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STUDIES ON C-SPACES

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CERTIFICATE

It is certified that, the reports of adjudicators for the thesis **"STUDIES ON C-SPACES"** in respect of **Ms. Sruthi A. K.**, Research Scholar, Department of Mathematics, University of Calicut have not suggested any modifications or corrections of the work.

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I hereby certify that the thesis entitled "STUDIES ON C-SPACES" is a bonafide work carried out by **Ms. SRUTHI A. K.**, initially under the guidance of Prof. Ramachandran P. T. and then after his demise, under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut. This work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Dr. Sini P.

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DECLARATION

I hereby declare that the work presented in the thesis entitled "**STUDIES ON C-SPACES**" is based on the original work done by me initially under the guidance of **Dr. Ramachandran P. T.**, Professor, Department of Mathematics, University of Calicut and then after his demise, under the guidance of **Dr. Sini P.**, Assistant Professor, Department of Mathematics, University of Calicut. This study has not been included in any other thesis submitted previously for the award of any degree. The contents of the thesis are undergone plagiarism check using iThenticate software at C.H.M.K. Library, University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.

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This is for you.

ABSTRACT

The concept of connectedness is discussed in numerous mathematical fields such as topology and graph theory and it is highly applicable in image filtering and segmentation, image compression and coding, motion analysis, pattern recognition etc. Both a topological and a graph theoretical framework are used to characterize connectivity. However, there are sometimes differences between topology and graph theory approaches to connectedness. But complications arise when these two approaches are used independently. In both theory and practice, a general description of connectedness that is applicable to both graph theory and topology is more beneficial. In 1983, R. Börger introduced an axiomatic approach to connectivity in order to standardize the definition of connectedness across these mathematical domains. The axioms were certain characteristics of connected sets such as empty set and singletons are connected and that union of connected sets having nonempty intersection is connected. A collection of subsets of a set satisfying these two axioms is called a c-structure and a set together with a c-structure on it is called a c-space.

The objective of this thesis is to present new contributions to the theory of c-spaces. Our primary focus is on the study of order-induced c-space, which is the c-space obtained from a linearly ordered set. Here, we discuss topological order induced c-spaces. We also characterize complete linearly ordered sets and dense linearly ordered sets in relation to order-induced c-space. Then we investigate the reversible property of c-spaces. The reversible c-spaces are characterized and prove the existence of non-reversible c-spaces with any infinite cardinality. Further, we define cut-point c-spaces and investigate the features of cut-point c-spaces. Moreover, we construct a cut-point c-structure on the union of an arbitrary family of mutually disjoint c-spaces if at least one of these c-spaces is a cut-point c-space. Finally, we associate c-spaces with hypergraphs. We discuss the properties of c-structures obtained from hypergraphs. Also, we prove that its members are the vertex sets of the connected hypersubgraphs of the given hypergraph.

Key Words: c-space, connective space, Order induced c-space, Reversible c-space, Cut-point c-space.

സംഗ്രഹം

കണക്ട്നെസ് എന്ന ആശയം, ടോപ്പോളജി, ഗ്രാഫ് തിയറി തുടങ്ങിയ നിരവധി ഗണിതശാസ്ത്ര ശാഖകളിൽ ഏറെ ചർച്ച ചെയ്യപ്പെടുന്നു. ഇമേജ് ഫിൽട്ടറിംഗ്, സെഗ്മെന്റേഷൻ, ഇമേജ് കംപ്രഷൻ, കോഡിംഗ്, മോഷൻ അനാലിസിസ്, പാറ്റേൺ റെക്കഗ്നിഷൻ മുതലായവയിൽ ഇത് വളരെ ഉപകാരപ്രദമാണ്. കണക്ടിവിറ്റിയെ ടോപ്പോളജിക്കൽ ആയും, ഗ്രാഫ് തിയററ്റിക്കൽ ആയും ചിത്രീകരിക്കാൻ സാധിക്കും. കണക്റ്റിവിറ്റിയിലേക്കുള്ള ഈ രണ്ട് സമീപനങ്ങളും തമ്മിൽ ചിലപ്പോൾ വ്യത്യാസങ്ങൾ ഉണ്ടാകാറുണ്ട്. എന്നാൽ ഇവയുടെ സ്വാതന്ത്രമായ ഉപയോഗം സങ്കീർണതകൾ ഉണ്ടാക്കുന്നു. ഗ്രാഫ് തിയറിക്കും ടോപ്പോളജിക്കും ബാധകമായ കണക്ടിവിറ്റിയുടെ പൊതുവായ ഗുണങ്ങൾ കൂടുതൽ പ്രയോജനകരമായതിനാൽ, ഗണിതശാസ്ത്ര ശാഖകളിലുട നീളമുള്ള കണക്ടിവിറ്റിയുടെ നിർവചനത്തെ ഏകീകരിക്കുന്നതിനായി 1983-ൽ, ആർ. ബോർഗർ എന്ന ഗണിതശാസ്ത്രജ്ഞൻ, കണക്റ്റിവിറ്റിക്ക് ഒരു ആക്സിയോമാറ്റിക് സമീപനം അവതരിപ്പിച്ചു.ശൂന്യഗണവും ഏകാംഗഗണവും കണക്റ്റഡ് ആണ്, സംഗമം ശൂന്യഗണമല്ലാത്ത ഗണങ്ങളുടെ യോഗം കണക്റ്റഡ് ആണ് എന്നിങ്ങനെയുള്ള കണക്ടിവിറ്റിയുടെ ചില സവിശേഷതകളായിരുന്നു പ്രമാണങ്ങളായി എടുത്തത്. ഈ രണ്ട് സവിശേഷതകളുള്ള ഒരു ഗണത്തിന്റെ ഉപഗണങ്ങളുടെ ശേഖരത്തെ സി- സ്ട്രക്ചർ എന്നും ഒരു ഗണവും അതിൽ നിന്ന് നിർവചിക്കുന്ന സി-സ്ട്രക്ചറും ചേർന്ന ജോഡിയെ സി-സ്പേസ് എന്നും പറയുന്നു.

സി-സ്പേസുകളുടെ സിദ്ധാന്തത്തിന് പുതിയ സംഭാവനകൾ നൽകുക എന്നതാണ് ഈ പ്രബന്ധം ലക്ഷ്യം വെക്കുന്നത്. ഞങ്ങളുടെ പ്രാഥമിക ശ്രദ്ധ രേഖീയമായി ഓർഡർ ചെയ്ത സെറ്റിൽ നിന്ന് ലഭിക്കുന്ന ഓർഡർ-ഇൻഡ്യൂസ്ഡ് സി-സ്പേസിന്റെ പഠനത്തിലാണ്. ടോപ്പോളജിക്കൽ ഓർഡർ ഇൻഡുസ്ഡ് സി-സ്പേസുകളും പഠനത്തിൽ ഉൾപ്പെടുത്തിയിട്ടുണ്ട്. ഇവിടെ ഞങ്ങൾ ഓർഡർ-ഇൻഡ്യൂസ്ഡ് സി-സ്പെയ്സുമായി ബന്ധപ്പെടുത്തി, കംപ്ലീറ്റ് ലീനിയർ ഓർഡർ സെറ്റുകളും ഡെൻസ് ലീനിയർ ഓർഡർ സെറ്റുകളും വിശേഷിപ്പിക്കുന്നു. തുടർന്ന് ഞങ്ങൾ സി-സ്പേസുകളുടെ റിവേഴ്സിബിൾ പ്രോപ്പർട്ടി അന്വേഷിക്കുന്നു. റിവേഴ്സിബിൾ സി-സ്പെയ്സുകളുടെ സ്വഭാവസവിശേഷതകൾ കണ്ടെത്തുകയും അനന്തമായ കാർഡിനാലിറ്റി യുള്ള നോൺ-റിവേഴ്സിബിൾ സി-സ്പെയ്സുകൾ ഉണ്ടെന്ന് തെളിയിക്കുകയും ചെയ്യുന്നു. കൂടാതെ, കട്ട്-പോയിന്റ് സി-സ്പേസുകൾ നിർവചിക്കുകയും കട്ട്-പോയിന്റ് സി-സ്പേസുകളുടെ സവിശേഷതകൾ അന്വേഷിക്കുകയും ചെയ്യുന്നു. മാത്രമല്ല, മ്യൂച്ചലി ഡിസ് ജോയിൻറ് സി-സ്പെയ്സുകളുടെ ശേഖരത്തിൽ ഒന്നെങ്കിലും കട്ട്-പോയിന്റ് സി-സ്പെയ്സ് ആണെങ്കിൽ, ഈ ശേഖരത്തിന്റെ യോഗത്തിൽ ഒരു കട്ട്-പോയിന്റ് സി-സ്ട്രക്ചർ ഞങ്ങൾ നിർമ്മിക്കുന്നു. അവസാനമായി, ഞങ്ങൾ ഹൈപ്പർഗ്രാഫുകളുമായി സി-സ്പേസുകളെ ബന്ധിപ്പിക്കുന്നു. ഹൈപ്പർഗ്രാഫുകളിൽ നിന്ന് ലഭിച്ച സി-സ്ട്രക്ചറുകളുടെ ഗുണങ്ങളെക്കുറിച്ച് ഞങ്ങൾ ചർച്ച ചെയ്യുന്നു. കൂടാതെ, നൽകിയിരിക്കുന്ന ഹൈപ്പർഗ്രാഫിന്റെ കണക്റ്റഡ് ആയിട്ടുള്ള ഹൈപ്പർ സബ്ഗ്രാഫുകളുടെ വെർട്ടെക്സ് സെറ്റുകളാണ് അതിലെ അംഗങ്ങൾ എന്ന് ഞങ്ങൾ തെളിയിക്കുന്നു.

Contents

Chapter 0

Introduction

Several mathematical fields of study, including topology, graph theory, and its fuzzy analogs, investigate the notion of connectedness. It is highly useful for motion analysis, pattern recognition, picture compression and coding, image filtering and segmentation etc [7, 14, 27, 28, 37, 38]. This becomes the motivation behind a thorough investigation of the collection of connected sets on a set, which is regarded as a structure on that set. The objective of this thesis is to present new contributions to the theory of connectivity. In the present study, we use set theoretical, topological, graph theoretical and order theoretical methods.

0.1 Motivation and Survey of Literature

The concept of connectivity can be introduced in different ways. In topology, connectedness is defined in terms of separation, which makes precise the instinctive idea that $[1,2)\cup(2,3]$ comprise of two pieces, while $[0,1)$ comprise of only one. According to the graph theoretic notion, a graph $\mathcal{G}$ is connected if there is a path in $\mathcal{G}$ with end vertices x and y for every vertex x and y in $\mathcal{G}$ [3].

The approaches to connectedness in topology and graph theory are not always synonymous. For example, the connectivity of the real line with the standard topology is not compatible with any graph. On the other hand, consider the well-known 4-adjacency and 8-adjacency connectivity for two dimensional discrete images. This connectivity is compatible with the connectivity on a graph with vertices in $\mathbb{Z}^2$ and edges given by an adjacency relation [7]; for (x, y) and (x', y') in $\mathbb{Z}^2$,

(i) (x, y) is 4-adjacent to (x', y') if $|x - x'| + |y - y'| = 1$,

(ii) (x, y) is 8-adjacent to (x', y') if $\max\{|x - x'|, |y - y'|\} = 1$.

Here, 8-adjacency connectivity is not equivalent to the connectivity of any topology on $\mathbb{Z}^2$. As a consequence, the adoption of topological and graph theoretical notions of connectedness independently has significant limitations; an integrated approach is more effective in theory as well as practice. In order to standardize the definition of connectedness across these mathematical domains, R. Börger [5] proposed an axiomatic approach to connectedness; the theory of connectivity classes or c-structures. The systematic study is on account of J. Serra [34, 35].

All meaningful notions of connectivity share the following properties [34]:

(i) empty set and singleton sets are connected,

(ii) union of connected sets with a nonempty intersection is connected.

A collection of subsets of X satisfying these two properties as axioms is called a *c-structure* [21]. A set together with a c-structure on it is called a *c-space* [21]. Following Serra, various mathematicians such as C. Ronse [26], J. Muscat and D. Buhagiar [21], S. Dugowson [12,13] etc. investigated it extensively [6,9,29–33,36,37,39]. B. M. Stadler and P. F. Stadler [43] condensed elementary results on connectivity spaces and their associated separation relations.

In [26], C. Ronse characterized connectivity in terms of separating pairs of sets. In that paper, he studied connectivity classes and systems of connectivity openings. In [14], H. J. A. M. Heijmans studied connected morphological operators for binary images.

The generation of connectivity structures were extensively studied by S. Dugowson in [13]. Also, he investigated the existence of limits and co-limits in the main categories of connectivity spaces. There, he characterized the finite connectivity structure by using irreducible connected subsets. He used the terminology integral connectivity spaces instead of c-spaces. Also, he introduced the connectivity tensor product of connective spaces.

J. Muscat and D. Buhagiar introduced a new category of connective spaces and they characterized topological connective spaces using compatible partial orders [21]. They proved that finite connective spaces are precisely simple graphs. He defined touching points, t-closed subsets of connective spaces and he characterized t-closed subsets of graphs.

In [25], K. P. Ratheesh and N. M. Madhavan Namboothiri discussed the concept of α-generated c-spaces and characterized finite topological c-spaces and connective spaces by using compatible transitive relations. They established that finite 2-generated c-spaces uniquely correspond to simple graphs. Also, they studied the lattice properties of c-spaces.

P. K. Santhosh [30] studied the product, the quotient, the sum of c-spaces and the strong and weak c-structures generated by a family of functions. He introduced a stronger form of connectedness, Y-connectedness, in c-spaces. He continued the works of S. Dugowson in [29–31, 33].

The notion of homogeneity in c-spaces is investigated by P. Sini and C. Darsana [9, 39]. In [39], completely homogeneous c-spaces are determined. She also characterized completely homogeneous connective spaces. Properties of hereditarily homogeneous c-structures are studied and characterised hereditarily homogeneous c-sructures in terms of group of c-automorphisms. [9]. They further proved that for connective spaces and finite c-spaces, the notion of complete homogeneity is the same as hereditarily homogeneity.

0.2 Organisation of the Thesis

This thesis comprises six chapters apart from the introduction, as follows:

The **first chapter** contains the preliminary definitions and results that are used in this thesis. It includes the basics of ordered set theory, topology, graph theory, hypergraph theory and the theory of c-spaces.

The **second chapter** is concerned with the study of order-induced c-space, which is the c-space obtained from a linearly ordered set. For a linearly ordered set $(X, \leq)$, a subset $A \subseteq X$ is said to be connected if for every $x, y \in A$ and $z \in X$ with $x \leq z \leq y$ implies $z \in A$. Then X, together with the collection of all connected sets, constitutes a c-space. Such c-space is known as the order induced c-space corresponding to the linearly ordered set $(X, \leq)$. Our focus is primarily on the properties of these c-spaces that are gained by the given order. We give a characterization of 2-generated order induced c-spaces. Further, we characterize finite order induced c-spaces that are homogeneous. Then we discuss the two relevant sub c-spaces of order induced c-space and give a necessary and sufficient condition for these to be identical.

Further, we discuss the t-closed subsets of an order induced c-space. Moreover, we characterize dense linearly ordered sets using t-closed sets. We also discuss the relation between order preserving and order reversing functions of a linearly ordered set and c-continuous functions of the corresponding order induced c-space. Then we move on to topological order induced c-spaces. Also, we proved that order induced c-space satisfied the fourth axiom of a connective space and give a necessary and sufficient condition for an order induced c-space to satisfy the third axiom. This leads to the characterization of a complete linearly ordered set.

In the **third chapter**, we investigate the reversible property of c-spaces. M. Rajagopalan and A. Wilansky in [23] introduced reversible topological spaces. Further analogous study of reversible properties of various structures on a set other than topology was carried out by several authors [8, 17, 18]. A c-space is reversible whenever every c-continuous bijection is a c-isomorphism. We show that an order induced c-space is reversible. Also, we characterize the reversible c-spaces in connection with stronger and weaker c-structures. We prove the existence of non-reversible c-spaces with any infinite cardinality. Also, we prove that c-isomorphism preserves reversible properties. Moreover, a c-space is reversible if and only if its Brunnian closure is reversible. Then we discuss the reversibility of sub c-spaces and the intersection, union, quotient space, product and sum of reversible c-spaces.

In [11] Muscat J. and Buhagiar D. mentioned homogeneously n-connected connective spaces. Analogous to this, we define homogeneously n-connected c-spaces. A c-space is called homogeneously n-connected when removing any n points disconnect it. In the **fourth chapter**, we investigate the features of homogeneously 1-connected c-spaces and rename it as cut-point c-spaces. The collection of all connected subsets of cut-point space discussed by B. Honari and Y. Bahrampour in [15] is a trivial example of cut-point c-space. Here, we construct a cut-point c-structure on the union of an arbitrary family of mutually disjoint c-spaces if at least one of these c-spaces is a cut-point c-space. Then we characterize the order induced c-spaces, which are cut-point c-spaces. Further, we define irreducible cut-point c-space where no proper sub c-space of it is a cut-point c-space and characterize irreducible cut-point c-space as order induced. Also, give a characterization of order induced c-spaces that are cut-point connective space.

In the **fifth chapter**, we associate c-spaces with hypergraphs. Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph and let $\mathfrak{C}_{\mathcal{H}} =< \mathcal{E} >$, the c-structure generated by the edge set $\mathcal{E}$. Then the c-space $(X, \mathfrak{C}_{\mathcal{H}})$ is known as the c-space induced by the hypergraph $\mathcal{H}$. We discuss the properties of c-spaces obtained from hypergraphs and prove that the members of $\mathfrak{C}_{\mathcal{H}}$ are the vertex sets of the connected hyper-subgraphs of $\mathcal{H}$. Moreover, we investigate the relation between the group of all automorphisms of the hypergraph $\mathcal{H} = (X, \mathcal{E})$ and the group of all automorphisms of the c-space $(X, \mathfrak{C}_{\mathcal{H}})$ induced by the hypergraph $\mathcal{H}$.

The **last chapter** contains the conclusion and some unsolved problems. A bibliography is also provided.

Chapter 1

Preliminaries

This chapter is devoted to the preliminary concepts that will be used in the forthcoming chapters. It includes the basics of set theory, order theory, topology, graph theory and hypergraph theory. Also, it covers some basic definitions and properties of c-spaces.

- In what follows, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all real numbers, the set of all rational numbers, the set of all integers and the set of all natural numbers respectively.
- If not mentioned $\leq$ on $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the usual ordering.
- Throughout our discussion, X denotes a non empty set.
- If A is a given set, then we use $|A|$ to denote the cardinality of A and $\mathcal{P}(A)$ the power set of A.

1.1 Ordered Set Theory

Basic set theoretic notions are adopted from [1, 22, 44].

Let $\leq$ be a relation [16] on a set X. Then it is said to be

1. *reflexive* if for all $x \in X$, $x \leq x$.
2. *anti-symmetric* if for all $x, y \in X$, $x \leq y$ and $y \leq x$ implies $x = y$.
3. *transitive* if for all $x, y, z \in X$, $x \leq y$ and $y \leq z$ implies $x \leq z$.

A relation that is reflexive, transitive and anti-symmetric is called a *partial order*. A *partially ordered set* (or a *poset*) [16] is an ordered pair $(X, \leq)$ where X is a set and $\leq$ is a partial order on X. The partial order $\leq$ is called *linear order* [16] if for every $a, b \in X$, either $a \leq b$ or $b \leq a$.

Let $(X, \leq)$ be a partially ordered set. Then $a \in X$ is the *first element* [1] of X if $a \leq x$ for every $x \in X$ and $b \in X$ is the *last element* [1]of X if $x \leq b$ for every $x \in X$. Further, $l \in X$ is the *lower bound* [1] of a subset A of X if $l \leq x$ for every $x \in A$ and $u \in X$ is the *upper bound* [1] of A if $x \leq u$ for every $x \in A$.

Let $(X, \leq)$ be a partially ordered set and $A \subseteq X$. Then A is said to be *bounded below* [1] whenever A has a lower bound. If A has an upper bound, then A is said to be *bounded above*. If A has both lower bound and upper bound, then A is called *bounded* [1]. An element $u \in X$ is said to be the *least upper bound* or *supremum* [16] of A (denoted by $\sup A$) if and only if u is an upper bound of A and $u \leq x$ for all upper bounds x of A. Similarly, an element $l \in X$ is said to be the *greatest lower bound* or *infimum* [16] of A (denoted by $\inf A$) if and only if l is a lower bound of A and $x \leq l$ for all lower bounds x of A.

Theorem 1.1.1. *[1] Let* $(X, \leq)$ *be a partially ordered set and let* k *be the greatest lower bound of the set* U *of all upper bounds of a subset* A *of* X*. Then* $k \in U$ *and* $k = \inf U = \sup A$*.*

Theorem 1.1.2. *[1] Let* $(X, \leq)$ *be a partially ordered set. Every nonempty subset of* X *that is bounded above has the least upper bound if and only if every nonempty subset of* X *that is bounded below has a greatest lower bound.*

A partially ordered set $(X, \leq)$ is said to be *complete* [16] if every nonempty subset which is bounded above has a supremum. A subset A of X is said to be *dense* in X if for every $x, y \in X$ with $x < y$, there exists $a \in A$ such that $x < a < y$. A complete dense linearly ordered set is called *linear continuum.*

Let $(X, \leq)$ and $(Y, \leq')$ be two partially ordered sets. A function $f : X \to Y$ is said to be *order preserving* if for every $x, y \in X$, $x \leq y \Rightarrow f(x) \leq' f(y)$. Now, f is said to be *order reversing* if for every $x, y \in X$, $x \leq y \Rightarrow f(y) \leq' f(x)$. The partially ordered sets X and Y are said to *order isomorphic* if there exists a bijection $f : X \to Y$ such that both f and f^{-1} are order preserving.

1.2 Topology

In this thesis, we use the connectedness property of topological spaces. So we begin with the definition of connectedness. Other basic definitions and notations are adopted from [19, 20, 46].

A topological space X is said to be *connected* [16] if it is impossible to find nonempty subsets A and B of it such that $X = A \cup B$ and $\overline{A} \cap \overline{B} = \emptyset$. Here, the subsets A and B are said to be *mutually separated*. A topological space which is not connected is called *disconnected*.

For a topological space X, the following are equivalent [16].

1. X is connected.
2. X cannot be written as the disjoint union of two nonempty closed subsets.
3. The only clopen subsets of X are $\emptyset$ and X.
4. Every nonempty proper subsets of X has a nonempty boundary.
5. X cannot be written as the disjoint union of two nonempty open subsets.

A subset C of X is said to be *connected* [16] whenever C with the relative topology is a connected space. Let X be a topological space and C be a connected subset of X such that $C \subseteq A \cup B$, where A, B are mutually separated subsets of X. Then either $C \subseteq A$ or $C \subseteq B$ [16]. The closure of a connected subset is connected. More generally, if C is a connected subset of a topological space X then any set D such that $C \subseteq D \subseteq \overline{C}$ is connected. Furthermore, the continuous image of a connected set is connected [16].

Let $\mathfrak{C}$ be a collection of connected subsets of a topological space X such that no two members of $\mathfrak{C}$ are mutually separated. Then $\bigcup\limits_{C \in \mathfrak{C}} C$ is also connected [16]. Let $\mathfrak{C}$ be a collection of all connected subsets of a space X and suppose K is a connected subset of X (not necessarily a member of $\mathfrak{C}$) such that $C \cap K \neq \emptyset$ for all $C \in \mathfrak{C}$. Then $(\bigcup\limits_{C \in \mathfrak{C}} C) \cup K$ is connected [16].

A maximally connected subset of a topological space X is called *component* [16]. The components are closed sets and two distinct components are mutually disjoint. Every nonempty connected subset is contained in a unique component. Moreover, every topological space is the disjoint union of its components.

1.3 Graph and Hypergraph Theory

In this section, we mainly give some basic hypergraph theoretical concepts which are essential for our study. The basic graph theoretic notions are adopted from [3].

A *graph* $\mathcal{G}$ is an ordered pair $(V, \mathcal{E})$, where V is a nonempty finite set and $\mathcal{E}$ is a set of two element subsets of V. The elements of V are called *vertices* [3] and the elements of $\mathcal{E}$ are called *edges*. The graph $\mathcal{G}$ is said to be *connected* [45] if any two vertices in it are connected by some path. Otherwise, it is called *disconnected*.

Now, we discuss the concept hypergraphs, the generalization of graphs. A *hypergraph* [45] is an ordered pair $\mathcal{H} = (X, \mathcal{E})$, where X is a set of elements called vertices and $\mathcal{E}$ is a family of subsets of X called edges. A hypergraph $\mathcal{H}' = (X', \mathcal{E}')$ is said to be a *hypersubgraph* [2] or *strong subhypergraph* [10] of $\mathcal{H}$ whenever $X' \subseteq X$ and $\mathcal{E}' \subseteq \mathcal{E}$. Two hypergraphs $\mathcal{H} = (X, \mathcal{E})$ and $\mathcal{H}' = (X', \mathcal{E}')$ are said to be *isomorphic* [4] if there exists a bijection $h : X \to X'$ such that for every $E \subseteq X$, $E \in \mathcal{E}$ if and only if $h(E) \in \mathcal{E}'$.

In a hypergraph $\mathcal{H} = (X, \mathcal{E})$, a *chain* [4] from the vertex x_1 to the vertex x_{q+1} is an alternated vertex-edge sequence $(x_1, E_1, x_2, E_2, \dots E_q, x_{q+1})$ of distinct vertices and edges of $\mathcal{H}$ such that for $i = 1, 2, \dots, q, \{x_i, x_{i+1}\} \subseteq E_i$, where q is called the length of the chain.

The two vertices a and b of a hypergraph $\mathcal{H} = (X, \mathcal{E})$ are said to be *connected* [4] in $\mathcal{H}$ if there exists a chain from a to b. A hypergraph $\mathcal{H}$ is said to be *connected* if every pair of distinct vertices are connected.

1.4 Theory of c-spaces

Here, we present certain definitions and notations related to c-spaces. For more details, see [13, 21].

A *c-structure* [21] on a set X is a collection $\mathfrak{C}_X$ of subsets of X such that the following axioms hold:

(i) $\emptyset \in \mathfrak{C}_X$ and $\{x\} \in \mathfrak{C}_X$ for every $x \in X$.

(ii) If $\{C_i : i \in I\}$ is a nonempty collection of subsets in $\mathfrak{C}_X$ with $\bigcap_{i \in I} C_i \neq \emptyset$, then $\bigcup_{i \in I} C_i \in \mathfrak{C}_X$.

The set X together with a c-structure $\mathfrak{C}_X$, that is $(X, \mathfrak{C}_X)$ is called a *c-space* and elements of $\mathfrak{C}_X$ are called connected sets in X with respect to $\mathfrak{C}_X$. Moreover, empty set and singleton sets are called *trivial* connected sets and others are called *non-trivial* connected sets. If $X \in \mathfrak{C}_X$, then $(X, \mathfrak{C}_X)$ is called a *connected c-space.*

The collection of all trivial connected sets of a set X is a c-structure on X and is denoted by $\mathfrak{D}_X$. Then c-space $(X, \mathfrak{D}_X)$ is called *discrete c-space* [21]. Let us go through some more examples.

Example 1.4.1. [30] For any set X, the c-space $(X, \mathfrak{C}_X)$ is known as

1. *indiscrete c-space* if $\mathfrak{C}_X = \mathcal{P}(X)$.
2. *rooted c-space rooted at* x if $\mathfrak{C}_X = \mathfrak{D}_X \cup \{A \subseteq X : x \in A\}$ for $x \in X$.
3. *co-finite c-space* if $\mathfrak{C}_X = \mathfrak{D}_X \cup \{A \subseteq X : A \text{ is infinite}\}$ and $|X|$ is infinite.
4. *co-countable c-space* if $\mathfrak{C}_X = \mathfrak{D}_X \cup \{A \subseteq X : A \text{ is uncountable}\}$ and X is an uncountable set.

Definition 1.4.2. Let $(X, \mathfrak{C}_X)$ be a c-space. Then the c-space $(X, \mathfrak{C}_X \cup \{X\})$ is called the *Brunnian closure* [13] of the given c-space.

The set of all c-structures on a set X can be partially ordered by set inclusion. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two c-structures on X such that $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$. Then $\mathfrak{C}_1$ is said to be *weaker* [30] than $\mathfrak{C}_2$ and $\mathfrak{C}_2$ is said to be *stronger* than $\mathfrak{C}_1$.

A c-space $(X, \mathfrak{C}_X)$ is said to be *topological* [25] if there exists a topology τ on X such that the associated c-space of (X, τ) is $(X, \mathfrak{C}_X)$. It is said to be *graphical* [30] if there exists a graph $\mathcal{G}$ such that the collection of all connected sets coincide with $\mathfrak{C}_X$.

Let $(X, \mathfrak{C}_X)$ be a c-space. For $Y \subseteq X$, define $\mathfrak{C}_Y = \{C \in \mathfrak{C}_X : C \subseteq Y\}$. Then the c-space $(Y, \mathfrak{C}_Y)$ is called *sub c-space* [21] of $(X, \mathfrak{C}_X)$ on Y.

Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$, then the intersection of all c-structures on X containing $\mathcal{B}$ is a c-structure on X and is called the *c-structure generated* *by* $\mathcal{B}$ [21] and is denoted by $< \mathcal{B} >$. A c-structure $\mathfrak{C}_X$ on X is said to be α-*generated* [25] if there is a sub collection $\mathcal{B} \subseteq \{A \in \mathfrak{C}_X : |A| \leq \alpha\}$ such that $\mathfrak{C}_X =< \mathcal{B} >$, where α is any cardinal with $\alpha \leq |X|$.

Remark 1.4.3. If the c-space $(X, \mathfrak{C}_X)$ is 2-generated [33] then for any $Y \subseteq X$, the sub c-space $(Y, \mathfrak{C}_Y)$ is 2-generated.

Proposition 1.4.4. *[21] The non-trivial connected sets of a c-structure generated by* $\mathcal{B}$ *are characterized by the condition that any two points of such a connected set* C *can be joined by a finite chain of basic connected sets (ie, elements of* $\mathcal{B}$*) in* $\mathfrak{C}_X$*. That is, for all* $x, y \in C$*, we can find elements* B_i*,* $i = 0$ *to* n *in* $\mathcal{B}$ *such that* $B_i \subseteq C$*,* $B_i \cap B_{i+1} \neq \emptyset$ *for* $i = 0$ *to* $n - 1$ *and* $x \in B_0$*,* $y \in B_n$*.*

Let $(X, \mathfrak{C}_X)$ be a c-space and $A \subseteq X$. A point $x \in X$ is said to *touch* [21] the set A if there is a nonempty subset C of A such that $\{x\} \cup C$ is connected. The set of all points touching the set A is denoted by $t(A)$. If $t(A) = A$ then A is said to be *t-closed* [21]. The smallest t-closed set containing A is called *connective closure* [21] of A, denoted by $\overline{\text{A}}$. That is, $\overline{\text{A}} = \bigcap\{E \subseteq X : t(E) = E \text{ and } A \subseteq E\}$.

Let $(X, \mathfrak{C}_X)$ and $(Y, \mathfrak{C}_Y)$ be two c-spaces. A function $f : X \to Y$ is called *c-continuous* [21] if $C \in \mathfrak{C}_X \Rightarrow f(C) \in \mathfrak{C}_Y$. Then f is said to be a *c-isomorphism* [21] if it is bijective and both f and f^{-1} are c-continuous. The c-spaces are said to be *c-isomorphic* [21] if there exists a c-isomorphism between them.

Remark 1.4.5. [33] Let $(X, \mathfrak{C}_X)$ and $(Y, \mathfrak{C}_Y)$ be c-isomorphic c-spaces. If $(X, \mathfrak{C}_X)$ is 2-generated, then $(Y, \mathfrak{C}_Y)$ is 2-generated.

A c-space $(X, \mathfrak{C}_X)$ is said to be *homogeneous* [21] if for any two points x and y in X, there is a c-isomorphism $f : X \to X$ such that $f(x) = y$. It is said to be *completely homogeneous* [39] if every bijection on X is a c-isomorphism on X.

A *connective structure* [21] on a set X is a c-structure $\mathfrak{C}_X$ on X such that the following axioms hold:

(iii) Given any nonempty sets $A, B \in \mathfrak{C}_X$ with $A \cup B \in \mathfrak{C}_X$, then there exists $x \in A \cup B$ such that $\{x\} \cup A \in \mathfrak{C}_X$ and $\{x\} \cup B \in \mathfrak{C}_X$,

(iv) If $A, B, C_i \in \mathfrak{C}_X$ $(i \in I)$ are disjoint subsets of X and $A \cup B \cup (\bigcup_{i \in I} C_i) \in \mathfrak{C}_X$, then there exists $J \subseteq I$, $A \cup (\bigcup_{j \in J} C_j) \in \mathfrak{C}_X$ and $B \cup (\bigcup_{i \in I \setminus J} C_i) \in \mathfrak{C}_X$.

The set X together with a connective structure $\mathfrak{C}_X$, that is $(X, \mathfrak{C}_X)$ is called a *connective space*.

Chapter 2

Order Induced c-spaces

2.1 Introduction

This chapter explores order induced c-space, which is the c-space obtained from a linearly ordered set. Our focus is primarily on the properties of these c-spaces that are gained by the given order. We also discuss topological order induced c-spaces. Furthermore, we characterize complete linearly ordered sets and dense linearly ordered sets in relation to order-induced c-space. A part of this chapter is published in the Palestine Journal of Mathematics [41].

2.2 c-structure on a Linearly Ordered Set

A subset I of a linearly ordered set $(X, \leq)$ is said to be an interval if for every $x, y \in I$ and $z \in X$ with $x \leq z \leq y$ implies $z \in I$. We start by convincing the reader that the collection of all intervals in a linearly ordered set satisfies all the axioms of a c-structure.

Theorem 2.2.1. *Let $(X, \leq)$ be a linearly ordered set and $\mathfrak{C}_X$ be the collection of all intervals in X. Then $\mathfrak{C}_X$ is a c-structure on X.*

Proof. It is clear that $\emptyset \in \mathfrak{C}_X$ and $\{x\} \in \mathfrak{C}_X$ for every $x \in X$. Let $\{C_i : i \in I\}$ be a nonempty collection of intervals in X with $\bigcap_{i \in I} C_i \neq \emptyset$. Let $x, y \in \bigcup_{i \in I} C_i$ and $z \in X$ be such that $x \leq z \leq y$. Then there exist $j, k \in I$ such that $x \in C_j$ and $y \in C_k$. Since $\bigcap_{i \in I} C_i \neq \emptyset$, there exists $w \in \bigcap_{i \in I} C_i$. If $w \leq z$, then $w, y \in C_k$ and $w \leq z \leq y$ implies that $z \in C_k$. Hence $z \in \bigcup_{i \in I} C_i$. If $z \leq w$, then $x \leq z \leq w$ and $x, w \in C_j$ implies that $z \in C_j$. Hence $z \in \bigcup_{i \in I} C_i$. This gives $\bigcup_{i \in I} C_i \in \mathfrak{C}_X$ whenever $\bigcap_{i \in I} C_i \neq \emptyset$. Therefore, $\mathfrak{C}_X$ is a c-structure on X. □

Now, we give a formal definition for the c-space obtained from a linearly ordered set.

Definition 2.2.2. Let $\mathfrak{C}_{(X,\leq)}$ be the collection of all intervals of a linearly ordered set $(X, \leq)$. Then $\mathfrak{C}_{(X,\leq)}$ is called the order induced c-structure and the ordered pair $(X, \mathfrak{C}_{(X,\leq)})$ is called the order induced c-space corresponding to the linearly ordered set $(X, \leq)$ or the c-space induced by the linearly ordered set $(X, \leq)$.

For an order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$, it is always true that $X \in \mathfrak{C}_{(X,\leq)}$. That is, an order induced c-space is always connected. Let us move through some examples of order induced c-spaces.

Example 2.2.3. Let $X = \{p_1, p_2, \ldots, p_n\}$ and the ordering $\leq$ on X is given by $p_1 \leq p_2 \leq \ldots \leq p_n$. Then the order induced c-structure corresponding to the linearly ordered set $(X, \leq)$ is $\mathfrak{C}_{(X,\leq)} =< \{\{p_k, p_{k+1}\} : k = 1, 2, \ldots n-1\} >$.

Example 2.2.4. Let $X = \{p_1, p_2, p_3, p_4\}$. The two linear orders $\leq$ and $\leq'$ on X are given by $p_1 \leq p_2 \leq p_3 \leq p_4$ and $p_4 \leq' p_2 \leq' p_3 \leq' p_1$. Then the c-structure induced by the order $\leq$ is

$$\mathfrak{C}_{(X,\leq)} = \mathfrak{D}_X \cup \{\{p_1, p_2\}, \{p_2, p_3\}, \{p_3, p_4\}, \{p_1, p_2, p_3\}, \{p_2, p_3, p_4\}, X\}.$$

The c-structure induced by the order $\leq'$ is

$$\mathfrak{C}_{(X,\leq')} = \mathfrak{D}_X \cup \{\{p_2, p_4\}, \{p_2, p_3\}, \{p_1, p_3\}, \{p_2, p_3, p_4\}, \{p_1, p_2, p_3\}, X\}.$$

In the above examples, the order induced c-spaces $(X, \mathfrak{C}_{(X,\leq)})$ and $(X, \mathfrak{C}_{(X,\leq')})$ are c-isomorphic. If we take any linear order other than $\leq$ on X, the c-space induced by that linear order is also c-isomorphic to $(X, \mathfrak{C}_{(X,\leq)})$. In general, for any finite set X, the c-structures induced by distinct linear orders are c-isomorphic. This need not be true for an infinite set. This is exhibited by the following example.

Example 2.2.5. For $X = \mathbb{N}$, let $\leq$ be the usual ordering of natural numbers and the order $\leq'$ on X is given by $2 \leq' 3 \leq' 4 \leq' \ldots \leq' 1$. Here, the order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$ is 2-generated since $\mathfrak{C}_{(X,\leq)} =< \{\{n, n+1\} : n \in \mathbb{N}\} >$. But the order induced c-space $(X, \mathfrak{C}_{(X,\leq')})$ is not 2-generated, because the conditions of Proposition 1.4.4 is not satisfied here for $x = 2$ and $y = 1$. Therefore, the order induced c-spaces $(X, \mathfrak{C}_{(X,\leq)})$ and $(X, \mathfrak{C}_{(X,\leq')})$ are not c-isomorphic.

There are instances where distinct linear orders on a set induce the same c-structure. One such example is given in the following theorem.

Theorem 2.2.6. *Let $(X, \leq)$ be a linearly ordered set. Then the c-structures induced by the order $\leq$ and its dual $\geq$ are the same.*

Proof. Let $\mathfrak{C}_{(X,\leq)}$ and $\mathfrak{C}_{(X,\geq)}$ be the c-structures induced by the order $\leq$ and its dual respectively and let $C \in \mathfrak{C}_{(X,\leq)}$. Suppose $x, y \in C$ and $z \in X$ be such that $x \geq z \geq y$. Then $y \leq z \leq x$ and $y, x \in C \in \mathfrak{C}_{(X,\leq)}$ gives $z \in C$. This implies $C \in \mathfrak{C}_{(X,\geq)}$. Therefore, $\mathfrak{C}_{(X,\leq)} \subseteq \mathfrak{C}_{(X,\geq)}$. Similarly, we can prove that $\mathfrak{C}_{(X,\geq)} \subseteq \mathfrak{C}_{(X,\leq)}$. □

2.3 Sub c-spaces of Order Induced c-spaces

Let $(X, \mathfrak{C}_{(X,\leq)})$ be an order induced c-space. Then corresponding to every subset $Y \subseteq X$, we get two relevant substructures of $\mathfrak{C}_{(X,\leq)}$. First one is the sub c-structure of $\mathfrak{C}_{(X,\leq)}$ on Y given by $\mathfrak{C}_Y = \{A \in \mathfrak{C}_{(X,\leq)} : A \subseteq Y\}$. Another one is the order induced c-structure $\mathfrak{C}_{(Y,\leq_Y)}$, where $\leq_Y$ is the order of Y inherited from the order $\leq$ on X.

It's not necessary for these two substructures to be identical. The example which follows illustrates this.

Example 2.3.1. Consider the linearly ordered set $(X, \leq)$, where $X = \{1, 2, 3, 4\}$ and $\leq$ is the usual ordering. The order induced c-structure on X is given by $\mathfrak{C}_{(X,\leq)} = \mathfrak{D}_X \cup \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3\},\{2,3,4\},X\}$. Let $Y \subseteq X$ be given by $Y = \{1,2,4\}$. Then the order induced c-structure on Y is $\mathfrak{C}_{(Y,\leq_Y)} = \mathfrak{D}_Y \cup \{\{1,2\},\{2,4\},Y\}$. The sub c-structure of $\mathfrak{C}_{(X,\leq)}$ on Y is $\mathfrak{C}_Y = \mathfrak{D}_Y \cup \{\{1,2\}\}$. Here, $\mathfrak{C}_{(Y,\leq_Y)} \neq \mathfrak{C}_Y$.

The next theorem gives a necessary and sufficient condition for $\mathfrak{C}_Y = \mathfrak{C}_{(Y,\leq_Y)}$.

Theorem 2.3.2. *Let $(X, \mathfrak{C}_{(X,\leq)})$ be an order induced c-space and $\mathfrak{C}_Y$ be the sub c-structure of $\mathfrak{C}_{(X,\leq)}$ on a subset Y of X. Then,*

(i) $\mathfrak{C}_Y \subseteq \mathfrak{C}_{(Y,\leq_Y)}$

(ii) $\mathfrak{C}_Y = \mathfrak{C}_{(Y,\leq_Y)}$ *if and only if Y is an interval in X*

where $\mathfrak{C}_{(Y,\leq_Y)}$ is the c-structure induced by the linearly ordered set $(Y, \leq_Y)$.

Proof. (i) Let $A \in \mathfrak{C}_Y$. Then $A \in \mathfrak{C}_{(X,\leq)}$ and $A \subseteq Y$. That is, A is an interval in X that completely contained in Y. Therefore, $A \in \mathfrak{C}_{(Y,\leq_Y)}$.

(ii) Suppose Y is an interval in X and $A \in \mathfrak{C}_{(Y,\leq_Y)}$. Let $x, y \in A$ and $z \in X$ be such that $x \leq z \leq y$. Since $A \subseteq Y$ and $Y \in \mathfrak{C}_{(X,\leq)}$, it follows that $z \in Y$. But $x, y \in A$ and $z \in Y$ implies $z \in A$. Thus we get if $x, y \in A$ and $z \in X$ be such that $x \leq z \leq y$ then $z \in A$. This implies $A \in \mathfrak{C}_{(X,\leq)}$. Also, we have $A \subseteq Y$, therefore $A \in \mathfrak{C}_Y$. This implies $\mathfrak{C}_{(Y,\leq_Y)} \subseteq \mathfrak{C}_Y$. Then from (i) we get $\mathfrak{C}_Y = \mathfrak{C}_{(Y,\leq_Y)}$. Conversely, suppose Y is not an interval in X. Then $Y \notin \mathfrak{C}_{(X,\leq)}$ implies that $Y \notin \mathfrak{C}_Y$. But we have $Y \in \mathfrak{C}_{(Y,\leq_Y)}$. This implies that $\mathfrak{C}_Y \neq \mathfrak{C}_{(Y,\leq_Y)}$.

□

Let $(X, \leq)$ be a linearly ordered set. For any $Y \subseteq X$, the order induced c-structure on $(Y, \leq_Y)$ need not be a subset of the order induced c-structure on $(X, \leq)$. This is interpreted in Example 2.3.1, where we get $\mathfrak{C}_{(Y,\leq_Y)} \nsubseteq \mathfrak{C}_{(X,\leq)}$.

Theorem 2.3.3. *For a linearly ordered set $(X, \leq)$, let $\mathfrak{C}_{(X,\leq)}$ and $\mathfrak{C}_{(Y,\leq_Y)}$ be the order induced c-structures on X and a subset Y of X respectively. Then $\mathfrak{C}_{(Y,\leq_Y)} \subseteq \mathfrak{C}_{(X,\leq)}$ if and only if Y is an interval in X.*

Proof. Suppose Y is an interval in X. Then by Theorem 2.3.2, $\mathfrak{C}_Y = \mathfrak{C}_{(Y,\leq_Y)}$. This implies $\mathfrak{C}_{(Y,\leq_Y)} \subseteq \mathfrak{C}_{(X,\leq)}$. Conversely, suppose $Y \notin \mathfrak{C}_{(X,\leq)}$. But we have $Y \in \mathfrak{C}_{(Y,\leq_Y)}$ and hence $\mathfrak{C}_{(Y,\leq_Y)} \nsubseteq \mathfrak{C}_{(X,\leq)}$. □

2.4 Properties of Order Induced c-spaces

Here we deduce a few special features of order-induced c-spaces. First, we prove a lemma, which is important in the subsequent sections.

Lemma 2.4.1. *Let $A, B \in \mathfrak{C}_{(X,\leq)}$ be disjoint. If $a_0 \leq b_0$ for some $a_0 \in A$ and $b_0 \in B$, then $a \leq b$ for every $a \in A$ and for every $b \in B$.*

Proof. Let $a_0 \in A$ and $b_0 \in B$ be such that $a_0 \leq b_0$. Then $a \leq b_0$ for every $a \in A$. Otherwise, there exists $a' \in A$ such that $b_0 \leq a'$. Then $a_0 \leq b_0 \leq a'$ follows $b_0 \in A$, which contradicts the fact that $A \cap B = \emptyset$. Now, we have $a \leq b_o$ for every $a \in A$. This gives $a \leq b$ for every $a \in A$ and for every $b \in B$. Otherwise, there exist $a' \in A$ and $b' \in B$ such that $b' \leq a'$. Then $b' \leq a' \leq b_0$ follows $a' \in B$, which contradicts the fact that $A \cap B = \emptyset$. □

Theorem 2.4.2. *For an order induced c-space* $(X, \mathfrak{C}_{(X,\leq)})$*, let* $A, B, C \in \mathfrak{C}_{(X,\leq)}$ *be disjoint and* $A \cup B \cup C \in \mathfrak{C}_{(X,\leq)}$*. Then either* $A \cup B \in \mathfrak{C}_{(X,\leq)}$ *or* $A \cup C \in \mathfrak{C}_{(X,\leq)}$*.*

Proof. Let A, B and C be disjoint subsets of X such that A, B, C and $A \cup B \cup C$ are connected in X. Suppose that $A \cup B \notin \mathfrak{C}_{(X,\leq)}$. Then there exist $x, y \in A \cup B$ and $z \in X \setminus (A \cup B)$ such that $x < z < y$. But $A \cup B \cup C \in \mathfrak{C}_{(X,\leq)}$ implies $z \in C$. Let $x \in A$ and $y \in B$. Since $x < z < y$ and $z \in C$, by Lemma 2.4.1 we have $a \leq c \leq b$, for every $a \in A$ for every $c \in C$ and for every $b \in B$. If $A \cup C \notin \mathfrak{C}_{(X,\leq)}$, then there exist $a_0 \in A$, $c_0 \in C$ and $z_0 \in X \setminus (A \cup C)$ such that $a_0 < z_0 < c_0$. But $A \cup B \cup C \in \mathfrak{C}_{(X,\leq)}$ gives $z_0 \in B$. That is, $z_0 < c_0$ for some $z_0 \in B$ and $c_0 \in C$, which is a contradiction. If $x \in B$ and $y \in A$, then also we get similar contradiction. Therefore, $A \cup C \in \mathfrak{C}_{(X,\leq)}$. □

Theorem 2.4.3. *For an order induced c-space* $(X, \mathfrak{C}_{(X,\leq)})$*, let* A*,* $B \in \mathfrak{C}_{(X,\leq)}$ *with* $|A \cup B| \geq 3$ *be disjoint and* $A \cup B \in \mathfrak{C}_{(X,\leq)}$*. Then there exists a nontrivial connected set* $C \subsetneq A \cup B$ *other than* A *and* B*.*

Proof. If $A = \emptyset$, choose $x_1, x_2, x_3 \in B$ with $x_1 < x_2 < x_3$, which always exist. Then $C = [x_1, x_2]$ satisfies the required conditions. Now, suppose $A \neq \emptyset$ and $B \neq \emptyset$. If $|A \cup B| = 3$, then take $A = \{x\}$ and $B = \{y, z\}$ with $y < z$. Then, either $x < y$ or $z < x$. Choose

$$C = \begin{cases} \{x, y\}, & \text{if } x < y. \\ \{z, x\}, & \text{if } z < x. \end{cases}$$

If $|A \cup B| > 3$, then choose x_1, x_2, x_3, $x_4 \in A \cup B$ with $x_1 < x_2 < x_3 < x_4$. Now, let $C = [x_2, x_3]$. In both cases, C satisfies all the required conditions. $\square$

Proposition 2.4.4. *Let $(X, \mathfrak{C}_{(X,\leq)})$ be an order induced c-space. If $C_i \in \mathfrak{C}_{(X,\leq)}$, for every $i \in I$, then $\bigcap_{i \in I} C_i \in \mathfrak{C}_{(X,\leq)}$.*

Proof. Let $x, y \in \bigcap_{i \in I} C_i$ and $z \in X$ be such that $x \leq z \leq y$. Then $x, y \in C_i$ and $C_i \in \mathfrak{C}_{(X,\leq)}$ implies $z \in C_i$. Since this is true for every $i \in I$, we get $z \in \bigcap_{i \in I} C_i$. Therefore $\bigcap_{i \in I} C_i \in \mathfrak{C}_{(X,\leq)}$. $\square$

Theorem 2.4.5. *Let $(X, \mathfrak{C}_{(X,\leq)})$ be an order induced c-space and $A \in \mathfrak{C}_{(X,\leq)}$. If $X \setminus A \notin \mathfrak{C}_{(X,\leq)}$, then $X \setminus A$ has two components.*

Proof. Consider the order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$ and let $A \in \mathfrak{C}_{(X,\leq)}$. Suppose $X \setminus A \notin \mathfrak{C}_{(X,\leq)}$. Then there exist $p, q \in X \setminus A$ and $r \in A$ such that $p \leq r \leq q$. Now, let $C_1 = \{x \in X \setminus A : x < r\}$ and $C_2 = \{x \in X \setminus A : r < x\}$. Then, clearly $X \setminus A = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$. Let $x, y \in C_1$ and $z \in X$ such that $x \leq z \leq y$. Then $y < r$ implies $z < r$. If $z \in A$, then $z < y < r$ and $z, r \in A \in \mathfrak{C}_{(X,\leq)}$ implies that $y \in A$, which is a contradiction. Thus, $z \in X \setminus A$ and hence $z \in C_1$. Therefore, $C_1 \in \mathfrak{C}_{(X,\leq)}$. Similarly, we can show that $C_2 \in \mathfrak{C}_{(X,\leq)}$. Thus, C_1 and C_2 are the components of $X \setminus A$. $\square$

Proposition 2.4.6. *Let $(X, \mathfrak{C}_{(X,\leq)})$ be an order induced c-space and $Y \subseteq X$ be such that $|Y| \geq 3$. Then there exists $y \in Y$ such that $X \setminus \{y\}$ has two components. Also, Y intersect with the two components of $X \setminus \{y\}$.*

Proof. Choose $y_1, y_2, y_3 \in Y$ such that $y_1 \leq y_2 \leq y_3$. Now, let $y = y_2$, $C_1 = \{x \in X : x < y\}$ and $C_2 = \{x \in X : y < x\}$. It is clear that C_1 and C_2 are the components of $X \setminus \{y\}$ and Y intersects C_1 and C_2. □

Theorem 2.4.7. *If an order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$ is 2-generated, then every element except the first element of the linearly ordered set $(X, \leq)$ has an immediate predecessor and every element except the last element has an immediate successor.*

Proof. Suppose that the order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$ is 2-generated. Then there exists $\mathcal{B} \subseteq \{A \in \mathfrak{C}_{(X,\leq)} : |A| = 2\}$ such that $\mathfrak{C}_{(X,\leq)} =< \mathcal{B} >$.

Let $y \in X$ be such that y is not the first element of X. Then there exists $y' \in X$ such that $y' < y$. Now, let $C_1 = [y', y]$. By Proposition 1.4.4, there exist $\{B_0, B_1, ..., B_n\} \subseteq \mathcal{B}$ such that $B_i \subseteq C_1$, $B_i \cap B_{i+1} \neq \emptyset$, $y' \in B_0$ and $y \in B_n$. Then $B_n = \{y'', y\}$ for some $y'' \in X$. Clearly, $y'' < y$ and there does not exist any $z \in X$ such that $y'' < z < y$. That is, y'' is the immediate predecessor of y.

Now, let $w \in X$ be such that w is not the last element of X. Then there exists $w' \in X$ such that $w < w'$. As in the previous part, corresponding to the set $C_2 = [w, w']$ there exist $\{B'_0, B'_1, ..., B'_n\} \subseteq \mathcal{B}$ such that $B'_i \subseteq C_2$, $B'_i \cap B'_{i+1} \neq \emptyset$, $w \in B'_0$ and $w' \in B'_n$. Then $B'_0 = \{w, w''\}$ for some $w'' \in X$. Clearly, $w < w''$

and there does not exist any $z \in X$ such that $w < z < w''$. That is, w'' is the immediate successor of w. $\square$

The converse of Theorem 2.4.7 is not true. The example that follows exemplifies this.

Example 2.4.8. Consider the linearly ordered set $(\mathbb{N}, \leq')$ where the order $\leq'$ is given be $2 \leq' 3 \leq' 4 \leq' \ldots \leq' 1$. Here, every element except 2 has an immediate predecessor and every element except 1 has an immediate successor. It is clear that $C = \{1, 2, 3, \ldots\} \in \mathfrak{C}_{(\mathbb{N}, \leq')}$. But the conditions of Proposition 1.4.4 is not satisfied for $x = 2$ and $y = 1$ and hence the order induced c-space $(\mathbb{N}, \mathfrak{C}_{(\mathbb{N}, \leq')})$ is not 2-generated.

Theorem 2.4.9. *An order induced c-space $(X, \mathfrak{C}_{(X, \leq')})$ is 2-generated if and only if it is c-isomorphic to a sub c-space of the order induced c-space $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z}, \leq)})$.*

Proof. Suppose the order induced c-space $(X, \mathfrak{C}_{(X, \leq')})$ is 2-generated. Choose $a_0 \in X$ such that a_0 is neither the first element nor the last element. If such an element does not exist, then $|X| \leq 2$ and clearly $(X, \mathfrak{C}_{(X, \leq')})$ is c-isomorphic to a sub c-space of $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z}, \leq)})$.

Let a_{-1} and a_1 denote the immediate predecessor and the immediate successor of a_0 respectively, which exist by Theorem 2.4.7. If a_{-1} is not the first element, then it has an immediate predecessor a_{-2}. Similarly, if a_1 is not the last element, then it has an immediate successor a_2. If we continue in this way, we will have four possibilities.

One possibility is that, the process of finding immediate predecessor and immediate successor will never stop. That is, $(X, \leq')$ neither has the first element nor has the last element. In this case, $(X, \mathfrak{C}_{(X,\leq')})$ is c-isomorphic to $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$.

Secondly, it may happen that both the process of finding immediate predecessor and process of finding immediate successor will stop. That is, $(X, \leq')$ has both the first element a_{-m} and the last element a_n for some positive integers m, n. Then $(X, \mathfrak{C}_{(X,\leq')})$ is c-isomorphic to the sub c-space $(Y, \mathfrak{C}_Y)$ of $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$, where $Y = \{-m, -(m-1), \ldots, -1, 0, 1, 2, \ldots, n\}$.

Another possibility is that $(X, \leq')$ has the first element a_{-m} but has no last element. Then $(X, \mathfrak{C}_{(X,\leq')})$ is c-isomorphic to a sub c-space $(Y, \mathfrak{C}_Y)$ of $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$, where $Y = \{-m, -(m-1), \ldots, -1, 0, 1, 2, \ldots\}$.

Last possibility is that $(X, \leq')$ has no first element but has the last element a_n. Then $(X, \mathfrak{C}_{(X,\leq')})$ is c-isomorphic to a sub c-space $(Y, \mathfrak{C}_Y)$ of $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$, where $Y = \{\ldots, -1, 0, 1, 2, \ldots, n\}$.

Conversely, suppose the order induced c-space $(X, \mathfrak{C}_{(X,\leq')})$ is c-isomorphic to a sub c-space $(Y, \mathfrak{C}_Y)$ of $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$. It is clear that the order induced c-space $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$ is 2-generated. As a sub c-space of a 2-generated c-space, $(Y, \mathfrak{C}_Y)$ is 2-generated. Then $(X, \mathfrak{C}_{(X,\leq')})$ is 2-generated since it is c-isomorphic to $(Y, \mathfrak{C}_Y)$. □

By Theorem 2.4.9, we can easily deduce that 2-generated order induced c-space is always countable.

Theorem 2.4.10. *A finite order induced c-space* $(X, \mathfrak{C}_{(X,\leq)})$ *is homogeneous if and only if* $|X| \leq 2$.

Proof. The proof is given in Remark 4.3.3. □

2.5 t-closed Subsets of Order Induced c-spaces

This section deals with t-closed subsets of an order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$. If $|X|$ is finite, then $t(A) \neq A$ for every proper nontrivial subset A of X. In other words, $\emptyset$ and X are the only t-closed subsets of $(X, \mathfrak{C}_{(X,\leq)})$. If $|X|$ is infinite then there may or may not exists proper nontrivial t-closed subsets. Let us go through the following example.

Example 2.5.1. Consider the order induced c-space $(\mathbb{N}, \mathfrak{C}_{(X,\leq')})$, where the ordering $\leq'$ is given by $1 \leq' 3 \leq' 5 \leq' \ldots \ \ldots 6 \leq' 4 \leq' 2$. Let $A = \{1, 3, 5, \ldots\}$. Then $t(A) = A$. That is, $(\mathbb{N}, \mathfrak{C}_{(\mathbb{N},\leq')})$ has a proper nontrivial t-closed subset. But $\emptyset$ and $\mathbb{Z}$ are the only t-closed sets of the order induced c-space $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$.

Theorem 2.5.2. *If the order induced c-space* $(X, \mathfrak{C}_{(X,\leq)})$ *is 2-generated, then no proper nontrivial subset of* X *is t-closed.*

Proof. Consider the 2-generated order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$. If $|X| \leq 2$, then there is nothing to prove.

Now, let $|X| > 2$ and A be a proper nontrivial subset of X. Choose $x \in A$ and $y \in X \setminus A$, which always exist. If $x \leq y$, then by Proposition 1.4.4, there exist finite number of elements $b_1, b_2, \ldots, b_n \in X$ such that $x = b_1 < b_2 < \ldots < b_n = y$ and b_{i+1} is the immediate successor of b_i for $i = 1, 2, \ldots, n-1$. Clearly, $b_n \in X \setminus A$. Let k be the smallest integer such that $b_k \in X \setminus A$. Then $\{b_{k-1}, b_k\} \in \mathfrak{C}_{(X,\leq)}$ and $b_{k-1} \in A$ implies $b_k \in t(A)$. Therefore, $t(A) \neq A$. If $y < x$, then also by similar arguments we can show that $t(A) \neq A$. That is, there does not exist any proper nontrivial t-closed sets in $(X, \mathfrak{C}_{(X,\leq)})$. $\square$

Remark 2.5.3. The converse of Theorem 2.5.2 is not true. Consider the order induced c-space $(X, \mathfrak{C}_{(X,\leq')})$, where $X = \{0, 1, 2, 3, \ldots\}$ and the ordering $\leq'$ is given by $1 \leq' 2 \leq' 3 \leq' \ldots \leq' 0$. Here, $\emptyset$ and X are the only t-closed sets of $(X, \mathfrak{C}_{(X,\leq')})$, but it is not a 2-generated c-space.

Remark 2.5.4. Theorem 2.5.2 is not true for every 2-generated c-space. Consider the 2-generated c-space $(X, \mathfrak{C}_X)$, where $X = \{1, 2, 3, 4\}$ and $\mathfrak{C}_X = \mathfrak{D}_X \cup \{\{1, 2\}\}$. Let $A = \{1, 2\}$. Then $t(A) = A$ and hence A is t-closed in $(X, \mathfrak{C}_X)$. Thus, a 2-generated c-space may have a proper nontrivial t-closed subset.

The closed and bounded interval in $(X, \leq)$ need not be t-closed in $(X, \mathfrak{C}_{(X,\leq)})$. For example, the interval $[m, n]$, where $m, n \in \mathbb{Z}$ is not a t-closed subset of $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$ since $m - 1$ and $n + 1$ touch $[m, n]$.

Next theorem provides a necessary and sufficient condition for every closed and bounded interval to be t-closed in $(X, \mathfrak{C}_{(X,\leq)})$. In order to prove this theorem, we need the following definition.

Definition 2.5.5. A c-space $(X, \mathfrak{C}_X$ is said to be C_1 [21] if distinct points do not touch.

Theorem 2.5.6. *Let $(X, \mathfrak{C}_{(X,\leq)})$ be an order induced c-space. Then the following are equivalent.*

(i) $(X, \leq)$ *is a dense linearly ordered set.*

(ii) *For every $a, b \in X$ with $a \leq b$, $[a, b]$ is t-closed in $(X, \mathfrak{C}_{(X,\leq)})$.*

(iii) *For every $x \in X$, $\{x\}$ is t-closed in $(X, \mathfrak{C}_{(X,\leq)})$.*

(iv) $(X, \mathfrak{C}_{(X,\leq)})$ *is C_1.*

Proof. ***(i) ⇒ (ii)***

Suppose $(X, \leq)$ is a dense linearly ordered set. Assume that there exist $a, b \in X$ with $a < b$ such that $[a, b]$ is not t-closed. So there is some $x \in X \setminus [a, b]$ such that $x \in t([a, b])$. Then there exists a nonempty subset C of $[a, b]$ such that $\{x\} \cup C \in \mathfrak{C}_{(X,\leq)}$. Choose an element $c_0 \in C$. Then clearly $a \leq c_0 \leq b$. Now, let $x < a$. Since $(X, \leq)$ is dense, there exists $z \in X$ such that $x < z < a$. This implies that $z \in \{x\} \cup C$. Since $z \neq x$, we have $z \in C$. Hence $z \in [a, b]$, which is a contradiction since $z < a$. Similarly, if $b < x$, we will get a contradiction. Therefore, $[a, b]$ is t-closed in $(X, \mathfrak{C}_{(X,\leq)})$.

(ii) ⇒ (iii)

Suppose $[a, b]$ is t-closed for every $a, b \in X$ with $a \leq b$. Take $a = b = x$, then we get $\{x\}$ is t-closed.

(iii)* ⇒ *(i)

Suppose $t(\{x\}) = \{x\}$ for every $x \in X$. Assume that $(X, \leq)$ is not a dense linearly ordered set. Then there exist $a, b \in X$ with $a < b$ such that there is no element in X between a and b. This implies $\{a, b\} \in \mathfrak{C}_{(X,\leq)}$. Then $b \in t(\{a\})$, which is a contradiction. Therefore, for every $a, b \in X$ with $a < b$, there exists $z \in X$ such that $a < z < b$. That is, $(X, \leq)$ is a dense linearly ordered set.

(iii)* ⇔ *(iv)

This follows from the definition of a C_1 c-space. □

Theorem 2.5.6 is a characterization of a dense linearly ordered set in terms of t-closed subsets of the corresponding order induced c-space.

Remark 2.5.7. If $(X, \leq)$ is a dense linearly ordered set, then the order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$ has proper nontrivial t-closed subsets.

Remark 2.5.8. The property of being dense in Theorem 2.5.7 cannot be replaced by completeness property. The order induced c-space $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$ corresponding to the complete linearly ordered set $(\mathbb{Z}, \leq)$ has no proper nontrivial t-closed subsets.

Definition 2.5.9. Let $(X, \mathfrak{C}_X)$ be a c-space and $Y \subseteq X$. Then Y is said to be c-dense [39] in X if $t(Y) = X$

Theorem 2.5.10. *Let $(X, \leq)$ be a dense linearly ordered set and $A \subseteq X$. If A is c-dense in $(X, \mathfrak{C}_{(X,\leq)})$, then A is dense in $(X, \leq)$.*

Proof. Suppose $A \subseteq X$ be such that $t(A) = X$. Let $x, y \in X$ be such that $x < y$. Since X is dense itself, there exists $z \in X$ such that $x < z < y$. If $z \in A$, then there is nothing to prove. Now, let $z \in X \setminus A$. Since $z \in t(A)$, there exists a nonempty subset $C \subseteq A$ such that $\{z\} \cup C \in \mathfrak{C}_{(X,\leq)}$. Choose an element $a_0 \in C \subseteq A$. If $x < a_0 < y$, then the proof is complete. Now, suppose $a_0 \leq x$. Since X is dense itself, there exists $p \in X$ be such that $x < p < z$. Then $a_0 \leq x < p < z$ and $a_0, z \in \{z\} \cup C$ implies $p \in \{z\} \cup C$. Hence $p \in C \subseteq A$, since $p \neq z$. Similarly, if $y \leq a_0$, there exists $p \in A$ such that $x < p < y$. Therefore, A is dense in $(X, \leq)$. □

The converse of Theorem 2.5.10 is not true. Consider the order induced c-space $(\mathbb{R}, \mathfrak{C}_{(\mathbb{R},\leq)})$ corresponding to the dense linearly ordered set $(\mathbb{R}, \leq)$. Here, $\mathbb{Q}$ is dense in $(\mathbb{R}, \leq)$, but $\mathbb{Q}$ is not c-dense in $(\mathbb{R}, \mathfrak{C}_{(\mathbb{R},\leq)})$.

2.6 Functions of Order Induced c-spaces

Here, we go through the characteristics of functions of order induced c-spaces. The fundamental concern is the relationship between order preserving and order reversing functions of a linearly ordered set and c-continuous functions of the corresponding order induced c-space.

Theorem 2.6.1. *Let $(X, \mathfrak{C}_{(X,\leq)})$ be an order induced c-space and f is a c-continuous function on X. If $a, b \in X$ and $y \in X$ satisfies $f(a) < y < f(b)$, then there exists $x \in X$ between a and b such that $f(x) = y$.*

Proof. Consider the c-continuous function f on $(X, \mathfrak{C}_{(X,\leq)})$. Let $a, b \in X$ and $y \in X$ be such that $f(a) < y < f(b)$. Now, let $C = [a, b]$ if $a < b$, otherwise take $C = [b, a]$. Then $C \in \mathfrak{C}_{(X,\leq)}$ implies $f(C) \in \mathfrak{C}_{(X,\leq)}$. Hence $f(a), f(b) \in f(C)$ and $f(a) < y < f(b)$ implies $y \in f(C)$. That is, there exists $x \in C \subseteq X$ such that $f(x) = y$. This completes the proof. □

Theorem 2.6.2. *If f is a c-continuous bijection on an order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$, then f is a c-isomorphism on $(X, \mathfrak{C}_{(X,\leq)})$.*

Proof. Let $f : (X, \mathfrak{C}_{(X,\leq)}) \to (X, \mathfrak{C}_{(X,\leq)})$ be a c-continuous bijection and $C \in \mathfrak{C}_{(X,\leq)}$ with $|C| > 1$. Let $a, b \in f^{-1}(C)$ and $z \in X$ be such that $a < z < b$. Consider $C_1, C_2 \in \mathfrak{C}_{(X,\leq)}$ given by $C_1 = [a, z]$ and $C_2 = [z, b]$. Since f is c-continuous, $f(C_1), f(C_2) \in \mathfrak{C}_{(X,\leq)}$. For $f(a), f(b) \in C$, either $f(a) < f(b)$ or $f(b) < f(a)$. Suppose $f(a) < f(b)$.

If $f(z) < f(a)$, then $f(a) \in f(C_2)$ as $f(z), f(b) \in f(C_2)$ and $f(z) < f(a) < f(b)$. This implies that $a \in C_2$, which is a contradiction. If $f(b) < f(z)$, then $f(b) \in f(C_1)$ as $f(a), f(z) \in f(C_1)$ and $f(a) < f(b) < f(z)$. This implies that $b \in C_1$, which is also a contradiction. Therefore, $f(a) < f(z) < f(b)$. It follows that $f(z) \in C$ and hence $f^{-1}(C) \in \mathfrak{C}_{(X,\leq)}$.

If $f(b) < f(a)$, then also by using similar arguments, it is easy to show that $f^{-1}(C) \in \mathfrak{C}_{(X,\leq)}$. This implies that f^{-1} is c-continuous. Thus, $f : (X, \mathfrak{C}_{(X,\leq)}) \to (X, \mathfrak{C}_{(X,\leq)})$ is a c-isomorphism. □

There are order preserving functions and order reversing functions on $(X, \leq)$

but not c-continuous on $(X, \mathfrak{C}_{(X,\leq)})$.

Example 2.6.3. Consider the linearly ordered set $(\mathbb{Z}, \leq)$ and the functions f, g on $(\mathbb{Z}, \leq)$ defined by $f(x) = 2x$ and $g(x) = -2x$. Then f is order preserving and g is order reversing. Here, $A = \{1, 2\} \in \mathfrak{C}_{(\mathbb{Z},\leq)}$. But $f(A) = \{2, 4\} \notin \mathfrak{C}_{(\mathbb{Z},\leq)}$ and $g(A) = \{-2, -4\} \notin \mathfrak{C}_{(\mathbb{Z},\leq)}$. Therefore, f and g are not c-continuous on $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$.

Theorem 2.6.4. *If a bijection $f : (X, \leq) \to (Y, \leq')$ is order preserving or order reversing, then $f : (X, \mathfrak{C}_{(X,\leq)}) \to (Y, \mathfrak{C}_{(Y,\leq')})$ is c-continuous.*

Proof. Let $f : X \to Y$ be an order preserving bijectin and let $C \in \mathfrak{C}_{(X,\leq)}$. Let $f(a), f(b) \in f(C)$ and $z \in Y$ such that $f(a) <' z <' f(b)$. Then $a, b \in C$ and $a \leq b$. Otherwise, $b < a$ implies $f(b) <' f(a)$, which is not true. Since f is surjective, there exists $w \in X$ such that $f(w) = z$. Then $a < w$. Otherwise, $z = f(w) <' f(a)$, which is not possible. Similarly, we can easily show that $w < b$. Therefore, $a < w < b$. Hence it follows that $w \in C$ and $z = f(w) \in f(C)$. Thus, $f(C) \in \mathfrak{C}_{(X,\leq')}$. Since C is arbitrary, f is c-continuous.

Now, suppose the bijection $f : X \to Y$ is order reversing. Consider the dual $(Y, \geq')$ of the linearly ordered set $(Y, \leq')$. If we consider f as a function from $(X, \leq)$ to $(Y, \geq')$, it is order preserving. Then we have $f : (X, \mathfrak{C}_{(X,\leq)}) \to (Y, \mathfrak{C}_{(Y,\geq')})$ is c-continuous. Since $\mathfrak{C}_{(Y,\leq')} = \mathfrak{C}_{(Y,\geq')}$, we get that $f : (X, \mathfrak{C}_{(X,\leq)}) \to (Y, \mathfrak{C}_{(Y,\leq')})$ is a c-continuous function. $\square$

There are c-continuous functions on $(X, \mathfrak{C}_{(X,\leq)})$ that are neither order preserving nor order reversing on $(X, \leq)$.

Example 2.6.5. Consider the linearly ordered set $(X, \leq)$ where $X = \{1, 2, 3, 4\}$ and $\leq$ is the usual ordering of numbers. Then the order induced c-structure on X is given by $\mathfrak{C}_{(X,\leq)} = \mathfrak{D}_X \cup \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. Let f be the function on X given by

$$f(x) = \begin{cases} 2, & \text{if } x = 1, 4. \\ 1, & \text{if } x = 2, 3. \end{cases}$$

Then $f(C) \in \mathfrak{C}_{(X,\leq)}$ for every $C \in \mathfrak{C}_{(X,\leq)}$ implies f is c-continuous. But f is neither order preserving nor order reversing.

Theorem 2.6.6. *If $f : (X, \mathfrak{C}_{(X,\leq)}) \to (Y, \mathfrak{C}_{(Y,\leq')})$ is a c-continuous one-one function then $f : (X, \leq) \to (Y, \leq')$ is either order preserving or order reversing.*

Proof. Let $f : (X, \mathfrak{C}_{(X,\leq)}) \to (Y, \mathfrak{C}_{(Y,\leq')})$ be a c-continuous one-one function such that $f : (X, \leq) \to (Y, \leq')$ is not order preserving. Then there exist $a, b \in X$ such that $a < b$ and $f(b) <' f(a)$.

Assume that f is not order reversing. Then there exist $x, y \in X$ with $x < y$ such that $f(x) <' f(y)$. Then we have the following cases.

Case(i) : $x < a$

Since $x \notin [a, b]$, we have $f(x) \notin [f(b), f(a)]$. Otherwise, $f(b) \leq' f(x) \leq' f(a)$ and $f(a), f(b) \in f([a, b]) \in \mathfrak{C}_{(Y,\leq')}$ implies $f(x) \in f([a, b])$. Since f is one-one, we get $x \in [a,b]$, which is not true. Therefore, either $f(x) <' f(b)$ or $f(a) <' f(x)$. If $f(x) <' f(b)$, then $f(x), f(a) \in f([x,a]) \in \mathfrak{C}_{(Y,\leq')}$ and $f(x) <' f(b) <' f(a)$ implies $f(b) \in f([x,a])$. This implies $b \in [x,a]$, which is a contradiction.

Now, let $f(a) <' f(x)$. If $y < a$, then $f(a) <' f(x) <' f(y)$ and $f(a), f(y) \in f([y,a]) \in \mathfrak{C}_{(Y,\leq')}$ implies $f(x) \in f([y,a])$. This implies $x \in [y,a]$, which is a contradiction. Similarly, if $a \leq y$ then we get the contradiction $x \in [a,y]$.

Case(ii) : $b < y$

As in case (i), $y \notin [a,b]$ implies $f(y) \notin [f(b), f(a)]$. Therefore, either $f(a) <' f(y)$ or $f(y) <' f(b)$. If $f(a) <' f(y)$, then $f(b) <' f(a) <' f(y)$ and $f(b), f(y) \in f([b,y]) \in \mathfrak{C}_{(Y,\leq')}$ implies $f(a) \in f([b,y])$. Hence $a \in [b,y]$, which is not true.

Now, let $f(y) <' f(b)$. If $x \leq b$, then $f(x) <' f(y) <' f(b)$ and $f(x), f(b) \in f([x,b]) \in \mathfrak{C}_{(Y,\leq')}$ implies $f(y) \in f([x,b])$. Hence $y \in [x,b]$, which is a contradiction. Similarly, if $b < x$ then we get the contradiction $y \in [b,x]$.

Case(iii) : $a \leq x < y \leq b$

Since $x \in [a,b]$, we have $f(x) \in [f(b), f(a)]$. Otherwise, either $f(x) <' f(b)$ or $f(a) <' f(x)$. If $f(x) <' f(b)$, then $f(x), f(a) \in f([a,x]) \in \mathfrak{C}_{(Y,\leq')}$ and $f(x) <' f(b) <' f(a)$ implies $f(b) \in f([a,x])$. This implies $b \in [a,x]$, which is not true. If $f(a) <' f(x)$, then $f(b), f(x) \in f([x,b]) \in \mathfrak{C}_{(Y,\leq')}$ implies $f(a) \in f([x,b])$. This follows that $a \in [x,b]$, which is not true.

Similarly, we have $f(y) \in [f(b), f(a)]$. Therefore, $f(b) \leq' f(x) <' f(y) \leq' f(a)$. Then $f(x) <' f(y) \leq' f(a)$ and $f(x), f(a) \in f([a,x]) \in \mathfrak{C}_{(Y,\leq')}$ implies $f(y) \in f([a,x])$. Therefore, $y \in [a,x]$, a contradiction.

Since we get contradictions in all cases, our assumption that $f(x) <' f(y)$ is wrong. Therefore, $f(y) \leq' f(x)$ whenever $x \leq y$. Thus, f is order reversing. $\square$

Now using Theorem 2.6.6, we can easily deduce that the only c-continuous bijections on a finite order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$, where $X = \{x_1, x_2, \ldots, x_n\}$ are the identity function and the function $f(x_r) = x_{n+1-r}$ for $r = 1, 2, \ldots, n$.

Remark 2.6.7. Theorem 2.6.6 is not true for c-continuous onto function. Consider the order induced c-spaces $(X, \mathfrak{C}_{(X,\leq)})$ and $(Y, \mathfrak{C}_{(Y,\leq')})$, where $X = \{1, 2, 3, 4\}$, $Y = \{1, 2\}$, $\leq$ and $\leq'$ are the usual ordering of integers on X and Y respectively. Consider the c-continuous onto function $f : X \to Y$ given by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, 4. \\ 2, & \text{if } x = 2, 3. \end{cases}$$

Here, f is neither order preserving nor order reversing X.

Theorem 2.6.8. *If $(X, \leq)$ and $(Y, \leq')$ are order isomorphic, then $(X, \mathfrak{C}_{(X,\leq)})$ and $(Y, \mathfrak{C}_{(Y,\leq')})$ are c-isomorphic. Conversely, if $(X, \mathfrak{C}_{(X,\leq)})$ and $(Y, \mathfrak{C}_{(Y,\leq')})$ are c-isomorphic, then $(X, \leq)$ is order isomorphic to either $(Y, \leq')$ or the dual of $(Y, \leq')$.*

Proof. Suppose the linearly ordered sets $(X, \leq)$ and $(Y, \leq')$ are order isomorphic. Then there exists a bijection $f : X \to Y$ such that both f and f^{-1} are order preserving. By Theorem 2.6.4, f and f^{-1} are c-continuous. Hence the order induced c-spaces $(X, \mathfrak{C}_{(X,\leq)})$ and $(Y, \mathfrak{C}_{(Y,\leq')})$ are c-isomorphic.

Conversely, suppose that the order induced c-spaces $(X, \mathfrak{C}_{(X,\leq)})$ and $(Y, \mathfrak{C}_{(Y,\leq')})$ are c-isomorphic. Then there exists a c-isomorphism $f : (X, \mathfrak{C}_{(X,\leq)}) \to (Y, \mathfrak{C}_{(Y,\leq')})$. By Theorem 2.6.6, f is either order preserving or order reversing.

Let $f : X \to Y$ be order preserving. Take $y_1, y_2 \in Y$ with $y_1 <' y_2$. Then there exist $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. If $x_2 < x_1$, then $f(x_2) <' f(x_1)$. That is, $y_2 <' y_1$, which is not true. Hence $f^{-1}(y_1) < f^{-1}(y_2)$. Therefore, $f^{-1} : Y \to X$ is order preserving and hence $f : (X, \leq) \to (Y, \leq')$ is an order isomorphism.

Now, let $f : Y \to X$ be order reversing. Then we can easily prove that, for every $y_1, y_2 \in Y$, $y_1 \leq' y_2$ implies $f^{-1}(y_2) \leq f^{-1}(y_1)$. Thus, f^{-1} is order reversing. That is, $f : (X, \leq) \to (Y, \leq')$ is a bijection such that f and f^{-1} are order reversing. Therefore, $(X, \leq)$ and the dual of $(Y, \leq')$ are order isomorphic. □

2.7 Topological Order Induced c-spaces

Here, we discuss which order induced c-spaces are topological. The order topology on a finite linearly ordered set is discrete topology and the corresponding c-structure is $\mathfrak{D}_X$. So the order induced c-space on a finite set is not order topological. Nevertheless, every finite order-induced c-space is topological. This is illustrated in the following theorem.

Theorem 2.7.1. *A finite order induced c-space is topological.*

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $(X, \mathfrak{C}_{(X,\leq)})$ be a finite order induced c-space. Now define $\mathcal{S}$ as follows.

$$\mathcal{S} = \begin{cases} \Big\{X \setminus \{x_1, x_2\}, X \setminus \{x_{n-1}, x_n\}, X \setminus \{x_{2p}\}, X \setminus \{x_{2q}, x_{2q+1}, x_{2q+2}\} : \\ \quad \text{where } p = 1, 2, \ldots, \frac{n-1}{2},\ q = 1, 2, \ldots, \frac{n-3}{2}\Big\}, & \text{if } n \text{ is odd.} \\ \Big\{X \setminus \{x_1, x_2\}, X \setminus \{x_{2p}\}, X \setminus \{x_{2q}, x_{2q+1}, x_{2q+2}\} : \\ \quad \text{where } p = 1, 2, \ldots, \frac{n}{2},\ q = 1, 2, \ldots, \frac{n-2}{2}\Big\} & \text{if } n \text{ is even.} \end{cases}$$

Let τ be the topology generated by $\mathcal{S}$ and $\mathfrak{C}_\tau$ be the corresponding c-structure.

Claim : $\{x_i, x_j\} \in \mathfrak{C}_\tau$ if and only if $j = i + 1$.

Since every open set in τ containing x_{i+1} contains x_i, we have $\{x_i, x_{i+1}\} \in \mathfrak{C}_\tau$. Now, let $C = \{x_i, x_j\}$, where $i < j$ and $j \neq i + i$. Let

$$G = \begin{cases} X \setminus \{x_{j-1}, x_j, x_{j+1}\}, & \text{if } j \text{ is odd.} \\ X \setminus \{x_j\}, & \text{if } j \text{ is even.} \end{cases}$$

and

$$H = \begin{cases} X \setminus \{x_i, x_{i+1}\}, & \text{if } i = 1. \\ X \setminus \{x_i\}, & \text{if } i \text{ is even.} \\ X \setminus \{x_{i-1}, x_i, x_{i+1}\}, & \text{otherwise.} \end{cases}$$

Obviously, G and H are open in τ. Note that $G \cap C = \{x_i\}$ and $H \cap C = \{x_j\}$. Also, $(G \cap C) \cup (H \cap C) = \{x_i, x_j\}$ implies that $\{x_i, x_j\} \notin \mathfrak{C}_\tau$. Therefore, $\{x_i, x_j\} \in \mathfrak{C}_\tau$ if and only if $j = i + 1$. It follows that $\mathfrak{C}_\tau = \mathfrak{C}_{(X,\leq)}$ and hence the c-space $(X, \mathfrak{C}_{(X,\leq)})$ is topological. $\square$

In the following theorem, we illustrate the relation between the order induced c-structure and the c-structure corresponding to the order topology. For this we need the following theorem.

Theorem 2.7.2. *Let X be an ordered set in the order topology. Then X is a linear continuum if and only if X is connected [20].*

Theorem 2.7.3. *For a linearly ordered set $(X, \leq)$, let $\mathfrak{C}_{(X,\leq)}$ and $\mathfrak{C}_{\tau}$ be the order induced c-structure and the c-structure corresponding to the order topology τ on X respectively. Then,*

(i) $\mathfrak{C}_{\tau} \subseteq \mathfrak{C}_{(X,\leq)}$

(ii) $\mathfrak{C}_{(X,\leq)} = \mathfrak{C}_{\tau}$ *if and only if X is a linear continuum.*

Proof. (i) Consider a subset A of X. Assume that $A \notin \mathfrak{C}_{(X,\leq)}$. Then there exist $a_1, a_2 \in A$ and $z \in X \setminus A$ such that $a_1 < z < a_2$. Now, let $G_1 = \{x \in X : x < z\}$ and $G_2 = \{x \in X : z < x\}$. Clearly, $G_1, G_2 \in \tau$ and $(G_1 \cap A) \cup (G_2 \cap A) = A$. This implies that $A \notin \mathfrak{C}_{\tau}$. Therefore, $\mathfrak{C}_{\tau} \subseteq \mathfrak{C}_{(X,\leq)}$.

(ii) Suppose that $\mathfrak{C}_{(X,\leq)} = \mathfrak{C}_{\tau}$. Then X is a linear continuum, since $X \in \mathfrak{C}_{\tau}$. Conversely, suppose X is a linear continuum. Then a subset of X is connected if and only if it is an interval. Therefore, $\mathfrak{C}_{(X,\leq)} = \mathfrak{C}_{\tau}$.

□

As a consequence of Theorem 2.7.3, we have order induced c-space corresponding to a linear continuum is always topological.

In [21], J. Muscat and D. Buhagiar just mentioned topologies associated with connective spaces. We extend these definitions to c-spaces and discuss the relation between the c-structures corresponding to these topologies and order induced c-structure. We need the following notation for further discussion.

Notation 2.7.4. For a c-space $(X, \mathfrak{C}_X)$, let

$$\mathcal{T} = \{A \subseteq X : t(X \setminus A) = X \setminus A\} \text{ and}$$

$$\mathcal{S} = \{A \subseteq X : X \setminus A \in \mathfrak{C}_X \text{ and } t(X \setminus A) = X \setminus A\}.$$

Then τ_t denotes the topology on X generated by $\mathcal{T}$ and τ_s denotes the topology generated by $\mathcal{S}$. These topologies are simply known as τ_t-topology and τ_s-topology respectively.

Clearly, τ_s-topology is weaker than τ_t-topology. But in the case of a finite order induced c-space, the τ_t-topology and τ_s-topology coincide with the indiscrete topology. Therefore, the c-structure corresponding to these topologies is not the same as the order induced c-structure on that set.

Theorem 2.7.5. *If a c-space is topological, then the corresponding topology is stronger than* τ_s.

Proof. Let τ be the topology corresponding to the topological c-space $(X, \mathfrak{C}_X)$ and let $A \subseteq X$. Suppose $A \in \tau_s$. Then, $X \setminus A \in \mathfrak{C}_X$ and $t(X \setminus A) = X \setminus A$. Let x be a limit point of $X \setminus A$. Then $\{x\} \cup (X \setminus A) \in \mathfrak{C}_X$. This implies $x \in t(X \setminus A)$ and hence $x \in X \setminus A$. It follows that $A \in \tau$. Therefore, $\tau_s \subseteq \tau$. □

The following theorem gives the relation between the connective closure of a c-space and the closure corresponding to the τ_t and τ_s topologies associated with the given c-space.

Theorem 2.7.6. *Let $(X, \mathfrak{C}_X)$ be a c-space and A be a subset of X. Then $\overline{\mathrm{A}}_{\tau_t} \subseteq \overline{\mathrm{A}} \subseteq \overline{\mathrm{A}}_{\tau_s}$, where $\overline{\mathrm{A}}$, $\overline{\mathrm{A}}_{\tau_s}$ and $\overline{\mathrm{A}}_{\tau_t}$ are the connective closure of A, the closure of A with respect to the topologies τ_s and τ_t respectively.*

Proof. It is clear that $\overline{\mathrm{A}}_{\tau_t} \subseteq \overline{\mathrm{A}}_{\tau_s}$ since $\tau_s \subseteq \tau_t$. Now, let $x \notin \overline{\mathrm{A}}$. Then $x \notin \bigcap\{E \subseteq X : t(E) = E \text{ and } A \subseteq E\}$. This implies there exists $E_1 \subseteq X$ such that $x \notin E_1$, $A \subseteq E_1$ and $t(E_1) = E_1$. Let $U = X \setminus E_1$. Then $U \in \tau_t$ and $x \in U$. If $U \cap A \neq \emptyset$, then there exists $z \in U \cap A$. But $z \in U$ implies $z \in X \setminus E_1 \subseteq X \setminus A$. This implies $z \notin A$, which is a contradiction. Therefore, $U \cap A = \emptyset$. That is, there exists a neighbourhood U of x in (X, τ_t) such that $U \cap A = \emptyset$. Therefore, $x \notin \overline{\mathrm{A}}_{\tau_t}$. Hence $\overline{\mathrm{A}}_{\tau_t} \subseteq \overline{\mathrm{A}}$.

Now, suppose $x \notin \overline{\mathrm{A}}_{\tau_s}$. Then there exists $U \in \tau_s$ such that $x \in U$ and $U \cap A = \emptyset$. Let $U = \bigcup\limits_{\alpha}(\bigcap\limits_{i_\alpha=1}^{n_\alpha} U_{i_\alpha})$, where $X \setminus U_{i_\alpha} \in \mathfrak{C}_{(X,\leq)}$ and $t(X \setminus U_{i_\alpha}) = X \setminus U_{i_\alpha}$. Then,

$$\begin{aligned} U \cap A = \emptyset \implies & A \subseteq X \setminus U \\ \implies & A \subseteq X \setminus \bigcup_{\alpha}(\bigcap_{i_\alpha=1}^{n_\alpha} U_{i_\alpha}) \\ \implies & A \subseteq \bigcap_{\alpha} X \setminus (\bigcap_{i_\alpha=1}^{n_\alpha} U_{i_\alpha}) \\ \implies & A \subseteq X \setminus (\bigcap_{i_\alpha=1}^{n_\alpha} U_{i_\alpha}) = \bigcup_{i_\alpha=1}^{n_\alpha} X \setminus U_{i_\alpha}. \end{aligned}$$

Let $F = \bigcup\limits_{i_\alpha=1}^{n_\alpha} X \setminus U_{i_\alpha}$. Then $A \subseteq F$ and

$$t(F) = t(\bigcup_{i_\alpha=1}^{n_\alpha} X \setminus U_{i_\alpha})$$

$$= \bigcup_{i_\alpha=1}^{n_\alpha} t(X \setminus U_{i_\alpha})$$
$$= \bigcup_{i_\alpha=1}^{n_\alpha} X \setminus U_{i_\alpha} = F.$$

That is, $t(F) = F$, $A \subseteq F$ and $x \notin F$. This implies $x \notin \overline{\mathrm{A}}$ and hence $\overline{\mathrm{A}} \subseteq \overline{\mathrm{A}}_{\tau_s}$. This completes the proof. □

Now, we discuss the relation between the c-continuous functions on a c-space and the continuous functions of the corresponding to the τ_t-topological space and τ_s-topological space.

Theorem 2.7.7. *Let $(X, \mathfrak{C}_X)$ be a c-space and (X, τ_t) be the corresponding τ_t-topological space. Then a function f is continuous on (X, τ_t) if it is c-continuous on $(X, \mathfrak{C}_X)$.*

Proof. Consider the c-space $(X, \mathfrak{C}_X)$ and let $f : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}_X)$ be a c-continuous function.

Claim : $f : (X, \tau_t) \to (X, \tau_t)$ is continuous.

Consider the collection $\mathcal{T} = \{A \subseteq X : t(X \setminus A) = X \setminus A\}$, the sub-base of the τ_t-topology. Let $A \in \mathcal{T}$. Then $X \setminus A$ is a t-closed subset of X. Now, consider $X \setminus (f^{-1}(A))$. Assume that there exists $z \in f^{-1}(A)$ such that z is a touching point of $X \setminus (f^{-1}(A))$. Then there exists a nonempty subset K of $X \setminus (f^{-1}(A))$ such that $K \cup \{z\} \in \mathfrak{C}_{(X,\leq)}$. This implies $f(K) \cup \{f(z)\} \in \mathfrak{C}_{(X,\leq)}$ since f is a c-continuous function. It is clear that $f(K)$ is a nonempty subset of $X \setminus A$. It follows that $f(z)$ is a touching point of $X \setminus A$. This is a contradiction since $t(X \setminus A) = X \setminus A$ and $f(z) \in A$. Therefore, $t(X \setminus (f^{-1}(A))) = X \setminus (f^{-1}(A))$.

Thus, $f^{-1}(A) \in \mathcal{T}$ whenever $A \in \mathcal{T}$. Since A is arbitrary, f is continuous on (X, τ_t). □

On the corresponding τ_s-topological space, the c-continuous functions of a c-space do not necessarily have to be continuous. The example that follows illustrates this.

Example 2.7.8. Consider the c-space $(X, \mathfrak{C}_X)$, where $X = \mathbb{N}$ and $\mathfrak{C}_X = \mathfrak{D}_X$. Here, empty set and singleton sets are the only connected t-closed subsets of X Therefore, the corresponding τ_s-topology is the co-finite topology. Now, consider the function f on X given by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, 3, 5, \ldots. \\ 2, & \text{if } x = 2, 4, 6, \ldots. \end{cases}$$

Clearly, f is a c-continuous function $(X, \mathfrak{C}_X)$. Let $U = \mathbb{N} \setminus \{1\}$. Then $U \in \tau_s$ and $f^{-1}(U) = \{2, 4, 6, \ldots\} \notin \tau_s$. It follows that f is not continuous on (X, τ_s).

In the following theorem, we prove that the c-isomorphism of a c-space is a homeomorphism of the corresponding τ_t and τ_s topological spaces.

Theorem 2.7.9. *Let $(X, \mathfrak{C}_X)$ be a c-space and let (X, τ_t) and (X, τ_s) be the corresponding τ_t-topological space and τ_s-topological space respectively. If f is a c-isomorphism on $(X, \mathfrak{C}_X)$, then it is a homeomorphism on both the topological spaces (X, τ_t) and (X, τ_s).*

Proof. Consider a c-space $(X, \mathfrak{C}_X)$ and let f be a c-isomorphism on $(X, \mathfrak{C}_X)$

Claim : f is a homeomorphism on (X, τ_s).

Clearly, f is a bijection on X. Let $U \in \tau_s$. This implies that $X \setminus U \in \mathfrak{C}_X$ and $t(X \setminus U) = X \setminus U$. It follows that $X \setminus f^{-1}(U) \in \mathfrak{C}_X$. Now, let a be a touching point of $X \setminus f^{-1}(U)$. Then,

$$\begin{aligned} a \notin X \setminus f^{-1}(U) &\implies a \in f^{-1}(U) \\ &\implies f(a) \in U \\ &\implies f(a) \notin X \setminus U \\ &\implies f(a) \text{ is not a touching point of } X \setminus U \\ &\implies f^{-1}(f(a)) \text{ is not a touching point of } f^{-1}(X \setminus U). \end{aligned}$$

This implies that $t(X \setminus f^{-1}(U)) = X \setminus f^{-1}(U)$ since $a = f^{-1}(f(a))$ and $X \setminus f^{-1}(U) = f^{-1}(X \setminus f^{-1}(U))$. It follows that $X \setminus f^{-1}(U)$ is a connected t-closed subset of X. Therefore, f is continuous since $f^{-1}(U) \in \tau_s$. Similarly, $V \in \tau_s$ implies that $X \setminus f(V)$ and

$$\begin{aligned} a \notin X \setminus f(V) &\implies a \in f(V) \\ &\implies f^{-1}(a) \in V \\ &\implies f^{-1}(a) \notin X \setminus V \\ &\implies f^{-1}(a) \text{ is not a touching point of } X \setminus V \\ &\implies a \text{ is not a touching point of } X \setminus f(V). \end{aligned}$$

That is, $X \setminus f(V)$ is a connected t-closed subset of X. Hence f^{-1} is continuous since $f(V) \in \tau_s$. Therefore, f is a homeomorphism on (X, τ_s). Similarly, we can prove that f is a homeomorphism on (X, τ_t). $\square$

2.8 Order Induced Connective Spaces

In this section, we examine whether the order induced c-space is a connective space. Let's look at a few examples.

Example 2.8.1. Consider the sets $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, $B = \{-\frac{1}{n} : n \in \mathbb{N}\}$ and let $X = A \cup B$. Now consider the order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$, where $\leq$ is the usual ordering of numbers. It is clear that $A, B, A \cup B \in \mathfrak{C}_{(X,\leq)}$. There does not exist any $x \in A \cup B$ such that $\{x\} \cup A$ and $\{x\} \cup B$ are members of $\mathfrak{C}_{(X,\leq)}$. Therefore $\mathfrak{C}_{(X,\leq)}$ is not a connective structure on X.

Example 2.8.2. Consider the linearly ordered set $(\mathbb{N}, \preceq)$, where $\preceq$ is the Sharkovsky ordering which is given by $3 \prec 5 \prec 7 \prec \ldots 2 \times 3 \prec 2 \times 5 \prec 2 \times 7 \prec \ldots 2^2 \times 3 \prec 2^2 \times 5 \prec 2^2 \times 7 \prec \ldots 2^3 \times 3 \prec 2^3 \times 5 \prec 2^3 \times 7 \prec \ldots 2^n \prec \ldots 2^3 \prec 2^2 \prec 2 \prec 1$. Let $A = \{2^{n-1} : n \in \mathbb{N}\}$ and $B = X \setminus A$. Then $A \cup B = \mathbb{N}$ and it is clear that $A, B, A \cup B \in \mathfrak{C}_{(\mathbb{N},\preceq)}$. But there does not exist any $x \in A \cup B$ such that $\{x\} \cup A \in \mathfrak{C}_{(\mathbb{N},\preceq)}$ and $\{x\} \cup B \in \mathfrak{C}_{(\mathbb{N},\preceq)}$. Thus, $\mathfrak{C}_{(\mathbb{N},\preceq)}$ is not a connective structure on $\mathbb{N}$.

The two instances above demonstrate that order-induced c-spaces do not always require the fulfillment of axiom (iii) of the connective space. However, we can show that order induced c-spaces satisfy the fourth axiom.

Theorem 2.8.3. *For an order induced c-space* $(X, \mathfrak{C}_{(X,\leq)})$*, let* $A, B, C_i \in \mathfrak{C}_{(X,\leq)}$ *for every* $i \in I$ *be disjoint and* $A \cup B \cup (\bigcup_{i \in I} C_i) \in \mathfrak{C}_{(X,\leq)}$*. Then there exists* $J \subseteq I$ *such that* $A \cup (\bigcup_{j \in J} C_j) \in \mathfrak{C}_{(X,\leq)}$ *and* $B \cup (\bigcup_{i \in I \setminus J} C_i) \in \mathfrak{C}_{(X,\leq)}$*.*

Proof. If $A = \emptyset$, then $J = \emptyset \subseteq I$ satisfies the required conditions. Similarly, if $A \neq \emptyset$, $B = \emptyset$ then we can take $J = I$. Now, suppose $A, B \neq \emptyset$ and choose $a_0 \in A$ and $b_0 \in B$. Then $a_0 \leq b_0$, otherwise interchange A and B. By Lemma 2.4.1, $a \leq b$ for every $a \in A$ and for every $b \in B$. Choose $A_1 = \{c \in \bigcup_{i \in I} C_i : c < b_0\}$ and $A_2 = \{c \in \bigcup_{i \in I} C_i : b_0 < c\}$. Clearly, A_1 and A_2 are disjoint. For $i \in I$, suppose $C_i \nsubseteq A_1$. Then there exists $c \in C_i$ such that $c \notin A_1$. Then $b_0 < c$ and this implies $b_0 \leq x$ for every $x \in C_i$. Therefore, $C_i \subseteq A_2$. That is, for each $i \in I$, either $C_i \subseteq A_1$ or $C_i \subseteq A_2$.

Let $J = \{i \in I : C_i \subseteq A_1\}$. We claim that $A \cup (\bigcup_{j \in J} C_j) \in \mathfrak{C}_{(X,\leq)}$ and $B \cup (\bigcup_{i \in I \setminus J} C_i) \in \mathfrak{C}_{(X,\leq)}$. Assume that $A \cup (\bigcup_{j \in J} C_j) \notin \mathfrak{C}_{(X,\leq)}$. Then there exist $x, y \in A \cup (\bigcup_{j \in J} C_j)$ and $z \in X$ such that $x \leq z \leq y$ and $z \notin A \cup (\bigcup_{j \in J} C_j)$. But $x, y \in A \cup B \cup (\bigcup_{i \in I} C_i) \in \mathfrak{C}_{(X,\leq)}$ implies $z \in A \cup B \cup (\bigcup_{i \in I} C_i)$ and hence $z \in B \cup (\bigcup_{i \in I \setminus J} C_i)$. Since $z \leq y < b_0$, $z \notin \bigcup_{i \in I \setminus J} C_i$ and hence $z \in B$. Since $z \leq y$, we have $z' \leq y$ for every $z' \in B$. This implies that $b_0 \leq y$, which is a contradiction. Hence $A \cup (\bigcup_{j \in J} C_j) \in \mathfrak{C}_{(X,\leq)}$.

Now, let $D = \{x \in A \cup B \cup (\bigcup_{i \in I} C_i) : b_0 \leq x\}$. Then $D \in \mathfrak{C}_{(X,\leq)}$ and $D \cap B \neq \emptyset$. Therefore, $D \cup B = B \cup (\bigcup_{i \in I \setminus J} C_i) \in \mathfrak{C}_{(X,\leq)}$. □

A necessary and sufficient condition for an order induced c-space to satisfy the third axiom of the connective space is given in the following theorem.

Theorem 2.8.4. *Let* $(X, \mathfrak{C}_{(X,\leq)})$ *be an order induced c-space. For any nonempty sets* $A, B \in \mathfrak{C}_{(X,\leq)}$ *with* $A \cup B \in \mathfrak{C}_{(X,\leq)}$ *there exists* $x \in A \cup B$ *such that* $\{x\} \cup A \in \mathfrak{C}_{(X,\leq)}$ *and* $\{x\} \cup B \in \mathfrak{C}_{(X,\leq)}$ *if and only if* $(X, \leq)$ *is a complete linearly ordered set.*

Proof. Assume that $(X, \leq)$ is not a complete linearly ordered set. Then there exists a nonempty subset $A \subseteq X$ which is bounded above and such that $\sup A$ does not exist. Let U be the set of all upper bounds of A and let V be the set of all lower bounds of U. Then $\sup V$ and $\inf U$ do not exist and $V \cap U = \emptyset$.

Claim 1 : $U \in \mathfrak{C}_{(X,\leq)}$

Let $u_1, u_2 \in U$ and $z \in X$ be such that $u_1 \leq z \leq u_2$. Then for every $a \in A$, $a \leq u_1 \leq z$ implies z is an upper bound of A. Therefore, $z \in U$ and hence $U \in \mathfrak{C}_{(X,\leq)}$.

Claim 2 : $V \in \mathfrak{C}_{(X,\leq)}$

Let $v_1, v_2 \in V$ and $z' \in X$ be such that $v_1 \leq z' \leq v_2$. Then for every $u \in U$, $z' \leq v_2 \leq u$ implies $z' \in V$. Hence $V \in \mathfrak{C}_{(X,\leq)}$.

Claim 3 : $V \cup U \in \mathfrak{C}_{(X,\leq)}$

Let $a, b \in V \cup U$ and $c \in X$ be such that $a \leq c \leq b$. If $a, b \in V$ then $c \in V \subseteq V \cup U$. Similarly, if $a, b \in U$ then $c \in U \subseteq V \cup U$.

Now, let $a \in V$ and $b \in U$. If $c \in U$, then clearly $c \in V \cup U$. If $c \notin U$, then c is not an upper bound of A. Then there exists $a' \in A$ such that $c \leq a'$.

Since $A \subseteq V$, $a' \in V$ and hence $a \leq c \leq a'$ implies $c \in V \subseteq V \cup U$. Hence $V \cup U \in \mathfrak{C}_{(X,\leq)}$.

Thus, we have $V, U, V \cup U \in \mathfrak{C}_{(X,\leq)}$.

Let $u \in U$. Then clearly u is an upper bound of V. Since $\sup V$ does not exist, $u \neq \sup V$. Then there exists $u' \in U$ such that $u' < u$. Take any $v \in V$, clearly $v \neq u'$. Then v, $u \in V \cup \{u\}$, $v < u' < u$ and $u' \notin V \cup \{u\}$ together implies $V \cup \{u\} \notin \mathfrak{C}_{(X,\leq)}$. That is, $V \cup \{u\} \notin \mathfrak{C}_{(X,\leq)}$ for any $u \in U$.

Now, let $v \in V$. It is clear that v is a lower bound of U. Since $\inf U$ does not exist, $v \neq \inf U$. Then there exist $v' \in V$ such that $v < v'$. Take any $u \in U$, clearly $u \neq v'$. Then v, $u \in \{v\} \cup U$, $v < v' < u$ and $v' \notin \{v\} \cup U$ together implies $\{v\} \cup U \notin \mathfrak{C}_{(X,\leq)}$. That is, $\{v\} \cup U \notin \mathfrak{C}_{(X,\leq)}$ for any $v \in V$.

Therefore, for nonempty sets $U, V \in \mathfrak{C}_{(X,\leq)}$ with $U \cup V \in \mathfrak{C}_{(X,\leq)}$ there does not exist any $x \in U \cup V$ such that $\{x\} \cup U \in \mathfrak{C}_{(X,\leq)}$ and $\{x\} \cup V \in \mathfrak{C}_{(X,\leq)}$. The proof of the sufficient part is finished with this.

Conversely, suppose that $(X, \leq)$ is a complete linearly ordered set. If $A \cap B \neq \emptyset$, then there is nothing to prove. Now, let $A \cap B = \emptyset$. Since $A \neq \emptyset$ and $B \neq \emptyset$, there exist $a_0, b_0 \in X$ such that $a_0 \in A$ and $b_0 \in B$. Then $a_0 \leq b_0$, otherwise interchange A and B. This implies $a \leq b_0$ for every $a \in A$, by Lemma 2.4.1. Thus, the set A is bounded above and $\sup A$ exists since X is complete. Let $x = \sup A$. Then clearly $\{x\} \cup A \in \mathfrak{C}_{(X,\leq)}$. It remains to show that $\{x\} \cup B \in \mathfrak{C}_{(X,\leq)}$.

Since $a \leq b_0$ for every $a \in A$, we have $x \leq b_0$. Hence $a_0 \leq x \leq b_0$ and

$a_0, b_0 \in A \cup B \in \mathfrak{C}_{(X,\leq)}$ together implies $x \in A \cup B$. Then either $x \in A$ or $x \in B$.

If $x \in B$, then clearly $\{x\} \cup B \in \mathfrak{C}_{(X,\leq)}$. Now, let $x \in A$. Consider $p, q \in \{x\} \cup B$ and let $z \in X$ be such that $p < z < q$. Then there are two possibilities; either $p, q \in B$ or $p = x$.

If $p, q \in B$, then clearly $z \in B \subseteq \{x\} \cup B$.

If $p = x$, then $x < z < q$ and $x, q \in A \cup B \in \mathfrak{C}_{(X,\leq)}$ implies $z \in A \cup B$. That is, either $z \in A$ or $z \in B$. But if $z \in A$ then x is not an upper bound of A. This is a contradiction. Therefore, $z \in B \subseteq \{x\} \cup B$.

Thus, we get if $p, q \in \{x\} \cup B$ and let $z \in X$ be such that $p < z < q$ then $z \in \{x\} \cup B$. Hence $\{x\} \cup B \in \mathfrak{C}_{(X,\leq)}$. This completes the proof. □

Remark 2.8.5. In the Theorem 2.8.4, we can also take $x = \inf B$, which always exists since B is bounded below by a_0.

Consequently, we have the following characterization of a complete linearly ordered set.

Theorem 2.8.6. *Let $(X, \mathfrak{C}_{(X,\leq)})$ be an order induced c-space. Then, $(X, \mathfrak{C}_{(X,\leq)})$ is a connective space if and only if $(X, \leq)$ is a complete linearly ordered set.*

Proof. The proof follows from Theorem 2.8.3 and Theorem 2.8.4. □

Chapter 3

Reversible c-spaces

3.1 Introduction

In [23], M. Rajagopalan and A. Wilansky introduced a new category of topological spaces, known as reversible topological spaces. Analogous to this, several authors investigated the reversible properties of various structures on a set other than topology [8, 17, 18]. Here, we propose to study the reversible properties of a c-space. We provide a characterization of reversible c-spaces in connection with a stronger and weaker c-structures. We demonstrate that, for any infinite cardinal α, there exists a non-reversible c-space with $|X| = \alpha$.

3.2 Reversible c-spaces

First and foremost, we define a reversible c-space.

Definition 3.2.1. A c-space $(X, \mathfrak{C}_X)$ is said to be reversible if every c-continuous bijection from $(X, \mathfrak{C}_X)$ onto itself is a c-isomorphism.

Obviously, every finite c-space, the discrete c-space, and the indiscrete c-space are trivial examples of reversible c-spaces. Furthermore, by Theorem 2.6.2, the order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$ is reversible.

Let us go through more examples of reversible c-spaces.

Proposition 3.2.2. *Let X be a nonempty set X and $x_0 \in X$. Then $\mathfrak{C}_X = \mathfrak{D}_X \cup \{A \subseteq X : x_0 \in A\}$ is a reversible c-structure on X. That is, the rooted c-structure is reversible.*

Proof. If f is a c-continuous bijection on $(X, \mathfrak{C}_X)$, then $f(x_0) = x_0$. Otherwise, $f(x_0) = y_0$ for some $y_0 \neq x_0 \in X$. Then $A = \{x_0, y_0\} \in \mathfrak{C}_X$ implies that $f(A) = \{y_0, f(y_0)\} \in \mathfrak{C}_X$. This follows that $f(y_0) = x_0$. Now, let $B = X \setminus \{y_0\}$. Then $B \in \mathfrak{C}_X$ implies $f(B) = X \setminus \{f(y_0)\} = X \setminus \{x_0\} \in \mathfrak{C}_X$, which is a contradiction.

If $(X, \mathfrak{C}_X)$ is not a reversible c-space, then there exists a c-continuous bijection g on $(X, \mathfrak{C}_X)$ such that g^{-1} is not c-continuous. That is, there exists $C \in \mathfrak{C}_X$ with $g^{-1}(C) \notin \mathfrak{C}_X$. This gives $x_0 \notin g^{-1}(C)$ and hence $x_0 = g(x_0) \notin C$, which is a contradiction. Therefore, $(X, \mathfrak{C}_X)$ is a reversible c-space. □

Proposition 3.2.3. *Let X be a nonempty set and $\{P_i : i \in I\}$ be a partition of X. Then $\mathfrak{C}_X = \mathfrak{D}_X \cup \{P_i : i \in I\}$ is a reversible c-structure on X.*

Proof. Let f be a c-continuous bijection on $(X, \mathfrak{C}_X)$. If f is not a c-isomorphism, there exists $P_k \in \mathfrak{C}_X$ with $|P_k| > 1$ such that $f^{-1}(P_k) \notin \mathfrak{C}_X$ for some $k \in I$. Since $\{P_i : i \in I\}$ is a partition of X and $f^{-1}(P_k) \subseteq X$, there always exists $j \in I$ such that $f^{-1}(P_k) \cap P_j \neq \emptyset$, where $|P_j| > 1$. This implies that $P_k \cap f(P_j) \neq \emptyset$. It follows that $P_k \cup f(P_j) \in \mathfrak{C}_X$ since P_k, $f(P_j) \in \mathfrak{C}_X$. Therefore, $P_k \cup f(P_j) = P_q$ for $q \in I$. This gives $P_k \cap P_q \neq \emptyset$, which contradict the fact that $\{P_i : i \in I\}$ is a partition. Therefore, f is a c-isomorphism on X. □

Proposition 3.2.4. *Every completely homogeneous c-spaces are reversible.*

Proof. Trivial. □

Let X be an infinite set and $|X| = k$. Then by Theorem 2.21 in [39], the following are reversible c-structures on X.

1. $\mathfrak{D}_X \cup \{B \subseteq X : |B| \geq n\}$, $1 < n \leq k$.
2. $\mathfrak{D}_X \cup \{B \subseteq X : |X \setminus B| \leq n\}$ where $n < k$.
3. $\mathfrak{D}_X \cup \{B \subseteq X : |X \setminus B| < n\}$ where $n \leq k$ and n is a limit cardinal.

Theorem 3.2.5. *Let $(X, \mathfrak{C}_X)$ and $(Y, \mathfrak{C}_Y)$ be c-isomorphic c-spaces. If $(X, \mathfrak{C}_X)$ is reversible then $(Y, \mathfrak{C}_Y)$ is reversible.*

Proof. Suppose that $(X, \mathfrak{C}_X)$ is a reversible c-space and let $g : (X, \mathfrak{C}_X) \to (Y, \mathfrak{C}_Y)$ be a c-isomorphism. If $(Y, \mathfrak{C}_Y)$ is not reversible, then there exists a c-continuous bijection $f : (Y, \mathfrak{C}_Y) \to (Y, \mathfrak{C}_Y)$ such that f^{-1} is not c-continuous. Therefore,

$f^{-1}(A) \notin \mathfrak{C}_Y$ for some $A \in \mathfrak{C}_Y$. Now, consider the function $h = g^{-1} \circ f \circ g$ on X. Being the composition of c-continuous bijections, h is a c-continuous bijection on X. As $(X, \mathfrak{C}_X)$ is a reversible c-space, h^{-1} is c-continuous. Therefore, $h^{-1}(g^{-1}(A)) \in \mathfrak{C}_X$ since $g^{-1}(A) \in \mathfrak{C}_X$. This implies that $g(h^{-1}(g^{-1}(A))) \in \mathfrak{C}_Y$. But $g(h^{-1}(g^{-1}(A))) = f^{-1}(A)$. Hence we have $f^{-1}(A) \in \mathfrak{C}_Y$, which is a contradiction. Therefore, $(Y, \mathfrak{C}_Y)$ is a reversible c-space. □

Proposition 3.2.6. *A c-space $(X, \mathfrak{C}_X)$ is reversible if and only if its Brunnian closure is reversible.*

Proof. Let $(X, \mathfrak{C}_X^{\circledast})$, where $\mathfrak{C}_X^{\circledast} = \mathfrak{C}_X \cup \{X\}$ be the Brunnian closure of the c-structure $\mathfrak{C}_X$. It is reasonable to take $X \notin \mathfrak{C}_X$, otherwise there is nothing to prove.

Suppose that the c-space $(X, \mathfrak{C}_X)$ is reversible. Let $f : (X, \mathfrak{C}_X^{\circledast}) \to (X, \mathfrak{C}_X^{\circledast})$ be a c-continuous bijection and $U \in \mathfrak{C}_X$. This implies that $U \in \mathfrak{C}_X^{\circledast}$, which gives $f(U) \in \mathfrak{C}_X^{\circledast}$. Since $X \notin \mathfrak{C}_X$, we have $U \neq X$. This gives $f(U) \neq X$ and hence $f(U) \in \mathfrak{C}_X$. Therefore, the bijection $f : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}_X)$ is c-continuous. Since $(X, \mathfrak{C}_X)$ is reversible, f is a c-isomorphism on $(X, \mathfrak{C}_X)$. Now, consider $f^{-1} : (X, \mathfrak{C}_X^{\circledast}) \to (X, \mathfrak{C}_X^{\circledast})$. Let $V \in \mathfrak{C}_X^{\circledast}$. If $V \neq X$, then $V \in \mathfrak{C}_X$. This gives $f^{-1}(V) \in \mathfrak{C}_X \subseteq \mathfrak{C}_X^{\circledast}$. If $V = X$, then $f^{-1}(V) = X \in \mathfrak{C}_X^{\circledast}$. Therefore, $f^{-1} : (X, \mathfrak{C}_X^{\circledast}) \to (X, \mathfrak{C}_X^{\circledast})$ is c-continuous. Thus, we get every c-continuous bijection on $(X, \mathfrak{C}_X^{\circledast})$ is a c-isomorphism. Therefore, $(X, \mathfrak{C}_X^{\circledast})$ is reversible.

Conversely, suppose that the Brunnian closure $(X, \mathfrak{C}_X^{\circledast})$ is reversible. Let

$g : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}_X)$ be a c-continuous bijection and $V \in \mathfrak{C}_X^{\circledast}$. If $V \neq X$, then $V \in \mathfrak{C}_X$. This gives $g(V) \in \mathfrak{C}_X \subseteq \mathfrak{C}_X^{\circledast}$. If $V = X$, then $g(V) = X \in \mathfrak{C}_X^{\circledast}$. Therefore, the bijection $g : (X, \mathfrak{C}_X^{\circledast}) \to (X, \mathfrak{C}_X^{\circledast})$ is c-continuous. Since $(X, \mathfrak{C}_X^{\circledast})$ is reversible, g is a c-isomorphism on $(X, \mathfrak{C}_X^{\circledast})$. Let $U \in \mathfrak{C}_X$. This implies that $U \in \mathfrak{C}_X^{\circledast}$, which gives $g^{-1}(U) \in \mathfrak{C}_X^{\circledast}$. Since $X \notin \mathfrak{C}_X$, we have $U \neq X$. This gives $g^{-1}(U) \neq X$ and hence $g^{-1}(U) \in \mathfrak{C}_X$. Therefore, $g^{-1} : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}_X)$ is c-continuous. Thus, we get every c-continuous bijection on $(X, \mathfrak{C}_X)$ is a c-isomorphism. Therefore, $(X, \mathfrak{C}_X)$ is reversible. □

3.3 Characterization of Reversible c-spaces

In this section, we characterize the reversible c-spaces in terms of stronger and weaker c-structures.

Theorem 3.3.1. *For any c-space $(X, \mathfrak{C}_X)$, the following conditions are equivalent.*

(i) $(X, \mathfrak{C}_X)$ *is reversible.*

(ii) *There does not exist any c-structure* $\mathfrak{C}''_X \supsetneq \mathfrak{C}_X$ *on* X *such that the c-spaces* $(X, \mathfrak{C}_X)$ *and* $(X, \mathfrak{C}''_X)$ *are c-isomorphic.*

(iii) *There does not exist any c-structure* $\mathfrak{C}'_X \subsetneq \mathfrak{C}_X$ *on* X *such that the c-spaces* $(X, \mathfrak{C}'_X)$ *and* $(X, \mathfrak{C}_X)$ *are c-isomorphic.*

Proof. ***(i)*** $\Rightarrow$ ***(ii)***

Suppose that the c-space $(X, \mathfrak{C}_X)$ is reversible. Assume that there exists a c-structure $\mathfrak{C}''_X \supsetneq \mathfrak{C}_X$ such that $(X, \mathfrak{C}_X)$ and $(X, \mathfrak{C}''_X)$ are c-isomorphic. Let $f : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}''_X)$ be the c-isomorphism. Now, consider the function $g : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}_X)$, where $g = f^{-1}$.

Claim 1 : g is c-continuous on $(X, \mathfrak{C}_X)$.

Let U be a subset of X. Then,

$$\begin{aligned} U \in \mathfrak{C}_X &\Longrightarrow U \in \mathfrak{C}''_X, \text{ since } \mathfrak{C}_X \subseteq \mathfrak{C}''_X \\ &\Longrightarrow f^{-1}(U) \in \mathfrak{C}_X, \text{ since } f : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}''_X) \text{ is a c-isomorphism} \\ &\Longrightarrow g(U) \in \mathfrak{C}_X. \end{aligned}$$

That is, $g(U) \in \mathfrak{C}_X$ whenever $U \in \mathfrak{C}_X$. Therefore, g is c-continuous.

Claim 2 : g^{-1} is not c-continuous.

Since $\mathfrak{C}''_X \supsetneq \mathfrak{C}_X$, there exists $U_0 \in \mathfrak{C}''_X$ such that $U_0 \notin \mathfrak{C}_X$. Let $V_0 = f^{-1}(U_0)$. Then we have $V_0 \in \mathfrak{C}_X$. But $g^{-1}(V_0) = f(V_0) = U_0 \notin \mathfrak{C}_X$. That is, there exists $V_0 \in \mathfrak{C}_X$ such that $g^{-1}(V_0) \notin \mathfrak{C}_X$. This implies g^{-1} is not c-continuous.

Thus, there exists a c-continuous bijection g on $(X, \mathfrak{C}_X)$ such that g^{-1} is not c-continuous. This is a contradiction since $(X, \mathfrak{C}_X)$ is reversible. Thus, there does not exist any c-structure $\mathfrak{C}''_X \supsetneq \mathfrak{C}_X$ on X such that the c-spaces $(X, \mathfrak{C}_X)$ and $(X, \mathfrak{C}''_X)$ are c-isomorphic.

(ii) $\Rightarrow$ ***(iii)***

Suppose there does not exist any c-structure $\mathfrak{C}''_X \supsetneq \mathfrak{C}_X$ on X such that the c-spaces $(X, \mathfrak{C}_X)$ and $(X, \mathfrak{C}''_X)$ are c-isomorphic. Assume that there exists a c-structure $\mathfrak{C}'_X \subsetneq \mathfrak{C}_X$ on X such that the c-spaces $(X, \mathfrak{C}'_X)$ and $(X, \mathfrak{C}_X)$ are c-isomorphic. Let $f : (X, \mathfrak{C}'_X) \to (X, \mathfrak{C}_X)$ be the c-isomorphism. Consider the collection

$$\mathfrak{C}^*_X = \{A \subseteq X : f^{-1}(A) \in \mathfrak{C}_X\}.$$

Claim 3 : $\mathfrak{C}^*_X$ is a c-structure on X.

It is clear that $\emptyset \in \mathfrak{C}^*_X$. We have for any $x \in X$, there exists $y \in X$ such that $f(y) = x$. Then $f^{-1}(\{x\}) = \{y\} \in \mathfrak{C}_X$ implies $\{x\} \in \mathfrak{C}^*_X$. Therefore, $\{x\} \in \mathfrak{C}^*_X$ for every $x \in X$. Now, let $\{U_i : i \in I\}$ be a nonempty collection of subsets in $\mathfrak{C}^*_X$ such that $\bigcap\limits_{i \in I} U_i \neq \emptyset$. Then $U_i \in \mathfrak{C}^*_X$ gives $f^{-1}(U_i) \in \mathfrak{C}_X$ for every $i \in I$ and $\bigcap\limits_{i \in I} U_i \neq \emptyset$ gives $\bigcap\limits_{i \in I} f^{-1}(U_i) = f^{-1}(\bigcap\limits_{i \in I} U_i) \neq \emptyset$. This implies $f^{-1}(\bigcup\limits_{i \in I} U_i) = \bigcup\limits_{i \in I} f^{-1}(U_i) \in \mathfrak{C}_X$ and hence $\bigcup\limits_{i \in I} U_i \in \mathfrak{C}^*_X$.

Claim 4 : $\mathfrak{C}^*_X \supsetneq \mathfrak{C}_X$.

Let U be a subset of X. Then,

$$\begin{aligned} U \in \mathfrak{C}_X &\Longrightarrow f^{-1}(U) \in \mathfrak{C}'_X, \text{ since } f : (X, \mathfrak{C}'_X) \to (X, \mathfrak{C}_X) \text{ is a c-isomorphism} \\ &\Longrightarrow f^{-1}(U) \in \mathfrak{C}_X, \text{ since } \mathfrak{C}'_X \subseteq \mathfrak{C}_X \\ &\Longrightarrow U \in \mathfrak{C}^*_X \end{aligned}$$

Therefore, $\mathfrak{C}_X \subseteq \mathfrak{C}^*_X$. Since $\mathfrak{C}'_X \subsetneq \mathfrak{C}_X$, there exists $U_0 \in \mathfrak{C}_X$ such that $U_0 \notin \mathfrak{C}'_X$. Now, let $V_0 = f(U_0)$. Then $f^{-1}(V_0) = f^{-1}(f(U_0)) = U_0 \in \mathfrak{C}_X$ implies $V_0 \in \mathfrak{C}^*_X$. Also, we have $V_0 \notin \mathfrak{C}_X$. Otherwise, $V_0 \in \mathfrak{C}_X$ gives $U_0 = f^{-1}(V_0) \in \mathfrak{C}'_X$.

This is a contradiction. Thus, we get $\mathfrak{C}_X \subsetneq \mathfrak{C}^*_X$.

Claim 5 : $f : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}^*_X)$ is a c-isomorphism

It is clear that f is a bijection on X. If $U \in \mathfrak{C}_X$, then $f(U) \in \mathfrak{C}^*_X$ since $f^{-1}(f(U)) = U \in \mathfrak{C}_X$. Now, let $V \in \mathfrak{C}^*_X$. Then by the definition of $\mathfrak{C}^*_X$ we get $f^{-1}(V) \in \mathfrak{C}_X$. Therefore, $f : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}^*_X)$ is a c-isomorphism.

Thus, there exists a c-structure $\mathfrak{C}^*_X \supsetneq \mathfrak{C}_X$ on X such that $(X, \mathfrak{C}_X)$ and $(X, \mathfrak{C}^*_X)$ are c-isomorphic. This is a contradiction. Therefore, there does not exist any c-structure $\mathfrak{C}'_X \subsetneq \mathfrak{C}_X$ on X such that the c-spaces $(X, \mathfrak{C}'_X)$ and $(X, \mathfrak{C}_X)$ are c-isomorphic.

(iii)* ⇒ *(i)

Suppose that there does not exist any c-structure $\mathfrak{C}'_X \subsetneq \mathfrak{C}_X$ on X such that the c-spaces $(X, \mathfrak{C}'_X)$ and $(X, \mathfrak{C}_X)$ are c-isomorphic. Assume that $(X, \mathfrak{C}_X)$ is not reversible. Then there exists a c-continuous bijection $h : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}_X)$ such that h^{-1} is not c-continuous. This implies $h^{-1}(C_0) \notin \mathfrak{C}_X$ for some $C_0 \in \mathfrak{C}_X$. Now, consider the collection $\mathfrak{C}^\star_X = \{A \subseteq X : h^{-1}(A) \in \mathfrak{C}_X\}$. It is clear that $\mathfrak{C}^\star_X$ is a c-structure on X. Let $U \in \mathfrak{C}^\star_X$. Then $h^{-1}(U) \in \mathfrak{C}_X$. This implies that $U = h(h^{-1}(U)) \in \mathfrak{C}_X$ since $h : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}_X)$ is c-continuous. Therefore, $\mathfrak{C}^\star_X \subseteq \mathfrak{C}_X$. We have $C_0 \in \mathfrak{C}_X$ but $C_0 \notin \mathfrak{C}^\star_X$ since $h^{-1}(C_0) \notin \mathfrak{C}_X$. Hence $\mathfrak{C}^\star_X \subsetneq \mathfrak{C}_X$.

Claim 6 : $h : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}^\star_X)$ is a c-isomorphism

It is clear that h is a bijection on X. Let $U \in \mathfrak{C}_X$. Since $h^{-1}(h(U)) = U \in \mathfrak{C}_X$, $h(U) \in \mathfrak{C}^\star_X$. Now, let $V \in \mathfrak{C}^\star_X$. Then clearly $h^{-1}(V) \in \mathfrak{C}_X$. Therefore,

$h : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}_X^\star)$ is a c-isomorphism.

Thus, there exists a c-structure $\mathfrak{C}_X^\star \subsetneq \mathfrak{C}_X$ on X such that the c-spaces $(X, \mathfrak{C}_X^\star)$ and $(X, \mathfrak{C}_X)$ are c-isomorphic. This is a contradiction. Therefore, $(X, \mathfrak{C}_X)$ is reversible. □

3.4 Non-reversible c-spaces

The c-spaces that are not reversible are called non-reversible. In this section, we discuss a few instances of non-reversible c-spaces.

Example 3.4.1. Let $X = \mathbb{N}$ and $\mathfrak{C}_X = \mathfrak{D}_X \cup \{\{1,2\},\{4,5\},\{7,8\},\ldots\}$. Then $(X, \mathfrak{C}_X)$ is a non-reversible c-space.

For $k \in \mathbb{N}$, let $C_k = \{3k-2, 3k-1\}$. Then $\mathfrak{C}_X = \mathfrak{D}_X \cup \{C_k : k \in \mathbb{N}\}$. Consider the bijection f on X which maps C_k onto C_{k+1} for $k = 2, 3, 4, \ldots$ and

$$f(3x) = \begin{cases} 1, & \text{if } x = 1. \\ 2, & \text{if } x = 2. \\ 3(x-2), & \text{if } x = 3, 4, 5, \ldots. \end{cases}$$

It is clear that f is a c-continuous function on X. Here, $C_1 = \{1,2\} \in \mathfrak{C}_X$ but $f^{-1}(C_1) = \{3,6\} \notin \mathfrak{C}_X$. This implies f is not a c-isomorphism on X.

Example 3.4.2. Let $X = \mathbb{N}$ and $\mathfrak{C}_X$ be the c-structure on X such that A is a nontrivial connected set in $(X, \mathfrak{C}_X)$ if and only if A contains all the even positive integers. Then $(X, \mathfrak{C}_X)$ is a non-reversible c-space.

Consider the bijection f on $(X, \mathfrak{C}_X)$ given by

$$f(x) = \begin{cases} x+2, & \text{if } x = 1, 3, 5, \ldots. \\ 1, & \text{if } x = 2. \\ x-2, & \text{if } x = 4, 6, 8, \ldots. \end{cases}$$

We claim that f is c-continuous.

Let $B = \{2, 4, 6, \ldots\}$. Then,

$$\begin{aligned} A \in \mathfrak{C}_X &\implies B \subseteq A \\ &\implies f(B) = \{1, 2, 4, 6, \ldots\} \subseteq f(A) \\ &\implies B \subseteq f(A) \\ &\implies f(A) \in \mathfrak{C}_X \end{aligned}$$

Thus, f is a c-continuous bijection on X. Here, $B = \{2, 4, 6, \ldots\} \in \mathfrak{C}_X$ but $f^{-1}(B) = \{4, 6, 8, \ldots\} \notin \mathfrak{C}_X$. Therefore, f is not a c-isomorphism on X.

Next theorem is the generalization of Example 3.4.2.

Theorem 3.4.3. *For any infinite cardinal α, there exists a non-reversible c-space $(X, \mathfrak{C}_X)$ with $|X| = \alpha$.*

Proof. Let X be an infinite set with $|X| = \alpha$. Choose an infinite subset A of X such that $X \setminus A$ is also infinite. Now, consider the c-structure on X given by $\mathfrak{C}_X = \mathfrak{D}_X \cup \{Y \subseteq X : A \subseteq Y\}$. Let $B = A \cup \{x_0\}$, where $x_0 \in X \setminus A$. Then consider another c-structure on X given by $\mathfrak{C}''_X = \mathfrak{D}_X \cup \{Y \subseteq X : B \subseteq Y\}$.

Claim 1 : $\mathfrak{C}''_X \subsetneq \mathfrak{C}_X$.

Let Y be a subset of X. Then,

$$\begin{aligned} Y \in \mathfrak{C}''_X &\Longrightarrow B \subseteq Y \\ &\Longrightarrow A \subseteq Y, \text{ since } A \subseteq B \\ &\Longrightarrow Y \in \mathfrak{C}_X. \end{aligned}$$

Therefore, $\mathfrak{C}''_X \subseteq \mathfrak{C}_X$. Choose $x_1 \in X \setminus A$ with $x_1 \neq x_0$ and let $E = A \cup \{x_1\}$. Then $E \in \mathfrak{C}_X$ and $E \notin \mathfrak{C}''_X$ implies $\mathfrak{C}''_X \subsetneq \mathfrak{C}_X$.

Since $|A| = |B|$ and $|X \setminus A| = |X \setminus B|$, there exists a bijection f on X such that $f(B) = A$.

Claim 2 : $f : (X, \mathfrak{C}''_X) \to (X, \mathfrak{C}_X)$ is a c-isomorphism

Let U and V be two subsets of X. Then,

$$\begin{aligned} U \in \mathfrak{C}''_X &\Longrightarrow B \subseteq U \\ &\Longrightarrow A = f(B) \subseteq f(U) \\ &\Longrightarrow f(U) \in \mathfrak{C}_X. \end{aligned}$$

Therefore, f is c-continuous. Similarly,

$$\begin{aligned} V \in \mathfrak{C}_X &\Longrightarrow A \subseteq V \\ &\Longrightarrow B = f^{-1}(A) \subseteq f^{-1}(V) \\ &\Longrightarrow f^{-1}(V) \in \mathfrak{C}''_X. \end{aligned}$$

Therefore, f^{-1} is also c-continuous. Hence $f : (X, \mathfrak{C}''_X) \to (X, \mathfrak{C}_X)$ is a c-isomorphism.

Thus, there exists a c-structure $\mathfrak{C}''_X \subsetneq \mathfrak{C}_X$ on X such that the c-spaces $(X, \mathfrak{C}''_X)$ and $(X, \mathfrak{C}_X)$ are c-isomorphic. Therefore, $(X, \mathfrak{C}_X)$ is a non reversible c-space. $\square$

3.5 Sub c-spaces of Reversible c-spaces

Here, we discuss the sub c-spaces of reversible c-spaces. Initially, we verify whether the sub c-space of a reversible c-space is also reversible. Let us go through the following example.

Example 3.5.1. Consider the order induced c-space $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$, where $\leq$ is the usual ordering of integers. It is reversible by Theorem 2.6.2. Let Y_0 be any finite subset of X. Then clearly the sub c-space $(Y_0, \mathfrak{C}_{Y_0})$ is reversible.

Now, let $Y = \{-(2n-1) : n \in \mathbb{N}\} \cup \mathbb{N}$. Then the sub c-structure of $\mathfrak{C}_{(\mathbb{Z},\leq)}$ on Y is given by $\mathfrak{C}_Y =< \{\{n, n+1\} : n \in \mathbb{N}\} >$. Here, the c-continuous bijection $f : (Y, \mathfrak{C}_Y) \to (Y, \mathfrak{C}_Y)$ given by

$$f(x) = \begin{cases} x+1, & \text{if } x \in \mathbb{N}. \\ x+2, & \text{if } x \in Y \setminus \mathbb{N}. \end{cases}$$

is not a c-isomorphism since $A = \{1, 2\} \in \mathfrak{C}_Y$ and $f^{-1}(A) = \{-1, 1\} \notin \mathfrak{C}_Y$. Therefore, $(Y, \mathfrak{C}_Y)$ is non-reversible.

Consequently, it is possible for the sub c-space of a reversible c-space to be non-reversible. We provide a sufficient condition in the following proposition for the sub c-space of an order induced c-space to be reversible.

Proposition 3.5.2. *Let* $(X, \mathfrak{C}_{(X,\leq)})$ *be an order induced c-space and* $Y \subseteq X$. *If* $Y \in \mathfrak{C}_{(X,\leq)}$, *then the sub c-space* $(Y, \mathfrak{C}_Y)$ *is reversible.*

Proof. If $Y \in \mathfrak{C}_{(X,\leq)}$, then the sub c-space $(Y, \mathfrak{C}_Y)$ is the same as the order induced c-space corresponding to the linearly ordered set $(Y, \leq)$. Hence it is reversible by Theorem 2.6.2. □

The forthcoming example demonstrate that the condition in Proposition 3.5.2 is not necessary for the sub c-space of an order induced c-space to be reversible.

Example 3.5.3. For an order induced c-space $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$, let $Y = \{-1\} \cup \mathbb{N}$. Then consider the sub c-space $(Y, \mathfrak{C}_Y)$, where $\mathfrak{C}_Y = < \{\{n, n+1\} : n \in \mathbb{N}\} >$.

Claim : Identity map is the only c-continuous bijection on $(Y, \mathfrak{C}_Y)$.

Assume that there exists a non-identity c-continuous bijection f on $(Y, \mathfrak{C}_Y)$. Let p be the smallest number such that $f(p) \neq p$. Then $f(x) = x$, for $x = -1, 1, 2, \ldots, p-1$ and $f(p) = q$ where $q > p$. Let $A = \{p-1, p\}$, then $f(A) = \{p-1, q\}$. Here, $A \in \mathfrak{C}_Y$ but $f(A) \notin \mathfrak{C}_Y$ This is a contradiction since f is c-continuous. Therefore, identity map is the only c-continuous bijection on $(Y, \mathfrak{C}_Y)$.

Thus, every c-continuous bijection on Y is a c-isomorphism. Hence $(Y, \mathfrak{C}_Y)$ is reversible but $Y \notin \mathfrak{C}_{(X,\leq)}$.

Definition 3.5.4. If every sub c-spaces of a reversible c-space $(X, \mathfrak{C}_X)$ is reversible, then it is called hereditarily reversible c-space.

Example 3.5.5. Consider the c-space $(X, \mathfrak{C}_X)$, where $X = \mathbb{N}$ and $\mathfrak{C}_X = \mathfrak{D}_X \cup \{\{1, 2, 3, \ldots, k\} : k \in \mathbb{N} \setminus \{1\}\}$. If f is a c-continuous bijection on X, then either f is the identity function or

$$f(x) = \begin{cases} 2, & \text{if } x = 1. \\ 1, & \text{if } x = 2. \\ x, & \text{otherwise.} \end{cases}$$

Clearly, f is a c-isomorphism on X. Therefore, the c-space $(X, \mathfrak{C}_X)$ is reversible. Let Y be any subset of X. Consider the sub c-space $(Y, \mathfrak{C}_Y)$. Then either $\mathfrak{C}_Y = \mathfrak{D}_Y$ or $\mathfrak{C}_Y = \mathfrak{D}_Y \cup \big\{\{1, 2\}, \{1, 2, 3\} \ldots, \{1, 2, \ldots, p\}\big\}$ for some $p \in \mathbb{N} \setminus \{1\}$. We can easily show that the sub c-space $(Y, \mathfrak{C}_Y)$ is reversible for every $Y \subseteq X$. Therefore, the c-space $(X, \mathfrak{C}_X)$ is hereditarily reversible.

Finite c-space, discrete c-space, and indiscrete c-space are trivial examples of hereditarily reversible c-spaces.

Proposition 3.5.6. *Every completely homogeneous c-spaces are hereditarily reversible.*

Proof. Trivial. □

In order to prove the next theorem, we need the following definition.

Definition 3.5.7. A subset A of a linearly ordered set is said to be *well* *ordered* [16] if every nonempty subset of A has a first element. It is *called co-well ordered* [24] if every nonempty subset of A has a last element.

Theorem 3.5.8. *Let $(X, \mathfrak{C}_{(X,\leq)})$ be an order induced c-space. If there exists an infinite subset $Y \in \mathfrak{C}_{(X,\leq)}$ such that Y is either well ordered or co-well ordered, then $(X, \mathfrak{C}_{(X,\leq)})$ is not hereditarily reversible.*

Proof. Suppose that there exists an infinite well ordered connected subset Y of X. Let y_1 be the first element of Y. Then clearly there exists $y_2 \in Y$ such that y_2 is the immediate successor of y_1. If we continue in the same way, we get $y_{i+1} \in Y$ such that y_{i+1} is the immediate successor of y_i for each $i \in \mathbb{N}$. Now consider $A \subseteq Y$ given by

$$\begin{aligned} A &= \{y_{1^4}, y_{1^4+1}, y_{1^4+3}, y_{2^4}, y_{2^4+1}, y_{2^4+3}, y_{3^4}, y_{3^4+1}, y_{3^4+3}, \ldots\} \\ &= \{y_{k^4}, y_{k^4+1}, y_{k^4+3} : k \in \mathbb{N}\} \end{aligned}$$

Let $\mathfrak{C}_A$ be the sub c-structure of $\mathfrak{C}_{(X,\leq)}$ on A. Since $Y \in \mathfrak{C}_{(X,\leq)}$, there does not exist any $x \in X$ such that $y_{k^4} < x < y_{k^4+1}$ for every $k \in \mathbb{N}$. Therefore, $\{y_{k^4}, y_{k^4+1}\} \in \mathfrak{C}_A$ for every $k \in \mathbb{N}$. It is clear that there does not exist any other nontrivial connected set $C \in \mathfrak{C}_{(X,\leq)}$ such that $C \subseteq A$. Therefore, $\mathfrak{C}_A = \mathfrak{D}_A \cup \{\{y_{k^4}, y_{k^4+1}\} : k \in \mathbb{N}\}$. For $k \in \mathbb{N}$, let $A_k = \{y_{k^4}, y_{k^4+1}\}$. Then we have $\mathfrak{C}_A = \mathfrak{D}_A \cup \{A_k : k \in \mathbb{N}\}$. Consider the bijection f on A which maps A_k onto

A_{k+1} for $k = 2, 3, 4, \ldots$ and

$$f(y_{k^4+3}) = \begin{cases} y_{1^4}, & \text{if } k = 1. \\ y_{1^4+1}, & \text{if } k = 2. \\ y_{(k-2)^4+3}, & \text{if } k = 3, 4, 5, \ldots. \end{cases}$$

It is clear that f is c-continuous on $(A, \mathfrak{C}_A)$. Here, $A_1 = \{y_{1^4}, y_{1^4+1}\} \in \mathfrak{C}_A$ but $f^{-1}(A_1) = \{y_{1^4+3}, y_{2^4+3}\} \notin \mathfrak{C}_A$. This implies the c-continuous bijection $f : (A, \mathfrak{C}_A) \to (A, \mathfrak{C}_A)$ is not a c-isomorphism. Therefore, the sub c-space $(A, \mathfrak{C}_A)$ is non-reversible and hence $(X, \mathfrak{C}_{(X,\leq)})$ is not hereditarily reversible. Similar will happen if there exists an infinite co-well ordered subset Y in $\mathfrak{C}_{(X,\leq)}$. $\square$

3.6 Operations on Reversible c-spaces

In this section, we investigate whether the intersection, union, quotient space, sum and product of reversible c-space is reversible.

The intersection of reversible c-structures on a set X need not be reversible. Consider the following example.

Example 3.6.1. Let $X = \{1, 2, 3, \ldots\}$ and $\mathfrak{C}_X^{(n)} = \mathfrak{D}_X \cup \{A \subseteq X : 2n \in A\}$ for $n \in \mathbb{N}$. Then by Proposition 3.2.2 $(X, \mathfrak{C}_X^{(n)})$ is reversible, for every $n \in \mathbb{N}$. Now, let $\mathfrak{C}_X = \bigcap_{n \in \mathbb{N}} \mathfrak{C}_X^{(n)}$. Then $(X, \mathfrak{C}_X)$ is the non-reversible c-space given in Example 3.4.2.

Similarly, the c-structure generated by the union of reversible c-structures on X need not be reversible. See the following example.

Example 3.6.2. Let $X = \{1, 2, 3, \ldots\}$ and $\mathfrak{C}_X^{(n)} = \mathfrak{D}_X \cup \{\{3n-2, 3n-1\}\}$ for $n \in \mathbb{N}$. Let $\mathfrak{C}_X$ be the c-structure generated by $\bigcup\limits_{n\in\mathbb{N}} \mathfrak{C}_X^{(n)}$. Here, $(X, \mathfrak{C}_X^{(n)})$ is reversible for every $n \in \mathbb{N}$ but $(X, \mathfrak{C}_X)$ is non-reversible (see Example 3.4.1).

Consider the c-spaces $(X, \mathfrak{C}_X)$ and $(Y, \mathfrak{C}_Y)$. An onto function $f : X \to Y$ is said to be a quotient map or Y is said to be a quotient space [30] of X with respect to f, if $\mathfrak{C}_Y$ is the smallest c-structure on Y which makes f c-continuous.

The quotient space of a reversible c-space need not be reversible. This is illustrated by the following example.

Example 3.6.3. For $k \in \mathbb{N}$, let

$$A_k = \{2k-1, 2k\},$$
$$B_k = \{-(2k-1), -2k\},$$
$$C_k = \{3k-2, 3k-1\}.$$

Consider the c-spaces $(X, \mathfrak{C}_X)$ and $(Y, \mathfrak{C}_Y)$, where

$$X = \mathbb{Z} \setminus \{0\}, \qquad \mathfrak{C}_X = \mathfrak{D}_X \cup \{A_k, B_k : k \in \mathbb{N}\}$$
$$Y = \mathbb{N}, \qquad \mathfrak{C}_Y = \mathfrak{D}_Y \cup \{C_k : k \in \mathbb{N}\}.$$

Now, consider the c-continuous function $f : X \to Y$ which maps A_k onto C_k and B_k onto $\{3k\}$ for every $k \in \mathbb{N}$. As $\mathfrak{C}_Y$ is the smallest c-structure on Y which make f is c-continuous, $(Y, \mathfrak{C}_Y)$ is the quotient space of $(X, \mathfrak{C}_X)$ with respect to f. By Proposition 3.2.3, the c-space $(X, \mathfrak{C}_X)$ is reversible. But the quotient space $(Y, \mathfrak{C}_Y)$ of X is non-reversible.

Proposition 3.6.4. *If the quotient map is one one then the quotient space of a reversible c-space is reversible.*

Proof. The proof follows from Theorem 3.2.5. □

Definition 3.6.5. A c-space $(X, \mathfrak{C}_X)$ is called the sum [30] of the family of c-spaces $\{(X_i, \mathfrak{C}_{X_i}) : i \in I\}$ whenever $X = \sum_{i \in I} X_i$ is the set theoretical sum of the sets $\{X_i : i \in I\}$ and $\mathfrak{C}_X$ is the smallest c-structure on X which make the family of one-one functions $\{f_i : X_i \to X : f_i(x) = (x, i)$ for every $i \in I\}$ c-continuous.

The sum of family of reversible c-spaces need not be reversible. This is illustrated by the following example.

Example 3.6.6. Consider the c-spaces $(X_1, \mathfrak{C}_{X_1})$ and $(X_2, \mathfrak{C}_{X_2})$, where

$$X_1 = \{-1, -3, -5, \ldots\}, \qquad \mathfrak{C}_{X_1} = \mathfrak{D}_{X_1},$$

$$X_2 = \{1, 3, 5 \ldots\}, \qquad \mathfrak{C}_{X_2} = \mathfrak{C}_{(X_2, \leq)}.$$

Here, $\leq$ is the usual ordering of numbers. It is clear that, $(X_i, \mathfrak{C}_{X_i})$ is reversible for $i = 1, 2$. The sum of c-spaces X_1 and X_2 is $(X, \mathfrak{C}_X)$, where $X = X_1 \cup X_2$ and $\mathfrak{C}_X = \mathfrak{C}_{X_1} \cup \mathfrak{C}_{X_2}$. Now, consider the c-continuous bijection f on $(X, \mathfrak{C}_X)$ defined by $f(x) = x + 2$, for every $x \in X$. Here, f is not a c-isomorphism, since $A = \{1, 3\} \in \mathfrak{C}_X$ and $f^{-1}(A) = \{-1, 1\} \notin \mathfrak{C}_X$. Therefore, $(X, \mathfrak{C}_X)$ is not reversible.

Theorem 3.6.7. *Let $(X, \mathfrak{C}_X)$ be the sum of family c-spaces $\{(X_i, \mathfrak{C}_{X_i}) : i \in I\}$. If $(X, \mathfrak{C}_X)$ is reversible then $(X_i, \mathfrak{C}_{X_i})$ is reversible for every $i \in I$.*

Proof. Suppose $(X, \mathfrak{C}_X)$ is reversible. Let $j \in I$ and f be a c-continuous bijection on $(X_j, \mathfrak{C}_{X_j})$. Assume that f is not a c-isomorphism. Then there exists $B \in \mathfrak{C}_{X_j}$ such that $f^{-1}(B) \notin \mathfrak{C}_{X_j}$. Now, consider the function $g : (X, \mathfrak{C}_X) \to (X, \mathfrak{C}_X)$ defined by

$$g(x) = \begin{cases} f(x), & \text{if } x \in X_j. \\ x, & \text{otherwise.} \end{cases}$$

It is clear that g is a bijection on X. Let $A \in \mathfrak{C}_X$. Then $A \in \mathfrak{C}_{X_i}$ for some $i \in I$. If $i \neq j$, then $g(A) = A \in \mathfrak{C}_X$. If $i = j$, then $g(A) = f(A) \in \mathfrak{C}_{X_j} \subseteq \mathfrak{C}_X$. Therefore, g is a c-continuous bijection on $(X, \mathfrak{C}_X)$. Since $f^{-1}(B) \notin \mathfrak{C}_{X_j}$, we have $g^{-1}(B) \notin \mathfrak{C}_X$. This implies that g is not a c-isomorphism on $(X, \mathfrak{C}_X)$. This is a contradiction since $(X, \mathfrak{C}_X)$ is reversible.

Therefore, every c-continuous bijection on $(X_j, \mathfrak{C}_{X_j})$ is a c-isomorphism. This is true for every $j \in I$. Thus $(X_i, \mathfrak{C}_{X_i})$ is reversible, for every $i \in I$. This completes the proof. □

Definition 3.6.8. A c-space $(X, \mathfrak{C}_X)$ is called the product space [30] of the family of c-spaces $\{(X_i, \mathfrak{C}_{X_i}) : i \in I\}$ whenever $X = \prod_{i \in I} X_i$ and $\mathfrak{C}_X = \{A \subseteq X : \pi_i(A) \in \mathfrak{C}_{X_i}, \text{ for every } i \in I\}$, where $\pi_i : X \to X_i$ are the projection functions for each $i \in I$.

The product of family of reversible c-spaces need not be reversible. Consider the following example.

Example 3.6.9. Let $(X, \mathfrak{C}_X)$ be the product of the c-spaces $(X_1, \mathfrak{C}_{X_1})$ and $(X_2, \mathfrak{C}_{X_2})$, where

$$X_1 = \mathbb{N}, \qquad \mathfrak{C}_{X_1} = \mathfrak{D}_{X_1}$$
$$X_2 = [-1, 1), \qquad \mathfrak{C}_{X_2} = \mathfrak{C}_{(X_2, \leq)}.$$

Here, $\leq$ is the usual ordering of numbers. It is clear that $(X_1, \mathfrak{C}_{X_1})$ and $(X_2, \mathfrak{C}_{X_2})$ are reversible c-spaces. Now, consider the bijection f on X which maps

$$\{1\} \times X_2 \quad \text{onto} \quad \{1\} \times [-1, 0),$$
$$\{2\} \times X_2 \quad \text{onto} \quad \{1\} \times [0, 1),$$
$$\{n\} \times X_2 \quad \text{onto} \quad \{n-1\} \times X_2, \text{ for } n = 3, 4, \ldots$$

We can easily show that f is c-continuous on $(X, \mathfrak{C}_X)$. Then $A = \{1\} \times X_2 \in \mathfrak{C}_X$. But $f^{-1}(A) = \{1, 2\} \times X_2 \notin \mathfrak{C}_X$ since $\{1, 2\} \notin \mathfrak{C}_{X_1}$. Therefore, f is not a c-isomorphism and hence the product c-space $(X, \mathfrak{C}_X)$ is not reversible.

Theorem 3.6.10. *Let $(X, \mathfrak{C}_X)$ be the product of family c-spaces $\{(X_i, \mathfrak{C}_{X_i}) : i \in I\}$. If $(X, \mathfrak{C}_X)$ is reversible then $(X_i, \mathfrak{C}_{X_i})$ is reversible for every $i \in I$.*

Proof. Suppose $(X, \mathfrak{C}_X)$ is reversible, where $X = \prod_{i \in I} X_i$ and $A \in \mathfrak{C}_X$ if and only if $\pi_i(A) \in \mathfrak{C}_{X_i}$, for every $i \in I$. Let $j \in I$ and f be a c-continuous bijection on $(X_j, \mathfrak{C}_{X_j})$. Assume that f is not a c-isomorphism. Then there exists $B_j \in \mathfrak{C}_{X_j}$

such that $f^{-1}(B_j) \notin \mathfrak{C}_{X_j}$. Define a function g on X as follows;

$$\pi_i(g(x)) = \begin{cases} f(\pi_j(x)), & \text{if } i = j. \\ \pi_i(x), & \text{if } i \neq j. \end{cases}$$

Clearly, g is a bijection on X. Let $A \in \mathfrak{C}_X$. Then $\pi_i(A) \in \mathfrak{C}_{X_i}$ for every $i \in I$. If $i \neq j$, then $\pi_i(g(A)) = \pi_i(A) \in \mathfrak{C}_{X_i}$. If $i = j$, then $\pi_j(g(A)) = f(\pi_j(A)) \in \mathfrak{C}_{X_j}$ since $\pi_j(A) \in \mathfrak{C}_{X_j}$. Thus, $\pi_i(g(A)) \in \mathfrak{C}_{X_i}$ for every $i \in I$. Hence $g(A) \in \mathfrak{C}_X$ whenever $A \in \mathfrak{C}_X$. That is, g is a c-continuous bijection on $(X, \mathfrak{C}_X)$.

Now, fix $a_i \in X_i$, for every $i \in I$ and consider $U \subseteq \prod_{i \in I} X_i$, where

$$\pi_i(U) = \begin{cases} B_j, & \text{if } i = j. \\ \{a_i\}, & \text{if } i \neq j. \end{cases}$$

Clearly, $U \in \mathfrak{C}_X$. But $g^{-1}(U) \notin \mathfrak{C}_X$ since $\pi_j(g^{-1}(U)) = f^{-1}(B_j) \notin \mathfrak{C}_{X_j}$. This implies that g is not a c-isomorphism on $(X, \mathfrak{C}_X)$. This is a contradiction, since $(X, \mathfrak{C}_X)$ is reversible. Therefore, every c-continuous bijection on $(X_j, \mathfrak{C}_{X_j})$ is a c-isomorphism. This is true for every $j \in I$. Thus, $(X_i, \mathfrak{C}_{X_i})$ is reversible for every $i \in I$. This completes the proof. □

Chapter 4

Cut-point c-spaces

4.1 Introduction

In [21], J. Muscat and D. Buhagiar mentioned n-connected and homogeneously n-connected connective spaces. A connective space X is said to be n-connected if X contains n points that disconnects X when they removed. If removing any n points disconnects X, it is said to be homogeneously n-connected. Analogous to this, we define homogeneously n-connected c-spaces. Here, we investigate the features of homogeneously 1-connected c-spaces, which we call cut-point c-spaces. Contents of this chapter is published in the Proceedings of ICETCMDS-24 [42].

4.2 Cut-point c-spaces

Observe that real line with standard c-structure has the following properties:

(i) $\mathbb{R}$ is connected,

(ii) the removal of any one of its points makes it disconnected.

Here, we study the c-spaces that satisfy conditions (i) and (ii).

Definition 4.2.1. A c-structure $\mathfrak{C}_X$ is said to be a cut-point c-structure on X if the following conditions are satisfied:

(i) $X \in \mathfrak{C}_X$,

(ii) $X \setminus \{x\} \notin \mathfrak{C}_X$ for every $x \in X$.

Then the c-space $(X, \mathfrak{C}_X)$ is called cut-point c-space.

Let $(X, \mathfrak{C}_X)$ be a connected c-space. A point $a \in X$ is called a cut-point [21] of X if $X \setminus \{a\} \notin \mathfrak{C}_X$. So $(X, \mathfrak{C}_X)$ is said to be a cut-point c-space if every $x \in X$ is a cut point of X.

Examples 4.2.2. Let X be any set with $|X| > 2$. Then the $(X, \mathfrak{C}_X)$ is a cut-point c-space, where

1. $\mathfrak{C}_X = \mathfrak{D}_X \cup \{A, X\}$, where $A \subseteq X$ such that $|A| \neq |X| - 1$.

2. $\mathfrak{C}_X = \mathfrak{D}_X \cup \{X\} \cup \{P_i : i \in I\}$, where $\{P_i : i \in I\}$ is a partition of X such that $|P_i| \neq |X| - 1$ for every $i \in I$.

The discrete c-space, indiscrete c-space, co-finite c-space and co-countable c-space are not cut-point c-spaces.

If $(X, \mathfrak{C}_X)$ is a cut-point c-space, then $|X| \geq 3$. Consequently, when we discuss cut-point c-space, the cardinality of X is assumed to be greater than or equal to 3.

Proposition 4.2.3. *Let $(X, \mathfrak{C}_X)$ and $(Y, \mathfrak{C}_Y)$ be c-isomorphic c-spaces. If $(X, \mathfrak{C}_X)$ is a cut-point c-space, then $(Y, \mathfrak{C}_Y)$ is a cut-point c-space.*

Proof. Let $f : (X, \mathfrak{C}_X) \to (Y, \mathfrak{C}_Y)$ be the c-isomorphism. Suppose $(X, \mathfrak{C}_X)$ is a cut-point c-space. Clearly, $Y = f(X) \in \mathfrak{C}_Y$. If $Y \setminus \{y\} \in \mathfrak{C}_Y$ for some $y \in Y$, then $X \setminus \{f^{-1}(y)\} = f^{-1}(Y \setminus \{y\}) \in \mathfrak{C}_X$. This is a contradiction. Thus, $Y \setminus \{y\} \notin \mathfrak{C}_Y$ for every $y \in Y$. Therefore, $(Y, \mathfrak{C}_Y)$ is a cut-point c-space. □

Remark 4.2.4. We can authenticate the following features of cut-point c-spaces via examples.

1. The intersection of cut-point c-structures on a set X need not be a cut-point c-structure on X.
2. The c-structure generated by the union of cut-point c-structures on X need not be a cut-point c-structure on X.
3. The quotient space of a cut-point c-space need not be a cut-point c-space.
4. The sum of the family of cut-point c-spaces need not be a cut-point c-space.
5. The product of the family of cut-point c-spaces need not be a cut-point c-space.

Proof. 1. Consider the set $X = \{1, 2, 3, \ldots\}$. For $n \in \mathbb{N}$, let $\mathfrak{C}_X^{(n)} = \mathfrak{D}_X \cup \{\{n, n+1\}, X\}$ and $\mathfrak{C}_X = \bigcap_{n\in\mathbb{N}} \mathfrak{C}_X^{(n)}$. Then $\mathfrak{C}_X = \mathfrak{D}_X \cup \{X\}$. It is clear that, $(X, \mathfrak{C}_X^{(n)})$ is a cut-point space for every $n \in \mathbb{N}$. But $(X, \mathfrak{C}_X)$ is not a cut-point c-space.

2. Consider the set $X = \{x \in \mathbb{R} : x \geq 0\}$. For $n \in \mathbb{N}$, let $\mathfrak{C}_X^{(n)} = \mathfrak{D}_X \cup \{X, \{x \in \mathbb{R} : x > \frac{1}{n}\}\}$. That is,

$$\begin{aligned} \mathfrak{C}_X^{(1)} &= \mathfrak{D}_X \cup \{\{x \in \mathbb{R} : x > 1\}, X\} \\ \mathfrak{C}_X^{(2)} &= \mathfrak{D}_X \cup \{\{x \in \mathbb{R} : x > \tfrac{1}{2}\}, X\} \\ \mathfrak{C}_X^{(3)} &= \mathfrak{D}_X \cup \{\{x \in \mathbb{R} : x > \tfrac{1}{3}\}, X\} \\ &\vdots \qquad\qquad \vdots \end{aligned}$$

Here, $(X, \mathfrak{C}_X^{(n)})$ is a cut-point c-space for every $n \in \mathbb{N}$. Now, let $\mathfrak{C}_X$ be the c-structure generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{C}_X^{(n)}$. Since $X \setminus \{0\} = \{x \in \mathbb{R} : x > 0\} \in \mathfrak{C}_X$, we get $(X, \mathfrak{C}_X)$ is not a cut-point c-space.

3. Consider the c-spaces $(X, \mathfrak{C}_X)$ and $(Y, \mathfrak{C}_Y)$ given by

$$\begin{aligned} X &= \mathbb{Z}, & \mathfrak{C}_X &= \mathfrak{D}_X \cup \{\mathbb{N}, X\} \\ Y &= \mathbb{N} \cup \{0\}, & \mathfrak{C}_Y &= \mathfrak{D}_Y \cup \{\mathbb{N}, Y\} \end{aligned}$$

Now, consider the function $f : X \to Y$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{N}. \\ 0, & \text{otherwise.} \end{cases}$$

As $\mathfrak{C}_Y$ is the smallest c-structure on Y which make f is c-continuous, $(Y, \mathfrak{C}_Y)$ is the quotient space of $(X, \mathfrak{C}_X)$ with respect to f. Here, the c-space $(X, \mathfrak{C}_X)$ is a cut-point c-space. But its quotient space $(Y, \mathfrak{C}_Y)$ is not a cut-point c-space since $Y \setminus \{0\} \in \mathfrak{C}_Y$.

4. Consider the cut-point c-spaces $(X_1, \mathfrak{C}_{X_1})$ and $(X_2, \mathfrak{C}_{X_2})$ given by

$$\begin{aligned} X_1 &= \mathbb{N}, & \mathfrak{C}_{X_1} &= \mathfrak{D}_{X_1} \cup \{X_1\} \\ X_2 &= \mathbb{Z} \setminus \mathbb{N}, & \mathfrak{C}_{X_2} &= \mathfrak{D}_{X_2} \cup \{X_2\} \end{aligned}$$

The sum of these c-spaces is given by $(X, \mathfrak{C}_X)$, where $X = \mathbb{Z}$ and $\mathfrak{C}_X = \mathfrak{D}_X \cup \{\mathbb{N}, \mathbb{Z} \setminus \mathbb{N}\}$. This is not a cut-point c-space since $\mathbb{Z} \notin \mathfrak{C}_X$.

5. The real line $\mathbb{R}$ is a cut-point c-space together with the c-structure obtained from the usual topology on $\mathbb{R}$. But the product c-space $\mathbb{R} \times \mathbb{R}$ is not a cut-point c-space since $\mathbb{R} \times \mathbb{R} \setminus (x, y)$ is connected for every $(x, y) \in \mathbb{R} \times \mathbb{R}$.

□

Evidently, the Brunnian closure of the sum of the family of cut-point c-spaces is a cut-point c-space.

In the next theorem, we construct a cut-point c-structure on the union of an arbitrary family of mutually disjoint c-spaces if at least one of these c-spaces is a cut-point c-space.

Theorem 4.2.5. *Let $\{X_i : i \in I\}$ be an arbitrary family of mutually disjoint sets and $\mathfrak{C}_{X_i}$ be a c-structure on X_i for $i \in I$. If $(X_p, \mathfrak{C}_{X_p})$ is a cut-point c-space* *for at least one* $p \in I$*, then there exists a c-structure* $\mathfrak{C}_X$ *on* $X = \bigcup_{i \in I} X_i$ *such that* $(X, \mathfrak{C}_X)$ *is a cut-point c-space.*

Proof. Let $\{(X_i, \mathfrak{C}_{X_i}) : i \in I\}$ be an arbitrary family of mutually disjoint c-spaces and $(X_p, \mathfrak{C}_{X_p})$ be a cut-point c-space for $p \in I$. Well order the set $I \setminus \{p\}$. By using this well ordering of $I \setminus \{p\}$, well order the set I in such a way that p is the last element of I. Let $X = \bigcup_{i \in I} X_i$ and $\mathfrak{C}'_{X_i} = \{(\bigcup_{j<i} X_j) \cup C_i : C_i \in \mathfrak{C}_{X_i}\}$.

Now, define

$$\mathfrak{C}_X = \mathfrak{D}_X \cup \bigcup_{i \in I} \mathfrak{C}'_{X_i}.$$

It is clear that $\emptyset \in \mathfrak{C}_X$ and $\{x\} \in \mathfrak{C}_X$ for every $x \in X$. Since $(X_p, \mathfrak{C}_{X_p})$ is a cut-point c-space, we have $X_p \in \mathfrak{C}_{X_p}$. This implies that $(\bigcup_{j<p} X_j) \cup X_p \in \mathfrak{C}'_{X_p}$. It follows that $X \in \mathfrak{C}_X$ since $X = (\bigcup_{j<p} X_j) \cup X_p$.

Let $\{C_\alpha \subseteq X : \alpha \in K\}$ be a nonempty collection of connected sets with $\bigcap_{\alpha \in K} C_\alpha \neq \emptyset$. We have $C_\alpha = (\bigcup_{j<i_\alpha} X_j) \cup C_{i_\alpha}$, where C_{i_α} is a connected subset of X_{i_α}. Consider the set

$$U = \{i_\alpha : \alpha \in K\}.$$

If U is an unbounded subset of I, then for every $i \in I$ there exists $\alpha \in K$ such that $i < i_\alpha$. Therefore, for every $i \in I$, $X_i \subseteq C_\alpha$ for some $\alpha \in K$. This implies that $\bigcup_{\alpha \in K} C_\alpha = \bigcup_{i \in I} X_i = X$. Hence $\bigcup_{\alpha \in K} C_\alpha \in \mathfrak{C}_X$ since $X \in \mathfrak{C}_X$.

Suppose U is a bounded subset of I. Let q be the least upper bound of U. Then either $q \in U$ or $q \notin U$. If $q \in U$, then $\bigcup_{\alpha \in K} C_\alpha = (\bigcup_{j<q} X_j) \cup C_q$ for some

$C_q \in \mathfrak{C}_{X_q}$. This implies that $\bigcup_{\alpha\in K} C_\alpha \in \mathfrak{C}_X$. If $q \notin U$, then for every α, we have $i_\alpha < q$. This implies that $\bigcup_{j<i_\alpha} X_j \subseteq \bigcup_{j<q} X_j$ and $C_{i_\alpha} \subseteq X_{i_\alpha} \subseteq \bigcup_{j<q} X_j$. It follows that $C_\alpha \subseteq \bigcup_{j<q} X_j$ for every $\alpha \in K$. Hence $\bigcup_{\alpha\in K} C_\alpha \subseteq \bigcup_{j<q} X_j$. Furthermore, we have $\bigcup_{j<i_\alpha} X_j \subseteq C_\alpha \subseteq \bigcup_{\alpha\in K} C_\alpha$ for every $i_\alpha \in U$. Thus, $\bigcup_{j<q} X_j \subseteq \bigcup_{\alpha\in K} C_\alpha$. It follows that $\bigcup_{\alpha\in K} C_\alpha \in \mathfrak{C}_X$ since $\bigcup_{\alpha\in K} C_\alpha = \bigcup_{j<q} X_j = (\bigcup_{j<q} X_j) \cup \emptyset$, where $\emptyset \in \mathfrak{C}_{X_q}$.

Therefore, the union of every nonempty collection of connected sets in $\mathfrak{C}_X$ with nonempty intersection is connected.

Now, let $x \in X$ and consider $X \setminus \{x\}$. Obviously, $X \setminus \{x\} \notin \mathfrak{C}_X$ for any $x \in X \setminus X_p$. Suppose $X \setminus \{x\} \in \mathfrak{C}_X$ for some $x \in X_p$. This implies $X_p \setminus \{x\} \in \mathfrak{C}_{X_p}$, which is a contradiction. Thus, $X \setminus \{x\} \notin \mathfrak{C}_X$ for any $x \in X$. Therefore, $(X, \mathfrak{C}_X)$ is a cut-point c-space. □

Remark 4.2.6. Cut-point c-spaces are not always reversible. The cut-point c-space $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$ is reversible. But $(\mathbb{Z}, \mathfrak{C}_\mathbb{Z})$, where $\mathfrak{C}_\mathbb{Z} = \mathfrak{D}_\mathbb{Z} \cup \{\mathbb{Z}, \{3k-2, 3k-1\} : k \in \mathbb{N}\}$ is not reversible.

4.3 Order Induced Cut-point c-spaces

Now we characterize order induced c-spaces that are cut-point c-spaces.

Theorem 4.3.1. *An order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$ is a cut-point c-space if and only if the linearly ordered set $(X, \leq)$ has neither the first element nor the last element.*

Proof. Suppose that the order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$ is a cut-point c-space. Assume that $(X, \leq)$ has the first element, say a. Now, consider $X \setminus \{a\}$. Let $x_1, x_2 \in X \setminus \{a\}$ and $z \in X$ be such that $x_1 \leq z \leq x_2$. Then clearly $z \in X \setminus \{a\}$. Thus, $X \setminus \{a\} \in \mathfrak{C}_{(X,\leq)}$. This is a contradiction since $(X, \mathfrak{C}_{(X,\leq)})$ is a cut-point c-space. Therefore, $(X, \leq)$ has no first element. Similarly, we can prove that $(X, \leq)$ has no last element.

Conversely, suppose the linearly ordered set $(X, \leq)$ has neither the first element nor the last element. Now, let $x \in X$. Then there exist $a, b \in X$ such that $a < x < b$. It is clear that $a, b \in X \setminus \{x\}$ and $x \notin X \setminus \{x\}$. Therefore, $X \setminus \{x\} \notin \mathfrak{C}_{(X,\leq)}$. This is true for every $x \in X$. Also, we have $X \in \mathfrak{C}_{(X,\leq)}$. Therefore, $(X, \mathfrak{C}_{(X,\leq)})$ is a cut-point c-space. □

Corollary 4.3.2. *(i) A finite order induced c-space cannot be a cut-point c-space. It has exactly two points that are not cut-points*

(ii) For an order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$, $X \setminus \{x\} \in \mathfrak{C}_{(X,\leq)}$ if and only if either x is the first element or the last element of $(X, \leq)$. Thus, an order induced c-space has at most two points that are not cut-points.

Remark 4.3.3. Consider a finite order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$ with $|X| > 2$. By Corollary 4.3.2 (i), there exist $a, b \in X$ such that a is a cut-point of X and b is not a cut-point of X. It is clear that there does not exist any c-isomorphism mapping a onto b. Therefore, the c-space $(X, \mathfrak{C}_{(X,\leq)})$ is not homogeneous.

Proposition 4.3.4. *If an order induced c-space* $(X, \mathfrak{C}_{(X,\leq)})$ *is a cut-point c-space then for every* $x \in X$*,* $X \setminus \{x\}$ *has* 2 *components.*

Proof. Take $A = \{x\}$ in the Theorem 2.4.5. □

Theorem 4.3.5. *If the order induced c-space* $(X, \mathfrak{C}_{(X,\leq')})$ *is a cut-point c-space, then there exists* $Y \subseteq X$ *such that the order induced c-space* $(Y, \mathfrak{C}_{(Y,\leq'_Y)})$ *is c-isomorphic to* $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$*.*

Proof. Suppose that $(X, \mathfrak{C}_{(X,\leq')})$ is a cut-point c-space. Let $a_0 \in X$, then clearly $X \setminus \{a_0\} \notin \mathfrak{C}_{(X,\leq')}$. By Corollary 4.3.2 (ii), a_0 is neither the first element nor the last element of X. Then there exists $a_{-1}, a_1 \in X$ such that $a_{-1} \leq' a_0 \leq' a_1$. Since a_{-1} is not the first element, there exists $a_{-2} \in X$ such that $a_{-2} \leq' a_{-1}$. Similarly, since a_1 is not the last element there exists $a_2 \in X$ such that $a_1 \leq' a_2$. If we continue in this way, we will have the elements $a_i \in X$ such that $a_i \leq' a_{i+1}$ for every $i \in \mathbb{Z}$. Take $Y = \{a_i : i \in \mathbb{Z}\}$. It is clear that, the c-spaces $(Y, \mathfrak{C}_{(Y,\leq'_Y)})$ and $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$ are c-isomorphic. □

The converse of Theorem 4.3.5 is not true. This is exhibited by the following example.

Example 4.3.6. Consider the linearly ordered set $(\mathbb{Z}, \leq')$. The order $\leq'$ is given by for every $x, y \in \mathbb{Z} \setminus \{0\}$, $x \leq' y$ if and only if $x \leq y$ and for every $x \in \mathbb{Z}$, $x \leq' 0$. That is, the ordering of $\mathbb{Z}$ is as follows:

$$\ldots \leq' -2 \leq' -1 \leq' 1 \leq' 2 \leq' \ldots \leq' 0.$$

Take $Y = \mathbb{Z} \setminus \{0\}$. Then the c-space $(Y, \mathfrak{C}_{(Y,\leq'_Y)})$ is c-isomorphic to $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$. But $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq')})$ is not a cut-point c-space since $\mathbb{Z} \setminus \{0\} \in \mathfrak{C}_{(\mathbb{Z},\leq')}$.

4.4 Irreducible Cut-point c-spaces

Obviously, a sub c-space of a cut point c-space need not be a cut point c-space. Here, we consider cut-point c-spaces whose proper sub c-spaces are not cut-point spaces.

Definition 4.4.1. A cut-point c-space is said to be an irreducible cut-point c-space if no proper sub c-space of it is a cut-point c-space.

Examples 4.4.2. The cut-point c-space $(X, \mathfrak{C}_X)$ is irreducible if

1. $\mathfrak{C}_X = \mathfrak{D}_X \cup \{A, X\}$, where $A \subseteq X$ such that $|A| \leq 2$.
2. $\mathfrak{C}_X = \mathfrak{D}_X \cup \{X\} \cup \{A_i : i \in I\}$, where $\{A_i : i \in I\} \subseteq \mathfrak{C}_X$ is such that $A_i \cap A_j = \emptyset$ and $|A_i| = 2$ for every $i, j \in I$ with $i \neq j$.

Proposition 4.4.3. *The order induced c-space $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$ is an irreducible cut-point c-space, where $\leq$ is the usual ordering of integers.*

Proof. Consider the order induced c-space $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$. By Theorem 4.3.1, $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$ is a cut-point c-space. Assume that it is not an irreducible cut-point c-space. Then there exists $Y \subsetneq \mathbb{Z}$ such that the sub c-space $(Y, \mathfrak{C}_Y)$ is a cut-point c-space. This implies $Y \in \mathfrak{C}_Y$ and hence $Y \in \mathfrak{C}_{(\mathbb{Z},\leq)}$. Then $(Y, \mathfrak{C}_Y)$ is coincide with

$(Y, \mathfrak{C}_{(Y,\leq)})$, the order induced c-space corresponding to the linearly ordered set $(Y, \leq)$. By Theorem 4.3.1, Y has neither the first element nor the last element. Since $Y \subsetneq X$, there exists $a_0 \in X$ such that $a_0 \notin Y$. Let $y_1 \in Y$ be such that $y_1 < a_0$, which always exists. Otherwise, $a_0 \leq y$ for every $y \in Y$ implies Y has a first element. Similarly, we can find $y_2 \in Y$ such that $a_0 < y_2$. Then $y_1 < a_0 < y_2$, and $y_1, y_2 \in Y \in \mathfrak{C}_{(\mathbb{Z},\leq)}$ implies $a_0 \in Y$. This is a contradiction. Therefore, $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$ is an irreducible cut-point c-space. □

Now, we characterize order induced irreducible cut-point c-spaces.

Theorem 4.4.4. *An order induced c-space $(X, \mathfrak{C}_{(X,\leq')})$ is an irreducible cut-point c-space if and only if it is c-isomorphic to the c-space $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$.*

Proof. By Proposition 4.4.3, we have $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$ is an irreducible cut-point c-space. Consider the order induced c-space $(X, \mathfrak{C}_{(X,\leq')})$ corresponding to the linearly ordered set $(X, \leq')$. Suppose $(X, \mathfrak{C}_{(X,\leq')})$ is an irreducible cut-point c-space. Choose $a_0 \in X$. By Proposition 4.3.4, $X \setminus \{a_0\}$ has two components say A_0 and B_0, where $a \leq' b$, for every $a \in A_0$ and $b \in B_0$. It is clear that the sub-c-space $(A_0, \mathfrak{C}_{A_0})$ is not a cut-point c-space. Then by Theorem 4.3.1, A_0 has either a first element or a last element. If A_0 has the first element p, then we have $p \leq x$ for every $x \in X$. This is a contradiction since X has no first element by Theorem 4.3.1. Therefore, A_0 has no first element, and hence A_0 has a last element, say a_{-1}. Similarly, we can prove that B_0 has the first element a_1. Assume that, there exists $x' \in X$ such that $a_{-1} <' x' <' a_0$. Then, $x' \in A_0$ and $a_{-1} <' x'$

implies a_{-1} is not the last element of A_0. This is a contradiction. Therefore, $\{a_{-1}, a_0\} = [a_{-1}, a_0]$ and hence $\{a_{-1}, a_0\} \in \mathfrak{C}_{(X,\leq')}$. Similarly, we can show that $\{a_0, a_1\} \in \mathfrak{C}_{(X,\leq')}$. Now, take $a_1 \in X$ instead of a_0 and continue the same process. Let A_1 and B_1 be the components of $X \setminus \{a_1\}$, where $a \leq' b$ for every $a \in A_1$ and $b \in B_1$. Then there exists $a_2 \in X$, which is the first element of B_1 such that $\{a_1, a_2\} \in \mathfrak{C}_{(X,\leq')}$. Similarly, let A_{-1} and B_{-1} be the components of $X \setminus \{a_{-1}\}$. Then there exists $a_{-2} \in X$, which is the last element of A_{-1} such that $\{a_{-2}, a_{-1}\} \in \mathfrak{C}_{(X,\leq')}$. If we continue like this, we get the set $Y = \{a_i \in X : i \in \mathbb{Z}\}$ such that $\{a_i, a_{i+1}\} \in \mathfrak{C}_{(X,\leq')}$ for every $i \in \mathbb{Z}$. It is clear that the sub c-space $(Y, \mathfrak{C}_Y)$ is c-isomorphic to $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$. Then by Proposition 4.2.3, $(Y, \mathfrak{C}_Y)$ is a cut point c-space. Since $(X, \mathfrak{C}_{(X,\leq')})$ is an irreducible cut-point c-space, $Y = X$. This completes the proof. □

4.5 Cut-point Connective Spaces

Here, we study cut-point connective spaces.

Definition 4.5.1. If a cut-point c-space $(X, \mathfrak{C}_X)$ is a connective space, then it is called cut-point connective space.

Example 4.5.2. The c-spaces $(\mathbb{R}, \mathfrak{C}_{(\mathbb{R},\leq)})$ and $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$ are cut-point connective spaces.

Theorem 4.5.3. *There does not exist any finite cut-point connective space.*

Proof. Suppose there exists a finite cut-point connective space $(X, \mathfrak{C}_X)$ with $|X| = n$. Take $a, b \in X$ and apply axiom (iv) of connective space for the connected sets $A = \{a\}$, $B = \{b\}$, $C_x = \{x\}$ for every $x \in X \setminus \{a, b\}$. Then there exists a subset C of X such that $C, X \setminus C \in \mathfrak{C}_X$ with $a \in C$ and $b \in X \setminus C$. That is, $\mathfrak{C}_X$ contain nontrivial connected sets other than X. Choose $P \in \mathfrak{C}_X$ such that there does not exist any $P' \in \mathfrak{C}_X$ with $|P| < |P'| < |X|$. Now, let $P = \{x_1, x_2, \ldots, x_k\}$ and $X \setminus P = \{x_{k+1}, x_{k+2}, \ldots, x_n\}$. Take $A = \{x_{k+1}\}$, $B = \{x_{k+2}\}$, $C_1 = P$, $C_i = \{x_{k+i+1}\}$ for $i = 2, 3, \ldots n - (k+1)$ in axiom (iv) of connectivity space. Then, we get a connected set $Q \in \mathfrak{C}_X$ such that $P \subsetneq Q \subsetneq X$. This is a contradiction. Therefore, $(X, \mathfrak{C}_X)$ can not be a connective space. □

Now we give a necessary and sufficient condition for an order induced c-space to be a cut-point connective space.

Theorem 4.5.4. *An order induced c-space $(X, \mathfrak{C}_{(X,\leq)})$ is a cut-point connective space if and only if the linearly ordered set $(X, \leq)$ is complete and has neither the first element or the last element.*

Proof. The proof follows from Theorem 2.8.6 and Theorem 4.3.1. □

Chapter 5

Hypergraphs and c-spaces

5.1 Introduction

Every finite order induced c-space is graphical. Moreover, an infinite order induced c-space is graphical if and only if it is c-isomorphic to either $(\mathbb{N}, \mathfrak{C}_{(\mathbb{N},\leq)})$ or $(\mathbb{Z}, \mathfrak{C}_{(\mathbb{Z},\leq)})$. It follows that there exist c-spaces such that their connected sets are incompatible with any graph's connected sets. Nevertheless, every c-space $(X, \mathfrak{C}_X)$ can be considered as a hypergraph having vertex set X and edge set $\mathfrak{C}_X$. In this chapter, we associate a hypergraph with a c-structure. Contents of this chapter is published in the Far East Journal of Mathematical Sciences [40].

5.2 c-spaces Induced by the Hypergraphs

Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph and $\mathfrak{C}$ be the collection of all vertex sets corresponding to the connected hypersubgraphs of $\mathcal{H}$. That is,

$$\begin{aligned}\mathfrak{C} =& \{A \subseteq X : \mathcal{H}_A = (A, \mathcal{E}_A) \text{ is a connected hypersubgraph of } \mathcal{H} \\ & \text{for some } \mathcal{E}_A \subseteq \mathcal{E}\}.\end{aligned}$$

Clearly, $\emptyset \in \mathfrak{C}$ and $\{x\} \in \mathfrak{C}$ for every $x \in X$. Let $\{A_i : i \in I\}$ be a nonempty collection in $\mathfrak{C}$ with $\bigcap\limits_{i \in I} A_i \neq \emptyset$. This implies that for every $i \in I$, there is a connected hypersubgraph $\mathcal{H}_{A_i} = (A_i, \mathcal{E}_{A_i})$ of $\mathcal{H}$. Now, consider the hypersubgraph $\mathcal{H}' = (\bigcup\limits_{i \in I} A_i, \bigcup\limits_{i \in I} \mathcal{E}_{A_i})$ of $\mathcal{H}$. Let $x, y \in \bigcup\limits_{i \in I} A_i$. Then, either $x, y \in A_k$ for some $k \in I$ or $x \in A_r$ and $y \in A_s$ for $r, s \in I$ with $r \neq s$.

Let $x, y \in A_k$. Since the hypersubgraph $\mathcal{H}_{A_k}$ is connected, there exists a chain from x to y in $\mathcal{H}_{A_k}$, say, $(x_1, E_1, x_2, E_2, \ldots, E_q, x_{q+1})$ where $x_1 = x$ and $x_{q+1} = y$. But this is a chain in $\mathcal{H}'$ since $x_i \in A_k \subseteq \bigcup\limits_{i \in I} A_i$ for $i = 1, 2, \ldots, q+1$ and $E_1 \in \mathcal{E}_{A_k} \subseteq \bigcup\limits_{i \in I} \mathcal{E}_{A_i}$ for $i = 1, 2, \ldots, q$. Therefore, x and y are connected in $\mathcal{H}'$.

Now, suppose $x \in A_r$ and $y \in A_s$ for $r \neq s$ in I. Since $\bigcap\limits_{i \in I} A_i \neq \emptyset$, choose $z \in A_r \cap A_s$. Then there exists a chain in $\mathcal{H}_{A_r}$ say, $(x_{r1}, E_{r1}, x_{r2}, E_{r2}, \ldots, E_{rp}, x_{r(p+1)})$ with $x_{r1} = x$ and $x_{r(p+1)} = z$, since $x, z \in A_r$ and $\mathcal{H}_{A_r}$ connected. But this is a chain in $\mathcal{H}'$. Similarly, $z, y \in A_s$ and $\mathcal{H}_{A_s}$ connected implies the existence of a chain in $\mathcal{H}_{A_s}$ say, $(x_{s1}, E_{s1}, x_{s2}, E_{s2}, \ldots, E_s, x_{s(q+1)})$ with $x_{s1} = z$ and $x_{s(q+1)} = y$.

This is also a chain in $\mathcal{H}'$.

Now, consider the sequence $(x_1, E_1, x_2, E_2, \ldots, E_{p+q}, x_{p+q+1})$, where

$$x_i = \begin{cases} x_{ri} & \text{if } 1 \leq i \leq p+1 \\ x_{s(i-p)} & \text{if } p+1 \leq i \leq p+q+1 \end{cases}$$

and

$$E_i = \begin{cases} E_{ri} & \text{if } 1 \leq i \leq p \\ E_{s(i-p)} & \text{if } p+1 \leq i \leq p+q \end{cases}$$

We can easily show that this sequence is a chain in $\mathcal{H}'$, which connects x and y. It follows that x and y are connected in $\mathcal{H}'$. This is true for every $x, y \in \bigcup\limits_{i\in I} A_i$. Hence $\mathcal{H}' = (\bigcup\limits_{i\in I} A_i, \bigcup\limits_{i\in I} \mathcal{E}_{A_i})$ is a connected hypersubgraph of $\mathcal{H}$. It follows that $\bigcup\limits_{i\in I} A_i \in \mathfrak{C}$.

Now, we provide a theorem that summarizes these observations.

Theorem 5.2.1. *Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph and $\mathfrak{C} = \{A \subseteq X : \mathcal{H}_A = (A, \mathcal{E}_A)$ is a connected hypersubgraph of $\mathcal{H}$ for some $\mathcal{E}_A \subseteq \mathcal{E}\}$. Then $\mathfrak{C}$ is a c-structure on X.*

The edge set of a hypergraph is not always a c-structure on that given set. But the edge set of the specified hypergraph always generates a c-structure. In the next theorem, we prove that the c-structure mentioned in the Theorem 5.2.1 of a hypergraph $\mathcal{H}$ is coincide with the c-structure generated by edge set of $\mathcal{H}$.

Theorem 5.2.2. *Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph and $\mathfrak{C} = \{A \subseteq X : \mathcal{H}_A = (A, \mathcal{E}_A)$ is a connected hypersubgraph of $\mathcal{H}$ for some $\mathcal{E}_A \subseteq \mathcal{E}\}$. Then $\mathfrak{C} =< \mathcal{E} >$.*

Proof. Suppose that $A \in \mathfrak{C}$. Then there exists a connected hypersubgraph $\mathcal{H}' = (A, \mathcal{E}')$ for some $\mathcal{E}' \subseteq \mathcal{E}$. Then for every $x, y \in A$, there exists a chain from x to y, say, $(x_1, E_1, x_2, E_2, \ldots, E_q, x_{q+1})$ where $x_1 = x$, $x_{q+1} = y$ and $x_k, x_{k+1} \in E_k$ for $k = 1, 2, \ldots, q$. Clearly, $E_i \in \mathcal{E}$ and $E_i \subseteq A$ for $i = 1, 2, \ldots, q$. Also, $E_i \cap E_{i+1} \neq \emptyset$ for $i = 1, 2, \ldots q-1$ since $x_{i+1} \in E_i \cap E_{i+1}$. Then by Proposition 1.4.4, we have A belongs to the c-structure generated by $\mathcal{E}$. Conversely, if C is a trivial connected set of the c-space $(X, <\mathcal{E}>)$, then clearly $C \in \mathfrak{C}$. Now, let C be a non-trivial connected set. Then consider the hypersubgraph $\mathcal{H}' = (C, \mathcal{E}')$ of $\mathcal{H}$, where $\mathcal{E}' = \{E_i \in \mathcal{E} : E_i \subseteq C\})$. Suppose that $a, b \in C$. Then by Proposition 1.4.4, there exist $\{E_{ki} : i = 1, 2, \ldots, m\} \subseteq \mathcal{E}'$ such that, for $i = 1, 2, \ldots, m-1$, $E_{ki} \cap E_{k(i+1)} \neq \emptyset$, $a \in E_{k1}$ and $b \in E_{km}$. Now, let $x_i \in E_{ki} \cap E_{k(i+1)}$. Then $(a, E_{k1}, x_1, E_{k2}, \ldots, E_{k(m-1)}, x_{m-1}, E_{km}, b)$ is a chain from a to b in $\mathcal{H}'$. Therefore, a and b are connected in $\mathcal{H}'$. This is true for every $a, b \in C$. Therefore, $C \in \mathfrak{C}$ and hence $\mathfrak{C} =< \mathcal{E} >$. □

Now, we give a formal definition for the c-space obtained from the hypergraph.

Definition 5.2.3. A c-space $(X, \mathfrak{C}_X)$ is called the c-space induced by the hypergraph $\mathcal{H} = (X, \mathcal{E})$ whenever $\mathfrak{C}_X =< \mathcal{E} >$. The c-structure induced by the hyperraph $\mathcal{H}$ is denoted by $\mathfrak{C}_{\mathcal{H}}$.

Let us go through the following example.

Example 5.2.4. Consider the hypergraph $\mathcal{H} = (X, \mathcal{E})$, where $X = \{1, 2, 3, 4, 5\}$ and $\mathcal{E} = \{\{1, 2\}, \{2, 3, 4\}, \{4, 5\}\}$ (see Figure 5.2.4). The c-space induced by the

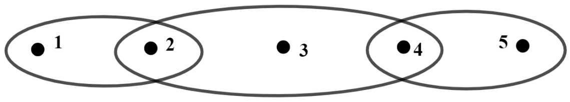

Figure 5.1: Hypergraph $\mathcal{H}$

hypergraph $\mathcal{H}$ is $(X, \mathfrak{C}_{\mathcal{H}})$, where $\mathfrak{C}_X =< \mathcal{E} >= \mathfrak{D}_X \cup \big\{\{1,2\},\{2,3,4\},\{4,5\},$ $\{1,2,3,4\},\{2,3,4,5\},\{1,2,3,4,5\}\big\}$.

Every c-space is induced by some hypergraph. Moreover, distinct hypergraphs may induce same c-space. This is illustrated by the following example.

Example 5.2.5. Consider the hypergraphs $\mathcal{H}_1 = (X, \mathcal{E}_1)$, $\mathcal{H}_2 = (X, \mathcal{E}_2)$, where $X = \{1,2,3,4\}$, $\mathcal{E}_1 = \{\{1,2\},\{2,3\}\}$ and $\mathcal{E}_2 = \{\{1,2\},\{2,3\},\{1,2,3\}\}$. The c-space induced by the hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ is $(X, \mathfrak{C}_X)$, where

$\mathfrak{C}_X = \mathfrak{D}_X \cup \{\{1,2\},\{2,3\},\{1,2,3\}\}$.

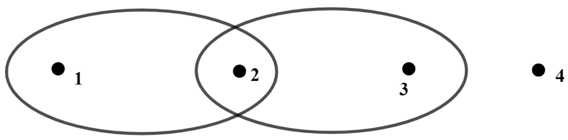

Figure 5.2: Hypergraph $\mathcal{H}_1$

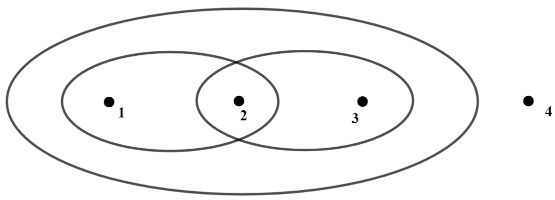

Figure 5.3: Hypergraph $\mathcal{H}_2$

5.3 Properties of c-spaces Induced by the Hypergraphs

In this section, we discuss some special properties of c-spaces induced by the hypergraphs.

Definition 5.3.1. A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is called *r-uniform* [4] whenever $|E| = r$ for all $E \in \mathcal{E}$.

Theorem 5.3.2. *The c-spaces induced by α-uniform hypergraphs are α-generated.*

Proof. Let $(X, \mathfrak{C}_X)$ be the c-space induced by the α-uniform hypergraph $\mathcal{H} = (X, \mathcal{E})$. Take $\mathcal{B} = \{A \in \mathfrak{C}_X : |A| = \alpha\}$. Then $< \mathcal{B} >=< \mathcal{E} >= \mathfrak{C}_X$ and hence the c-space $(X, \mathfrak{C}_X)$ is α-generated. □

Not every α-generated c-space is induced by an α-uniform hypergraph. This is illustrated by the following example.

Example 5.3.3. Let us consider the hypergraph $\mathcal{H} = (X, \mathcal{E})$, where $X = \{x_1, x_2, \ldots, x_{10}\}$ and $\mathcal{E} = \{\{x_1, x_2\}, \{x_3, x_6, x_7\}, \{x_4, x_9\}, \{x_5, x_6\}, \{x_8, x_9, x_{10}\}\}$. The c-structure induced by the hypergraph $\mathcal{H}$ is given by $\mathfrak{C}_X = \mathfrak{D}_X \cup \{\{x_1, x_2\}, \{x_3, x_6, x_7\}, \{x_4, x_9\}, \{x_5, x_6\}, \{x_8, x_9, x_{10}\}, \{x_3, x_5, x_6, x_7\}, \{x_4, x_8, x_9, x_{10}\}\}$. Here, the c-space $(X, \mathfrak{C}_{\mathcal{H}})$ is 3-generated. But the hypergraph $\mathcal{H}$ is not 3-uniform.

Theorem 5.3.4. *If $(X, \mathfrak{C}_X)$ is 2-generated c-space, then there exists a 2-uniform hypergraph $\mathcal{H}$ such that the c-space induced by the hypergraph is $(X, \mathfrak{C}_X)$.*

Proof. Consider the 2-generated c-space $(X, \mathfrak{C}_X)$. If $\mathfrak{C}_X = \mathfrak{D}_X$, then choose $\mathcal{E} = \emptyset$. Otherwise, there exists $\mathcal{B} \subseteq \{A \in \mathfrak{C}_X : |A| \leq 2\}$ such that $< \mathcal{B} >= \mathfrak{C}_X$. Then, choose $\mathcal{E} = \{B \in \mathcal{B} : |B| = 2\}$. It is clear that $< \mathcal{E} >= \mathfrak{C}_X$ and the hypergraph $(X, \mathcal{E})$ is 2-uniform. □

Definition 5.3.5. Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph and $E \in \mathcal{E}$. Then E is said to be an *isolated edge* [11] if for all $E' \in \mathcal{E}$ with $E' \neq E$, $E \cap E' \neq \emptyset$ implies that $E' \subseteq E$.

Theorem 5.3.6. *Let $(X, \mathfrak{C}_\mathcal{H})$ be the c-space induced by the hypergraph $\mathcal{H} = (X, \mathcal{E})$. If E is an isolated edge of $\mathcal{H}$, then it is a t-closed subset of $(X, \mathfrak{C}_\mathcal{H})$.*

Proof. Suppose E is an isolated edge of the hypergraph $\mathcal{H}$. If E is not t-closed, then there exists $x \in t(E)$ such that $x \notin E$. But $x \in t(E)$ implies there exists a nonempty subset C of E such that $\{x\} \cup C \in \mathfrak{C}_\mathcal{H}$. Let $A = \{x\} \cup C$. Then, $A \cap E \neq \emptyset$ and $A \cap (X \setminus E) \neq \emptyset$ implies E is not an isolated edge. This is a contradiction. Therefore, E is t-closed in $(X, \mathfrak{C}_\mathcal{H})$. □

Not every t-closed subset of a hypergraph induced c-space is an isolated edge.

Example 5.3.7. Let $\mathcal{H} = (X, \mathcal{E})$, where $X = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{E} = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_2, x_3, x_4\}\}$ be a hypergraph. Then the c-structure induced by $\mathcal{H}$ is given by $\mathfrak{C}_\mathcal{H} = \mathfrak{D}_X \cup \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}$. Here, $\{x_1, x_2\}$ is a t-closed subset of the c-space $(X, \mathfrak{C}_\mathcal{H})$, but $\{x_1, x_2\}$ is not an isolated edge of the hypergraph $\mathcal{H}$.

Definition 5.3.8. A vertex u of a hypergraph $\mathcal{H} = (X, \mathcal{E})$ is called *strong cut vertex* [10] of $\mathcal{H}$ if the hypergraph $\mathcal{H}' = (X', \mathcal{E}')$, where $X' = X \setminus \{u\}$ and $\mathcal{E}' = \{E \in \mathcal{E} : u \notin E\}$ has more connected components than $\mathcal{H}$.

Theorem 5.3.9. *Let $\mathcal{H} = (X, \mathcal{E})$ be a connected hypergraph and $(X, \mathfrak{C}_{\mathcal{H}})$ be the hypergraph induced c-space. Then $u \in X$ is a cut-point of $(X, \mathfrak{C}_{\mathcal{H}})$ if and only if u is a strong cut vertex of the hypergraph $\mathcal{H}$.*

Proof. Let $u \in X$ be a cut-point of the c-space $(X, \mathfrak{C}_{\mathcal{H}})$. Assume that u is not a strong cut vertex of the hypergraph $\mathcal{H}$. This implies that the hypergraph $\mathcal{H}' = (X', \mathcal{E}')$, where $X' = X \setminus \{u\}$ and $\mathcal{E}' = \{E \in \mathcal{E} : u \notin E\}$ is connected. It follows that there exists a connected hypersubgraph $\mathcal{H}'$ of $\mathcal{H}$ having vertex set $X \setminus \{u\}$. Thus, $X \setminus \{u\} \in \mathfrak{C}_{\mathcal{H}}$, which is a conradiction. Therefore, u is a strong cut vertex of $\mathcal{H}$.

Conversely, suppose u is a strong cut vertex of the hypergraph $\mathcal{H}$. If u is not a cut-point of the c-space $(X, \mathfrak{C}_{\mathcal{H}})$, then $X \setminus \{u\} \in \mathfrak{C}_{\mathcal{H}}$. Then there exists a connected hypersubgraph $\mathcal{H}_u = (X \setminus \{u\}, \mathcal{E}_u)$ of $\mathcal{H}$ for some $\mathcal{E}_u \subseteq \mathcal{E}$. Now, consider the hypergraph $\mathcal{H}' = (X', \mathcal{E}')$, where $X' = X \setminus \{u\}$ and $\mathcal{E}' = \{E \in \mathcal{E} : u \notin E\}$. Then every x and y in $X \setminus \{u\}$ are connected by a chain in $\mathcal{H}'$, since $\mathcal{E}_u \subseteq \mathcal{E}'$. It follows that the hypergraph $\mathcal{H}'$ is connected and hence u is a strong cut vertex of $\mathcal{H}$. This is a contradiction. Therefore, u is a cut-point of the c-space $(X, \mathfrak{C}_{\mathcal{H}})$. □

In the following theorem, we prove that the c-spaces induced by isomorphic hypergraphs are c-isomorphic.

Theorem 5.3.10. *Let $(X, \mathfrak{C}_{\mathcal{H}})$ and $(Y, \mathfrak{C}_{\mathcal{G}})$ be the c-spaces induced by the hypergraphs $\mathcal{H} = (X, \mathcal{E})$ and $\mathcal{G} = (Y, \mathcal{F})$ respectively. If the hypergraphs $\mathcal{H}$ and $\mathcal{G}$ are isomorphic, then the c-spaces $(X, \mathfrak{C}_{\mathcal{H}})$ and $(Y, \mathfrak{C}_{\mathcal{G}})$ are c-isomorphic.*

Proof. Suppose the hypergraphs $\mathcal{H}$ and $\mathcal{G}$ are isomorphic. Let $h : (X, \mathcal{E}) \to (Y, \mathcal{F})$ be the hypergraph isomorphism. Consider h as a function from $(X, \mathfrak{C}_{\mathcal{H}})$ onto $(Y, \mathfrak{C}_{\mathcal{G}})$ and let $C \in \mathfrak{C}_{\mathcal{H}}$. If C is a trivial connected subset of X, then clearly $h(C) \in \mathfrak{C}_{\mathcal{G}}$. Now, consider a nontrivial connected set C of X. Let $y, y' \in h(C)$. Then there exist $x, x' \in C$ such that $h(x) = y$ and $h(x') = y'$. By Proposition 1.4.4, there exist $\{E_i \subseteq C : i = 0, 1, \ldots, n\} \subseteq \mathcal{E}$ such that $E_i \cap E_{i+1} \neq \emptyset$ for $i = 0, 1, \ldots, n-1$, $x \in E_0$ and $x' \in E_n$ for some $n \in \mathbb{N}$. For $i = 0, 1, \ldots, n$, take $F_i = h(E_i)$. Then $\{F_i \subseteq h(C) : i = 0, 1, \ldots, n\} \subseteq \mathcal{F}$, $F_i \cap F_{i+1} \neq \emptyset$ for $i = 0, 1, \ldots, n-1$, $y \in F_0$ and $y' \in F_n$. This implies $h(C) \in \mathfrak{C}_{\mathcal{G}}$ and hence h is c-continuous. Similarly, we can prove that h^{-1} is c-continuous. Therefore, $h : (X, \mathfrak{C}_{\mathcal{H}}) \to (Y, \mathfrak{C}_{\mathcal{G}})$ is a c-isomorphism. □

Converse of the Theorem 5.3.10 is not true (see the Example 5.2.5).

Recall that an isomorphism from a hypergraph onto itself is called automorphism [4] and a c-isomorphism from a c-space onto itself is called c-automorphism [21].

Proposition 5.3.11. *Let $Aut(\mathcal{H})$ be the group of all automorphisms of the hypergraph $\mathcal{H}$ onto itself and $Aut(\mathfrak{C}_{\mathcal{H}})$ be the group of all c-automorphisms of the* *c-space* $(X, \mathcal{C}_{\mathcal{H}})$ *onto itself. Then* $Aut(\mathcal{H}) \subseteq Aut(\mathfrak{C}_{\mathcal{H}})$.

Proof. Let $f \in Aut(\mathcal{H})$. Then by Theorem 5.3.10, $f \in Aut(X, C_H)$. This implies that $Aut(\mathcal{H}) \subseteq Aut(\mathfrak{C}_{\mathcal{H}})$. □

The inclusion may proper in the Proposition 5.3.11. This is exhibited in the following example.

Example 5.3.12. Consider the hypergraph $\mathcal{H} = (X, \mathcal{E})$, where $X = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{E} = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_1, x_4\}, \{x_1, x_2, x_3\}\}$. Then the c-structure induced by the hypergraph $\mathcal{H}$ is

$$\mathfrak{C}_{\mathcal{H}} = \mathfrak{D}_X \cup \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_1, x_4\}, \{x_1, x_2, x_3\},$$
$$\{x_1, x_2, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}.$$

Consider the function f on X given by $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_4$, $f(x_4) = x_1$. Then $f \in Aut(\mathcal{H})$, but $f \notin Aut(\mathfrak{C}_{\mathcal{H}})$. Therefore, here we have $Aut(\mathcal{H}) \subsetneq Aut(\mathfrak{C}_{\mathcal{H}})$.

Chapter 6

Conclusion and Recommendations

6.1 Conclusion

In this thesis, we developed the theory of c-spaces. The properties of order-induced c-spaces that are obtained from linearly ordered sets are studied. We gave a characterization of 2-generated order induced c-spaces. Further, dense linearly ordered sets are characterized using t-closed subsets of the corresponding order induced c-space. We discussed the relation between order preserving and order reversing functions of a linearly ordered set and c-continuous functions of the corresponding order induced c-space. Moreover, topological order induced c-spaces are studied. Then, we prove that an order induced c-space is a connective space if and only if the corresponding linearly ordered set is complete.

Further, we studied the reversible property of c-spaces. A c-space is said to be reversible if every c-continuous bijection is a c-isomorphism. The reversible c-spaces are characterized in connection with stronger and weaker c-structures and proved that, for any infinite cardinal β, there exists a non-reversible c-space with $|X| = \beta$. The features of cut-point c-spaces are studied and characterized in order induced c-spaces, which are cut-point c-spaces. Furthermore, we defined irreducible cut-point c-spaces and gave a characterization of irreducible cut-point c-spaces, which are order induced. Also, we studied the properties of c-spaces obtained from hypergraphs. We proved that the c-structure genetated by the edge set of a hypergraph is same as the collection of vertex sets of the connected hypersubgraphs of the given hypergraph. Moreover, the group of all automorphisms of the hypergraph $\mathcal{H}$ is a subgroup of the group of all c-automorphisms of the c-space induced by the hypergraph $\mathcal{H}$.

6.2 Recommendations

Here, we describe some open problems that are to be solved.

Some of the problems studied in this thesis has obtained only a partial solution. We have every order induced c-space on a finite set is topological. But in the case of infinite set, we get only partial answer. We studied only order induced c-spaces on linearly ordered sets. There is a scope for studying c-spaces formed from partially ordered sets.

We characterised reverible c-spaces. One of the problem in this area is the following. Let $\mathfrak{C}$ be the collection of all connected subsets of a reversible topological space (X, τ). Is the c-space $(X, \mathfrak{C})$ reversible? We didn't draw any conclusions about this problem. Hypergraphs have uses in the study of mathematical morphology, just like connective spaces do. Our initial focus is on examining c-spaces derived from hypergraphs. Therefore, there is a lot of scope for research in that field.

The characterisation of topological and graphical c-structures are still open.

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Appendix

List of Publications

1. A. K. Sruthi and P. T. Ramachandran : *On c-spaces and Hypergraphs*, Far East Journal of Mathematical Sciences, 2019 ; 110(1), 83–92.

2. A. K. Sruthi and P. Sini : *Some Order Properties of c-spaces*, Palestine Journal of Mathematics, 2023 Jul 1;12(3).

3. A. K. Sruthi and P. sini : *Cut-point c-spaces*, Proceedings of the International Conference on Emerging Trends in Computational Mathematics and Data Science, ISBN 978-81-962641-6-1, 2024 ;137-144.

List of Presentations

1. A. K. Sruthi : *Reversible c-spaces*, International Conference on Mathematical sciences (ICMS-2021), Sardar Vallabhbhai National Institute of Technology, Surat, Gujarat, India.

2. A. K. Sruthi : *Cut-point c-spaces*, Proceedings of the International Conference on Emerging Trends in Computational Mathematics and Data Science (ICETCMDS-24), Sri Krishna Arts and Science College, Coimbatore, Tamil Nadu, India.

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