^{3n-4} + nx^{3n-5}$. This completes the proof. □

Theorem 2.2.22. *For a book graph Bk_n, where $n \geq 1$, we have the following*

$$CD[Bk_n, x] = 2nx^{2n-1} + 2x^n.$$

Proof. Note that book graph Bk_n is the Cartesian product of the star graph $K_{1,n}$ and path graph P_2 and let $v_0, v_1, \ldots, v_n, u_0, u_1, u_2, \ldots, u_n$ be the vertices of the book graph Bk_n (where $v_0, v_1, \ldots, v_n$ are the vertices of one star graph with v_0 as a central vertex and $u_0, u_1, u_2, \ldots, u_n$ are the vertices of another star graph with u_0 as the central vertex). Since v_i is adjacent to u_i and v_0, $i = 1, 2, \ldots, n$ in Bk_n, then the degree of v_i is 2 in Bk_n. Therefore, the degree of v_i is $2n - 1$ in $\overline{Bk_n}$. Similarly, the degree of u_i, $i = 1, 2, \ldots, n$, is $2n - 1$ in $\overline{Bk_n}$. The central vertex

v_0 is adjacent to $v_1, v_2, \ldots, v_n$ and u_0 in Bk_n, that is, the degree of v_0 is $n+1$ in Bk_n. Thus the degree of v_0 is $2n+1-n-1=n$ in $\overline{Bk_n}$. Similarly, the degree of u_0 is n in $\overline{Bk_n}$. Therefore, $CD[Bk_n, x] = 2nx^{2n-1} + 2x^n$. This completes the proof. □

Theorem 2.2.23. *For a crown graph Cr_n, where $n \geq 2$, we have the following*

$$CD[Cr_n, x] = 2nx^n.$$

Proof. Observe that the crown graph Cr_n is $(n-1)$- regular. Then by Theorem 2.1.6, $CD[Cr_n, x] = 2nx^{2n-(n-1)-1} = 2nx^n$. This completes the proof. □

Theorem 2.2.24. *For a banana tree graph, where $m \geq 1$ and $n \geq 2$, we have the following*

$$CD[BT_{m,n}, x] = x^{mn}(x^{-m} + mx^{1-n} + mx^{-2} + m(n-2)x^{-1}).$$

Proof. Since degree of the root vertex in the banana tree graph $BT_{m,n}$ is m, the degree of this vertex in $\overline{BT_{m,n}}$ is $mn-m$. Similarly, the degree of the central vertex of each star graph is $n-1$ in $BT_{m,n}$, the degree of these m vertices is $mn-1+n$ in $\overline{BT_{m,n}}$. Also, the degree of each neighbor of the root vertex is 2 in $BT_{m,n}$, it follows that the degree of these m vertices is $mn-2$ in $\overline{BT_{m,n}}$. The remaining leaf of each star graph is adjacent to $mn-1$ in $\overline{BT_{m,n}}$. Therefore, $CD[BT_{m,n}, x] = x^{mn}(x^{-m} + mx^{1-n} + mx^{-2} + m(n-2)x^{-1})$. This completes the proof. □

Theorem 2.2.25. *For a sun flower graph Sf_n, where $n \geq 3$, we have the fol-*

lowing

$$CD[Sf_n, x] = nx^{2n-2} + nx^{2n-5} + x^n.$$

Proof. Let $w, v_1, v_2 \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of the sun flower graph Sf_n, where w is the central vertex, $v_1, v_2, \ldots, v_n$ are the vertices of the inner ring of Sf_n, and $u_1, u_2, \ldots,$
u_n are the vertices of the outer ring of Sf_n. Since w is adjacent to the vertices $v_1, v_2, \ldots, v_n$ in Sf_n, then w is adjacent to the n vertices $u_1, u_2, \ldots, u_n$. Also, each v_i is adjacent to w, v_{i-1}, v_{i+1}, u_i and u_{i+1} in Sf_n, v_i is adjacent to $n-2$ vertices $u_1, u_2, \ldots, u_{i-1}, u_{i+2}, \ldots, u_n$ and $n-3$ vertices $v_1, v_2, \ldots, v_{i-2}, v_{i+2}, \ldots, v_n$. Thus $deg(v_i) = n-3+n-2 = 2n-5$. Similarly, u_i is adjacent to v_i and v_{i+1} in Sf_n, u_i is adjacent to $n-1$ vertices. $u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n, w$ and $n-2$ vertices $v_1, v_2, \ldots, v_{i-1}, v_{i+2}, \ldots, v_n$ in $\overline{Sf_n}$. Thus $deg(u_i) = n-1+n-2+1 = 2n-2$. Therefore, $CD[Sf_n, x] = nx^{2n-2} + nx^{2n-5} + x^n$. This completes the proof. □

Theorem 2.2.26. *For a triangular snake graph TS_n, where $n \geq 5$, we have the following*

$$CD[TS_n, x] = \frac{(n+3)}{2}x^{n-3} + \frac{(n-3)}{2}x^{n-5}.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the triangular snake graph TS_n. We have the degrees of $v_3, v_5, \ldots, v_{n-2}$ is 4 in TS_n. Thus the degree of these $\frac{n-3}{2}$ vertices is $n-5$ in $\overline{TS_n}$. Similarly, the degree of $v_1, v_2, v_4, \ldots, v_{n-1}, v_n$ is 2 in TS_n. Then the degree of these $\frac{n+3}{2}$ vertices is $n-3$ in $\overline{TS_n}$. Therefore, $CD[TS_n, x] = \frac{(n+3)}{2}x^{n-3} + \frac{(n-3)}{2}x^{n-5}$. This completes the proof. □

Theorem 2.2.27. *For a quadrilateral snake graph Q_n, where $n \geq 3$, we have* *the following*

$$CD[Q_n, x] = (n-2)x^{3n-7} + 2nx^{3n-5}.$$

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}$ be the vertices of the quadrilateral snake graph Q_n, where $v_1, u_1, w_1, v_2, u_2, w_2, \ldots, u_{n-1}, w_{n-1}, v_n$ is the largest path in Q_n and v_i and v_{i+1} is adjacent. Note that the degrees of $v_2, v_3, \ldots, v_{n-1}$ are 4 in Q_n. Then the degree of these $n-2$ vertices is $3n-2-1-4 = 3n-7$ in $\overline{Q_n}$. Similarly, the degree of $v_1, v_n, u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}$ is 2 in Q_n. Then the degree of these $2n$ vertices is $3n-2-1-2 = 3n-5$ in $\overline{Q_n}$. Therefore, $CD[Q_n, x] = (n-2)x^{3n-7} + 2nx^{3n-5}$. This completes the proof. □

Theorem 2.2.28. *For an alternate quadrilateral graph $A(Q_n)$, where $n \geq 3$, we have the following*

$$CD[A(Q_n), x] = \begin{cases} (n-2)x^{2n-5} + nx^{2n-4} + x^{2n-3}, & if\ n\ is\ odd \\ (n-2)x^{2n-4} + (n+2)x^{2n-3}, & if\ n\ is\ even. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_k$ (where $k = n-1$ if n is odd or $k = n$ if n is even) be the vertices of alternate quadrilateral graph $A(Q_n)$ by joining v_i and v_{i+1} (alternatively) to new vertices u_i and u_{i+1}, respectively, and joining the vertices u_i and u_{i+1}. We consider the following cases.

Case(i) If n is odd: Note that the degree of $v_2, v_3, \ldots, v_{n-1}$ is 3 in $A(Q_n)$. Then the degree of these $n-2$ vertices is $2n-1-1-3 = 2n-5$ in $\overline{A(Q_n)}$. Similarly, the degree of $v_1, u_1, u_2, \ldots, u_{n-1}$ is 2 in $A(Q_n)$, and the degree of these n vertices $v_1, u_1, u_2, \ldots, u_{n-1}$ is $2n-1-1-2 = 2n-4$ in $\overline{A(Q_n)}$. If n

is odd, then $A(Q_n)$ has a pendant vertex v_n. The $deg(v_n) = 2n-1-1-1 = 2n-3$. Therefore, $CD[A(Q_n), x] = (n-2)x^{2n-5} + nx^{2n-4} + x^{2n-3}$.

Case(ii) If n is even: Note that $A(Q_n)$ has no pendant vertex. The vertices $v_1, v_n, u_1, u_2, \ldots, u_n$ have the degree 2 in $A(Q_n)$. Then the degree of $v_1, v_n, u_1, u_2, \ldots, u_n$ is $2n-3$ in $\overline{A(Q_n)}$. Similarly, the degree of $v_2, v_3, \ldots, v_{n-1}$ is 3 in $A(Q_n)$. Then the degree of these $n-2$ vertices is $2n-4$ in $\overline{A(Q_n)}$. Therefore, $CD[A(Q_n), x] = (n-2)x^{2n-4} + (n+2)x^{2n-3}$.

This completes the proof. □

Corollary 2.2.29. *For a double quadrilateral snake graph $D(Q_n)$, where $n \geq 3$, we have the following*

$$CD[D(Q_n), x] = (n-2)x^{5n-11} + 2x^{5n-8} + (4n-4)x^{5n-7}.$$

Proof. Note that $D(Q_n)$ has $5n-4$ vertices. Of these $n-2$ vertices have degree 6, $4n-4$ vertices have degree 2 and 2 vertices have degree 3 in $D(Q_n)$. Then the $n-2$ vertices have degree $5n-11$, $4n-4$ vertices are degree $5n-7$ and the remaining 2 vertices have degree $5n-8$ vertices in $\overline{A(Q_n)}$. Therefore, $CD[D(Q_n), x] = (n-2)x^{5n-11} + 2x^{5n-8} + (4n-4)x^{5n-7}$. This completes the proof. □

Corollary 2.2.30. *For an alternate double quadrilateral snake graph $A(D(Q_n))$, where $n \geq 3$, we have the following*

$$CD[A(D(Q_n)), x] = \begin{cases} (n-2)x^{3n-7} + x^{3n-6} + 2(n-1)x^{3n-5} & \\ +x^{3n-4}, & if\ n\ is\ odd \\ (n-2)x^{3n-5} + 2x^{3n-4} + 2nx^{3n-3}, & if\ n\ is\ even. \end{cases}$$

Proof. Note that in $A(D(Q_n))$, there are $3n$ vertices if n is an even number and there are $3n-2$ vertices if n is an odd number. The rest of the proof is the same as the proof of Theorem 2.2.28. □

Theorem 2.2.31. *For an alternate triangular snake graph $A(TS_n)$, where $n \geq 3$, we have the following*

$$CD[A(TS_n), x] = \begin{cases} (n-2)x^{\frac{3n-9}{2}} + \frac{n+1}{2}x^{3n-7} + x^{\frac{3n-5}{2}}, & if\ n\ is\ odd \\ (n-2)x^{\frac{3n-8}{2}} + (\frac{n+4}{2})x^{\frac{3n-6}{2}}, & if\ n\ is\ even. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_k$ (where $k = \frac{n-1}{2}$ if n is odd or $k = \frac{n}{2}$ if n is even) be the vertices of $A(TS_n)$ joining v_i, v_{i+1} to $u_{\frac{i+1}{2}}$. We consider the following cases.

Case(i) If n is odd: Note that $A(TS_n)$ has $n + \frac{n-1}{2} = \frac{3n-1}{2}$ vertices. The $n-2$ vertices $v_2, v_3, \ldots, v_{n-1}$ have degree 3, $\frac{n-1}{2} + 1 = \frac{n+1}{2}$ vertices $v_1, u_1, u_2, \ldots,$ $u_{\frac{n-1}{2}}$ has degree 2 and the pendant vertex v_n has degree 1 in $A(TS_n)$. Then these $n-2$ vertices $v_2, v_3, \ldots, v_{n-1}$ have degree $\frac{3n-1}{2} - 1 - 3 = \frac{3n-9}{2}$ and $\frac{n+1}{2}$ vertices $v_1, u_1, u_2, \ldots,$ $u_{\frac{n-1}{2}}$ have degree $\frac{3n-1}{2} - 1 - 2 = \frac{3n-7}{2}$, and the degree of v_n is $\frac{3n-5}{2}$ in $\overline{A(TS_n)}$. Therefore, $CD[A(TS_n), x] = (n-2)x^{\frac{3n-9}{2}} + \frac{n+1}{2}x^{3n-7} + x^{\frac{3n-5}{2}}$.

Case(ii) If n is even: Note that $A(TS_n)$ has $n + \frac{n}{2} = \frac{3n}{2}$ vertices, and $A(TS_n)$ has no pendant vertices. Thus, as in case (i), the degree of $n-2$ vertices is $\frac{3n-8}{2}$, and that of $\frac{n+4}{2}$ is $\frac{3n-6}{2}$ in $\overline{A(TS_n)}$. Therefore, $CD[A(TS_n), x] = (n-2)x^{\frac{3n-8}{2}} + (\frac{n+4}{2})x^{\frac{3n-6}{2}}$.

This completes the proof. □

Corollary 2.2.32. *For a double triangular snake graph $D(TS_n)$, where $n \geq 3$, we have the following*

$$CD[D(TS_n), x] = (n-2)x^{3n-9} + 2x^{3n-6} + 2(n-1)x^{3n-5}.$$

Theorem 2.2.33. *For a ladder graph L_n, where $n \geq 2$, we have the following*

$$CD[L_n, x] = 4x^{2n-3} + (2n-4)x^{2n-4}.$$

Proof. Let $v_1, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of the ladder graph L_n (where $v_1, \ldots, v_n$ are the vertices of one path graph P_n, $u_1, u_2, \ldots, u_n$ are the vertices of another path graph P_n and v_i is adjacent to u_i, $i = 1, 2, \ldots, n$). Note that v_1 is adjacent to v_2 and u_1 in L_n. Therefore, v_1 is adjacent to the other $2n-3$ vertices $v_3, v_4, \ldots, v_n, u_2, u_3, \ldots, u_n$ in $\overline{L_n}$. Similarly, u_1, v_n and u_n are adjacent to $2n-3$ vertices in $\overline{L_n}$. Also we have v_2 is adjacent to v_1, v_3, and u_2 in L_n. Therefore, v_2 is adjacent to the other $2n-4$ vertices in $\overline{L_n}$. Similarly, the vertices $v_3, v_4, \ldots, v_{n-1}, u_2, u_3, \ldots u_{n-1}$ are adjacent to $2n-4$ vertices in $\overline{L_n}$. Thus $CD[L_n, x] = 4x^{2n-3} + (2n-4)x^{2n-4}$. This completes the proof. □

Theorem 2.2.34. *For a circular ladder graph CL_n, where $n \geq 3$, we have the following*

$$CD[CL_n, x] = 2nx^{2n-4}.$$

Proof. Note that CL_n is a 3-regular graph. Hence the result follows from Theorem 2.1.6. □

Theorem 2.2.35. *For a Mobius ladder graph ML_n, where $n \geq 3$, we have the following $CD[ML_n, x] = 2nx^{2n-4}$.*

Proof. Note that ML_n is a 3-regular graph. The result follows from Theorem 2.1.6 . □

Remark 2.2.36. *The circular ladder graph and the Mobius ladder graph have the same complement degree polynomial.*

Theorem 2.2.37. *For a triangular ladder graph TL_n, where $n \geq 2$, we have the following*

$$CD[TL_n, x] = 2x^{2n-3} + 2x^{2n-4} + (2n-4)x^{2n-5}.$$

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of TL_n . In the triangular ladder graph, there are four corner vertices. The two of the opposite corner vertices have degree 2 in TL_n, and then the degree of these two vertices is $2n-3$ in $\overline{TL_n}$. The other corner vertices have degree 3 because the vertices u_i and v_{i+1} are joined by an edge in TL_n, then the degree of these two vertices is $2n-4$ in $\overline{TL_n}$. The remaining inner vertices have degree four in TL_n, the degree of these $2n-4$ inner vertices is $2n-5$ in $\overline{TL_n}$. Hence $CD[TL_n, x] = 2x^{2n-3} + 2x^{2n-4} + (2n-4)x^{2n-5}$. This completes the proof. □

Theorem 2.2.38. *For a diagonal ladder graph DL_n, where $n \geq 2$, we have the following*

$$CD[DL_n, x] = 4x^{2n-4} + (2n-4)x^{2n-6}.$$

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of DL_n. The triangular ladder graph has four corner vertices, each of degree 3 in DL_n. Thus the degree of these 4 corner vertices is $2n-4$ in $\overline{DL_n}$. The remaining vertices have degree 5 in DL_n, the degree of these $2n-4$ vertices is $2n-6$. Therefore, $CD[DL_n, x] = 4x^{2n-4} + (2n-4)x^{2n-6}$. This completes the proof. □

Theorem 2.2.39. *For a step ladder graph SL_n, where $n \geq 2$, we have the following*

$$CD[SL_n, x] = (n+2)x^{p-3} + (2n-4)x^{p-4} + \left(\frac{n^2-3n+2}{2}\right)x^{p-5}$$

where $p = \frac{n^2+3n-2}{2}$.

Proof. Note that the graph SL_n has $n+2$ corner vertices, each of degree 2; then the degree of these $n+2$ vertices is $p-3$ in $\overline{SL_n}$. The vertices in boundary lines have degree 3 except the corner vertices in SL_n; the degree of these $2n-4$ vertices is $p-4$ in $\overline{SL_n}$. Since remaining inner vertices have degree 4 in SL_n, the degree of these $\frac{n^2-3n+2}{2}$ vertices is $p-5$ in $\overline{SL_n}$. Hence the result follows. □

Theorem 2.2.40. *For a double sided step ladder graph DSL_n, where $n \geq 1$, we have the following*

$$CD[DSL_n, x] = (2n+2)x^{n^2+3n-3} + (2n-2)x^{n^2+3n-4} + (n^2-n)x^{n^2+3n-5}.$$

Proof. The graph DSL_n has $2n+2$ corner vertices, and each of them has degree 2. Hence the degree of these $2n+2$ vertices is n^2+3n-3 in $\overline{DSL_n}$. The total number of vertices in the base of the graph, except the corner vertices, is $2n-2$. Note taht each of which has degree 3 in DSL_n. Thus the degree of these $2n-2$ vertices is n^2+3n-4 in $\overline{DSL_n}$. Since remaining n^2-n vertices have degree 4 in DSL_n, the degree of these vertices is n^2+3n-5 in $\overline{DSL_n}$, which gives the result. □

Theorem 2.2.41. *For a bipartite cocktail party graph B_n, where $n \geq 2$, we have* *the following* $CD[B_n, x] = 2nx^n$.

Proof. Note that $\overline{B_n}$ is the graph obtained by joining perfect match to the two disconnected graph K_n. The degree of each $2n$ vertex is $n - 1 + 1 = n$ in $\overline{B_n}$. Thus $CD[B_n, x] = 2nx^n$. □

Theorem 2.2.42. *For a cocktail party graph* CP_n*, where* $n \geq 2$*, we have the following*

$$CD[CP_n, x] = 4x^2 + 2(n-2)x^3.$$

Proof. Observe that $\overline{CP_n} = L_n$. Therefore, $CD[CP_n, x] = V(L_n, x) = 4x^2 + 2(n-2)x^3$. This completes the proof. □

2.3 Complement degree polynomial of some graph operations

Theorem 2.3.1. *Let* G *be a graph with order* n *and* $\boldsymbol{G} = G \cup G \cup \ldots \cup G$ *(*m *times). Then* $CD[\boldsymbol{G}, x] = mx^{(m-1)n}CD[G, x]$.

Proof. Observe that the order of G is mn. Let $v \in V(G)$ and the degree of v is d, then $deg(v) = n - 1 - d$ in $\overline{G}$. Since v is adjacent to each of the n vertices of $(m-1)$ copies of $\overline{G}$ in $\overline{\mathbf{G}}$ and v adjacent to the vertices $V - N(v)$ in $\overline{G}$, $deg(v) = n - 1 - d + (m-1)n$. Since v is an arbitrary vertex, it follows that $CD[\mathbf{G}, x] = mx^{(m-1)n}CD[G, x]$. This completes the proof. □

Theorem 2.3.2. *Let* G *be a graph with order* n *and* $H = G+G+\ldots+G$*(*m *times) . Then* $CD[H, x] = mCD[G, x]$.

Proof. Note that each vertex of H is adjacent to all vertices of $(m-1)$ copies of G. Therefore, each vertex $v \in V(\overline{mG})$ is adjacent to all vertices $V - N(v)$. It follows that for each i, $Cd_i(H) = mCd_i(G)$. Hence $CD[H, x] = mCD[G, x]$. This completes the proof. □

Theorem 2.3.3. *Let G be a graph of order n and H be a graph of order m, then*

$$CD[G \circ H, x] = x^{(n-1)m} CD[G, x] + n x^{(n-1)(m+1)} CD[H, x].$$

Proof. Let G be a graph of order n and H be a graph of order m. Note that each vertex of G is adjacent to m vertices of one copy of H. Let $v \in V(G)$, then v is adjacent to $V(G) - N(v)$ vertices in $\overline{G}$ and $(n-1)m$ vertices of $(n-1)$ copy of H in $\overline{G \circ H}$. Therefore, $deg(v) = deg(v)$ in $\overline{G} + (n-1)m$ in $\overline{G \circ H}$. Let u be a vertex in the i^{th} copy of H, then u is adjacent to $N(u)$ vertices in H and the i^{th} vertex in G. Therefore, u is adjacent to $V(H) - N(u)$ vertices in the i^{th} copy of H, m vertices of $n-1$ copies of H and $n-1$ vertices of G. That is,

$$\begin{aligned} deg(u) &= (deg(u) in \overline{H}) + m(n-1) + (n-1) \\ &= (deg(u) in \overline{H}) + (n-1)(m+1). \end{aligned}$$

Therefore, $CD[G \circ H, x] = x^{(n-1)m} CD[G, x] + n x^{(n-1)(m+1)} CD[H, x]$. This completes the proof. □

Theorem 2.3.4. *If G be a graph having two components G_1 and G_2 with n and m vertices respectively, then*

$$CD[G, x] = \sum_{i=\delta(\overline{G_1})}^{\Delta(\overline{G_1})} Cd_i(G_1) x^{i+m} + \sum_{i=\delta(\overline{G_2})}^{\Delta(\overline{G_2})} Cd_i(G_2) x^{i+n}.$$

Proof. Let $v \in V(G_1)$ and $u \in V(G_2)$ be the vertices of G with degree d_1 and d_2, respectively. Since v is adjacent to $n-1-d_1$ vertices in $\overline{G_1}$, v is adjacent to $n-1-d_1+m$ vertices in $\overline{G}$. Therefore, $deg(v) = n-1-d_1+m$ in $\overline{G}$. Similarly, $deg(u) = n-1-d_1+n$ in $\overline{G}$. Since v and u are arbitrary vertices in G_1 and G_2, respectively, then the degree of any vertex v in G_1 is ($deg(v)$ in $\overline{G_1}$)+m in $\overline{G}$. Similarly, the degree of any vertex u in G_2 is ($deg(u)$ in $\overline{G_2}$)+n in $\overline{G}$. Hence the result follows. □

Corollary 2.3.5. *Let G be a graph having m components $G_1, G_2, \ldots, G_m$ where $|V(G_i)| = n_i$ for $i = 1, 2, \ldots, m$. Then*

$$CD[G, x] = \sum_{k=1}^{m} \sum_{i=\delta(\overline{G_k})}^{\Delta(\overline{G_k})} Cd_i(G_k)\ x^{\alpha}, \quad where\ \alpha = (i + \sum_{\substack{r=1 \\ r\neq k}}^{m} n_r).$$

.

Proof. The proof follows from Theorem 2.3.5 using mathematical induction on the number of components m of G. □

Theorem 2.3.6. *If G is a graph with n vertices, then the complement degree polynomial of Mycielski graph is given by,*

$$CD[\mu(G), x] = x^2 CD[G, x^2] + x^n CD[G, x] + x^n.$$

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, w$ be the vertices of $\mu(G)$ (where $v_1, v_2, \ldots,$ v_n are the vertices of G, u_i corresponds to each v_i, and w is the vertex is adjacent to each u_i, $i = 1, 2, \ldots, n$). Note that each v_i adjacent to $2|V(G) - N(v_i)|$ vertices, u_i and w in $\overline{\mu(G)}$. That is, if $deg(v_i) = d$ in $\mu(G)$, then $deg(v_i) =$

$2(n-1-d)+2$ in $\overline{\mu(G)}$. Similarly, each u_i is adjacent to $|V(G)-N(v_i)|$ vertices, $u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n$ and v_i. That is $deg(u_i) = (n-1-d)+n-1+1 = n-1-d+n$ in $\overline{\mu(G)}$. Finally, the vertex w adjacent to $v_1, v_2, \ldots, v_n$ in $\overline{\mu(G)}$. That is $deg(w) = n$ in $\overline{\mu(G)}$. Therefore, $CD[\mu(G), x] = x^2 CD[G, x^2] + x^n CD[G, x] + x^n$. This completes the proof. □

Theorem 2.3.7. *If G is a graph with n vertices, the complement degree polynomial of the shadow graph of G is given by $CD[Sh(G), x] = 2xCD[G, x^2]$.*

Proof. Let G_1 and G_2 be the two copies of G in $Sh(G)$, and let $v \in G_1$ and $deg(v) = d$ in G, then v is adjacent to $V(G_1)-N(v), V(G_2)-N(v)$ and $v \in G_2$ in $\overline{Sh(G)}$. That is, the degree of v in $\overline{Sh(G)}$ is $n-1+d+n-1+d+1 = 2(n-1-d)+1$. It follows that $CD[Sh(G), x] = 2xCD[G, x^2]$. This completes the proof. □

Theorem 2.3.8. *The complement degree polynomial of the graph K'_n obtained by the duplication of one of the vertices of the complete graph K_n is given by*

$$CD[K'_n, x] = 2x + (n-1).$$

Proof. Let $v \in V(K_n)$ and $v' \in V(K'_n)$ where v' is the duplicate vertex of v. Then v' is adjacent to all vertices in K_n other than v. Thus v' is adjacent to only the vertex v in $\overline{K'_n}$. Similarly, v is adjacent to only v' in $\overline{K'_n}$. All other $n-1$ vertices are isolated vertices in $\overline{K'_n}$. Therefore, $CD[K'_n, x] = 2x + (n-1)$. This completes the proof. □

Theorem 2.3.9. *If V_1 and V_2 is a bi-partition of the vertex set of $K_{m,n}$ with cardinalities m and n respectively, and v' is the duplication of a vertex v of*

$K_{m,n}$, *then*

$$CD[K'_{m,n}, x] = \begin{cases} (m+1)x^m + nx^{n-1}, & if v \in V_1 \\ mx^{m-1} + (n+1)x^n, & if v \in V_2 \end{cases}.$$

Proof. Let V_1 and V_2 be a bi-partition of the vertex set of $K_{m,n}$ with order m and n, respectively, and let v' be the duplication of a vertex v of $K_{m,n}$. If $v \in V_1$, then $K'_{m,n}$ is another complete bipartite graph $K_{m+1,n}$. Similarly, if $v \in V_2$, then $K'_{m,n}$ is $K_{m,n+1}$. Therefore,

$$CD[K'_{m,n}, x] = \begin{cases} (m+1)x^m + nx^{n-1}, & if v \in V_1 \\ mx^{m-1} + (n+1)x^n, & if v \in V_2 \end{cases}.$$

This completes the proof. □

Theorem 2.3.10. *For a chaplet graph $C_p \bigodot {C_q}^t$, where $p, q, t \geq 3$, we have the following*

$$CD[C_p \bigodot {C_q}^t, x] = px^{p(q-1)t+p-2t-3} + p(q-1)tx^{p(q-1)t+p-3}.$$

Proof. Let $u_1, u_2, \ldots, u_p$ be the vertices of the cycle C_p. For $j \in \{1, 2, \ldots, t\}$ and $k \in \{1, 2, \ldots, p\}$ let $u_k, u^j_{k_1}, u^j_{k_2}, \ldots, u^j_{k_{(q-1)}}$ be the vertices of j^{th} copy of the cycle C_q attached to the vertex u_k of C_p. Note that the degree of any vertex of $C_p \bigodot {C_q}^t$ other than $u_1, u_2, \ldots, u_p$ is 2. That is the degree of these $p(q-1)t$ vertices is $p(q-1)t + p - -3$ in $\overline{C_p \bigodot {C_q}^t}$. But the degree of $u_1, u_2, \ldots, u_p$ is $2t + 2$ in $C_p \bigodot {C_q}^t$. Thus, the degree of $u_1, u_2, \ldots, u_p$ is $p(q-1)t - 3$ in $\overline{C_p \bigodot {C_q}^t}$. Therefore, $CD[C_p \bigodot {C_q}^t, x] = px^{p(q-1)t+p-2t-3} + p(q-1)tx^{p(q-1)t+p-3}$.

This completes the proof. □

Theorem 2.3.11. *For a armed crown graph $C_n \bigodot P_m$, where $n \geq 3$ and $m \geq 1$, we have the following*

$$CD[C_n \bigodot P_m, x] = nx^{n(m+1)-2} + n(m-1)x^{n(m+1)-3} + nx^{n(m+1)-4}.$$

Proof. The degree of end vertices of each n-path graph is 1 in $C_n \bigodot P_m$. Then the degree of these n vertices is $n(m+1)-2$ in $\overline{C_n \bigodot P_m}$. Similarly, the degree of the other $m-1$ vertices of each n-path graph is 2 in $C_n \bigodot P_m$. Thus the degree of these $n(m-1)$ vertices is $n(m+1)-3$ in $\overline{C_n \bigodot P_m}$. The vertices of the cycle graph are adjacent to two vertices of the cycle graph and one vertex of the path graph. That is, the degree of n- vertices of the cycle graph is 3 in $C_n \bigodot P_m$. Thus the degree of these n-vertices is $n(m+1)-4$ in $\overline{C_n \bigodot P_m}$. Therefore, $CD[C_n \bigodot P_m, x] = nx^{n(m+1)-2}+n(m-1)x^{n(m+1)-3}+nx^{n(m+1)-4}$. This completes the proof. □

Results on path graphs

Theorem 2.3.12. *If $M(P_n)$ is the middle graph of path graph P_n, then*

$$CD[M(P_n), x] = \begin{cases} 2x+1, & n = 2 \\ 2x^{2n-2} + (n-2)x^{2n-3} + 2x^{2n-4} + (n-3)x^{2n-5}, & n \geq 3. \end{cases}$$

Proof. If $n = 2$, then $M(P_2)$ is a path graph P_3. Thus $CD[M(P_2), x] = CD[P_3, x]$ $= 2x + 1$. When $n > 2$, since P_n has $n-1$ edges, $M(P_n)$ has $2n-1$ vertices, say $v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}$, where $v_1, v_2, \ldots, v_n$ are the vertices of P_n and $e_1, e_2, \ldots, e_{n-1}$ are the edges of P_n. Note that e_1 is adjacent to e_2, v_1 and v_2.

Similarly, e_{n-1} is adjacent to e_{n-2}, v_{n-1}, and v_n. Thus $deg(e_1) = deg(e_{n-1} = 2n-1-3 = 2n-4$ in $\overline{M(P_n)}$. Note that v_1 and v_n are adjacent to only e_1 and e_{n-1}, respectively. Therefore, $deg(v_1) = deg(v_n) = 2n-2$ in $\overline{M(P_n)}$. The vertex v_i is adjacent to e_{i-1} and e_i, then $deg(v_i) = 2n-3$, $i = 2, 3, \ldots, n-1$ in $\overline{M(P_n)}$. The remaining vertices e_i, $i = 2, 3, \ldots, n-2$ are adjacent to e_{i-1}, e_{i+1}, v_i and v_{i+1}, then $deg(e_i) = 2n-5$, $i = 2, 3, \ldots, n-2$ in $\overline{M(P_n)}$. Hence the result follows. □

Theorem 2.3.13. *If $S(P_n)$ is the splitting graph of path graph P_n, where$n \geq 2$, then*

$$CD[S(P_n), x] = \begin{cases} 2x + 2x^2, & n = 2 \\ 2x^{2n-2} + nx^{2n-3} + (n-2)x^{2n-5}, & n \geq 3. \end{cases}.$$

Proof. Since for $n = 2$, $S(P_2)$ is another path graph P_4, then $CD[S(P_2), x] = CD[P_4, x] = 2x+2x^2$. When $n > 2$, let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of $S(P_n)$, where $v_1, v_2, \ldots, v_n$ are the vertices of P_n and u_i corresponds to v_i in $S(P_n)$, $i = 1, 2, \ldots, n$. Since v_1 and v_n are pendant vertices in P_n, then u_1 and u_n are pendant vertices in $S(P_n)$. Therefore, $deg(u_1) = deg(u_n) = 2n-2$ in $\overline{S(P_n)}$. Note that $deg(v_i) = 2, i = 2, 3, \ldots, n-1$ in P_n, then $deg(u_i) = 2, i = 2, 3, \ldots, n-1$ in $S(P_n)$. Thus $deg(v_i) = 2n-3, i = 2, 3, \ldots, n-1$ in $\overline{S(P_n)}$. Also $deg(v_1) = deg(v_n) = 2$ in $S(P_n)$, then $deg(v_1) = deg(v_n) = 2n-3$ in $\overline{S(P_n)}$. Similarly, the $deg(v_i) = 4$ in $S(P_n)$. Then $deg(v_i) = 2n-5$, $i = 2, 3, \ldots, n-1$ in $\overline{S(P_n)}$. Therefore, $CD[S(P_n), x] = 2x^{2n-2} + nx^{2n-3} + (n-2)x^{2n-5}$. This completes the proof. □

Theorem 2.3.14. *If G_n be a graph of order n, then $CD[S(G_n), x]$ do not have a constant term.*

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of $S(G_n)$, where $v_1, v_2, \ldots,$ v_n are the vertices of G_n and u_i corresponds to v_i in $S(G_n)$, $i = 1, 2, \ldots, n$. Note that u_i is not adjacent to v_i, $i = 1, 2, \ldots, n$ in $S(G_n)$. Therefore, $\Delta(G) \leq 2n - 2$. That is, there is no isolated vertex in $\overline{S(G_n)}$. Thus $CD[S(G_n), x]$ does not have a constant term. This completes the proof. □

Theorem 2.3.15. *If $CS(P_n)$ is the cosplitting graph of path graph P_n, then*

$$CD[CS(P_n), x] = \begin{cases} 2x + 2x^2, & n = 2 \\ nx^{n-1} + 2x^n + (n-2)x^{n+1}, & n \geq 3. \end{cases}.$$

Proof. Since for $n = 2$, $CS(P_2)$ is another path graph P_4, then $CD[CS(P_2), x] = CD[P_4, x] = 2x + 2x^2$. When $n > 2$, let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of $CS(P_n)$, where $v_1, v_2, \ldots, v_n$ are the vertices of P_n and u_i corresponds to v_i in $CS(P_n)$, $i = 1, 2, \ldots, n$. Note that $deg(v_i) = n$ in $CS(P_n)$, $i = 1, 2, \ldots, n$. Therefore, $deg(v_i) = n - 1$ in $\overline{CS(P_n)}$, $i = 1, 2, \ldots, n$. The vertex u_1 is adjacent to each u_i, $i = 2, \ldots, n$ and v_2 in $\overline{CS(P_n)}$. Thus $deg(u_1) = n - 1 + 1 = n$ in $\overline{CS(P_n)}$. Similarly, the degree of u_n is n in $\overline{CS(P_n)}$. The remaining vertices u_i, $i = 2, 3, \ldots, n-1$ are adjacent to each u_j, $j = 1, \ldots, i-1, i+1, \ldots, n$, v_{i-1} and v_{i+1}. Therefore, $deg(u_i) = n - 1 + 2 = n + 1$, $i = 2, 3, \ldots, n-1$ in $\overline{CS(P_n)}$. Hence for $n > 2$, $CD[CS(P_n), x] = nx^{n-1} + 2x^n + (n-2)x^{n+1}$. This completes the proof. □

Theorem 2.3.16. *If $d(P_n)$ is the derivative of path graph P_n, where $n \geq 5$, then $CD[d(P_n), x] = CD[P_{n-2}, x]$.*

Proof. Note that the derivative of a path graph P_n is again a path graph with

$n-2$ vertices. Thus $CD[d(P_n), x] = CD[P_{n-2}, x]$. This completes the proof. □

Theorem 2.3.17. *If G has no pendant vertices , then $CD[G, x] = CD[d(G), x]$.*

Proof. Let G be a graph, and if G has no pendant vertices, then $G \cong d(G)$. Therefore, $CD[G, x] = CD[d(G), x]$. This completes the proof. □

Theorem 2.3.18. *The complement degree polynomial of the graph P'_n obtained by the duplication of the pendant vertex of P_n $(n \geq 3)$ is given by*

$$CD[P'_n, x] = x^{n-3} + (n-3)x^{n-2} + 3x^{n-1}.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and v' be the duplication of v_1; then v_1 is a pendant vertex. That is, v_1, v_2 and v' are pendant vertices of P'_n. Thus the degree of these 3 vertices is $n-1$ in $\overline{P'_n}$. Since v_1, v_3 and v' are adjacent to v_2, then $deg(v_2) = 3$ in P'_n. Thus $deg(v_2) = n-3$ in $\overline{P'_n}$. The other $n-3$ inner vertices of P_n have degree 2 in P'_n. This implies that the degree of those $n-3$ vertices is $n-2$ in $\overline{P'_n}$. Therefore, $CD[P'_n, x] = x^{n-3} + (n-3)x^{n-2} + 3x^{n-1}$. This completes the proof. □

Theorem 2.3.19. *The complement degree polynomial of the graph P'_n obtained by the duplication of the vertex of P_n $(n \geq 4)$ which is not a pendant vertex but the neighbor of a pendant vertex is given by*

$$CD[P'_n, x] = x^{n-3} + (n-1)x^{n-2} + x^{n-1}.$$

Proof. Let v be the pendant vertex of P_n, $u \in N(v)$, and let v' be the duplication of u. Then $P'_n \cong T_{3,n-2}$. By Theorem 2.2.9, we have $CD[T_{m,n}, x] = x^{m+n-2} +$

$(m+n-2)x^{m+n-3} + x^{m+n-4}$. Thus

$$\begin{aligned} CD[P_n', x] &= x^{3+n-2-2} + (3+n-2-2)x^{3+n-2-3} + x^{3+n-2-4} \\ &= x^{n-1} + (n-1)x^{n-2} + x^{n-3}. \end{aligned}$$

This completes the proof. □

Theorem 2.3.20. *The complement degree polynomial of the graph P_n' $(n \geq 5)$obtained by the duplication of the inner vertex of P_n which is not a neighbor of the pendant vertex is given by*

$$CD[P_n', x] = 2x^{n-3} + (n-3)x^{n-2} + 2x^{n-1}.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and v' be the duplication of the inner vertex $v_i, i = 3, 4, \ldots, n-2$. Then $deg(v_{i-1}) = deg(v_{i+1}) = 3$ in P_n'. Thus $deg(v_{i-1}) = deg(v_{i+1}) = n-3$ in $\overline{P_n'}$. Note that $deg(v_1) = deg(v_n) = n-1$ in $\overline{P_n'}$ and the degree of the other $n-3$ vertices is $n-2$ in $\overline{P_n'}$, including v' other than $v_1, v_n, v_{i-1}, v_{i+1}$. Therefore, $CD[P_n', x] = 2x^{n-3} + (n-3)x^{n-2} + 2x^{n-1}$. This completes the proof. □

Theorem 2.3.21. *If $L(P_n)$ is the line graph of path graph P_n, where $n \geq 3$, then*

$$CD[L(P_n), x] = \begin{cases} 2x^{n-3}, & n = 3 \\ (n-3)x^{n-4} + 2x^{n-3}, & n \geq 4. \end{cases}$$

.

Proof. Since the line graph of P_n is another path graph P_{n-1}. Therefore,

$CD[L(P_n), x] = CD[P_{n-1}, x]$. This completes the proof. □

Theorem 2.3.22. *If $T(P_n)$ is the total graph of path graph P_n, then*

$$CD[T(P_n), x] = \begin{cases} 3, & n = 2 \\ 2x^{2n-4} + 2x^{2n-5} + (2n-5)x^{2n-6}, & n \geq 3. \end{cases}$$

Proof. Since $T(P_2)$ is C_3, we have $CD[T(P_2), x] = CD[C_3, x] = 3$. Assume that $n \geq 3$. Let $v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}$ be the vertices of $T(P_n)$ where $v_1, v_2, \ldots, v_n$ are the vertices of P_n and $e_1, e_2, \ldots, e_{n-1}$ are the edges of P_n. Since v_1 is adjacent to v_2 and e_1 in $T(P_n)$, $deg(v_n) = 2n - 4$ in $\overline{T(P_n)}$. Similarly, $deg(v_n) = 2n - 4$ in $\overline{T(P_n)}$. Note that e_1 is adjacent to v_1, v_2 and e_2, then $deg(e_1) = 2n - 5$ in $\overline{T(P_n)}$. Similarly, $deg(e_{n-1}) = 2n - 5$ in $\overline{T(P_n)}$. The degree of remaining vertices $v_2, \ldots, v_{n-1}, e_2, \ldots, e_{n-2}$ is 4 in $T(P_n)$. Thus the degree of these $2n - 5$ vertices is $2n - 6$ in $\overline{T(P_n)}$. Therefore, for $n \geq 3$, $CD[T(P_n), x] = 2x^{2n-4} + 2x^{2n-5} + (2n-5)x^{2n-6}$. This completes the proof. □

Theorem 2.3.23. *If $C(P_n)$ is the central graph of P_n, where $n \geq 2$, then*

$$CD[C(P_n), x] = (n-1)x^{2n-3} + nx^{n-1}.$$

Proof. Let $v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}$ be the vertices of $C(P_n)$, where $v_1, v_2, \ldots,$ v_n are the vertices of P_n and $e_1, e_2, \ldots, e_{n-1}$ are the new vertex for subdividing each edge. Since the degree of each new vertices e_i, $i = 1, 2, \ldots, n-1$ is 2 in $C(P_n)$, $deg(e_i) = 2n-4$ in $\overline{C(P_n)}$. Note that v_1 is adjacent to $v_3, v_4, \ldots, v_n$ and e_1 in $C(P_n)$, that is, $deg(v_1) = n-1$ in $C(P_n)$. Then $deg(v_1) = 2n-1-1-(n-1) = n-1$ in $\overline{C(P_n)}$. Similarly, $deg(v_n) = n-1$ in $\overline{C(P_n)}$. But the remaining vertices

v_i, $i = 2, 3, \ldots, n-1$ are adjacent to $v_1, \ldots, v_{i-2}, v_{i+2}, \ldots, v_n, e_{i-1}$ and e_i, that is, $deg(v_i) = n-3+2 = n-1$ in $C(P_n)$. Therefore, $deg(v_i) = n-1$, $i = 2, 3, \ldots, n-1$ in $\overline{C(P_n)}$. Hence

$$\begin{aligned} CD[C(P_n), x] &= (n-1)x^{2n-3} + 2x^{n-1} + (n-2)x^{n-1} \\ &= (n-1)x^{2n-3} + nx^{n-1}. \end{aligned}$$

This completes the proof. □

Results on regular graphs

Theorem 2.3.24. *Let G be the r-regular graph of order n, then complement degree polynomial splitting graph of G is*

$$CD[S(G), x] = nx^{(2n-1-r)} + nx^{(2n-1-2r)}.$$

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of the splitting graph $S(G)$ of the regular graph G where $v_1, v_2, \ldots, v_n$ are the vertices of G and $u_1, u_2, \ldots, u_n$ are the corresponding vertices . Note that u_i, $i = 1, 2, \ldots, n$, is adjacent to $n-r$ vertices in $\{v_1, v_2, \ldots, v_n\}$ and $n-1$ vertices $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n$ in $\overline{S(G)}$. That is, $deg(u_i) = n-1+n-r = 2n-1-r$ in $\overline{S(G)}$. Note that the degree of v_i, $i = 1, 2, \ldots, n$ in $S(G)$ is twice the degree in G. Therefore, the degree of each v_i is $2n-1-2r$ $\overline{S(G)}$. Hence $CD[S(G), x] = nx^{(2n-1-r)} + nx^{(2n-1-2r)}$. This completes the proof. □

Theorem 2.3.25. *Let G be the r-regular graph of order n, then complement* *degree polynomial of cosplitting graph of G is*

$$CD[CS(G),x] = nx^{n-1}(1+x^r).$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the regular graph G and $w_1, w_2, \ldots, w_n$ be the new vertices of $CS(G)$. Let $v_i \in V(G)$, then $deg(v_i) = n$ in $CS(G)$, and then $deg(v) = n-1$ in $\overline{CS(G)}$. Since each w_i is adjacent to $n-r$ vertices in $CS(G)$, w_i is adjacent to r vertices in $V(G)$ and $n-1$ vertices $w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n$ in $\overline{CS(G)}$. That is $deg(w_i) = n-1+r$ in $\overline{CS(G)}$. Therefore, $CD[CS(G),x] = nx^{n-1} + nx^{n-1+r} = nx^{n-1}(1+x^r)$.. This completes the proof. □

Theorem 2.3.26. *If G is a r-regular graph of order n and G' is a graph obtained by duplication of a vertex of G, then*

$$CD[G',x] = (n+1-r)x^{n-r} + rx^{n-r-1}.$$

Proof. Let G be the r- regular graph of order n, and let v' be the duplication of a vertex v in G. Note that v' is adjacent to r vertices in G', then v' is adjacent to $n-r$ vertices in $\overline{G'}$. Similarly, the vertices in $V-N(v)$ in G are adjacent to $n-r$ vertices in $\overline{G'}$. The vertices in $N(v)$ in G are adjacent to r vertices in G and v'. That is, the degree of vertices in $N(v)$ in G are $r+1$ in G'. Then the degree of that r vertices is $n-r-1$ in $\overline{G'}$. Therefore, $CD[G',x] = (n+1-r)x^{n-r} + rx^{n-r-1}$. This completes the proof. □

2.4 Complement degree polynomial of some chemical graphs

Alkanes

Alkanes are organic compounds that consist of single- bonded carbon and hydrogen atoms. The formula for alkanes is C_nH_{2n+2} [3].

Theorem 2.4.1. *If* $n \geq 1$, *then* $CD[C_nH_{2n+2}, x] = nx^{3n-3} + (2n+2)x^{3n}$.

Proof. Let $C_1, C_2, \ldots, C_n$ and $H_1, H_2, \ldots, H_{2n+2}$ be the vertices of the C_nH_{2n+2} graph, where C_i and H_i represent carbon and hydrogen atoms, respectively. Note that all C_i has degree 4, then C_i have degree $3n+2-5 = 3n-3$ in $\overline{C_nH_{2n+2}}$. Similarly, if all H_i is a pendant vertex, then H_i has degree $3n+2-2 = 3n$ in $\overline{C_nH_{2n+2}}$. Thus $CD[C_nH_{2n+2}, x] = nx^{3n-3} + (2n+2)x^{3n}$. This completes the proof. □

The complement degree polynomial of first 5 alkanes are provided in the following table.

Names	Molecular Formula	Complement degree Polynomial
Methane	CH_4	$4x^3 + 1$
Ethane	C_2H_6	$2x^3 + 6x^6$
Propane	C_3H_8	$3x^6 + 8x^9$
Butane	C_4H_{10}	$4x^9 + 10x^{12}$
Pentane	C_5H_{12}	$5x^{12} + 12x^{15}$

Hexagonal system

A hexagonal system HS_n is a connected plane graph without cut-vertices in which all inner faces are hexagons (and all hexagons are faces), such that two hexagons are either disjoint or have exactly one common edge, and no three hexagons share a common edge [13].

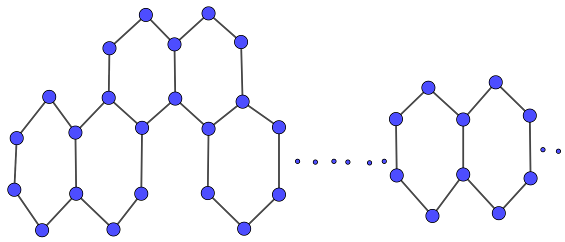

Figure 2.2: Hexagonal System

Theorem 2.4.2. *If* $n \geq 1$, *then*

$$CD[HS_n, x] = \begin{cases} 6x^3, & n = 1; \\ 4x^{4n-1} + 2x^{4n-2}, & n \geq 2. \end{cases}$$

Proof. Observe that HS_n has $4n+2$ vertices where $2n+4$ vertices have degree 2, and $2n-2$ vertices have degree 3 in HS_n. Therefore, $2n+4$ vertices have degree $4n-1$, and $2n-2$ vertices have degree $4n-2$. If $n = 1$, $HS_1 = C_6$, then $CD[F, x] = 6x^3$. Therefore,

$$CD[HS_n, x] = \begin{cases} 6x^3, & n = 1; \\ 4x^{4n-1} + 2x^{4n-2}, & n \geq 2. \end{cases}$$

This completes the proof. □

Star like tree graph

A star like tree graph $S(n_1, n_2, \ldots, n_k)$ is a graph having only one vertex w of degree greater than 2 such that deletion of w results in disjoint union of the path graphs $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$.

Stand A of human insulin

Stand A of human insulin has 21 amino acids of kinds is usually represented by a star like tree graph with branches [4].

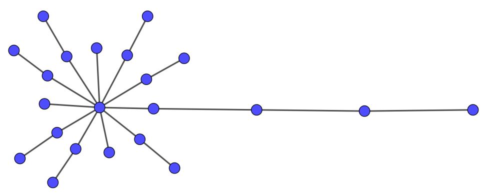

Figure 2.3: Stand A of human insulin graph

Theorem 2.4.3. *Let S be the graphical representation of stand A human insulin, then* $CD[S, x] = 11x^{20} + 10x^{19} + x^{10}$.

Proof. Note that S has 11 leaves; then the degree of these 11 vertices is 20 in $\overline{S}$. The degree of the other 10 vertices other than the central vertex is 2 in S. Thus, the degree of these 10 vertices is 19 in $\overline{S}$. Similarly, if the central vertex is adjacent to 11 vertices in S, then the degree of this vertex is 10 in $\overline{S}$. Hence $CD[S, x] = 11x^{20} + 10x^{19} + x^{10}$. This completes the proof. □

Generalized Hierarchial Product

The generalized hierarchial product of G and G' with $\phi \neq U \subseteq V(G)$ is represented by $G(U) \sqcap G'$. It has vertex set $V(G) \times V(G')$ and $(a_1, b_1)(a_2, b_2) \in E(G(U) \sqcap G')$ if $a_1 = a_2 \in U$ and $b_1 b_2 \in E(G')$ or $b_1 = b_2$ and $a_1 a_2 \in E(G)$ [5].

Linear Phenylene

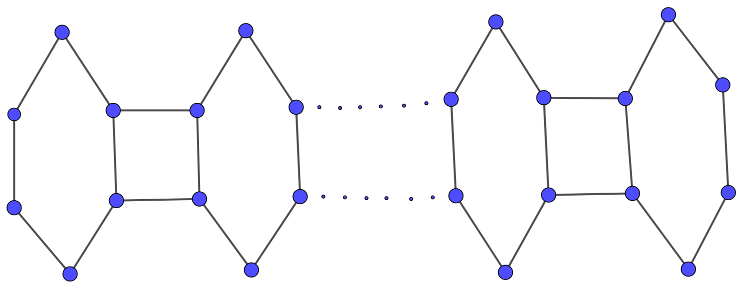

Figure 2.4: Linear Phenylene graph

In organic chemistry, the phenylene group (C_6H_4) is based on a di-substituted benezene ring. The linear phenylene is one whic the phenyles are arranged in a stright line (less than 180 degrees). Graphical structure of linear phenylene LP_{6n} $(n > 1)$ is $LP_{6n} = P_{3n}(U) \sqcap P_2$, where $U = \{a_{3k} : 1 \leq k \leq n\} \cup \{a_{3k+1} : 0 \leq k \leq n-1\}$ [5].

Theorem 2.4.4. *If $n \geq 1$, then $CD[LP_{6n}, x] = (2n+4)x^{6n-3} + (4n-4)x^{6n-4}$.*

Proof. Note that LP_{6n} consists of n cycles C_6. In each $n-2$ cycle, 4 vertices have degree 3, and 2 vertices have degree 2. But in the end cycles, 2 vertices have degree 3 and 4 vertices have degree 2. Therefore, in the case of $\overline{LP_{6n}}$, $2n+4$

vertices have degree $6n-3$ and $4n-4$ vertices have degree $6n-4$. Hence the result follows. □

Dopamine

A dopamine molecule consists of a catechole structure (a benzene ring with two hydroxyle side groups) with one amine group attached via an ethyl chain. The graphical structure of dopamine graph D is in Figure 2.5.

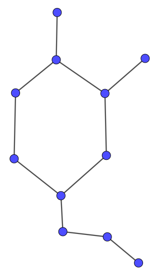

Figure 2.5: Dopamine graph

Theorem 2.4.5. *If D is a dopamine graph, then* $CD[D,x] = 3x^9 + 5x^8 + 3x^7$.

Proof. Note that the dopamine graph has 11 vertices, 5 vertices have degree 2, 3 vertices have degree 3 and 3 vertices are leaves. Then in the case of $\overline{D}$. 5 vertices have degree 8, 3 vertices have degree 9 and 3 vertices have degree 7. Hence the result follows. □

Caffeine

Caffeine is a bitter white crystalline purine, where purine has two cycles: a six-membered pyrimidine ring and a five membered imidazole ring fused together. The graphical structure of caffeine graph C consists 14 vertices.

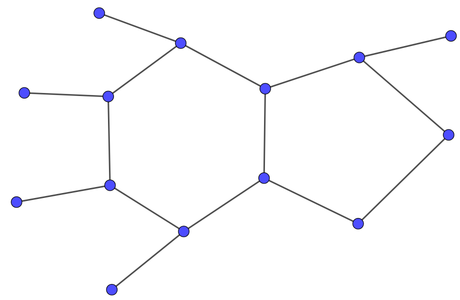

Figure 2.6: Caffeine graph

Theorem 2.4.6. *If C is a caffeine graph, then* $CD[C,x] = 5x^{12} + 2x^{11} + 7x^{10}$.

Proof. Note that the caffeine graph has 14 vertices where 5 vertices are leaves, 2 vertices have degree 2 and 7 vertices have degree 3. Therefore, 5 vertices have degree 12, 2 vertices have degree 11 and 7 vertices have degree 10 in $\overline{C}$. Hence $CD[C,x] = 5x^{12} + 2x^{11} + 7x^{10}$. □

Ribonucleic acid (RNA)

RNA is typically single stranded and is made of ribonucleotides that are linked by phosphodiester bonds. A ribonucleotide in the RNA chain contains ribose (the pentose sugar), one of the four nitrogeneouse bases (A,U,G and C), and a phosphate group. The graphical structure of RNA graph R is in Figure 2.7.

Theorem 2.4.7. *If R is the RNA graph, then* $CD[R,x] = 2x^{40} + 12x^{39} + 28x^{38}$.

Proof. Note that the graph R has only two leaves, 12 vertices have degree 2 and 28 vertices have degree 3. 2 vertices have degree 40, 12 vertices have degree 39 and 28 vertices have degree 38 in $\overline{R}$. Hence the result follows. □

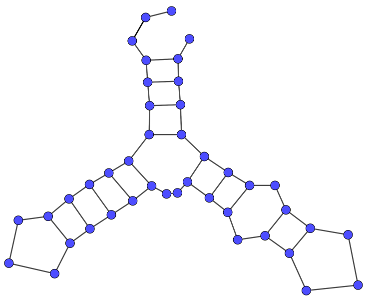

Figure 2.7: Ribonucleic acid graph

Conditional Random Field (CRF)

Conditional Random Field (CRF) model is a new probabilistic model for segmenting and labeling sequence data. CRF is an undirected graphical model that encodes a conditional probability distribution with a given set of features.The Figure 2.8 shows the graphical structure of a chain structured CRF [6].

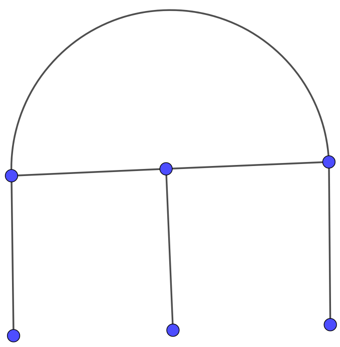

Figure 2.8: Conditional Random Field graph

Theorem 2.4.8. *If CR is the CRF graph, then $CD[CR,x] = 3x^4 + 3x^2$.*

Proof. Note that CRF is a sunlet graph Sl_n with $n = 3$. Also note that $CD[Sl_n,x] = nx^{2n-2}+nx^{2n-4}$. Therefore, $CD[CR,x] = 3x^4+3x^2$. This completes the proof. □

Chapter 3

Stability and Real Roots of Complement Degree Polynomial of Graphs

This chapter includes four sections. The first section deals with basic definitions and results of polynomials. In the second section we study the stability of the complement degree polynomial of graphs. In the third section, we study the real roots of the complement degree polynomials and define $cd - roots$ of a graph; also we investigate the location of the roots of some complement degree polynomials in the fourth section.

3.1 Polynomials

A polynomial is said to be stable if either:

- all its roots lie in the open left half-plane, or
- all its roots lie in the open unit disk.

A polynomial with the first property is called at times a Hurwitz polynomial and with the second property a Schur polynomial . The Routh-Hurwitz theorem provides an algorithm for determining if a given polynomial is stable.

Theorem 3.1.1. *(Routh-Hurwitz Criteria [11]) Given a polynomial,* $P(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_0$*, where the coefficients* a_i *are real constants,* $i = 1, 2, \ldots, n$ *define the* n *Hurwitz matrices using the coefficients* a_i *of the above polynomial as*

$$H_1 = [a_1] \qquad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} \quad \cdots \quad H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

where $a_j = 0$ *if* $j > n$*. All the roots of the polynomial* $P(x)$ *are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive: det* $H_j > 0$*,* $j = 1, 2, \ldots, n$*.*

Theorem 3.1.2. *[2] Let* $f(z) = z^n + a_1z^{n-1} + \ldots + a_n$*, where* $a_i \in \mathbb{C}$*. Then, inside the circle* $|z| = 1 + max|a_i|$*, there are exactly* n *roots of* f*, multiplicities counted.*

3.2 Stability of complement degree polynomial of graphs

Definition 3.2.1. *A polynomial $f(x_1, x_2, \ldots, x_n)$ with real coefficients is called stable if all of its roots lie in the open left half plane.*

Theorem 3.2.2. *If G is a regular graph, then $CD[G, x]$ is stable.*

Proof. Let G be a r-regular graph; then by Theorem 2.1.6 $CD[G, x] = nx^{n-1-r}$. Note that $x = 0$ is the only root of this polynomial, and hence $CD[G, x]$ is stable. □

Corollary 3.2.3. *We have the following:*

(1) If $n \geq 3$, then $CD[C_n, x]$ is stable,

(2) $CD[P, x]$ is stable, where P is the Petersen graph,

(3) $CD[K_{n,n}x]$ is stable,

(4) $CD[Cr_n, x]$ is stable, where Cr_n is the crown graph,

(5) $CD[B_n, x]$ is stable, where B_n is the bipartite cocktail graph,

(6) $CD[CL_n, x]$ is stable, where CL_n is the circular ladder graph,

(7) $CD[ML_n, x]$ is stable, where ML_n is the Mobius ladder graph.

Corollary 3.2.4. *If $S(G)$ is the splitting graph of a regular graph G , then $CD[S(G), x]$ is stable.*

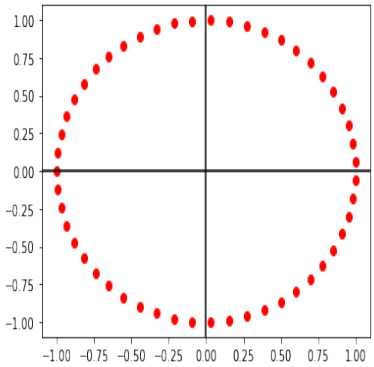

Figure 3.1: Plot of roots of $x^{50}+1$ in the complex plane

Theorem 3.2.5. *If $CS(G)$ is a cosplitting graph of r-regular graph G with order n, then $CD[CS(G),x]$ is stable if and only if $r=1$.*

Proof. Let G be a r-regular graph. Note that $CD[CS(G),x] = nx^{n-1}(1+x^r)$(by Theorem 2.3.25). Observe that $CD[CS(G),x]$ is stable if $r=1$. Also note that when $r=2$, $CD[CS(G),x]$ is not stable. For $r \geq 2$, this polynomial has real roots and complex roots with positive and negative real parts. Hence $CD[CS(G),x]$ is not stable. A plot of roots of $x^{50}+1$ in the complex plane is shown in figure 3.1. □

Theorem 3.2.6. *Let G be a graph of order n, Then $CD[G,x]$ is stable if and only if $CD[mG,x]$ is stable.*

Proof. Note that $CD[mG,x] = mCD[G,x]$(by Theorem 2.3.2). This implies that $CD[G,x]$ is stable if and only if $CD[mG,x]$ is stable. □

Theorem 3.2.7. *Let G be a graph with order n and $\mathbf{G}$=$G \cup G \cup \ldots \cup G$ (m times). Then $CD[G,x]$ is stable if and only if $CD[\mathbf{G},x]$ is stable.*

Proof. Let G be a graph of order n and $\mathbf{G}$=$G \cup G \cup \ldots \cup G$ (m times), then $CD[\mathbf{G}, x] = mx^{(m-1)n}CD[G, x]$ (by Theorem 2.3.1). This implies that $CD[G, x]$ is stable if and only if $CD[\mathbf{G}, x]$ is stable. □

Theorem 3.2.8. *If G' is a graph obtained by duplication of a vertex of regular graph G, then $CD[G', x]$ is stable.*

Proof. Let G be a r-regular graph. Note that $CD[G', x] = (n + 1 - r)x^{n-r} + rx^{n-r-1}$ (by Theorem 2.3.26). Note that the roots of $CD[G', x]$ are $x = 0$ and $x = -\frac{r}{n+1-r}$ which lie in the open left half plane, hence the result follows. □

Corollary 3.2.9. *If $K'_{n,n}$ is a graph obtained by duplication of a vertex of complete bipartite graph $K_{n,n}$, then $K'_{n,n}$ is stable.*

Theorem 3.2.10. *For a path P_n ($n \geq 2$), $CD[P_n, x]$ is stable unless $n = 2$.*

Proof. From Theorem 2.2.1, we have

$$CD[P_n, x] = \begin{cases} 2x^{n-2}, & n \leq 2 \\ (n-2)x^{n-3} + 2x^{n-2}, & n \geq 3 \end{cases}$$

If $n \neq 2$, $CD[P_n, x] = (n - 2)x^{n-3} + 2x^{n-2} = x^{n-3}(n - 2 + 2x)$. In this case $x = 0$ and $x = -\frac{n-2}{2}$ are the roots of $CD[P_n, x]$ which lie in the left half plane, it follows that $CD[P_n, x]$ is stable. If $n = 2$, $CD[P_2, x] = 2$. Hence $CD[P_2, x]$ is not stable. □

Theorem 3.2.11. *If L_n is the ladder graph with $n \geq 2$ vertices, then $CD[L_n, x]$ is stable.*

Proof. Note that $CD[L_n, x] = 4x^{2n-3} + (2n-4)x^{2n-4} = x^{2n-4}(4x + 2n - 4)$(by Theorem 2.2.33). Observe that the roots of $CD[L_n, x]$ are $x = 0, -\frac{n-2}{2}$ which lie in the open left half plane. Hence the result follows. □

Theorem 3.2.12. *Let K'_n be the graph obtained by the duplication of one of the vertices of the complete graph K_n with $n \geq 2$, then $CD[K'_n, x]$ is stable.*

Proof. Note that $CD[K'_n, x] = 2x + n - 1$ (by Theorem 2.3.8). Also observe that $CD[K'_n, x]$ has a single root $x = -(n-1)/2$ which lie in the open left half plane and hence the result follows. □

Theorem 3.2.13. *If $T_{m,n}$ is a tadpole graph with $m \geq 3$ and $n \geq 1$ vertices, then $CD[T_{m,n}, x]$ is stable.*

Proof. Note that $CD[T_{m,n}, x] = x^{m+n-4}(x^2 + (m+n-2)x + 1)$ (by Theorem 2.2.9). The roots of $x^2 + (m+n-2)x + 1$ are $\frac{-(m+n-2)\pm\sqrt{(m+n-2)^2-4}}{2}$. For $m \geq 3$ and $n > 1$, we have $(m+n-2) > 0$, $(m+n-2)^2 - 4 > 0$ and $\sqrt{(m+n-2)^2-4} - (m+n-2) < 0$, the result follows. □

Theorem 3.2.14. *If $A(Q_n)$ is a alternating quadrilateral snake graph with $n \geq 3$ vertices, then $CD[A(Q_n), x]$ is stable.*

Proof. From Theorem 2.2.28, we have

$$CD[A(Q_n), x] = \begin{cases} (n-2)x^{2n-5} + nx^{2n-4} + x^{2n-3}, & if\ n\ is\ odd \\ (n-2)x^{2n-4} + (n+2)x^{2n-3}, & if\ n\ is\ even. \end{cases}$$

Case(i)If n is odd: In this case $CD[A(Q_n), x] = x^{2n-5}(x^2 + nx + n - 2)$. The roots of $x^2+nx+n-2$ are $(-n\pm\sqrt{n^2-4(n-2)})/2$. Since $\sqrt{n^2-4(n-2)} <$

n, it follows that $CD[A(Q_n), x]$ is stable.

Case(ii)If n is even: In this case, $CD[A(Q_n), x] = x^{2n-4}(n - 2 + (n + 2)x)$. Observe that the roots of $CD[A(Q_n), x]$ are $x = 0$ and $x = -(n-2)/(n + 2)$ which lie in the open left half plane. Hence for n is even, $CD[A(Q_n), x]$ is stable.

This completes the proof. □

Theorem 3.2.15. *If CP_n is a cocktail party graph with $n \geq 2$, then $CD[CP_n, x]$ is stable.*

Proof. Note that $CD[CP_n, x] = 4x^2+2(n-2)x^3 = 2x^2(2+(n-2)x)$ (by Theorem 2.2.42). Then the roots of $CD[CP_n, x]$ are $x = 0$ and $x = -2/(n - 2)$ which lie in the open left half plane. Hence the proof. □

Theorem 3.2.16. *If $n \geq 3$, G_n is the gear graph of order $2n+1$, then $CD[G_n, x]$ is stable if and only if $n = 3, 4$.*

Proof. Note that $CD[G_n, x] = nx^{2n-2}+nx^{2n-3}+x^n = x^n(nx^{n-2}+nx^{n-3}+1)$ (by Theorem 2.2.6). When $n = 3$, the roots of $CD[G_3, x]$ are $x = 0$ and $x = -4/3$. When $n = 4$, the roots of $CD[G_4, x]$ are $x = 0$ and $x = -1/2$. Hence $CD[G_n, x]$ is stable for $n = 3, 4$. When $n > 4$, the result follows from the fact that the determinant of second Hurwitz matrix of $CD[G_n, x]$ is zero or negative. □

Theorem 3.2.17. *If Sh_n denotes the shell graph, then $CD[Sh_n, x]$ is stable if and only if $n = 4, 5$.*

Proof. Note that when $n \geq 4$, $CD[Sh_n, x] = 2x^{n-3}+(n-3)x^{n-4}+1$ (by Theorem 2.2.12). For $n = 4, 5$ the result is trivial. When $n > 5$, consider the polynomial

$x^{n-3} + \frac{n-3}{2}x^{n-4} + \frac{1}{2}$. Observe that determinant of second Hurwitz matrix is $|H_2| = 0$. Hence the result follows from the fact that the determinant of second Hurwitz matrix is zero. □

Theorem 3.2.18. *If $L_{n,n}$ is the lollipop graph, then $CD[L_{m,n}, x]$ is stable if and only if $n = 2, 3, 4$.*

Proof. Note that $CD[L_{n,n}, x] = x^{2n-2} + (n-1)x^{2n-3} + (n-1)x^n + x^{n-1}$ (by Theorem 2.2.8). For $n = 1$, $CD[L_{1,1}, x] = 2$ is a constant polynomial and for $n = 2$, the roots of $CD[L_{2,2}, x]$ are $x = 0, -1$, for $n = 3$, the roots of $CD[L_{3,3}, x]$ are $x = \frac{-4\pm\sqrt{12}}{2}$ and for $n = 4$, the roots of $CD[L_{4,4}, x]$ are $x = 0, -1$. Thus for $n = 2, 3, 4$, $CD[L_{n,n}, x]$ is stable. When $n = 5$, $CD[L_{5,5}, x] = x^8 + 4x^7 + 4x^5 + x^4$, the determinant of Hurwitz matrices are $|H_1| = 4$ and $|H_2| = -4$. It follows that $CD[L_{5,5}, x]$ is not stable. When $n > 5$ the result follows from the fact that the determinant of second Hurwitz matrix of $CD[L_{n,n}, x]$ is zero. □

Theorem 3.2.19. *If $F_{1,n}$ is the fan graph ($n \geq 3$), then $CD[F_{1,n}, x]$ is stable if and only if $n = 3, 4$.*

Proof. Note that $CD[F_{1,n}, x] = 2x^{n-2} + (n-2)x^{n-3} + 1$ (by Theorem 2.2.15). The result is trivial for $n = 3$ and $n = 4$. When $n > 4$, the result follows from the fact that the determinant of second Hurwitz matrix of $CD[F_{1,n}, x]$ is zero. □

Theorem 3.2.20. *If $F_{2,n}$ ($n \geq 3$) is the double fan graph , then $CD[F_{2,n}, x]$ is stable if and only if $n = 3, 4, 5$.*

Proof. Note that $CD[F_{2,n}, x] = 2x^{n-2} + (n-2)x^{n-3} + 2x$ (by Theorem 2.2.16). For $n = 3, 4$ the result follows from elementary algebra. For $n = 5$, the roots of

$CD[F_{2,5}, x]$ are $x = 0$ and $x = \frac{-3 \pm i\sqrt{7}}{4}$ which lie in the open left half plane. It follows that $CD[F_{2,5}, x]$ is stable. When $n > 5$, the result follows from the fact that the determinant of second Hurwitz matrix of $CD[F_{2,n}, x]$ is zero. □

Theorem 3.2.21. *For a armed crown graph $C_n \bigodot P_m$ $(n \geq 3,\ m \geq 1)$, $CD[C_n \bigodot P_m, x]$ is stable unless $m = 1$.*

Proof. Note that $CD[C_n \bigodot P_m, x] = nx^{n(m+1)-4}(x^2 + (m-1)x + 1)$ (by Theorem 2.3.11). Observe that the determinants of the Hurwitz matrices of the polynomial $x^2 + (m-1)x + 1$ are $|H_1| = m - 1$ and $|H_2| = m - 1$. Since all the determinants are positive except when $m = 1$, the result follows. □

Theorem 3.2.22. *If Bk_n is the book graph with $2n+2$ vertices, then $CD[Bk_n, x]$ is stable if and only if $n = 1, 2$.*

Proof. Note that $CD[Bk_n, x] = 2nx^{2n-1} + 2x^n$ (by Theorem 2.2.22). When $n = 1$, $CD[Bk_1, x] = 4x$ and when $n = 2$,$CD[Bk_2, x] = 4x^3 + 2x^2$. Observe that the roots of $CD[Bk_1, x]$ and $CD[Bk_2, x]$ are lie in the open left half plane. It follows that for $n = 1, 2$, $CD[Bk_n, x]$ is stable. When $n > 2$, the result follows from the fact that the determinant of first Hurwitz matrix of the polynomial $x^{2n-1} + \frac{1}{n}x^n$ are zero. □

Theorem 3.2.23. *If Bl is the bull graph, then $CD[Bl, x]$ is stable.*

Proof. Note that $CD[Bl, x] = 2x^3 + x^2 + 2x = x(2x^2 + x + 2)$ and its roots are $x = 0, \frac{-1 \pm i\sqrt{15}}{4}$ which lie in open left half plane. Hence $CD[Bl, x]$ is stable. □

Theorem 3.2.24. *If Fr is the fork graph, then $CD[Fr, x]$ is stable.*

Proof. Obviously, $CD[Fr,x] = 3x^3 + x^2 + x = x(3x^2 + x + 1)$. The roots of $CD[Fr,x]$ are $x = 0, \frac{-1\pm i\sqrt{11}}{6}$. Hence $CD[Fr,x]$ is stable. □

Theorem 3.2.25. *If TL_n is the triangular ladder graph, then $CD[TL_n,x]$ is stable for $n \geq 2$.*

Proof. Note that $CD[TL_n,x] = 2x^{2n-3}+2x^{2n-4}+(2n-4)x^{2n-5} = 2x^{2n-5}(x^2+x+n-2)$ (by Theorem 2.2.37). For $n = 2$, the result follows from simple elementary algebra. For $n > 2$ consider the polynomial $x^2 + x + n - 2$. The determinants of Hurwitz matrices are $|H_1| = 1$ and $|H_2| = n - 2$ and so on. Since all the determinants are positive when $n > 2$, the results follows. □

Theorem 3.2.26. *If DSL_n is the double sided step ladder graph, then $CD[DSL_n,x]$ is stable.*

Proof. Note that $CD[DSL_n,x] = (2n+2)x^{n^2+3n-3} + (2n-2)x^{n^2+3n-4} + (n^2-n)x^{n^2+3n-5} = x^{n^2+3n-5}((2n+2)x^2 + (2n-2)x + n^2 - n)$ (by Theorem 2.2.40). For n=1, the result follows from elementary algebra. For $n > 1$, consider the polynomial

$$x^2 + \frac{2n-2}{2n+2}x + \frac{n^2-n}{2n+2}.$$

Observe that the determinants of Hurwitz matrices are $|H_1| = (2n-2)/(2n+2)$ and $|H_2| = (2n-2)(n^2-n)/(2n+2)^2$. Since all the determinants are positive when $n > 1$, it follows that $CD[DSL_n,x]$ is stable. □

3.3 Real roots of complement degree polynomial of graphs

Definition 3.3.1. *The roots of complement degree polynomial of a graph G are called* cd-roots *of G. The number of real* cd-roots *of a graph G where the multiplicities counted, is denoted by $cd(G)$.*

Theorem 3.3.2. *Zero is a* cd-root *of a complement degree polynomial of a graph G with n vertices if and only if $\Delta(G) \leq n-2$.*

Proof. Let G be a graph of order n and zero is a *cd-root* of the polynomial $CD[G,x]$. If G has a vertex, say v which is adjacent to all other vertices, then v is an isolated vertex in $\overline{G}$. This implies that $CD[G,x]$ has a constant term. This is a contradiction because zero is a *cd-root* of $CD[G,x]$. Therefore, G has no vertices adjacent to all other vertices.

Conversely, assume that $\Delta(G) \leq n-2$. Then $\delta(\overline{G}) \geq 1$. Equivalently, $Cd_0(G) = 0$. This tells us that the constant term of $CD[G,x]$ is zero, and hence the result follows. □

Corollary 3.3.3. *If $\overline{G}$ has no isolated vertices, then zero is a root of $CD[G,x]$ with multiplicity $\delta(\overline{G})$.*

Theorem 3.3.4. *If G is a non complete graph of order n, then zero is the only* cd-root *of $CD[G,x]$ if and only if G is a r-regular graph.*

Proof. First, assume that zero is the only *cd*-root of a graph G with n vertices. Then it follows that the complement degree polynomial of G is $CD[G,x] = nx^t$

for $t = n - r - 1$ and $0 < r < n$. This implies that the degree of every vertex in $\overline{G}$ is the same. Equivalently, G is r-regular.

Conversely, assume that G is a r-regular graph. Then we have $CD[G, x] = nx^{n-r-1}$. It follows that zero is only cd-root of $CD[G, x]$. □

Corollary 3.3.5. *If G is an r-regular graph with n vertices, then $cd(G) = n - r - 1$.*

Theorem 3.3.6. *Let G be a graph with n vertices. Then*

(1) $CD[G, x]$ is a strictly increasing function in $[0, \infty)$.

(2) Let G be a graph and H be any spanning subgraph of G. Then the degree of $CD[G, x])$ is less than or equal to the degree of $CD[H, x]$.

(3) Let G be a graph and H be any induced subgraph of G. Then the degree of $CD[G, x]$ greater than or equal to the degree of $CD[H, x]$.

(4) Let G be a graph of order n with t isolated vertices in G and r isolated vertices in $\overline{G}$. Then $Cd_0(G) = r$ and $Cd_{n-1}(G) = t$.

Proof. Proof of the above result follows from the definition of complement degree polynomial of a graph. □

Theorem 3.3.7. *For a cosplitting graph $CS(G)$ of an r-regular graph G with n vertices,*

$$cd(CS(G)) = \begin{cases} n - 1 & \text{if } r \text{ is even,} \\ n & \text{if } r \text{ is odd.} \end{cases}$$

Proof. Observe that $CD[CS(G), x] = nx^{n-1}(1 + x^r)$ (by Theorem 2.3.25). It is clear that $x = 0$ is a *cd*-root of G with multiplicity $n - 1$. Note that the polynomial $1+x^r$ has no real roots if r is even and one real root if r is odd. Thus we have,

$$cd(CS(G)) = \begin{cases} n-1 & \text{if } r \text{ is even,} \\ n & \text{if } r \text{ is odd.} \end{cases}$$

This completes the proof. □

Theorem 3.3.8. *For a path graph P_n,*

$$cd(P_n) = \begin{cases} 0, & n = 2 \\ n-2, & n \geq 3. \end{cases}$$

Proof. For a path graph P_n, we have Theorem 2.2.1:

$$CD[P_n, x] = \begin{cases} 2x^{n-2}, & n = 2 \\ (n-2)x^{n-3} + 2x^{n-2}, & n \geq 3. \end{cases}$$

Here we consider two cases:

If $n = 2$, then $CD[P_2, x] = 2$, which has no zeros. If $n > 2$, then we have $CD[P_n, x] = x^{n-3}(2x + n - 2)$. Obviously, $x = 0$ is the *cd*-root of $CD[P_n, x]$ with multiplicity $n-3$ and $x = -(n-2)/2$ is the another *cd*-root of $CD[P_n, x]$. Thus

$$cd(P_n) = \begin{cases} 0, & n = 2 \\ n-2, & n \geq 3. \end{cases}$$

This completes the proof. □

Theorem 3.3.9. *Let G be a graph with order n and $\boldsymbol{G}$=$G \cup G \cup \ldots \cup G$ (m times). Then $cd(\boldsymbol{G}) = n(m-1) + cd(G)$.*

Proof. Let G be a graph of order n and $\mathbf{G}$=$G \cup G \cup \ldots \cup G$ (m times). Then, $CD[\mathbf{G}, x] = mx^{(m-1)n} CD[G, x]$. Observe that $x = 0$ is a zero of $CD[\mathbf{G}, x]$ of multiplicity $n(m-1)$. Consequently, $cd(\mathbf{G}) = n(m-1) + cd(G)$. This completes the proof. □

Theorem 3.3.10. *For a ladder graph L_n, $cd(L_n) = 2n - 3$ for $n \geq 2$.*

Proof. Obviously, *cd-roots* of $CD[L_n, x]$ are $x = 0$ with multiplicity $2n - 4$ and $x = -(n-2)/2$ with multiplicity one (by Theorem 2.2.33). Hence the result follows. □

Theorem 3.3.11. *For a cocktail party graph CP_n, $cd(CP_n) = 3$ for $n \geq 2$.*

Proof. In Theorem 2.2.42, $CD[CP_n, x] = 2x^2(2 + (n-2)x)$. It follows that $CD[CP_n, x]$ has *cd-roots* $x = 0$ with multiplicity 2 and $x = -2/(n-2)$ with multiplicity one. Thus $cd(CP_n) = 3$. Hence the proof follows. □

Theorem 3.3.12. *If $S(G)$ is a splitting graph of a graph G with n vertices, then $cd(S(G)) \geq 1$.*

Proof. Observe that $CD[S(G), x]$ do not have a constant term (see Theorem 2.3.13). Hence the result follows. □

Theorem 3.3.13. *For a bull graph Bl, $cd(Bl) = 1$.*

Proof. Note that $CD[Bl,x] = x(2x^2+x+2)$. The roots of this polynomial are $x = 0, \frac{-1\pm i\sqrt{15}}{4}$. Obviously, $x = 0$ is the only real root of $CD[Bl,x]$. Hence the result follows. □

Theorem 3.3.14. *For a sunlet graph Sl_n, $cd(Sl_n) = 2n-4$ for $n \geq 3$.*

Proof. Note that $CD[Sl_n,x] = nx^{2n-4}(1+x^2)$ (by Theorem 2.2.13). Then the *cd-roots* are $x = 0$ with multiplicity $2n-4$ and $x = \pm i$. Thus $cd(Sl_n) = 2n-4$. □

Theorem 3.3.15. *For a tadpole graph $T_{m,n}$, $cd(T_{m,n}) = m+n-2$ for $m \geq 3$ and $n \geq 1$.*

Proof. Note that $CD[T_{m,n},x] = x^{m+n-4}(x^2+(m+n-2)x+1)$ (by Theorem 2.2.9). Since the discriminant of the polynomial $x^2+(m+n-2)x+1$ is always greater than or equal to zero, it follows that $cd(T_{m,n}) = m+n-2$. This completes the proof. □

Theorem 3.3.16. *For a bistar graph $B_{n,n}$ $(n \geq 1)$,*

$$cd(B_{n,n}) = \begin{cases} n & \textit{if } n \textit{ is even} \\ n+1 & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Note that $CD[B_{n,n},x] = 2x^n(nx^n+1)$ (by Theorem 2.2.7). If n is even, then nx^n+1 has only complex roots. If n is odd, then nx^n+1 has only one real root and $n-1$ complex roots. Hence,

$$cd(B_{n,n}) = \begin{cases} n & \text{if n is even} \\ n+1 & \text{if n is odd.} \end{cases}$$

This completes the proof. □

Theorem 3.3.17. *For a web graph Wb_n, $cd(Wb_n) = 3n - 4$ for $n \geq 3$.*

Proof. Note that $CD[Wb_n, x] = nx^{3n-5}(x^3 + x + 1)$ (by Theorem 2.2.21). Obviously, the *cd-roots* of $CD[Wb_n, x]$ are 0 with multiplicity $3n-5$, $-0.68233, 0.34116 \pm 1.16154i$. Hence $cd(Wb_n) = 3n - 4$. □

Theorem 3.3.18. *For a armed crown graph $C_n \bigodot P_m$,*

$$cd(C_n \bigodot P_m) = \begin{cases} n(m+1) - 4, & \textit{if m=1,2} \\ n(m+1) - 2, & \textit{if m} \geq \textit{3.} \end{cases}$$

Proof. Note that $CD[C_n \bigodot P_m, x] = x^2 + (m-1)x + 1$ (by Theorem 2.3.11). For $m = 1, 2$, the zeros of $x^2 + (m-1)x + 1$ are complex numbers. If $m > 2$, the zeros of $x^2 + (m-1)x + 1$ are real numbers. Thus

$$cd(C_n \bigodot P_m) = \begin{cases} n(m+1) - 4 & \text{if m=1,2} \\ n(m+1) - 2 & \text{if m} \geq 3. \end{cases}$$

This completes the proof. □

Theorem 3.3.19. *For a sun graph S_n, $n \geq 3$,*

$$cd(S_n) = \begin{cases} n - 2, & \textit{if } n \textit{ is odd} \\ n - 1, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Note that $CD[S_n, x] = nx^{n-2}(x^{n-1}+1)$ (by Theorem 2.2.14). Since $x^{n-1} + 1$ has real roots if and only if n is even. This tells us that the real *cd-roots* of

$CD[S_n, x]$ are 0 and -1 if n is even. If n is odd $x = 0$ is the only real root of $CD[S_n, x]$. Therefore, we have

$$cd(S_n) = \begin{cases} n-2, & \text{if n is odd} \\ n-1, & \text{if n is even.} \end{cases}$$

This completes the proof. □

Theorem 3.3.20. *For a bipartite cocktaill party graph $b_n(n \geq 2)$, we have $cd(B_n) = n$.*

Proof. The result follows from the fact that $CD[B_n, x] = 2nx^n$. □

3.4 Location of the *cd*-roots of the some graphs

Theorem 3.4.1. *All the* cd-roots *of the gear graph G_n lie inside the circle with center $(0, 0)$ and radius $n + 1$.*

Proof. Observe that $CD[G_n, x] = x^n + nx^{2n-3} + nx^{2n-2}$. In this case $max|a_i| = n$, where $a_i's$ are the coefficients of $CD[G_n, x]$ for $i = 1, 2, \ldots, 2n-2$. Then by Theorem 2.2.25, the result follows. □

Theorem 3.4.2. *All the* cd-roots *of the wheel graph W_n lie inside the circle with center $(0, 0)$ and radius n.*

Proof. It follows from the fact that $CD[W_n, x] = (n-1)x^{n-4} + 1$. □

Theorem 3.4.3. *All the* cd-roots *of the bull graph Bl lie on the unit circle centered at the origin.*

Proof. Note that the *cd-roots* of Bl are $x = 0, \frac{-1 \pm i\sqrt{15}}{4}$. These three roots lie on the unit circle centered at the origin. □

Theorem 3.4.4. *All the* cd-roots *of the sunlet graph Sl_n lies in the disk $|z| \leq 1$.*

Proof. The *cd-roots* of the sunlet graph $CD[Sl_n, x]$ are $x = 0$ and $x = \pm i$. Hence the result follows. □

Theorem 3.4.5. *All the* cd-roots *of the sun graph S_n lies in the disk $|z| \leq 1$.*

Proof. Note that $CD[S_n, x] = nx^{2n-3} + nx^{n-2} = nx^{n-2}(x^{n-1} + 1)$. Obviously, roots of $x^{n-1} + 1$ lie on the unit circle. Hence the result follows. □

Chapter 4

CD-Equivalent Classes of Graphs

Some times two non-isomorphic graphs have same complement degree polynomial. From these polynomials the CD-equivalent classes of graphs are defined. In this chapter, CD-equivalent classes of graphs are defined and studied.

4.1 Main Results

Definition 4.1.1. *Let G be the graph of order n, and the CD- equivalent class of the graph G is defined as $\mathcal{CD}[G] := \{H : CD[G, x] = CD[H, x]\}$.*

Theorem 4.1.2. *Let G be a graph with n vertices, then $\overline{G} \in \mathcal{CD}[G]$ if and only if G is a self-complementary graph.*

Proof. Let G be a self-complementary graph that is $G \cong \overline{G}$. Note that isomorphic graphs will be the same complement degree polynomial. Conversely, suppose that $\overline{G} \in \mathcal{CD}[G]$; then G and $\overline{G}$ have the same degree sequence, that is G is a self-complementary graph. □

Theorem 4.1.3. *If $\overline{G}$ is a connected graph and $H \in \mathcal{CD}[G]$, then $\overline{H}$ has more than one component.*

Proof. Let G be a graph and let $\overline{G}$ be connected. Let $H \in \mathcal{CD}[G]$, then $\overline{G}$ and $\overline{H}$ will have the same degree sequence. But $\overline{G} \ncong \overline{H}$. Therefore, $\overline{H}$ must have more than one component. □

Theorem 4.1.4. *Let H and G be the graph of order n, H has the same degree sequence as G and $G \ncong H$, then $H \in \mathcal{CD}[G]$.*

Proof. Observe that non isomorphic graphs have the same degree sequence. The complement graphs of those graphs have the same degree sequence. That is, let G_1 and G_2 be two graphs with the same degree sequence of G, and $G_1 \ncong G_2$, then $CD[G_1, x] = CD[G_2, x]$. Hence the result follows. □

4.2 Some CD-Equivalent Classes of Graphs

Theorem 4.2.1. *If $n \geq 2$, then $\mathcal{CD}[K_n]$ is a singleton.*

Proof. Note that $\overline{K_n} = N_n$ and let $G \in \mathcal{CD}[K_n]$. Suppose $G \ncong K_n$. The degree sequence of $\overline{K_n}$ and $\overline{G}$ are the same; that is, the degree sequence of $\overline{G}$ is $0, 0, \ldots, 0$. Therefore, the degree sequence of G is $n-1, n-1, \ldots, n-1$. Since $G \ncong K_n$, G has more than one component. Let G_1 and G_2 be two components of G, then $|V(G_1)| \leq n-1$. Let $v \in V(G_1)$, $deg(v) \leq n-2$, we obtain a contradiction. Thus $G \cong K_n$. This completes the proof. □

Lemma 4.2.2. *If G_1 is a graph of order n and G_2 is a graph of order m, then $CD[G_1 \cup G_2, x] = x^m CD[G_1, x] + x^n CD[G_2, x]$.*

Proof. Let $v \in V(G_1)$ and $deg(v) = d_1$. Consider the graph $\overline{G_1 \cup G_2}$. Then v is adjacent to each m vertices of $\overline{G_2}$, and also v is adjacent to $V(G_1) - N(v)$ vertices in $\overline{G_1}$. Therefore, $deg(v) = n - 1 - d_1 + m$. Similarly, let $u \in V(G_2)$ and $deg(u) = d_2$ in G_2, then $deg(u) = m - 1 - d_2 + n$. Thus $CD[G_1 \cup G_2, x] = x^m CD[G_1, x] + x^n CD[G_2, x]$. This completes the proof. □

Theorem 4.2.3. *If $n \geq 5$, then $C_n \cup P_m \in \mathcal{CD}[P_{m+n}]$.*

Proof. We will show that $CD[C_n \cup P_m, x] = CD[P_{m+n}, x]$. Note that $CD[P_{m+n}, x] = (m + n - 2)x^{m+n-3} + 2x^{m+n-2}$. By the lemma

$$\begin{aligned} CD[C_n \cup P_m, x] &= x^m CD[C_n, x] + x^n CD[P_m, x] \\ &= x^m n x^{n-3} + x^n[(m-2)x^{m-3} + 2x^{m-2}] \\ &= (m + n - 2)x^{m+n-3} + 2x^{m+n-2} \\ &= CD[P_{m+n}, x]. \end{aligned}$$

This completes the proof. □

The house graph Hu is a simple graph on five vertices and six edges, illustrated in Figure 4.1.

Lemma 4.2.4. *If Hu is the house graph, then $CD[Hu, x] = 3x^2 + 2x$.*

Proof. Note that the house graph has five vertices, out of which two have three degrees and three have two degrees. Note that $\overline{Hu} = P_5$. P_5 has five vertices, out

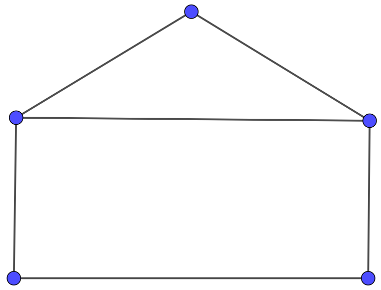

Figure 4.1: House graph Hu

of which two have one degree and three have two degrees. Thus $CD[Hu, x] = 3x^2 + 2x$. This completes the proof. □

Theorem 4.2.5. *The House graph Hu, $Hu \in \mathcal{CD}[K_{3,2}]$.*

Proof. The degree sequences of Hu and $K_{3,2}$ are the same. The complement of Hu and $K_{3,2}$ are P_5 and $K_3 \cup K_2$, respectively. Thus $\overline{Hu} \ncong \overline{K_{3,2}}$. Note that $CD[K_{3,2}, x] = 3x^2+2x$. By Theorem 4.2.4, $CD[Hu, x] = 3x^2+2x = CD[K_{3,2}, x]$. This completes the proof. □

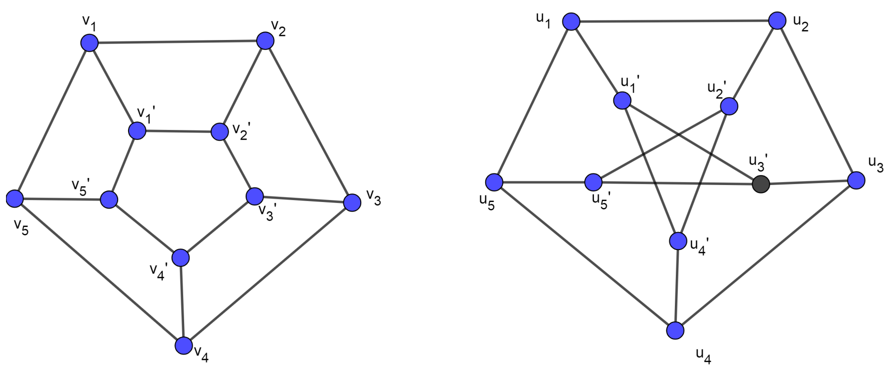

Figure 4.2: Circular ladder graph CL_5 and Petersen graph P

Theorem 4.2.6. *If CL_5 is the circular ladder graph with 10 vertices and P is the Petersen graph, then $CL_5 \in \mathcal{CD}[P]$.*

Proof. We will show that $CD[CL_5, x] = CD[P, x]$ and $CL_5 \ncong P$. Note that the circular ladder graph CL_5 and the Petersen graph P have the same degree sequence (3,3,...,3), and both are $3-$ regular graphs. Hence $CD[CL_5, x] = 10x^6 = CD[P, x]$.

Next we prove that $CL_5 \ncong P$. Let $V(CL_5) = \{v_i, v_i' : i = 1, \ldots, 5\}$ and $V(P) = \{u_i, u_i' : i = 1, \ldots, 5\}$, where v_i and u_i are in the outer vertices, v_i' and u_i' are the inner vertices, and v_i and v_i', u_i and u_i' are adjacent as in Figure 4.2. Note that $|N(v_i) \cap N(v_j')| \leq 2$ and $|N(v_i) \cap N(v_j')| \leq 1$. Then clearly $CL_5 \ncong P$. Hence $CL_5 \in \mathcal{CD}[P]$. This completes the proof. □

Chapter 5

Vertex Cut Polynomial of Graphs

In this chapter we introduce another polynomial viz vertex cut polynomial of a graph related to vertex connectivity. In the first section we define and provide an example of the vertex cut polynomial of graphs. In the second section, we derive the vertex cut polynomial of some well-known graphs in the second section.

5.1 Vertex cut polynomial of graphs

Definition 5.1.1. *Let G be a non-complete simple graph of order n, and let $V(G,i)$ denote the family of vertex cuts with cardinality i and let $d_v(G,i) = |V(G,i)|$. Then the polynomial*

$$V[G;x] := \sum_{i=\kappa(G)}^{(n-2)} d_v(G,i)x^i.$$

is defined as the vertex cut polynomial of the graph G.

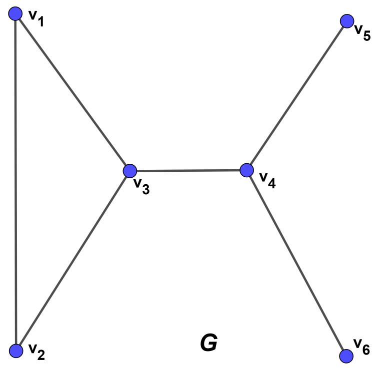

Figure 5.1: The graph G

Example 5.1.2. *Consider the graph G shown in Figure 5.1 . Note that*

$$\begin{aligned}
V(G,1) &= \{\{v_3\},\{v_4\}\},\\
V(G,2) &= \{\{v_1,v_3\},\{v_1,v_4\},\{v_2,v_3\},\{v_2,v_4\},\{v_3,v_4\},\{v_3,v_5\},\{v_3,v_6\},\\
&\quad \{v_4,v_5\},\{v_4,v_6\}\},\\
V(G,3) &= \{\{v_1,v_2,v_4\},\{v_1,v_3,v_4\},\{v_1,v_3,v_5\},\{v_1,v_3,v_6\},\{v_1,v_4,v_5\},\\
&\quad \{v_1,v_4,v_6\},\{v_2,v_3,v_4\},\{v_2,v_3,v_5\},\{v_2,v_3,v_6\},\{v_2,v_4,v_5\},\\
&\quad \{v_2,v_4,v_6\},\{v_3,v_4,v_5\},\{v_3,v_4,v_6\},\{v_3,v_5,v_6\}\},\\
V(G,4) &= \{\{v_1,v_2,v_3,v_4\},\{v_1,v_2,v_4,v_5\},\{v_1,v_2,v_4,v_6\},\{v_1,v_3,v_4,v_5\},\\
&\quad \{v_1,v_3,v_4,v_6\},\{v_1,v_3,v_5,v_6\},\{v_2,v_3,v_4,v_5\},\{v_2,v_3,v_4,v_6\},\\
&\quad \{v_2,v_3,v_5,v_6\}\}.
\end{aligned}$$

Therefore, $V[G;x] = 2x + 9x^2 + 14x^3 + 9x^4$.

Theorem 5.1.3. *If the smallest power of x in $V[G;x]$ is greater than or equal to 2,then G has no pendant vertex.*

Proof. Let $V[G;x]$ be the vertex cut polynomial of a graph G and the smallest power of x in $V[G;x]$ is greater than or equal to 2. Then $d_v(G,1)=0$. Suppose G has a pendant vertex v adjacent to the vertex u. If we delete u in $V(G)$, then G becomes disconnected and $d_v(G,1)\geq 1$. This contradiction shows that G has no pendant vertex. This completes the proof. $\square$

Remark 5.1.4. *The converse of above result need not be true. For example* $V(F_2,x)=x+4x^2+4x^3$*. The smallest power of* x *in* $V[F_2;x]$ *is* 1 *but* F_2 *has no pendant vertex.*

5.2 Vertex cut polynomial of some graphs

Theorem 5.2.1. *For a path graph* P_n*, where* $n\geq 3$*, then*

$$V[P_n;x]=\sum_{i=1}^{(n-2)}[\binom{n}{i}-i-1]x^i.$$

Proof. Let $v_1,v_2,\ldots,v_n$ be the vertices of the path graph P_n. Observe that if we delete i vertices in $V(P_n)$ other than $\{v_1,v_2,\ldots,v_i\}$, $\{v_1,v_2,\ldots,v_{i-1},v_n\}$, $\{v_1,v_2,\ldots,v_{i-2},v_{n-2},v_{n-1},v_n\}$, …, $\{v_1,v_{n-i+2},\ldots,v_n\}$, $\{v_{n-i+1},v_{n-i+2},\ldots,v_n\}$, then P_n becomes disconnected. Thus $d_v(P_n,1)=n-2=\binom{n}{1}-2$, $d_v(P_n,2)=\binom{n}{2}-3,\ldots,d_v(P_n,i)=\binom{n}{i}-(i+1)$. Therefore, $V[P_n;x]=\sum_{i=1}^{(n-2)}[\binom{n}{i}-i-1]x^i$. This completes the proof. $\square$

Theorem 5.2.2. *For a cycle graph* C_n*, where* $n\geq 4$*, then*

$$V[C_n;x]=\sum_{i=1}^{(n-2)}[\binom{n}{i}-n]x^i.$$

Proof. Deleting a vertex in $V(C_n)$, we get a path P_{n-1}. Thus $V(C_n,1)=\phi$. Next, deleting an adjacent pair $\{v_i, v_j\}$, again we get a path P_{n-2}. Thus we obtain a disconnected graph by the removal of any non adjacent pairs of vertices. Observe that C_n has n adjacent pairs of vertices. Thus $d_v(C_n, 2) = \binom{n}{2} - n$. Similarly the removal of i adjacent vertices disconnects C_n. Since C_n has n adjacent vertices of cardinality i, we have $d_v(C_n, i) = \binom{n}{i} - n$. Therefore, $V[C_n; x] = \sum_{i=1}^{(n-2)}[\binom{n}{i} - n]x^i$. This completes the proof. □

Corollary 5.2.3. *For a wheel graph W_n, where $n \geq 5$, then*

$$V[W_n; x] = \sum_{i=1}^{(n-3)} [\binom{n-1}{i} - n + 1]x^{i+1}.$$

Proof. Let $v_0, v_1, v_2, \ldots, v_{n-1}$ be the vertices of the wheel graph W_n, where v_0 is the central vertex. Since v_0 is adjacent to all vertices, v_0 will be contained in every vertex cut. We delete v_0 in $V(W_n)$ and get a cycle C_{n-1}. By Theorem 5.2.2, $V[C_{n-1}; x] = \sum_{i=1}^{(n-3)}[\binom{n-1}{i} - n+1]x^i$. Thus $V[W_n; x] = \sum_{i=1}^{(n-3)}[\binom{n-1}{i} - n+1]x^{i+1}$. This completes the proof. □

Theorem 5.2.4. *For a star graph $K_{1,n}$, where $n \geq 2$, then*

$$V[K_{1,n}; x] = \sum_{i=0}^{(n-2)} \binom{n}{i} x^{i+1}.$$

Proof. If we delete any vertex other than central vertex, then we obtain star graph $K_{1,n-1}$. But with removal of central vertex, we get n isolated vertices. Thus $d_v(K_{1,n}, 1) = 1$. Similarly, the removal of any i vertices other than the central vertex does not decompose the graph. Thus any vertex cut contain central vertex. It follows that $d_v(K_{1,n}, i+1) = \binom{n}{i}$. Therefore, $V[K_{1,n}; x] = \sum_{i=0}^{(n-2)} \binom{n}{i} x^{i+1}$. This completes the proof. □

Theorem 5.2.5. *For a complete bipartite graph $K_{m,n}$, where $n \geq 1, m \geq 2$ or $n \geq 2, m \geq 1$, then*

$$V[K_{m,n}; x] = x^q + \sum_{i=1}^{(p-2)} \binom{p}{i} x^{q+i}, \text{ where } q = min\{m,n\} \text{ and } p = max\{m,n\}.$$

Proof. Note that the vertex connectivity of the complete bipartite graph $K_{m,n}$ is $min\{m,n\}$. Let $q = min\{m,n\}$, $p = max\{m,n\}$ and $v_1, v_2, \ldots, v_q, u_1, u_2, \ldots, u_p$ be the vertices of $K_{m,n}$. We delete the vertices $v_1, v_2, \ldots, v_q$, to get a disconnected graph. Thus $d_v(K_{m,n}, q) = 1$. In general, if we delete $q + i$ vertices (the removed vertices are $v_1, v_2, \ldots, v_q$ and i vertices from $\{u_1, u_2, \ldots, u_p\}$), then we have $d_v(K_{m,n}, q+i) = \binom{p}{i}$. Therefore, $V[K_{m,n}; x] = x^q + \sum_{i=1}^{(p-2)} \binom{p}{i} x^{q+i}$. □

Theorem 5.2.6. *The vertex cut polynomial of a Petersen graph P is given by*

$$V[P; x] = 10 \sum_{i=0}^{5} \binom{6}{i} x^{i+3}.$$

Proof. Observe that Petersen graph is a 3-regular graph. Let $v \in V(P)$. If we delete the vertices the neighbors of v, $N(v)$, then we get an isolated vertex and a component G having 6 vertices. Next, we delete 3 vertices v_1, v_2, v_3 ($v_1, v_2 \in N(v)$, $v_3 \notin N(v)$), and get a connected graph. Thus $d_v(P, 3) = 10 = 10\binom{6}{0}$. Similarly, the graph can be disconnected by the removal of $N(v)$ and i vertices in G. Thus $d_v(P, i+3) = 10\binom{6}{i}$. Therefore, $V[P; x] = 10 \sum_{i=0}^{5} \binom{6}{i} x^{i+3}$. □

Theorem 5.2.7. *For a windmill graph $W_{n+1}^{(m)}$, where $n \geq 2, m \geq 2$, then*

$$V[W_{n+1}^{(m)}; x] = \sum_{i=0}^{(m-1)n-1} \binom{mn}{i} x^{i+1} + \sum_{i=(m-1)n}^{mn-2} [\binom{mn}{i} - m\binom{n}{i-(m-1)n}] x^{i+1}.$$

Proof. We have a windmill graph having m copies of K_n adjacent to a single vertex. Note that the removal of central vertex disconnects the graph. Thus $d_v(W_n^{(m)}, 1) = 1$. Observe that any other vertex cut contains the central vertex. Therefore, $d_v(W_n^{(m)}, i+1) = \binom{mn}{i}, i = 1, 2, \ldots, (m-1)n - 1$. Since after the removal of vertices, the remaining vertices lie in the same complete graph, some subsets of $V(W_n^{(m)})$ with cardinality greater than or equal to $(m-1)n$ are not vertex cuts. Thus $d_v(W_n^{(m)}, i+1) = \binom{mn}{i} - m\binom{n}{i-(m-1)n}$, $i \geq (m-1)n$. Therefore, $V[W_{n+1}^{(m)}; x] = \sum_{i=0}^{(m-1)n-1} \binom{mn}{i} x^{i+1} + \sum_{i=(m-1)n}^{mn-2} [\binom{mn}{i} - m\binom{n}{i-(m-1)n}] x^{i+1}$. This completes the proof. □

Corollary 5.2.8. *For a friendship graph F_n, where $n \geq 2$, then*

$$V[F_n; x] = \sum_{i=0}^{(2n-3)} \binom{2n}{i} x^{i+1} + [\binom{2n}{2n-1} - n] x^{2n-1}.$$

Proof. Note that the removal of a common vertex of F_n disconnects the graph. But the removal of any other vertex does not disconnect the graph. Thus $d_v(F_n, 1) = 1$. Since any vertex cut contains the central vertex, we have $d_v(F_n, i+1) = \binom{2n}{i}$, $\quad i = 1, 2, \ldots, 2n-3$. Note that $V(F_n)$ has $\binom{2n}{2n-1}$ subsets of cardinality $2n-1$. But n subsets of cardinality $2n-1$ are not vertex cuts. Thus $d_v(F_n, 2n-1) = \binom{2n}{2n-1} - n$. Therefore, $V[F_n; x] = \sum_{i=0}^{(2n-3)} \binom{2n}{i} x^{i+1} + [\binom{2n}{2n-1} - n] x^{2n-1}$. This completes the proof. □

Theorem 5.2.9. *For a fan graph $F_{1,n}$, where $n \geq 3$, then*

$$V[F_{1,n}; x] = \sum_{i=1}^{(n-2)} [\binom{n}{i} - i - 1] x^{i+1}.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a path graph in a fan graph $F_{1,n}$ and

v_0 be the vertex of $F_{1,n}$ adjacent to all vertices $v_1, v_2, \ldots, v_n$. By Theorem 5.2.1 $V[P_n; x] = \sum_{i=1}^{(n-2)}[\binom{n}{i} - i - 1]x^i$. Since every vertex cut contains v_0, we have $V[F_{1,n}; x] = \sum_{i=1}^{(n-2)}[\binom{n}{i} - i - 1]x^{i+1}$. This completes the proof. □

Theorem 5.2.10. *For a double fan graph $F_{2,n}$, where $n \geq 1$, then*

$$V[F_{2,n}; x] = \sum_{i=1}^{(n-3)}[\binom{n}{i} - i - 1]x^{i+2} + [\binom{n}{i} - i]x^n.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a path graph in a double fan graph $F_{2,n}$ and u, w be the vertices of $F_{2,n}$ adjacent to all vertices $v_1, v_2, \ldots, v_n$. Note that every vertex cut in $V(F_{2,n}, i)$, $\quad i = 1, 2, \ldots, n-1$ contains u and w. Thus $d_v(F_{2,n}, i+2) = \binom{n}{i} - i - 1$, $\quad i = 1, 2, \ldots, n-3$. But $V(F_{2,n}, n)$ contains a vertex cut $\{v_1, v_2, \ldots, v_n\}$. Thus $d_v(F_{2,n}, n) = \binom{n}{i} - i$. Therefore, $V[F_{2,n}; x] = \sum_{i=1}^{(n-3)}[\binom{n}{i} - i - 1]x^{i+2} + [\binom{n}{i} - i]x^n$. This completes the proof. □

Theorem 5.2.11. *For a shell graph Sh_n, where $n \geq 4$, then*

$$V[Sh_n; x] = \sum_{i=1}^{(n-3)}[\binom{n-1}{i} - i - 1]x^{i+1}.$$

Proof. Note that every vertex cut of shell graph contains the apex. If we delete apex, then we get a path P_{n-1}. By Theorem 5.2.1, $V[P_n; x] = \sum_{i=1}^{(n-2)}[\binom{n}{i} - i - 1]x^i$. Therefore, $V[Sh_n; x] = \sum_{i=1}^{(n-3)}[\binom{n-1}{i} - i - 1]x^{i+1}$. This completes the proof. □

Theorem 5.2.12. *For a bow graph $Sh_{n,n}$, where $n \geq 1$, then*

$$V[Sh_{n,n}; x] = \sum_{i=0}^{(n-1)}\binom{2n}{i}x^{i+1} + \sum_{i=n}^{(2n-2)}[\binom{2n}{i} - i - 1 + n]x^{i+1}.$$

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, w$ be the vertices of $Sh_{n,n}$ (where $v_1, v_2, \ldots, v_n$ are the vertices of left shell graph, $u_1, u_2, \ldots, u_n$ are the vertices of right shell graph and w is the apex). Note that $Sh_{n,n}$ has only one cut vertex. Thus $d_v(BF_n, 1) = 1$. Since every vertex cut contains the apex w, we have $d_v(Sh_{n,n}, i+1) = \binom{2n}{i}, \quad i = 1, 2, \ldots, n-1$. Also, we have some subsets $S \subset V(Sh_{n,n})$ with cardinality greater than or equal to $n+1$ that are not vertex cuts. If we delete $w, v_1, v_2, \ldots, v_n$, then we get a path $u_1, u_2, \ldots, u_n$. By Theorem 5.2.1, $d_v(P_n, i) = n-i-1$. Similarly, we delete $w, u_1, u_2, \ldots, u_n$ and again we get a path $v_1, v_2, \ldots, v_n$. Since $i \geq n$, the path $v_1, v_2, \ldots, v_n$ or $u_1, u_2, \ldots, u_n$ has only n vertices . Therefore, $d_v(Sh_{n,n}, i+1) = \binom{2n}{i} - 2(i+1-n), \quad i = n, n+1, \ldots, 2n-2$. Thus $V[Sh_{n,n}; x] = \sum_{i=0}^{(n-1)} \binom{2n}{i} x^{i+1} + \sum_{i=n}^{(2n-2)} [\binom{2n}{i} - i - 1 + n] x^{i+1}$. This completes the proof. □

Corollary 5.2.13. *For a butterfly graph BF_n, where $n \geq 2$, then*

$$V[BF_n; x] = \sum_{i=0}^{(n+1)} \binom{2n+2}{i} x^{i+1} + \sum_{i=n+2}^{(2n)} [\binom{2n+2}{i} - 2(i+1-n] x^{i+1}.$$

Proof. Observe that the butterfly graph has only one cut vertex. Thus $d_v(BF_n, 1) = 1$. Since every vertex cut contains the apex, $d_v(BF_n, i+1) = \binom{2n+2}{i}, \quad i = 1, 2, \ldots, n+1$. We have some subsets of $V(BF_n)$ with cardinality greater than or equal to $n+2$ which are not vertex cuts like $Sh_{n,n}$. By Theorem 5.2.12, $d_v(BF_n, i+1) = \binom{2n+2}{i} - 2(i+1-n), \quad i = n+1, n+2, \ldots, 2n$. Thus $V[BF_n; x] = \sum_{i=0}^{(n+1)} \binom{2n+2}{i} x^{i+1} + \sum_{i=n+2}^{(2n)} [\binom{2n+2}{i} - 2(i+1-n] x^{i+1}$. This completes the proof. □

Theorem 5.2.14. *For a sun graph S_n, where $n \geq 2$, then*

$$V[S_n; x] = n \sum_{i=0}^{(2n-3)} \binom{2n-3}{i} x^{i+2} + \binom{n}{2} x^{2n-2}.$$

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of sun graph where $v_1, v_2, \ldots,$ v_n are the vertices of a complete graph and $u_1, u_2, \ldots, u_n$ are the vertices of outer ring of sun graph. If we delete $N(u_1)$, then we get an isolated vertex and a connected component G of $2n-3$ vertices. Next we delete any two vertices $v_i, v_j \in \{v_1, v_2, \ldots, v_n\}$ other than $N(u_i)$, $i = 1, 2, \ldots, n$ and get a connected graph. It follows that $d_v(S_n, 2) = n$. Similarly, the graph can be disconnected with i vertices only by the removal of $N(u_i)$ and $i-2$ vertices in G. Therefore, $d_v(S_n, i+2) = n\binom{2n-3}{i}$, $\quad i = 1, 2, \ldots, 2n-3$. Vertex cuts in $V(S_n, 2n-2)$ are contained in all vertices of complete graph and $n-2$ vertices of the outer ring of a sun graph. Therefore, $d_v(S_n, 2n-2) = \binom{n}{2}$. Thus $V[S_n; x] = n \sum_{i=0}^{(2n-3)} \binom{2n-3}{i} x^{i+2} + \binom{n}{2} x^{2n-2}$. This completes the proof. □

Theorem 5.2.15. *For a pan graph $T_{n,1}$, where $n \geq 3$, then*

$$V[T_{n,1}; x] = x + \sum_{i=2}^{(n-1)} (\binom{n-1}{i-1} + \binom{n}{i-1} - n) x^i.$$

Proof. Let $v_0, v_1, \ldots, v_n$ be the vertices of pan graph $T_{n,1}$, where $v_1, v_2, \ldots, v_n$ are the vertices of cycle and v_0 is the singleton graph with v_0 adjacent to v_1. Note that pan graph has only one cut vertex v_1. Therefore, $d_V(T_{n,1}, 1) = 1$. The number of vertex cut sets with cardinality i of the cycle graph are $\binom{n}{i} - n$. But in the case of pan graph, the removal of v_1 disconnects the graph by a singleton graph v_0 and a connected component. Note that some subsets of $V(T_{n,1})$ are not vertex cut. Also we have vertex cuts S_1 with cardinality i of two types (first types being $v_0 \in S_1$ and the other on when $v_0 \notin S_1$). Since v_1 belongs to i vertex cut sets with cardinality i that do not contain v_0. By Theorem 5.2.2, the number of vertex cut sets not including v_0 with cardinality i is $\binom{n-1}{i-1}$ and including v_0 is $\binom{n}{i-1} - n$. Therefore, $d_v(T_{n,1}, i) = \binom{n-1}{i-1} + \binom{n}{i-1} - n$. Thus $V[T_{n,1}; x] = x + \sum_{i=2}^{(n-1)} (\binom{n-1}{i-1} + \binom{n}{i-1} - n)x^i$. This completes the proof. $\square$

Chapter 6

Vertex Cut Polynomial of some Graph Operations

In this chapter, we discuss and derive the vertex cut polynomial of some graph operations. In the first section, we derive the vertex cut polynomial of some unary graph operations. The second section is devoted to the vertex cut polynomial of some binary graph operations.

6.1 Vertex cut polynomial of some unary graph operations

Theorem 6.1.1. *If $n \geq 2$, then the vertex cut polynomial of splitting graph of star graph is given by*

$$V[S(K_{1,n});x] = x + \sum_{i=1}^{n} \binom{2n+1}{i} x^{i+1} + \sum_{i=n+1}^{2n-2} \left[\binom{2n+1}{i} - \binom{n}{i-n} \right] x^{i+1} +$$

$$\left[\binom{2n}{2}+1\right]x^{2n}.$$

Proof. Let $v_0, v_1, v_2 \ldots, v_n$ be the vertices of $K_{1,n}$ where v_0 is the central vertex and $u_0, u_1, u_2, \ldots, u_n$ be the corresponding vertices of $v_0, v_1, v_2 \ldots, v_n$, respectively, in $S(K_{1,n})$. Note that if we delete the vertex u_0, then $u_1, u_2, \ldots, u_n$ become isolated vertices. Thus $d_v(S(K_{1,n}), i) = \binom{2n+1}{i}$, $i = 2, 3, \ldots, n+1$. But the set that contains $u_1, u_2, \ldots, u_n$ is not a vertex cut set. Therefore, deleting any of $v_1, v_2 \ldots, v_n$ along with $u_1, u_2, \ldots, u_n$ to not increase the number of components. Thus $d_v(S(K_{1,n}), i) = \binom{2n+1}{i} - \binom{n}{i-n}$, $i = n+2, n+3, \ldots, 2n-1$. Similarly, selecting the $2n-2$ vertices from the vertices after deleting v_0 and u_0 will still be the vertex cut set. Also, $\{v_1, v_2 \ldots, v_n, u_1, u_2, \ldots, u_n\}$ includes $V(S(K_{1,n}), 2n)$. Therefore, $d_v(S(K_{1,n}), 2n) = \binom{2n}{2} + 1$. Hence the result follows. □

Theorem 6.1.2. *If $n \geq 2$, then the vertex cut polynomial of cosplitting graph of star graph is given by*

$$V[CS(K_{1,n}; x] = \sum_{i=1}^{n-2}\binom{2n}{i-1}x^i + \left[\binom{2n}{n-1}+1\right]x^n + \sum_{i=n+1}^{2n-1}\left[\binom{2n}{i-1}+\binom{n+2}{i-n}\right] x^i + \left[\binom{2n}{2n-1}+\binom{n+2}{n}-1\right]x^{2n}.$$

Proof. Let $v_0, v_1, v_2 \ldots, v_n, u_0, u_1, \ldots, u_n$ be the vertices of $CS(K_{1,n})$, where $v_0, v_1, v_2 \ldots, v_n$ be the vertices of $K_{1,n}$ with v_0 is the central vertex and u_i be the vertices corresponding to v_i, $i = 0, 1, \ldots, n$. We consider several cases for $d_v(CS(K_{1,n}), i)$, $i = 1, 2, \ldots, 2n$.

Case(i) $i = 1, 2, \ldots, n-1$**:** Note that $CS(K_{1,n})$ has only one cut vertex v_0. If we delete v_0, then we get two components, one is an isolated vertex and other one is a complete bipartite graph $K_{n,n}$. Let $U = \{v_1, v_2 \dots, v_n, u_1, \dots, u_n\}$ and $S \subset U$, then the union $S \cup \{v_0\}$ is a vertex cut. Therefore, $d_v(CS(K_{1,n}), i) = \binom{2n}{i-1}$, $i = 1, 2, \dots, n-1$.

Case(ii) $i = n$**:** The set $\{v_1, v_2, \dots, v_n\}$ is a vertex cut. Thus, in this case, these vertex cuts will be the vertex cut with the vertex cuts mentioned in Case(i). Therefore, $d_v(CS(K_{1,n}), n) = \binom{2n}{n-1} + 1$.

Case(iii) $i = n+1, n+2, \dots, 2n-1$**:** Let $S' \subset \{v_0, u_0.u_1, u_2, \dots, u_n\}$, then the union $S' \cup \{v_1, v_2, \dots, v_n\}$ is a vertex cut. Thus these vertex cuts will be the vertex cuts with the vertex cuts $S \cup \{v_0\}$ mentioned in Case(i). Therefore, $d_v(CS(K_{1,n}), i) = \sum_{i=n+1}^{2n-1}(\binom{2n}{i-1} + \binom{n+2}{i-n})$.

Case(iv) $i = 2n$**:** If the set of vertices $S'' = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ are deleted, then we are left with two vertices v_0 and u_0, it is a connected graph, the set S'' is not a vertex cut. In this case all other vertex cuts mentioned in Case(iii) are vertex cuts. Therefore, $d_v(CS(K_{1,n}, i) = \binom{2n}{2n-1} + \binom{n+2}{n} - 1$.

This completes the proof. □

Theorem 6.1.3. *If $n \geq 1$ and $K'_{1,n}$ is a graph obtained by duplication of the central vertex of $K_{1,n}$, then*

$$V[K'_{1,n}; x] = \sum_{i=0}^{n-3} \binom{n}{i} x^{i+2} + \left[\binom{n}{2} + 1\right] x^n.$$

Proof. Let $v_0, v_1, v_2 \dots, v_n$ be the vertices of $K_{1,n}$ and v' be the duplication of v_0, where v_0 be the center vertex. Note that $K'_{1,n}$ has no cut vertex. Thus $d_v(K'_{1,n}, 1) = 0$. If we delete v_0 and v', then we obtain n isolated vertices.

Thus $d_v(K'_{1,n}, i+1) = \binom{n}{i}$, $\quad i = 0, 1, 2, \ldots, n-1$. All vertex cuts in $V(K'_{1,n}, i)$, $i = 2, 3, \ldots, n-1$ include v_0 and v', but there is a vertex cut $\{v_1, v_2 \ldots, v_n\}$ in $V(K'_{1,n}, n)$ which does not contain v_0 and v'. Thus $d_v(K'_{1,n}, n) = \binom{n}{n-2} + 1 = \binom{n}{2} + 1$. Hence $V[K'_{1,n}; x] = \sum_{i=0}^{n-3} \binom{n}{i} x^{i+2} + (\binom{n}{2} + 1)x^n$. This completes the proof. □

Theorem 6.1.4. *If $n \geq 1$ and $K'_{1,n}$ is a graph obtained by duplication of one of the leaves of $K_{1,n}$, then*

$$V[K'_{1,n}; x] = \sum_{i=0}^{n-1} \binom{n+1}{i} x^{i+1}.$$

Proof. Let v' be the duplication of one of the leaves of $K_{1,n}$, then $K_{1,n}$ will be a star graph with $n+1$ leaves. Hence $V[K'_{1,n}; x] = \sum_{i=0}^{n-1} \binom{n+1}{i} x^{i+1}$. This completes the proof. □

Theorem 6.1.5. *If $n \geq 1$, then the vertex cut polynomial of shadow graph of star graph is given by*

$$V[Sh(K_{1,n}); x] = \sum_{i=0}^{2n-1} \binom{2n}{i} x^{i+2} + \left[\binom{2n}{2} + 1\right] x^{2n}.$$

Proof. Note that $Sh(K_{1,n}) = K'_{1,2n}$, where $K_{1,2n}$ is a graph obtained by duplication of a central vertex of $K_{1,2n}$. Hence by Theorem 6.1.3, $V[Sh(K_{1,n}); x] = V[K'_{1,2n}; x]$. This completes the proof. □

Theorem 6.1.6. *If $n \geq 1$, then the vertex cut polynomial of Mycielski graph of star graphis given by*

$$V[\mu(K_{1.n}); x] = 2x^2 + \sum_{i=3}^{n} \left[2\binom{2n}{i-2} + \binom{2n}{i-3}\right] x^i + [2\binom{2n}{n-1} + \binom{2n}{n-2}$$

$$+ 2]x^{n+1} + \left[2\binom{2n}{n} + \binom{2n}{n-1} - 2\right] x^{n+2} + \sum_{i=n+3}^{2n-1} [2\binom{2n}{i-2}$$
$$+ \binom{2n}{i-3} - 2\binom{n}{i-n-2}]x^i + [2\binom{2n}{2n-2} + \binom{2n}{2n-3} -$$
$$2\binom{n}{n-2} + 1]x^{2n} + [2\binom{2n}{n-1} + \binom{2n}{n-2} - 2\binom{n}{n-1}$$
$$+ 2]x^{2n+1}.$$

Proof. Let $v_0, v_1, \ldots, v_n, u_0, u_1, \ldots, u_n, w$ be the vertices of $\mu(K_{1.n})$, where $v_0, v_1, \ldots, v_n$ are the vertices of $K_{1,n}$ with central vertex v_0, u_i corresponding to each v_i and w is the vertex to each u_i, $i = 0, 1, \ldots, n$. Let $S = \{w, u_0, v_0\}$, $V = \{v_1, \ldots, v_n\}$ and $U = \{u_1, \ldots, u_n\}$. Note that $\mu(K_{1.n})$ has no cut vertex. We consider several cases for $d_v(\mu(K_{1,n}), i)$, $i = 2, 3, \ldots, 2n + 1$.

Case(i) $i = 2$**:** Note that $V(G, 2) = \{\{v_0, w\}, \{u_0, v_0\}\}$. Thus $d_v(G, 2) = 2$.

Case(ii) $i = 3, 4, \ldots, n$**:** Note that the set S itself is a vertex cut. The unions, any subset of $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ with cardinality $i - 2$ and any element in $V(G, 2)$ is a vertex cut. Similarly, the unions, any subset of $\{v_1, v_2, \ldots, v_n, u_1, u_2,$
$\ldots, u_n\}$ with cardinality $i-3$ and S is a vertex cut. Therefore, $d_v(\mu(K_{1,n}), i) = 2\binom{2n}{i-2} + \binom{2n}{i-3}$, $i = 3, 4, \ldots n$.

Case(iii) $i = n + 1$**:** Note that $V \cup \{w\}$ and $U \cup \{u_0\}$ are the vertex cuts. Thus in this case, these two vertex cuts will be the vertex cuts with the vertex cuts mentioned in Case(ii). Therefore, $d_v(\mu(K_{1,n}), n + 1) = 2\binom{2n}{n-1} + \binom{2n}{n-2} + 2$.

Case(iv) $i = n + 2$**:** he unions $V \cup \{v_0, u_0\}$ and $U \cup \{v_0, w\}$ are not vertex cuts. Thus these two subsets of $V(\mu(K_{1,n}))$ will not be vertex cuts mentioned in

Case(ii). Therefore, $d_v(\mu(K_{1,n}), n+2) = 2\binom{2n}{n} + \binom{2n}{n-1} - 2$.

Case(v) $i = n+3, \ldots, 2n-1$**:** Let $U' \subset U$, $V' \subset V$. The unions $V \cup \{v_0, u_0\} \cup U'$ and $U \cup \{v_0, w\} \cup V'$ are not vertex cuts. Thus these $2\binom{n}{i-n}$ subsets of $V(\mu(K_{1,n}))$ will not be vertex cuts mentioned in Case(ii). Therefore, $d_v(\mu(K_{1,n}), i) = 2\binom{2n}{i-2} + \binom{2n}{i-3} - 2\binom{n}{i-n-2}$, $i = n+3, \ldots, 2n-1$.

Case(vi) $i = 2n$**:** The subset $V \cup U$ of $V(\mu(K_{1,n}))$ is the vertex cut set with the vertex cut sets mentioned in Case(v). Therefore, $d_v(\mu(K_{1,n}), 2n) = 2\binom{2n}{2n-2} + \binom{2n}{2n-3} - 2\binom{n}{n-2} + 1$.

Case(vii) $i = 2n+1$**:** The subsets $V \cup U \cup \{w\}$ and $V \cup U \cup \{u_0\}$ of $V(\mu(K_{1,n}))$ are the vertex cut sets with the vertex cut sets mentioned in Case(v). Therefore, $d_v(\mu(K_{1,n}), 2n+1) = 2\binom{2n}{n-1} + \binom{2n}{n-2} - 2\binom{n}{n-1} + 2$.

This completes the proof. □

Theorem 6.1.7. *If $n \geq 1$, then the vertex cut polynomial of complement graph of star graph is given by*

$$V[\overline{K_{1,n}}, x] = \sum_{i=1}^{n-1} \binom{n}{i} x^i.$$

Proof. Note that $\overline{K_{1,n}}$ has two components, one is an isolated vertex and the other is a complete graph with n vertices. If we delete any vertex that is not an isolated vertex from $\overline{K_{1,n}}$, give more than one component. Therefore, $V[\overline{K_{1,n}}, x] = \sum_{i=1}^{n-1} \binom{n}{i} x^i$. □

6.2 Vertex cut polynomial of some binary graph operations

Theorem 6.2.1. *If G_1 is a graph having n vertices and G_2 is a graph having m vertices, $m > n$, then vertex cut polynomial of join of G_1 and G_2 is given by*

$$V[G_1 \vee G_2; x] = \sum_{i=\kappa(G_2)}^{m-2} d_v(G_2, i)x^{n+i}.$$

Proof. Let G_1 and G_2 be two disjoint graphs with n and m vertices, respectively. Since every vertex in G_1 is adjacent to every vertex in G_2, we get more than one component only if all the vertices in G_1 and the vertex cuts in G_2 are removed. Hence the result follows. □

Corollary 6.2.2. *If Sh_n is the shell graph with $n+1 (n \geq 4)$ vertices, then*

$$V[Sh_n; x] = \sum_{i=1}^{n-3} \left[\binom{n-1}{i} - i - 1 \right] x^{i+1}.$$

Proof. Note that $Sh_n = P_{n-1} \vee K_1$ and $V[P_n; x] = \sum_{i=1}^{n-2} \left[\binom{n}{i} - i - 1 \right] x^i$. Therefore, $V[P_{n-1} \vee K_1; x] = \sum_{i=1}^{n-3} \left[\binom{n-1}{i} - i - 1 \right] x^{i+1}$. This completes the proof. □

Theorem 6.2.3. *If G_1 is a graph having n vertices and G_2 is a graph having m vertices, $m > n$, then the vertex cut polynomial of union of G_1 and G_2 is given by*

$$\begin{aligned} V[G_1 \cup G_2; x] =& (m+n)x + \sum_{i=2}^{n-1} \left[\sum_{j=0}^{i} \binom{m}{j} \binom{n}{i-j} \right] x^i + (\sum_{j=0}^{n} \binom{m}{j} \\ & \binom{n}{n-j})x^n + (\sum_{i=n+1}^{m-1} d_v(G_2, i-n) + \binom{m}{i} + \sum_{j=1}^{n-1} \binom{m}{i-j} \end{aligned}$$

$$\binom{n}{j})x^i + \left[\sum_{j=1}^{n-1}\binom{m}{m-j}\binom{n}{j}\right]x^m + \sum_{i=m+1}^{m+n-2}(d_v(G_1, i-m)+$$
$$d_v(G_2, i-n) + \sum_{j=i-m+1}^{n-1}\binom{n}{j}\binom{m}{i-j})x^i.$$

Proof. Since $G_1 \cup G_2$ has two components, all vertices of $G_1 \cup G_2$ are in $V(G_1 \cup G_2, 1)$. Therefore, $d_v(G_1 \cup G_2, 1) = m + n$. Similarly, any subset of $V(G_1 \cup G_2)$ is contained in $V(G_1 \cup G_2, i)$, $i = 2, 3, \ldots, n-1$. Therefore, $d_v(G_1 \cup G_2, i) = \sum_{j=0}^{i}\binom{m}{j}\binom{n}{i-j}$. But if G_1 is deleted, a connected component will be obtained. Therefore, $d_v(G_1 \cup G_2, n) = \sum_{j=1}^{n}\binom{m}{j}$.

After deleting G_1, if we want to get more than one component, then we have to delete the vertex cuts of G_2. Therefore, $d_v(G_1 \cup G_2, i) = d_v(G_2, i-n) + \binom{m}{i} + \sum_{j=1}^{n-1}\binom{m}{i-j}\binom{n}{j}$, $i = n+1, n+2, \ldots, m-1$. Similarly, if we remove G_2, we get a connected component; thus $V(G_1 \cup G_2, m)$ does not have a vertex cut that only contains vertices in G_2. Therefore, $d_v(G_1 \cup G_2, m) = \sum_{j=1}^{n-1}\binom{m}{m-j}\binom{n}{j}$. After deleting G_2, we have to remove the vertex cuts of G_1. Thus $d_v(G_1 \cup G_2, i) = d_v(G_2, i-n) + \binom{m}{i} + \sum_{j=1}^{n-1}\binom{m}{i-j}\binom{n}{j}$, $i = m+1.m+2, \ldots, m+n-2$. Hence the result follows. □

Theorem 6.2.4. *If G_1 is a graph having n vertices and G_2 is a graph having m vertices,$m > n$, then the vertex cut polynomial of corona of G_1 and G_2 is given by*

$$V[G_1 \circ G_2; x] = nx + \sum_{i=2}^{n}\left[\sum_{j=1}^{n}\binom{n}{j}\binom{mn}{i-j} + \binom{n}{i}\right]x^i + \sum_{i=n+1}^{m}\sum_{j=1}^{n}\binom{n}{j}\binom{mn}{i-j}$$
$$x^i + \sum_{k=1}^{n-1}\sum_{i=m+1}^{km+k-1}\left[\sum_{j=1}^{n}\binom{n}{j}\binom{mn}{i-j} - \binom{n}{k}\binom{m(n-k)}{i-km-k}\right]x^i$$

$$+\sum_{i=(n-1)(m+1)}^{n(m+1)-2}\left[\sum_{j=1}^{n}\binom{n}{j}\binom{mn}{i-j}+nd_v(G_2,r)-\binom{mn}{r-1}\right]x^i.$$

Proof. Let G_1 and G_2 be two disjoint graphs with n and m $(n < m)$ vertices, respectively. We consider several cases for $d_v(G_1 \circ G_2, i)$, $i = 1, 2, \ldots, n(m+1)-2$.

Case(i) $i = 1$**:** Since the i^{th} vertex in G_1 is connected to all vertices in the i^{th} copy of G_2, the cut vertices in the corona graph will be vertices in G_1. If we delete a vertex in G_1, then we get a copy of G_2 and a component with $(n-1)(m+1)$ vertices. Thus every vertex in G_1 is cut vertex of $G_1 \circ G_2$. Therefore, $d_v(G_1 \circ G_2, 1) = n$.

Case(ii) $i = 2, 3, \ldots, n$**:** All subsets of $V(G_1 \circ G_2)$ that include at-least one vertex in G_1 are vertex cuts of $G_1 \circ G_2$. Thus there are $\binom{n}{i}$ vertex cuts with only vertices in G_1 and $\sum_{j=1}^{n}\binom{n}{j}\binom{mn}{i-j}$ vertex cuts with vertices in G_1 and G_2. Therefore, $d_v(G_1 \circ G_2, i) = \sum_{j=1}^{n}\binom{n}{j}\binom{mn}{i-j} + \binom{n}{i}$, $i = 2, 3, \ldots, n$.

Case(iii) $i = n+1, n+2, \ldots, m$**:** In this case there will be no vertex cuts with only vertices in G_1. All the vertex cuts have vertices in both G_1 and G_2. Therefore, $d_v(G_1 \circ G_2, i) = \sum_{j=1}^{n}\binom{n}{j}\binom{mn}{i-j}$, $i = n+1, n+2, \ldots, m$.

Case(iv) $i = m+1, m+2, \ldots, (n-1)m+n$**:** Deleting the i^{th} vertex of G_1 and the i^{th} copy of G_2 together leaves only one component. Deleting the i^{th} and j^{th} vertices of G_1 and the i^{th} and j^{th} copies of G_2 together leaves only one component. Similarly, if k vertices and k copies of G_2 are deleted, then only one component remains. Therefore, $d_v(G_1 \circ G_2, i) = \sum_{k=1}^{n-1}\left[\sum_{j=1}^{n}\binom{n}{j}\binom{mn}{i-j}\right] - \binom{n}{k}\binom{m(n-k)}{i-km-k}$, $i = m+1, m+2, \ldots, (n-1)m+n$.

Case(v) $i = (n-1)(m+1), \ldots, n(m+1)-2$**:** In this case, more than one component is obtained only if all vertex cuts of G_2 are deleted along with $(n-1)m+n$ vertices when only the i^{th} copy of G_2 remains. All subsets of $V(G_2)$ cannot be considered in the case where only the i^{th} copy of G_2 remains. All subsets of $V(G_2)$ cannot be considered in the case where only the i^{th} copy of G_2 remains among vertex cuts involving vertices in G_1 and G_2 as stated in case(ii). Therefore, $d_v(G_1 \circ G_2, i) = \sum_{j=1}^{n} \binom{n}{j}\binom{mn}{i-j} + nd_v(G_2, r) - \binom{mn}{r-1}$, $i = (n-1)(m+1), \ldots, n(m+1)-2$.

This completes the proof. □

Theorem 6.2.5. *If $n \geq 1$, $m \geq 3$, then the vertex cut polynomial of rooted product graph of path graph P_n and cycle graph C_m is given by*

$$\begin{aligned}V[P_n \circ_v C_m; x] =& nx + \sum_{i=2}^{m-1} n\left[\binom{(n-1)m}{i-1} + \binom{m}{i} - m + i\right] x^i + \\ & \sum_{k=1}^{n-2}\sum_{i=km}^{(k+1)m-1}\left[\sum_{j=1}^{i}\binom{n}{j}\binom{mn-i}{i-j} - (k+1)\right]x^i + \\ & \left[\sum_{j=1}^{m}\binom{n}{j}\binom{mn-m}{m-j} - n\right]x^{(n-1)m} + \\ & \sum_{i=(n-1)m+1}^{nm-2}\left[\sum_{j=1}^{i}\binom{n}{j}\binom{mn-i}{i-j} - nm\right]x^i.\end{aligned}$$

Proof. Consider four cases for $d_v(P_n \circ_v C_m, i)$, $i = 1, 2, \ldots, nm-2$.

Case(i) $i = 1$**:** Observe that root vertices are cut vertices; the remaining $(n-1)m$ vertices are not cut vertices. Therefore, $d_v(P_n \circ_v C_m, 1) = n$.

Case(ii) $i = 2, 3, \ldots, m-1$**:** Note that $V[C_m; x] = \sum_{i=2}^{m-2}[\binom{m}{i} - m]x^i$. That is in a cycle graph; we obtain a disconnected graph by the removal of any

i non-adjacent vertices. But in the case of $P_n \circ_v C_m$, i adjacent vertices, including the root vertex, will be vertex cut. Similarly, a vertex cut consists of $i-1$ vertices in $n-1$ copies of C_m that are not the i^{th} copy of C_m and the root vertex of the i^{th} copy of C_m. Therefore, $d_v(P_n \circ_v C_m, i) = \binom{(n-1)m}{i-1} + \binom{m}{i} - m + i$, $i = 2, 3, \ldots, m-1$.

Case(iii) $i = m, m+1 \ldots, (n-1)m-1$**:** Note that P_n has $\binom{n}{i} - i - 1$ vertex cuts with cardinality i. Similarly, the removal of $i+1$ copies of C_m cannot disconnect $P_n \circ_v C_m$. Thus, for $k = 1, 2, \ldots, n-2$, $d_v(P_n \circ_v C_m, i) = \sum_{j=1}^{i} \binom{n}{j}\binom{mn-i}{i-j} - (k+1)$, $i = km, km+1, \ldots, km-1$.

Case(iv) $i = (n-1)m$**:** Note that we get a connected graph if only one cycle is left when $(n-1)m$ vertices are deleted. We have there are n cycles. Therefore, $d_v(P_n \circ_v C_m, (n-1)m) = \sum_{j=1}^{m} \binom{n}{j}\binom{mn-m}{m-j} - n$.

Case(v) $i = (n-1)m+1, (n-1)m+2 \ldots, nm-2$**:** As stated in the previous case, when there is only one cycle left, deleting m adjacent vertices in that cycle does not result in a disconnected graph. Therefore, $d_v(P_n \circ_v C_m, i) = \sum_{j=1}^{i} \binom{n}{j}\binom{mn-i}{i-j} - nm.$, $i = (n-1)m+1, (n-1)m+2, \ldots, nm-2$.

This completes the proof. □

Theorem 6.2.6. *If $n \geq 1$, $m \geq 3$, then the vertex cut polynomial of Cartesian product of path graph P_n and cycle graph C_m is given by*

$$\begin{aligned} V[P_n \square C_m; x] =& (n-2)x^m + \sum_{i=m+1}^{2m-1} (n-2)\binom{(n-1)m}{i-m} x^i + \\ & \sum_{k=2}^{n-2} \sum_{i=km}^{(k+1)m-1} \left[(n-2)\binom{(n-1)m}{i-m} - (k+1)\binom{(n-k)m}{i-km} \right] x^i + \end{aligned}$$

$$\binom{(n-1)m}{(n-2)m}x^{(n-1)m} + \sum_{i=(n-1)m+1}^{mn-2}\left[(n-2)\binom{(n-1)m}{i-m} - nm\right]x^i.$$

Proof. Note that $\kappa(P_n \Box C_m) = m$. We consider several cases for $d_v(P_n \Box C_m, i)$, $i = m, m+1, \ldots, mn-2$.

Case(i) $i = m$**:** Observe that P_n has $n-2$ cut vertices. We obtain a disconnected graph by removal of the i^{th} copy of C_m, $i = 2, 3, \ldots, n-1$. The removal of other vertices cannot disconnect $P_n \Box C_m$. Therefore, $d_v(P_n \Box C_m, m) = n-2$.

Case(ii) $i = m+1, m+2, \ldots, 2m-1$**:** In this case, if we delete the j^{th} copy of C_m and $i-m$ vertices in the other $n-1$ copies of C_m, $j = 2, 3, \ldots, n-1$, $i = m+1, m+2, \ldots, 2m-1$, then we get a disconnected graph. Therefore, $d_v(P_n \Box C_m, i) = (n-2)\binom{(n-1)m}{i-m}$, $i = m+1, m+2, \ldots, 2m-1$.

Case(iii) $i = 2m, 2m+1 \ldots, (n-1)m-1$**:** For a fixed k, we obtain a disconnected graph by removal of k copies of C_m and $i-km$ vertices in other copies of C_m. Note that $d_v(P_n, i) = \binom{n}{i} - i - 1$. We cannot obtain a disconnected graph in P_n by the removal of $(i+1)$ set of vertices. Similarly, the removal of $(k+1)$ sets of k copies of C_m does not decompose the graph. Thus we cannot obtain a disconnected graph by the removal of $k+1$ set of k copies of C_m and $i-km$ vertices in other copies of C_m, where $i = km, km+1, \ldots, (k+1)m-1$.

Case(iv) $i = (n-1)m+1, \ldots, mn-2$**:** In this case, if the remaining vertex after deletion is a copy of a cycle, then to obtain a disconnected graph when the vertices are removed from it, the vertices that are not adjacent to the vertex cut of the cycle graph must be deleted. Each cycle has m adjacent sets of vertices. Therefore, $d_v(P_n \Box C_m, i) = (n-2)\binom{(n-1)m}{i-m} - nm$, $i = (n-1)m+1, \ldots, mn-2$.

This completes the proof. $\square$

Chapter 7

VC-Equivalent Classes of Graphs

The coefficients of some vertex cut polynomials are entirely the same. Similarly, vertex cut polynomials of non-isomorphic graphs are the same. In this chapter, we discuss those types of vertex cut polynomials of graphs, and we define and study VC-equivalent graphs and VC-equivalent classes of graphs.

7.1 VC-Equivalent Graphs

Definition 7.1.1. *Two graphs G and H with order n and m, respectively, are VC-equivalent if and only if $d_v(G,i) = d_v(H,i+j)$, $i = 1,2,\ldots,n-2$, $j = 0,1,2,\ldots,m-i-2$.*

Theorem 7.1.2. *Let W_n $(n \geq 3)$ is the wheel graph, C_n is the cycle graph, $F_{1,n}$ $(n \geq 3)$ is the fan graph, P_n is the path graph, $K_{1,n}$ $(n \geq 2)$ is the star graph, then*

1. *W_{n+1} and C_n are $VC-equivalent$,*

2. *$F_{1,n}$ and P_n are $VC -$ equivalent,*

3. *$\overline{K_{1,n}}$ and $K_{1,n}$ are $VC -$ equivalent.*

Proof

1. Note that $V(C_n, i) = \binom{n}{i} - n$, $i = 2, \ldots, n-2$ and $V(W_{n+1}, i) = \binom{n}{i-1} - n$, $i = 3, \ldots, n-3$. Then clearly $d_v(C_n, i) = d_v(W_{n+1}, i+1)$, $i = 2, \ldots, n-2$. Hence W_{n+1} and C_n are *VC − equivalent*.

2. Note that $V[P_n; x] = \sum_{i=1}^{n-2} \left[\binom{n}{i} - i - 1\right] x^i$ and $V[F_{1,n}; x] = \sum_{i=1}^{n-2} \left[\binom{n}{i} - i - 1\right] x^{i+1}$, then clearly $d_v(P_n, i) = d_v(F_{1,n}, i+1)$, $i = 1, 2, \ldots, n-2$. Hence $F_{1,n}$ and P_n are *VC − equivalent*.

3. Note that $V(K_{1,n}, i+1) = \binom{n}{i}$ and $V(\overline{K_{1,n}}, i) = \binom{n}{i}$, $i = 1, 2, \ldots, n-2$. Then $d_v(K_{1,n}, i+1) = d_v(\overline{K_{1,n}}, i)$, $i = 1, 2, \ldots, n-2$. Hence $K_{1,n}$ and $\overline{K_{1,n}}$ are *VC − equivalent*.

This completes the proof.

Theorem 7.1.3. *If $H \in VC[G]$, then H and G are $VC -$ equivalent.*

Proof. If $H \in VC[G]$, then $V[G; x] = V[H; x]$ that is $d_v(G, i) = d_v(H, i)$, $\forall i \geq 1$. Hence G and H are *VC − equivalent*. □

Remark 7.1.4. *The converse of the above theorem only true if G and H are $VC -$ equivalent, then $d_v(G, i) = d_v(H, i + j)$, $j = 0$.*

Lemma 7.1.5. *If $n \geq 3$, then*

$$V[CL_n; x] = 2n \sum_{i=0}^{2n-5} \binom{2n-4}{i} x^{i+3}.$$

Proof. Note that a circular ladder graph is a 3-regular graph. Let $v \in V(CL_n)$. We delete the vertices of the neighbors of v, that is, $N(v)$ (like the vertex cut polynomial of the Peterson graph); the remaining will be an isolated vertex and a component G having $2n-4$ vertices. Next, we delete 3 vertices not in $N(v)$, and then we get a connected graph. Thus the graph can be disconnected by the removal of $N(v)$ and i vertices in G. Thus $d_v(CL_n, i+3) = 2n\binom{2n-4}{i}$. Hence $V[CL_n; x] = 2n\sum_{i=0}^{2n-5}\binom{2n-4}{i}x^{i+3}$. This completes the proof. □

7.2 VC-Equivalent Classes of Graphs

Definition 7.2.1. *Let G be the graph; the VC-equivalent class of the graph G is defined as $VC[G] := \{H : V[G; x] = V[H; x]\}$.*

Theorem 7.2.2. *If CL_5 is the circular ladder graph and P is the Petersen graph, then $CL_5 \in VC[P]$.*

Proof. Note that $V[P; x] = 10\sum_{i=0}^{5}\binom{6}{i}x^{i+3}$ and $V[CL_5; x] = 10\sum_{i=0}^{5}\binom{6}{i}x^{i+3}$. Also $CL_5 \ncong P$ (by Theorem 4.2.6). Hence $CL_5 \in VC[P]$. This completes the proof. □

Theorem 7.2.3. *Let H and G be the graph of order m and n respectively. If H and G are $VC-equivalent$ and $H \notin VC[G]$, then $m \neq n$.*

Proof. Suppose $m = n$, then the largest power of $V[G; x]$ and $V[H; x]$ are $n-2$. Since G and H are $VC-equivalent$ and $H \notin VC[G]$, then $d_v(G, i) = d_v(H, i+j)$, $j \neq 0$, $i = 1, 2, \ldots, n-2$. Note that $d_v(G, n-2)$ exist. Thus $d_v(H, n-2+j)$

exists. Then we get a contradiction because the largest power of $V[H;x]$ is $n-2$. Hence $m \neq n$. This completes the proof. $\square$

Chapter 8

Conclusion and further scope of research

First section of this chapter is summary of the thesis and the next section includes some guidelines for future research work to explore more areas.

8.1 Summary of the thesis

In this thesis, we introduced two new graph polynomials named the complement degree polynomial of graphs and the vertex cut polynomial of graphs. These polynomials of many well-known graphs and graph operations are derived.

The stability, number of real roots and location of roots in the complex plane of the complement degree polynomial of graphs are also studied. In this thesis the concept of CD-equivalent classes of graphs and VC-equivalent classes of graphs are also introduced.

Complement degree polynomial of some graphs in chemical graph theory, biological graph theory and network theory are derived. In addition, we conclude $CD[G,x] = V(\overline{G},x)$ ($V(G,x) = \sum_{k=0}^{\Delta(G)} v_k x^k$, where $\Delta(G) = max\{deg(v) : v \in V\}$ and v_k is the number of vertices of degree k) [14].

8.2 Further scope of research

1. Study the stability of the vertex cut polynomial of graphs.
2. Study the roots of the vertex cut polynomial of graphs.
3. Identify more CD-equivalent polynomials.
4. Identify more VC-equivalent polynomials.
5. Investigate vertex cut polynomials in other branches like chemical graph theory, biological graph theory etc.

Bibliography

[1] Harary F, Graph Theory, Adison-Wesley.Springer, New York, 1969.

[2] Victor V. Prasolov, Polynomioals, Springer Science and Business Media, 2009.

[3] Smith, Michaei B, March, Jerry(2007). Advanced Organic Chemistry: Reactions, Mechanisms, and Structurew(6th ed.). New York:Wiley-interscience. p. 23. ISBN 978-0-471-72091-1.

[4] Shikhi.M, A Study on common Neighbor Polynomial of Graphs, Ph.D. Thesis, 2019.

[5] Muhammed imran, shehgnaz Akhter, Zahid Iqbal., On the Eccentric Connectivity polynomial of F-Sum of Connected graphs. Hindawi Complexity, volume 8 , 2020, Article ID 5061682, 9 pages.

[6] Chengzhi ZHANG, Huilin WANG, Yao LIU, Dan WU, Yi LIAO and Bo WANG, Automatic Key word Extraction from Documents using Conditional

Random Fields, Journal of computational information systems 4: 3(2008) 1169-1180.

[7] S. Alikhani, Dominating sets and domination polynomials of graphs, Ph.D. Thesis, 2009.

[8] Swali, Dr.Chinta mani, Tiwari, Stochastic modeling and applications, An introduction of graph theory in applied Mathematics, vol 25, No.1, ISSN:0972-3641.

[9] J.J.Sylvester: On an application of the new atomic theory to the graphical presentation of the invariants and covariants of binary quantics, with three appendices, American journal of Mathematics, Vol. 1, 161-228,1878.

[10] E.Sampathkumar and H.B. Walikar, On Splitting graph of a graph, Karnataka University Journal-Vol.XXV,1980, pages 13-16.

[11] Gajendar,Gaurav and Himanshu Sharma, Hurwitz polinomial, International journal of innovative research in technology,Vol.1(7),2014.

[12] M.P.Shyama and V.Anilkumar, On the roots of Hosoya polynomials, Journal of Discrete Mathematical Sciences and Criptography, Volume 19, 2016, pages 199-219.

[13] Ivan Gutman and Sandi Klavzar, Chemical Graph Theory of Fibonacences, MATCH Communications in Mathematical and Computer Chemistry, 55(2006) 39-54.

[14] A.M.Anto, Vertex Polynomial of Cycle related graphs, International Journal of Pure and Applied Mathematics, Volume 117,No.5,2017, 83-87.

[15] Weistein, Eric W."Wheel graph", Math world.

[16] Bondy, John Adrian; Murty, U.S.R(1976), Graph theory with applications, North Holland,P.ISBN 0-444-19451-7.

[17] Weistein,Eric W."Gear graph" Math World.

[18] Weistein,Eric W."Gear graph" Math World.

[19] Weistein,Eric W."Lollipop graph" Math World.

[20] DeMaio,Joe; Jocobson, John(2014)."Fibonacci number of the tadpole grah". Electronic Journal of Graph theory and Applications.

[21] V.Jeba Rani, S. Sundar Raj and T. Shyla Isaac Mary, Vertex Polynomial of Ladder graphs,Infokara Research, Volume 8, 2019,169-179.

[22] Wolfram Mathworld.https://mathworld.wolfram.com.

[23] V.P SWEDHA. R.VANITHASREE,"A Study on combination of shell graph and path P_2 graph". Journal of Emerging Technologies and Innovation Research(JETIR),May 2019,volume 6, issue 5.ISSN:2349-5162, 578-580.

[24] G.Sunkari, S.Lavanya, "Odd-Even Graceful Labeling of Umbrella and Tadpole Graphs", Interational Journalof Pure and Applied Mathematics, 114(6),2017,139-14.

[25] Ezhilasseri Hilda Stanky and JJeb Jesintha,"Butterfly Graphs with shell orders m and 2m+1 are Graceful", Bonfring International Journal of Research in Communication Engineering, vol 2,No 2, June 2012.

[26] Agarval, Pankaj K; Alone, Noga; aronov, Boris, Suri, Subhash(1994,"Can Geometry, 12(1): 347-365, doi:10.1007/BF02574385,MR 12991.

[27] P.Sumathi and B.Fathima,"Mod(k) Vertex Magic Labeling of Quadrilateral Snake Graphs", Advances and Applications in Mathematical Sciences,Volume 20, Issue 9,July 2021,Pages 1865-1877.

Appendix I

List of Publications

1. Safeera.K and Anilkumar V, *Vertex cut polynomial of graphs*, Advances and Applications in Discrete Mathematics, Volume 32, pages 1-12,2022.

2. Safeera.K and Anilkumar V, *Vertex cut polynomial of some unary graph operations*, Advances and Applications in Discrete Mathematics, Volume 37, pages 95-103,2023.

3. Safeera.K and Anilkumar V, *Stability of Complement Degree polynomial of graphs*, Baghdad Science Journal, 20(1 special issue)ICAAM: 300-304,2023.

4. Safeera.K and Anilkumar V, *On Real Roots of Complement Degree polynomial of graphs*, Ratio Mathematica, Volume 47,2023.

5. Safeera K and Anil Kumar V., Complement Degree Polynomial of Some Graph Operations , Global Scientific and academic Research Journal of Multidisciplinary Studies, Vol-3,Issue-10(2024), PP:60-64.

6. Safeera.K and Anilkumar V, *Complement Degree polynomial of graphs*, Southeast Asian Bulletin of Mathematics (Communicated).

7. Safeera.K, *Vertex Cut Polynomial of Some Binary Graph Operations*, Gulf Journal of Mathematics (Communicated).

8. Safeera.K, *CD-Equivalent and VC-Equivalent Classes of Graphs*, The Mathematics Student (Communicated).

9. Safeera.K, *Complement Degree Polynomial of Some Chemical Graphs*, Southeast Asian Bulletin of Mathematics (Communicated).

Appendix II

List of Presentations

1. The paper 'Stability of Complement Degree Polynomial of Graphs' presented in International Conference on Analysis and Applied Mathematics (ICAAM-2022) conducted by Ayya Nadar Janaki Ammal College.

2. The paper 'On Real Roots of complement Degree polynomial of Graphs' presented in International Conferenceon Advances in Mathematics Computer Engineering 2022 conducted by Nesamony Memorial Christian College.

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