ON THE STUDY OF p-SPECTRAL SETS AND GENERALIZED SPECTRAL SETS

Thesis Submitted to the University of Calicut in partial fulfilment of the requirements for the degree of
DOCTOR OF PHILOSOPHY
IN MATHEMATICS

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To My Father

CERTIFICATE

This is to certify that the material presented in the thesis "**ON THE STUDY OF p- SPECTRAL SETS AND GENERALIZED SPECTRAL SETS**", of Smt. Raji Pilakkat is a bonafide record of the work done by her under my supervision and no part of this has been presented elsewhere for any degree or prize.

8.10.1999

Research Supervisor

DECLARATION

I do hereby declare that the present work is original and has not been published or submitted in part or full for any degree or prize.

Raji 14/10/

8.10.1999 Raji Pilakkat

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Introduction

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Introduction

While investigating the properties of bounded linear operators on Hilbert spaces, we came across a small section on spectral sets in Berberian's ' Lecturers in Functional Analysis and Operator Theory' [21]. The most striking property that we noticed was : "T is Hermitian if and only if R is a spectral set".

Going back to the beginning of the idea of spectral sets we found that the definition involved supremum norm and rational functions. More investigation revealed that von Neumann introduced the notion of spectral sets in 1951. He took a compact set Q of C containing the spectrum $\sigma(T)$ of an operator T on a Hilbert space and called it a spectral set for T if

$$||f(T)|| \leq ||f||_\infty$$

for all f in the closure of the set of all rational functions having poles off Q. First of all he proved that an operator T on a Hilbert space H is a contraction if and only if $||P(T)|| \leq \text{Sup} \{ |P(z)| : |z| \leq 1 \}$ .... (*) for all polynomials P. In other words the unit disc of C is a spectral set for T if and only if T is a contraction. As we have mentioned above we started looking at replacing the supremum norm by p-norm ($1 \leq p \leq \infty$) and simultaneously enlarging the collection of functions involved to include analytic functions and vector valued analytic functions.

The theory of spectral set has very close relationship with the theory of dilation. An operator T has a normal (unitary) dilation if there exists a normal (unitary) operator N on a Hilbert space K containing H as a subspace and such that

$$T = PNP|H,$$

where P is the orthogonal projection of K onto H. Sz- Nagy [19- 1970] proved that if T is a contraction on a Hilbert space H then there exists a unitary operator U on a Hilbert space K containing the space H such that U^n is a unitary dilation of T^n, for $n \geq 0$. D.Sarason [7- 1965], Foias [8- 1959], and A. Lebow [1- 1963] proved independently that: "If Q is a compact spectral set for an operator T having connected complement in the complex plane then T has a normal dilation N such that the spectrum $\sigma(N)$ of N is contained in the boundary ∂Q of Q. In other words T has a normal ∂Q dilation." Further in the case of normal operators, the spectrum is always a spectral set. We noted that in all these only supremum norm was used.

In this thesis we have introduced the notion of p- spectral sets and generalized spectral sets. The p- spectral sets are obtained by changing the supremum norm in the definition of a spectral sets by a p-norm on the class of analytic functions over a suitable domain and the generalized spectral sets are obtained by enlarging the class of rational functions to include operator valued analytic functions. The primary objective of the thesis is to find out whether our definitions leads to a more natural extension of the existing theory and secondly to search for 'new' things this generalization will throw up.

We are happy that our attempt has been somewhat fruitful and we are convinced that this work can be extended beyond this thesis- which we are unable to carry out due to time limitations.

We begin our thesis by giving all the necessary preliminary results in Chapter 1.

In the second chapter we have introduced p-spectral sets. We have studied their properties in two sections. In the case of spectral sets the spectrum of normal operators are spectral sets. We have found that in the general case it is not true. We have overcome this difficulty by introducing a new class of functions called nice p-functions. We have defined nice p-spectral sets and have shown in 2.1.12, that an operator is normal if and only if $\sigma(T)$ is a nice p-spectral set.

In Chapter 3 we have defined f(T) for a wider class of functions, namely for A valued analytic functions, where A is a sub algebra of B(H). Here instead of writing f(T) we write $\hat{f}(T)$. This chapter is also divided into two sections. The first section deals with the definition and properties of $\hat{f}(T)$. Using $\hat{f}(T)$ we have introduced a new concept called generalized spectral set. The main result in this chapter is the following. "If A is a commutative closed sub algebra of B(H), and T an element of A then $\sigma(T)$ is a generalized spectral set of T relative to A provided that $\|\hat{f}(T)\| = r(\hat{f}(T))$ for every A valued analytic function f on $\sigma(T)$."

Special mention should be made of Mergelayan's theorem, 1.2.5, which has helped us a great deal in our attempt to seek natural extension of the idea of spectral sets.

Chapter 4 has two sections. In this chapter we have studied p-spectral sets for operators with finite spectrum and denumerable spectrum. First section deals with the p-spectral sets of operators having a finite spectrum. Here we define p- spectral sets relative to some open set Ω in C. Using the Residue theorem we define f(T), for a complex analytic function f on $\sigma(T)$. The most important result in this chapter is the spectral splitting theorem 4.1.7. Some other properties we have proved are: Every spectral set is a p-spectral set and every p-spectral set is a K spectral set. From this we have deduced that the spectrum of a normal operator is a p- spectral set. All these results are proved relative to some open set Ω. Choosing Ω in a suitable manner we have proved the converse also. We have also proved that if a finite set Q containing the spectrum $\sigma(T)$ of T is a p- spectral set for T relative to an open set Ω satisfying certain conditions, then with every pair of points (x, y) in the Hilbert space H we can associate a measure $\mu(x, y)$ with the property that

$$(f(T)x, y) = \int_{\partial\Omega} f(\lambda)\, d\mu(x, y)$$

for every continuous function f on $\partial\Omega$. Using this theorem we have deduced that if Q is a p- spectral set for every p, $1 \leq p \leq \infty$, then it is a spectral set for T. We have also given Characterization theorems for normal and unitary operators.

Second section of chapter 4 talks about denumerable p-spectral sets. Denumerable p- spectral sets are also defined relative to some open set in C. Here also we have given a characterization theorem for denumerable p- spectral sets. The important result in this section is that: "If Ω is an open set disjoint from the origin and if T is a compact self adjoint invertible operator then $\sigma(T)$ is a p- spectral set for T".

We would like to mention that the study p- spectral sets of operators with finite spectrum has been more illustrative as we could do some computations and these threw more light in our study in the general situation.

We conclude our thesis by returning to von Neumann's definition of spectral sets (chapter 5). We have introduced minimal spectral sets and proved that every operator T has a minimal spectral set. Defining pseudo spectral radius as supremum of $|\lambda|$, (where λ varies over the minimal spectral set of T), we have deduced some relation between spectral radius and pseudo spectral radius.

1. *Preliminary Results*

This chapter deals with the basics needed for the study of the p- spectral sets and generalized spectral sets. The first section contains some important definitions and results about vector valued functions.

1.1 *Vector valued integration*

H will always denote a complex Hilbert space, B(H) the algebra of all bounded linear operators on H and $B(H)^*$ the space of all bounded linear functionals on B(H).

Let μ be a measure on a compact subset Q of C and f be a function from Q into B(H) such that $\Lambda \circ f$ is integrable with respect μ for every Λ in $B(H)^*$. With such a function f it is possible to associate an element $\int_Q f\, d\mu$ of B(H) called integral of f with respect to the measure μ on Q (where μ may be real or complex) as follows:

1.1.1 ***Definition*** [3.26; 30] :

Suppose μ is a real or a complex measure on a measurable space Q. X is a real or complex topological vector space on which X^* separates points and f is a measurable function from Q into X such that scalar functions $\Lambda_\circ f$ are integrable with respect to μ,for every $\Lambda \in X^*$, where $\Lambda_\circ f$ is given by

$$\Lambda_\circ f(q) = \Lambda(f(q)) \qquad (q \in Q) .$$

If there exists a vector $y \in X$ such that

$$\Lambda y = \int_Q \Lambda f \, d\mu$$

for every $\Lambda \in B(H)^*$,then we define

$$\int_Q f \, d\mu = y$$

The existence of y is proved in certain special cases. The uniqueness of y (whenever such a y exists) follows from the separability of $B(H)^*$.

1.1.2 ***Theorem*** [3.27; 30] :

Suppose X is a complex topological vector space on which the dual space separates points and μ is a real or complex Borel measure on a compact Hausdorff space Q. If $f: Q \to X$ is continuous, and if the convex hull K of f(Q) has compact closure $\bar{K}$ in X, then there exists a y in X such that

$$y = \int_Q f \, d\mu$$

in the sense of definition 1.1.1.

Remark **:**

The requirement that the closed convex hull of f(Q) should be compact is automatically satisfied when X is a Frechet space. For example B(H) is a Frechet space. In addition to that B(H)* separates points of B(H). Thus for every continuous function $f:Q \to B(H)$ and for every real or complex Borel measure μ on Q, the integral $\int_Q f \ d\mu$ exists in a unique manner.

Analytic Functions

Our theory depends mainly on the ideas of analytic functions and some of their properties. So we list some of the important properties of analytic functions that we have used in the course of this thesis. We start from the definition of an analytic function. This part of the discussion depends mostly on Ahlfors[14].

1.1.3 ***Definition*** [Chap- 3; Sec- 2.2; 14] **:**

A complex valued function f defined on some open subset Ω of C is said to be analytic if

$$\underset{\lambda \to \lambda_0}{\text{Limt}} \frac{f(\lambda)-f(\lambda_0)}{\lambda-\lambda_0}$$

exists, for every $\lambda_0 \in \Omega$.

1.1.4 ***Definition*** [Chap-3; Sec- 2.2; 14] **:**

A complex valued function defined on an arbitrary set A of C is said to be analytic if there exists an open set Ω containing A and an analytic function g on Ω such that the restriction g to A is f.

1.1.5 ***Definition*** [Chap- 0; Sec-6; 17] **:**

If γ is a continuous function from $[0,1] \to C$, then γ is said to be a path or arc in C. An arc γ is said to be closed if the initial point $\gamma(0)$ = the terminal point $\gamma(1)$ of γ.

1.1.6 *Definition* [Chap- 3; Sec- 2.1; 14] :

An arc γ is said to be differentiable if $\gamma'(t)$ exists and is continuous. If in addition $\gamma'(t) \neq 0$, for $t \in [0,1]$, the arc γ is said to be regular.

Here after we consider only differentiable arcs.

1.1.7 *Definition* [Chap- 4; Sec- 1.1; 14] :

If γ is a path in the open set Ω and f is a complex valued analytic function defined on Ω then the integral of f over γ, i.e., $\int_\gamma f\, d\lambda$, is defined as

$$\int_0^1 f(\gamma(t))\gamma'(t)\, dt$$

and integral of f along the arc length, $\int_\gamma f\, |d\lambda|$, is defined as

$$\int_0^1 f(\gamma(t))|\gamma'(t)|\, dt$$

If $\gamma_1, \gamma_2, \ldots \gamma_n$ are n arcs then their formal sum $\gamma_1 + \gamma_2 + \ldots \gamma_n$ is called a chain. A chain is a cycle if it can be represented as a sum of closed curves. An arc is simple or Jordan if $\gamma(t_1) = \gamma(t_2)$ only for $t_1 = t_2$.

1.1.8 ***Definition*** [Chap- 4; Sec- 4.2; 14] :

A region (that is a connected open set) is said to be simply connected if it contains no holes. In other words a region is simply connected if its complement with respect to the extended plane is connected.

Two important results in the study of analytic functions are Cauchy's theorem and Cauchy's integral formula.

1.1.9 ***Definition*** [Chap- 4; Sec- 2.1; 30] :

The index or the winding number of a point a with respect to a closed curve γ is defined as $n(\gamma,a) = 1/2\pi i \int_{\gamma} 1/(z-a)\, dz$.

Suppose K is a compact subset of an open set Ω in C and γ is a cycle in Ω disjoint from K, then γ is said to surround K in Ω, if $n(\gamma,a) = 1$ for $a \in K$ and $n(\gamma,a) = 0$ for $a \notin \Omega$.

A cycle γ in an open set Ω in C is said to be homologous to zero with respect to Ω if $n(\gamma,a) = 0$ for all points a in the complement of Ω.

1.1.10 ***Theorem*** [3.31; 30] :

If f(z) is analytic in a region Ω , γ is a cycle in Ω and a is a point in Ω whose winding number is one with respect to γ then $1/2\pi i \int_\gamma f(z)/(z-a)\, dz = f(a)$.

1.1.11 ***Cauchy's Theorem*** [Chap- 4; Sec- 4.4; 14] :

If f is analytic in a region Ω and γ is any cycle homologous to zero in Ω then

$$\int_\gamma f\, d\lambda = 0$$

Analogous to analyticity of complex functions one can define the analyticity of the vector valued functions.

1.1.12 ***Definition*** [3.30; 30] :

Let Ω be an open set in C. A function $f:\Omega \rightarrow B(H)$ is said to be analytic in Ω if the complex function $\Lambda \circ f$ is analytic in Ω in the sense of definition 1.1.3, for every Λ in $B(H)^*$.

1.1.13 ***Lemma*** [10.24; 30] :

Let $T \in B(H), \alpha \in C$, $\alpha \notin \sigma(T)$, Ω be the complement of α in C and Γ surrounds $\sigma(T)$ in Ω. Then

$1/2\pi i \int_\Gamma (\alpha-\lambda)^n (\lambda I-T)^{-1} d\lambda = (\alpha I - T)^n$, $(n = \ldots -3, -2, -1, 0, 1, 2, 3 \ldots)$.

If $p(\lambda)$ is a polynomial, the function $R(\lambda) = P(\lambda) + \sum_{m,k} c_{m,k}(\lambda-\alpha_m)^{-k}$ is a rational function with poles at the point α_m. Let $T \in B(H)$ be such that the spectrum $\sigma(T)$ contains no pole of R then using the above lemma one can deduce that

$$R(T) = P(T) + \sum_{m,k} c_{m,k}(T-\alpha_m I)^{-k} = 1/2\pi i \int_\Gamma R(\lambda)(\lambda I-T)^{-1} d\lambda. \qquad (*)$$

Which is just the Cauchy's integral formula for the vector valued functions.

This motivates the following definition.

1.1.14 ***Definition*** [10.26; 30] :

Let Ω be an open set in C and $H(\Omega)$ be the algebra of all complex valued analytic functions on Ω. Let T be an element in B(H) such that the spectrum $\sigma(T)$ of T is contained in Ω and Γ be a cycle in

Ω which surrounds $\sigma(T)$ of T. Then for any complex analytic function f on Ω, we define

$$\bar{f}(T) = 1/2\pi i \int_\Gamma f(\lambda)(\lambda I - T)^{-1}\, d\lambda$$

in the sense that

$$\Lambda\ \bar{f}(T) = 1/2\pi i \int_\Gamma \Lambda(f(\lambda)(\lambda I - T)^{-1})d\lambda$$

for every Λ in B(H)*.

Example :

If $f(\lambda) = 1$, for $\lambda \in \Omega$, then $\bar{f}(T) = I$ and if $f(\lambda) = \lambda$, for $\lambda \in \Omega$ then $\bar{f}(T) = T$ for any operator T in B(H).

By the choice of T and Γ the integrand in the definition of $\bar{f}(T)$ is continuous. So that the integral exists and defines $\bar{f}(T)$ as an element of B(H) as in the definition 1.1.2.

The integrand is actually an analytic vector valued function in the complement of $\sigma(T)$. Cauchy's theorem therefore implies that $\bar{f}(T)$ is independent of the choice of Γ, provided Γ surrounds $\sigma(T)$ in Ω.

For an open set Ω in C, let us denote A_Ω to be the set of all $T \in B(H)$ for which $\sigma(T) \subset \Omega$, and $H(A_\Omega)$ the set of all vector valued functions $\bar{f}(T)$ with domain A_Ω that arise from an $f \in H(\Omega)$ by the formula $\bar{f}(T) = 1/2\pi i\int_\Gamma f(\lambda)(\lambda I-T)^{-1}\, d\lambda$, where Γ is a cycle in Ω which surrounds $\sigma(T)$.

1.1.15 ***Theorem*** [10.27; 30] :

Suppose T ,$H(\Omega)$ and $H(A_\Omega)$ are as in the preceding paragraph then $H(A_\Omega)$ is a complex algebra. The mapping $f \rightarrow \bar{f}(T)$ is an algebra isomorphism of $H(\Omega)$ into $H(A_\Omega)$ which is continuous in the following sense.

If $\{f_n\}$ is a sequence of functions in $H(\Omega)$ and if f_n converges uniformly on compact subsets of Ω, then $\bar{f}_n(T)$ converge to $\bar{f}(T)$ in B(H) with respect to the norm.

One of the most important results in functional calculus is the spectral mapping theorem.

1.1.16 ***Theorem*** [10.28; 30] :

Suppose $T \in A_\Omega$ and $f \in H(\Omega)$.

(1) $\tilde{f}(T)$ is invertible in B(H) if and if only if $f(\lambda) \neq 0$ for every $\lambda \in \sigma(T)$

(2) $\sigma(\tilde{f}(T)) = f(\sigma(T))$, which is called spectral mapping theorem.

With the help of the above theorem one can introduce the composition of functions in the operations of functional calculus.

1.1.17 ***Theorem*** [10.29; 30] :

Suppose $T \in A_\Omega$, $f \in H(\Omega)$, Ω_1 is an open set containing $f(\sigma(x))$, $g \in H(\Omega_1)$ and $h(\lambda) = g(f(\lambda))$ in Ω_0, the set of λ in Ω with $f(\lambda)$ in Ω_1. Then $\tilde{f}(T) \in A_\Omega$ and $\tilde{h}(T) = \tilde{g}(\tilde{f}(T))$. (This means $\tilde{h} = \tilde{g} \circ \tilde{f}$ if $h = g \circ f$.)

Henceforth for an $f \in H(\Omega)$ we shall denote $\tilde{f}(T)$ as $f(T)$ itself.

1.2 *Approximation theorems*

This section is devoted to approximation theorems that we have used in the course of this thesis.

1.2.1 *Theorem* [7.33; 31] :

Suppose that A is a self adjoint algebra of complex continuous functions on a compact set K. A separates points on K and vanishes at no point of K. Then the uniform closure B of A consists of all complex continuous functions on K.

1.2.2 *Theorem* [13.6; 32] :

Suppose K is a compact set in the plane C and (α_j) is a set which contains one point in each component of C\K. If Ω is open, $K \subset \Omega$, $f \in H(\Omega)$ and $\in > 0$, then there exists a rational function R, all of whose poles lie in the prescribed set (α_j) such that

$$|f(z) - R(z)| < \in$$

for all $z \in K$.

As a consequence of this we get the following important result.

1.2.3 *Theorem* [13.7; 32] :

Suppose K is a compact set in the plane, C \ K is connected and $f \in H(\Omega)$ (where Ω is some open set containing K). Then there is a sequence (p_n) of polynomials such that $p_n(z) \to p(z)$ uniformly on K. The above theorem is usually known as Runge's theorem. Mergelyan improved Runge's theorem, as 'the set of polynomials is dense in the space of all complex continuous functions on K which are holomorphic in the interior of K'.

1.2.4 ***Theorem*** [20.5; 32] :

If K is a compact set in the plane whose complement is connected, f is a continuous complex function on K which is holomorphic in the interior of K and if $\in > 0$ then there exists a polynomial P such that $|f(z)-P(z)| < \in$, $\forall$ $z \in K$.

The above version of Mergelyan's theorem can be extended as:

1.2.5 ***Theorem*** [Chap-20; Sec- 3; 32] :

If K is a compact set in the plane whose complement has finitely many components then every continuous function on K which is holomorphic in the interior of K can be uniformly approximated on K by rational functions.

1.3 *Spectral sets*

In this section we discuss the main ideas concerning spectral sets which owes its origin to von-Neumann and some results due to Lebow, Paulsen and others. Throughout this section Q will denote a compact subset of the complex plane and R(Q) the uniform closure of the rational functions with poles off Q and C(Q) the collection of all complex continuos functions on Q.

1.3.1 ***Definition*** [24] :

A compact set Q in C is a K- spectral set for an operator T $\in$ B(H) if the spectrum $\sigma(T)$ of T is contained in Q and

$$\|f(T)\| \leq K \, \|f\|_Q = \sup_{\lambda \in Q} |f(\lambda)|$$

for, f $\in$ R(Q), where K is a real constant.

When K = 1, Q is called a spectral set for T.

From the very definition of a spectral set it follows that the correspondence $f \to f(T)$ defines a unital algebra homomorphism from $R(Q) \to B(H)$, whenever Q is a spectral set for T.

A necessary and sufficient condition for a compact set Q of C containing the spectrum $\sigma(T)$ of $T \in B(H)$ to be a spectral set for T is the following.

1.3.2 ***Theorem*** [66.6; 21] :

Let Q be a compact subset of C and $T \in B(H)$. Suppose $\sigma(T) \subset Q$. The following conditions on Q are equivalent.

(1) Q is a spectral set for T.

(2) If $f \in R(Q)$ and $\|f\|_Q \leq 1$ then $\|f(T)\| \leq 1$.

An immediate consequence of this result is that every compact superset of a spectral set is a spectral set.

For certain operators the spectrum itself could be a spectral set. For example if T is normal, then its spectrum is a spectral set, which is a consequence of the following result.

1.3.3 ***Theorem*** [66.9; 21] :

The following conditions on T are equivalent.

(1) $\sigma(T)$ is a spectral set for T.

(2) $\|f(T)\| = r(f(T))$, (where $r(f(T))$ is the spectral radius of $f(T)$), for all $f \in R(\sigma(T))$.

1.3.4 ***Theorem*** [66.11; 21] :

An operator T is unitary if and only if the unit circle S^1 in the complex plane is a spectral set for T.

Consider the operator T on C^2 defined by

$$T(x_1, x_2) = (x_1, x_1+x_2), \qquad (x_1, x_2) \in C^2.$$

Even though it is not self adjoint it has the real spectrum {0, 2}. But in the case of spectral sets we have the following result.

1.3.5 ***Theorem*** [66.13; 21] :

A compact subset Q of the real line R is a spectral set for T if and only if T is self adjoint.

We already remarked that complex function theory has an important role in the theory of spectral sets. The following lemma is one such instance.

1.3.6 ***Lemma*** [22] :

Let Q be a compact subset of C containing the spectrum $\sigma(T)$ of T. If Q does not separate the plane and

$$\|P(T)\| \leq \|P\|_Q$$

for all complex polynomials P then Q is a spectral set for the operator T.

Arnold Lebow [1] in his doctoral dissertation has given a measure theoretic characterization of spectral sets.

1.3.7 ***Theorem*** [1] :

A compact set Q of C is a spectral set for an operator T if and only if for every pair of vectors x and y in H there exist a regular Borel measure $\mu(x,y)$ with support in the boundary ∂Q of Q such that

(1) $\int f \, d\mu(x,y) = (f(T)x,y)$ for all $f \in R(Q)$ and

(2) $\|\mu(x,y)\| \leq \|x\| \, \|y\|$.

Consequences of this important characterization are:

1.3.8 ***Corollary*** [1] :

(1) The measure $\mu[x,x]$ is positive whenever $x \neq 0$

(2) If $\mathrm{Re} f(z) \geq 0$ for $z \in \partial Q$ then $\mathrm{Re}\, f(T) = \frac{1}{2}(f(T)+f(T)^*) \geq 0$

(3) For $f \in R(Q)$, $\|\mathrm{Re}\, f(T)\| \leq \|\mathrm{Re}\, f\|_{\partial Q}$.

1.3.9 ***Corollary*** [1] :

If R(Q) = C(Q) and Q is a spectral set for T, then T is normal.

Suppose that A is an operator on the Hilbert space K and also suppose that H is a closed subspace of K. If $Ax \in H$ whenever $x \in H$, then H is called an invariant subspace of A. If P is the orthogonal projection of K onto H, then the equality, PAP = AP is equivalent to H being an invariant subspace of A. If H is an invariant subspace of both A and A* then H is called a reducing subspace of A. Thus a subspace H is reducing if and only if PA = AP. Now if H is an invariant subspace of A then there is an operator T on the Hilbert space H such that Tx = Ax , for all x in H. The operator T is the restriction of A and A is an extension of T. When A is a normal operator its restriction T is called a subnormal operator.

Suppose H is not an invariant subspace of A we can still form an operator on H that is derived from A. For example the operator PAP is an operator having H as a reducing subspace. If T is the restriction of PAP to H then T is called the compression of A to H, A is called the dilation of T and the domain of A is called the dilation space. Halmos introduced the terminology subnormal dilation and compression in [11] wherein he also proved that every contraction has a unitary dilation. Sz Nagy [19] improved this result by showing that if T is a contraction then there is a unitary operator U such that T_n is the compression of U_n for all $n \geq 0$. Thus for every polynomial P, P(T) is the compression of P(U).

If for every $f \in R(Q)$, f(T) is the compression of f(N) then T is called the Q compression of N and N is a Q dilation of T. Here $\sigma(N)$ and $\sigma(T)$ must be subsets of Q. Now if Q is a spectral set for N and T is the Q- compression of N then

$$\|f(T)\| = \|P\, f(N)P\| \leq \|f(N)\| \leq \|f\|_Q.$$

So that Q is a spectral set for T. That is if T is a Q- compression of an operator N and if Q is a spectral set for N then Q is a spectral set for T also. Now let us suppose N is normal. By the spectral theorem, there is a spectral measure E on $\sigma(T)$ such that

$$f(N) = \int_{\sigma(T)} f(\lambda)\, d\, E(\lambda).$$

Letting $F(\delta) = PE(\delta)P$, f(T) becomes

$$f(T) = Pf(N)P = \int_{\sigma(T)} f(\lambda)\, dF(\lambda)$$

for all f in R(Q).

The operator measure F thus obtained is called a Q-measure for T. By Neumark's theorem [1], every Q –measure is the compression of a spectral measure. If E is a spectral measure whose compression is a Q measure for T then

$$\begin{aligned} f(T) &= \int_{\sigma(T)} f(\lambda)\, dF(\lambda) \\ &= \int_{\sigma(T)} f(\lambda)\, dPE(\lambda)P \\ &= P\left(\int_{\sigma(T)} f(\lambda)\, dE(\lambda)\right)P \\ &= Pf(N)P. \end{aligned}$$

Where $N = \int \lambda\, dE(\lambda)$. Thus T is the Q- compression of the normal operator N.

1.3.10 ***Definition*** [8] :

The set R(Q) is said to be a Dirichlet algebra if the real parts of the functions in R(Q) are dense in the real valued continuous functions on the boundary ∂Q of Q.

1.3.11 ***Theorem*** [1] :

If R(Q) is a Dirichlet Algebra and Q is a spectral set for an operator T on H then T is the compression of a normal operator N with $\sigma(N) \subset \partial Q$.

1.3.12 ***Corollary*** [1] :

If ∂Q is the boundary of one component of the complement of Q then every T having Q as a spectral set has a normal Q -dilation.

1.3.13 ***Corollary*** [22] :

If Q is any compact spectral set for T then there is a normal dilation N such that N^n is a dilation of T^n and $\sigma(N)$ is contained in ∂Q.

1.3.14 ***Corollary*** [1] :

Let R(Q) be a Dirichlet Algebra and Q be a spectral set for T. Let N be a normal dilation of T. If $\sigma(N)$ does not separate the plane then T is normal.

Donald Sarason [7] proved the following result about the normal dilation.

1.3.15 ***Theorem*** [7] :

If Q is a compact set of the complex plane with a connected complement then every Hilbert space operator having Q as a spectral set has a normal dilation whose spectrum is contained in ∂Q.

Remark :

This can be treated as a corollary of 1.3.12, for such a condition on Q, R(Q) is a Dirichlet algebra.

Another result which concern the dilation theory is due to Jim Agler.

1.3.16 ***Theorem*** [10] :

Let $T \in B(H)$, let $0 < r < 1$ and let $Q = \{z \in C : r \leq |z| \leq 1\}$. If Q is a spectral set for T then there exists a Hilbert space K containing H and a normal operator $N \in B(K)$ such that N is a normal ∂Q dilation of T.

Misra [15] and Paulsen [26] also have their contributions in this area We just state these results also.

1.3.17 ***Theorem*** [15] :

If T is a 2×2 matrix with one eigen value and Q is a compact spectral set for T then T has a normal ∂Q dilation.

1.3.18 ***Theorem*** [26] :

If T is a 2×2 matrix and Q is a spectral for T then T has a normal ∂Q dilation.

p-Spectral Sets Nice p-spectral sets and Kp- Spectral sets

Raji Pilakkat "On the study of p-spectral sets and generalized spectral sets "
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2 p-Spectral Sets, Nice p-Spectral Sets, and Kp-Spectral Sets

2.1 *p- Spectral Sets, Nice p- Spectral Sets*

We begin our work with the introduction of p- spectral sets, where p is an extended real number greater than or equal to one. We take an arbitrary compact set containing the spectrum of the given operator T.

Most of the time Q will denote a compact set in the complex plane whose boundary ∂Q is a path or a cycle. As in section- 1 of Chapter-1 H(Q) will denote the class of all complex valued analytic functions on Q.

The main result of this section is 2.1.12, which says that the spectrum of a normal operator is a nice p- spectral set

2.1.1 ***Definition :***

Let T be a bounded linear operator on a Hilbert space H. Let Q be a compact subset of C. Then Q is said to be a p-spectral set for T if

(1) $\sigma(T) \subset Q$.

(2) $\|f(T)\| \leq \|f\|_p$

for all $f \in H(Q)$, where $f(T)$ is as in definition 1.1.14 and

$$\|f\|_p = \left(\int_{\partial Q} |f(\lambda)|^p \, |d\lambda|\right)^{1/p} \qquad (1 \leq p < \infty)$$

$$\|f\|_\infty = \sup \{|f(\lambda)| \; : \; \lambda \in Q\}.$$

When $p = \infty$ we call a p-spectral set, a spectral set.

Suppose Q is a p-spectral set for T and suppose $f \in H(Q)$. By the maximum property of an analytic function the maximum of f is attained at some point $\lambda_0 \in \partial Q$. Hence if Q is a p-spectral set for T then

$$\begin{aligned} \|f(T)\| &\leq \|f\|_p \\ &\leq K\,|f(\lambda_0)| \\ &= K\,\|f\|_\infty, \end{aligned}$$

where $K = \left(\int_{\partial Q} d|\lambda|\right)^{1/p}$

We summarize this as follows.

2.1.2 ***Theorem*** **:**

Every p-spectral set for T is a K spectral set. (For K spectral set see Definition 1.3.1.)

Henceforth we shall take Q to be a compact set of C containing the spectrum $\sigma(T)$ of the operator T.

We now give a characterization theorem for p-spectral sets, which is analogous to the characterization theorem for spectral sets given in [1.3.2]

2.1.3 ***Theorem*** **:**

The following conditions on Q are equivalent.

(1) Q is a p-spectral set for T.

(2) If $f \in H(Q)$ and if $\|f\|_p \leq 1$ then $\|f(T)\| \leq 1$

Proof **:**

Suppose Q is a p-spectral set for T. Then by the definition we have that $\|f(T)\| \leq 1$, for every complex analytic function f on Q for which $\|f\|_p \leq 1$.

Conversely suppose that the condition of the theorem holds. Let f be a complex analytic function on Q. Suppose $||f||_p = 0$ then $f = 0$ almost everywhere on ∂Q. The analyticity of f implies that f is identically zero on ∂Q. But the maximum of an analytic function is attained on the boundary of its domain. Thus we have that f is identically zero on Q. This will imply that $f(T) = 0$. On the other hand if $||f||_p \neq 0$ Then we have $||(1/||f||_p)f||_p = 1$ and hence we have $||(1/||f||_p)f(T)|| \leq 1$. □

Now look at some cases where a compact set happens to be a spectral set.

If Q is a compact set which does not separate the plane, then by the Mergelyan's theorem every analytic function on Q can be approximated uniformly by polynomials. This strong result immediately gives:

2.1.4 *Theorem* :

Suppose Q does not separate the plane. If $||P(T)|| \leq ||P||_p$, for all complex polynomials then Q is a p-spectral set.

Going a step further and using the theorem 1.2.5, we get the following result.

2.1.5 ***Theorem*** **:**

Suppose Q is finitely connected and

$$\|R(T)\| \leq \|R\|_p$$

for all rational functions having poles off Q. Then Q is a p-spectral set for T.

Let Q be a p-spectral set for T and $f \in H(Q)$. Now we ask the question "whether f(Q) is a p-spectral set for f(T) or not?." – analogue of the spectral mapping theorem.

2.1.6 ***Theorem*** **:**

Let $f \in H(Q)$ be such that the derivative f' of f carries the boundary ∂Q of Q into the unit circle in the complex plane and if Q is a p-spectral set for T then f(Q) is a p-spectral set for f(T).

Proof **:**

By the spectral mapping theorem, for every $f \in H(Q)$

$$\sigma(f(T)) = f(\sigma(T)) \subset f(Q).$$

Hence for $g \in H(f(Q))$ we have

$$\begin{aligned}
||g(f(T))|| &= ||g_\circ f(T)|| \\
&\leq ||g_\circ f||_p \\
&= \left(\int_{\partial Q} |g(f(\lambda))|^p \, |d\lambda|\right)^{1/p} \\
&= \left(\int_{\partial f(Q)} |g(\mu)|^p (1/|f'(\lambda)|)|d\mu|\right)^{1/p} \\
&= \left(\int_{\partial f(Q)} |g(\mu)|^p \, |d\mu|\right)^{1/p} \\
&= ||g||_p, \text{ the p-norm of g taken on } \partial f(Q). \quad \square
\end{aligned}$$

On the other hand if $\gamma : [a,b] \to C$ is a continuously differentiable path with $\gamma'(t) \neq 0$, for $t \in [a,b]$ then there exists an equivalent path $\delta : [c,d] \to C$ such that $|\delta'(t)| = 1$, for $t \in [c,d]$, where $[a,b]$ and $[c,d]$ are intervals in the real line R [17]. Using this result we can strengthen the preceding theorem as follows.

2.1.7 ***Theorem*** **:**

Let Q be a compact set in C with boundary ∂Q and let $f \in H(Q)$ be such that $f'(\partial Q)$ is disjoint from the origin. Then $f(Q)$ is a p-spectral set for $f(T)$.

Analogous to the result on spectral sets [1.3.7], we have the following result.

2.1.8 ***Theorem :***

If Q is a p-spectral set for an operator T. Then for every pair of vectors $(x,y) \in H$ there exists a regular Borel measure $\mu(x,y)$ with support in ∂Q and a constant K such that for every pair (x,y) in H

(1) $\int_{\partial Q} f \, d\mu(x,y) = (f(T)x,y), \; f \in H(Q)$

(2) $\|\mu(x,y\| \leq K \|x\| \|y\|$

Proof **:**

The map $f \rightarrow (f(T)x,y)$ from $H(Q) \rightarrow C$ is linear and bounded. For $f \in H(Q)$

$$|(f(T)x,y)| \leq \|f(T)\| \|x\| \|y\|$$
$$\leq (\int_{\partial Q} (|f(\lambda)|^p |d\lambda|)^{1/p} \|x\| \|y\|$$
$$\leq K \|f\|_\infty \|x\| \|y\|,$$

where $K = (\int_{\partial Q} |d\lambda|)^{1/p}$. Hahn Banach theorem extends this functional to a continuous linear functional on C(Q) with norm $\leq K \|x\| \|y\|$. At this stage we can apply the Riesz representation theorem to get the measure stated in the theorem. □

Now we have an interesting situation, rather the deviation one expects.

In the case of a spectral sets [Chap- 1; Sec- 3] we have that if T is normal then $\sigma(T)$ is a spectral set. However in this general situation this is not true. This happens because if T is normal then for any analytic function f, $\|f(T)\| = \sup_{\lambda \in \sigma(T)} |f(\lambda)|$. But even for certain polynomials P we may have

$$\left(\int_{\partial\sigma(T)} |P(z)|^p |dz|\right)^{1/p} < \sup_{\lambda \in \sigma(T)} |P(\lambda)|$$

and this means that the condition $\|f(T)\| \leq \|f\|_F$ may not be satisfied for every $f \in H(Q)$.

For example if T is the multiplication operator on $L^2([0,1])$ defined by $Tx = zx$, $(x \in L^2([0,1]))$, where $z(t) = t$, $(t \in [0,1])$, then T is normal and $\sigma(T) = [0,1]$, the unit interval in the real line R. Suppose $f(\lambda) = \lambda$, $\lambda \in [0,1]$ and $p = 1$

$$\left(\int_{[0,1]} |f(\lambda)|^p |d\lambda|\right) = 1/2$$

$$< 1 = \sup_{\lambda \in \sigma(T)} |f(\lambda)|$$

We have overcome this pathology by introducing a new class of analytic functions.

2.1.9 ***Definition* :**

An analytic function f on a compact set Q is said to be a nice p-function if $\sup |f(\lambda)| \leq 1$, $(\lambda \in \partial Q)$ whenever $(\int_{\partial Q} |f(z)|^p |dz|)^{1/p} \leq 1$.

For example the function $f(\lambda) = \lambda/2\pi$ on the unit disc D of C is a nice p-function.

For a compact set Q, we denote the class of all nice p-functions on Q by $HB_p(Q)$, the subclass consisting of all rational functions having poles outside Q by $RB_p(Q)$ and that consisting of all polynomials by $PB_p(Q)$.

We now prove the following results using Mergelyan's theorem.

2.1.10 ***Theorem* :**

If Q is a compact set in C whose complement is connected then $PB_p(Q)$ is dense in $HB_p(Q)$ and if the complement of Q has only finitely many components then the class $RB_p(Q)$ is dense in $HB_p(Q)$.

***Proof* :**

Assume that the compact set Q has a connected complement. To prove that $PB_p(Q)$ is dense in $HB_p(Q)$, let $f \in HB_p(Q)$. Then f is an analytic function on Q and hence by the Mergelayan's theorem there exists a sequence of polynomials (P_n) which converges to the function f uniformly on Q.

If $(\int |f|^p |d\lambda|)^{1/p} \leq 1$ then, since $f \in HB_p(Q)$, $\sup_{\lambda \in \partial Q} |f(\lambda)| \leq 1$.

Now by passing to a subsequence we can assume that, the supremum of P_n over ∂Q, that is $\|P_n\|_\infty \leq 1+1/n$ and $\|P_n\|_p \leq 1+1/n$, for $n = 1,2,3......$ Now replacing P_n by $P_n/(1+1/n)$ we have $\|P_n/(1+1/n)\|_\infty \leq 1$ and $\|P_n/(1+1/n)\|_p \leq 1$ and $P_n/(1+1/n)$ converges uniformly to f on Q.

Now suppose that the complement of Q has a finite number of components in C. Then we can prove that $RB_p(Q)$ is dense in $HB_p(Q)$.

For if $f \in HB_p(Q)$ is such that $||f||_p \leq 1$ .Then again by the Mergelayan's theorem there exists a sequence of rational functions (R_n) with poles off Q and converges uniformly on Q to f. Then as above by passing to a subsequence (R_n) and multiplying each R_n by a suitable constant we can assume that both $||R_n||_p$ and $||R_n||_\infty$ are ≤ 1. Hence each such $R_n \in RB_p(Q)$ and the sequence R_n converges to f on Q uniformly. □

This leads to the following version of spectral sets.

2.1.11 ***Definition*** **:**

A compact set Q is said to be a nice p-spectral set for T if $\sigma(T) \subset Q$ and

$$||f(T)|| \leq ||f||_p$$

for all $f \in HB_p(Q)$.

Remark **:**

(1) Every p-spectral set is a nice p-spectral set.

(2) Theorem 2.1.3 is also true in the case of nice p-spectral sets.

Now the following theorems follow easily.

2.1.12 *Theorem* :

If T is normal then its spectrum $\sigma(T)$ is a nice p-spectral set.

***Proof* :**

For a normal operator T, $\|f(T)\| = \sup_{\lambda\in\sigma(T)} |f(\lambda)|$

Now suppose that $\|f\|_p \leq 1$ for some $f \in HBp(Q)$.

$$\Rightarrow \sup_{\lambda \in Q} |f(\lambda)| \leq 1$$

$$\Rightarrow \|f(T)\| \leq 1.$$

$$\Rightarrow \sigma(T) \text{ is a nice p-spectral set.}$$ □

2.1.13 *Theorem* :

If Q is a simply connected compact set in C and if $\|P(T)\| \leq \|P\|_p$ for every polynomial $P \in PB_p(Q)$ then Q is a nice p-spectral set for T.

***Proof* :**

If $f \in HB_p(Q)$, then by the theorem 2.1.10, there exists a sequence of polynomials $(P_n) \in PB_p(Q)$ such that P_n converges uniformly to f on Q. Hence by the theorem 1.1.15 we have

$$\|f(T)\| = \lim_{n\to\infty} \|P_n(T)\| \leq \|P_n\|_p.$$

□

From the second part of the theorem 2.1.10 it follows that:

2.1.14 ***Theorem*** **:**

If Q is finitely connected compact set in C and if

$$\|R(T)\| \leq \|R\|_p$$

for every rational function $R \in RB_p(Q)$ then Q is a nice p-spectral set.

Proof **:**

If $f \in HB_p(Q)$ then by the theorem 2.1.10, there exists a sequence (R_n) in $RB_p(Q)$ such that R_n converges to f uniformly on Q. Hence by the theorem 1.1.15 we have that

$$\|f(T)\| = \lim_{n\to\infty} \|R_n(T)\|$$

$$\leq \lim_{n\to\infty} \|R_n\|_p = \|f\|_p.$$

□

2.2 *Kp-Spectral Sets*

In this section we study some conditions under which a collection of operators have the same spectral sets.

2.2.1 ***Definition*** **:**

Q is said to be a Kp-spectral set for T if $\sigma(T) \subset Q$ and

$$||f(T)|| \leq K\, ||f||_p$$

for all $f \in H^{\infty}(Q)$, where K is a real positive constant.

Recall that the operators S and T are similar if there exists an invertible operator R such that $S = RTR^{-1}$.

2.2.2 ***Theorem*** **:**

Let Q be a compact p-spectral set of T whose complement is connected and if S is an operator on H similar to T then Q is a Kp-spectral set for S.

Proof **:**

Since S is similar to T, $S = RTR^{-1}$ for some invertible operator R on H. Now for polynomials P we have that

$$\begin{aligned} ||P(S)|| &\leq ||P(RTR^{-1})|| = ||RP(T)R^{-1}|| \\ &\leq ||R||\, ||P(T)||\, ||R^{-1}|| \\ &\leq K\, ||P(T)||. \end{aligned}$$

$$\leq K\,\|P\|_p, \qquad (1)$$

where $K = \|R\|\,\|R^{-1}\|$

Since the complement of Q is connected the class of all polynomials are dense in the class of all analytic functions on Q and from (1) it follows that

$$\|f(T)\| \leq K\|f\|_p$$

for every analytic function f on Q. □

Remark :

(1) If Q is as in the previous theorem then corresponding to every operator S similar to T there exists a real number K^s such that Q is a K^sp- spectral set for S.

(2) As in the theorem 2.1.16 we can prove that if Q is a compact p-spectral set for T whose complement has only finitely many components and if S is similar to T then Q is a Kp-spectral set for S. Since in this case rational functions having no poles on Q are dense in

the class of all analytic functions on Q. For such rational functions R there exists a constant K such that $||R(S)|| \leq K\ ||R||_p$.

2.2.3 *Theorem* :

If Q is a p-spectral set for T_1 and if T_2 is unitarily equivalent to T_1 then Q is a p-spectral set for T_2, provided that the complement of Q has at most a finite number of components.

***Proof* :**

Since T_2 is unitarily equivalent to T_1 there exists a unitary operator U such that

$$T_2 = UT_1U^{-1}$$

Hence for every $f \in R(Q)$

$$f(T_2) = Uf(T_1)U^{-1}$$

$$\Rightarrow ||f(T_2)|| \leq ||U||\ ||f(T_1)||\ ||U^{-1}||$$

$$= ||f(T)||$$

$$\leq ||f||_p.$$

and hence for every analytic function on Q. □

Remark **:**

The above result shows that the unitarily equivalent class of operators have the same class of finitely connected spectral sets. Generalizing this correspondence we have:

2.2.4 ***Definition*** **:**

Let A be a subalgebra of B(H). An automorphism G from A to A is said to preserve the p-spectral sets if Q is a p-spectral set for T then Q is also a p-spectral set for G(T), for all $T \in A$.

2.2.5 ***Theorem*** **:**

Let G be an automorphism from B(H) into B(H) . Then G preserves simply connected and finitely connected p-spectral sets.

Proof **:**

Let $T \in A$ and Q be a simply or finitely connected p-spectral set for T. Let $f \in R(Q)$. Then $f(\lambda) = P(\lambda) + \Sigma C_{m,k} (\lambda - \alpha_m)^{-k}$ for some polynomial P and for some scalars $C_{m,k}$.

Consider f(G(T))

$$f(G(T)) = P(G(T)) + \Sigma C_{m,n}(G(T)-\alpha_m I)^{-k}$$

$$= G(f(T))$$

$$\Rightarrow \text{f and G commute.}$$

Hence for every $f \in H(Q)$ we have

$$\|f(G(T))\| = \|G(f(T))\|$$

$$\leq \|f(T)\|, \text{ since } \|G\| \leq 1$$

$$\leq \|f\|_p.$$ □

Remark :

(1) Let U be a unitary operator in B(H). Theorem 2.1.17 shows that the inner automorphism $T \to UTU^{-1}$ from $B(H) \to B(H)$ preserves simply connected p-spectral sets as well as p-spectral sets having finite components.

(2) The idea of spectral sets as we have discussed above is some what 'collective' in comparison to the highly ' individualistic' nature of spectrum. This, we feel, is a plus point of the theory we have developed.

Generalized Spectral Sets

Raji Pilakkat "On the study of p-spectral sets and generalized spectral sets "
Thesis. Department of Mathematics , University of Calicut, 1999

3. *Generalized Spectral Sets*

3.1 *Vector Valued Functional Calculus*

Let H be a complex Hilbert space, B(H) the space of all bounded linear operators on H, A a closed subalgebra of B(H) and Ω be an open set in the complex plane C. A function $f : \Omega \rightarrow A$ is said to be analytic, if for every $\wedge$ in B(H)*, the dual of B(H), the complex valued function $\wedge \circ f: \Omega \rightarrow C$ is analytic in the usual sense.

3.1.1 *Definition* :

Let Q be an arbitrary subset of C and let A be a closed subalgebra of B(H). A function $f: Q \rightarrow A$ is said to be analytic if there exists an open set Ω in C containing Q and an analytic function $g: \Omega \rightarrow A$ such that the restriction of g to Q is f.

Notation :

Throughout this chapter A will denote a closed subalgebra of B(H) containing the identity element I of B(H) unless otherwise specified.

For a subset Ω of C we denote the class of all analytic functions from Ω into A by $H(\Omega,A)$.

3.1.2 ***Theorem :***

Let Q be a compact subset of Ω. With respect point wise operations H(Q,A) is a linear space. For $f \in H(Q,A)$ define

$$\| f \|_Q = \sup_{\lambda \in Q} \|f(\lambda)\|.$$

With $\|.\|$, H(Q,A) is a normed linear space.

(In this Chapter Q in $\|.\|_Q$ will always denote a set)

3.1.3 ***Definition :***

A sequence (f_n) in H(Q,A) is said to converge to an element f of H(Q,A) if $\| f_n - f \|_Q \to 0$ as $n \to \infty$.

Now by 1.1.2, if X is a topological vector space on which X^* separates points and if f is a continuous function from Q into X such that the convex hull H of f(Q) has compact closure in X then for any real or complex Borel measure μ on Q the integral

$$y = \int_Q f \, d\mu$$

exists in the sense that

$$\wedge y = \int_Q \wedge f \, d\mu$$

for every $\wedge \in X^*$.

3.1.4 ***Definition*** :

Let $T \in A$ and let Q be a compact subset of C which contains the spectrum $\sigma(T)$ of T. Suppose $f \in H(Q,A)$ then there exists an open set $\Omega \supset Q$ such that $f \in H(\Omega,A)$. Let Γ be a cycle in Ω, which surrounds $\sigma(T)$. Suppose Γ is given by $\lambda = \lambda(t)$, $a \leq t \leq b$. We define $\hat{f}(T)$ as follows.

$$\hat{f}(T) = 1/2\pi i \int_\Gamma f(\lambda)\,(\lambda I - T)^{-1}\, d\lambda$$

again with the understanding that

$$\wedge\hat{f}(T) = 1/2\pi i \int_\Gamma \wedge f(\lambda)\,(\lambda I - T)^{-1}\, d\lambda$$

$$= 1/2\pi i \int_a^b (\wedge\ f(\lambda)\,(\lambda I - T)^{-1})(t)\,\lambda'(t)\,dt$$

for every $\wedge \in A^*$.

Remark **:**

For $T \in A$ and $f \in H(Q,A)$, $\hat{f}(T)$ is well defined by the above definition.

Recall that if $P(t) = a_0 + a_1 t + \ldots + a_n t^n$ is a polynomial then for an operator T, $P(T) = a_0 + a_1 T + \ldots + a_n T^n$

If $f(\lambda)$ is a rational function having no poles in Q, then $g(\lambda) = f(\lambda)I$ is an element of $H(Q,A)$ and for such a g

$$\hat{g}(T) = (1/2\pi i) \int_\Gamma g(\lambda)\,(\lambda I - T)^{-1} d\lambda$$

$$= (1/2\pi i) \int_\Gamma f(\lambda)\,(\lambda I - T)^{-1} d\lambda$$

By theorem 10-25 of Rudin [30] the above integral is f(T). Thus by identifying g with f, we can write $\hat{f}(T) = f(T)$.

In particular if $f(\lambda) = I, \quad \forall\ \lambda \in Q$, then

$$\hat{f}(T) = I$$

and if $\quad f(\lambda) = \lambda I, \quad \forall\ \lambda \in Q$; then

$$\hat{f}(T) = T.$$

If Q is simply connected and $f(\lambda) = T$, $\lambda \in Q$, then

$$\hat{f}(T) = (1/2\pi i) \int_\Gamma T(\lambda I - T)^{-1} d\lambda$$

$$= (-1/2\pi i) \int_\Gamma (\lambda I - T)(\lambda I - T)^{-1} d\lambda \; +$$

$$(1/2\pi i) \int_\Gamma \lambda I(\lambda I - T)^{-1} d\lambda .$$

Now by the preceding remark and by the Cauhy's theorem we can conclude that

$$\hat{f}(T) = T.$$

These show that the definition 3.1.4 is a natural extension of definition 1.1.14

3.1.5 ***Theorem*** **:**

Let $T \in A$ and Q be a compact subset of C containing $\sigma(T)$ such that the boundary ∂Q of Q is a cycle which surrounds the spectrum of T. If $\{f_n\}$ is a sequence in H(Q,A) which converges to f in H(Q,A) then $\hat{f}_n(T)$ converges to $\hat{f}(T)$ in A.

***Proof*:**

For $\wedge \in A^*$

$$|\wedge(\hat{f}_n(T) - \hat{f}(T))| \leq (1/2\pi) \int_{\partial Q} |\wedge[(f_n(\lambda) - f(\lambda))(\lambda I - T)^{-1}]| \, |d\lambda|$$

$$\leq \alpha \, \| f_n - f \|_Q \, . \, \|\wedge\|$$

for a suitable constant α.

Hence

$$\|\hat{f}_n(T) - \hat{f}(T)\| = \sup_{\substack{\wedge \in A^* \\ \|\wedge\| \leq 1}} |\wedge(f_n(T) - f(T))|$$

$$\to 0 \text{ as } n \to \infty.$$ □

Now suppose that $\hat{f}(T) = 0$, then this

$$\Rightarrow \quad \wedge\hat{f}(T) = 0 \quad \forall \wedge \in A^*$$

$$\Rightarrow \quad \wedge(f(\lambda))(\lambda I - T)^{-1} = 0, \lambda \in \partial Q \text{ and } \wedge \in A^* \qquad (*)$$

If A is a semi-simple commutative closed subalgebra of B(H) then (*) implies that $f(\lambda) = 0$, $\lambda \in \partial Q$. The maximum principle now shows that $f(\lambda) = 0$, $\lambda \in Q$. We summarize these results as follows:

3.1.6 *Theorem* :

Let A be a semi-simple commutative closed subalgebra of B(H) and T be an element of A. Let Q be a compact set in C whose boundary surrounds $\sigma(T)$. Then the map

$$f \rightarrow \hat{f}(T)$$

from H(Q,A) into A is a continuous injection.

3.2 *Generalized Spectral Sets*

Having introduced vector valued functional calculus to suit our needs in the previous section we now pass on to the study Generalized spectral sets.

3.2.1 *Definition* :

A compact subset Q of C is said to be a generalized spectral set for an operator $T \in A$ relative to A if the spectrum $\sigma(T)$ of T is contained in Q and if

$$\|\hat{f}(T)\| \leq \| f \|_Q = \sup_{\lambda \in Q} \|f(\lambda)\|$$

for all $f \in H(Q,A)$.

3.2.2 ***Theorem*** **:**

Every generalized spectral set Q of T relative to A is a spectral set for T.

Proof **:**

If f: $Q \rightarrow C$ is a rational function having poles off Q, then f can be considered as an A valued analytic function by identifying f with the function $f(\lambda)I$, $\lambda \in Q$ and we have already noted that for such functions $\hat{f}(T) = f(T)$. □

Remark **:**

The above result shows that the definition 3.2.1 is quite meaningful and more general as it covers a wide collection of sets.

We note from the definition of generalized spectral sets that every compact superset of a generalized spectral set is also a generalized spectral set.

We now give a necessary and sufficient condition (similar to 2.1.3) for a non empty compact subset Q of C to be a generalized spectral set for an operator T of A.

3.2.3 *Theorem* :

Let $T \in A$ and Q be a compact set which contains the spectrum $\sigma(T)$ of T. Then the following conditions on Q are equivalent:

(a) Q is a generalized spectral set for T relative to A.

(b) If $f \in H(Q,A)$ and $\| f \|_Q \leq 1$ then $\|\hat{f}(T)\| \leq 1$.

Proof **:**

(a) implies (b): If $f \in H(Q,A)$ and $\| f \|_Q \leq 1$ then $\|\hat{f}(T)\| \leq \| f \|_Q \leq 1$

(b) implies (a): If $\| f \|_Q = 0$, then $n\| f \|_Q = 0$ which implies that $\|n\hat{f}(T)\| \leq 1$ for $n = 1, 2, 3, \ldots$ and hence $f(T) = 0$. On the other hand $\| f \|_Q^{-1} f \in H(Q,A)$. Now apply (b) to $\| f \|_Q^{-1} f$. □

We make the following important observations.

1. For $T \in A$ and for any closed subalgebra B of B(H) containing A, if $\sigma_B(T)$, the spectrum of T relative to B, does not separate the complex

plane then $\sigma_A(T)$, the spectrum of T relative to A $= \sigma_B(T)$, by the theorem 10.18 of Rudin [30]. An immediate consequence is that:

If B is any closed subalgebra of B(H) containing A and if $\sigma_B(T)$ does not separate the complex plane for an operator $T \in A$, then every generalized spectral set of T relative to A is a generalized spectral set of T relative to B.

2. von-Neumann [29] proved that if T is a contraction then the unit disc D of C is a spectral set for T. Here also we can prove that if A is a commutative closed subalgebra of B(H) such that $\|T^2\| = \|T\|^2$ for every T in A and if T is a contraction in A then D is a generalized spectral set for T relative to A.

3.2.4 *Definition* :

A sum of the form $a_0 + a_1\lambda + \dots. + a_n \lambda^n$ is called polynomial over A,

where λ is a complex variable and $a_0, a_1 \dots a_n$ are elements of A.

Suppose f is a polynomial over A, then

$$\varphi(\hat{f}(T)) = (1/2\pi i) \int_\Gamma \varphi(f)\, \varphi(\lambda I - T)^{-1}\, d\lambda$$

for $\varphi \in \Delta(A)$, the space of all complex homomorphisms on A and for some cycle Γ which surrounds $\sigma(T)$.

$$\varphi(\hat{f}(T)) = (1/2\pi i) \int_\Gamma (\varphi(\Sigma\, a_i\, \lambda^i))\ \varphi\, (\lambda I - T)^{-1}\, d\lambda$$

$$= (1/2\pi i) \int_\Gamma (\Sigma\lambda^i\, \varphi(a_i))\, \varphi\, (\lambda I - T)^{-1}\, d\lambda$$

$$= \varphi\, (\Sigma\lambda^i\, \varphi(a_i)(T))$$

$$= \Sigma\, \varphi(a_i))\, \varphi\, (T^i)$$

Thus

$$\varphi(\hat{f}(T)) = \varphi\, (\Sigma a_i T^i)$$

for every $\varphi \in \Delta(A)$.

3.2.5 ***Definition*** **:**

We say that a function $f \in H(Q,A)$ is the uniform limit of a sequence of polynomials over Q if for every $\varphi \in \Delta(A)$, there exists a sequence of complex valued polynomials $(P_n^{\varphi}(\lambda))$ such that $\varphi \circ f$ is the uniform limit of (P_n^{φ}) on Q.

If Q is a compact set having connected complement then by Mergelyan's theorem every $f \in H(Q,A)$ is the uniform limit of a sequence of polynomials over Q.

3.2. 6 ***Theorem*** **:**

Let **A** be a commutative closed subalgebra of B(H) in which every $T \in A$ satisfies the identity $\|T^2\| = \|T\|^2$. Let T be a contraction in A such that $\sigma(T) \cap \partial D = \Phi$. Then the unit disc D of C is a generalized spectral set of T relative to A.

Proof **:**

For every polynomial $f = \sum a_i \lambda^i$ over A and for every $\varphi \in \Delta(A)$.

$$\varphi(\hat{f}(T)) = \varphi\left(\sum_{i=1}^{n} a_i T^i\right)$$

By the definition of **A**

$$\begin{aligned}
\|\hat{f}(T)\| &= \sup_{\varphi \in \Delta(A)} |\varphi(\hat{f}(T))| \\
&= \sup_{\varphi \in \Delta(A)} |\sum \varphi(a_i)\, \varphi(T^i)| \\
&\leq \sup_{\varphi \in \Delta(A)} \sup_{\lambda \in D} |\sum \varphi(a_i) \lambda^i| \\
&\leq \sup_{\lambda \in D} \|\sum a_i \lambda^i\| \\
&= \|f\|_D.
\end{aligned}$$

Let $f \in H(D,A)$ be arbitrary. Then by the Mergelyan's theorem, for every $\varphi \in \Delta(A)$ there exists a sequence of complex polynomials (P_n^{φ}) on D such that (P_n^{φ}) converges to $\varphi \circ f$ uniformly on D. Hence

$$\begin{aligned}
\varphi\, \hat{f}(T) &= (1/2\pi i) \int_{\Gamma} \varphi(f(\lambda))\, \varphi(\lambda I - T)^{-1}\, d\lambda \\
&= \varphi\, (\varphi \circ f)\, (T) \\
&= \varphi \lim_{n\to\infty} P_n^{\varphi}\, (T) \\
&= \lim_{n\to\infty} \varphi\, P_n^{\varphi}\, (T)
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\hat{f}(T)\| &= \sup_{\varphi \in \Delta(A)} |\varphi\, (\hat{f}(T))| \\
&= \sup_{\varphi} \lim_{n\to\infty} |\varphi\, P_n^{\varphi}\, (T)| \\
&\le \lim_{n\to\infty} \|P_n^{\varphi}\, (T)\| \\
&\le \lim_{n\to\infty} \|P_n \varphi\|_D \\
&= \|\varphi \circ f\|_D. \\
&\le \|f\|_D.
\end{aligned}$$

Thus D is a generalized spectral set for T relative to A. □

3.2.7 ***Corollary :***

If A is as in the previous theorem and if T is a strict contraction then D is a generalized spectral set for T relative to A.

Remark :

If Q is a compact set in C whose complement in C is connected and if

$$\|\hat{f}(T)\| \leq \|f\|_Q$$

for every polynomial $f \in H(Q,A)$ then Q is a generalized spectral set for T provided ∂Q surrounds $\sigma(T)$.

3.2.8 ***Theorem*** :

If A is a commutative closed self adjoint subalgebra of B(H) and if $T \in A$ then $\sigma(T)$ is a generalized spectral set for T relative to A.

Proof :

By the definition of A, every element of A is normal. Therefore for every $T \in A$.

$$\| T\| = \sup_{\varphi \in \Delta (A)} | \varphi(T) |$$

Now for any $f \in H(\sigma(T),A)$ and for any $\varphi \in \Delta (A)$

$$| \varphi (\hat{f}(T))| = | (1/2\pi i) \int_\Gamma \varphi(f) \varphi(\lambda I - T)^{-1} d\lambda |$$

$$= | \varphi (\varphi \circ f (T))|$$

Since $\hat{f}(T)$ is a normal,

$$\| \hat{f} (T)\| = \sup_{\varphi \in \Delta(A)} | \varphi(\hat{f}(T))|$$

$$= \sup_{\varphi\in\Delta(A)} | \varphi(\varphi \circ f) (T) |$$

$$\leq \sup_{\varphi\in\Delta(A)} \| (\varphi \circ f) (T) \|$$

$$\leq \| \varphi \circ f \|_{\sigma(T)}$$

$$\leq \|f\|_{\sigma(T)}$$

Thus for every $f \in H (\sigma(T), A)$

$$\|\hat{f}(T)\| \leq \|f\|_{\sigma(T)}.$$ □

3.2.9 ***Corollary*** **:**

If T is normal and A is the subalgebra generated by T, T* and I, then $\sigma(T)$ is a generalized spectral set for T relative to A and $\overline{\sigma(T)}$, which consists of the set of all complex conjugates of elements of $\sigma(T)$ is a generalized spectral set for T* relative to A.

The crucial point in the proof of the preceding theorem is that $\|T\| = r(T)$, the spectral radius of T, for every $T \in A$. This in turn determine the most important result in this work. We can treat Theorems 3.2.6 and 3.2.8 as its corollaries.

3.2.10 ***Theorem*** **:**

Let A be a commutative closed subalgebra of B(H) and T be an element of A. If $\|\hat{f}(T)\| = r(\hat{f}(T))$, for every $f \in H(\sigma(T),A)$ then $\sigma(T)$ is a generalized spectral set of T relative to A.

Proof **:**

Every rational function f with poles off $\sigma(T)$, determines an element of $H(\sigma(T),A)$, namely f I. For such a function we have $\hat{f}(T) = f(T)$. Thus for such functions the relation $\|\hat{f}(T)\| = r(\hat{f}(T))$ becomes $\|f(T)\| = r(f(T))$. The proposition 66.9 of Berberian [21] shows that $\sigma(T)$ is then a spectral set for T.

Now for any complex homomorphism φ on A and f $\in$ $H(\sigma(T),A)$, $\varphi \circ f$ is a complex valued analytic function and hence by the spectral mapping theorem $\sigma(\varphi \circ f(T)) = \varphi \circ f(\sigma(T))$. Hence we have

$$\begin{aligned} \|(\varphi \circ f)(T)\| &= \|\varphi \circ f\|_{\sigma(T)} \\ &\leq \|f\|_{\sigma(T)}. \end{aligned}$$

Further,

$$\begin{aligned} \varphi(\hat{f}(T)) &= (1/2\pi i) \int_\Gamma \varphi(f(\lambda))\, \varphi(\lambda I - T)^{-1}\, d\lambda \\ &= \varphi[\,(\varphi \circ f)(T)\,] \end{aligned}$$

Thus,

$$\begin{aligned} \|\hat{f}(T)\| &= r(\hat{f}(T)) \\ &= \sup_{\varphi \in \Delta(A)} |\varphi(\hat{f}(T))|. \\ &= \sup_{\varphi \in \Delta(A)} |\varphi(\varphi \circ f)(T)| \\ &\leq \sup_{\substack{\wedge \in A^* \\ \|\wedge\| \leq 1}} |\wedge(\varphi \circ f)(T)| \\ &= \|\varphi \circ f(T)\| \\ &= \|\varphi \circ f\|_{\sigma(T)} \\ &\leq \|f\|_{\sigma(T)}. \end{aligned}$$

□

***Remark* :**

The converse of this result is true if every element of **A** is invertible. This follows from Gelfand-Mazur theorem and theorem 66.9 of Berberian [21].

Particular Cases

Raji Pilakkat “On the study of p-spectral sets and generalized spectral sets ”
Thesis. Department of Mathematics , University of Calicut, 1999

4. *Particular Cases*

Now we study p-spectral set of certain operators. Here instead of taking an arbitrary operator we take an operator T with a finite or a denumerable spectrum. The consequent development of this idea is done in the following two sections. In the first section we deal with finite p-spectral sets. The second section is devoted for denumerable p-spectral sets.

4.1 *Finite p-spectral sets*

We consider this theory to be very important since it has been possible to obtain a characterization for normal operators on Hilbert spaces having finite spectrum in terms of p-spectral sets. Further we have introduced a somewhat natural way of associating an operator with an analytic function and this association lead to a very simple way of expressing an operator as a linear combination which we have termed as the spectral splitting theorem. We start with a definition from the theory of complex analysis.

4.1.1 ***Definition*** [Chap-4; Sec-5.1; 14] :

The residue of f(z) at an isolated singularity a of f(z) is the unique complex number R which makes f(z)-R/(z-a) the derivative of a single valued analytic function in an annulus $0 < |z-a| < \delta$ which is contained in the domain of f.

If a is an isolated singularity of an analytic function f in a region Ω and if the annulus $0 < |z-a| < \delta$ is contained in Ω then the period, P of f is defined as $P = \int_{|z-a|=\delta^*} f(z)\, dz$, where δ^* is a positive real number < δ. The period of 1/(z-a) is $2\pi i$. This in turn implies that, when $R=P/2\pi i$, the period of f(z)-R/(z-a) is equal to zero. This again implies that f(z)-R/(z-a) is the derivative of a single valued analytic function in the annulus $0 < |z-a| < \delta$. Hence the residue of f(z) at z = a is

$$1/2\pi i\int_{|z-a|=\delta^*} f(z)\, dz.$$

(**) Suppose f is analytic in a region Ω except at the point a in Ω and if g is analytic in Ω then, from the definition of residues, it follows that the residue of f(z)g(z) at z = a is Rg(a), where R is the residue of f(z) at z = a.

4.1.2 ***Theorem*** [Chap-4; Sec-5.1; 14] :

Let f(z) be analytic except for isolated singularities a_j in a region Ω. Then

$$1/2\pi i \int_\Gamma f(z)\, dz = \sum_{z=a_j} \text{Res}\, f(z)$$

where Γ is a cycle in Ω, which surrounds the points a_j.

Using the theory of residues, for an operator T we can define the operator f(T) relative to a complex analytic function f on the open set Ω containing the spectrum of T as follows.

H will denote a complex Hilbert space, T a bounded linear operator on the Hilbert space H having a finite spectrum $\sigma(T)=\{a_1,a_2,a_3, \ldots, a_n\}$ and Ω an open set containing the spectrum $\sigma(T)$ of T.

4.1.3 ***Definition*** :

If $f(\lambda)$ is a complex analytic function defined on $\sigma(T)$ then a spectral value of T is called an isolated singularity of the vector valued function $f(\lambda)(\lambda I-T)^{-1}$.

We now introduce the idea of residue of an operator at an isolated singularity.

4.1.4 ***Definition :***

Let a be a spectral value of T or an isolated singularity of $(\lambda I-T)^{-1}$ and Ω^* be the complement of $\sigma(T)$ in the complex plane C. Let D be the annulus $o < |z-a| \leq \delta$ in Ω^*. Then the residue of $(\lambda I-T\ )^{-1}$ at a is defined as the unique operator R for which,

$$\Lambda R = 1/2\pi i\int_{|z-a|=\delta} \Lambda(\lambda I-T)^{-1}dz,$$

for every Λ in $B(H)^*$. We denote this R by Res T at a.

This definition requires some comments.

(a) Since the integrand is continuous on $|z-a| = \delta$, the integral exists as per the definiton 1.1.2.

(b) Since the function $\Lambda(\lambda I-T)^{-1}$ is analytic in any annulus like $0 < |z-a| \leq \delta$, the Cauchy's theorem implies that R is independent of the choice of such an annulus.

4.1.5 *Lemma* :

Let T be an operator on H with spectrum $\sigma(T)$. Suppose f is a complex valued analytic function on Ω. Then $f(\lambda)(\lambda I-T)^{-1}$ is an operator valued analytic function defined on Ω with $\sigma(T)$ as the set of all isolated singularities. Suppose a is a singularity of $(\lambda I-T)^{-1}$ with the residue R and δ is a real number such that the annulus $0 < |z-a| \leq \delta$ lies completely in Ω and contains no spectral value of T. Then there exists a unique operator R* on H such that

$$\Lambda R^* = 1/2\pi i\int_{|z-a|=\delta} f(\lambda)\Lambda(\lambda I-T)^{-1}d\lambda = f(a)\Lambda R,$$

for every $\Lambda \in B(H)^*$.

***Proof* :**

Since the annulus $0 < |z-a| \leq \delta$ contains no singularities of the vector valued function $f(\lambda)(\lambda I-T)^{-1}$, by the theorem 1.1.2, there exists a unique operator R* on H for which

$$\Lambda R^* = 1/2\pi i\int_{|z-a|=\delta} f(\lambda)\Lambda(\lambda I-T)^{-1} d\lambda,$$

for every $\Lambda \in B(H)^*$. But by (**) following 4.1.1, the right hand side

$$= f(a)\Lambda R. \qquad \square$$

4.1.6 *Definition* :

The unique operator R* defined in the above lemma is called the residue of $f(\lambda)(\lambda I-T)^{-1}$ at a.

4.1.7 *Theorem* (Spectral Splitting Theorem) **:**

If T is an operator on H with spectrum $\sigma(T) = \{a_1,a_2,\ldots,a_n\}$ then there exists unique operators $R_1,R_2,\ldots R_n$ on H such that

$$T = a_1R_1 + a_2R_2 + \ldots + a_nR_n.$$

***Proof* :**

Let Ω be a region in C which contains the spectrum $\sigma(T)$ of T and Γ a cycle in Ω which surrounds the spectrum $\sigma(T)$ of T. Then we have

$$T = 1/2\pi i\int_\Gamma \lambda(\lambda I- T)^{-1}\, d\lambda$$

in the sense that

$$\Lambda T = 1/2\pi i\int_\Gamma \Lambda(\lambda(\lambda I- T)^{-1})\, d\lambda$$

for every $\Lambda \in B(H)^*$.

The function $\Lambda(\lambda(\lambda I-T)^{-1})$ is analytic in Ω with the isolated singularities $a_1,a_2,\ldots.a_n$. Suppose the residues of T at these isolated singularities $a_1,a_2,\ldots.a_n$ are R_1, R_2, ….R_n respectively. Then that of the function $\Lambda(\lambda(\lambda I-T)^{-1})$ at these points respectively are given by $\Lambda(a_1R_1)$, $\Lambda(a_2R_2)$, …$\Lambda(a_nR_n)$. This is true for every $\Lambda \in B(H)^*$. Hence by the theorem 4.1.2, for every $\Lambda \in B(H)^*$, we have

$$\Lambda T = 1/2\pi i\int \Lambda(\lambda(\lambda I-T)^{-1})\, d\lambda$$

$$= \text{Sum of the residues of } \Lambda(\lambda(\lambda I-T)^{-1}) \text{ at } a_1,a_2,\ldots.a_n$$

$$= \Lambda(a_1R_1)+ \Lambda(a_2R_2) + \ldots.+ \Lambda(a_nR_n)$$

$$= \Lambda(a_1R_1+ a_2R_2 + \ldots.+ a_nR_n).$$

$$\Rightarrow T = a_1R_1+ a_2R_2 + \ldots.+ a_nR_n.$$ □

We generalize the theorem as follows.

4.1.8 ***Theorem*** (Reside theorem for vector valued functions) :

Suppose f is a complex valued analytic function on a connected open set Ω containing the spectrum $\sigma(T)$ of T. Then

$$f(T) = \sum_{1}^{n} f(a_i) \operatorname*{Res}_{\lambda=a_i} T = \text{Sum of the residues of } f(\lambda)(\lambda I-T)^{-1}.$$

***Proof*:**

Let f be a complex analytic function on Ω and Γ a cycle in Ω which surrounds the spectrum $\sigma(T)$ of T. Then for every Λ in B(H)*, by 1.1.14, there exists an operator f(T) in B(H) such that

$$\Lambda f(T) = 1/2\pi i \int_\Gamma \Lambda(f(\lambda)(\lambda I-T)^{-1})\, d\lambda$$

$$=1/2\pi i \int_\Gamma f(\lambda)\Lambda(\lambda I-T)^{-1}\, d\lambda$$

The function $\Lambda(\lambda I-T)^{-1}$ has isolated singularities at the points a_j. Hence by the theorem 4.1.2 we have that

$$\Lambda f(T) = \sum_{1}^{n} f(a_i) \operatorname{Res}_{\lambda=a_i} \Lambda(\lambda I-T)^{-1}.$$

$$f(T) = \sum_{1}^{n} f(a_i) \operatorname{Res}_{a_i} T$$

This leads to the following definition. □

***4.1.9 Definition* :**

Let f be a complex analytic function on some open set containing the spectrum $\sigma(T)$ of T. We define the operator f(T) as

$$f(T) = \sum_{1}^{n} f(a_i) \operatorname{Res}_{a_i} T,$$

with the understanding that,

$$\Lambda f(T) = \sum_{1}^{n} f(a_i) \Lambda \operatorname{Res}_{a_i} T$$

for every $\Lambda \in B(H)^*$.

Remark **:**

Suppose T is an operator with the spectrum $\sigma(T)=\{a_1,a_2,a_3\ldots a_n\}$. Since $\sigma(T) = \{a_1,a_2,a_3\ldots a_n\}$ is finite, for each point $a_i \in \sigma(T)$ we can choose a neighborhood U_{ai} of a_i in such a way that the collection of all neighborhoods $\{U_{ai}: a_i \in \sigma(T)\}$ is pair wise disjoint. Now let Ω^* be the union of all these neighborhoods. Define a complex function f on Ω^* as

$$f(t) = c_i, \quad t \in U_{ai}, \ i=1,2,3,\ldots n$$

where c_i's are complex numbers.

In this particular case the function f is analytic on Ω^* and hence f(T) is defined. Thus every n-tuple determines an operator f(T) of B(H).

4.1.10 ***Definition*** **:**

A finite subset $Q = \{b_1, b_2, b_3, \ldots b_m\}$ containing $\sigma(T)$ and contained in an open set Ω is said to be a p-spectral set $(1 \leq p \leq \infty)$ relative to Ω if for every complex analytic function f on Ω

$$\|f(T)\| \leq \|f\|_p$$

where f(T) has the meaning as in definition 1.1.14 and

$$\|f\|_p = \left(\Sigma\, |f(b_j)|^p\right)^{1/p}, \qquad (1 \leq p < \infty)$$

$$= \sup_{j=1,2,..n} |f(b_j)|\,, \qquad p = \infty$$

When $p = \infty$, we simply call a p-spectral set as a spectral set.

Remark **:**

From the very definition of a p-spectral set it follows that every finite super set of a p-spectral set is also a p-spectral set.

4.1.11 ***Definition*** **:**

If the norms of residues of an operator T are less than or equal to 1 then it is called a unit residual operator.

4.1.12 ***Proposition*** **:**

If T is a unit residual operator then its spectrum is a 1-spectral set for T relative to a connected open set Ω containing the spectrum of T.

Proof **:**

Let $a_1, a_2, a_3, \ldots a_n$ be the isolated singularities of $(\lambda I - T)^{-1}$. Then by the theorem 4.1.8

$$f(T) = \sum_{1}^{n} f(a_i) \operatorname{Res}_{a_i} T.$$

Hence

$$\|f(T)\| \leq \sum_{1}^{n} |f(a_i)| \, \|\operatorname{Res}_{a_i} T\|$$

$$\leq \sum_{1}^{n} |f(a_i)|$$

since T is a unit residual operator. □

4.1.13 ***Theorem*** **:**

The spectrum $\sigma(T)$ of a normal operator T is always a p-spectral set relative to any open set Ω containing $\sigma(T)$ of T, $(1 \leq p \leq \infty)$.

***Proof*:**

For if T is normal then f(T) is normal for any complex analytic function f. Hence for such functions f , $\|f(T)\| = \sup |f(\lambda)|$, where supremum is taken over the spectrum $\sigma(T)$ of T. □

We give a very simple yet an illustrative example.

$$\|f(T)\| = \|f\|_{\sigma(T)} = \sup_{\lambda\in\sigma(T)} |f(\lambda)|$$

$$\leq \|f\|_p \qquad (1 \leq p \leq \infty).$$

Consider the operator

$$T(x, y) = (x + y, x + y)$$

on C^2, which has spectrum $\{0,2\}$. This operator is normal and hence its spectrum is a spectral set.

Consider the basis $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ of $B(C^2)^*$, where

$$\Lambda_i(x_1, x_2, x_3, x_4) = x_i, \quad i = 1, 2, 3, 4.$$

We have

$$(\lambda I-T)^{-1} = (1/(\lambda^2-2\lambda))(\lambda-1,-1,-1,\lambda-1).$$

Hence the residue of $\Lambda_i(\lambda I-T)^{-1}$ at 0 or 2 is ½ (i = 1,2,3,4).

Let $\Lambda = c_1\Lambda_1+c_2\Lambda_2+c_3\Lambda_3+c_4\Lambda_4$. Then for any complex analytic function f on Ω,

$$|\Lambda f(T)| = \tfrac{1}{2}|f(2)+f(0)||c_1+c_2+c_3+c_4|$$

$$\leq |f(0)+f(2)|||\Lambda||$$

$$\Rightarrow ||f(T)|| \leq ||f||_1$$

Thus {0,2} is a 1-spectral set. □

4.1.14 ***Theorem*** **:**

If Q is a spectral set for T relative to Ω then Q is a p-spectral set for T relative to Ω.

Proof **:**

This result follows immediately from the fact that $||f||_\infty \leq ||f||_p$ for $(1 \leq p < \infty)$.

Thus in the example cited above {0,2} is a p-spectral set relative to any open set Ω containing the set {0,2}, $(1 \leq p \leq \infty)$.

We recall that a compact subset Q of C is a K spectral set for an operator T if

$$||f(T)|| \leq K\ ||f||_\infty$$

for all rational functions having poles off Q, where K is a real constant.

Since Q is a finite set, we have $||f||_\infty \leq ||f||_p \leq K\ ||f||_\infty$, for $1 \leq p < \infty$.

Thus we have the following result.

4.1.15 ***Theorem*** **:**

Suppose Q is a p-spectral set of T relative to every open set Ω. Then it is a K spectral set for T.

We now give a characterization theorem for p-spectral sets. Which is analogous to the characterization theorem for spectral sets given in 3.2.3.

4.1.16 ***Theorem*** **:**

Q is a p-spectral set relative to Ω if and only if $||f(T)|| \leq 1$ for every complex analytic function f on Ω for which $||f||_p \leq 1$.

Proof **:**

Suppose Q is a p-spectral set for T relative to Ω then by the definition we have that $||f(T)|| \leq 1$ for every complex analytic function f on Ω for which $||f||_p \leq 1$.

Conversely suppose the condition of the theorem holds. Let f be a complex analytic function on Ω. Suppose $\|f\|_p = 0$ then $f(T) = 0$. On the other hand if $\|f\|_p \neq 0$ then we have $\|(1/\|f\|_p)f\|_p = 1$ and hence we have $\|(1/\|f\|_p)f(T)\| \leq 1$. □

We know that one of the important results in the development of functional calculus is the spectral mapping theorem which says that if $\sigma(T)$ is the spectrum of T and if f is an analytic function on $\sigma(T)$ then the spectrum of f(T) is nothing but $f(\sigma(T))$. It is quite interesting to note that this result is true for p-spectral sets also.

4.1.17 ***Theorem*** **:**

Let Q be a p-spectral set for T relative to Ω then f(Q) is a p-spectral set for f(T) relative to $f(\Omega)$ and this is true for every non constant complex analytic function f on Ω.

Proof **:**

By the definition of a p-spectral set, $\sigma(T) \subset Q$. Now by the spectral mapping theorem $\sigma(f(T)) \subset f(Q)$ and since f is a non constant analytic function $f(\Omega)$ is open. Suppose g is a complex analytic function on $f(\Omega)$ then $g_o f$ is analytic on Ω and hence

$$\|g_o f(T)\| \leq \|g_o f\|_p$$

$$= \text{the p-norm of g on } f(Q).$$

Theorem 1.1.17 implies that

$$g_o f(T) = g(f(T))$$

Hence we have

$$\|g(f(T))\| \leq \|g_o f\|_p$$

$$= \text{the p-norm of g on } f(Q). \qquad \square$$

Remark :

This theorem is not true in the general case.

Next result is the crux of this section. We consider this result to be very important because it enables us to determine normal operators with a finite spectrum. Suppose Q is a p-spectral set for T relative to Ω. If we choose Ω in a suitable manner we can attach measures corresponding to every pair of vectors in H. Using these measures first of all we can prove that if Q is a p-spectral set for T relative to Ω for every p, ($1 \leq p < \infty$), then Q is a spectral set for T relative to Ω. From this we can deduce that an operator T is normal if and only if its spectrum is a p-spectral set for T for every p ($1 \leq p < \infty$) relative to Ω.

Consider an open set Ω for which the boundary $\partial\Omega$ of Ω is a cycle which surrounds the spectrum $\sigma(T)$ of T. Then as per definition 1.1.14 we can define f(T) as follows.

$$f(T) = (1/2\pi i)\int_{\partial\Omega}(f(\lambda)(\lambda I-T)^{-1})d\lambda\ ,$$

again with the understanding that,

$$\Lambda f(T) = (1/2\pi i)\int_{\partial\Omega}\Lambda(f(\lambda)(\lambda I-T)^{-1})d\lambda$$

for every $\Lambda \in B(H)^*$.

Notation :

$H^*(\bar{\Omega})$ denotes the class of all complex functions which are continuous on $\bar{\Omega}$ and analytic on Ω and $C(\bar{\Omega})$ denotes the class of all complex continuous functions on $\bar{\Omega}$.

4.1.18 ***Theorem*** **:**

Let Ω be an open set containing the spectrum $\sigma(T)$ of T such that $H^*(\bar{\Omega}) = C(\bar{\Omega})$ and the closure of Ω is compact. Let Q be a p-spectral set for T relative to Ω. Then for every pair of points (x, y) in H there exists a measure $\mu(x,y)$ on $\partial\Omega$ such that

$$(f(T)x,y) = \int_{\partial\Omega} f(\lambda)\, d\mu(x,y)$$

for every complex continuous function f on $\partial\Omega$.

Further $\|\mu(x,y)\| \leq n^{1/p} \|x\| \, \|y\|$, for every pair (x,y) in H, where n is the cardinality of Q.

Proof **:**

For every complex analytic function f on Ω, define

$$\varphi(x,y)(f) = (f(T)x,y) \qquad (1)$$

$$\begin{aligned} \|\varphi(x,y)(f)\| &= \|(f(T)x,y)\| \\ &\leq \|f(T)\| \, \|x\| \, \|y\| \\ &\leq \|f\|_p \, \|x\| \, \|y\| \\ &\leq n^{1/p} \|f\|_\infty \|x\| \, \|y\| \qquad (2) \end{aligned}$$

Since $H^*(\bar{\Omega}) = C(\bar{\Omega})$

$$\|\varphi(x,y)(f)\| \leq n^{1/p} \|f\|_\infty \|x\| \|y\| \tag{3}$$

for every $f \in C(\bar{\Omega})$.

Now we can prove that the relations (1) and (2) are also true for every continuous function f on $\partial\Omega$. For this we can use Tietz extension theorem. Which implies that every function continuous on the boundary of Ω has a continuous extension to the closure of Ω. Hence every continuous function on $\partial\Omega$ can be identified with a continuous function on Ω. This identification together with (3) imply that $\varphi(x,y)$ is a bounded linear functional on $C(\partial\Omega)$, such that

$$\varphi(x,y)(f) = (f(T)x,y) \text{ and}$$

$$|\varphi(x,y)| \leq n^{1/p} \|f\|_\infty \|x\| \|y\|$$

for every $f \in C(\partial\Omega)$. Hence by the Riesz Representation Theorem there exists a measure $\mu(x,y)$ on $\partial\Omega$ such that $\|\mu(x,y)\| \leq n^{1/p} \|x\| \|y\|$ and

$$(f(T)x,y) = \int_{\partial\Omega} f(\lambda) d\mu(x,y). \qquad \square$$

4.1.19 ***Corollary*** **:**

Let Ω be as in the theorem 4.1.18 and Q be a finite subset Ω containing the spectrum of T. If in addition Ω be such that the complement of the closure of Ω is connected and Q be such that $||P(T)|| \leq ||P||_p = (\sum |f(b_i)|^p)^{1/p}$, (where the summation is taken over Q) for every polynomial P then all the conclusions of the theorem 4.1.18 are satisfied.

Proof **:**

Define

$$\varphi(x,y)(P) = (P(T)x,y) \tag{1}$$

for every polynomial P. Hence for every polynomial P we have

$$\begin{aligned} ||\varphi(x,y)(P)|| &= ||(P(T)x,y)|| \\ &\leq ||P(T)||\ ||x||\ ||y|| \\ &\leq ||P||_p\ ||x||\ ||y|| \\ &\leq n^{1/p}\ ||P||_\infty ||x|\ ||y|| \end{aligned} \tag{2}$$

where n is the cardinality of Q. (2) shows that φ is continuous on the class of all polynomial functions with respect to the supremum norm.

By the Mergelyan's theorem the set of polynomials is dense in $H^*(\bar{\Omega})$. But as per the definition Ω, $H^*(\bar{\Omega}) = C(\bar{\Omega})$. Now using the continuity of $\varphi(x, y)$ and that of inner product we can extend $\varphi(x, y)$ to $C(\bar{\Omega})$, the class of all continuous functions on the closure of Ω, such that $\varphi(x, y)(f)$ to be equal to $(f(T)x, y)$ for every $f \in C(\bar{\Omega})$ and

$$|\varphi(x,y)(f)| \leq n^{1/p}\|f\|_\infty\|x\|\|y\| \tag{3}$$

for every $f \in C(\Omega)$. The rest of the proof is similar to the proof of the theorem 4.1.18. □

The converse of this theorem is valid if Q is a p-spectral set for every $p < \infty$ relative to an open set Ω as in 4.1.18.

Suppose Q is a p-spectral set for T relative to an open set Ω as in the theorem 4.1.18, for every $p < \infty$. Then we have

$$\begin{aligned} |(f(T)x,y)| &= |\varphi(x,y)(f)| \\ &\leq n^{1/p}\|f\|_\infty \|x\| \|y\| \end{aligned}$$

for every $p < \infty$. Letting $p \to \infty$ in the above inequality we get,

$$|(f(T)x,y)| \leq \|f\|_\infty \|x\| \|y\| \tag{*}$$

$$\|f(T)\| \leq \|f\|_\infty$$

for every complex continuous function f on Ω. Which implies that under the above condition Q is a spectral set for T relative to Ω. (*) implies that the measure corresponding to such a φ (x, y) has norm $\leq \|x\| \, \|y\|$.

With this development the analogue of Corollary 1.3.8 is true in our situation also.

4.1.20 ***Corollary*** **:**

If Ω is as in the theorem 4.1.18 and Q is a p-spectral set for T relative to Ω, for every $p < \infty$ then

(1) The measure $\mu(x,x)$ is positive whenever $x \neq 0$.

(2) If Re $f(\lambda) \geq 0$, for all λ in $\partial\Omega$ then Re $f(T) = \frac{1}{2}(f(T)+f(T)^*) \geq 0$ for every $f \in H(\Omega)$ and

$$\|\mu(x,y)\| \leq \|x\| \, \|y\|$$

for every pair of vectors (x, y) in H.

Proof **:**

Since

$$\|x\|^2 = \int_{\partial\Omega} 1 \, d\mu(x,x)$$

$$= \mu(x,x)(\partial\Omega).$$

$$\leq \|\mu(x,x)\|$$

$$\leq \|x\|^2$$

But any measure μ such that $\int 1\, d\mu = \|\mu\|$ is positive. This proves (1).

To prove (2), consider,

$$(\mathrm{Re}\ f(T)x,x) = \int_{\partial\Omega} \mathrm{Re} f\, d\mu(x,x) \geq 0$$

for every $x \in H$. But

$$(\mathrm{Re} f(T)x, x) = \mathrm{Re} \int_{\partial\Omega} f\, d\mu(x, x)$$

$$= \mathrm{Re}(f(T)x, x)$$

$$= (1/2(f(T)+f(T)^*)x, x)$$

Where $f(T)^*$ is the adjoint of $f(T)$. This implies that

$$(1/2(f(T)+f(T)^*) \geq 0.$$

4.1.21 ***Theorem* :**

Let Ω be as in the theorem 4.1.18 and Q be a spectral set relative to Ω. Then T is normal. In particular T is normal if $\sigma(T)$ is a spectral set relative to Ω.

Proof **:**

Let $f(\lambda) = \bar{\lambda}$,$\lambda \in \partial\Omega$. Then for $x \in H$,

$$(f(T)x,x) = \int_{\partial\Omega} f \, d\mu(x,x)$$

$$= \int_{\partial\Omega} \bar{\lambda} \, d\mu\,(x, x)$$

$$= \overline{(T\,x,\,x)}$$

$$= (T^{*}x,\,x)$$

$$\Rightarrow f(T) = T^{*}$$

$\Rightarrow$ T is normal. □

Remark **:**

The theorem 4.1.21 is in general is not true for an arbitrary operator. For example the unilateral right shift operator T on the space of all square summable complex sequences is not normal but its spectrum is a spectral set for T relative to any open set Ω. This shows that this theory works only for operators having finite spectrum.

If T is unitary then a subset of the unit circle is a p-spectral set for T relative to every open set Ω.

4.1.22 ***Theorem* :**

Let Ω be an open set in C disjoint from the origin. Then T is unitary if and if only if it has a spectral set Q relative to Ω on the unit circle.

***Proof* :**

If T is unitary then it is normal and its spectrum lies on the unit circle and hence we have the result. On the other hand suppose that a subset Q of the unit circle is a spectral set for T relative to Ω. Then since the function $f(t) = 1/t$ is analytic on Ω and for this function, $f(T)$ is T^{-1} and hence we have that

$$\| T^{-1} \| \leq \|f\|_{\infty} = 1$$

Also the function $f(\lambda) = \lambda$ is analytic on Ω and for this function f, $f(T) = T$. Hence

$$\|T\| \leq \|f\|_{\infty} = 1$$

Thus $\|Tx\| \leq \|x\| = \| T^{-1}(Tx)\| \leq \|Tx\|$ for every $x \in H$. Hence T is unitary. □

If T is self adjoint then T has a p-spectral set which consists of only real numbers and if T is positive then T has a p-spectral set which consists of only non negative real numbers relative to every open set Ω.

4.1.23 ***Theorem*** **:**

Let Ω be an open set disjoint from i and –i then T is self adjoint if and if only if there exists a real spectral set for T relative to Ω.

Proof **:**

If T is self adjoint then its spectrum is real and hence it has a real spectral set relative to Ω. On the other hand if T has a real spectral set Q relative to Ω then the function $f(t) = (t-i)(t+i)^{-1}$ is analytic on Ω and it carries Q into the unit circle. Hence f(T) has a spectral set f(Q) on the unit circle relative to $f(\Omega)$. By the definition of Ω, $f(\Omega)$ is disjoint from the origin. Hence by the theorem 4.1.22 f(T) is unitary. Now $f(T) = (T-iI)\,(T+iI)^{-1}$. It can be proved that I-f(T) is invertible and $T = i(I+f(T))(I-f(T)^{-1})$. From this we can deduce that $T^* = T$. □

4.1.25 ***Corollary*** **:**

Let Ω be an open set which is disjoint from i and –i. Then an operator T is positive if and only if there exists a spectral set for T relative to Ω which consists of only positive reals.

4.2 Denumerable p-spectral set

In this section we consider only those operators on a Hilbert space H having only a denumerable spectrum.

Notation **:**

Through out this section T will denote an operator on the Hilbert space H having a denumerable spectrum.

4.2.1 ***Definition*** **:**

Let Ω be an open set containing the spectrum $\sigma(T)$ of T.A denumerable set $Q = \{a_1, a_2, a_3, \ldots\}$ in Ω is said to be a p-spectral set for T relative to Ω $(1 \le p \le \infty)$ if the following conditions are true .

(1) $\sigma(T) \subset Q$

(2) $\|f(T)\| \le \|f\|_p$

for every $f \in H(\Omega)$. Where

$$\|f\|_p = (\Sigma |f(a_j)|^p)^{1/p}, \qquad (1 \le p < \infty)$$

$$\|f\|_\infty = \sup \{|f(a_j)|\}, j = 1,2,3\ldots$$

Here also when $p = \infty$ we call Q simply a spectral set for T relative to Ω.

From the very definition of a denumerable p-spectral set it follows that every countable super set of a denumerable p-spectral set is a p-spectral set relative to any open set.

4.2.2 *Theorem* :

Let Ω be an open set disjoint from the origin. If T is a compact self-adjoint invertible operator on a Hilbert space H then $\sigma(T)$ is a p-spectral set for T relative to Ω $\quad (1 \le p \le \infty)$.

***Proof* :**

Let $t_1, t_2, t_3, \ldots$ be distinct eigen values of T with $|t_1| \ge |t_2| \ge \ldots$ and $P_1, P_2 \ldots$ be the orthogonal projections with $R(P_j) = \{ x \in H : Tx = t_j x\}$, $j = 1, 2, \ldots$

Then by the spectral theorem

$$T = \sum_{n\geq 1} t_n P_n$$

Suppose

$$T_m = \sum_{n=1}^{m} t_n P_n$$

Since t_n's are real and P_n's are self- adjoint, each T_m is self-adjoint. Hence the spectrum of each T_m is a p-spectral set for T_m relative to Ω. Now for $f \in H(Q)$,

$$\|f(T)\| = \lim_{n\to\infty} \|f(T_n)\|$$

$$\leq \|f\|_p$$

4.2.3 ***Definition*** **[28.1; 4] :**

A bounded operator A on H is called Hilbert–Schmidt if $\sum_j \|A(v_j)\|^2 < \infty$ for some orthonormal basis $\{v_j\}$ for H.

4.2.4 ***Corollary*** **:**

Let Ω be an open set disjoint from the origin and T a self adjoint, invertible Hilbert-Schmidt operator. Then $\sigma(T)$ of T is a p-spectral set relative to Ω.

Proof **:**

Every Hilbert-Schmidt operator is compact [28.2; 4].

4.2.5 ***Corollary*** **:**

Let Ω be as in the corollary 2.2.4. Let K(s,t) be an element of $L^2([a,b])$ such that $K(s,t) = \overline{K(t,s)}$ for every pair (s,t) in [a,b].Then the spectrum of the integral operator

$$T(x)(s) = \int_{[a,b]} K(s,t)x(t)\, dt \qquad (a \leq s \leq b)$$

for $x \in L^2([a,b])$,

is a p-spectral set relative to Ω.

Proof **:**

The operator T is a Hilbert-Schmidt operator.

Relations Between Spectral Radius And PseudoSpectral Radius

Raji Pilakkat "On the study of p-spectral sets and generalized spectral sets "
Thesis. Department of Mathematics , University of Calicut, 1999

5. *Relations Between Spectral Radius And PseudoSpectral Radius*

In this chapter we consider von-Neumann's definition of spectral sets and try to get some relations between spectral sets and spectral radius. For this we develop the following.

According to von-Neumann a compact set Q of C is said to be a spectral set for T if it contains the spectrum of T and for every rational function f having poles off Q,

$$\|f(T)\| \leq \|f\|_\infty = \sup_{\lambda \in Q} |f(\lambda)|.$$

He proved that if $M_j \supset M_{j+1}$, j=1,2,... are spectral sets for T then $\cap M_j$ was also a spectral set for T [29]. But in general intersection of two spectral sets is not a spectral set. [34] contains examples for this fact. Here we introduce the idea of minimal spectral sets and prove that every operator T in B(H) has a minimal spectral set.

5.1.1 ***Theorem*** **:**

Let $T \in B(H)$ and M be the intersection of all spectral sets for T. Then M is a spectral set for T.

Proof **:**

Let C_T be the collection of all spectral sets for T. Define a partial order relation on C_T as follows. Let M_1, M_2 be any two elements of C_T. We say that $M_1 \leq M_2$ if $M_2 \subset M_1$. Then $\leq$ is a partial order relation on C_T. We claim that every totally ordered set in C_T has a maximal element. Let **F** be a totally ordered subset of C_T. Let N be the intersection of all elements of **F**. Being the intersection of collection of compact sets satisfying finite intersection condition, N is compact.

Now consider the complement N^c of N. Being a subspace of the second countable space C, N^c is second countable. Hence it is Lindelof. The collection $\{M_F{}^c : M_F \in \mathbf{F}\}$ forms an open covering of N^c. Hence there exists a countable subcovering $\{ M_1{}^c, M_2{}^c, \ldots \}$ of N^c in the above collection. Since F is totally ordered, the collection $\{M_1, M_2, ..\}$ is also totally ordered . Hence by the von Neumann's result [29], $M^* = \cap M_i$, $i = 1, 2, 3\ldots$ is a spectral set for T and hence it $\in C_T$.

We assert that $M^* = N$. Since N is the intersection of all elements of F, we have that $N \subset M^*$ and since $\{M_1^c, M_2^c, ..\}$ is a covering of N^c we have that $N^c \subset M^{*c}$. Hence we have that $M^* \subset N$. Hence $M^* = N$. Clearly it is the maximal element of **F** in C_T. Thus every totally ordered subset of C_T has a maximal element in C_T. Therefore by Zorn's lemma C_T has a maximal element. Let it be M^{**}. We claim that $M^{**} = M$. Since by the definition M^{**}, it is contained in all elements of C_T. Hence we have that $M^{**} \subset M$. Since M^{**} is a spectral set we have that $M \subset M^{**}$. Hence $M = M^{**}$. Which is clearly minimal among all spectral set for T. □

5.1.2 ***Definition*** **:**

We denote the minimal spectral set of T by M_T and define the radius of M_T, denoted by $|M_T|$, as

$$|M_T| = \sup_{\lambda \in M_T} |\lambda|$$

and call it as pseudo spectral radius

5.1.3 ***Corollary*** **:**

(1) $M_T \subset R$ if and only if T is Hermition.

(2) $M_T \subset R^+$ if and only if T is positive.

(3) T is normal then $|M_T| = \rho =$ the spectral radius of $T = ||T||$.

(4) $||T^2|| = |M_{TT^*}|$ for any $T \in B(H)$.

Proof **:**

The proof follows from the fact that, by the theorem 1.3.3, if T is normal then its spectrum is a spectral set. □

Remark **:**

In general, the converse of the result (3) in the corollary 5.1.3 is not true. But it is true when H is finite dimensional.

Using the theorem 11.31 of [30] we can prove the following.

5.1.4 ***Corollary*** **:**

If F is positive linear on B(H) then

(1) $|F(T)|^2 \leq (F(I))^2 \, |M_{TT^*}|$, for all T in B(H).

(2) $|F(T)| \leq F(I)\,|M_T|$, for every normal T in B(H).

von Neumann proved that for every operator T, $\{\lambda \in C: |\lambda| \leq ||T||\}$ is a spectral set for T. Hence we have that

$$|M_T| \leq ||T||$$

On the other hand $\sigma(T)$ is contained in M_T and hence we have that,

$$\rho \leq |M_T|$$

Thus in general we have that

$$\rho \leq |M_T| \leq ||T||.$$

Remark **:**

(1) If T is normal then $\rho = |M_T| = ||T||$.

(2) There are non-normal operator for which $\rho = |M_T| = ||T||$. For example the unilateral right shift operator on the class of all square summable scalar sequences.

Example of an operator T for which $\rho < |M_T|$

Let H be a separable Hilbert space with an orthonormal basis $\{e_n: n = 1,2,\ldots\}$. Define an operator T on H as

$$Te_n = (1/n)e_{n+1}$$

$n = 1,2,\ldots$ For this operator T, $\sigma(T) = \{0\}$. Here $\sigma(T)$ is not a spectral set, since T is not self adjoint. Thus in this case $|M_T| > \rho$.

Now we recall the definition of a subnormal operator. An operator T is subnormal on H if there exists a normal operator U on a Hilbert space K containing H such that T = U on H.

It should be noted that the power of a subnormal operator T is subnormal and every polynomial in T is subnormal. Hence for such operator T

$$||P(T)|| = |M_{P(T)}| = \rho(P(T))$$

for every polynomial P.

Remark :

For a Hyponormal operator T (that means operators for which $T^*T-TT^* > 0$), $||T^n|| = ||T||^n$, for all n and hence we have

$$||T|| = \rho = |M_T|.$$

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