REGULARIZATIONS AND DIVERGENT DIAGRAMS IN GAUGE THEORIES

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CERTIFICATE

Certified that this is the bonafide work carried out by Ms.P.C.Raje Bhageerathi under my supervision and guidance during 1994-1999 for the award of the degree of Doctor of Philosophy in Physics, of University of Calicut, Kerala and that no part of this work has been presented for any other examination or degree of this or any other University.

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DECLARATION

I hereby declare that this thesis entitled "Regularizations and Divergent diagrams in Gauge Theories" is a bonafide record of research work done by me and that no part of this thesis has been presented before for the award of any degree or diploma.

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Abstract

The quantum theory of fields was invented many years ago to give expression to the relativistic quantum mechanics of the radiation field. Within one or two decades, the theory was extended to include the nuclear as well as the weak interactions.

In the past, there were several occasions when the quantum theory of fields appeared to be inconsistent and were at the verge of abandonment. But with the advent of the renormalization theory, this problem was more or less solved. Instead of completely eliminating the inconsistencies, one was able to reformulate the theory in such a way, that the embarrassing infinities did not appear explicitly.

Several regularization techniques were devised earlier to eliminate the infinities appearing in the field theories, the most important ones being the Pauli-Villars and the dimensional regularizations. Recently, a new regularization technique was developed by Evens etal, which they have named nonlocal regularization. Chapter 1 describes the methods of dimensional and nonlocal regularizations as applied to QED. In chapter 2, the nonlocal regularization method is extended to evaluate the vertex part of QED. Chapter 3 gives the Ward identity satisfied by nonlocal QED. In chapter 4, the Yang-Mills theory is studied using the nonlocal regularization method. This method of regularization being new offers immense scope for development.

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Raje Bhageerathi

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P. C. Raje Bhageerathi "Regularizations and divergent diagram in gauge theories" Thesis. Department of Physics, University of Calicut, 1999

Chapter – 1

Introduction

1.1 Primitive divergences in Gauge Theories

It has been observed for a long time that there arise certain divergences while tackling many problems in field theories. Many methods were devised to isolate the divergences appearing in the field theories. This was necessary for the theories to be considered as physically acceptable ones.

The divergences that are chiefly encountered while considering any field theory — the Abelian, non-Abelian or ϕ^4 theory — are the ultraviolet and the infrared ones. Consider an integral of the form $g \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - m^2)}$ [1]. There are four powers of q in the numerator and two in the denominator, so the integral diverges quadratically at large q. It is ultraviolet divergent. The problems of infrared divergence arise when $q \to 0$. g is the order of the particular diagram represented by the above integral.

A lot of effort has gone into the study of ultraviolet divergences in Abelian and non-Abelian gauge theories during the last four decades. To tackle divergent diagrams in these theories, it is necessary to regularize the divergences in suitable ways. Regularization is a method of isolating the divergences in Feynman integrals. It makes the task of renormalization much more explicit. There are several techniques of regularization. The most intuitive one is to introduce a cut-off Λ in the momentum integrals. In QED, it is required to modify the free photon propagator

$$\frac{1}{k^2} \rightarrow \frac{1}{k^2} - \frac{1}{(k^2 - \Lambda^2)} = - \frac{\Lambda^2}{k^2 (k^2 - \Lambda^2)}$$

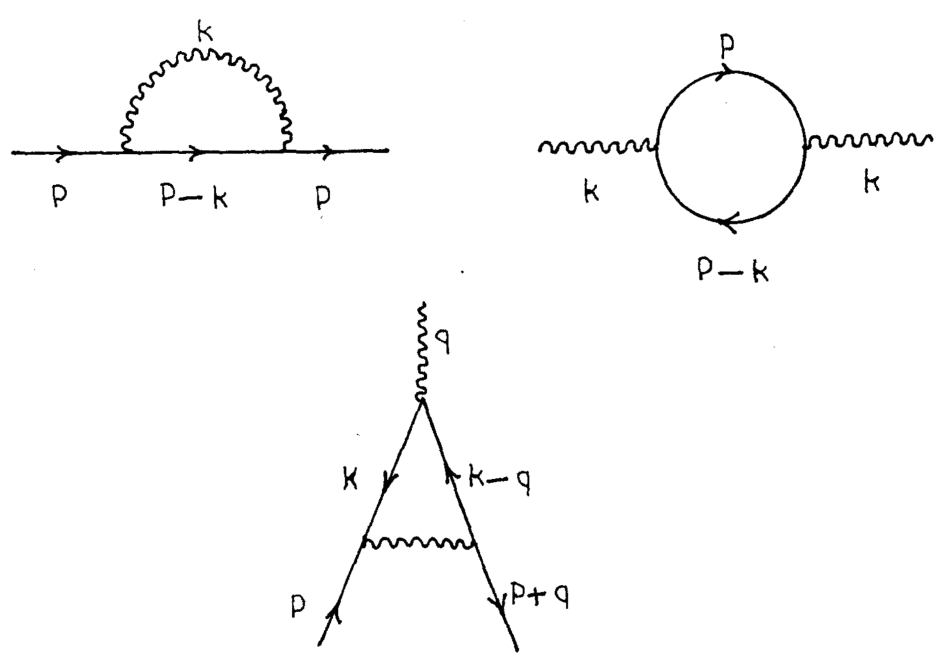
The Pauli–Villars regularization [2] is also similar to the one above. Here, a fictitious field of mass M is introduced. In both cases the limit $\Lambda \rightarrow \infty$ ($M \rightarrow \infty$) is taken. The renormalized quantities then become independent of Λ (M).

These methods create problems while dealing with non-Abelian gauge theories. The Pauli–Villars regularization does not preserve gauge invariance in non-Abelian gauge theories. So inorder to tackle these problems, the technique of dimensional regularization [3] was developed mainly by 't Hooft and Veltman [4], (see Ashmore [5], C.G. Bollini and J.J. Giambiagi [6]). The idea is to treat the loop integrals which cause the divergences as integrals over d – dimensional momenta, and then take the limit $d \rightarrow 4$. It turns out that the singularities of one – loop graphs are simple poles in d – 4. In the next section the method of dimensional regularization is given in the context of QED.

1.2 The Technique of Dimensional Regularization

In this section, one can see how the method of dimensional regularization is applied to a field theory, by considering QED as an example. QED is the theory of interaction of light and matter. It is also one of the rare parts of physics which is known for sure, a theory that has stood the test of time. It is an Abelian gauge theory which is renormalizable [7-11]. It comes under the U(1) group — the unitary group of order one [12, 13].

The three primitively divergent diagrams in QED are the electron and photon self energy graphs and the vertex graph [1, 14, 15] shown in fig. 1.1.



(a) Electron self-energy diagram (b) Photon self-energy diagram (c) Vertex graph.

Fig.1.1.

The general formula for the superficial degree of divergence D of a Feynman graph in d – dimensional space – time is [1],

$$D = d + n\left(\frac{d-2}{2}\right) - \left(\frac{d-1}{2}\right) E_e - \left(\frac{d-2}{2}\right) P_e \qquad (1.2.1)$$

where

n = number of vertices,

P_e = number of external photon lines,

E_e = number of external electron lines,

d = dimension of space – time

When d = 4 this yields

$$D = 4 - \frac{3E_e}{2} - P_e \qquad (1.2.2)$$

showing that D is independent of n.

Consider the two self–energy diagrams in fig.1.1. The electron self–energy diagram has $E_e = 2$, $P_e = 0$, so D = 1. The Feynman rules give

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i}{\not p - \not k - m} \frac{-ig_{\mu\nu}}{k^2} \gamma^\nu \qquad (1.2.3)$$

Here there are four powers of k in the numerator and three in the denominator. So D = 1, as predicted. The photon self – energy, which is also called vacuum polarization denoted as $\Pi_{\mu\nu}$ is

$$i\,\Pi^{\mu\nu}(k) = -(-ie)^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\gamma^\mu \frac{i}{\not p - m} \gamma^\nu \frac{i}{\not p - \not k - m}\right) \qquad (1.2.4)$$

This integral is quadratically divergent, as anticipated. Now for the vertex graph, we have $E_e = 2$, $P_e = 1$. Hence D = 0 which indicates a logarithmic divergence. The Feynman rules give

$$-ie\ \Lambda_\mu\,(p,q,\,p+q) = (-ie)^3 \int \frac{d^4k}{(2\pi)^4}\ \frac{-ig_{\rho\sigma}}{(p+k)^2}\ \gamma^\rho\ \frac{i}{\not{k}-\not{q}-m}\ \gamma^\mu\ \frac{i}{\not{k}-m}\ \gamma^\sigma \qquad (1.2.5)$$

Although the electron and photon self energy graphs are superficially linearly and quadratically divergent respectively, they both turn out to be logarithmically divergent only.

The two self energy graphs and the vertex graph, all have the property that the removal of their infinities results in a redefinition of various physical quantities, ie; electron mass and wave function normalization, and electric charge. ie; no extra terms in the Lagrangian are required of a type which are not there already.

Having isolated the three primitive divergences of QED, one can calculate them using dimensional regularization. For extending to d dimensions, we have to define the algebra of Dirac matrices in d dimensions. We have

$$\{\gamma_\mu, \gamma_\nu\} = 2\, g_{\mu\nu} \qquad (1.2.6)$$

where $g_{\mu\nu}$ is the metric tensor in d - dimensional Minkowski space.

Also $\gamma^\mu \gamma_\mu = d$ and $\gamma_\mu \gamma_\nu \gamma^\mu = (2-d)\,\gamma_\nu$ (1.2.7)

In addition,

Tr (odd no. of γ - matrices) = 0, (1.2.8)

$\mathrm{Tr}\, I = f(d)$, $\mathrm{Tr}\, \gamma_\mu \gamma_\nu = f(d)\, g_{\mu\nu}$, (1.2.9)

$\mathrm{Tr}\, \gamma_\mu \gamma_\kappa \gamma_\nu \gamma_\lambda = f(d)\,(g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\nu} g_{\kappa\lambda} + g_{\mu\lambda} g_{\kappa\nu})$ (1.2.10) where f(d) is an arbitrary function with f(4) = 4. With all these necessary facts one can proceed to calculate the three primitive divergences in QED.

First, start with the electron self-energy graph of fig.1.1a. The Lagrangian for the system is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}\,(i\partial\!\!\!/ + m)\,\psi + e\,\bar{\psi} A\!\!\!/ \psi \tag{1.2.11}$$

Generalizing expression (1.2.3) to d dimensions,

$$\Sigma(p) = -ie^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \gamma_\mu \frac{1}{p\!\!\!/ - k\!\!\!/ - m} \gamma_\nu \frac{g^{\mu\nu}}{k^2}$$

$$= -ie^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu (p\!\!\!/ - k\!\!\!/ + m)\gamma^\mu}{[(p-k)^2 - m^2]\,k^2} \tag{1.2.12}$$

where μ is an arbitrary mass to be multiplied with the interaction term inorder to have the correct dimension. The denominators in the integrand are combined by using the Feynman formula [1, 16 - 18],

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2} \tag{1.2.13}$$

Introducing the Feynman parameter z in expression (1.2.12) yields

$$\Sigma(p) = -ie^2 \mu^{4-d} \int_0^1 dz \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu (p\!\!\!/ - k\!\!\!/ + m)\gamma^\mu}{[(p-k)^2 z - m^2 z + k^2 (1-z)]^2}$$

Define $k' = k - pz$. This gives

$$\Sigma(p) = -ie^2 \mu^{4-d} \int_0^1 dz \int \frac{d^dk'}{(2\pi)^d} \frac{\gamma_\mu(\not p - \not p z - \not k' + m)\gamma^\mu}{[k'^2 - m^2 z + p^2 z(1-z)]^2}$$

The term linear in k′ integrates to zero. So

$$\Sigma(p) = -ie^2\mu^{4-d} \int_0^1 dz\, \gamma_\mu(\not p - \not p z + m)\gamma^\mu \int \frac{d^dk'}{(2\pi)^d} \frac{1}{[k'^2 - m^2 z + p^2 z(1-z)]^2}$$

This integral is performed with the help of

$$\int \frac{d^dp}{[p^2 + 2pq - m^2]^\alpha} = i\,\pi^{d/2} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)} \frac{1}{[-q^2 - m^2]^{\alpha - d/2}}$$

giving

$$\Sigma(p) = e^2\mu^{4-d} \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 dz\, \gamma_\mu[\not p(1-z) + m]\gamma^\mu\, [m^2 z - p^2 z(1-z)]^{d/2-2}$$

As $d \to 4$, $\Gamma(2 - d/2)$ develops a pole. Putting $\epsilon = 4 - d$,

$$\Gamma(2 - d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + O(\epsilon).$$ Using this and eq.(1.2.7),

$$\Sigma(p) = -\frac{e^2}{16\pi^2} \Gamma(\epsilon/2) \int_0^1 dz \{2\not p(1-z) - 4m - \epsilon[\not p(1-z) - m]\}$$

$$\times \left(\frac{m^2 z - p^2 z(1-z)}{4\pi\mu^2}\right)^{-\epsilon/2}$$

$$= \frac{e^2}{8\pi^2\epsilon}(-\not p + 4m) + \frac{e^2}{16\pi^2}\left\{\not p(1+\gamma) - 2m(1+2\gamma) + 2\int_0^1 dz[\not p(1-z) - 2m]\right.$$

$$\left. \times\ \ln\left(\frac{m^2 z - p^2 z(1-z)}{4\pi\mu^2}\right)\right\}$$

$$= \frac{e^2}{8\pi^2\epsilon}\ (-\not p + 4m) + \text{finite} \qquad (1.2.14)$$

Next, we calculate the vacuum polarization graph of fig.1.1b. Extending expression (1.2.4) to d - dimensions,

$$\Pi_{\mu\nu}(k) = i\mu^{4-d} e^2 \int \frac{d^d p}{(2\pi)^d} \mathrm{Tr}\left(\gamma_\mu \frac{1}{\not p - m} \gamma_\nu \frac{1}{\not p - \not k - m} \right)$$

$$= ie^2\mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{\mathrm{Tr}\,[\gamma_\mu (\not p + m)\, \gamma_\nu\, (\not p - \not k + m)]}{(p^2-m^2)[(p^2-k^2)-m^2]} \qquad (1.2.15)$$

Introducing the Feynman parameter z, and putting $p' = p-kz$ gives,

$$\Pi_{\mu\nu}(k) = -\, ie^2\mu^{4-d} \int_0^1 dz \int \frac{d^d p'}{(2\pi)^d} \frac{\mathrm{Tr}[\gamma_\mu(\not p' + \not k z + m)\, \gamma_\nu\, (\not p' - \not k(1-z) + m)]}{[\, p'^2 - m^2 + k^2 z\,(1-z)]^2}$$

Since the trace of odd number of γ - matrices is zero, and the terms odd in p' give no contribution to the integral, the numerator N in the momentum integral is

$$N = [p'^\kappa p'^\lambda - k^\kappa k^\lambda z(1-z)]\, \mathrm{Tr}(\gamma_\mu \gamma_\kappa \gamma_\nu \gamma_\lambda) + m^2\, \mathrm{Tr}\,(\gamma_\mu \gamma_\nu)$$

$$= f(d)\, \{2p'_\mu p'_\nu - 2z(1-z)\, (k_\mu k_\nu - k^2 g_{\mu\nu}) - g_{\mu\nu}[\, p'^2 - m^2 + k^2 z(1-z)]\}$$

Putting $p' \to p$,

$$\Pi_{\mu\nu}(k) = ie^2\mu^{4-d} f(d) \int_0^1 dz \int \frac{d^d p}{(2\pi)^d} \left\{ \frac{2p_\mu p_\nu}{[p^2 - m^2 + k^2 z(1-z)]^2} - \frac{2z(1-z)[k_\mu k_\nu - k^2 g_{\mu\nu}]}{[p^2 - m^2 + k^2 z(1-z)]^2} - \frac{g_{\mu\nu}}{[p^2 - m^2 + k^2 z(1-z)]} \right\}$$

The contributions of the first and the third terms in the integrand cancel. The second term leads to a logarithmically divergent integral.

$$\Pi_{\mu\nu}(k) = \frac{e^2}{2\pi^2} (k_\mu k_\nu - g_{\mu\nu} k^2)$$

$$\times \left\{ \frac{1}{3\epsilon} - \frac{\gamma}{6} - \int_0^1 dz\, z(1-z) \ln\left(\frac{m^2 - k^2 z(1-z)}{4\pi\mu^2} \right) + O(\epsilon) \right\}$$

The divergent part is a pole in ϵ. The finite part contains terms depending on k^2, so for small k^2 we have

$$\Pi_{\mu\nu}(k) = \frac{e^2}{6\pi^2} (k_\mu k_\nu - g_{\mu\nu} k^2) \left(\frac{1}{\epsilon} + \frac{k^2}{10m^2} + \ldots\ldots\ldots\ldots \right)$$

$$= \frac{e^2}{6\pi^2\epsilon} (k_\mu k_\nu - g_{\mu\nu} k^2) + \text{finite} \qquad (1.2.16)$$

Finally, we evaluate the vertex graph of fig.1.1c. In d–dimensions equation (1.2.5) becomes

$$-ie\,\mu^{2-d/2} \Lambda_\mu(p,q,p') = (-ie\mu^{2-d/2})^3 \int \frac{d^d k}{(2\pi)^d} \gamma_\nu \frac{i}{\not p' - \not k - m} \gamma_\mu \frac{i}{\not p - \not k - m} \gamma_\rho \frac{-ig^{\nu\rho}}{k^2}$$

$$= -(e\mu^{2-d/2})^3 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\nu (\not p' - \not k + m) \gamma_\mu (\not p - \not k + m) \gamma^\nu}{k^2[(p-k)^2 - m^2][(p'-k)^2 - m^2]} \qquad (1.2.17)$$

Here, the two-parameter Feynman formula which is analogous to (1.2.13) is introduced (1,16 -18) ·

$$\frac{1}{abc} = 2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{[a(1-x-y) + bx + cy]^3} \qquad (1.2.18)$$

This gives

$$\Lambda_\mu(p, q, p') = \frac{2ie^2\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int_0^{1-x} dy\int d^dk$$

$$\times\frac{\gamma_\nu(\not{p}'-\not{k}+m)\gamma_\mu(\not{p}-\not{k}+m)\gamma^\nu}{[k^2-m^2(x+y)-2k(px+p'y)+p^2x+p'^2y]^3}$$

Defining $k' = k-px-p'y$ gives (with $k'\to k$)

$$\Lambda_\mu(p, q, p') = \frac{2ie^2\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int_0^{1-x} dy\int d^dk$$

$$\times\frac{\gamma_\nu[\not{p}'(1-y)-\not{p}x-\not{k}+m]\gamma_\mu[\not{p}(1-x)-\not{p}'y-\not{k}+m]\gamma^\nu}{[k^2-m^2(x+y)+p^2x(1-x)+p'^2y(1-y)-2p.p'xy]^3} \quad (1.2.19)$$

This integral contains convergent and divergent terms. The part of the numerator quadratic in k is divergent, whereas the rest is convergent. So we can write

$$\Lambda_\mu = \Lambda_\mu^{(1)} + \Lambda_\mu^{(2)} \quad (1.2.20)$$

where $\Lambda_\mu^{(1)}$ is the divergent part and $\Lambda_\mu^{(2)}$,the convergent one.

$\Lambda_\mu^{(1)}$ may be written as

$$\Lambda_\mu^{(1)}(p,q,p') = \frac{e^2}{2}\mu^{4-d}\left(\frac{1}{4\pi}\right)^{d/2}\Gamma(2-d/2)\int_0^1 dx\int_0^{1-x} dy$$

$$\times\frac{\gamma_\nu\gamma_\rho\gamma_\mu\gamma^\rho\gamma^\nu}{[m^2(x+y)+p^2x(1-x)+p'^2y(1-y)+2p.p'xy]^{2-d/2}}$$

Here, the expression

$$\int d^dp \frac{p_\mu p_\nu}{[p^2 + 2pq - m^2]^\alpha} = \frac{i\pi^{d/2}}{\Gamma(\alpha)} \frac{1}{(-q^2 - m^2)^{\alpha - d/2}}$$

$$\times \left[q_\mu q_\nu \Gamma(\alpha - d/2) + \frac{1}{2} g_{\mu\nu} (-q^2 - m^2) \Gamma(\alpha - 1 - d/2) \right]$$

is used.

Also, $\gamma_\nu \gamma_\rho \gamma_\mu \gamma_\sigma \gamma^\nu = (2-d)\, \gamma_\rho \gamma_\mu \gamma_\sigma + 2(\gamma_\mu \gamma_\sigma \gamma_\rho - \gamma_\rho \gamma_\sigma \gamma_\mu)$

So $\gamma_\nu \gamma_\rho \gamma_\mu \gamma^\rho \gamma^\nu = (2-d)^2 \gamma_\mu$. Putting $\in = 4-d$, so that $(2-d)^2 = 4 - 2\in$, we get

$$\Lambda_\mu^{(1)} (p, q, p') = \frac{e^2}{8\pi^2 \in} \gamma_\mu + \text{finite.} \qquad (1.2.21)$$

The convergent part $\Lambda_\mu^{(2)}(p, q, p')$ does not contain k in the numerator of the integrand. Putting d=4 and performing the integration over k, gives

$$\Lambda_\mu^{(2)} (p, q, p') = \frac{e^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy.$$

$$\times \frac{\gamma_\nu [\not{p}'(1-y) - \not{p}x + m] \gamma_\mu [\not{p}(1-x) - \not{p}'y + m] \gamma^\nu}{m^2(x+y) - p^2 x(1-x) - p'^2 y(1-y) + 2p.p'xy} \qquad (1.2.22)$$

1.3 Ward Identity

The Ward identities [1,19] and its generalization by Takahashi [1, 20, 21] are crucial in proving the renormalizability of gauge theories, and renormalizability, in turn, is crucial in order that these theories make sense and are believable.

We have for the electron self-energy, in 4-dimensions,

$$\frac{\Sigma}{i} = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\kappa\lambda}}{k^2} \gamma_\kappa \frac{i}{\not{p}-\not{k}-m} \gamma^\lambda$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \gamma_\lambda S_F(p-k) \gamma^\lambda \qquad (1.3.1)$$

$S_F(p-k)$ is the virtual electron propagator. Differentiating (1.3.1) with respect to p_μ,

$$\frac{\partial \Sigma}{\partial p_\mu} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \gamma_\lambda \frac{\partial}{\partial p_\mu} S_F(p-k) \gamma^\lambda.$$

$$= ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \gamma_\lambda S_F(p-k) \gamma_\mu S_F(p-k) \gamma^\lambda. \qquad (1.3.2)$$

The vertex graph in 4-dimensions gives (with $q = 0$),

$$-ie\Lambda_\mu(p,0.p) = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\kappa\lambda}}{k^2} \gamma_\kappa \frac{i}{\not{p}-\not{k}-m} \gamma_\mu \frac{i}{\not{p}-\not{k}-m} \gamma^\lambda$$

$$= -e^3 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \gamma_\lambda S_F(p-k) \gamma_\mu S_F(p-k) \gamma^\lambda.$$

Hence,

$$\Lambda_\mu(p,0,p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \gamma_\lambda S_F(p-k) \gamma_\mu S_F(p-k) \gamma^\lambda. \qquad (1.3.3)$$

From expressions (1.3.2) and (1.3.3) it is clear that

$$\frac{\partial \Sigma(p)}{\partial p_\mu} = -\Lambda_\mu(p,0.p) \qquad (1.3.4)$$

This is the Ward identity to order e^2 for QED. The Ward identity is satisfied to any order in perturbation theory.

1.4 Limitations of the technique of Dimensional Regularization

In four dimensions, we have

$$\gamma^5 = \frac{i}{4!} \in_{\mu\nu\alpha\beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \tag{1.4.1}$$

We cannot define the analogue of γ^5 in d dimensions. The Levi-Civita symbol $\in_{\mu\nu\alpha\beta}$ is specific to d = 4. So the method of dimensional regularization fails if in the Ward identities there appear quantities that have the desired properties only in the four dimensional space[3, 4]. One cannot generalize $\in_{\mu\nu\alpha\beta}$ to a tensor satisfying the required properties for non-integer n. Similarly for γ^5. One can insert (1.4.1) wherever γ^5 occurs and take the $\in$-tensor outside of the expression to be generalized to non-integer n. If we are dealing with Ward identities that depend on $\{\gamma^5, \gamma^\alpha\} = 0$ for $\alpha = 1,2,\ldots\ldots\ldots,n$ and $\mathrm{Tr}\,\{\gamma^5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta\} = i4\in_{\mu\nu\alpha\beta}$ then this method breaks down. This is the case of the axial vector anomaly.

1.5 Nonlocal Regularization

Quantum field theories are plagued with ultraviolet divergences which posed difficulties to the theorists. Recently it has been found that the superstring theories are finite [22 –24]. The vertices of the string field theory contain nonlocal factors of exp ($-\alpha' p^2$) which causes loops to converge in Euclidean space [25, 26].

The phenomenon is not restricted to strings, attaching such factors to the interactions of any otherwise local Lagrangian gives an ultraviolet – finite theory. The perturbative S-matrix resulting from the nonlocalization is finite, unitary, and Lorentz invariant. This sacrifices certain other benefits too [27]. But they pose no serious obstacle.

This scheme of regularization is termed "nonlocal regularization". This was developed recently by Evens etal [28]. This has several advantages over conventional methods. It lacks the notorious "automatic subtractions" present in both dimensional regularization [3, 4] and the ζ - function method [29]. It does not sacrifice perturbative unitarity as does the Pauli-Villars method [2]. Nonlocal regularization is operationally very similar to Schwinger's proper time method [30, 31].

Nonlocalization is supposed to cure only ultraviolet divergences. The fact that ultraviolet divergences can be cured by nonlocalization was realized long ago [32-37]. There were certain problems which inhibited its early application. First, it was unknown how to canonically formulate nonlocal actions. Although procedures valid to low order were known for certain theories, it was believed that obstacles must appear at two loops [38 - 41]. By exploiting the functional formalism directly Polchinski [42] exhibited a satisfactory model in 1984 in the context of ϕ^4 theory. The procedure for deriving the associated operator formalism was finally given by Jaén etal. In 1986 [43].

The second obstruction was the belief that gauge invariance is inconsistent with any sort of nonlocalization. This can be explained in the context of QED:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \bar{\psi}\,(i\not{\mathcal{D}} + m)\,\psi \tag{1.5.1}$$

where $\mathcal{D}_\mu$ is the covariant derivative operator ∂_μ + ie A_μ. The γ matrices are in the Euclidean space and obey the commutation relation [44]

$$\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu} \tag{1.5.2}$$

Now simply nonlocalizing the interaction gives,

$$e\,\bar{\psi}\not{A}\,\psi \rightarrow e\bar{\psi}^\Lambda \not{A}^\Lambda \psi^\Lambda \tag{1.5.3}$$

where $\bar{\psi}^\Lambda \equiv \varepsilon_m\bar{\psi}$, $\psi^\Lambda \equiv \varepsilon_m\psi$, and $A^\Lambda \equiv \varepsilon_0$ A. The nonlocal smearing operator ε_m is defined, for any mass m, as follows:

$$\varepsilon_m \equiv \exp\left(\frac{\partial^2 - m^2}{2\Lambda^2}\right) \tag{1.5.4}$$

The resulting theory although free of ultraviolet divergences, is not gauge invariant since we have broken the covariant derivative and used ordinary derivatives in smearing charged fields. Losing gauge invariance is not acceptable physically. So, if we attempt to avoid these problems by covariantly nonlocalizing the entire covariant – derivative term,

$$-i\bar{\psi}\not{D}\psi \rightarrow i\left[\exp\left(\frac{\mathcal{D}^2 - m^2}{2\Lambda^2}\right)\bar{\psi}\right](\not{\partial} + ie\not{A}^{\Lambda})\left[\exp\left(\frac{\mathcal{D}^2 - m^2}{2\Lambda^2}\right)\psi\right], \tag{1.5.5}$$

then the resulting theory is invariant, but not completely finite. This is because, the factor of $1/\varepsilon^2_m$ carried by the electron propagator, cancels the convergence factors on the vertices.

This problem cannot apply to all nonlocal gauge theories. To understand this, the notion of "gauge invariance" is extended to include nonlocal transformation laws. In fact, invariant string field theory possesses a nonlocal gauge invariance [45 - 48].

In ref. 28, it has been shown that any local gauge theory can be generalized to a finite, nonlocal theory endowed with a nonlocal gauge symmetry that reconciles unitarity and Poincare′ invariance [49].

Nonlocalization can be viewed as a regularization of the original local theory. The results obtained by this method can be read off from the analogous result of dimensional regularization by replacing the Γ function with an incomplete Γ function. The method of nonlocal regularization reproduces the logarithmic divergences of dimensional regularization simultaneously picking out other divergences that dimensional regularization misses.

The nonlocal regularization method has been applied to a very different number of theories. The renormalization of ϕ^4 and ϕ^3 scalar theories is carried out by Kleppe and Woodard [50], in yet another work, the lowest order measure factor is constructed for pure Yang-Mills [51]. The method of nonlocal regularization is applied to string field theory [52, 53] and also to a simple, globally supersymmetric model [54]. The nonlocal field theory formalism is also applied to study a variety of problems in gravitation [55 - 61]. This scheme of regularization being new, offers immense scope for development.

1.6 Nonlocal Quantum Electrodynamics

The first step is to introduce nonlocal convergence factors onto the interaction term in the manner of (1.5.3) so as to make the Euclidean loop integrals finite. The nonlocalized version of QED thus obtained is invariant at order e under the transformation [28].

$$\delta A_\mu = - \partial_\mu \theta \tag{1.6.1a}$$

$$\delta \psi = ie\, \varepsilon_m \theta^\Lambda \psi^\Lambda \tag{1.6.1b}$$

Here $\theta(x)$ is an arbitrary scalar field and $\Box\, \phi(x) = 0$. Also $\theta^\Lambda \equiv \varepsilon_0 \theta$. The explicit operator ε_m in (1.6.1b) acts on everything to its right, whereas the implicit operators in θ^Λ and ψ^Λ are expected to act only on θ and ψ respectively. But the invariance is lost at order e^2. So one must add higher order terms to the interaction [59, 62, 63] to restore gauge invariance.

A curious feature of fundamentally nonlocal field theories is that they are perturbatively acausal [27, 64]. One consequence is that one cannot quantize them in such a way as to preserve simultaneously Lorentz invariance of the functional formalism and the operator formalism. So, a choice is made to preserve the Lorentz invariance in the functional formalism.

The problem of quantization amounts to finding an acceptable measure factor [28] which makes the functional formalism invariant under the classical gauge transformation. The measure factor interactions obey the same restrictions as the classical ones. If such a functional exists, then the resulting perturbation theory is Poincare′ invariant, and finite.

To see how the technique of nonlocal regularization is applied to QED, first define a smearing operator [28] ε_m :

$$\varepsilon_m \equiv \exp\left(\frac{[\partial^2 - m^2]}{2\Lambda^2}\right) \qquad (1.6.2)$$

The superscript Λ denotes the smearing of the unsuperscripted field with appropriate mass:

$$\psi^\Lambda \equiv \varepsilon_m \psi, \qquad (1.6.3a)$$

$$A^\Lambda{}_\mu \equiv \varepsilon_0 A_\mu \qquad (1.6.3b)$$

The free Lagrangian takes the form

$$\mathcal{L}_0 \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi} (i\partial\!\!\!/ + m) \psi \qquad (1.6.4)$$

With the interaction term, the Lagrangian becomes

$$\mathcal{L}_{0+1} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi} (i\partial\!\!\!/ + m) \psi + e\bar{\psi} A\!\!\!/ \psi. \qquad (1.6.5)$$

The initial nonlocalization of QED is performed by nonlocalizing the interaction part of the Lagrangian:

$$\mathcal{L}_{0+1} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi} (i\partial\!\!\!/ + m) \psi + e\bar{\psi}^{\Lambda} A\!\!\!/^{\Lambda} \psi^{\Lambda} \qquad (1.6.6)$$

This Lagrangian is invariant upto order e^2 (not including e^2) under the transformation:

$$\delta A_\mu = \delta_0 A_\mu = -\partial_\mu \theta \qquad (1.6.7a)$$

$$\delta_1 \psi = ie\varepsilon_m \theta^{\Lambda} \psi^{\Lambda} \qquad (1.6.7b)$$

where $\theta^{\Lambda} \equiv \varepsilon_0 \theta$. From ε_m we form the operator ϑ;

$$\vartheta \equiv \frac{(\varepsilon_m)^2 - 1}{\partial^2 - m^2} = \int_0^1 \frac{d\tau}{\Lambda^2} \exp\left(\tau \frac{\partial^2 - m^2}{\Lambda^2} \right) \qquad (1.6.8)$$

The simplest of the four point interactions can be expressed in terms of the operator ϑ :

$$\mathcal{L}_2 = -e^2 \bar{\psi}^{\Lambda} A\!\!\!/^{\Lambda} (i\partial\!\!\!/ - m) \vartheta A\!\!\!/^{\Lambda} \psi^{\Lambda} \qquad (1.6.9)$$

The process can be extended to higher photon amplitudes with interactions of the form

$$\mathcal{L}_n = -(-e)^n \bar{\psi}^{\Lambda} A\!\!\!/^{\Lambda} [(i\partial\!\!\!/ - m)\, \vartheta A\!\!\!/^{\Lambda}]^{(n-1)} \psi^{\Lambda} \tag{1.6.10}$$

Summing up all the terms, the total lagrangian is :

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}(i\partial\!\!\!/ + m)\psi + e\bar{\psi}^{\Lambda} A\!\!\!/^{\Lambda} [1 + e(i\partial\!\!\!/ - m)\, \vartheta A\!\!\!/^{\Lambda}]^{-1} \psi^{\Lambda} \tag{1.6.11}$$

The action is kept invariant under a transformation of the form

$$\delta A_\mu (x) = - \partial_\mu \theta(x) \tag{1.6.12a}$$

$$\delta \psi (x) = ie \int d^4y\, d^4z\, \mathcal{T}[eA] (x,y,z)\, \theta (y)\, \psi (z) \tag{1.6.12b}$$

where $\mathcal{T}[eA] (x,y,z)$ the representation operator [28] $\mathcal{T} \sim 1+ eA + \ldots$ . . . . is a spinorial matrix as well as a functional of the vector potential. Equation (1.6.12b) shows that the fermionic transformation must be modified at each order, for the action to remain invariant. That is,

$$\delta_n \psi = -i (-e)^n \varepsilon_m \theta^{\Lambda} [(i\partial\!\!\!/ - m)\, \vartheta A\!\!\!/^{\Lambda}]^{(n-1)} \psi^{\Lambda} \tag{1.6.13}$$

Summing up all terms the transformation could be read as :

$$\delta A_\mu = - \partial_\mu \theta \tag{1.6.14a}$$

$$\delta\psi = ie\, \varepsilon_m \theta^{\Lambda} [1 + e (i\partial\!\!\!/ - m)\, \vartheta A\!\!\!/^{\Lambda}]^{-1} \psi^{\Lambda} \tag{1.6.14b}$$

From expressions (1.6.12b) and (1.6.14b) $\mathcal{T}[eA] (x,y,z)$ can be obtained. (1.6.14b) can be written as

$$\delta\psi (x) = ie\, \varepsilon_m \int d^4y\, \theta(y)[\delta^4(x-y)]^{\Lambda} [1+e(i\partial\!\!\!/ - m)\, \vartheta A\!\!\!/^{\Lambda}]^{-1} \int d^4z \psi(z)[\delta^4(x-z)]^{\Lambda} \tag{1.6.15}$$

Comparing (1.6.12b) and (1.6.15),

$$\mathcal{T}[e A] (x,y,z,) \equiv \varepsilon_m[\delta^4(x-y)]^{\Lambda} [1 + e (i\partial\!\!\!/ - m)\, \vartheta A\!\!\!/^{\Lambda}]^{-1} \varepsilon_m \delta^4(x-z) \tag{1.6.16}$$

This representation matrix is endowed with certain transformation properties. Also, the gauge field fails to commute with it:

$$\varepsilon_m \not{A}^{\Lambda} [1 + e (i\not{\partial}-m) \vartheta \not{A}^{\Lambda}]^{-1} \varepsilon_m = \varepsilon_m [1 + e \not{A}^{\Lambda} \vartheta (i\not{\partial}-m)]^{-1} \not{A}^{\Lambda} \varepsilon_m \qquad (1.6.17)$$

Also (1.6.17) is not Hermitian. Hence it follows that $\bar{\psi}\psi$ is not an invariant. The consequence of this is that the fermion measures of the functional formalism is not invariant. Another is that the natural covariant kinetic operator

$$(i\not{\partial}+m) + e\, \varepsilon_m \not{A}^{\Lambda} [1 + e (i\not{\partial}-m) \vartheta \not{A}^{\Lambda}]^{-1} \varepsilon_m$$

transforms one way on the right and another way on the left. This implies that the products of such derivatives are not covariant.

In the nonlocal QED theory [28, 59] the classical action was constructed to possess gauge invariance. The vector transformation rule is the same as in the local QED. Only the fermion transformation rule is modified as given by (1.6.12b). Hence the functional measure [dA] remains the same and the problem arises only from the fermionic functional measures $[d\psi]$ and $[d\bar{\psi}]$. Since [dA] is invariant, gauge fixing can be done in terms of the vector potential just as in the local theory. We have to consider the behaviour of $[d\psi]$ and $[d\bar{\psi}]$ under the transformation (1.6.12b). It is useful to introduce the "dot product" at this stage:

$$(\vartheta \mathscr{T}[e A])(x,z) \equiv \int d^4y\, \theta(y) \mathscr{T}[e A](x,y,z) \qquad (1.6.18)$$

To lowest order in θ, we obtain,

$$[d\psi'] = [d\psi] \det{}^{-1}(1 + ie\,\theta.\mathscr{T}[e A]) \quad (1.6.19a)$$

$$= [d\psi] \exp[-\, ie \operatorname{Tr}(\theta.\mathscr{T}[e A])] \quad (1.6.19b)$$

A similar argument for $[d\bar{\psi}]$ gives the result

$$[d\bar{\psi}'] = [d\bar{\psi}] \exp[ie \operatorname{Tr}(\theta.\bar{\mathscr{T}}[e A])] \quad (1.6.19c)$$

The complete result is:

$$[d\psi'][d\bar{\psi}'] = [d\psi][d\bar{\psi}] \exp[-\, ie \operatorname{Tr}(\theta.\mathscr{T}[e A]) + ie \operatorname{Tr}(\theta.\bar{\mathscr{T}}[e A])] \quad (1.6.20)$$

The "trace" in (1.6.20) involves both summing over spinor indices and integrating over space time co-ordinates. Owing to the peculiar mixing between gauge and space time indicates, the two traces in general need not cancel; hence the fermion measures are generally not invariant. The condition for successful quantization is the existence of an acceptable measure factor $\mu[e A]$ [28] which absorbs this non invariance:

$$\mu[e A] = \exp(S_{meas}[e A]) \quad (1.6.21a)$$

$$\partial_\mu \theta \quad \frac{\delta S_{meas}[e A]}{\delta A_\mu} = -\, e \operatorname{Tr}(\theta.\mathscr{T}[e A]) + e \operatorname{Tr}(\theta.\bar{\mathscr{T}}[e A]) \quad (1.6.21b)$$

Odd powers of e cancel. Only even powers of e contribute to the measure factor $S_{meas}[e A]$. Consider the order e^N contribution to $e \operatorname{Tr}(\theta.\bar{\mathscr{T}})$:

$$-\ (-e)^N \operatorname{Tr}\{\varepsilon_m[\not{A}^\Lambda \vartheta(i\not{\partial}-m)]^{N-1}\, \theta^\Lambda \varepsilon_m\}$$

$$= -\ (-e)^N \operatorname{Tr}\{\varepsilon_m[C^{-1}C\, \not{A}^\Lambda\, C^{-1}C\, \vartheta(i\not{\partial}-m)\, C^{-1}C]^{N-1}\, \theta^\Lambda \varepsilon_m\} \quad (1.6.22)$$

C is the charge – conjugation matrix. We have

$$C(\gamma^{\mu})^{tr}\, C^{-1} = -\,\gamma^{\mu} \qquad (1.6.23)$$

Hence, $-\,(-e)^N\, \mathrm{Tr}\, \{\varepsilon_m [A\!\!\!/^{\Lambda}\, \vartheta (i\partial\!\!\!/ - m)]^{N-1}\, \theta^{\Lambda} \varepsilon_m\}$

$$= -\,(-e)^N\, \mathrm{Tr}\, \{\varepsilon_m [-(A\!\!\!/^{\Lambda})^{tr}\, \vartheta (\overleftarrow{i\partial\!\!\!/} - m)^{tr}]^{N-1}\, \theta^{\Lambda} \varepsilon_m\} \qquad (1.6.24)$$

$$= e^N\, \mathrm{Tr}\, \{\varepsilon_m\, \theta^{\Lambda}\, [(i\partial\!\!\!/ - m)\, \vartheta\, A\!\!\!/^{\Lambda}]^{N-1} \varepsilon_m\} \qquad (1.6.25)$$

For N odd this cancels the analogous contribution from $-\, e\, \mathrm{Tr}\, (\theta . \mathscr{T})$, while for even N they add.

1.7 The Electron self-energy and Vacuum Polarization in Nonlocal Regularization

Evens etal [28] have explicitly calculated the electron self–energy and vacuum polarization in this scheme of regularization. The Feynman diagrams corresponding to the electron self–energy in nonlocal regularization are given in fig.1.2. The one-loop correction to the electron self–energy which derives from joining two V_1's [fig. 1.2(a)] is :

$$-i\Sigma_1 (p) \equiv \int \frac{d^4 k}{(2\pi)^4}\, (ie\, \gamma^{\mu}) \left[\frac{-\,i}{q\!\!\!/ + m - i\epsilon} \right] (ie\, \gamma^{\nu}) \left[\frac{-i\eta^{\mu\nu}}{k^2 - i\epsilon} \right]$$

$$\times \quad \exp \left[-\, \frac{p^2 + m^2}{\Lambda^2} - \frac{q^2 + m^2}{\Lambda^2} - \frac{k^2}{\Lambda^2} \right] \qquad (1.7.1)$$

where $q\!\!\!/ \equiv p\!\!\!/ - k\!\!\!/$. Promoting the propagators to Schwinger integrals and performing the momentum integral

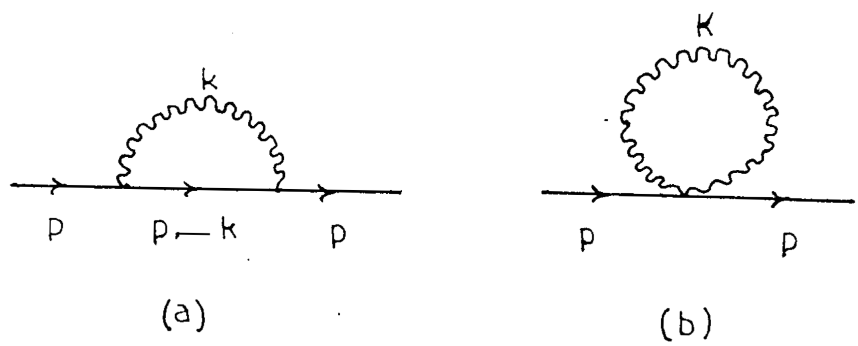
$$-i\,\Sigma_1(p) = -e^2 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int_1^{\infty}\frac{d\tau_1}{\Lambda^2}\int_1^{\infty}\frac{d\tau_2}{\Lambda^2}\int\frac{d^4k}{(2\pi)^4}\,(2\not{q}+4m)$$

$$\times \exp\left[-\tau_1\frac{q^2+m^2}{\Lambda^2} - \tau_2\frac{k^2}{\Lambda^2}\right] \qquad (1.7.2a)$$

$$= -\frac{ie^2}{8\pi^2}\exp\left[-\frac{p^2+m^2}{\Lambda^2}\right]\int_1^{\infty}d\tau_1\int_1^{\infty}d\tau_2\left[\frac{\tau_2\not{p}}{(\tau_1+\tau_2)^3}\right.$$

$$\left.+\frac{2m}{(\tau_1+\tau_2)^2}\right]\exp\left[-\frac{\tau_1\tau_2}{\tau_1+\tau_2}\frac{p^2}{\Lambda^2}-\tau_1\frac{m^2}{\Lambda^2}\right] \qquad (1.7.2b)$$

The factor of i comes from the rotation to Euclidean space [28, 44].



(a) Electron self-energy at one loop;contribution of V_1

(b) Electron self-energy at one loop;contribution of V_2

Fig.1.2

The other correction at the same order comes from a single V_2 [fig. 1.2(b)]. This comes from the Lagrangian in expression (1.6.9).

$$-i\,\Sigma_2(p) \equiv \int \frac{d^4k}{(2\pi)^4}\,(-ie^2)\,\gamma^\mu(\not{q}-m)\,\gamma^\nu\left[\frac{-i\eta_{\mu\nu}}{k^2-i\epsilon}\right]$$

$$\times \int_0^1 \frac{d\tau}{\Lambda^2}\exp\left[-\frac{p^2+m^2}{\Lambda^2}-\tau\frac{q^2+m^2}{\Lambda^2}-\frac{k^2}{\Lambda^2}\right] \qquad (1.7.3)$$

The result of performing the momentum integration is the same as expression (1.7.2b) except that the Schwinger parameters are integrated over $0 \le \tau_1 \le 1 \le \tau_2 < \infty$. Σ_1 and Σ_2 obviously add to give the expression

$$\Sigma(p) = \frac{e^2}{8\pi^2}\exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int_0^\infty d\tau_1\int_1^\infty d\tau_2\left[\frac{\tau_2}{(\tau_1+\tau_2)^3}\not{p}+\right.$$

$$\left.+\frac{2}{(\tau_1+\tau_2)^2}m\right]\exp\left[-\frac{\tau_1\tau_2}{\tau_1+\tau_2}\frac{p^2}{\Lambda^2}-\tau_1\frac{m^2}{\Lambda^2}\right] \qquad (1.7.4a)$$

$$= \frac{e^2}{8\pi^2}\exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int_0^1 dx\;(x\not{p}+2m)$$

$$\times E_1\left[(1-x)\frac{p^2}{\Lambda^2}+\left(\frac{1-x}{x}\right)\frac{m^2}{\Lambda^2}\right] \qquad (1.7.4b)$$

where E_1 is the exponential integral [28, 65]:

$$E_1(z) \equiv \int_z^\infty dt\,\frac{\exp(-t)}{t} = -\ln(z)-\gamma-\sum_{n=1}^{\infty}\frac{(-z)^n}{n\,n!} \qquad (1.7.5)$$

For asymptotically large values of Λ, the expression for electron self – energy takes the form,

$$\Sigma(p) = \frac{e^2}{8\pi^2}\left[\left(\frac{1}{2}\not{p} + 2m\right)\ln(\Lambda^2) - \left(\frac{1}{2}\not{p} + 2m\right)\gamma + \frac{1}{2}\not{p}\right.$$

$$\left. - \int_0^1 dx\,(x\not{p} + 2m)\ln(xp^2 + m^2) + O\left(\frac{\ln(\Lambda^2)}{\Lambda^2}\right)\right] \qquad (1.7.6)$$

Comparison with the result of dimensional regularization in D dimensions [66] and scale μ gives

$$\Sigma(p) = e^2\,\frac{\Gamma(2 - D/2)}{2^D\,\pi^{D/2}}\int_0^1 dx\;[D - 2)\,x\not{p} + Dm]\left[x(1-x)\frac{p^2}{\mu^2} + (1-x)\frac{m^2}{\mu^2}\right]^{D/2-2} \qquad (1.7.7a)$$

$$= \frac{e^2}{8\pi^2}\left[\left(\frac{1}{2}\not{p} + 2m\right)\frac{2}{4 - D} + \left(\frac{1}{2}\not{p} + m\right)\left(\ln 4\pi - \gamma + \frac{1}{2}\right)\right.$$

$$\left. - \int_0^1 dx\;(x\not{p} + 2m)\ln(xp^2 + m^2) + O(4 - D)\right] \qquad (1.7.7b)$$

This suggests the correspondence,

$$\frac{2}{4 - D} \sim \ln(\Lambda^2) \qquad (1.7.8)$$

for the co-efficients of logarithmic divergences in the two methods.

Consider the nonlocally regulated electron self-energy in D dimensions:

$$\Sigma(p) = -ie^2 \exp\left(-\frac{p^2 + m^2}{\Lambda^2}\right)\int_0^\infty \frac{d\tau_1}{\Lambda^2}\int_1^\infty \frac{d\tau_2}{\Lambda^2}\int \frac{d^Dk}{(2\pi)^D}$$

$$\times\;[(D - 2)\not{q} + Dm]\exp\left[-\tau_1\frac{q^2 + m^2}{\Lambda^2} - \tau_2\frac{k^2}{\Lambda^2}\right] \qquad (1.7.9a)$$

$$\Sigma(p) = \frac{e^2}{2^D \pi^{D/2}} \exp\left[-\frac{p^2+m^2}{\Lambda^2}\right] \int_0^\infty d\tau_1 \int_1^\infty d\tau_2 \left[\frac{(D-2)\,\tau_2 \not{p}}{(\tau_1+\tau_2)^3} + \frac{Dm}{(\tau_1+\tau_2)^2}\right]$$

$$\times \left(\frac{\Lambda^2}{\tau_1+\tau_2}\right)^{D/2-2} \exp\left[-\frac{\tau_1\tau_2}{\tau_1+\tau_2}\frac{p^2}{\Lambda^2} - \tau_1 \frac{m^2}{\Lambda^2}\right] \qquad (1.7.9b)$$

$$= \frac{e^2}{2^D \pi^{D/2}} \exp\left[-\frac{p^2+m^2}{\Lambda^2}\right] \int_0^1 dx \quad [(D-2)x\not{p} + Dm]$$

$$\times \quad [x(1-x)p^2 + (1-x)m^2]^{D/2-2}\; \Gamma\left[2-\frac{D}{2}, \frac{(1-x)\,p^2}{\Lambda^2} + \left(\frac{1-x}{x}\right)\frac{m^2}{\Lambda^2}\right] \qquad (1.7.9c)$$

where $\Gamma(n,z)$ is the incomplete γ - function [28, 50, 65]:

$$\Gamma(n,z) \equiv \int_z^\infty dt \quad t^{n-1} \exp(-t) \qquad (1.7.10)$$

Obtaining asymptotic expansions for large Λ^2 depends heavily upon the loop order. If the degree n of the incomplete gamma function is positive then the result is finite in the unregulated limit and one can take Λ^2 to infinity directly. If the degree is negative, it can be raised either to zero or to one half by means of the recursion relation [28, 50]:

$$\Gamma(n,z) = -\frac{1}{n} z^n \; e^{-z} + \frac{1}{n}\; \Gamma(n+1, z) \qquad (1.7.11)$$

This may reach to either

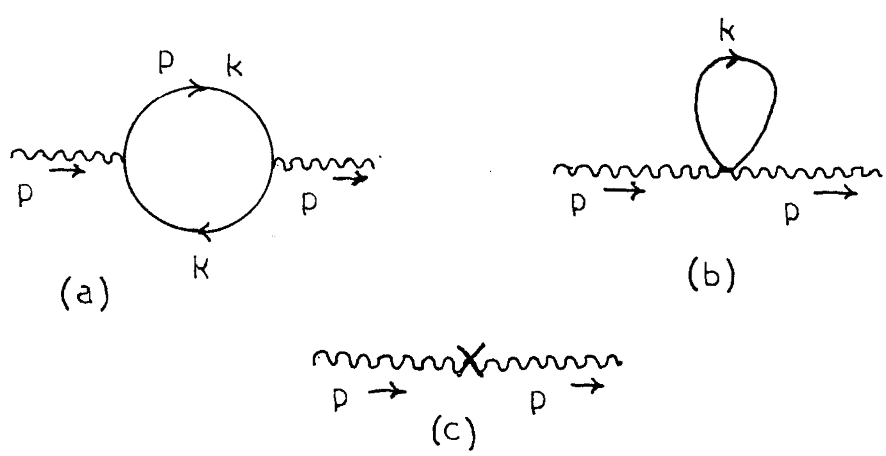
$$\Gamma(0,z) \equiv Ei(z) = -\ln(z) - \gamma - \sum_{n=1}^{\infty} \frac{(-z)^n}{n\, n!} \quad \text{for D even} \qquad (1.7.12a)$$

$$\text{or } \Gamma(1/2, z) \equiv \sqrt{\pi}\ \text{erfc}(\sqrt{z}) = \sqrt{\pi} - \sqrt{4z} \sum_{n=0}^{\infty} \frac{(-z)^n}{(2n+1)\, n!}$$

for D odd (1.7.12b)

Here erfc (x) is the complementary error function [28, 65]. Since z goes like Λ^{-2}, and since the loop parameter integrals harbour no divergences for positive m^2, one can integrate termwise to obtain the desired expansion. The logarithmic divergences of dimensional regularization are captured by the first term of expansion (1.7.12a); the first term in (1.7.11) gives a power–law divergence for negative n, which is lost in the automatic subtraction of dimensional regularization [4].

The Feynman diagrams corresponding to vacuum polarization in nonlocal regularization [28] are as follows:



(a) Vacuum Polarization at one loop;contribution of V_1
(b) Vacuum Polarization at one loop:Contribution of V_2
(c) Vacuum Polarization at one loop:measure factor contribution

Fig.1.3

The one loop correction to the vacuum polarization which derives from joining the two V_1's [fig.1.3(a)] is :

$$i\Pi_1^{\mu\nu}(p) \equiv - \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left[(ie\gamma^\mu)\left\{\frac{-i}{\not{q} + m - i\epsilon}\right\}(ie\gamma^\nu)\left\{\frac{-i}{\not{k} + m - i\epsilon}\right\}\right.$$

$$\times \exp\left[-\frac{q^2+m^2}{\Lambda^2} - \frac{k^2+m^2}{\Lambda^2} - \frac{p^2}{\Lambda^2}\right] \qquad (1.7.13a)$$

$$\equiv i\,\Pi_1^T(p^2)\left[\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right] + i\,\Pi_1^L(p^2)\frac{p^\mu p^\nu}{p^2} \qquad (1.7.13b)$$

where the transverse and longitudinal coefficients are

$$i\,\Pi_1^T(p^2) = \frac{ie^2}{4\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)\int_1^\infty d\tau_1\int_1^\infty d\tau_2\left[\frac{\Lambda^2}{(\tau_1+\tau_2)^3} - \frac{\tau_1\tau_2\,p^2}{(\tau_1+\tau_2)^4} + \frac{m^2}{(\tau_1+\tau_2)^2}\right]$$

$$\times \exp\left[-\frac{\tau_1\tau_2}{\tau_1+\tau_2}\frac{p^2}{\Lambda^2} - (\tau_1+\tau_2)\frac{m^2}{\Lambda^2}\right] \qquad (1.7.13c)$$

$$i\,\Pi_1^L(p^2) = \frac{ie^2}{4\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)\int_1^\infty d\tau_1\int_1^\infty d\tau_2\left[\frac{\Lambda^2}{(\tau_1+\tau_2)^3} + \frac{\tau_1\tau_2 p^2}{(\tau_1+\tau_2)^4} + \frac{m^2}{(\tau_1+\tau_2)^2}\right]$$

$$\times \exp\left[-\frac{\tau_1\tau_2}{\tau_1+\tau_2}\frac{p^2}{\Lambda^2} - (\tau_1+\tau_2)\frac{m^2}{\Lambda^2}\right] \qquad (1.7.13d)$$

A similar contribution comes from a single V_2 [fig.1.3(b)]:

$$i\Pi_2^{\mu\nu}(p) \equiv - \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left[-2ie^2\ \gamma^\mu\ (\not{q} - m)\ \gamma^\nu \left\{\frac{-i}{\not{k} + m - i\epsilon}\right\}\right]$$

$$\times \int_0^1 \frac{d\tau}{\Lambda^2}\exp\left[-\tau\frac{q^2+m^2}{\Lambda^2} - \frac{k^2+m^2}{\Lambda^2} - \frac{p^2}{\Lambda^2}\right] \qquad (1.7.14a)$$

$$= i\Pi^{T}_{2}(p^2)\left[\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right] + i\,\Pi^{L}_{2}(p^2)\,\frac{p^\mu p^\nu}{p^2} \qquad (1.7.14b)$$

where the transverse and the longitudinal coefficients are

$$i\,\Pi^{T}_{2}(p^2) = \frac{ie^2}{4\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)\left[\int_0^1 d\tau_1 \int_1^\infty d\tau_2 + \int_1^\infty d\tau_1 \int_0^1 d\tau_2\right]$$

$$\times\left[\frac{\Lambda^2}{(\tau_1+\tau_2)^3} - \frac{\tau_1\tau_2 p^2}{(\tau_1+\tau_2)^4} + \frac{m^2}{(\tau_1+\tau_2)^2}\right]$$

$$\times \exp\left[-\frac{\tau_1\tau_2}{\tau_1+\tau_2}\frac{p^2}{\Lambda^2} - (\tau_1+\tau_2)\frac{m^2}{\Lambda^2}\right] \qquad (1.7.14c)$$

$$i\,\Pi^{L}_{2}(p^2) = \frac{ie^2}{4\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)\left[\int_0^1 d\tau_1 \int_1^\infty d\tau_2 + \int_1^\infty d\tau_1 \int_0^1 d\tau_2\right]$$

$$\times\left[\frac{\Lambda^2}{(\tau_1+\tau_2)^3} + \frac{\tau_1\tau_2 p^2}{(\tau_1+\tau_2)^4} + \frac{m^2}{(\tau_1+\tau_2)^2}\right]$$

$$\times\left[\exp - \frac{\tau_1\tau_2}{\tau_1+\tau_2}\frac{p^2}{\Lambda^2} - (\tau_1+\tau_2)\frac{m^2}{\Lambda^2}\right] \qquad (1.7.14d)$$

From the expressions (1.7.13d) and (1.7.14d), it is clear that the two classical vertex contributions to $\Pi^{\mu\nu}$ do not sum to give zero longitudinal part: neither do the transverse parts sum to zero on shell. If these were the only contributions, we would have lost both decoupling and the photon's masslessness. Both these features were restored by the measure factor contribution [28] [fig.1.3(c)] :

$$i\Pi^{\mu\nu}{}_3 \equiv -\frac{ie^2}{2\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)\int_0^1 d\tau\,\frac{\Lambda^2}{(\tau+1)^3}\exp\left[-\frac{\tau}{\tau+1}\frac{p^2}{\Lambda^2}-(\tau+1)\frac{m^2}{\Lambda^2}\right]\eta^{\mu\nu} \tag{1.7.15}$$

To see transversality, change variab!e in (1.7.13d) from τ_1 to $x=\frac{\tau_1}{\tau_2}$:

$$i\Pi^L{}_1(p^2) = \frac{ie^2}{4\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)\left[\int_1^\infty dx\int_1^\infty d\tau_2 + \int_0^1 dx\int_{1/x}^\infty d\tau_2\right]$$

$$\times\left[\frac{\Lambda^2}{\tau_2{}^2(x+1)^3}+\frac{x}{\tau_2(x+1)^4}p^2+\frac{m^2}{\tau_2(x+l)^2}\right]$$

$$\times\ \exp\left[-\tau_2\frac{x}{x+1}\frac{p^2}{\Lambda^2}-\tau_2(x+1)\frac{m^2}{\Lambda^2}\right] \tag{1.7.16a}$$

$$= \frac{ie^2}{4\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)\left[\int_1^\infty dx\int_1^\infty d\tau_2 + \int_0^1 dx\int_{1/x}^\infty d\tau_2\right]$$

$$\times\ \frac{\partial}{\partial\tau_2}\left[-\frac{1}{\tau_2}\frac{\Lambda^2}{(x+1)^3}\exp\left\{-\tau_2\frac{x}{(x+1)}\frac{p^2}{\Lambda^2}-\tau_2(x+1)\frac{m^2}{\Lambda^2}\right\}\right] \tag{1.7.16b}$$

$$= \frac{ie^2}{2\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)\int_1^\infty d\tau\,\frac{\Lambda^2}{(\tau+1)^3}\exp\left[-\frac{\tau}{\tau+1}\frac{p^2}{\Lambda^2}-(\tau+1)\frac{m^2}{\Lambda^2}\right] \tag{1.7.16c}$$

A similar set of manipulation gives

$$i\Pi^L{}_2(p^2) = \frac{ie^2}{4\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)\left[\int_0^1 - \int_1^\infty\right]d\tau\,\frac{\Lambda^2}{(\tau+1)^3}$$

$$\times\ \exp\left[-\frac{\tau}{\tau+1}\frac{p^2}{\Lambda^2}-(\tau+1)\frac{m^2}{\Lambda^2}\right] \tag{1.7.17}$$

Now $\Pi^L_1 + \Pi^L_2 + \Pi^L_3 = 0$ as required. The total transverse part is:

$$\Pi^T(p^2) = -\frac{e^2p^2}{2\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)\left[\int_0^\infty d\tau_1\int_1^\infty d\tau_2 + \int_1^\infty d\tau_1\int_0^\infty d\tau_2\right]\frac{\tau_1\tau_2}{(\tau_1+\tau_2)^4}$$

$$\times \exp\left[-\frac{\tau_1\tau_2}{\tau_1+\tau_2}\frac{p^2}{\Lambda^2} - (\tau_1+\tau_2)\frac{m^2}{\Lambda^2}\right] \qquad (1.7.18a)$$

Changing the parameters to $\tau_2 = \tau x$ and $\tau_1 = \tau(1\text{-}x)$ where $\tau = \tau_1 + \tau_2$, expression (1.7.18a) becomes

$$\Pi^T(p^2) = -\frac{e^2p^2}{2\pi^2}\exp\left(-\frac{p^2}{\Lambda^2}\right)2\int_0^{1/2} dx\, x(1\text{-}x)\, E_1\left[x\frac{p^2}{\Lambda^2} + \frac{1}{1\text{-}x}\frac{m^2}{\Lambda^2}\right]$$

(1.7.18b)

It can be seen that the gauge invariance has absorbed the simple quadratic divergence . The factor of p^2 guarantees masslessness, since $\Pi^T(p^2) \to 0$ as $p^2 \to 0$.

We can develop an asymptotic expansion in Λ [65]. The exponential integral $E_1(z)$ is:

$$E_1(z) = -\ln(z) - \gamma - \sum_{n=1}^{\infty}\frac{(-z)^n}{n\,n!} \qquad (1.7.19)$$

In (1.7.18b), $z = \frac{xp^2}{\Lambda^2} + \frac{1}{1\text{-}x}\frac{m^2}{\Lambda^2}$. So,

$$\Pi^T(p^2) = -\frac{e^2p^2}{2\pi^2}\,2\int_0^{1/2} dx\; x(1\text{-}x)\left[-\ln\left(\frac{xp^2}{\Lambda^2} + \frac{1}{1\text{-}x}\frac{m^2}{\Lambda^2}\right) - \gamma\right]$$

(1.7.20a)

$$= -\frac{e^2p^2}{2\pi^2}\left[\frac{1}{6}\ln(\Lambda^2) + \frac{1}{6}\ln(2) - \frac{1}{6}\gamma - \frac{13}{72}\right.$$

$$\left. - \int_0^1 dx\, x\,(1-x)\, \ln\{x\,(1-x)\,p^2 + m^2\} + O\left(\frac{\ln(\Lambda^2)}{\Lambda^2}\right)\right] \quad (1.7.20b)$$

The result of dimensional regularization in D dimensions with scale μ [66] gives:

$$\Pi^T(p^2) = -e^2p^2\, 2^{D/2+1}\frac{\Gamma(2-D/2)}{2^D\,\Pi^{D/2}}\int_0^1 dx\, x(1-x)\left[x\,(1-x)\frac{p^2}{\mu^2} + \frac{m^2}{\mu^2}\right]^{D/2-2} \quad (1.7.21a)$$

$$= -\frac{e^2p^2}{2\pi^2}\left[\frac{1}{6}\,\frac{2}{4-D} - \frac{1}{6}\gamma - \frac{1}{6}\ln(2\pi)\right.$$

$$\left. - \int_0^1 dx\; x\,(1-x)\ln\left[x(1-x)\frac{p^2}{\mu^2} + \frac{m^2}{\mu^2}\right] + O(4-D)\right] \quad (1.7.21b)$$

Comparison between the two results suggests the correspondence

$$\frac{2}{4-D} \sim \ln(\Lambda^2), \quad (1.7.22)$$

for the co-efficients of logarithmic divergences in the two methods.

To find the measure factor contribution [eq. (1.7.15)] [28] to the vacuum polarization, consider the field dependent representation matrix $\mathscr{T}[eA](x, y, z)$:

$$\mathscr{T}[eA](x,y,z) \equiv \varepsilon_m[\delta^4(x-y)]^\Lambda\,[1 + e(i\partial\!\!\!/ - m)\vartheta A\!\!\!/^\Lambda]^{-1}\varepsilon_m\delta^4(x-z) \quad (1.7.23)$$

Substituting this into the defining relation (1.6.21b) and expanding to order e^2, one obtains :

$$\partial_\mu \theta \frac{\delta S_{meas}[eA]}{\delta A_\mu} = 2e^2 \mathrm{Tr}\ \{ \varepsilon_m \theta^\Lambda (i\partial\!\!\!/ - m)\, \vartheta A\!\!\!/^\Lambda \varepsilon_m \} \tag{1.7.24}$$

It will be easier if everything is converted into the momentum space.

Let the momentum of the gauge parameter be k_1^μ, that of the vector potential k_2^μ, and that of the delta function k_3^μ [67]:

$$\theta(x) = \int \frac{d^4k_1}{(2\pi)^4} \exp(ik_1.x)\, \tilde{\theta}(k_1) \tag{1.7.25a}$$

$$A_\mu(x) = \int \frac{d^4k_2}{(2\pi)^4} \exp(ik_2.x)\, \tilde{A}_\mu(k_2) \tag{1.7.25b}$$

$$\delta^4(x - z) = \int \frac{d^4k_3}{(2\pi)^4} \exp[ik_3.(x - z)] \tag{1.7.25c}$$

The argument of the trace in the order e^2 term above is:

$$\begin{aligned}
\varepsilon_m \theta^\Lambda (i\partial\!\!\!/ - m)\, \vartheta A\!\!\!/^\Lambda \varepsilon_m \delta^4(x - z) = & \int \frac{d^4k_1}{(2\pi)^4} \exp\left[- \frac{k_1^2 + m^2}{2\Lambda^2} + ik_1.x \right] \tilde{\theta}(k_1) \\
& \times \int \frac{d^4k_2}{(2\pi)^4} \exp\left[- \frac{k_2^2 + m^2}{2\Lambda^2} + ik_2.x \right] \tilde{A}_\mu(k_2) \\
& \times \int \frac{d^4k_3}{(2\pi)^4} \exp\left[- \frac{k_3^2 + m^2}{2\Lambda^2} + ik_3.(x-z) \right] \\
& \times \exp\left[- \frac{(k_1+k_2+k_3)^2 + m^2}{2\Lambda^2} \right] \mathrm{Tr}\,(-k\!\!\!/_2 - k\!\!\!/_3 - m) \\
& \times \int_0^1 \frac{d\tau}{\Lambda^2} \exp\left[-\tau \frac{(k_2 + k_3)^2 + m^2}{\Lambda^2} \right] \gamma^\mu
\end{aligned} \tag{1.7.26}$$

The trace is over spinor indices and position. The spinor trace affects only the following terms:

$$\mathrm{Tr}\ [(-\not{k}_2-\not{k}_3-m)\,\gamma^{\mu}] \quad = \quad 4\,(k_2+k_3)^{\mu} \tag{1.7.27}$$

The position trace is accomplished by setting $z^{\mu} = x^{\mu}$ and integrating over x^{μ}. This gives :

$$\int d^4x^{\mu} \exp[i\,(k_1+k_2)\cdot x] \;=\; (2\pi)^4\,\delta^4\,(k_1+k_2) \tag{1.7.28}$$

The result is :

$$\begin{aligned}2e^2\,\mathrm{Tr}\ [\varepsilon_m\,\theta^{\Lambda}\,(i\not{\partial}-m)\,\vartheta\,\not{A}^{\Lambda}\varepsilon_m] \;=\; & 8e^2\int\frac{d^4k_2}{(2\pi)^4}\ \exp\left[-\,\frac{k_2^2+m^2}{2\Lambda^2}\right]\tilde{\theta}\,(-k_2)\\ & \times\ \exp\left[-\,\frac{k_2^2+m^2}{2\Lambda^2}\right]\ \tilde{A}_{\mu}(k_2)\\ & \times\int\frac{d^4k_3}{(2\pi)^4}\ \exp\left[-\,\frac{k_3^2+m^2}{\Lambda^2}\right](k_2+k_3)^{\mu}\\ & \times\int_0^1\frac{d\tau}{\Lambda^2}\ \exp\left[-\tau\,\frac{(k_2+k_3)^2+m^2}{\Lambda^2}\right]\end{aligned} \tag{1.7.29}$$

Consider the exponentials $\exp\left(-\,\dfrac{k_3^2+m^2}{\Lambda^2}\right)\ \exp\left(-\tau\,\dfrac{(k_2+k_3)^2+m^2}{\Lambda^2}\right)$.

Completing the square in the exponentials, we get:

$$\exp\left[-\,\frac{(\tau+1)}{\Lambda^2}\left(k_3+\frac{\tau}{\tau+1}\,k_2\right)^2\right]\exp\left[-\,\frac{\tau}{\tau+1}\,\frac{k_2^2}{\Lambda^2}\ -(\tau+1)\,\frac{m^2}{\Lambda^2}\right]$$

Expression (1.7.29) becomes:

$$2e^2\,\mathrm{Tr}\ [\varepsilon_m\,\theta^\Lambda\,(i\partial\!\!\!/ - m)\,\vartheta\,A\!\!\!/^\Lambda\varepsilon_m] = 8e^2\int\frac{d^4k_2}{(2\pi)^4}\exp\left[-\frac{k_2^2+m^2}{2\Lambda^2}\right]\widetilde{\theta}(-k_2)$$

$$\times\ \exp\left[-\frac{k_2^2+m^2}{2\Lambda^2}\right]\widetilde{A}_\mu(k_2)$$

$$\times\int\frac{d^4k_3}{(2\pi)^4}\exp\left[-\frac{k_3^2+m^2}{\Lambda^2}\right](k_2+k_3)^\mu$$

$$\times\int_0^1\frac{d\tau}{\Lambda^2}\exp\left\{-\frac{\tau+1}{\Lambda^2}\left[k_3+\frac{\tau}{\tau+1}k_2\right]^2\right\}$$

$$\times\ \exp\left[-\frac{\tau}{\tau+1}\frac{k_2^2}{\Lambda^2}-(\tau+1)\frac{m^2}{\Lambda^2}\right]$$

(1.7.30)

Make a shift $k_3 \rightarrow k_3 - \frac{\tau}{\tau+1}k_2$. (1.7.31)

Then (1.7.30) becomes :

$$2e^2\,\mathrm{Tr}\ [\varepsilon_m\,\theta^\Lambda\,(i\partial\!\!\!/ - m)\,\vartheta\,A\!\!\!/^\Lambda\varepsilon_m] = 8e^2\int\frac{d^4k_2}{(2\pi)^4}\,\widetilde{\theta}^\Lambda(-k_2)\ \widetilde{A}_\mu^{\ \Lambda}(k_2)$$

$$\times\int_0^1\frac{d\tau}{\Lambda^2}\int\frac{d^4k_3}{(2\pi)^4}\left[k_3+\frac{1}{\tau+1}k_2\right]^\mu$$

$$\times\exp\left[-(\tau+1)\frac{k_3^2}{\Lambda^2}\right]$$

$$\times\exp\left[-\frac{\tau}{\tau+1}\frac{k_2^2}{\Lambda^2}-(\tau+1)\frac{m^2}{\Lambda^2}\right]$$

(1.7.32)

Performing the trivial Gaussian integral over k_3^μ gives :

$$2e^2 \,\mathrm{Tr}\, \left[\varepsilon_m \theta^\Lambda (i\partial\!\!\!/ - m)\, \vartheta\, A\!\!\!/^\Lambda \varepsilon_m\right] = \frac{e^2}{2\pi^2} \int \frac{d^4k_2}{(2\pi)^4}\, k_2^\mu\, \tilde{\theta}^\Lambda(-k_2)$$

$$\times \int_0^1 d\tau \frac{\Lambda^2}{(\tau+1)^3} \exp\left[-\frac{\tau}{\tau+1} \frac{k_2^2}{\Lambda^2} - (\tau+1)\frac{m^2}{\Lambda^2} \right]$$

$$\times\, \tilde{A}_\mu^\Lambda(k_2) \qquad (1.7.33)$$

Converting back to position space we have :

$$-i \int d^4x\, \partial_\mu \theta(x) \frac{\delta S_{meas}[eA]}{\delta A_\mu(x)} = \frac{ie^2}{2\pi^2} \int d^4x\, \partial^\mu \theta^\Lambda(x)\, \vartheta_1\, A_\mu^\Lambda(x) + O(e^4)$$

(1.7.34a)

$$\text{where } \vartheta_1 \equiv \int_0^1 d\tau \frac{\Lambda^2}{(\tau+1)^3} \exp\left[\frac{\tau}{\tau+1} \frac{\partial^2}{\Lambda^2} - (\tau+1)\frac{m^2}{\Lambda^2} \right] \qquad (1.7.34b)$$

This gives,

$$\delta S_{meas}[eA] = \frac{e^2}{2\pi^2} \int d^4x\, \partial^\mu \theta^\Lambda(x)\, \vartheta_1\, A_\mu^\Lambda(x) + O(e^4) \qquad (1.7.35a)$$

or

$$S_{meas}[eA] = -\frac{e^2}{4\pi^2} \int d^4x\, A_\mu^\Lambda(x) \eta^{\mu\nu} \vartheta_1 A_\nu^\Lambda(x) + O(e^4) \qquad (1.7.36)$$

At one loop there are no simpler diagrams than two - point insertions. Simply dropping the two vector potentials in (1.7.36) and multiplying by a factor of 2 gives the measure factor contribution to vacuum polarization at one loop,

$$i\Pi_3^{\mu\nu} = -\frac{ie^2}{2\pi^2} \exp\left(-\frac{p^2}{\Lambda^2} \right) \int_0^1 d\tau \frac{\Lambda^2}{(\tau+1)^3} \exp\left[-\frac{\tau}{\tau+1} \frac{p^2}{\Lambda^2} - (\tau+1)\frac{m^2}{\Lambda^2} \right]$$

which is equation (1.7.15).

Chapter 2

QED Vertex Part In Nonlocal Regularization

2.1 Introduction

The method of nonlocal regularization has been developed by Evens etal [28]. They have applied it to obtain the QED electron self-energy and vacuum polarization, and they have compared their result with that obtained using dimensional regularization [66]. We have extended the method of nonlocal regularization to evaluate the QED vertex part [68]. Here also, the result obtained agrees perfectly with that of the result of dimensional regularization. The divergent parts may be equated using

$$\ln \Lambda^2 = \frac{1}{\epsilon} .$$

The finite parts are the same apart from trivial numerical constants.

The Furry's theorem [69 - 71] is also proved for nonlocal QED. The content of the Furry's theorem is that Feynman diagrams containing a closed fermion loop with an odd number of photon vertices can be omitted in the calculation of physical processes.

2.2 Feynman rules for nonlocal QED

In the nonlocal theory, first a smearing operator ε_m is defined :

$$\varepsilon_m \equiv \exp\left(\frac{\partial^2 - m^2}{2\Lambda^2} \right) \tag{2.2.1}$$

From this operator another operator ϑ is defined :

$$\vartheta \equiv \frac{\varepsilon_m^2 - 1}{\partial^2 - m^2} = \int_0^1 \frac{d\tau}{\Lambda^2} \exp\left(\tau \frac{\partial^2 - m^2}{\Lambda^2}\right) \qquad (2.2.2)$$

Using this operator the Lagrangian for the higher interactions can be expressed as

$$\mathcal{L}_n = -(-e)^n \bar{\psi}^{\Lambda} \not{A}^{\Lambda} [(i\not{\partial} - m)\, \vartheta \not{A}^{\Lambda}]^{n-1} \psi^{\Lambda} \qquad (2.2.3)$$

The total Lagrangian then has the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}(i\not{\partial} + m)\psi + e \bar{\psi}^{\Lambda} \not{A}^{\Lambda} [1 + e(i\not{\partial} - m)\, \vartheta \not{A}^{\Lambda}]^{-1} \psi^{\Lambda} \qquad (2.2.4)$$

The nonlocalized Feynman rules for any theory are a trivial extension of the local ones [16, 72 – 74]. There are two kinds of propagators —— an unbarred one and a barred one. The second one can be expressed diagrammatically either as a propagator with a bar or by just joining the two vertices [28]. Here the latter convention is adopted. The unbarred electron propagator is expressed as,

$$\frac{-i\varepsilon^2}{p^2 + m^2 - i\epsilon} = -i \int_1^{\infty} \frac{d\tau}{\Lambda^2} \exp\left(-\tau \frac{p^2 + m^2}{\Lambda^2}\right) \qquad (2.2.5a)$$

and the barred one as,

$$-i \frac{\varepsilon^2 - 1}{p^2 + m^2 - i\epsilon} = -i \int_0^1 \frac{d\tau}{\Lambda^2} \exp\left(-\tau \frac{p^2 + m^2}{\Lambda^2}\right) \qquad (2.2.5b)$$

where ε is as given in expression (2.2.1).

Each vertex is associated with a factor $(-ie\gamma^{\mu})$.

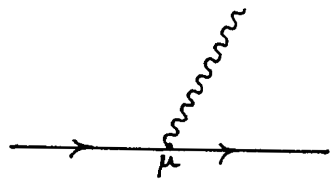
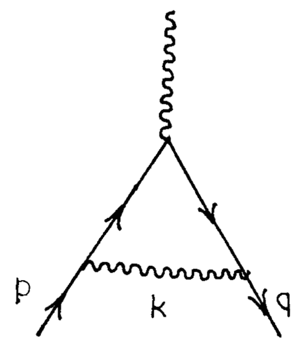


Fig .2.1

A nonlocal vertex of order e^n is associated with a factor $-ie^n$.

2.3 The QED vertex part in nonlocal regularization

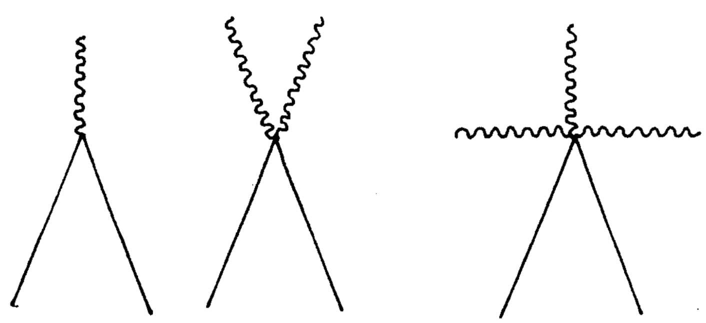
The diagram corresponding to the vertex part in QED is as follows:



QED vertex graph

Fig.2.2

But in nonlocal QED, there are four diagrams contributing to the vertex part [68]. The types of vertices which contribute to the vertex part in nonlocal QED to order e^2 are shown in fig.2.3.



First, second and third order vertices in nonlocal regularization.

Fig.2.3.

The vertices in figs.2.3(a) to (c) come from the interaction part of the Lagrangian :

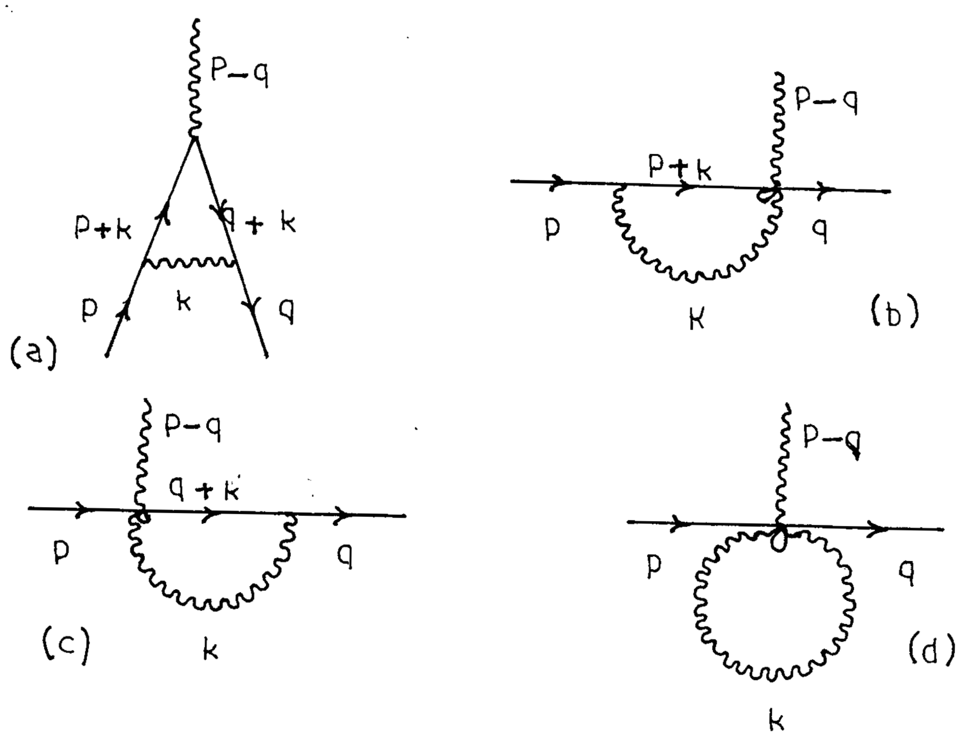
$$\mathcal{L}_n = -(-e)^n\, \bar{\psi}^\Lambda \not{A}^\Lambda \,[(i\not{\partial}-m)\, \vartheta \not{A}^\Lambda\,]^{n-1}\psi^\Lambda \qquad (2.3.1)$$

where n = 1,2,3..................

When n =1 , this reduces to the usual interacting Lagrangian for QED,

$$\mathcal{L}_1 = e\, \bar{\psi}^\Lambda \not{A}^\Lambda \psi^\Lambda \ . \qquad (2.3.2)$$

The four diagrams which contribute to the vertex part in nonlocal regularization are given in fig.2.4.



The four diagrams contributing to the QED vertex part in nonlocal regularization.

Fig.2.4.

The contribution from fig.2.4(a) is

$$-ie\Lambda_a^{\rho}(q,p) \equiv \int \frac{d^4k}{(2\pi)^4} (ie\gamma^t) \left[\frac{-i}{\not p + \not k + m}\right] (ie\gamma^{\rho}) \left[\frac{-i}{\not q + \not k + m}\right] (ie\gamma^{\sigma}) \frac{-i\eta^{t\sigma}}{k^2}$$

$$\times \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{k^2}{\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} - \frac{(p+k)^2+m^2}{\Lambda^2}\right.$$

$$\left. - \frac{(q+k)^2+m^2}{\Lambda^2}\right] \qquad (2.3.3)$$

Promoting the propagators to Schwinger integrals, one obtains,

$$-ie\Lambda_a^{\rho}(q,p) = -i\, e^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_1^{\infty} \frac{d\tau_1}{\Lambda^2} \int_1^{\infty} \frac{d\tau_2}{\Lambda^2} \int_1^{\infty} \frac{d\tau_3}{\Lambda^2} \gamma^t (\not p + \not k - m)\, \gamma^{\rho} (\not q + \not k - m)\, \gamma_t$$

$$\times \exp\left[-\tau_1 \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} - \tau_3 \frac{k^2}{\Lambda^2}\right] \qquad (2.3.4)$$

The factor of i comes from the rotation to the Euclidean space. The contribution from fig.2.4(b) comes from a vertex of the first order and a vertex of the second order. The contribution is,

$$-ie\Lambda_b^{\rho}(q,p) \equiv \int \frac{d^4k}{(2\pi)^4} (ie\gamma^t) \left[\frac{-i}{\not p + \not k + m}\right] (-ie^2)\, \gamma^{\rho}\, (\not q + \not k - m)\gamma^{\sigma} \frac{-i\eta^{t\sigma}}{k^2}$$

$$\times \int_0^1 \frac{d\tau_2}{\Lambda^2} \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{k^2}{\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2}\right.$$

$$\left. - \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2}\right] \qquad (2.3.5a)$$

$$= \quad -i\, e^3 \exp\left[- \frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_1^\infty \frac{d\tau_1}{\Lambda^2} \int_0^1 \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2} \; \gamma^t (\not p + \not k - m)\, \gamma^\rho\, (\not q + \not k - m)\, \gamma_t$$

$$\times \exp\left[-\tau_1 \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} - \tau_3 \frac{k^2}{\Lambda^2} \right] \qquad (2.3.5b)$$

The contribution from fig.2.4(c) also comes from a vertex of the first order and a vertex of the second order. The contribution is,

$$-ie\Lambda_c^\rho (q,p) \quad = -i\, e^3 \exp\left[- \frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{d\tau_1}{\Lambda^2} \int_1^\infty \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2} \; \gamma^t (\not p + \not k - m)\, \gamma^\rho\, (\not q + \not k - m)\, \gamma_t$$

$$\times \exp\left[-\tau_1 \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} - \tau_3 \frac{k^2}{\Lambda^2} \right] \qquad (2.3.6)$$

The contribution from fig.2.4(d) comes from a single vertex of third order. The contribution is,

$$-ie\Lambda_d^\rho (q,p) \quad = -i\, e^3 \exp\left[- \frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{d\tau_1}{\Lambda^2} \int_0^1 \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2} \; \gamma^t (\not p + \not k - m)\, \gamma^\rho\, (\not q + \not k - m)\, \gamma_t$$

$$\times \exp\left[-\tau_1 \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} - \tau_3 \frac{k^2}{\Lambda^2} \right] \qquad (2.3.7)$$

The total contribution is

$$\Lambda^\rho(q,p) = \Lambda_a^\rho(q,p) + \Lambda_b^\rho(q,p) + \Lambda_c^\rho(q,p) + \Lambda_d^\rho(q,p) \qquad (2.3.8)$$

Adding all the four contributions, we get, for $\Lambda^\rho(q,p)$,

$$-ie\Lambda^\rho(q,p) = -i\,e^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2}\right]$$
$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^\infty \frac{d\tau_1}{\Lambda^2} \int_0^\infty \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2}\; \gamma^t (\not{p}+\not{k}-m)\, \gamma^\rho (\not{q}+\not{k}-m)\, \gamma_t$$
$$\times \exp\left[-\tau_1 \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} - \tau_3 \frac{k^2}{\Lambda^2}\right]$$

(2.3.9)

Consider only the exponentials in the second line of (2.3.9). Completing the squares in the exponentials one gets,

$$-ie\Lambda^\rho(q,p) = -i\,e^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2}\right]$$
$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^\infty \frac{d\tau_1}{\Lambda^2} \int_0^\infty \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2}\; \gamma^t (\not{p}+\not{k}-m)\, \gamma^\rho (\not{q}+\not{k}-m)\, \gamma_t$$
$$\times \exp\left[-\frac{\tau_1+\tau_2+\tau_3}{\Lambda^2}\left(k + \frac{\tau_1}{\tau_1+\tau_2+\tau_3}\,p + \frac{\tau_2}{\tau_1+\tau_2+\tau_3}\,q\right)^2\right]$$
$$\times \exp\left[-\frac{\tau_1(\tau_2+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2} - \frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2} - (\tau_1+\tau_2)\frac{m^2}{\Lambda^2}\right.$$
$$\left. + \frac{2\tau_1\tau_2}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}\right] \qquad (2.3.10)$$

Introducing the new integration variable,

$$k \to k - \frac{\tau_1 p + \tau_2 q}{\tau_1 + \tau_2 + \tau_3} \tag{2.3.11}$$

expression (2.3.10) becomes

$$-ie\Lambda^\rho(q,p) = -i\, e^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^\infty \frac{d\tau_1}{\Lambda^2} \int_0^\infty \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2} \exp\left[-(\tau_1+\tau_2+\tau_3)\frac{k^2}{\Lambda^2}\right]$$

$$\times \exp\left[-\frac{\tau_1(\tau_2+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2} - \frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2} - (\tau_1+\tau_2)\frac{m^2}{\Lambda^2}\right.$$

$$\left. + \frac{2\tau_1\tau_2}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}\right]$$

$$\times \gamma^t \left[\not{k} + \frac{(\tau_2+\tau_3)\not{p} - \tau_2\not{q}}{\tau_1+\tau_2+\tau_3} - m\right] \gamma^\rho \left[\not{k} + \frac{(\tau_1+\tau_3)\not{q} - \tau_1\not{p}}{\tau_1+\tau_2+\tau_3} - m\right] \gamma_t \tag{2.3.12}$$

Substitute for τ_1, τ_2 and τ_3 as follows:

$$\tau_1 = \lambda x,\ \tau_2 = \lambda y \text{ and } \tau_3 = \lambda(1 - x - y) \tag{2.3.13}$$

such that $\tau_1 + \tau_2 + \tau_3 = \lambda$. Now, (2.3.12) becomes,

$$-ie\,\Lambda^\rho(q,p) = -ie^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{dx}{\Lambda^2} \int_0^{1-x} \frac{dy}{\Lambda^2} \int_{\frac{1}{1-x-y}}^\infty \frac{d\lambda}{\Lambda^2}\, \lambda^2 \exp\left(-\lambda\frac{k^2}{\Lambda^2}\right)$$

$$\times\ \gamma^t[\not{k} + \not{p}(1-x) - \not{q}y - m]\,\gamma^\rho\,[\not{k} + \not{q}(1-y) - \not{p}x - m]\,\gamma_t$$

$$\times \exp\left[-\frac{\lambda}{\Lambda^2}\left\{x(1-x)\,p^2 + y(1-y)\,q^2 - 2p\,qxy + m^2(x+y)\right\}\right] \tag{2.3.14}$$

Λ^ρ (q, p) can be split into two parts–one term quadratic in k and the other which does not contain k. The term which is quadratic in k is the divergent part whereas the other is convergent.

$$\Lambda^\rho (q, p) = \Lambda_1^\rho (q, p) + \Lambda_2^\rho (q, p), \tag{2.3.15}$$

where Λ_1^ρ (q, p) contains the divergent part and Λ_2^ρ (q, p) the part which is not divergent. For Λ_1^ρ (q, p) we obtain

$$-ie\, \Lambda_1^\rho (q, p) = -\, ie^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right] \int_0^1 \frac{dx}{\Lambda^2}$$

$$\times \int_0^{1-x} \frac{dy}{\Lambda^2} \int_{\frac{1}{1-x-y}}^{\infty} \frac{d\lambda}{\Lambda^2}\, \lambda^2 \int \frac{d^4k}{(2\pi)^4}\, k^2 \,(\gamma^t\gamma^\sigma\gamma^\rho\gamma_\sigma\gamma_t) \exp\left(-\lambda \frac{k^2}{\Lambda^2}\right)$$

$$\times \exp\left[-\frac{\lambda}{\Lambda^2} \left\{ m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy \right\} \right] \tag{2.3.16}$$

For Λ_2^ρ (q, p), we have,

$$-ie\, \Lambda_2^\rho (q, p) = -\, ie^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int_0^1 \frac{dx}{\Lambda^2} \int_0^{1-x} \frac{dy}{\Lambda^2} \int_{\frac{1}{1-x-y}}^{\infty} \frac{d\lambda}{\Lambda^2}\, \lambda^2 \int \frac{d^4k}{(2\pi)^4} \exp\left(-\lambda \frac{k^2}{\Lambda^2}\right)$$

$$\times \gamma^t\, [\not p(1-x) - \not q y - m]\, \gamma^\rho\, [\not q(1-y) - \not p x - m]\, \gamma_t$$

$$\times \exp\left[-\frac{\lambda}{\Lambda^2} \left\{ m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy \right\} \right] \tag{2.3.17}$$

The terms odd in k, vanish on integration with respect to k. Performing the momentum integration, expression (2.3.16) becomes,

$$-ie\,\Lambda_1{}^{\rho}(q,p) = -\frac{ie^3}{8\pi^2}\gamma^{\rho}\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int_0^1 dx\int_0^{1-x} dy\int_{\frac{1}{1-x-y}}^{\infty}\frac{d\lambda}{\lambda}$$

$$\times\exp\left[-\frac{\lambda}{\Lambda^2}\left\{m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy\right\}\right] \tag{2.3.18}$$

Here, we have used the identity

$$\gamma^t\gamma^{\sigma}\gamma^{\rho}\gamma_{\sigma}\gamma_t = 4\gamma^{\rho} \tag{2.3.19}$$

Expression (2.3.18) can be written as

$$-ie\,\Lambda_1{}^{\rho}(q,p) = -\frac{ie^3}{8\pi^2}\gamma^{\rho}\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int_0^1 dx\int_0^{1-x} dy\; E_1\left[\frac{m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy}{\Lambda^2(1-x-y)}\right] \tag{2.3.20}$$

where $E_1(z)$ is the exponential integral,

$$E_1(z) = \int_z^{\infty} dt\,\frac{\exp(-t)}{t} = -\ln z - \gamma - \sum_{n=1}^{\infty}\frac{(-z)^n}{n\,n!} \tag{2.3.21}$$

Although the final parameter integral cannot be evaluated in terms of elementary functions, we can develop an asymptotic expansion in Λ by expanding the exponential integral :

$$-ie\,\Lambda_1{}^{\rho}(q,p) = -\frac{ie^3}{8\pi^2}\gamma^{\rho}\int_0^1 dx\int_0^{1-x} dy$$

$$\times\left[-\ln\left\{\frac{m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy}{\Lambda^2(1-x-y)}\right\}-\gamma\right] \tag{2.3.22a}$$

$$= -\frac{ie^3}{16\pi^2}\gamma^\rho\left[\ln(\Lambda^2) - \gamma + \frac{3}{2}\right] + \frac{ie^3}{8\pi^2}\gamma^\rho\int_0^1 dx\int_0^{1-x} dy$$

$$\times \ln\ [m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy] \qquad (2.3.22b)$$

For Λ_2^ρ (q, p), we obtain in the limit $\Lambda \to \infty$ the following expression:

$$-ie\,\Lambda_2^\rho(q,p) = -\frac{ie^3}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\gamma^t[\not p(1-x)-\not q y - m]\gamma^\rho$$

$$\times [\not q(1-y)-\not p x - m]\gamma_t \int_{\frac{1}{1-x-y}}^{\infty}\frac{d\lambda}{\Lambda^2}$$

$$\times \exp\left[-\frac{\lambda}{\Lambda^2}\left\{m^2(x+y) + p^2x(1-x) + q^2y(1-y) - 2\,p.qxy\right\}\right]$$

(2.3.23a)

$$= -\frac{ie^3}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{\gamma^t[\not p(1-x)-\not q y-m]\gamma^\rho[\not q(1-y)-\not p x-m]\gamma_t}{[m^2(x+y)+p^2x(1-x)+q^2y(1-y) - 2\,p.qxy]}$$

(2.3.23b)

We compare our result with the expression for the vertex obtained under dimensional regularization [66].

$$-ie\,\Lambda_1^\rho(q,p) = -\frac{ie^3}{16\pi^2}\gamma^\rho\left[\frac{1}{\epsilon} - \gamma - \ln 4\pi\mu^2\right] + \frac{ie^3}{8\pi^2}\gamma^\rho\int_0^1 dx\int_0^{1-x} dy$$

$$\times \ln\ [m^2(x+y) + p^2x(1-x) + q^2y(1-y) - 2\,p.qxy]$$

(2.3.24)

and

$$-ie\,\Lambda_2^\rho(q,p) = -\frac{ie^3}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{\gamma^t[\not p(1-x)-\not q y-m]\gamma^\rho[\not q(1-y)-\not p x-m]\gamma_t}{[m^2(x+y)+p^2x(1-x)+q^2y(1-y) - 2\,p.qxy]}$$

(2.3.25)

The two expressions agree if for the divergent parts, we use the correspondence:

$$\ln \Lambda^2 \sim \frac{1}{\epsilon} \qquad (2.3.26)$$

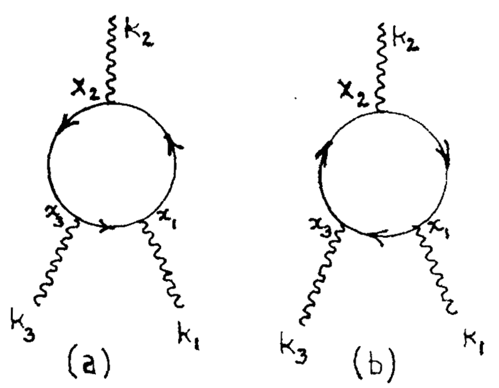
as derived previously by Evens etal [28] for self–energy and vacuum polarization.

2.4 Furry's Theorem

Furry's theorem [69 - 71] states that the Feynman diagrams containing a closed fermion loop with an odd number of photon vertices can be omitted in the calculation of physical processes. In a closed loop there can be an electron as well as a positron circling around. If the number of photon vertices is even, then the two contributions just get added.

Proof:

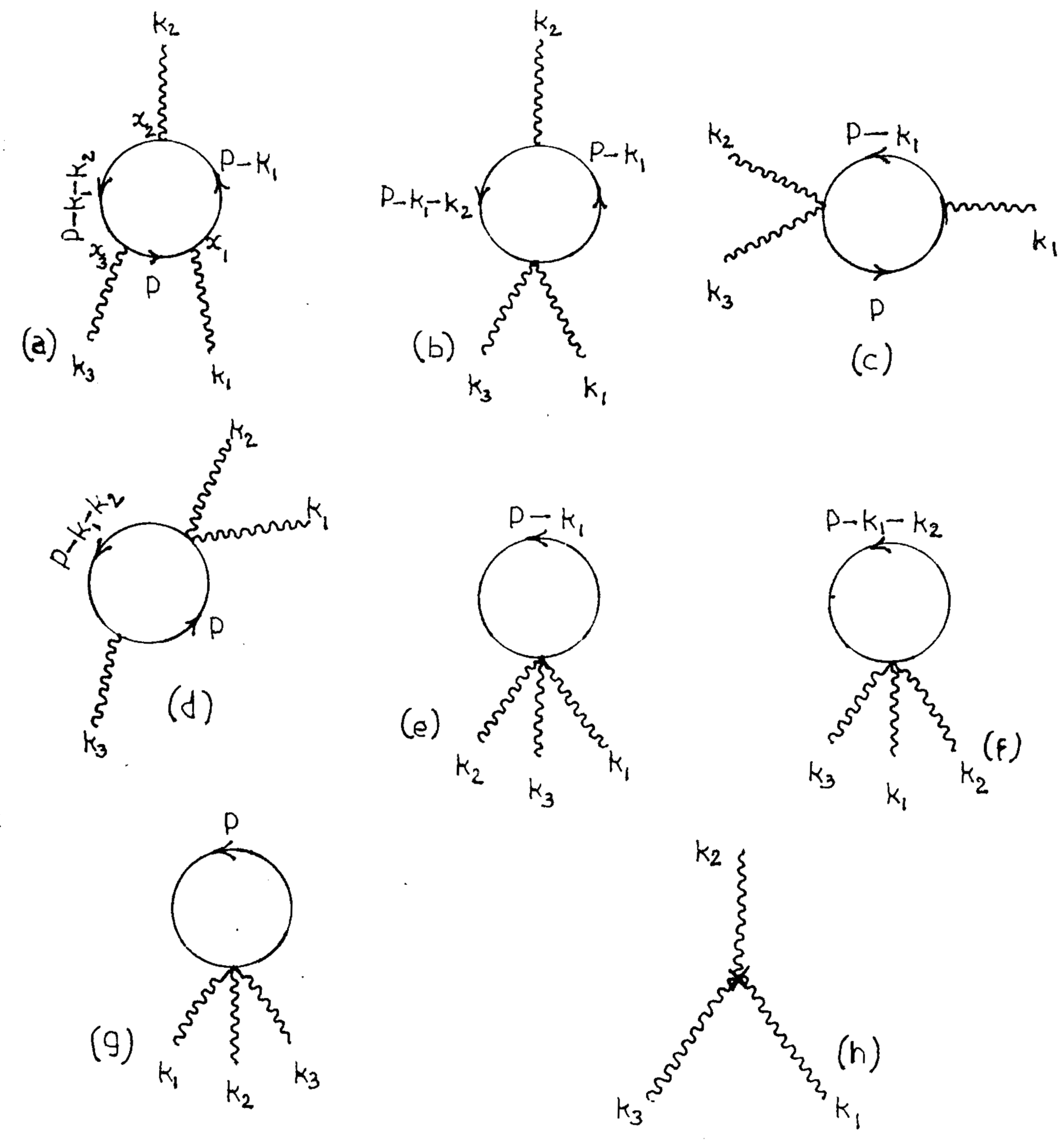
Consider a process that can be described by a graph containing an electron loop with three vertices. In QED the two graphs corresponding to the process are



Two graphs with opposite directions of the internal fermion loop.

Fig.2.5

But in nonlocal QED, there are eight diagrams corresponding to each one of these graphs. They are given in figs.2.6 and 2.7 respectively.



Nonlocal graphs corresponding to fig. 2.5(a)

Fig.2.6

The relevant contribution to the S – matrix element describing the loops 2.6(a) to (g) are

$$M_{2.6a} = \mathrm{Tr}\left[(-ie\,\gamma_{\mu1})\frac{i}{\not p+m-i\epsilon}(-ie\,\gamma_{\mu3})\frac{i}{\not p-\not k_1-\not k_2+m-i\epsilon}(-ie\,\gamma_{\mu2})\frac{i}{\not p-\not k_1+m-i\epsilon}\right] \quad (2.4.1a)$$

$$M_{2.6b} = \mathrm{Tr}\left[(-ie^2\,\gamma_{\mu1})\,[\not p-m]\,(\gamma_{\mu3})\frac{i}{\not p-\not k_1-\not k_2+m-i\epsilon}(-ie\,\gamma_{\mu2})\frac{i}{\not p-\not k_1+m-i\epsilon}\right] \quad (2.4.1b)$$

$$M_{2.6c} = \mathrm{Tr}\left[(-ie\,\gamma_{\mu1})\frac{i}{\not p+m-i\epsilon}(-ie^2\,\gamma_{\mu3})\,[\not p-\not k_1-\not k_2-m](\gamma_{\mu2})\frac{i}{\not p-\not k_1+m-i\epsilon}\right] \quad (2.4.1c)$$

$$M_{2.6d} = \mathrm{Tr}\left[(-ie^2\,\gamma_{\mu1})\frac{i}{\not p+m-i\epsilon}(-ie\,\gamma_{\mu3})\frac{i}{\not p-\not k_1-\not k_2+m-i\epsilon}(\gamma_{\mu2})\,[\not p-\not k_1-m]\right] \quad (2.4.1d)$$

$$M_{2.6e} = \mathrm{Tr}\left[(-ie^3\,\gamma_{\mu1})\,[\not p-m]\,(\gamma_{\mu3})\,[\not p-\not k_1-\not k_2-m](\gamma_{\mu2})\frac{i}{\not p-\not k_1+m-i\epsilon}\right] \quad (2.4.1e)$$

$$M_{2.6f} = \mathrm{Tr}\left[(-ie^3\,\gamma_{\mu1})\,[\not p-m]\,(\gamma_{\mu3})\frac{i}{\not p-\not k_1-\not k_2+m-i\epsilon}(\gamma_{\mu2})\,[\not p-\not k_1-m]\right] \quad (2.4.1f)$$

$$M_{2.6g} = \mathrm{Tr}\left[(-ie^3\,\gamma_{\mu1})\frac{i}{\not p+m-i\epsilon}(\gamma_{\mu3})\,[\not p-\not k_1-\not k_2-m]\,(\gamma_{\mu2})\,[\not p-\not k_1-m]\right] \quad (2.4.1g)$$

Fig.2.6 (h) is the measure factor contribution. The measure factor absorbs the noninvariance due to the fermion measures. The measure factor $\mu[e\,A]$ is defined as follows:

$$\mu[e\,A] = \exp\left(S_{meas}\,[e\,A]\right), \quad (2.4.2a)$$

$$\partial_\mu \theta \quad \frac{\delta S_{meas}[eA]}{\delta A_\mu} = -\, e\, \mathrm{Tr}\, (\theta . \mathscr{T}\, [eA] + e\mathrm{Tr}\, (\theta . \bar{\mathscr{T}}[eA]) \qquad (2.4.2b)$$

In the case of odd number of photon vertices, the two contributions given in (2.4.2b) cancel. Consider the order e^3 contribution to $e\,\mathrm{Tr}\,(\theta . \bar{\mathscr{T}})$:

$$e^3\ \mathrm{Tr}\, \{\varepsilon_m\, [\not{A}^\Lambda \vartheta\, (i\not{\partial} - m)\, \not{A}^\Lambda \vartheta (\, i\not{\partial} - m)]\, \theta^\Lambda \varepsilon_m\}$$

$$= e^3\ \mathrm{Tr}\, \{\varepsilon_m\, [C^{-1} C \not{A}^\Lambda\, C^{-1} C\, \vartheta\, (i\not{\partial} - m)\, C^{-1} C \not{A}^\Lambda\, C^{-1} C\, \vartheta\, (i\not{\partial} - m)\, C^{-1} C]\, \theta^\Lambda \varepsilon_m\} \qquad (2.4.3)$$

Here C is the charge – conjugation matrix:

$$C\, \gamma_\mu\, C^{-1} = -\ \gamma_\mu{}^T \qquad (2.4.4)$$

Hence (2.4.3) becomes,

$$e^3\, \mathrm{Tr}\, \{\varepsilon_m\, [\not{A}^\Lambda \vartheta\, (i\not{\partial} - m)\, \not{A}^\Lambda \vartheta (\, i\partial - m)]\, \theta^\Lambda \varepsilon_m\}$$

$$= e^3\, \mathrm{Tr}\, \{\varepsilon_m\, [-(\not{A}^\Lambda)^T] \vartheta\, (\overleftarrow{i\not{\partial}} - m)^T\, [-(\not{A}^\Lambda)^T] \vartheta (\,\overleftarrow{i\not{\partial}} - m)^T]\, \theta^\Lambda \varepsilon_m\}. \qquad (2.4.5a)$$

$$= e^3\, \mathrm{Tr}\, \{\varepsilon_m\, \theta^\Lambda\, (i\not{\partial} - m)\, \vartheta \not{A}^\Lambda\, (i\not{\partial} - m)\, \vartheta \not{A}^\Lambda\, \varepsilon_m\}^T \qquad (2.4.5b)$$

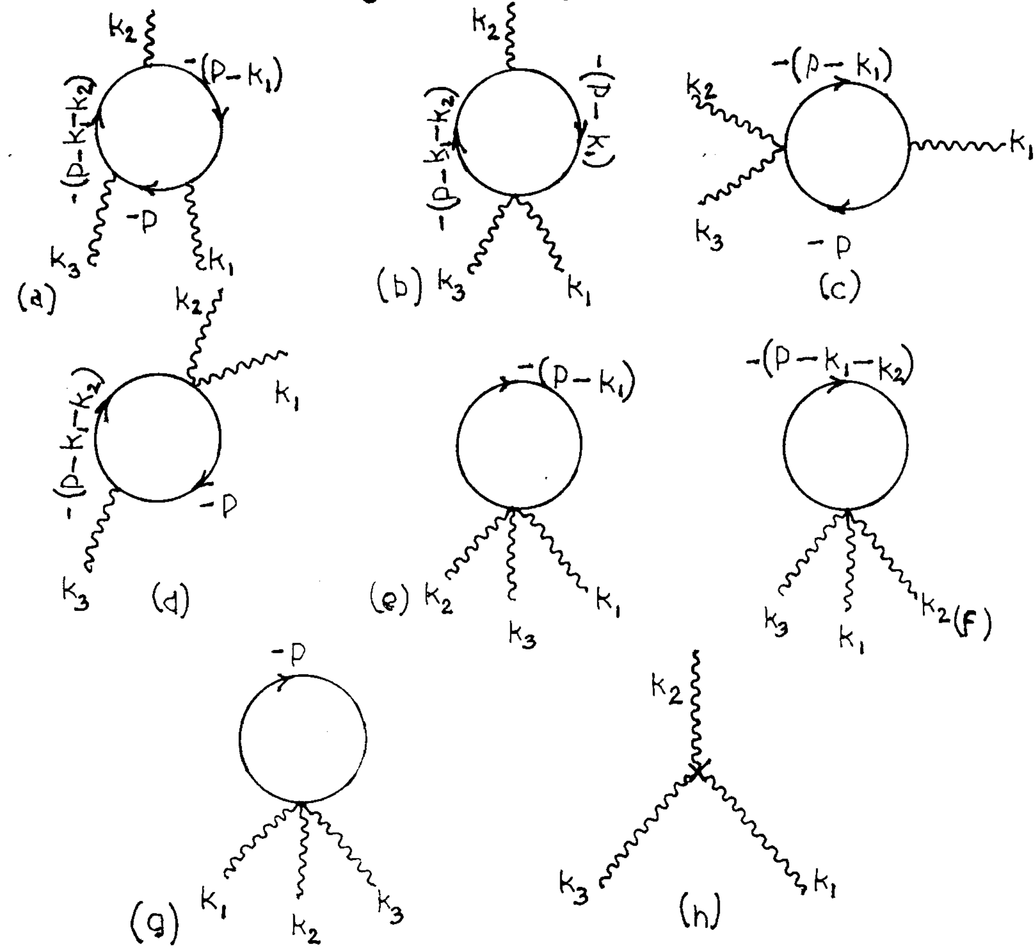
But $\mathrm{Tr}\, \{(AB)^T\} = \mathrm{Tr}\, \{AB\}$

$$e^3\, \mathrm{Tr}\, \{\varepsilon_m\, [\not{A}^\Lambda \vartheta\, (i\not{\partial} - m) \not{A}^\Lambda \vartheta\, (i\not{\partial} - m)] \theta^\Lambda \varepsilon_m\}$$

$$= e^3\, \mathrm{Tr}\, \{\varepsilon_m \theta^\Lambda\, (i\not{\partial} - m)\, \vartheta \not{A}^\Lambda\, (i\not{\partial} - m)\, \vartheta \not{A}^\Lambda\, \varepsilon_m\} \qquad (2.4.6)$$

The e^3 contribution to $-e\mathrm{Tr}(\theta . \mathscr{T})$ is the same as (2.4.6) with a negative sign. Hence for a diagram with odd number of photon vertices (2.4.6) cancels with the analogous contribution from $-e\mathrm{Tr}\, (\theta . \mathscr{T})$, while for even number of photon vertices, they add. So in the case of a fermion loop with odd number of photon vertices, the contribution from the measure factor is zero.

The nonlocal diagrams corresponding to fig. 2.5(b) are:



Nonlocal graphs corresponding to fig. 2.5(b)

Fig.2.7.

The relevant contribution to the S-matrix element describing the loops 2.7(a) to (g) are :

$$M_{2.7a} = Tr\left[(-ie\gamma_{\mu 1})\frac{i}{-(\not{p}-\not{k}_1)+m-i\epsilon}(-ie\gamma_{\mu 2})\frac{i}{-(\not{p}-\not{k}_1-\not{k}_2)+m-i\epsilon}(-ie\gamma_{\mu 3})\frac{i}{-\not{p}+m-i\epsilon}\right] \quad (2.4.7a)$$

$$M_{2.7b} = Tr\left[(-ie^2\gamma_{\mu 1})\frac{i}{-(\not{p}-\not{k}_1)+m-i\epsilon}(-ie\gamma_{\mu 2})\frac{i}{-(\not{p}-\not{k}_1-\not{k}_2)+m-i\epsilon}(\gamma_{\mu 3})[-\not{p}-m]\right] \quad (2.4.7b)$$

$$M_{2.7c} = Tr\left[(-ie\gamma_{\mu 1})\frac{i}{-(\not{p}-\not{k}_1)+m-i\epsilon}(-ie^2\gamma_{\mu 2})[-(\not{p}-\not{k}_1-\not{k}_2)-m](\gamma_{\mu 3})\frac{i}{-\not{p}+m-i\epsilon}\right] \quad (2.4.7c)$$

$$M_{2.7d} = \mathrm{Tr}\left[(-ie^2\gamma_{\mu 1})[-(\not{p}-\not{k}_1)-m]\,(\gamma_{\mu 2})\frac{i}{-(\not{p}-\not{k}_1-\not{k}_2)+m-i\epsilon}(-ie\gamma_{\mu 3})\frac{i}{-\not{p}+m-i\epsilon}\right] \quad (2.4.7d)$$

$$M_{2.7e} = \mathrm{Tr}\left[(-ie^3\gamma_{\mu 1})\frac{i}{-(\not{p}-\not{k}_1)+m-i\epsilon}(\gamma_{\mu 2})\,[-(\not{p}-\not{k}_1-\not{k}_2)-m]\,(\gamma_{\mu 3})\,[-\not{p}-m]\right] \quad (2.4.7e)$$

$$M_{2.7f} = \mathrm{Tr}\left[(-ie^3\gamma_{\mu 1})\,[-(\not{p}-\not{k}_1)-m]\,(\gamma_{\mu 2})\frac{i}{-(\not{p}-\not{k}_1-\not{k}_2)+m-i\epsilon}(\gamma_{\mu 3})\,[-\not{p}-m]\right] \quad (2.4.7f)$$

$$M_{2.7g} = \mathrm{Tr}\left[(-ie^3\gamma_{\mu 1})\,[-(\not{p}-\not{k}_1)-m](\gamma_{\mu 2})\,[-(\not{p}-\not{k}_1-\not{k}_2)+m-i\epsilon]\;(\gamma_{\mu 3})\frac{i}{-\not{p}+m-i\epsilon}\right]$$

(2.4.7g)

As before the measure factor contribution vanishes. Now insert factors of $C^{-1}C = I$ in (2.4.7a).

$$M_{2.7a} = \mathrm{Tr}\left[C^{-1}C\,(-ie\gamma_{\mu 1})\,C^{-1}C\frac{i}{-(\not{p}-\not{k}_1)+m-i\epsilon}C^{-1}C\,(-ie\gamma_{\mu 2})\,C^{-1}C\right.$$

$$\left.\frac{i}{-(\not{p}-\not{k}_1-\not{k}_2)+m-i\epsilon}C^{-1}C\,(-ie\gamma_{\mu 3})\,C^{-1}C\frac{i}{-\not{p}+m-i\epsilon}C^{-1}C\right] \quad (2.4.8)$$

$$\text{Now,}\quad C\frac{1}{-(\not{p}-k_1)+m-i\epsilon}C^{-1} = C\frac{-(\not{p}-\not{k}_1)-m}{(p-k_1)^2+m^2-i\epsilon}C^{-1} \quad (2.4.9a)$$

$$= C\frac{-\gamma_\mu\,(p^\mu-k_1^\mu)-m}{(p-k_1)^2+m^2-i\epsilon}C^{-1} \quad (2.4.9b)$$

But $C\gamma_\mu C^{-1} = -\gamma_\mu^T$.

$$\therefore C\frac{1}{-(\not{p}-\not{k}_1)+m-i\epsilon}C^{-1} = \frac{\gamma_\mu^T\,(p^\mu-k_1^\mu)-m}{(p-k_1)^2+m^2-i\epsilon} \quad (2.4.10)$$

Hence (2.4.8) becomes,

$$M_{2.7a} = (-1)^3\, Tr\left[(-ie\gamma_{\mu 1}{}^T) \frac{i\gamma_\mu{}^T(p^\mu - k_1{}^\mu) - m}{(p-k_1)^2 + m^2 - i\epsilon} (-ie\gamma_{\mu 2}{}^T) \frac{i\gamma_\mu{}^T(p^\mu - k_1{}^\mu - k_2{}^\mu) - m}{(p-k_1-k_2)^2 + m^2 - i\epsilon} \right.$$

$$\left. \times\ (-ie\gamma_{\mu 3}{}^T) \frac{i\gamma_\mu{}^T(p^\mu - m)}{(p^2 + m^2 - i\epsilon)} \right] \qquad (2.4.11a)$$

$$= (-1)^3\, Tr\left[\frac{(i)\not{p} - m}{(p^2 + m^2 - i\epsilon)} (-ie\gamma_{\mu 3}) \frac{(i)\not{p} - \not{k}_1 - \not{k}_2 - m}{(p-k_1-k_2)^2 + m^2 - i\epsilon} (-ie\gamma_{\mu 2}) \right.$$

$$\left. \times\ \frac{(i)\not{p} - \not{k}_1 - m}{(p-k_1)^2 + m^2 - i\epsilon} (-ie\gamma_{\mu 1}) \right]^T \qquad (2.4.11b)$$

$$= (-1)^3\, Tr\left[\frac{i}{\not{p} + m - i\epsilon} (-ie\gamma_{\mu 3}) \frac{i}{\not{p} - \not{k}_1 - \not{k}_2 + m - i\epsilon} (-ie\gamma_{\mu 2}) \frac{i}{\not{p} - \not{k}_1 + m - i\epsilon} (-ie\gamma_{\mu 1}) \right] \qquad (2.4.11c)$$

since trace is invariant under transportation ·

So, $M_{2.7a} = (-1)^3 M_{2.6a}$ or, $M_{2.7a} + M_{2.6a} = 0$ (2.4.11d)

Similarly for fig.2.7(b),

$$M_{2.7b} = Tr\left[C^{-1}C\ (-ie^2\gamma_{\mu 1})\ C^{-1}C \frac{i}{-(\not{p} - \not{k}_1) + m - i\epsilon} C^{-1} C\ (-ie\gamma_{\mu 2})\ C^{-1}C \right.$$

$$\left. \frac{i}{-(\not{p} - \not{k}_1 - \not{k}_2) + m - i\epsilon} C^{-1}C\ (\gamma_{\mu 3})\ C^{-1}C\ [-\not{p} - m]\ C^{-1}C \right] \qquad (2.4.12a)$$

$$= (-1)^3 \left[Tr\ \ (-ie^2\gamma_{\mu 1}{}^T) \frac{i\{(\not{p} - \not{k}_1)^T - m\}}{(p-k_1)^2 + m^2 - i\epsilon} (-ie\gamma_{\mu 2}{}^T) \frac{i\{(\not{p} - \not{k}_1 - \not{k}_2)^T - m\}}{(p-k_1-k_2)^2 + m^2 - i\epsilon} \right.$$

$$\left. (\gamma_{\mu 3}{}^T)\ \ [\not{p}^T - m] \right] \qquad (2.4.12b)$$

$$= (-1)^3\ \ Tr\left[[\not{p} - m]\ (\gamma_{\mu 3}) \frac{i\{(\not{p} - \not{k}_1 - \not{k}_2) - m\}}{(p-k_1-k_2)^2 + m^2 - i\epsilon} (-ie\gamma_{\mu 2}) \right.$$

$$\left. \times \frac{i\{(\not{p} - \not{k}_1) - m\}}{(p-k_1)^2 + m^2 - i\epsilon} (-ie^2\gamma_{\mu 1}) \right]^T \qquad (2.4.12c)$$

$$= (-1)^3 \mathrm{Tr}\left[(\not{p}-m)(\gamma_{\mu 3}) \frac{i}{\not{p}-\not{k}_1-\not{k}_2+m-i\epsilon} (-ie\gamma_{\mu 2}) \frac{i}{\not{p}-\not{k}_1+m-i\epsilon} (-ie^2\gamma_{\mu 1}) \right] \tag{2.4.12d}$$

$$= (-1)^3 M_{2.6b}$$

Therefore, $M_{2.7b} + M_{2.6b} = 0$ (2.4.12e)

Similarly for the other diagrams also thus proving the Furry's theorem.

2.5 Discussion

The technique of nonlocal regularization [28] is applied to evaluate the QED vertex part [68]. Corresponding to the single Feynman diagram of local QED, there are four Feynman diagrams in nonlocal QED, which arises from the consideration of two kinds of propagators differing only in the limits of integration. The result predicted by the nonlocal theory is exactly the same as that given by the dimensional regularization method [66].The divergent parts in the two methods may be equated using $\ln(\Lambda^2) \sim \frac{2}{4-D}$ where D is the number of dimensions. The finite parts are the same apart from trivial numerical constants.

The Furry's theorem is also proved for nonlocal QED, by considering a particular example of a closed fermion loop with three photon vertices. Here, there will be eight Feynman diagrams in nonlocal QED, corresponding to the single one in local QED. There can be an electron as well as a positron circling around in a closed loop. The contribution from one loop can be seen to be cancelled by the contribution from the other loop, thus enabling to omit the closed fermion loops with an odd number of photon vertices while evaluating the physical processes .

QED Vertex Part in Nonlocal Regularization

P. C. Raje Bhageerathi "Regularizations and divergent diagram in gauge theories" Thesis. Department of Physics, University of Calicut, 1999

Chapter 2

QED Vertex Part In Nonlocal Regularization

2.1 Introduction

The method of nonlocal regularization has been developed by Evens etal [28]. They have applied it to obtain the QED electron self-energy and vacuum polarization, and they have compared their result with that obtained using dimensional regularization [66]. We have extended the method of nonlocal regularization to evaluate the QED vertex part [68]. Here also, the result obtained agrees perfectly with that of the result of dimensional regularization. The divergent parts may be equated using

$$\ln \Lambda^2 = \frac{1}{\epsilon} .$$

The finite parts are the same apart from trivial numerical constants.

The Furry's theorem [69 - 71] is also proved for nonlocal QED. The content of the Furry's theorem is that Feynman diagrams containing a closed fermion loop with an odd number of photon vertices can be omitted in the calculation of physical processes.

2.2 Feynman rules for nonlocal QED

In the nonlocal theory, first a smearing operator ε_m is defined :

$$\varepsilon_m \equiv \exp\left(\frac{\partial^2 - m^2}{2\Lambda^2} \right) \qquad (2.2.1)$$

From this operator another operator ϑ is defined :

$$\vartheta \equiv \frac{\varepsilon_m^2 - 1}{\partial^2 - m^2} = \int_0^1 \frac{d\tau}{\Lambda^2} \exp\left(\tau \frac{\partial^2 - m^2}{\Lambda^2}\right) \tag{2.2.2}$$

Using this operator the Lagrangian for the higher interactions can be expressed as

$$\mathcal{L}_n = -(-e)^n \bar{\psi}^{\Lambda} A\!\!\!/^{\Lambda} [(i\partial\!\!\!/ - m)\, \vartheta A\!\!\!/^{\Lambda}]^{n-1} \psi^{\Lambda} \tag{2.2.3}$$

The total Lagrangian then has the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}(i\partial\!\!\!/ + m)\psi + e\bar{\psi}^{\Lambda} A\!\!\!/^{\Lambda} [1 + e(i\partial\!\!\!/ - m)\,\vartheta A\!\!\!/^{\Lambda}]^{-1} \psi^{\Lambda} \tag{2.2.4}$$

The nonlocalized Feynman rules for any theory are a trivial extension of the local ones [16, 72 – 74]. There are two kinds of propagators —— an unbarred one and a barred one. The second one can be expressed diagrammatically either as a propagator with a bar or by just joining the two vertices [28]. Here the latter convention is adopted. The unbarred electron propagator is expressed as,

$$\frac{-i\varepsilon^2}{p^2 + m^2 - i\epsilon} = -i\int_1^{\infty} \frac{d\tau}{\Lambda^2} \exp\left(-\tau \frac{p^2 + m^2}{\Lambda^2}\right) \tag{2.2.5a}$$

and the barred one as,

$$-i\frac{\varepsilon^2 - 1}{p^2 + m^2 - i\epsilon} = -i\int_0^1 \frac{d\tau}{\Lambda^2} \exp\left(-\tau \frac{p^2 + m^2}{\Lambda^2}\right) \tag{2.2.5b}$$

where ε is as given in expression (2.2.1).

Each vertex is associated with a factor $(-ie\gamma^{\mu})$.

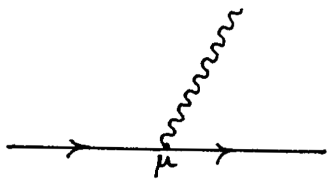
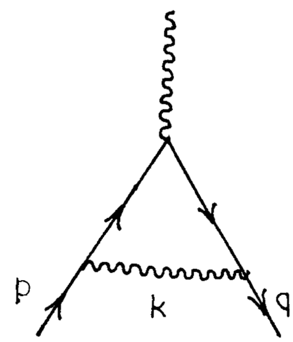


Fig .2.1

A nonlocal vertex of order e^n is associated with a factor $-ie^n$.

2.3 The QED vertex part in nonlocal regularization

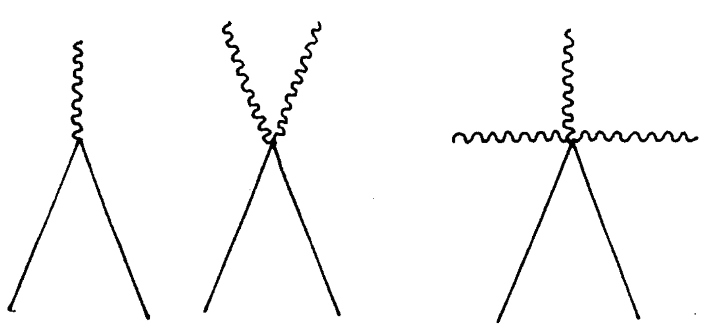
The diagram corresponding to the vertex part in QED is as follows:



QED vertex graph

Fig.2.2

But in nonlocal QED, there are four diagrams contributing to the vertex part [68]. The types of vertices which contribute to the vertex part in nonlocal QED to order e^2 are shown in fig.2.3.



First, second and third order vertices in nonlocal regularization.

Fig.2.3.

The vertices in figs.2.3(a) to (c) come from the interaction part of the Lagrangian :

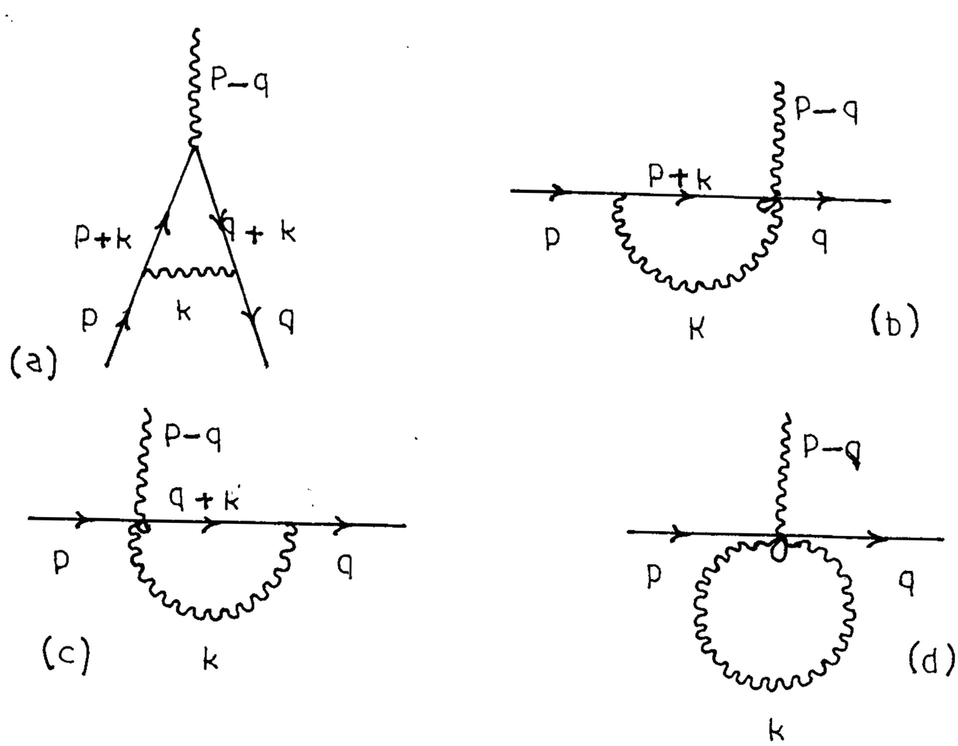
$$\mathcal{L}_n = -(-e)^n\, \bar{\psi}^\Lambda \not{A}^\Lambda \left[(i\not{\partial}-m)\, \vartheta \not{A}^\Lambda\right]^{n-1} \psi^\Lambda \qquad (2.3.1)$$

where n = 1,2,3..................

When n =1 , this reduces to the usual interacting Lagrangian for QED,

$$\mathcal{L}_1 = e\, \bar{\psi}^\Lambda \not{A}^\Lambda \psi^\Lambda \; . \qquad (2.3.2)$$

The four diagrams which contribute to the vertex part in nonlocal regularization are given in fig.2.4.



The four diagrams contributing to the QED vertex part in nonlocal regularization.

Fig.2.4.

The contribution from fig.2.4(a) is

$$-ie\Lambda_a^{\rho}(q,p) \equiv \int \frac{d^4k}{(2\pi)^4} (ie\gamma^t) \left[\frac{-i}{\not p + \not k + m} \right] (ie\gamma^{\rho}) \left[\frac{-i}{\not q + \not k + m} \right] (ie\gamma^{\sigma}) \frac{-i\eta^{t\sigma}}{k^2}$$

$$\times \exp \left[- \frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{k^2}{\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} - \frac{(p+k)^2+m^2}{\Lambda^2} - \frac{(q+k)^2+m^2}{\Lambda^2} \right] \quad (2.3.3)$$

Promoting the propagators to Schwinger integrals, one obtains,

$$-ie\Lambda_a^{\rho}(q,p) = -i e^3 \exp \left[- \frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_1^{\infty} \frac{d\tau_1}{\Lambda^2} \int_1^{\infty} \frac{d\tau_2}{\Lambda^2} \int_1^{\infty} \frac{d\tau_3}{\Lambda^2} \gamma^t (\not p + \not k - m) \gamma^{\rho} (\not q + \not k - m) \gamma_t$$

$$\times \exp \left[-\tau_1 \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} - \tau_3 \frac{k^2}{\Lambda^2} \right] \quad (2.3.4)$$

The factor of i comes from the rotation to the Euclidean space. The contribution from fig.2.4(b) comes from a vertex of the first order and a vertex of the second order. The contribution is,

$$-ie\Lambda_b^{\rho}(q,p) \equiv \int \frac{d^4k}{(2\pi)^4} (ie\gamma^t) \left[\frac{-i}{\not p + \not k + m} \right] (-ie^2) \gamma^{\rho} (\not q + \not k - m) \gamma^{\sigma} \frac{-i\eta^{t\sigma}}{k^2}$$

$$\times \int_0^1 \frac{d\tau_2}{\Lambda^2} \exp \left[- \frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{k^2}{\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} - \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} \right] \quad (2.3.5a)$$

$$= -i\, e^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_1^\infty \frac{d\tau_1}{\Lambda^2} \int_0^1 \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2} \; \gamma^t (\not{p}+\not{k}-m)\, \gamma^\rho\, (\not{q}+\not{k}-m)\, \gamma_t$$

$$\times \exp\left[-\tau_1 \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} - \tau_3 \frac{k^2}{\Lambda^2} \right] \qquad (2.3.5b)$$

The contribution from fig.2.4(c) also comes from a vertex of the first order and a vertex of the second order. The contribution is,

$$-ie\Lambda_c^\rho(q,p) = -i\, e^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{d\tau_1}{\Lambda^2} \int_1^\infty \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2} \; \gamma^t (\not{p}+\not{k}-m)\, \gamma^\rho\, (\not{q}+\not{k}-m)\, \gamma_t$$

$$\times \exp\left[-\tau_1 \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} - \tau_3 \frac{k^2}{\Lambda^2} \right] \qquad (2.3.6)$$

The contribution from fig.2.4(d) comes from a single vertex of third order. The contribution is,

$$-ie\Lambda_d^\rho(q,p) = -i\, e^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{d\tau_1}{\Lambda^2} \int_0^1 \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2} \; \gamma^t (\not{p}+\not{k}-m)\, \gamma^\rho\, (\not{q}+\not{k}-m)\, \gamma_t$$

$$\times \exp\left[-\tau_1 \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} - \tau_3 \frac{k^2}{\Lambda^2} \right] \qquad (2.3.7)$$

The total contribution is

$$\Lambda^{\rho}(q, p) = \Lambda_a^{\rho}(q, p) + \Lambda_b^{\rho}(q, p) + \Lambda_c^{\rho}(q, p) + \Lambda_d^{\rho}(q, p) \qquad (2.3.8)$$

Adding all the four contributions, we get, for $\Lambda^{\rho}(q, p)$,

$$-ie\Lambda^{\rho}(q,p) = -i\, e^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^{\infty} \frac{d\tau_1}{\Lambda^2} \int_0^{\infty} \frac{d\tau_2}{\Lambda^2} \int_1^{\infty} \frac{d\tau_3}{\Lambda^2}\ \gamma^t (\not{p}+\not{k}-m)\, \gamma^{\rho} (\not{q}+\not{k}-m)\, \gamma_t$$

$$\times \exp\left[-\tau_1 \frac{(p+k)^2+m^2}{\Lambda^2} - \tau_2 \frac{(q+k)^2+m^2}{\Lambda^2} - \tau_3 \frac{k^2}{\Lambda^2} \right]$$

(2.3.9)

Consider only the exponentials in the second line of (2.3.9). Completing the squares in the exponentials one gets,

$$-ie\Lambda^{\rho}(q,p) = -i\, e^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^{\infty} \frac{d\tau_1}{\Lambda^2} \int_0^{\infty} \frac{d\tau_2}{\Lambda^2} \int_1^{\infty} \frac{d\tau_3}{\Lambda^2}\ \gamma^t (\not{p}+\not{k}-m)\, \gamma^{\rho} (\not{q}+\not{k}-m)\, \gamma_t$$

$$\times \exp\left[-\frac{\tau_1+\tau_2+\tau_3}{\Lambda^2} \left(k + \frac{\tau_1}{\tau_1+\tau_2+\tau_3} p + \frac{\tau_2}{\tau_1+\tau_2+\tau_3} q \right)^2 \right]$$

$$\times \exp\left[-\frac{\tau_1(\tau_2+\tau_3)}{\tau_1+\tau_2+\tau_3} \frac{p^2}{\Lambda^2} - \frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3} \frac{q^2}{\Lambda^2} - (\tau_1+\tau_2) \frac{m^2}{\Lambda^2} \right.$$

$$\left. + \frac{2\tau_1\tau_2}{\tau_1+\tau_2+\tau_3} \frac{p.q}{\Lambda^2} \right] \qquad (2.3.10)$$

Introducing the new integration variable,

$$k \to k - \frac{\tau_1 p + \tau_2 q}{\tau_1 + \tau_2 + \tau_3} \tag{2.3.11}$$

expression (2.3.10) becomes

$$-ie\Lambda^{\rho}(q,p) = -i e^3 \exp\left[- \frac{p^2 + m^2}{2\Lambda^2} - \frac{q^2 + m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^\infty \frac{d\tau_1}{\Lambda^2} \int_0^\infty \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2} \exp\left[- (\tau_1+\tau_2+\tau_3) \frac{k^2}{\Lambda^2} \right]$$

$$\times \exp\left[- \frac{\tau_1(\tau_2+\tau_3)}{\tau_1+\tau_2+\tau_3} \frac{p^2}{\Lambda^2} - \frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3} \frac{q^2}{\Lambda^2} - (\tau_1+\tau_2) \frac{m^2}{\Lambda^2} \right.$$

$$\left. + \frac{2\tau_1\tau_2}{\tau_1+\tau_2+\tau_3} \frac{p.q}{\Lambda^2} \right]$$

$$\times \gamma^t \left[\not{k} + \frac{(\tau_2+\tau_3)\not{p} - \tau_2 \not{q}}{\tau_1+\tau_2+\tau_3} - m \right] \gamma^{\rho} \left[\not{k} + \frac{(\tau_1+\tau_3)\not{q} - \tau_1 \not{p}}{\tau_1+\tau_2+\tau_3} - m \right] \gamma_t \tag{2.3.12}$$

Substitute for τ_1, τ_2 and τ_3 as follows:

$$\tau_1 = \lambda x,\ \tau_2 = \lambda y \text{ and } \tau_3 = \lambda(1 - x - y) \tag{2.3.13}$$

such that $\tau_1 + \tau_2 + \tau_3 = \lambda$. Now, (2.3.12) becomes,

$$-ie\,\Lambda^{\rho}(q, p) = - ie^3 \exp\left[- \frac{p^2 + m^2}{2\Lambda^2} - \frac{q^2 + m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right]$$

$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{dx}{\Lambda^2} \int_0^{1-x} \frac{dy}{\Lambda^2} \int_{\frac{1}{1-x-y}}^\infty \frac{d\lambda}{\Lambda^2} \lambda^2 \exp\left(- \lambda \frac{k^2}{\Lambda^2} \right)$$

$$\times \gamma^t [\not{k} + \not{p}(1 - x) - \not{q}y - m] \gamma^{\rho} [\not{k} + \not{q}(1 - y) - \not{p}x - m] \gamma_t$$

$$\times \exp\left[- \frac{\lambda}{\Lambda^2} \left\{ x(1-x)p^2 + y(1-y)q^2 - 2p\,qxy + m^2(x+y) \right\} \right] \tag{2.3.14}$$

Λ^ρ (q, p) can be split into two parts–one term quadratic in k and the other which does not contain k. The term which is quadratic in k is the divergent part whereas the other is convergent.

$$\Lambda^\rho (q, p) = \Lambda_1{}^\rho (q, p) + \Lambda_2{}^\rho (q, p), \tag{2.3.15}$$

where $\Lambda_1{}^\rho$ (q, p) contains the divergent part and $\Lambda_2{}^\rho$ (q, p) the part which is not divergent. For $\Lambda_1{}^\rho$ (q, p) we obtain

$$-ie\, \Lambda_1{}^\rho (q, p) = -\, ie^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2}\right] \int_0^1 \frac{dx}{\Lambda^2}$$

$$\times \int_0^{1-x} \frac{dy}{\Lambda^2} \int_{\frac{1}{1-x-y}}^{\infty} \frac{d\lambda}{\Lambda^2}\, \lambda^2 \int \frac{d^4k}{(2\pi)^4}\, k^2 (\gamma^t\gamma^\sigma\gamma^\rho\gamma_\sigma\gamma_t) \exp\left(-\lambda \frac{k^2}{\Lambda^2}\right)$$

$$\times \exp\left[-\frac{\lambda}{\Lambda^2} \left\{ m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy \right\}\right] \tag{2.3.16}$$

For $\Lambda_2{}^\rho$ (q, p), we have,

$$-ie\, \Lambda_2{}^\rho (q, p) = -\, ie^3 \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times \int_0^1 \frac{dx}{\Lambda^2} \int_0^{1-x} \frac{dy}{\Lambda^2} \int_{\frac{1}{1-x-y}}^{\infty} \frac{d\lambda}{\Lambda^2}\, \lambda^2 \int \frac{d^4k}{(2\pi)^4} \exp\left(-\lambda \frac{k^2}{\Lambda^2}\right)$$

$$\times \gamma^t [\not p(1-x) - \not q y - m]\, \gamma^\rho\, [\not q (1-y) - \not p x - m]\, \gamma_t$$

$$\times \exp\left[-\frac{\lambda}{\Lambda^2} \left\{ m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy \right\}\right] \tag{2.3.17}$$

The terms odd in k, vanish on integration with respect to k. Performing the momentum integration, expression (2.3.16) becomes,

$$-ie\,\Lambda_1^{\rho}(q,p) = -\frac{ie^3}{8\pi^2}\gamma^{\rho}\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int_0^1 dx\int_0^{1-x}dy\int_{\frac{1}{1-x-y}}^{\infty}\frac{d\lambda}{\lambda}$$

$$\times\exp\left[-\frac{\lambda}{\Lambda^2}\left\{m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy\right\}\right] \quad (2.3.18)$$

Here, we have used the identity

$$\gamma^t\gamma^{\sigma}\gamma^{\rho}\gamma_{\sigma}\gamma_t = 4\gamma^{\rho} \quad (2.3.19)$$

Expression (2.3.18) can be written as

$$-ie\,\Lambda_1^{\rho}(q,p) = -\frac{ie^3}{8\pi^2}\gamma^{\rho}\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int_0^1 dx\int_0^{1-x}dy\; E_1\left[\frac{m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy}{\Lambda^2(1-x-y)}\right] \quad (2.3.20)$$

where $E_1(z)$ is the exponential integral,

$$E_1(z) = \int_z^{\infty} dt\,\frac{\exp(-t)}{t} = -\ln z - \gamma - \sum_{n=1}^{\infty}\frac{(-z)^n}{n\,n!} \quad (2.3.21)$$

Although the final parameter integral cannot be evaluated in terms of elementary functions, we can develop an asymptotic expansion in Λ by expanding the exponential integral :

$$-ie\,\Lambda_1^{\rho}(q,p) = -\frac{ie^3}{8\pi^2}\gamma^{\rho}\int_0^1 dx\int_0^{1-x}dy$$

$$\times\left[-\ln\left\{\frac{m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy}{\Lambda^2(1-x-y)}\right\}-\gamma\right] \quad (2.3.22a)$$

$$= - \frac{ie^3}{16\pi^2}\gamma^\rho \left[\ln(\Lambda^2) - \gamma + \frac{3}{2}\right] + \frac{ie^3}{8\pi^2}\gamma^\rho \int_0^1 dx \int_0^{1-x} dy$$

$$\times \ln \; [m^2(x+y)+p^2x(1-x)+q^2y(1-y)-2p.qxy] \qquad (2.3.22b)$$

For Λ_2^ρ (q, p), we obtain in the limit $\Lambda \to \infty$ the following expression:

$$- ie\, \Lambda_2^\rho (q, p) = - \frac{ie^3}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy\, \gamma^t [\not p (1-x) - \not q y - m]\, \gamma^\rho$$

$$\times \; [\not q(1-y) - \not p x - m]\, \gamma_t \int_{\frac{1}{1-x-y}}^{\infty} \frac{d\lambda}{\Lambda^2}$$

$$\times \exp\left[- \frac{\lambda}{\Lambda^2} \left\{ m^2 (x+y) + p^2x(1-x) + q^2y(1-y) - 2\, p.qxy \right\} \right]$$

(2.3.23a)

$$= - \frac{ie^3}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \; \frac{\gamma^t [\not p(1-x) - \not q y - m]\, \gamma^\rho\, [\not q(1-y) - \not p x - m]\, \gamma_t}{[m^2(x+y)+p^2x(1-x)+q^2y(1-y) - 2\, p.qxy]}$$

(2.3.23b)

We compare our result with the expression for the vertex obtained under dimensional regularization [66].

$$- ie\, \Lambda_1^\rho (q,p) = - \frac{ie^3}{16\pi^2}\gamma^\rho \left[\frac{1}{\epsilon} - \gamma - \ln 4\pi\mu^2 \right] + \frac{ie^3}{8\pi^2}\gamma^\rho \int_0^1 dx \int_0^{1-x} dy$$

$$\times \ln \; [m^2 (x+y) + p^2x(1-x) + q^2y(1-y) - 2\, p.qxy]$$

(2.3.24)

and

$$- ie\, \Lambda_2^\rho (q,p) = - \frac{ie^3}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \; \frac{\gamma^t [\not p(1-x) - \not q y - m]\, \gamma^\rho\, [\not q(1-y) - \not p x - m]\, \gamma_t}{[m^2(x+y)+p^2x(1-x)+q^2y(1-y) - 2\, p.qxy]}$$

(2.3.25)

The two expressions agree if for the divergent parts, we use the correspondence:

$$\ln \Lambda^2 \sim \frac{1}{\epsilon} \tag{2.3.26}$$

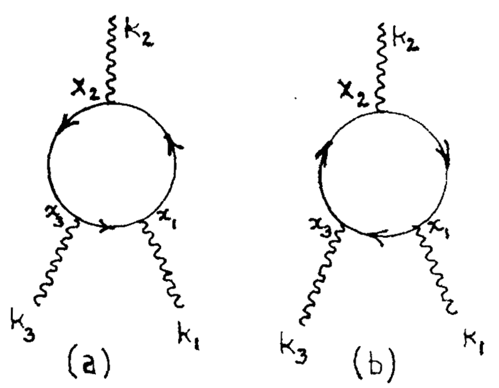
as derived previously by Evens etal [28] for self–energy and vacuum polarization.

2.4 Furry's Theorem

Furry's theorem [69 - 71] states that the Feynman diagrams containing a closed fermion loop with an odd number of photon vertices can be omitted in the calculation of physical processes. In a closed loop there can be an electron as well as a positron circling around. If the number of photon vertices is even, then the two contributions just get added.

Proof:

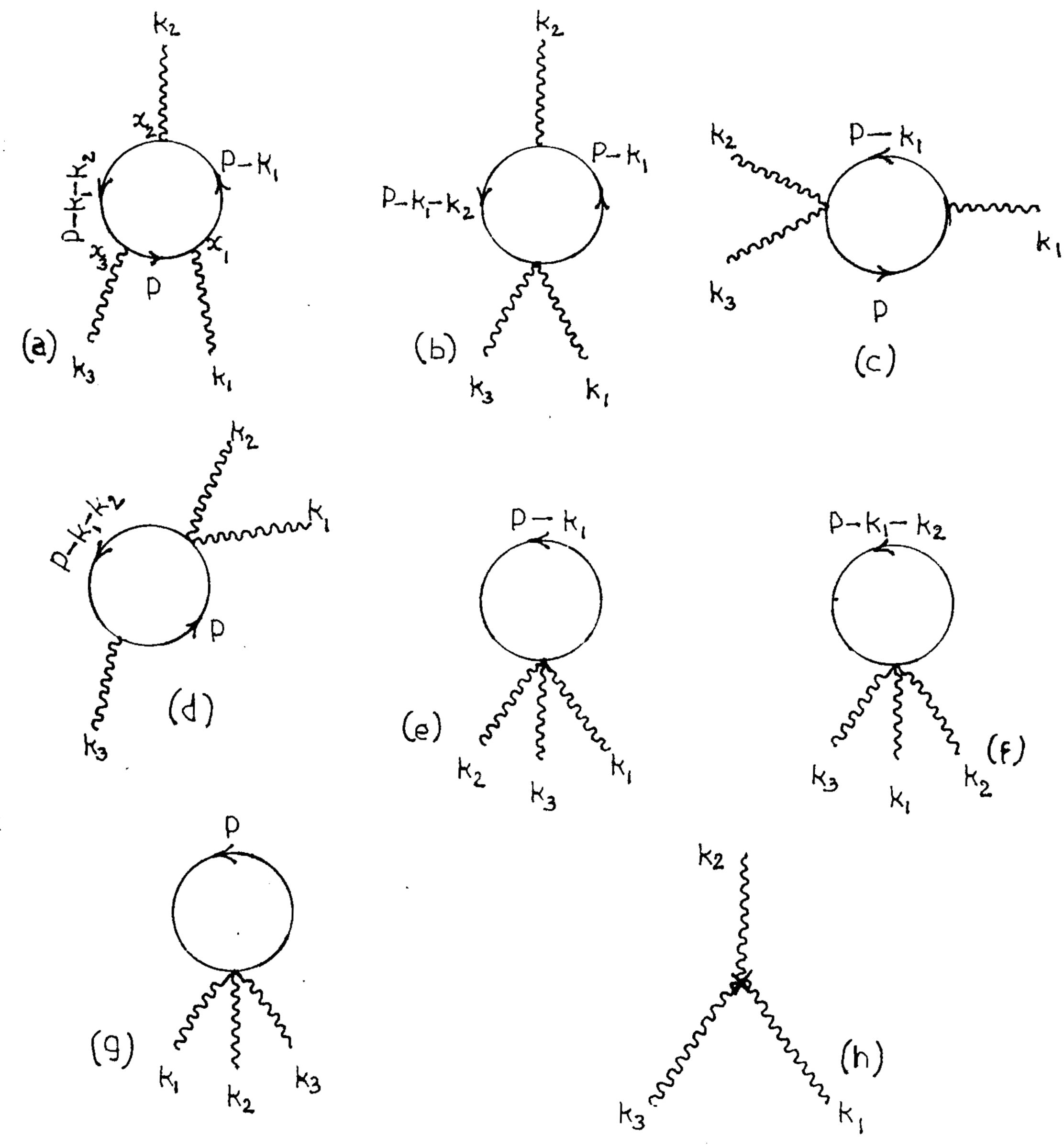
Consider a process that can be described by a graph containing an electron loop with three vertices. In QED the two graphs corresponding to the process are



Two graphs with opposite directions of the internal fermion loop.

Fig.2.5

But in nonlocal QED, there are eight diagrams corresponding to each one of these graphs. They are given in figs.2.6 and 2.7 respectively.



Nonlocal graphs corresponding to fig. 2.5(a)

Fig.2.6

The relevant contribution to the S – matrix element describing the loops 2.6(a) to (g) are

$$M_{2.6a} = \mathrm{Tr}\left[(-ie\,\gamma_{\mu 1})\frac{i}{\not p+m-i\epsilon}(-ie\,\gamma_{\mu 3})\frac{i}{\not p-\not k_1-\not k_2+m-i\epsilon}(-ie\,\gamma_{\mu 2})\frac{i}{\not p-\not k_1+m-i\epsilon}\right] \quad (2.4.1a)$$

$$M_{2.6b} = \mathrm{Tr}\left[(-ie^2\,\gamma_{\mu 1})\,[\not p-m]\,(\gamma_{\mu 3})\frac{i}{\not p-\not k_1-\not k_2+m-i\epsilon}(-ie\,\gamma_{\mu 2})\frac{i}{\not p-\not k_1+m-i\epsilon}\right] \quad (2.4.1b)$$

$$M_{2.6c} = \mathrm{Tr}\left[(-ie\,\gamma_{\mu 1})\frac{i}{\not p+m-i\epsilon}(-ie^2\,\gamma_{\mu 3})\,[\not p-\not k_1-\not k_2-m](\gamma_{\mu 2})\frac{i}{\not p-\not k_1+m-i\epsilon}\right] \quad (2.4.1c)$$

$$M_{2.6d} = \mathrm{Tr}\left[(-ie^2\,\gamma_{\mu 1})\frac{i}{\not p+m-i\epsilon}(-ie\,\gamma_{\mu 3})\frac{i}{\not p-\not k_1-\not k_2+m-i\epsilon}(\gamma_{\mu 2})\,[\not p-\not k_1-m]\right] \quad (2.4.1d)$$

$$M_{2.6e} = \mathrm{Tr}\left[(-ie^3\,\gamma_{\mu 1})\,[\not p-m]\,(\gamma_{\mu 3})\,[\not p-\not k_1-\not k_2-m](\gamma_{\mu 2})\frac{i}{\not p-\not k_1+m-i\epsilon}\right] \quad (2.4.1e)$$

$$M_{2.6f} = \mathrm{Tr}\left[(-ie^3\,\gamma_{\mu 1})\,[\not p-m]\,(\gamma_{\mu 3})\frac{i}{\not p-\not k_1-\not k_2+m-i\epsilon}(\gamma_{\mu 2})\,[\not p-\not k_1-m]\right] \quad (2.4.1f)$$

$$M_{2.6g} = \mathrm{Tr}\left[(-ie^3\,\gamma_{\mu 1})\frac{i}{\not p+m-i\epsilon}(\gamma_{\mu 3})\,[\not p-\not k_1-\not k_2-m]\,(\gamma_{\mu 2})\,[\not p-\not k_1-m]\right] \quad (2.4.1g)$$

Fig.2.6 (h) is the measure factor contribution. The measure factor absorbs the noninvariance due to the fermion measures. The measure factor $\mu[e\,A]$ is defined as follows:

$$\mu[e\,A] = \exp\left(S_{meas}\,[e\,A]\right), \quad (2.4.2a)$$

$$\partial_\mu \theta \frac{\delta S_{meas}[eA]}{\delta A_\mu} = - e\,Tr\,(\theta.\mathscr{T}[eA] + eTr\,(\theta.\bar{\mathscr{T}}[eA]) \tag{2.4.2b}$$

In the case of odd number of photon vertices, the two contributions given in (2.4.2b) cancel. Consider the order e^3 contribution to $e\,Tr\,(\theta.\bar{\mathscr{T}})$:

$$e^3\ Tr\,\{\varepsilon_m [\not{A}^\Lambda \vartheta\,(i\not{\partial} - m)\,\not{A}^\Lambda \vartheta(\,i\not{\partial} - m)]\,\theta^\Lambda \varepsilon_m\}$$

$$= e^3\ Tr\,\{\varepsilon_m [C^{-1}C\not{A}^\Lambda\ C^{-1}C\ \vartheta\,(i\not{\partial} - m)\ C^{-1}C\not{A}^\Lambda\ C^{-1}C\ \vartheta\,(i\not{\partial} - m)\ C^{-1}C]\ \theta^\Lambda \varepsilon_m\} \tag{2.4.3}$$

Here C is the charge – conjugation matrix:

$$C\,\gamma_\mu\,C^{-1} = -\ \gamma_\mu{}^T \tag{2.4.4}$$

Hence (2.4.3) becomes,

$$e^3\,Tr\,\{\varepsilon_m [\not{A}^\Lambda \vartheta\,(i\not{\partial} - m)\,\not{A}^\Lambda \vartheta(\,i\partial - m)]\,\theta^\Lambda \varepsilon_m\}$$

$$= e^3\,Tr\,\{\varepsilon_m [-(\not{A}^\Lambda)^T]\vartheta\,(\overleftarrow{i\not{\partial}} - m)^T\,[-(\not{A}^\Lambda)^T]\vartheta(\,\overleftarrow{i\not{\partial}} - m)^T]\,\theta^\Lambda \varepsilon_m\}. \tag{2.4.5a}$$

$$= e^3\,Tr\,\{\varepsilon_m\,\theta^\Lambda\,(i\not{\partial} - m)\ \vartheta\not{A}^\Lambda\,(i\not{\partial} - m)\ \vartheta\not{A}^\Lambda\ \varepsilon_m\}^T \tag{2.4.5b}$$

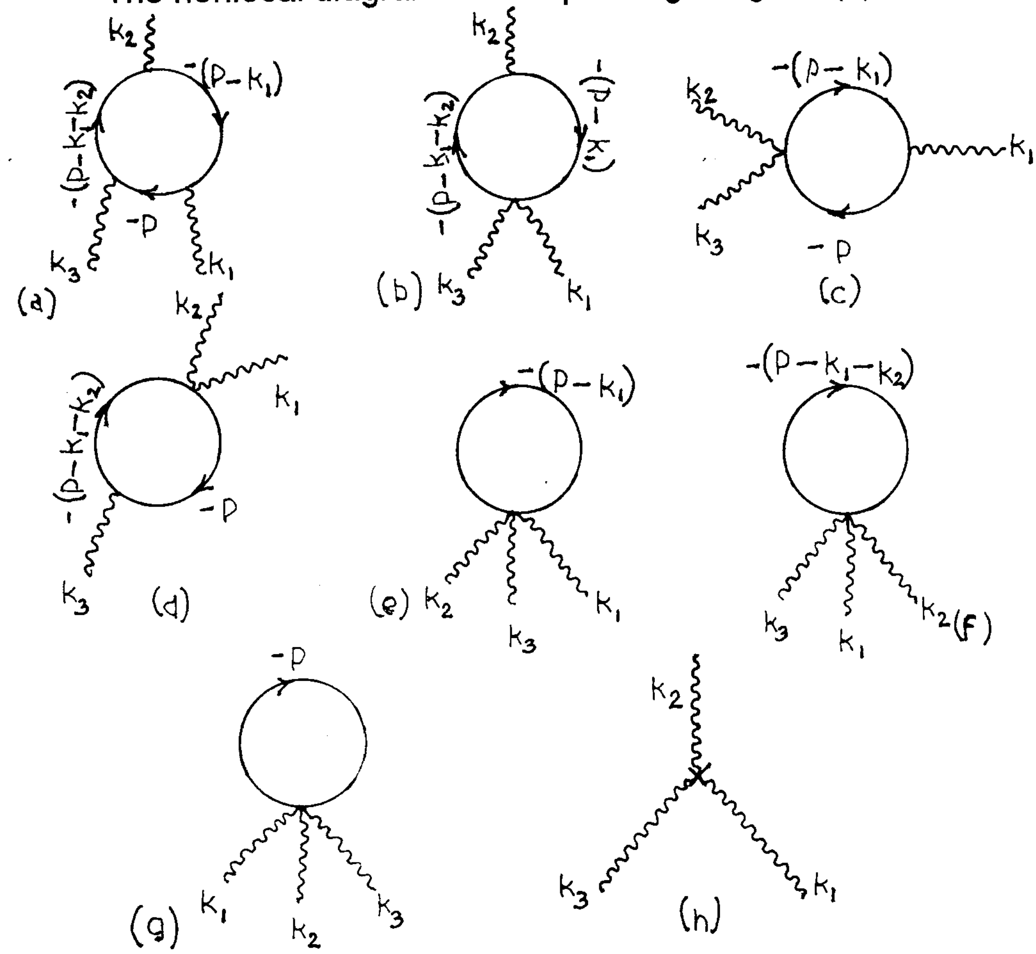
But $Tr\,\{(AB)^T\} = Tr\,\{AB\}$

$$e^3\,Tr\,\{\varepsilon_m [\not{A}^\Lambda \vartheta\,(i\not{\partial} - m)\,\not{A}^\Lambda \vartheta\,(i\not{\partial} - m)]\theta^\Lambda \varepsilon_m\}$$

$$= e^3\,Tr\,\{\varepsilon_m \theta^\Lambda\,(i\not{\partial} - m)\ \vartheta\not{A}^\Lambda\,(i\not{\partial} - m)\ \vartheta\not{A}^\Lambda\ \varepsilon_m\} \tag{2.4.6}$$

The e^3 contribution to $-eTr(\theta.\,\mathscr{T})$ is the same as (2.4.6) with a negative sign. Hence for a diagram with odd number of photon vertices (2.4.6) cancels with the analogous contribution from $-eTr\,(\theta.\mathscr{T})$, while for even number of photon vertices, they add. So in the case of a fermion loop with odd number of photon vertices, the contribution from the measure factor is zero.

The nonlocal diagrams corresponding to fig. 2.5(b) are:



Nonlocal graphs corresponding to fig. 2.5(b)

Fig.2.7.

The relevant contribution to the S-matrix element describing the loops 2.7(a) to (g) are :

$$M_{2.7a} = \mathrm{Tr}\left[(-ie\gamma_{\mu 1})\frac{i}{-(\not p-\not k_1)+m-i\epsilon}(-ie\gamma_{\mu 2})\frac{i}{-(\not p-\not k_1-\not k_2)+m-i\epsilon}(-ie\gamma_{\mu 3})\frac{i}{-\not p+m-i\epsilon}\right] \tag{2.4.7a}$$

$$M_{2.7b} = \mathrm{Tr}\left[(-ie^2\gamma_{\mu 1})\frac{i}{-(\not p-\not k_1)+m-i\epsilon}(-ie\gamma_{\mu 2})\frac{i}{-(\not p-\not k_1-\not k_2)+m-i\epsilon}(\gamma_{\mu 3})[-\not p-m]\right] \tag{2.4.7b}$$

$$M_{2.7c} = \mathrm{Tr}\left[(-ie\gamma_{\mu 1})\frac{i}{-(\not p-\not k_1)+m-i\epsilon}(-ie^2\gamma_{\mu 2})[-(\not p-\not k_1-\not k_2)-m](\gamma_{\mu 3})\frac{i}{-\not p+m-i\epsilon}\right] \tag{2.4.7c}$$

$$M_{2.7d} = Tr\left[(-ie^2\gamma_{\mu 1})[-(\not p-\not k_1)-m]\,(\gamma_{\mu 2})\,\frac{i}{-(\not p-\not k_1-\not k_2)+m-i\epsilon}\,(-ie\gamma_{\mu 3})\,\frac{i}{-\not p+m-i\epsilon}\right] \quad (2.4.7d)$$

$$M_{2.7e} = Tr\left[(-ie^3\gamma_{\mu 1})\,\frac{i}{-(\not p-\not k_1)+m-i\epsilon}\,(\gamma_{\mu 2})\,[-(\not p-\not k_1-\not k_2)-m]\,(\gamma_{\mu 3})\,[-\not p-m]\right] \quad (2.4.7e)$$

$$M_{2.7f} = Tr\left[(-ie^3\gamma_{\mu 1})\,[-(\not p-\not k_1)-m]\,(\gamma_{\mu 2})\,\frac{i}{-(\not p-\not k_1-\not k_2)+m-i\epsilon}\,(\gamma_{\mu 3})\,[-\not p-m]\right] \quad (2.4.7f)$$

$$M_{2.7g} = Tr\left[(-ie^3\gamma_{\mu 1})\,[-(\not p-\not k_1)-m](\gamma_{\mu 2})\,[-(\not p-\not k_1-\not k_2)+m-i\epsilon]\,(\gamma_{\mu 3})\,\frac{i}{-\not p+m-i\epsilon}\right] \quad (2.4.7g)$$

As before the measure factor contribution vanishes. Now insert factors of $C^{-1}C = I$ in (2.4.7a).

$$M_{2.7a} = Tr\left[C^{-1}C\,(-ie\gamma_{\mu 1})\,C^{-1}C\,\frac{i}{-(\not p-\not k_1)+m-i\epsilon}\,C^{-1}C\,(-ie\gamma_{\mu 2})\,C^{-1}C\right.$$

$$\left.\frac{i}{-(\not p-\not k_1-\not k_2)+m-i\epsilon}\,C^{-1}C\,(-ie\gamma_{\mu 3})\,C^{-1}C\,\frac{i}{-\not p+m-i\epsilon}\,C^{-1}C\right] \quad (2.4.8)$$

Now, $$C\,\frac{1}{-(\not p-k_1)+m-i\epsilon}\,C^{-1} = C\,\frac{-(\not p-\not k_1)-m}{(p-k_1)^2+m^2-i\epsilon}\,C^{-1} \quad (2.4.9a)$$

$$= C\,\frac{-\gamma_\mu\,(p^\mu-k_1{}^\mu)-m}{(p-k_1)^2+m^2-i\epsilon}\,C^{-1} \quad (2.4.9b)$$

But $C\gamma_\mu C^{-1} = -\gamma_\mu{}^T$.

$$\therefore C\,\frac{1}{-(\not p-\not k_1)+m-i\epsilon}\,C^{-1} = \frac{\gamma_\mu{}^T\,(p^\mu-k_1{}^\mu)-m}{(p-k_1)^2+m^2-i\epsilon} \quad (2.4.10)$$

Hence (2.4.8) becomes,

$$M_{2.7a} = (-1)^3\, Tr\left[(-ie\gamma_{\mu 1}{}^T) \frac{i\gamma_\mu{}^T(p^\mu - k_1{}^\mu) - m}{(p-k_1)^2 + m^2 - i\epsilon} (-ie\gamma_{\mu 2}{}^T) \frac{i\gamma_\mu{}^T(p^\mu - k_1{}^\mu - k_2{}^\mu) - m}{(p-k_1-k_2)^2 + m^2 - i\epsilon} \right.$$

$$\left. \times\ (-ie\gamma_{\mu 3}{}^T) \frac{i\gamma_\mu{}^T(p^\mu - m)}{(p^2 + m^2 - i\epsilon)} \right] \qquad (2.4.11a)$$

$$= (-1)^3\, Tr\left[\frac{(i)\not{p} - m}{(p^2 + m^2 - i\epsilon)} (-ie\gamma_{\mu 3}) \frac{(i)\not{p} - \not{k}_1 - \not{k}_2 - m}{(p-k_1-k_2)^2 + m^2 - i\epsilon} (-ie\gamma_{\mu 2}) \right.$$

$$\left. \times \frac{(i)\not{p} - \not{k}_1 - m}{(p-k_1)^2 + m^2 - i\epsilon} (-ie\gamma_{\mu 1}) \right]^T \qquad (2.4.11b)$$

$$= (-1)^3\, Tr\left[\frac{i}{\not{p} + m - i\epsilon} (-ie\gamma_{\mu 3}) \frac{i}{\not{p} - \not{k}_1 - \not{k}_2 + m - i\epsilon} (-ie\gamma_{\mu 2}) \frac{i}{\not{p} - \not{k}_1 + m - i\epsilon} (-ie\gamma_{\mu 1}) \right] \qquad (2.4.11c)$$

since trace is invariant under transportation ·

So, $M_{2.7a} = (-1)^3 M_{2.6a}$ or, $M_{2.7a} + M_{2.6a} = 0$ (2.4.11d)

Similarly for fig.2.7(b),

$$M_{2.7b} = Tr\left[C^{-1}C\ (-ie^2\gamma_{\mu 1})\ C^{-1}C \frac{i}{-(\not{p} - \not{k}_1) + m - i\epsilon} C^{-1}C\ (-ie\gamma_{\mu 2})\ C^{-1}C \right.$$

$$\left. \frac{i}{-(\not{p} - \not{k}_1 - \not{k}_2) + m - i\epsilon} C^{-1}C\ (\gamma_{\mu 3})\ C^{-1}C\ [-\not{p} - m]\ C^{-1}C \right] \qquad (2.4.12a)$$

$$= (-1)^3 \left[Tr\quad (-ie^2\gamma_{\mu 1}{}^T) \frac{i\{(\not{p} - \not{k}_1)^T - m\}}{(p-k_1)^2 + m^2 - i\epsilon} (-ie\gamma_{\mu 2}{}^T) \frac{i\{(\not{p} - \not{k}_1 - \not{k}_2)^T - m\}}{(p-k_1-k_2)^2 + m^2 - i\epsilon} \right.$$

$$\left. (\gamma_{\mu 3}{}^T)\quad [\not{p}^T - m] \right] \qquad (2.4.12b)$$

$$= (-1)^3\quad Tr\left[[\not{p} - m]\ (\gamma_{\mu 3}) \frac{i\{(\not{p} - \not{k}_1 - \not{k}_2) - m\}}{(p-k_1-k_2)^2 + m^2 - i\epsilon} (-ie\gamma_{\mu 2}) \right.$$

$$\left. \times \frac{i\{(\not{p} - \not{k}_1) - m\}}{(p-k_1)^2 + m^2 - i\epsilon} (-ie^2\gamma_{\mu 1}) \right]^T \qquad (2.4.12c)$$

$$= (-1)^3 \mathrm{Tr} \left[(\not{p}-m)(\gamma_{\mu 3}) \frac{i}{\not{p}-\not{k}_1-\not{k}_2+m-i\epsilon} (-ie\gamma_{\mu 2}) \frac{i}{\not{p}-\not{k}_1+m-i\epsilon} (-ie^2\gamma_{\mu 1}) \right] \qquad (2.4.12d)$$

$$= (-1)^3 M_{2.6b}$$

Therefore, $M_{2.7b} + M_{2.6b} = 0$ (2.4.12e)

Similarly for the other diagrams also thus proving the Furry's theorem.

2.5 Discussion

The technique of nonlocal regularization [28] is applied to evaluate the QED vertex part [68]. Corresponding to the single Feynman diagram of local QED, there are four Feynman diagrams in nonlocal QED, which arises from the consideration of two kinds of propagators differing only in the limits of integration. The result predicted by the nonlocal theory is exactly the same as that given by the dimensional regularization method [66].The divergent parts in the two methods may be equated using $\ln(\Lambda^2) \sim \frac{2}{4-D}$ where D is the number of dimensions. The finite parts are the same apart from trivial numerical constants.

The Furry's theorem is also proved for nonlocal QED, by considering a particular example of a closed fermion loop with three photon vertices. Here, there will be eight Feynman diagrams in nonlocal QED, corresponding to the single one in local QED. There can be an electron as well as a positron circling around in a closed loop. The contribution from one loop can be seen to be cancelled by the contribution from the other loop, thus enabling to omit the closed fermion loops with an odd number of photon vertices while evaluating the physical processes .

Ward Identity For Nonlocal QED

P. C. Raje Bhageerathi "Regularizations and divergent diagram in gauge theories" Thesis. Department of Physics, University of Calicut, 1999

CHAPTER - 3

Ward Identity For Nonlocal QED

3.1 Introduction

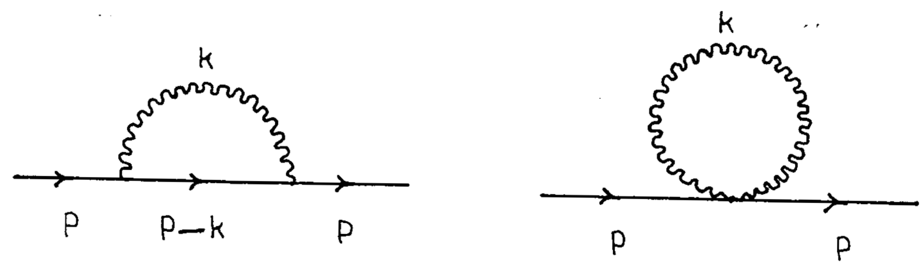
The Ward identity [19] and its generalization by Takahashi [20, 21] are true to all orders of perturbation theory in QED. They follow from the gauge invariance of QED.

The nonlocal regularization is essentially a perturbative one. But the addition of nonlocal terms to the Lagrangian destroys the local gauge invariance of the theory. The gauge invariance is restored by the modification of the fermionic transformation rule at each order. Hence it is necessary to prove the Ward identity at each order [75].

The Ward identity for nonlocal QED upto the order of two loops (order e^4) has been developed in ref. 75. It can be seen that in the limit of QED ($\Lambda \to \infty$), the usual expression for the Ward identity is retained. Higher order calculations involve higher order measure factors which are not yet evaluated to our knowledge.

3.2 The Ward Identity to order e^2

In nonlocal QED, at order e^2, there are two contributions to the electron self-energy — one is the usual QED self-energy graph and the other is obtained by joining the two vertices.



(a) (b)

Electron self-energy graphs
at one loop in nonlocal QED
Fig 3.1

The electron self energy at one loop has been worked out by Evens etal [28]. It is given by

$$-i\Sigma\,(p) = -\,e^2 \exp\left(-\,\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\int_0^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}\qquad \gamma^t\,(\not{q}-m)\,\gamma_t$$

$$\times\ \exp\left[-\,\tau_1\,\frac{q^2+m^2}{\Lambda^2}\ -\ \tau_2\,\frac{k^2}{\Lambda^2}\right] \qquad (3.2.1)$$

where $\not{q} = \not{p} - \not{k}$. Now change the variable τ_2 to $\tau_2' + 1$.

That is, $\tau_2 \rightarrow \tau_2' + 1$ (3.2.2)

Expression (3.2.1) becomes

$$-i\ \Sigma\,(p) = -\,e^2 \exp\left(-\,\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\int_0^\infty\frac{d\tau_1}{\Lambda^2}\int_0^\infty\frac{d\tau_2'}{\Lambda^2}\qquad \gamma^t\,(\not{q}-m)\,\gamma_t$$

$$\times\ \exp\left[-\ \tau_1\,\frac{q^2+m^2}{\Lambda^2}\ -\ (\tau_2'+1)\,\frac{k^2}{\Lambda^2}\right] \qquad (3.2.3)$$

$$-i\Sigma(p) = -e^2 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4} \exp\left(-\frac{k^2}{\Lambda^2}\right)\int_0^\infty\frac{d\tau_1}{\Lambda^2}\int_0^\infty\frac{d\tau_2}{\Lambda^2}$$

$$\times\, \gamma^t (\not{q} - m)\, \gamma_t \exp\left[-\tau_1 \frac{q^2+m^2}{\Lambda^2} - \tau_2 \frac{k^2}{\Lambda^2}\right] \tag{3.2.4}$$

where

$$\int_0^\infty \frac{d\tau_2'}{\Lambda^2} \exp\left[-(\tau_2'+1)\frac{k^2}{\Lambda^2}\right] = \exp\left(-\frac{k^2}{\Lambda^2}\right)\int_0^\infty\frac{d\tau_2}{\Lambda^2}\exp\left(-\tau_2 \frac{k^2}{\Lambda^2}\right) \tag{3.2.5}$$

Depromoting the exponentials involving τ_1 and τ_2 to the denominator using the Schwinger integrals,

$$\int_0^\infty \frac{d\tau_1}{\Lambda^2} \exp\left(-\tau_1 \frac{q^2+m^2}{\Lambda^2}\right) = \frac{1}{q^2+m^2} \tag{3.2.6}$$

and similarly for the integral involving τ_2. Then we obtain for Σ (p),

$$-i\Sigma(p) = -e^2 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\exp\left(-\frac{k^2}{\Lambda^2}\right)\frac{\gamma^t(\not{q}-m)\gamma_t}{(q^2+m^2)\,k^2} \tag{3.2.7a}$$

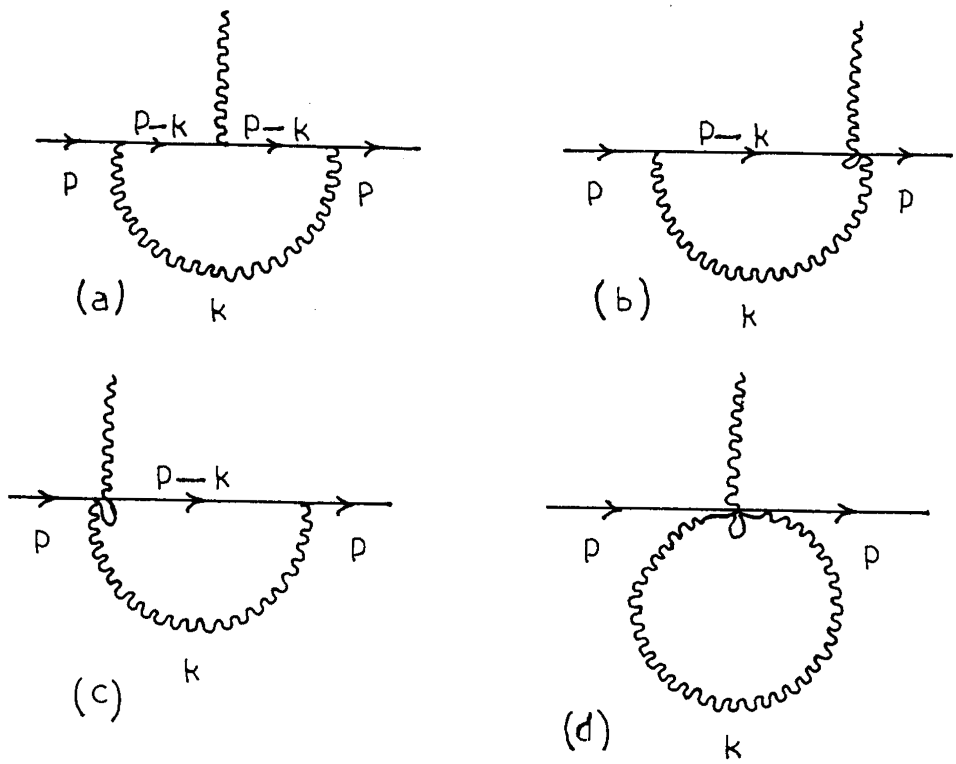
$$= -e^2 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\exp\left(-\frac{k^2}{\Lambda^2}\right)\gamma^t \frac{1}{\not{q}+m-i\epsilon}\,\gamma_t\,\frac{1}{k^2-i\epsilon} \tag{3.2.7b}$$

The vertex part for nonlocal QED has been worked out in ref. 68. The diagrams contributing to the nonlocal QED vertex part are given in fig.3.2. The nonlocal QED vertex part is :

$$-ie\,\Lambda^\mu(p,p) = -e^3 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\int_0^\infty\frac{d\tau_1}{\Lambda^2}\int_0^\infty\frac{d\tau_2}{\Lambda^2}\int_1^\infty\frac{d\tau_3}{\Lambda^2}$$

$$\times\, \gamma^t(\not{q}-m)\,\gamma^\mu(\not{q}-m)\,\gamma_t$$

$$\times \exp\left[-\tau_1\frac{q^2+m^2}{\Lambda^2} - \tau_2\frac{q^2+m^2}{\Lambda^2} - \tau_3\frac{k^2}{\Lambda^2}\right] \tag{3.2.8}$$



Insertions of photon into second order
self-energy graph. (Nonlocal QED vertex part)
Fig.3.2

Following the same line of arguments which led to (3.2.7b), we get

for $\Lambda^{\mu}(p,p)$ as follows:

$$-ie\,\Lambda^{\mu}(p,p) = -\,e^{3}\exp\left(-\,\frac{p^{2}+m^{2}}{\Lambda^{2}}\right)\int\frac{d^{4}k}{(2\pi)^{4}}\,\exp\left(-\,\frac{k^{2}}{\Lambda^{2}}\right)$$

$$\times\,\gamma^{t}\,\frac{1}{\not{q}+m-i\epsilon}\,\gamma^{\mu}\,\frac{1}{\not{q}+m-i\epsilon}\,\gamma_{t}\,\frac{1}{k^{2}-i\epsilon} \qquad (3.2.9)$$

From equations (3.2.7b) and (3.2.9), we can see that

$$- \frac{\partial}{\partial p_\mu} \left[\exp\left(\frac{p^2+m^2}{\Lambda^2} \right) \Sigma(p) \right] = \exp\left(\frac{p^2+m^2}{\Lambda^2} \right) \Lambda^\mu(p,p) \qquad (3.2.10)$$

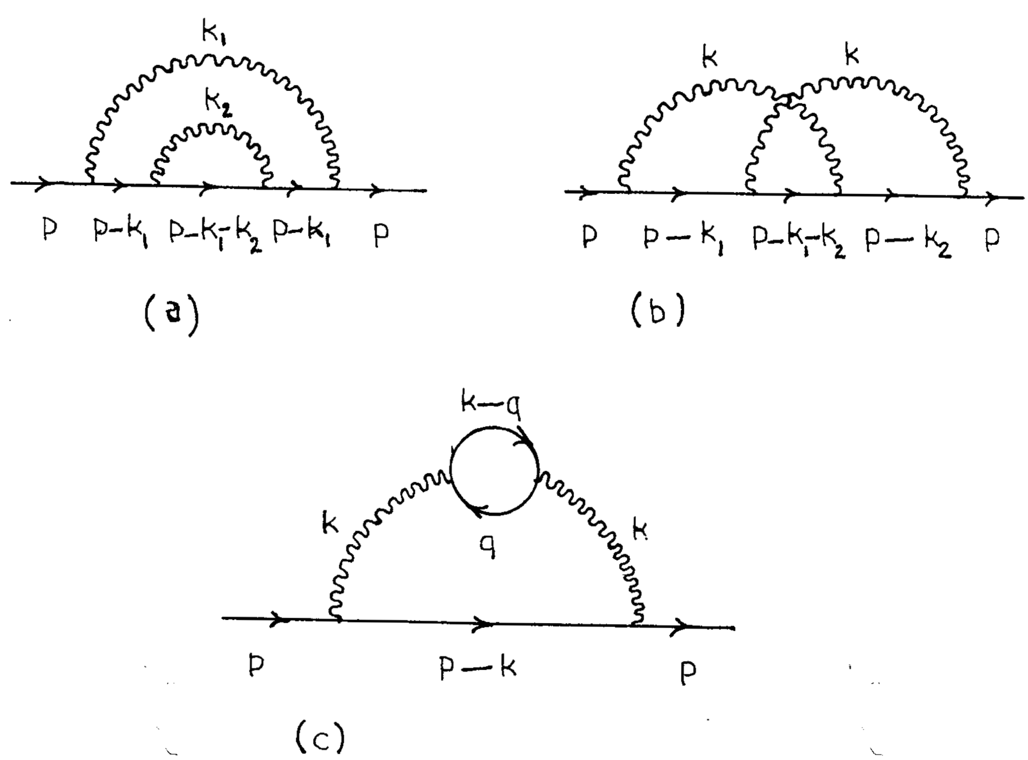
This is the Ward identity for nonlocal QED to order e^2. In the limit of nonlocal regularization ($\Lambda \to \infty$), this becomes

$$- \frac{\partial}{\partial p_\mu} \Sigma(p) = \Lambda^\mu(p,p) \qquad (3.2.11)$$

which is the form of the Ward identity for QED.

3.3 Fourth order electron self-energy

There are three different Feynman diagrams in QED for the electron self – energy at two loops. They are given in fig.3.3.



Electron self-energy graphs at two loops in QED
Fig 3.3

Fig.3.3 (b) is an example of overlapping divergences in QED. Fig.3.3(a) gives a contribution to $\Sigma(p)$ of

$$\Sigma_a(p) = -e^4 \int d^4k_1 \int d^4k_2 \frac{1}{k_1^2 - i\epsilon} \gamma^\mu \frac{1}{\not p - \not k_1 - m - i\epsilon} \gamma^\rho \frac{1}{\not p - \not k_1 - \not k_2 - m - i\epsilon} \gamma_\rho$$

$$\times \frac{1}{k_2^2 - i\epsilon} \frac{1}{\not p - \not k_1 - m - i\epsilon} \gamma_\mu \qquad (3.3.1)$$

Fig 3.3(b) gives a contribution

$$\Sigma_b(p) = -e^4 \int d^4k_1 \int d^4k_2 \frac{1}{k_1^2 - i\epsilon} \gamma^\mu \frac{1}{\not p - \not k_1 - m - i\epsilon} \gamma^\rho \frac{1}{\not p - \not k_1 - \not k_2 - m - i\epsilon} \gamma_\mu$$

$$\times \frac{1}{k_2^2 - i\epsilon} \frac{1}{\not p - \not k_2 - m - i\epsilon} \gamma_\rho \qquad (3.3.2)$$

Fig 3.3 (c) contains photon self–energy as a part. It gives a contribution

$$\Sigma_c(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - i\epsilon} \gamma^\sigma [\Pi^{\sigma\nu}(q^2)] \frac{1}{\not p - \not k - m - i\epsilon} \gamma^\nu \frac{1}{k^2 - i\epsilon} \eta^{\nu\sigma} \qquad (3.3.3)$$

where $\Pi^{\sigma\nu}(q^2) = \left[\eta^{\sigma\nu} - \dfrac{q^\sigma q^\nu}{q^2}\right] \Pi(q^2)$. (3.3.4)

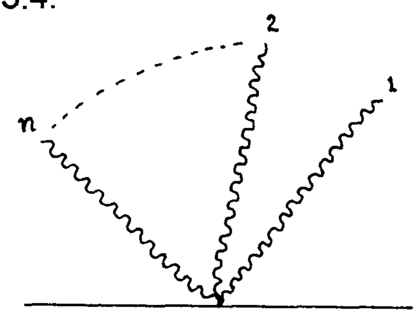
is the polarization tensor .

3.4 Fourth order electron self-energy in nonlocal QED

The Feynman diagrams for nonlocal QED are from the interaction part of the Lagrangian

$$\mathcal{L}_n = -(-e)^n \bar\psi^\Lambda \not A^\Lambda [(i\not\partial - m) \vartheta \not A^\Lambda]^{n-1} \psi^\Lambda, \qquad (3.4.1)$$

where n = 1, 2, 3, ... The Feynman rules are easily read off from the Lagrangian. The Feynman graphs of the vertex has photon lines on one side of the two fermion lines. A nonlocal vertex of order e^n is given in fig 3.4.



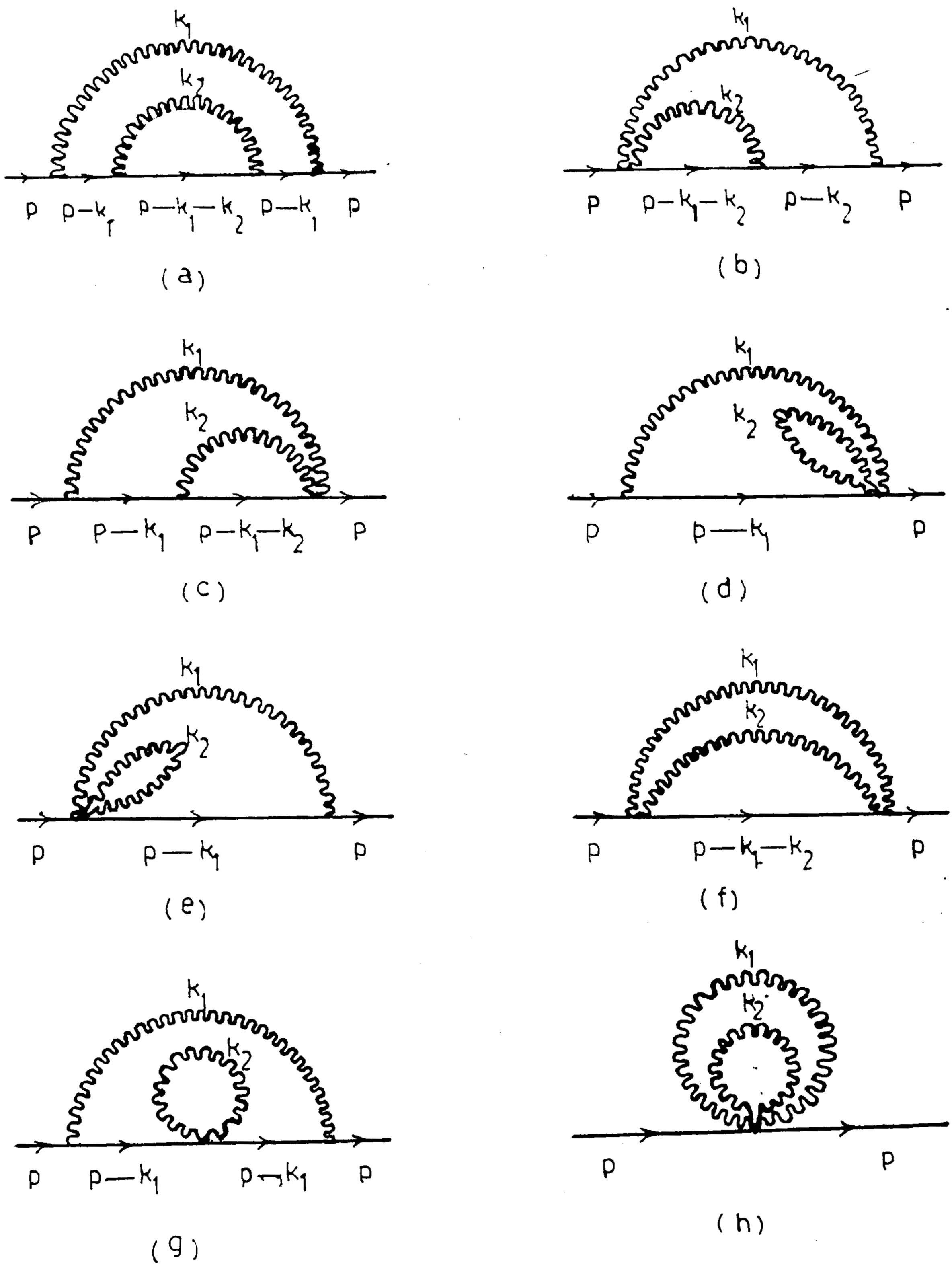
A nonlocal vertex of order e^n
Fig 3.4

The diagram for nonlocal QED may be obtained from the diagram for QED by contracting two or more QED vertices of order e to form a nonlocal QED vertex of order e^2 or more in the corresponding order. The nonlocal QED diagrams corresponding to the QED diagram [fig.3.3(a)] are given in fig. 3.5. There are eight such diagrams. Each figure contribute to the electron self–energy. The individual contributions are:

$$\Sigma_a(p) = -e^4 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k_1}{2\pi^4}\int\frac{d^4k_2}{2\pi^4}\int_1^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}\int_1^\infty\frac{d\tau_3}{\Lambda^2}\int_1^\infty\frac{d\tau_4}{\Lambda^2}\int_1^\infty\frac{d\tau_5}{\Lambda^2}$$

$$\times\, \gamma^t (\not{p} - \not{k}_1 - m)\, \gamma^\rho\, (\not{p} - \not{k}_1 - \not{k}_2 - m)\, \gamma_\rho\, (\not{p} - \not{k}_1 - m)\, \gamma_t$$

$$\times \exp\left[-\tau_1\frac{k_1^2}{\Lambda^2} - \tau_2\frac{k_2^2}{\Lambda^2} - \tau_3\frac{(p-k_1)^2+m^2}{\Lambda^2} - \tau_4\frac{(p-k_1-k_2)^2+m^2}{\Lambda^2} - \tau_5\frac{(p-k_1)^2+m^2}{\Lambda^2}\right] \qquad (3.4.1a)$$



Electron self-energy graphs at two loops in nonlocal QED corresponding to fig.3.3(a)

Fig.3.5.

$$\Sigma_a(p) = -e^4 \int_1^\infty \frac{d\tau_3}{\Lambda^2} \int_1^\infty \frac{d\tau_4}{\Lambda^2} \int_1^\infty \frac{d\tau_5}{\Lambda^2} I_1 \qquad (3.4.1b)$$

where

$$I_1 = \exp\left[-\frac{(p^2+m^2)}{\Lambda^2}\right] \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int_1^\infty \frac{d\tau_1}{\Lambda^2} \int_1^\infty \frac{d\tau_2}{\Lambda^2}$$

$$\times \gamma^t (\not p - \not k_1 - m) \gamma^\rho (\not p - \not k_1 - \not k_2 - m) \gamma_\rho (\not p - \not k_1 - m) \gamma_t$$

$$\times \exp\left[-\tau_1 \frac{k_1^2}{\Lambda^2} - \tau_2 \frac{k_2^2}{\Lambda^2} - \tau_3 \frac{(p-k_1)^2+m^2}{\Lambda^2}\right.$$

$$\left. - \tau_4 \frac{(p-k_1-k_2)^2+m^2}{\Lambda^2} - \tau_5 \frac{(p-k_1)^2+m^2}{\Lambda^2}\right] \qquad (3.4.2)$$

Figs. 3.5(b) and (c) comes from a vertex of order e^2 and two vertices of order e. Their contributions are:

$$\Sigma_b(p) = -e^4 \int_0^1 \frac{d\tau_3}{\Lambda^2} \int_1^\infty \frac{d\tau_4}{\Lambda^2} \int_1^\infty \frac{d\tau_5}{\Lambda^2} I_1, \qquad (3.4.3a)$$

$$\Sigma_c(p) = -e^4 \int_1^\infty \frac{d\tau_3}{\Lambda^2} \int_1^\infty \frac{d\tau_4}{\Lambda^2} \int_0^1 \frac{d\tau_5}{\Lambda^2} I_1, \qquad (3.4.3b)$$

where I_1 is as given in (3.4.2). Figs. 3.5(d) and (e) comes from a vertex of order e^3 and another one of order e. The contributions are:

$$\Sigma_d(p) = -e^4 \int_1^\infty \frac{d\tau_3}{\Lambda^2} \int_0^1 \frac{d\tau_4}{\Lambda^2} \int_0^1 \frac{d\tau_5}{\Lambda^2} I_1, \qquad (3.4.4a)$$

$$\Sigma_e(p) = -e^4 \int_0^1 \frac{d\tau_3}{\Lambda^2} \int_0^1 \frac{d\tau_4}{\Lambda^2} \int_1^\infty \frac{d\tau_5}{\Lambda^2} I_1. \qquad (3.4.4b)$$

Fig. 3.5(f) is as a result of contracting two vertices of order e to form a nonlocal vertex of order e^2. Hence there are two vertices of order

e^2. The contribution is:

$$\Sigma_f(p) = -e^4 \int_0^1 \frac{d\tau_3}{\Lambda^2} \int_1^\infty \frac{d\tau_4}{\Lambda^2} \int_0^1 \frac{d\tau_5}{\Lambda^2} \quad I_1. \tag{3.4.5}$$

Fig.3.5(g) has a vertex of e^2 and two vertices of order e. Its contribution to the self–energy is,

$$\Sigma_g(p) = -e^4 \int_1^\infty \frac{d\tau_3}{\Lambda^2} \int_0^1 \frac{d\tau_4}{\Lambda^2} \int_1^\infty \frac{d\tau_5}{\Lambda^2} \quad I_1. \tag{3.4.6}$$

Fig.3.5(h) is a single vertex of order e^4. Its contribution is,

$$\Sigma_h(p) = -e^4 \int_0^1 \frac{d\tau_3}{\Lambda^2} \int_0^1 \frac{d\tau_4}{\Lambda^2} \int_0^1 \frac{d\tau_5}{\Lambda^2} \quad I_1, \tag{3.4.7}$$

Summing up all the terms [ie, $\Sigma_a(p) + \ldots\ldots + \Sigma_h(p)$] one gets,

$$\Sigma_A(p) = -e^4 \exp\left(- \frac{p^2+m^2}{\Lambda^2}\right) \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int_1^\infty \frac{d\tau_1}{\Lambda^2} \int_1^\infty \frac{d\tau_2}{\Lambda^2} \int_0^\infty \frac{d\tau_3}{\Lambda^2}$$

$$\times \int_0^\infty \frac{d\tau_4}{\Lambda^2} \int_0^\infty \frac{d\tau_5}{\Lambda^2} \exp\left[-\tau_1 \frac{k_1^2}{\Lambda^2} - \tau_2 \frac{k_2^2}{\Lambda^2}\right] \gamma^t(\not p - \not k_1 - m)\, \gamma^\rho(\not p - \not k_1 - \not k_2 - m)$$

$$\times\ \gamma_\rho (\not p - \not k_1 - m)\, \gamma_t \ \exp\left[-\tau_3 \frac{(p-k_1)^2+m^2}{\Lambda^2} - \tau_4 \frac{(p-k_1-k_2)^2+m^2}{\Lambda^2} - \tau_5 \frac{(p-k_1)^2+m^2}{\Lambda^2}\right] \tag{3.4.8a}$$

$$= - e^4 \exp\left(- \frac{p^2+m^2}{\Lambda^2}\right) \int \frac{d^4k_1}{(2\pi)^4} \exp\left(- \frac{k_1^2}{\Lambda^2}\right) \int \frac{d^4k_2}{(2\pi)^4} \exp\left(- \frac{k_2^2}{\Lambda^2}\right)$$

$$\times \frac{1}{k_1^2 - i\epsilon}\, \gamma^t \frac{1}{\not p - \not k_1 + m - i\epsilon}\, \frac{1}{k_2^2 - i\epsilon}\, \gamma^\rho \frac{1}{\not p - \not k_1 - \not k_2 + m - i\epsilon}\, \gamma_\rho \frac{1}{\not p - \not k_1 + m - i\epsilon}\, \gamma_t \tag{3.4.8b}$$

The nonlocal diagrams corresponding to fig. 3.3 (b), the overlapping divergence, are given in fig. 3.6. Here also there are eight diagrams. The individual contributions are:

$$\Sigma_a(p) = -e^4 \int_1^\infty \frac{d\tau_3}{\Lambda^2} \int_1^\infty \frac{d\tau_4}{\Lambda^2} \int_1^\infty \frac{d\tau_5}{\Lambda^2} I_2 \tag{3.4.9}$$

where $I_2 = \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int_1^\infty \frac{d\tau_1}{\Lambda^2} \int_1^\infty \frac{d\tau_2}{\Lambda^2}$

$$\times\ \gamma^t (\not{p} - \not{k}_1 - m)\, \gamma^\rho\, (\not{p} - \not{k}_1 - \not{k}_2 - m)\, \gamma_t\, (\not{p} - \not{k}_2 - m)\, \gamma_\rho$$

$$\times \exp\left[-\tau_1 \frac{k_1^2}{\Lambda^2} - \tau_2 \frac{k_2^2}{\Lambda^2} - \tau_3 \frac{(p-k_1)^2 + m^2}{\Lambda^2} - \tau_4 \frac{(p-k_1-k_2)^2+m^2}{\Lambda^2} - \tau_5 \frac{(p-k_2)^2 + m^2}{\Lambda^2}\right] \tag{3.4.10}$$

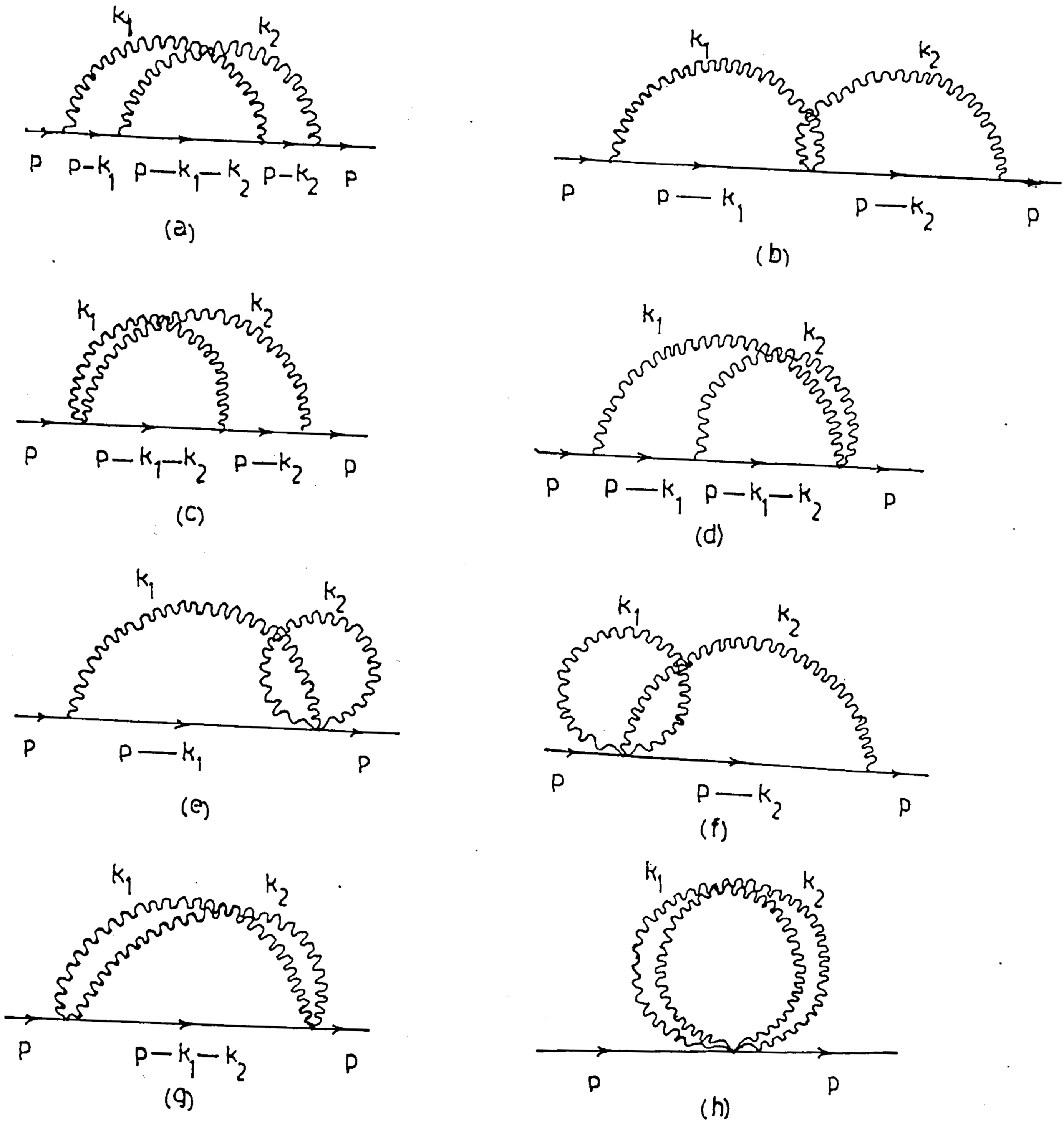
$$\Sigma_b(p) = -e^4 \int_1^\infty \frac{d\tau_3}{\Lambda^2} \int_0^1 \frac{d\tau_4}{\Lambda^2} \int_1^\infty \frac{d\tau_5}{\Lambda^2} I_2 \tag{3.4.11}$$

$$\Sigma_c(p) = -e^4 \int_0^1 \frac{d\tau_3}{\Lambda^2} \int_1^\infty \frac{d\tau_4}{\Lambda^2} \int_1^\infty \frac{d\tau_5}{\Lambda^2} I_2 \tag{3.4.12}$$

$$\Sigma_d(p) = -e^4 \int_1^\infty \frac{d\tau_3}{\Lambda^2} \int_1^\infty \frac{d\tau_4}{\Lambda^2} \int_0^1 \frac{d\tau_5}{\Lambda^2} I_2 \tag{3.4.13}$$

$$\Sigma_e(p) = -e^4 \int_1^\infty \frac{d\tau_3}{\Lambda^2} \int_0^1 \frac{d\tau_4}{\Lambda^2} \int_0^1 \frac{d\tau_5}{\Lambda^2} I_2 \tag{3.4.14}$$

$$\Sigma_f(p) = -e^4 \int_0^1 \frac{d\tau_3}{\Lambda^2} \int_0^1 \frac{d\tau_4}{\Lambda^2} \int_1^\infty \frac{d\tau_5}{\Lambda^2} I_2 \tag{3.4.15}$$



Electron self-energy graphs at two loops in nonlocal QED corresponding to fig.3.3(b)

Fig.3.6.

$$\Sigma_g(p) = -e^4\int_0^1\frac{d\tau_3}{\Lambda^2}\int_1^\infty\frac{d\tau_4}{\Lambda^2}\int_0^1\frac{d\tau_5}{\Lambda^2}\, I_2 \tag{3.4.16}$$

$$\Sigma_h(p) = -e^4\int_0^1\frac{d\tau_3}{\Lambda^2}\int_0^1\frac{d\tau_4}{\Lambda^2}\int_0^1\frac{d\tau_5}{\Lambda^2}\, I_2 \tag{3.4.17}$$

Adding all the individual contributions,

$$\Sigma_B(p) = -e^4\exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k_1}{(2\pi)^4}\exp\left(-\frac{k_1^2}{\Lambda^2}\right)\int\frac{d^4k_2}{(2\pi)^4}\exp\left(-\frac{k_2^2}{\Lambda^2}\right)$$

$$\times\frac{1}{k_1^2-i\epsilon}\gamma^t\frac{1}{\not p-\not k_1+m-i\epsilon}\frac{1}{k_2^2-i\epsilon}\gamma^\rho\frac{1}{\not p-\not k_1-\not k_2+m-i\epsilon}\gamma_t\frac{1}{\not p-\not k_2+m-i\epsilon}\gamma_\rho \tag{3.4.18}$$

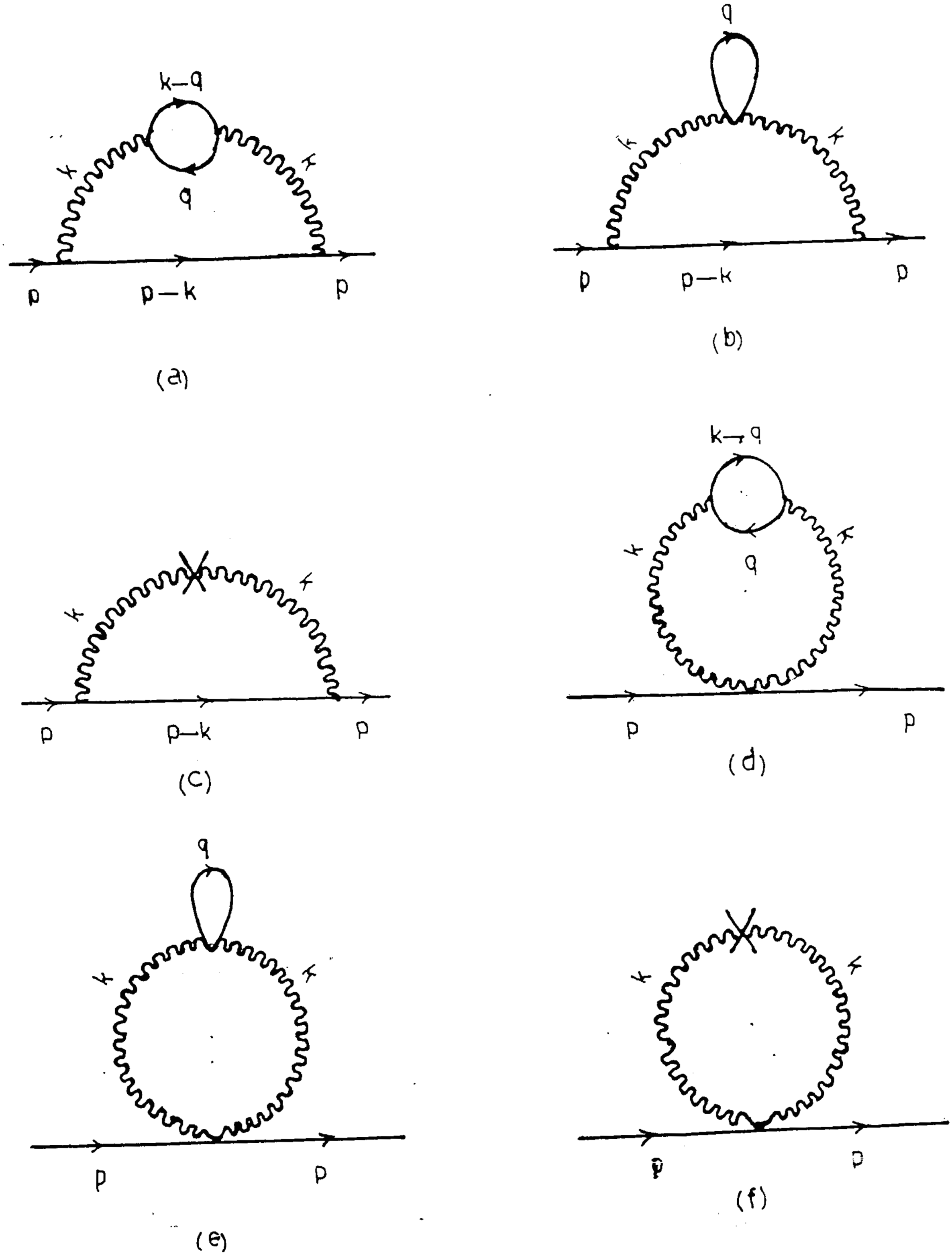
The nonlocal diagrams corresponding to fig.3.3(c), which contain vacuum polarization as a part, are given in fig. 3.7. There are six diagrams corresponding to fig.3.3(c).The individual contributions are:

$$\Sigma_a(p) = -e^2\exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\int_1^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}\int_1^\infty\frac{d\tau_3}{\Lambda^2}$$

$$\times\gamma^\sigma\{\Pi_1^{\sigma\rho}(q)\}(\not p-\not k-m)\gamma^\rho\eta^{\sigma\rho}\exp\left[-\tau_1\frac{k^2}{\Lambda^2}-\tau_2\frac{k^2}{\Lambda^2}-\tau_3\frac{(p-k)^2+m^2}{\Lambda^2}\right] \tag{3.4.19a}$$

$$\Sigma_b(p) = -e^2\exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\int_1^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}\int_1^\infty\frac{d\tau_3}{\Lambda^2}$$

$$\times\gamma^\sigma\{\Pi_2^{\sigma\rho}(q)\}(\not p-\not k-m)\gamma^\rho\eta^{\sigma\rho}\exp\left[-\tau_1\frac{k^2}{\Lambda^2}-\tau_2\frac{k^2}{\Lambda^2}-\tau_3\frac{(p-k)^2+m^2}{\Lambda^2}\right] \tag{3.4.19b}$$



Electron self-energy graphs at two loops in nonlocal QED corresponding ti fig. 3.3(c)

Fig.3.7.

$\Sigma_c(p)$ contains the measure factor [28] contribution.

$$\Sigma_c(p) = -e^2 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\int_1^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}\int_1^\infty\frac{d\tau_3}{\Lambda^2}$$

$$\times\,\gamma^\sigma\{\Pi_3^{\sigma\rho}(q)\}\,(\not p-\not k-m)\,\gamma^\rho\eta^{\sigma\rho}\exp\left[-\tau_1\frac{k^2}{\Lambda^2}-\tau_2\frac{k^2}{\Lambda^2}-\tau_3\frac{(p-k)^2+m^2}{\Lambda^2}\right] \quad (3.4.19c)$$

where $\Pi_1^{\sigma\rho}$, $\Pi_2^{\sigma\rho}$ and $\Pi_3^{\sigma\rho}$ are as given in chapter 1. Adding the three contributions, as the longitudinal parts sum to zero, only the transverse parts remain [28] which can be written as:

$$\Sigma_I(p) = -e^2 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\int_1^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}\int_1^\infty\frac{d\tau_3}{\Lambda^2}$$

$$\times\,\gamma^\sigma\{\Pi^{\sigma\rho}(q)\}\,(\not p-\not k-m)\,\gamma^\rho\,\eta^{\sigma\rho}\exp\left[-\tau_1\frac{k^2}{\Lambda^2}-\tau_2\frac{k^2}{\Lambda^2}-\tau_3\frac{(p-k)^2+m^2}{\Lambda^2}\right] \quad (3.4.20)$$

where $\Pi^{\sigma\rho}(q) = \left[\eta^{\sigma\rho} - \frac{q^\sigma q^\rho}{q^2}\right]\Pi^T(q^2)$. $\Pi^T(q^2)$ is the transverse part of $\Pi^{\sigma\rho}(q)$. Similarly the other three contributions add up to :

$$\Sigma_{II}(p) = -e^2 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\int_1^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}\int_0^1\frac{d\tau_3}{\Lambda^2}$$

$$\times\,\gamma^\sigma\{\Pi^{\sigma\rho}(q)\}\,(\not p-\not k-m)\,\gamma^\rho\,\eta^{\sigma\rho}\exp\left[-\tau_1\frac{k^2}{\Lambda^2}-\tau_2\frac{k^2}{\Lambda^2}-\tau_3\frac{(p-k)^2+m^2}{\Lambda^2}\right] \quad (3.4.21)$$

Therefore, the total contribution from all the six graphs is :

$$\Sigma_C(p) = -e^2 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\int_1^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}\int_0^\infty\frac{d\tau_3}{\Lambda^2}$$

$$\times\, \gamma^\sigma\{\Pi^{\sigma\rho}(q)\}\,(\not{p}-\not{k}-m)\,\gamma^\rho\,\eta^{\sigma\rho}\exp\left[-\tau_1\frac{k^2}{\Lambda^2}-\tau_2\frac{k^2}{\Lambda^2}\right.$$

$$\left.-\tau_3\frac{(p-k)^2+m^2}{\Lambda^2}\right] \qquad (3.4.22)$$

Again converting the Schwinger integrals to propagators,

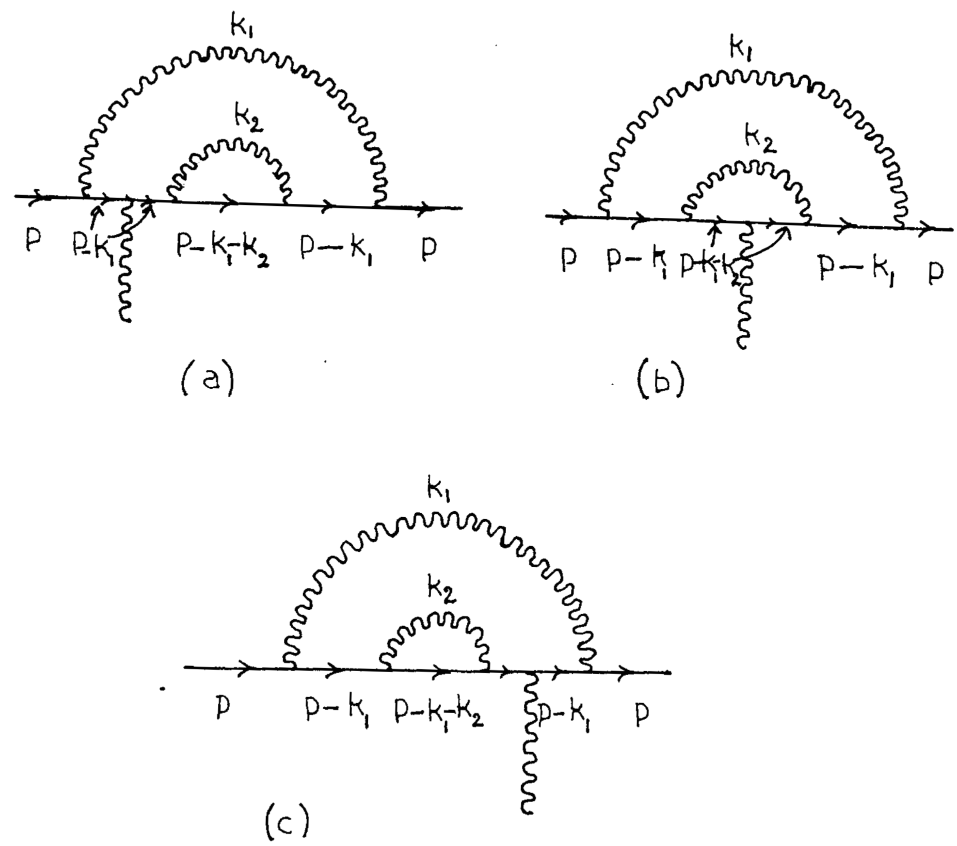
$$\Sigma_C(p) = -e^2 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4}\exp\left(-\frac{k^2}{\Lambda^2}\right)\frac{1}{k^2-i\epsilon}$$

$$\times\, \gamma^\sigma\{\Pi^{\sigma\rho}(q)\}\frac{1}{\not{p}-\not{k}+m-i\epsilon}\gamma^\rho\frac{1}{k^2-i\epsilon}\eta^{\sigma\rho} \qquad (3.4.23)$$

$\Sigma_A(p)$, $\Sigma_B(p)$ and $\Sigma_C(p)$ give the fourth order electron self–energy in nonlocal QED.

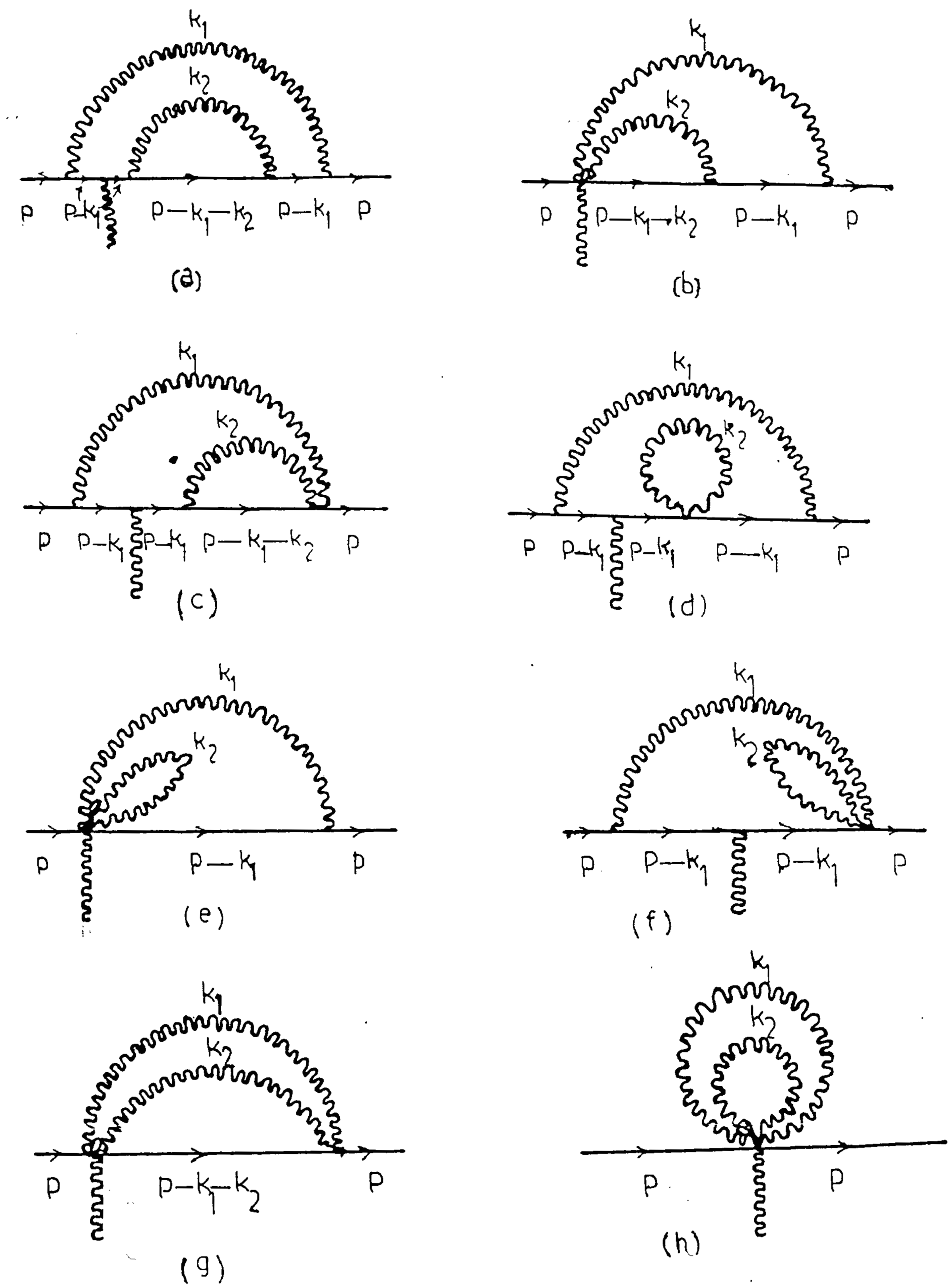
3.5 Ward Identity to order e^4 in Nonlocal Regularization

The vertex graphs to order e^4 in QED, corresponding to the electron self-energy given in fig.3.3(a) are given in fig.3.8.



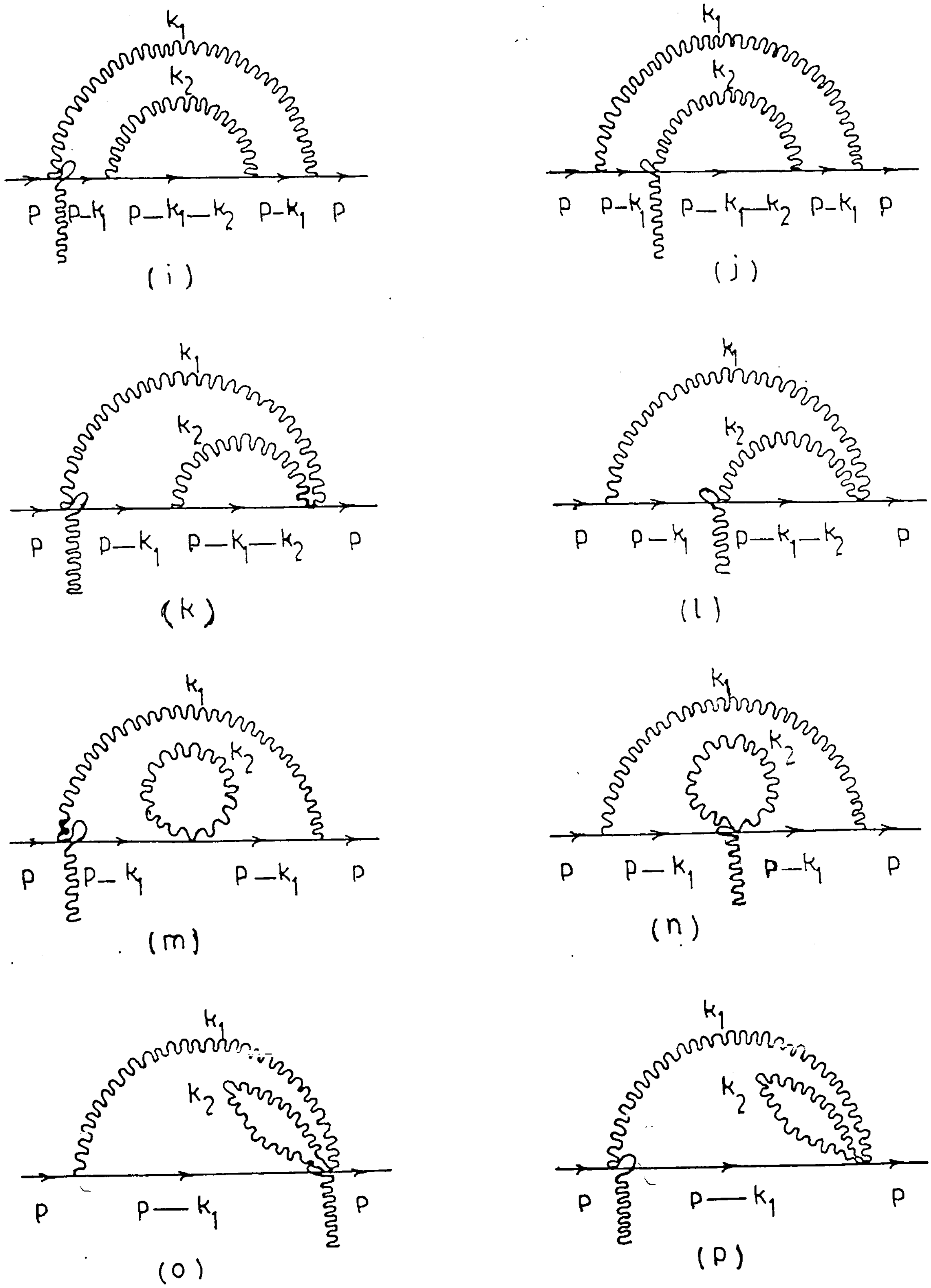
Insertions of photon to second order self-energy
graph fig.3.3(a)(QED vertex part)
Fig. 3.8

For nonlocal QED, the vertex graphs may be obtained by contracting two or more QED vertices of order e to form a nonlocal QED vertex of order e^2 or more in the corresponding order. The photon lines are on one side of the two fermion lines. Corresponding to each diagram 3.8(a), (b) and (c), there are a set of sixteen vertex diagrams in nonlocal QED. The nonlocal QED graphs corresponding to fig. 3.8(a) are given in fig. 3.9.



Fourth order vertex graphs corresponding to fig. 3.8(a) in nonlocal QED.

Fig.3.9.



k_1
k_2
p $p-k_1$ $p-k_1-k_2$ $p-k_1$ p
(i)
k_1
k_2
p $p-k_1$ $p-k_1-k_2$ $p-k_1$ p
(j)
k_1
k_2
p $p-k_1$ $p-k_1-k_2$ p
(k)
k_1
k_2
p $p-k_1$ $p-k_1-k_2$ p
(l)
k_1
k_2
p $p-k_1$ $p-k_1$ p
(m)
k_1
k_2
p $p-k_1$ $p-k_1$ p
(n)
k_1
k_2
p $p-k_1$ p
(o)
k_1
k_2
p $p-k_1$ p
(p)

Fig.3.9.(contd).

The individual contributions are given below:

$$e\,\Lambda_a^{\mu}(p,p) = -e^5 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int_1^{\infty}\frac{d\tau_1}{\Lambda^2}\int_1^{\infty}\frac{d\tau_2}{\Lambda^2}\int_1^{\infty}\frac{d\tau_3}{\Lambda^2}\int_1^{\infty}\frac{d\tau_4}{\Lambda^2}$$

$$\times\int_1^{\infty}\frac{d\tau_5}{\Lambda^2}\int_1^{\infty}\frac{d\tau_6}{\Lambda^2}\gamma^t(\not p-\not k_1-m)\,\gamma^{\mu}\,(\not p-\not k_1-m)\,\gamma^{\rho}\,(\not p-\not k_1-\not k_2-m)\gamma_{\rho}(\not p-\not k_1-m)\,\gamma_t$$

$$\times\exp\left[-\tau_1\frac{k_1^2}{\Lambda^2}-\tau_2\frac{k_2^2}{\Lambda^2}-\tau_3\frac{(p-k_1)^2+m^2}{\Lambda^2}-\tau_4\frac{(p-k_1-k_2)^2+m^2}{\Lambda^2}\right.$$

$$\left.-\tau_5\frac{(p-k_1)^2+m^2}{\Lambda^2}-\tau_6\frac{(p-k_1)^2+m^2}{\Lambda^2}\right] \qquad (3.5.1\ )$$

$$\Lambda_a^{\mu}(p,p) = -e^4\int_1^{\infty}\frac{d\tau_3}{\Lambda^2}\int_1^{\infty}\frac{d\tau_4}{\Lambda^2}\int_1^{\infty}\frac{d\tau_5}{\Lambda^2}\int_1^{\infty}\frac{d\tau_6}{\Lambda^2}\ I_3, \qquad (3.5.2a)$$

$$\text{where } I_3 = \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int_1^{\infty}\frac{d\tau_1}{\Lambda^2}\int_1^{\infty}\frac{d\tau_2}{\Lambda^2}$$

$$\times\gamma^t(\not p-\not k_1-m)\,\gamma^{\mu}(\not p-\not k_1-m)\gamma^{\rho}(\not p-\not k_1-\not k_2-m)\,\gamma_{\rho}\,(\not p-\not k_1-m)\,\gamma_t$$

$$\times\exp\left[-\tau_1\frac{k_1^2}{\Lambda^2}-\tau_2\frac{k_2^2}{\Lambda^2}-\tau_3\frac{(p-k_1)^2+m^2}{\Lambda^2}-\tau_4\frac{(p-k_1-k_2)^2+m^2}{\Lambda^2}\right.$$

$$\left.-\tau_5\frac{(p-k_1)^2+m^2}{\Lambda^2}-\tau_6\frac{(p-k_1)^2+m^2}{\Lambda^2}\right] \qquad (3.5.2b)$$

$$\Lambda_b^{\mu}(p,p) = -e^4\int_0^{1}\frac{d\tau_3}{\Lambda^2}\int_1^{\infty}\frac{d\tau_4}{\Lambda^2}\int_1^{\infty}\frac{d\tau_5}{\Lambda^2}\int_0^{1}\frac{d\tau_6}{\Lambda^2}\ I_3 \qquad (3.5.3)$$

$$\Lambda_c^{\mu}(p,p) = -e^4\int_1^{\infty}\frac{d\tau_3}{\Lambda^2}\int_1^{\infty}\frac{d\tau_4}{\Lambda^2}\int_0^{1}\frac{d\tau_5}{\Lambda^2}\int_1^{\infty}\frac{d\tau_6}{\Lambda^2}\ I_3. \qquad (3.5.4)$$

$$\Lambda_d^{\mu}(p,p) = -e^4 \int_1^{\infty} \frac{d\tau_3}{\Lambda^2} \int_0^{1} \frac{d\tau_4}{\Lambda^2} \int_1^{\infty} \frac{d\tau_5}{\Lambda^2} \int_1^{\infty} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.5)$$

$$\Lambda_e^{\mu}(p,p) = -e^4 \int_0^{1} \frac{d\tau_3}{\Lambda^2} \int_0^{1} \frac{d\tau_4}{\Lambda^2} \int_1^{\infty} \frac{d\tau_5}{\Lambda^2} \int_0^{1} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.6)$$

$$\Lambda_f^{\mu}(p,p) = -e^4 \int_1^{\infty} \frac{d\tau_3}{\Lambda^2} \int_0^{1} \frac{d\tau_4}{\Lambda^2} \int_0^{1} \frac{d\tau_5}{\Lambda^2} \int_1^{\infty} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.7)$$

$$\Lambda_g^{\mu}(p,p) = -e^4 \int_0^{1} \frac{d\tau_3}{\Lambda^2} \int_1^{\infty} \frac{d\tau_4}{\Lambda^2} \int_0^{1} \frac{d\tau_5}{\Lambda^2} \int_0^{1} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.8)$$

$$\Lambda_h^{\mu}(p,p) = -e^4 \int_0^{1} \frac{d\tau_3}{\Lambda^2} \int_0^{1} \frac{d\tau_4}{\Lambda^2} \int_0^{1} \frac{d\tau_5}{\Lambda^2} \int_0^{1} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.9)$$

$$\Lambda_i^{\mu}(p,p) = -e^4 \int_0^{1} \frac{d\tau_3}{\Lambda^2} \int_1^{\infty} \frac{d\tau_4}{\Lambda^2} \int_1^{\infty} \frac{d\tau_5}{\Lambda^2} \int_1^{\infty} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.10)$$

$$\Lambda_j^{\mu}(p,p) = -e^4 \int_1^{\infty} \frac{d\tau_3}{\Lambda^2} \int_1^{\infty} \frac{d\tau_4}{\Lambda^2} \int_1^{\infty} \frac{d\tau_5}{\Lambda^2} \int_0^{1} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.11)$$

$$\Lambda_k^{\mu}(p,p) = -e^4 \int_0^{1} \frac{d\tau_3}{\Lambda^2} \int_1^{\infty} \frac{d\tau_4}{\Lambda^2} \int_0^{1} \frac{d\tau_5}{\Lambda^2} \int_1^{\infty} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.12)$$

$$\Lambda_l^{\mu}(p,p) = -e^4 \int_1^{\infty} \frac{d\tau_3}{\Lambda^2} \int_1^{\infty} \frac{d\tau_4}{\Lambda^2} \int_0^{1} \frac{d\tau_5}{\Lambda^2} \int_0^{1} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.13)$$

$$\Lambda_m^{\mu}(p,p) = -e^4 \int_0^{1} \frac{d\tau_3}{\Lambda^2} \int_0^{1} \frac{d\tau_4}{\Lambda^2} \int_1^{\infty} \frac{d\tau_5}{\Lambda^2} \int_1^{\infty} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.14)$$

$$\Lambda_n^{\mu}(p,p) = -e^4 \int_1^{\infty} \frac{d\tau_3}{\Lambda^2} \int_0^{1} \frac{d\tau_4}{\Lambda^2} \int_1^{\infty} \frac{d\tau_5}{\Lambda^2} \int_0^{1} \frac{d\tau_6}{\Lambda^2} \; I_3 \qquad (3.5.15)$$

$$\Lambda_o^{\mu}(p,p) = -e^4 \int_1^{\infty} \frac{d\tau_3}{\Lambda^2} \int_0^1 \frac{d\tau_4}{\Lambda^2} \int_0^1 \frac{d\tau_5}{\Lambda^2} \int_0^1 \frac{d\tau_6}{\Lambda^2} \, I_3 \qquad (3.5.16)$$

$$\Lambda_p^{\mu}(p,p) = -e^4 \int_0^1 \frac{d\tau_3}{\Lambda^2} \int_0^1 \frac{d\tau_4}{\Lambda^2} \int_0^1 \frac{d\tau_5}{\Lambda^2} \int_1^{\infty} \frac{d\tau_6}{\Lambda^2} \, I_3 \qquad (3.5.17)$$

Summing the above terms, one obtains,

$$e\Lambda_I^{\mu}(p,p) = -e^5 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int_1^{\infty} \frac{d\tau_1}{\Lambda^2} \int_1^{\infty} \frac{d\tau_2}{\Lambda^2} \int_0^{\infty} \frac{d\tau_3}{\Lambda^2} \int_0^{\infty} \frac{d\tau_4}{\Lambda^2}$$

$$\times \int_0^{\infty} \frac{d\tau_5}{\Lambda^2} \int_0^{\infty} \frac{d\tau_6}{\Lambda^2} \gamma^t (\not{p}-\not{k}_1-m)\, \gamma^{\mu}\, (\not{p}-\not{k}_1-m)\, \gamma^{\rho} (\not{p}-\not{k}_1-\not{k}_2-m)\, \gamma_{\rho}\, (\not{p}-\not{k}_1-m)\, \gamma_t$$

$$\times \exp\left[-\tau_1 \frac{k_1^2}{\Lambda^2} - \tau_2 \frac{k_2^2}{\Lambda^2} - \tau_3 \frac{(p-k_1)^2+m^2}{\Lambda^2} - \tau_4 \frac{(p-k_1-k_2)^2+m^2}{\Lambda^2}\right.$$

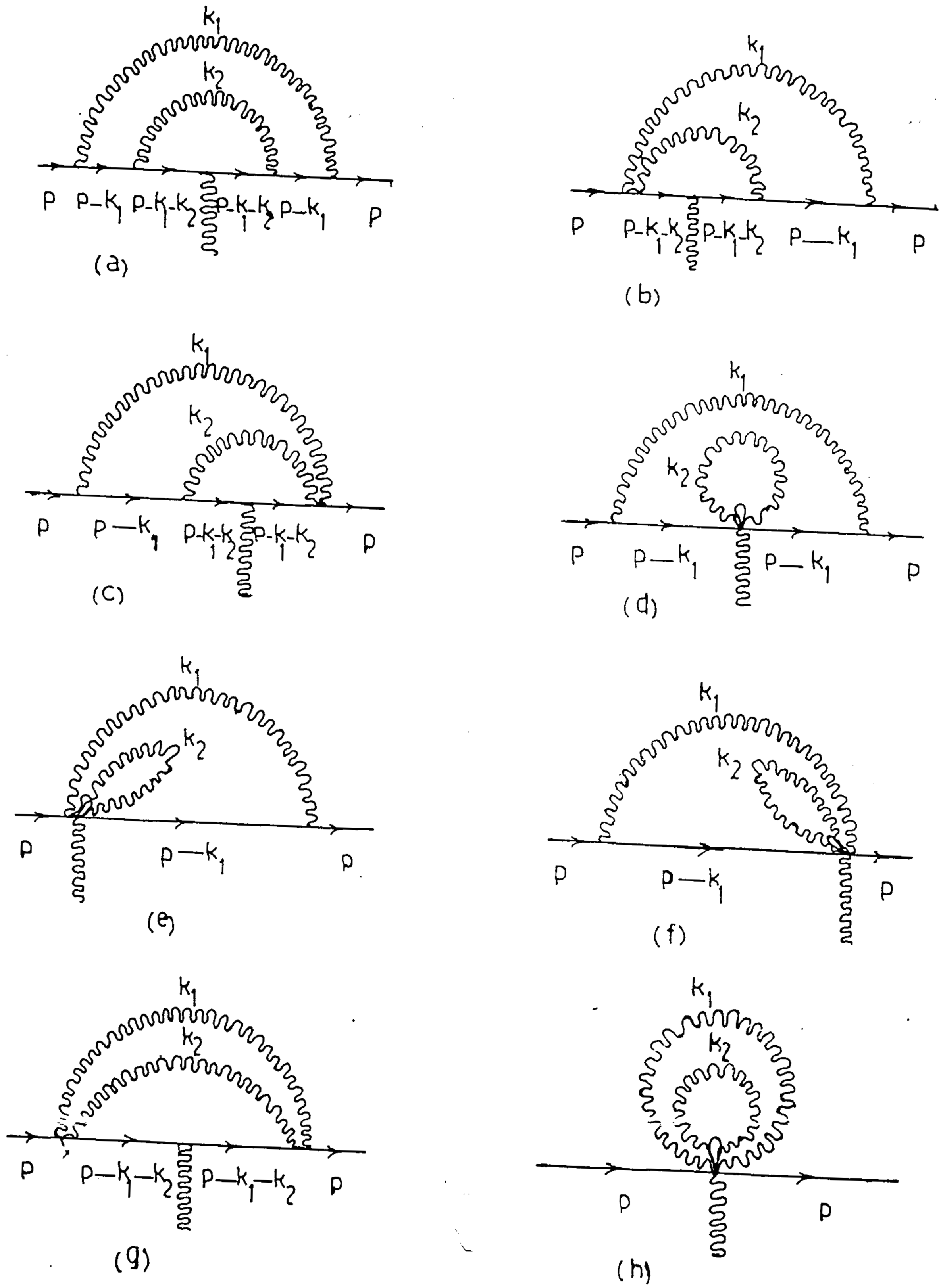
$$\left. - \tau_5 \frac{(p-k_1)^2+m^2}{\Lambda^2} - \tau_6 \frac{(p-k_1)^2+m^2}{\Lambda^2}\right] \qquad (3.5.18a)$$

$$= -e^5 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \int \frac{d^4k_1}{(2\pi)^4} \exp\left(-\frac{k_1^2}{\Lambda^2}\right) \int \frac{d^4k_2}{(2\pi)^4} \exp\left(-\frac{k_2^2}{\Lambda^2}\right)$$

$$\times \frac{1}{k_1^2-i\epsilon}\, \gamma^t\, \frac{1}{\not{p}-\not{k}_1+m-i\epsilon}\, \frac{1}{k_2^2-i\epsilon}\, \gamma^{\mu}\, \frac{1}{\not{p}-\not{k}_1+m-i\epsilon}\, \gamma^{\rho}$$

$$\times \frac{1}{\not{p}-\not{k}_1-\not{k}_2+m-i\epsilon}\, \gamma_{\rho}\, \frac{1}{\not{p}-\not{k}_1+m-i\epsilon}\, \gamma_t \qquad (3.5.18b)$$

The nonlocal QED vertex graphs corresponding to fig. 3.8(b) are given in fig 3.10. There are sixteen such graphs.



FourthOorder vertex graphs corresponding to fig.3.8(b) in nonlocal QED.

Fig.3.10.

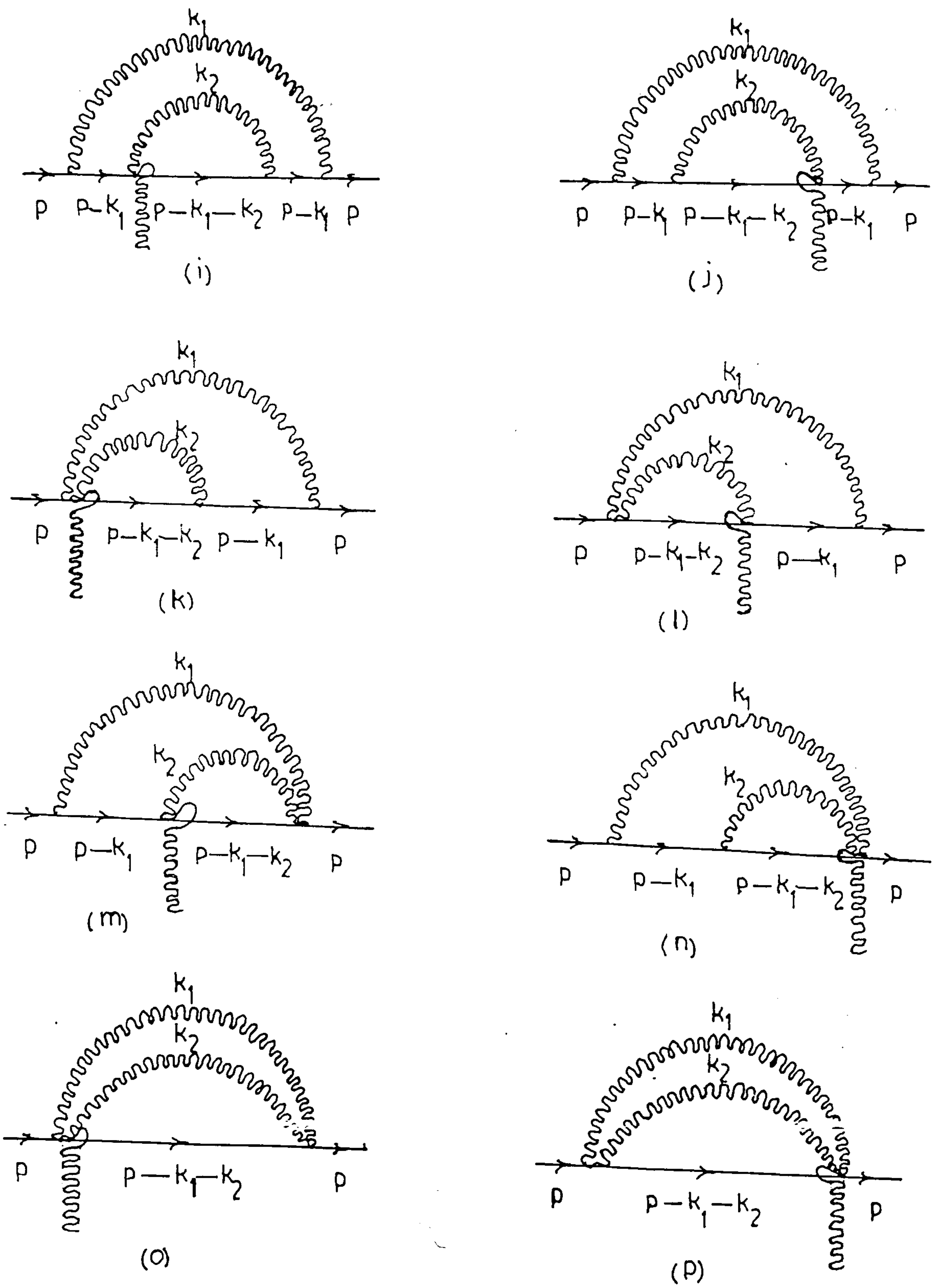


Fig.3.10.(contd).

Following the same method as done above we obtain:

$$e\Lambda^{\mu}_{II}(p,p) = -e^5 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \int \frac{d^4k_1}{(2\pi)^4} \exp\left(-\frac{k_1^2}{\Lambda^2}\right) \int \frac{d^4k_2}{(2\pi)^4} \exp\left(-\frac{k_2^2}{\Lambda^2}\right)$$

$$\times \frac{1}{k_1^2-i\epsilon}\gamma^t \frac{1}{\not{p}-\not{k}_1+m-i\epsilon} \frac{1}{k_2^2-i\epsilon}\gamma^\rho \frac{1}{\not{p}-\not{k}_1-\not{k}_2+m-i\epsilon}\gamma^\mu$$

$$\times \frac{1}{\not{p}-\not{k}_1-\not{k}_2+m-i\epsilon}\gamma_\rho \frac{1}{\not{p}-\not{k}_1+m-i\epsilon}\gamma_t \qquad (3.5.19\ )$$

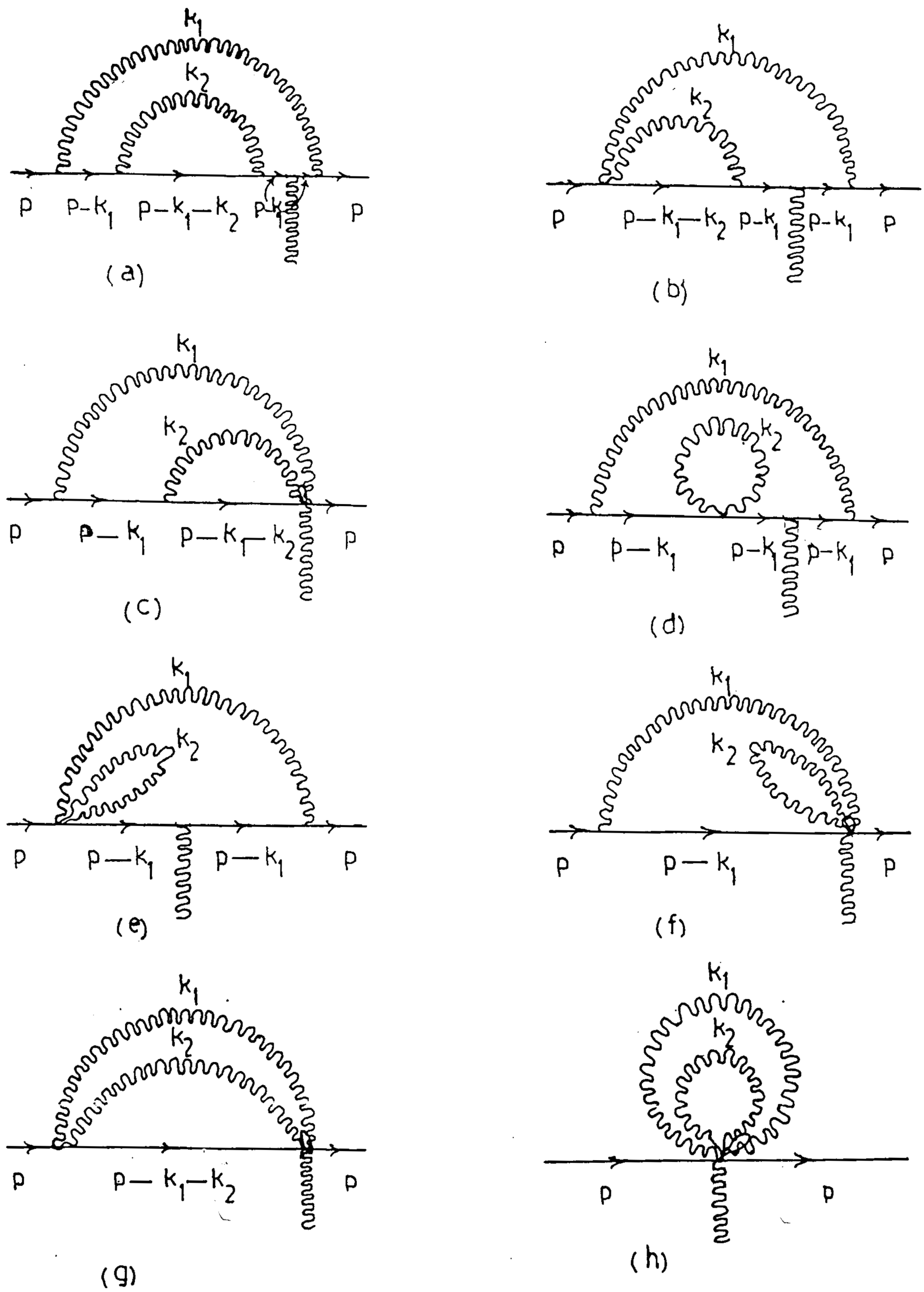
The nonlocal QED vertex graphs corresponding to fig.3.8(c) are given in fig.3.11. They are sixteen in number. The individual contributions sum upto

$$e\Lambda^{\mu}_{III}(p,p) = -e^5 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \int \frac{d^4k_1}{(2\pi)^4} \exp\left(-\frac{k_1^2}{\Lambda^2}\right) \int \frac{d^4k_2}{(2\pi)^4} \exp\left(-\frac{k_2^2}{\Lambda^2}\right)$$

$$\times \frac{1}{k_1^2-i\epsilon}\gamma^t \frac{1}{\not{p}-\not{k}_1+m-i\epsilon} \frac{1}{k_2^2-i\epsilon}\gamma^\rho \frac{1}{\not{p}-\not{k}_1-\not{k}_2+m-i\epsilon}\gamma_\rho$$

$$\times \frac{1}{\not{p}-\not{k}_1+m-i\epsilon}\gamma^\mu \frac{1}{\not{p}-\not{k}_1+m-i\epsilon}\gamma_t \qquad (3.5.20)$$

The total contribution from all the diagrams of figs.3.9 to 3.11 is:



Fourth-order vertex graphs corresponding to fig. 3.8(c) in nonlocal QED.

Fig.3.11.

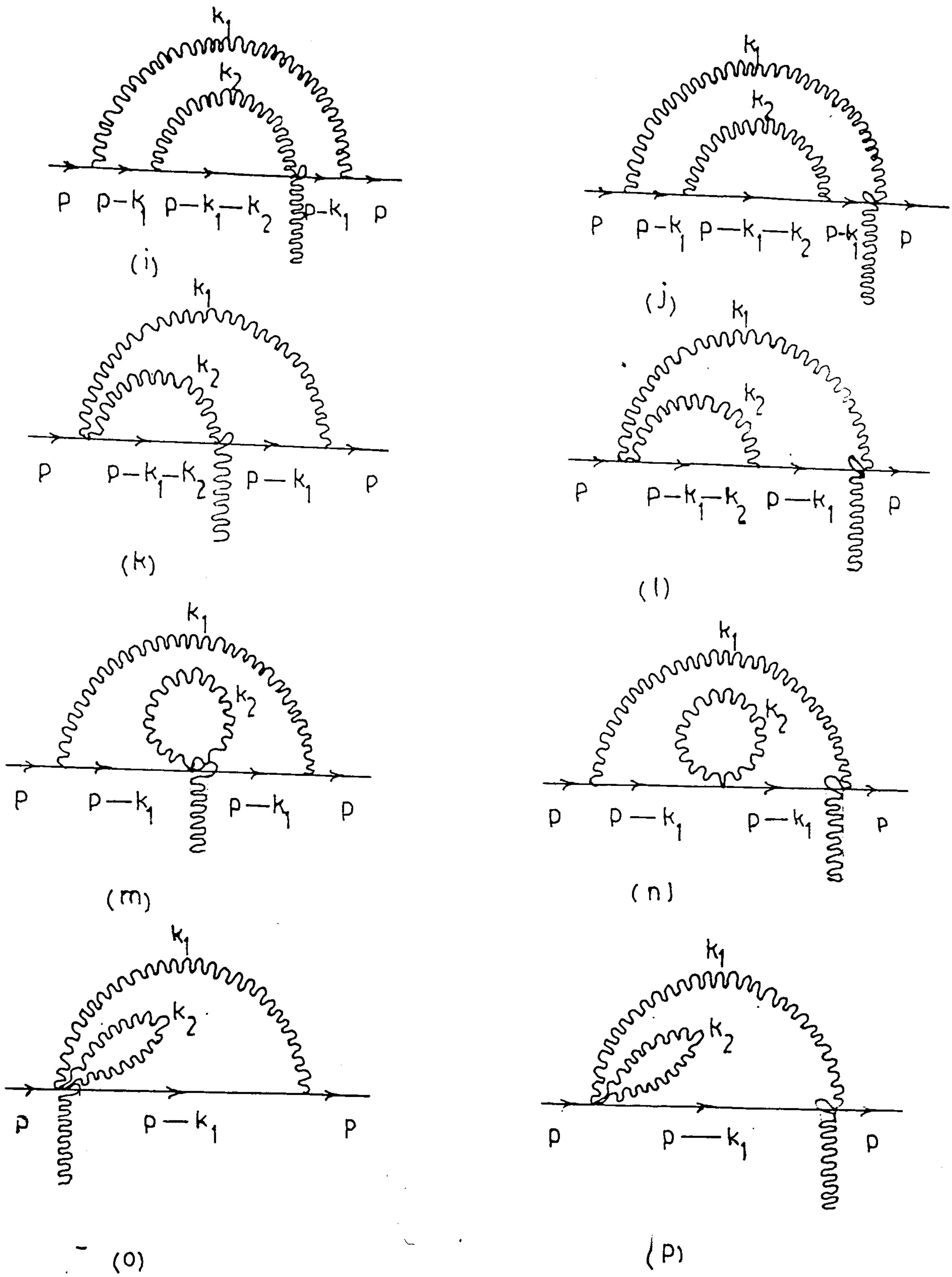


Fig.3.11.(contd)

$$e\,\Lambda_A^{\mu}(p,p) = -e^5 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k_1}{(2\pi)^4}\exp\left(-\frac{k_1^2}{\Lambda^2}\right)\int\frac{d^4k_2}{(2\pi)^4}\exp\left(-\frac{k_2^2}{\Lambda^2}\right)$$

$$\times\frac{1}{k_1^2-i\epsilon}\gamma^t\frac{1}{\not p-\not k_1+m-i\epsilon}\;\frac{1}{k_2^2-i\epsilon}\left[\gamma^{\mu}\frac{1}{\not p-\not k_1+m-i\epsilon}\gamma^{\rho}\frac{1}{\not p-\not k_1-\not k_2+m-i\epsilon}\right.$$

$$\gamma_{\rho}+\gamma^{\rho}\frac{1}{\not p-\not k_1-\not k_2+m-i\epsilon}\gamma^{\mu}\frac{1}{\not p-\not k_1-\not k_2+m-i\epsilon}\gamma_{\rho}+\gamma^{\rho}\frac{1}{\not p-\not k_1-\not k_2+m-i\epsilon}\gamma_{\rho}$$

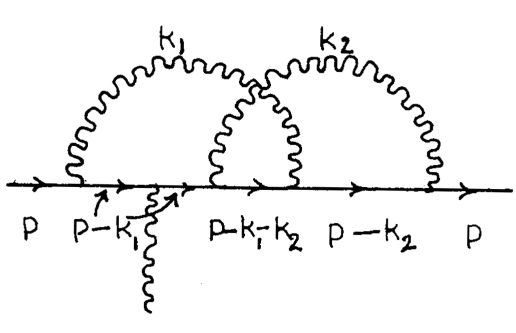
$$\left.\frac{1}{\not p-\not k_1+m-i\epsilon}\gamma^{\mu}\right]\frac{1}{\not p-\not k_1+m-i\epsilon}\gamma_t \qquad (3.5.21)$$

From equations (3.4.8b) and (3.5.21), we obtain

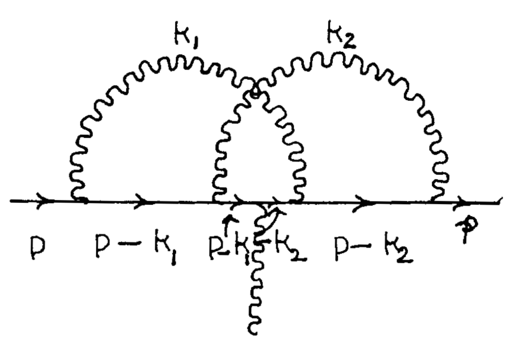
$$\exp\left(\frac{p^2+m^2}{\Lambda^2}\right)\Lambda^{\mu}{}_A(p,p) = -\frac{\partial}{\partial p_{\mu}}\left[\exp\left(\frac{p^2+m^2}{\Lambda^2}\right)\Sigma_A(p)\right], \qquad (3.5.22)$$

which is the Ward identity for nonlocal QED.

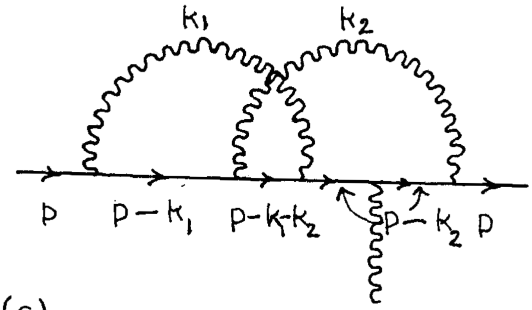
The QED vertex graphs to order e^4 corresponding to the electron self-energy given in fig. 3.3(b) (the case of overlapping divergence) are given below:



(a)



(b)



Insertions of photon into second-order self-energy of fig.3.3(b).

Fig 3.12

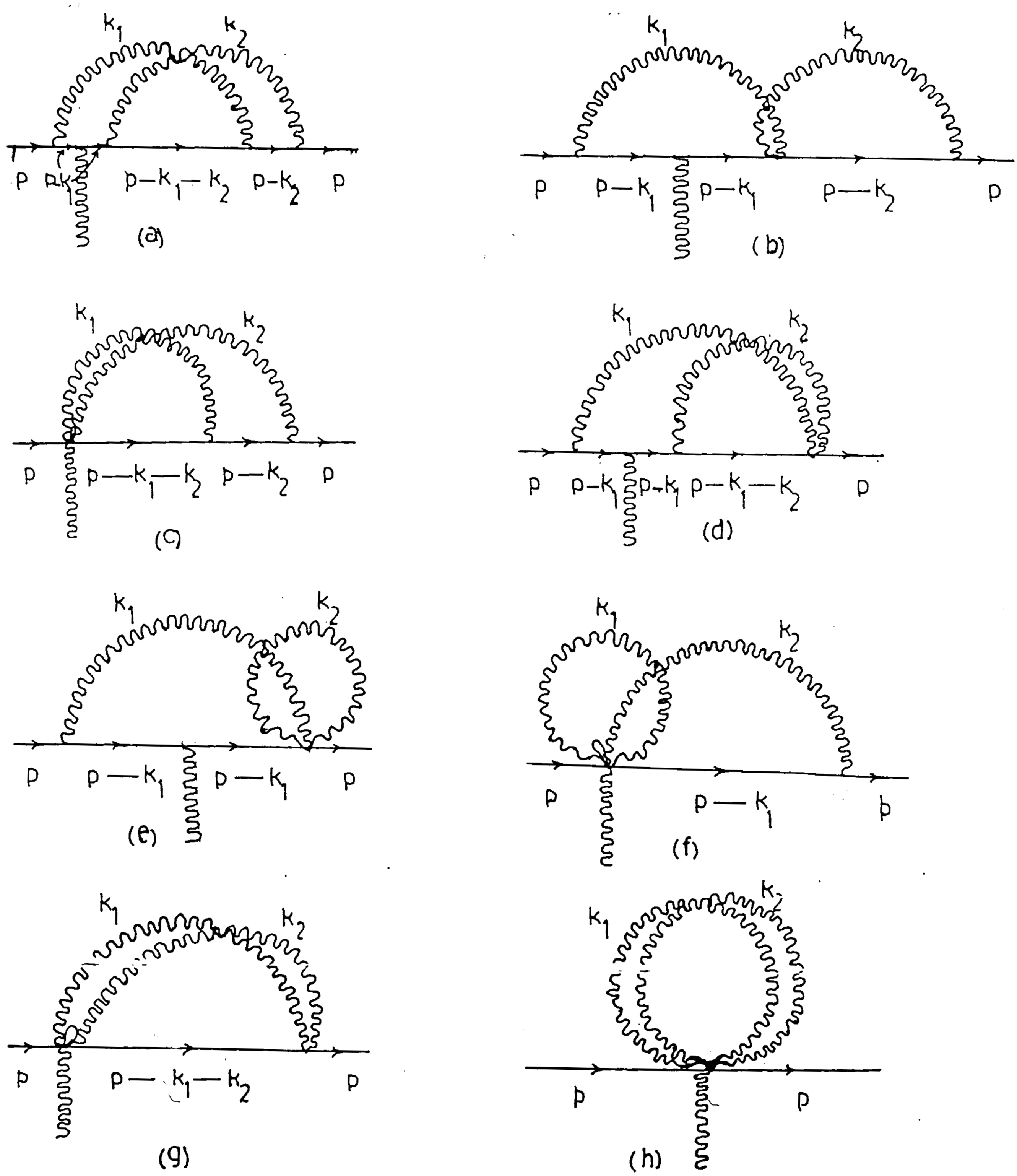
A photon of zero momentum can be inserted to any of the three electron propagators. Corresponding to each of the above diagrams, there are a set of sixteen vertex diagrams in nonlocal QED. The nonlocal QED graphs corresponding to fig.3.12(a) are given in fig.3.13. The individual contributions can be summed as before to give the total contribution,

$$e\Lambda_I^{\mu}(p,p) = -e^5 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int \frac{d^4k_1}{(2\pi)^4} \exp\left(-\frac{k_1^2}{\Lambda^2}\right)\int \frac{d^4k_2}{(2\pi)^4} \exp\left(-\frac{k_2^2}{\Lambda^2}\right)$$

$$\times \frac{1}{k_1^2-i\epsilon}\gamma^t \frac{}{\not p-\not k_1+m-i\epsilon}\gamma^{\mu}\frac{1}{\not p-\not k_1+m-i\epsilon}\gamma^{\rho}\frac{1}{\not p-\not k_1-\not k_2+m-i\epsilon}\gamma_t$$

$$\times \frac{1}{k_2^2-i\epsilon}\frac{1}{\not p-\not k_2+m-i\epsilon}\gamma_{\rho} \qquad (3.5.23)$$

The nonlocal QED graphs belonging to fig.3.12(b) are given in fig 3.14. The total contribution from these sixteen diagrams is:



Fourth-order vertex graphs corresponding to fig.3.12(a) in nonlocal QED.

Fig.3.13.

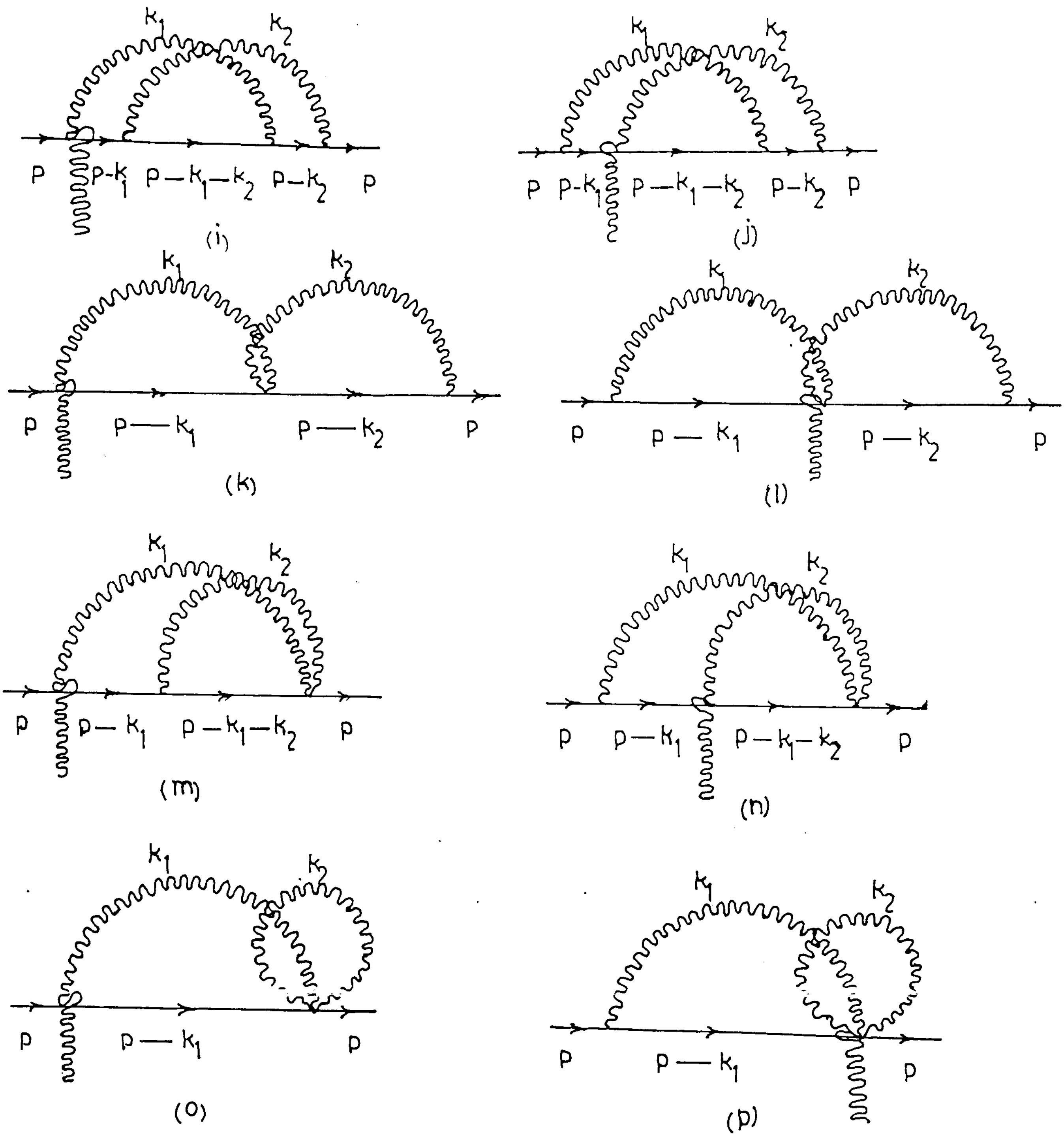
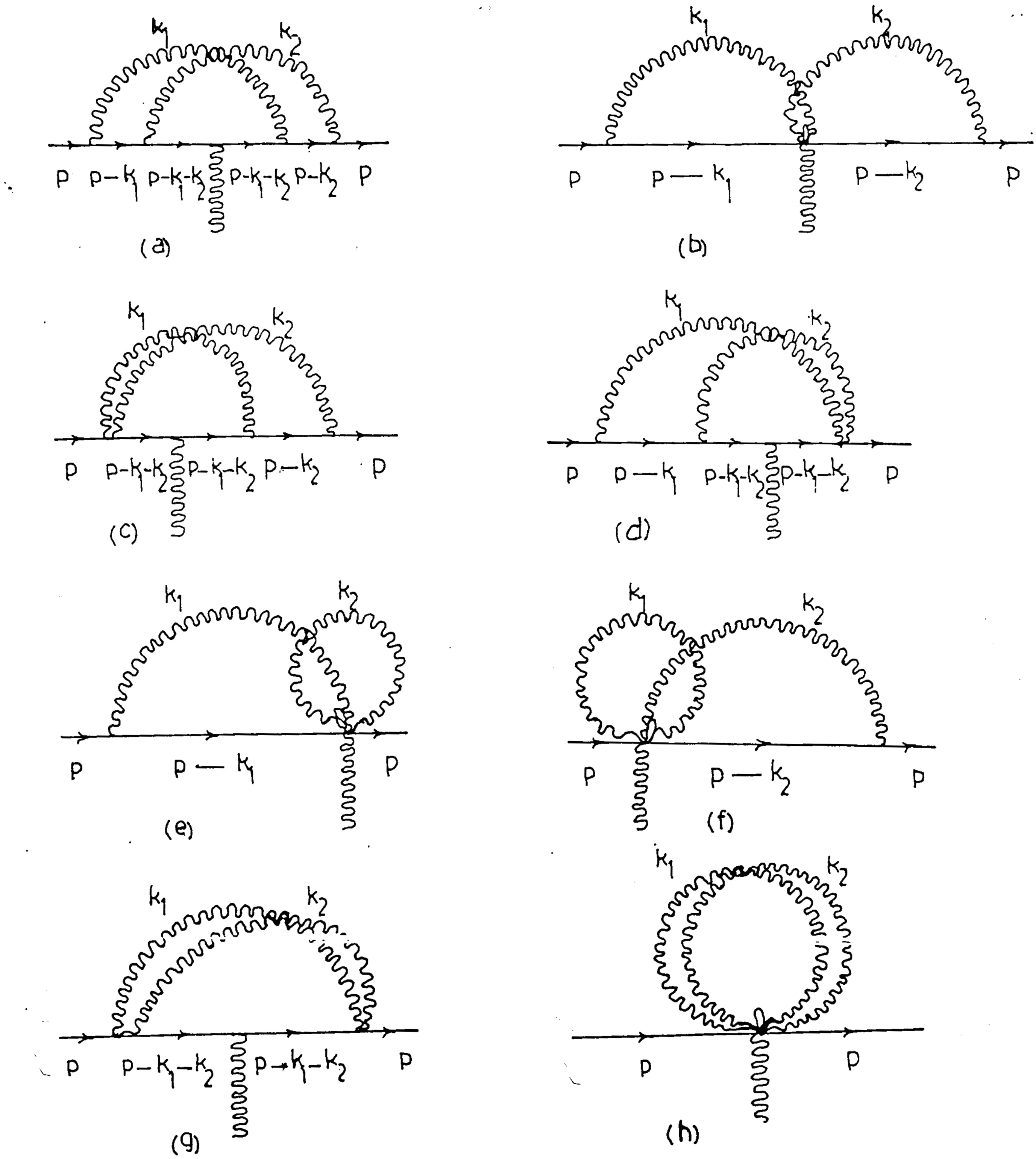


Fig.3.13.(contd)



Fourth-order vertex graphs corresponding to fig. 3.12(b) in nonlocal QED.

Fig.3.14.

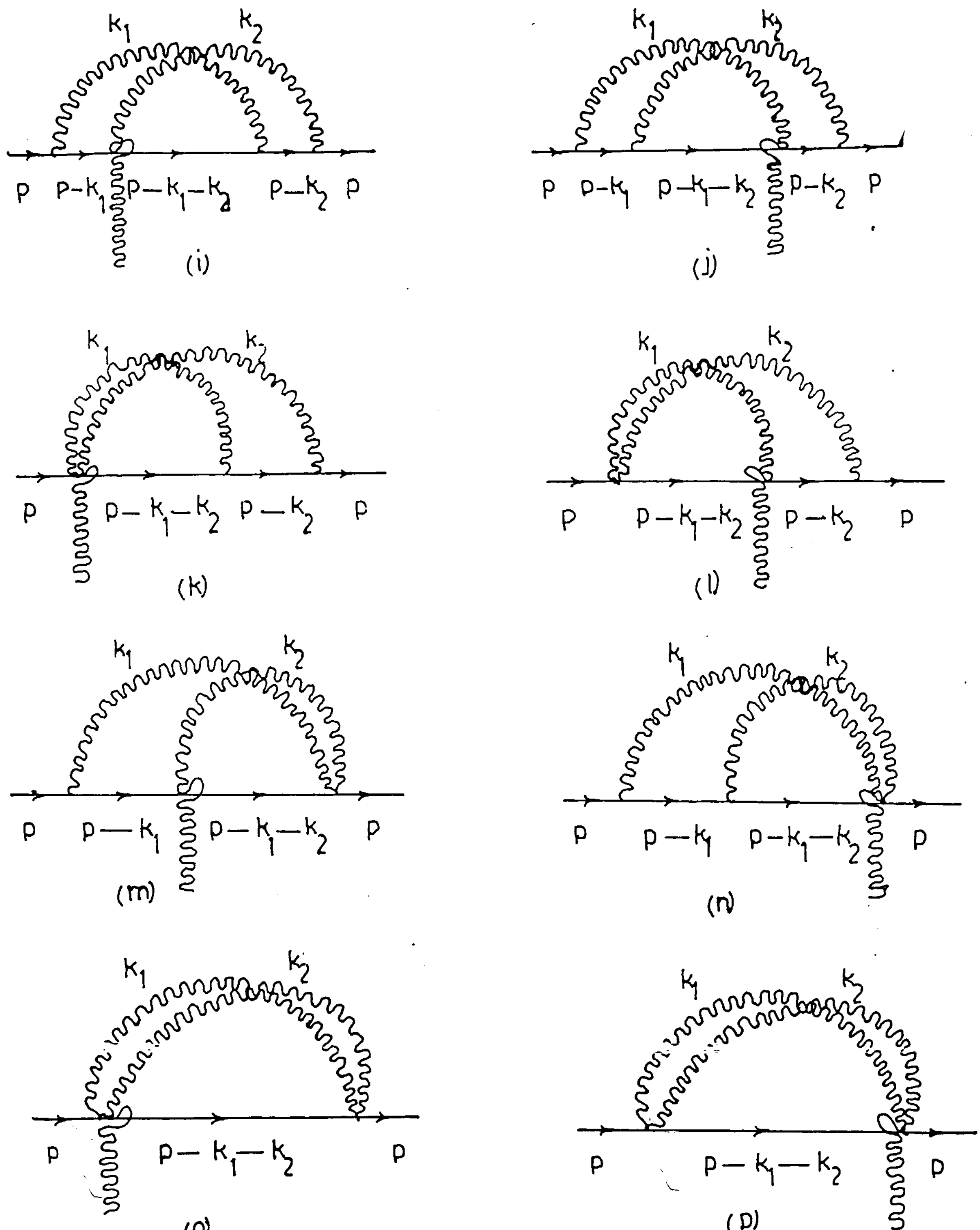


Fig.3.14.(contd)

$$e\Lambda_{II}^{\mu}(p,p) = -e^5 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k_1}{(2\pi)^4}\exp\left(-\frac{k_1^2}{\Lambda^2}\right)\int\frac{d^4k_2}{(2\pi)^4}\exp\left(-\frac{k_2^2}{\Lambda^2}\right)$$

$$\times\ \frac{1}{k_1^2-i\epsilon}\gamma^t\frac{1}{\not{p}-\not{k}_1+m-i\epsilon}\gamma^\rho\frac{1}{\not{p}-\not{k}_1-\not{k}_2+m-i\epsilon}\gamma^\mu\frac{1}{\not{p}-\not{k}_1-\not{k}_2+m-i\epsilon}\gamma_t$$

$$\times\ \frac{1}{k_2^2-i\epsilon}\ \frac{1}{\not{p}-\not{k}_2+m-i\epsilon}\gamma_\rho \qquad (3.5.24)$$

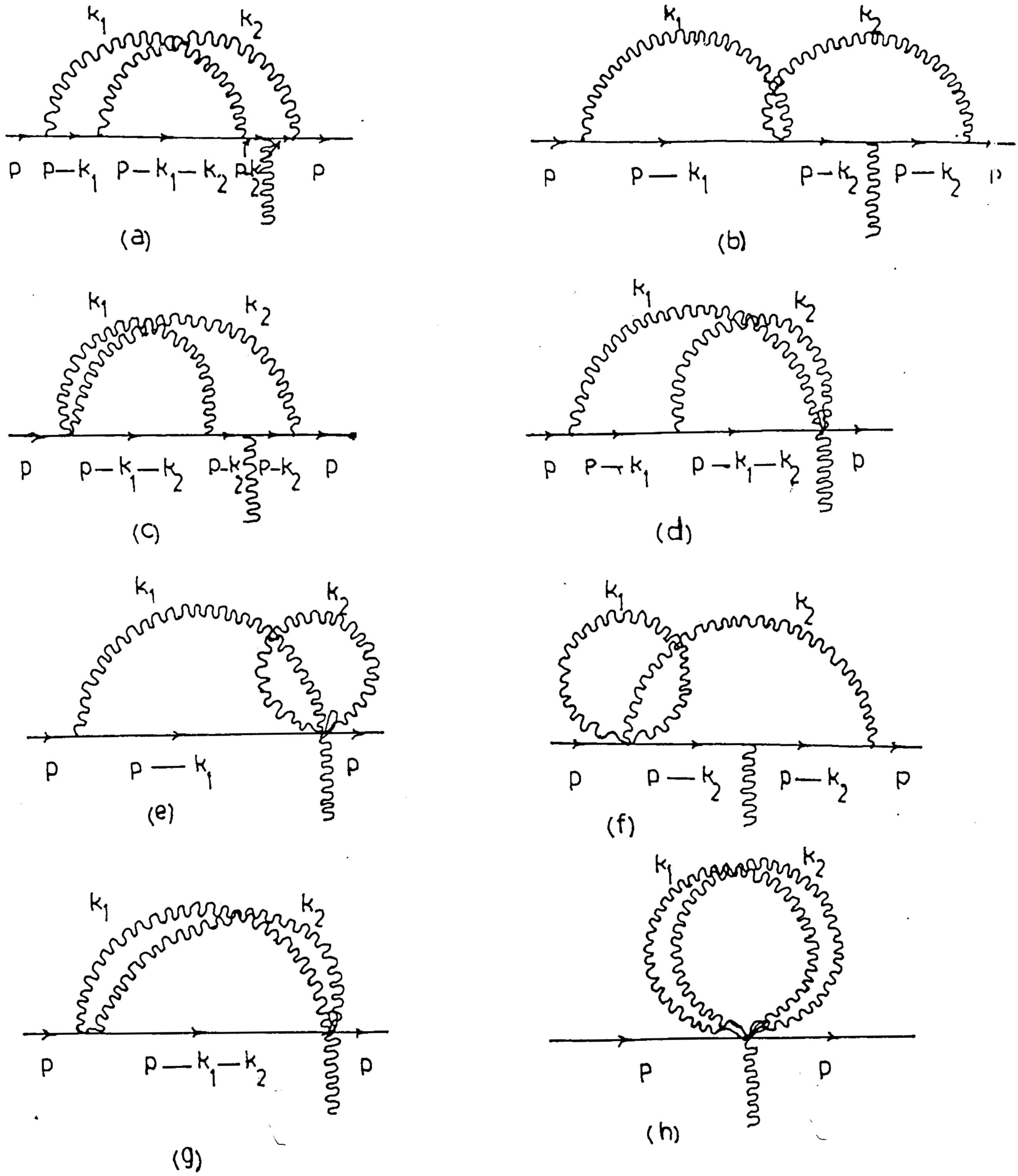
The nonlocal QED graphs belonging to fig. 3.12(c) are given in fig. 3.(15). The total contribution from this set of diagrams is given by:

$$e\Lambda_{III}^{\mu}(p,p) = -e^5 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k_1}{(2\pi)^4}\exp\left(-\frac{k_1^2}{\Lambda^2}\right)\int\frac{d^4k_2}{(2\pi)^4}\exp\left(-\frac{k_2^2}{\Lambda^2}\right)$$

$$\times\ \frac{1}{k_1^2-i\epsilon}\gamma^t\frac{1}{\not{p}-\not{k}_1+m-i\epsilon}\gamma^\rho\frac{1}{\not{p}-\not{k}_1-\not{k}_2+m-i\epsilon}\gamma_t\frac{1}{\not{p}-\not{k}_2+m-i\epsilon}\gamma^\mu$$

$$\times\ \frac{1}{k_2^2-i\epsilon}\ \frac{1}{\not{p}-\not{k}_2+m-i\epsilon}\gamma_\rho \qquad (3.5.25)$$

The total contribution from figs.3.13 to 3.15 is :



Fourth-order vertex graphs corresponding to fig.3.12(c) in nonlocal QED.

Fig.3.15.

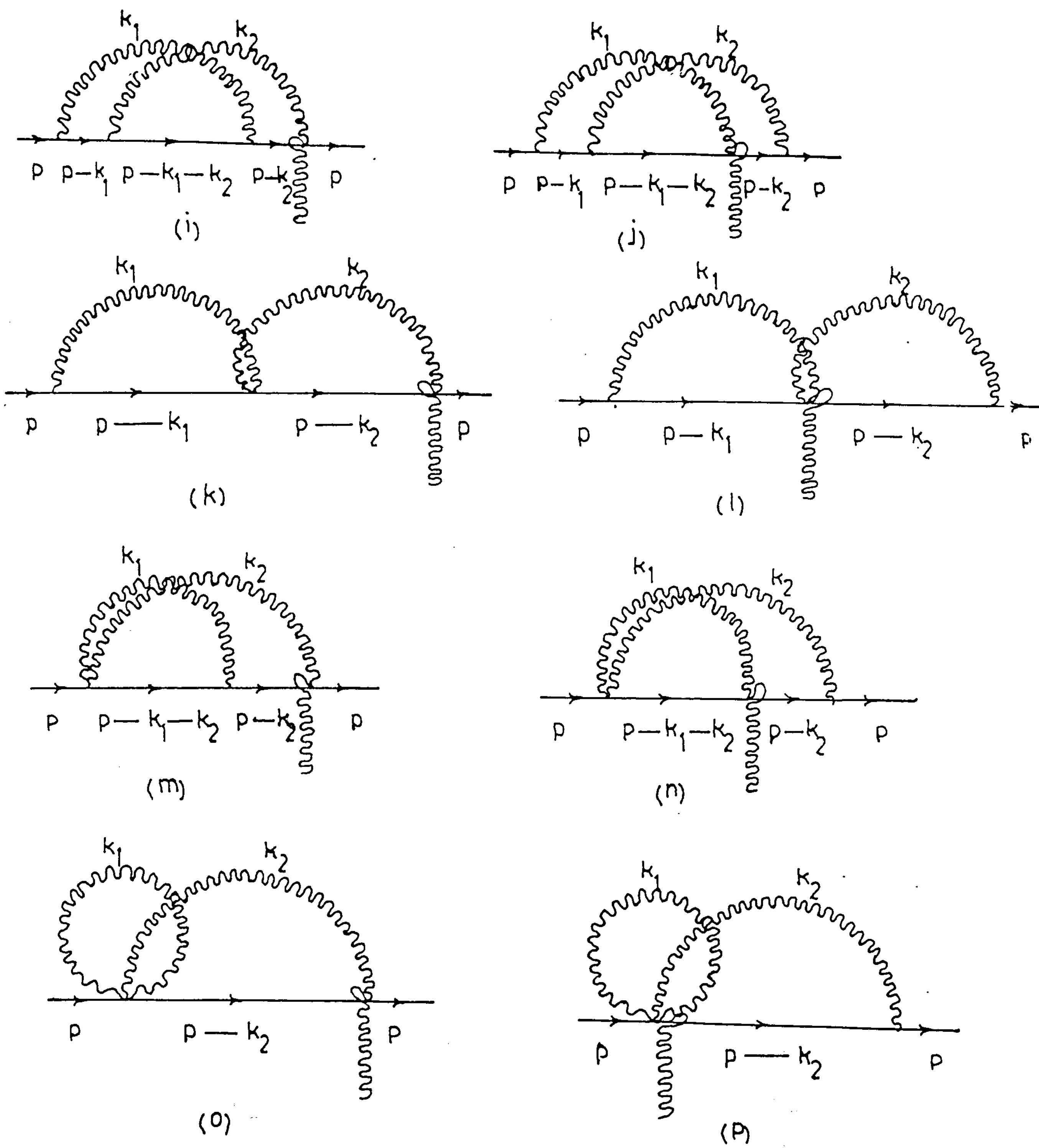


Fig.3.15.(contd)

$$e\,\Lambda_B^{\mu}(p,p) = -e^5 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k_1}{(2\pi)^4}\exp\left(-\frac{k_1^2}{\Lambda^2}\right)\int\frac{d^4k_2}{(2\pi)^4}\exp\left(-\frac{k_2^2}{\Lambda^2}\right)$$

$$\times\frac{1}{k_1^2-i\epsilon}\gamma^t\frac{1}{\not p-\not k_1+m-i\epsilon}\left[\gamma^{\mu}\frac{1}{\not p-\not k_1+m-i\epsilon}\gamma^{\rho}\frac{1}{\not p-\not k_1-\not k_2+m-i\epsilon}\gamma_t\frac{1}{k_2^2-i\epsilon}\right.$$

$$+\gamma^{\rho}\frac{1}{\not p-\not k_1-\not k_2+m-i\epsilon}\gamma^{\mu}\frac{1}{\not p-\not k_1-\not k_2+m-i\epsilon}\gamma_t\frac{1}{k_2^2-i\epsilon}$$

$$\left.+\gamma^{\rho}\frac{1}{\not p-\not k_1-\not k_2+m-i\epsilon}\gamma_t\frac{1}{k_2^2-i\epsilon}\,\frac{1}{\not p-\not k_2+m-i\epsilon}\gamma^{\mu}\right]\frac{1}{\not p-\not k_2+m-i\epsilon}\gamma_{\rho}$$

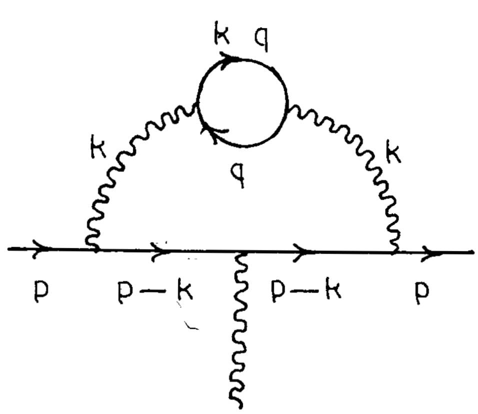
(3.5.26)

This expression together with expression (3.4.18) satisfy the nonlocal Ward identity

$$\exp\left(\frac{p^2+m^2}{\Lambda^2}\right)\Lambda_B^{\mu}(p,p) = -\frac{\partial}{\partial p_{\mu}}\left[\exp\left(\frac{p^2+m^2}{\Lambda^2}\right)\Sigma_B(p)\right]$$

(3.5.27)

The QED vertex graph to order e^4 corresponding to the self-energy given in fig3.3(c) is given in fig.3.16 [76].



Insertion of photon into second-order self-energy of fig.3.3(c)

Fig 3.16.

The nonlocal QED graphs corresponding to this are given in fig.3.17. There are twelve of them. The individual contributions total up to:

$$e\Lambda^{\mu}{}_{C}(p,p) = -e^3 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\int\frac{d^4k}{(2\pi)^4} \exp\left(-\frac{k^2}{\Lambda^2}\right)\frac{1}{k^2}$$

$$\times\ \gamma^{\sigma}\ [\Pi^{\sigma\rho}(q)]\ \frac{1}{\not p-\not k+m-i\epsilon}\gamma^{\mu}\frac{1}{\not p-\not k+m-i\epsilon}\gamma^{\rho}\frac{1}{k^2-i\epsilon}\exp\left(-\frac{k^2}{\Lambda^2}\right)$$

$$\times\ \eta^{\sigma\rho} \tag{3.5.28}$$

Combining expressions (3.4.27) and (3.5.28), one gets,

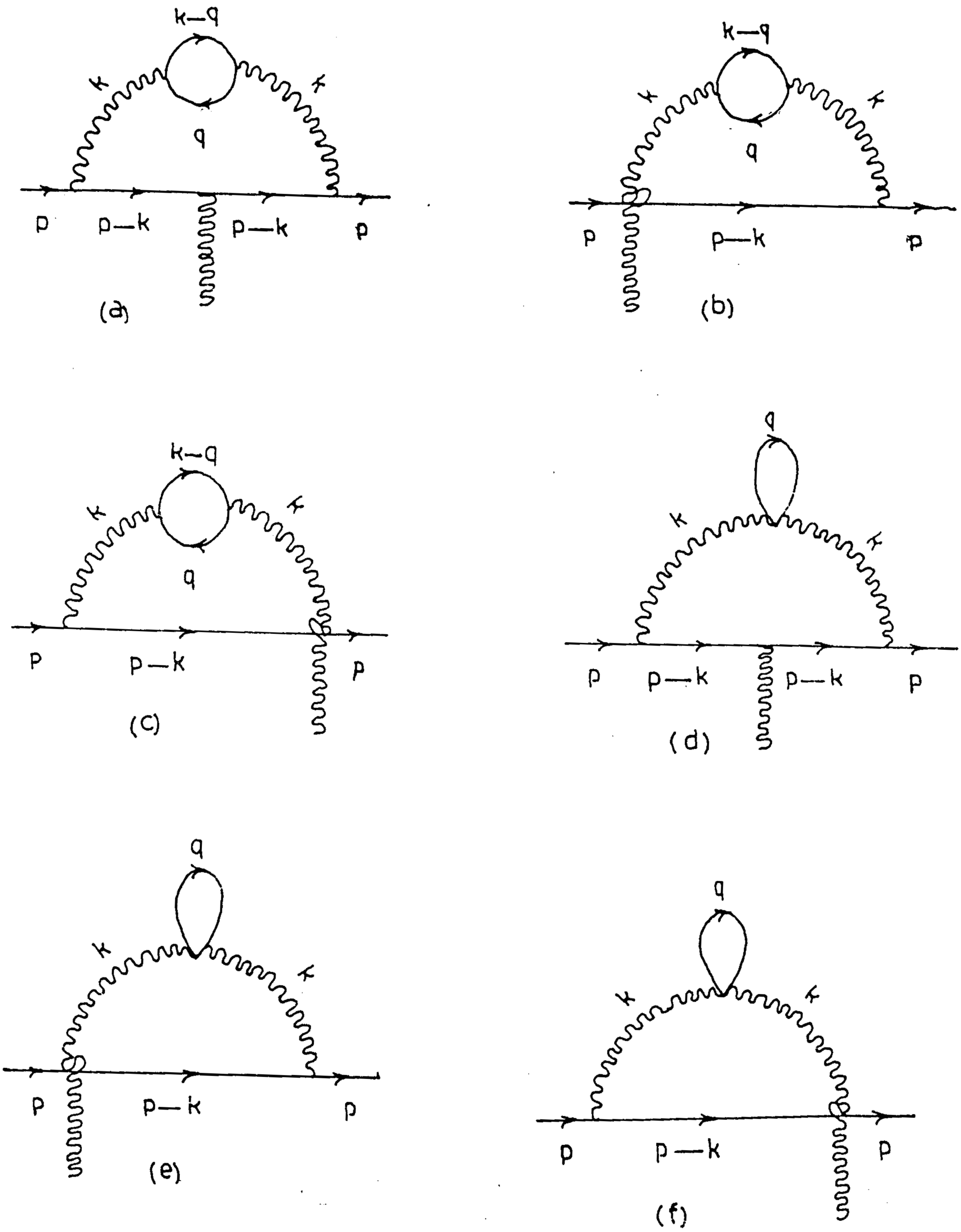
$$\exp\left(\frac{p^2+m^2}{\Lambda^2}\right)\Lambda^{\mu}{}_{C}(p,p) = -\frac{\partial}{\partial p_{\mu}}\left[\exp\left(\frac{p^2+m^2}{\Lambda^2}\right)\ \Sigma_C(p)\right] \tag{3.5.29}$$

the nonlocal Ward identity [75] .

We have shown that the Ward Identity is valid to second order. We conjecture that the Ward identity has the form

$$-\frac{\partial}{\partial p_{\mu}}\left[\exp\left(\frac{p^2+m^2}{\Lambda^2}\right)\Sigma(p)\right] = \exp\left(\frac{p^2+m^2}{\Lambda^2}\right)\ \Lambda^{\mu}(p,p)$$

to all orders.



Fourth-order vertex graphs corresponding to fig.3.16 in nonlocal QED.

Fig.3.17.

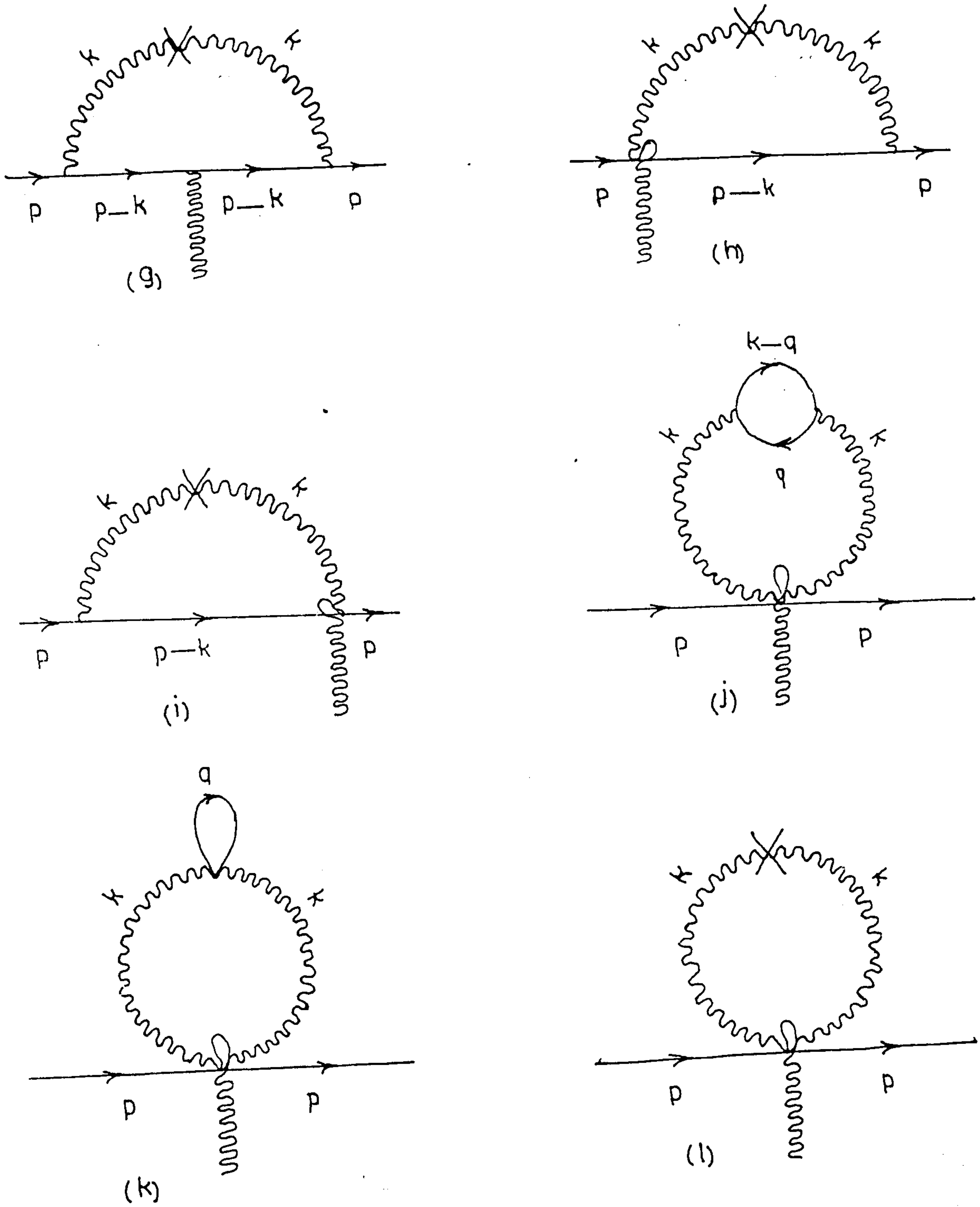


Fig.3.17(contd).

3.6 Discussion

The addition of the nonlocal terms to the Lagrangian destroys the local gauge invariance of the theory. The gauge invariance can be restored by modifying the fermionic transformation rule at each order. Hence it is necessary to prove the Ward identity for nonlocal QED. The Ward identity for nonlocal QED has been evaluated [75] upto the order of two loops. At orders e^2 and e^4 the Ward identity for nonlocal QED is of the form

$$\exp\left(\frac{p^2+m^2}{\Lambda^2}\right)\Lambda^{\mu}(p,p) = -\frac{\partial}{\partial p_{\mu}}\left[\exp\left(\frac{p^2+m^2}{\Lambda^2}\right)\Sigma(p)\right]$$

Higher order calculations involve higher order measure factors, which to our knowledge has not been evaluated. But we conjecture that the Ward identity has the same form to all orders. In the limit of QED ($\Lambda \rightarrow \infty$), we obtain the usual expression for the Ward identity.

$$\Lambda^{\mu}(p,p) = -\frac{\partial}{\partial p_{\mu}}\Sigma(p).$$

Nonlocal Yang - Mills

P. C. Raje Bhageerathi “Regularizations and divergent diagram in gauge theories” Thesis. Department of Physics, University of Calicut, 1999

Chapter 4

Nonlocal Yang - Mills

4.1 Introduction

The method of nonlocal regularization as applied to QED can be extended to Yang–Mills theories [71, 77-79] also. This is a non–Abelian gauge theory which is renormalizable [1,80-82]. Here, no effort is made to study the general case. Because of the physical relevance of QCD, we are only confining ourselves to SU(3) [13,83] gauge symmetry — Special Unitary group of order three — There are eight generators for the SU(3) group. – The eight matrix generators λ_a obey the commutation relations.

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = if^{abc} \frac{\lambda_c}{2} \qquad (4.1.1)$$

with summation over c from 1 to 8. The quantities if^{abc} are the structure constants of the group.

Here the particles coming into play are quarks and gluons. [84,85]. Quarks interact via gluons. Unlike in QED, here the gluons interact among themselves. Hence there will be a difference while studying the vacuum polarization and the vertex graphs in QCD. In QCD, there will also be a ghost field–a scalar field with Fermi statistics. The gluons will interact with quarks, ghosts and themselves. The Ward identity is found to hold for the simplest gauge theory QED. Analogous identities hold for the non-Abelian gauge theories also. They are the Slavnov-Taylor identities [86-90].

4.2 Nonlocalization of QCD

A smearing operator ε_m is defined as follows:

$$\varepsilon_m \equiv \exp\left(\frac{\partial^2 - m^2}{2\Lambda^2}\right) \tag{4.2.1}$$

For gluons and ghosts, m = 0 in the above expression. Just as in nonlocal QED, we have

$$\phi^\Lambda \equiv \varepsilon_m\, \phi \tag{4.2.2}$$

where ϕ may be fields which are commuting or anti-commuting, or there may be some of each type. Another operator ϑ is defined as:

$$\vartheta \equiv \frac{\varepsilon_m{}^2 - 1}{\partial^2 - m^2} = \int_0^1 \frac{d\tau}{\Lambda^2} \exp\left(\tau\, \frac{\partial^2 - m^2}{\Lambda^2}\right) \tag{4.2.3}$$

The Lagrangian for QCD [80,91] is :

$$\mathcal{L}_{0+1} = -\frac{1}{4} F_{\mu\nu}{}^a F^{\mu\nu a} - \bar{\psi}(i\partial\!\!\!/ + m)\psi + gT^a \bar{\psi}\, A\!\!\!/^a\, \psi \tag{4.2.4}$$

The initial nonlocalization of QCD follows from nonlocalizing the interaction part of the Lagrangian:

$$\mathcal{L}_{0+1} = -\frac{1}{4} F_{\mu\nu}{}^a F^{\mu\nu a} - \bar{\psi}(i\partial\!\!\!/ + m)\psi + gT^a \hat{\bar{\psi}}\, \hat{A\!\!\!/}^a\, \hat{\psi} \tag{4.2.5}$$

where $F_{\mu\nu}{}^a$ is the field tensor for the non–Abelian fields given as:

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g f^{abc} A_\mu{}^b A_\nu{}^c \tag{4.2.6}$$

f^{abc} are the structure constants of the group. Also

$$T^a = \frac{1}{2}\lambda^a, \; (a = 1, 2, \ldots\ldots\ldots\ldots 8) \text{ for SU (3).} \qquad (4.2.7)$$

The operators T^a are the generators of the group and they obey the commutation relation

$$[T^a, T^b] = i\, f^{abc}\, T^c \qquad (4.2.8)$$

4.3 Feynman rules for nonlocal QCD

The nonlocalized Feynman rules are a trivial extension of the local ones [71,92]. Except for the measure factor the vertices are unchanged, but every leg can now connect either to a smeared propagator [50,51,93] :

$$\frac{i\varepsilon_m^2}{p^2+m^2+i\epsilon} = -i\int_1^\infty \frac{d\tau}{\Lambda^2}\exp\left[-\tau\,\frac{p^2+m^2}{\Lambda^2}\right] \qquad (4.3.1a)$$

or to a shadow propagator:

$$-i\,\vartheta = \frac{i(1-\varepsilon_m^2)}{p^2+m^2} = -i\int_0^1 \frac{d\tau}{\Lambda^2}\exp\left[-\tau\,\frac{p^2+m^2}{\Lambda^2}\right] \qquad (4.3.1b)$$

The unbarred (smeared) and the barred (shadow) matter propagators are represented graphically in fig. 4.1.

Smeared and shadow matter propagators of nonlocal Yang-Mills.

Fig 4.1

The smeared matter propagator is given by

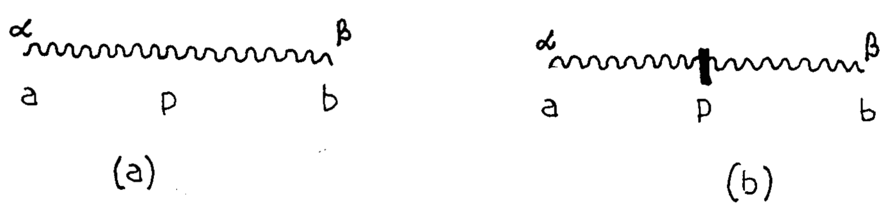
$$- \frac{i\delta_{ab}}{\not{p}+m-i\epsilon} \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) = -\frac{i\delta_{ab}(\not{p}-m)}{p^2+m^2-i\epsilon} \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)$$

$$= -i\,\delta_{ab}(\not{p}-m) \int_1^{\infty} \frac{d\tau}{\Lambda^2} \exp\left(-\tau \frac{p^2+m^2}{\Lambda^2}\right) \tag{4.3.2a}$$

The shadow matter propagator is :

$$- \frac{i\delta_{ab}}{\not{p}+m-i\epsilon}\left[1-\exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)\right] = -i\delta_{ab}(\not{p}-m) \int_0^{1} \frac{d\tau}{\Lambda^2} \exp\left(-\tau \frac{p^2+m^2}{\Lambda^2}\right) \tag{4.3.2b}$$

The unbarred and the barred gluon propagators [51] are given in fig. 4.2.



Smeared and shadow gluon propagators
of nonlocal Yang – Mills.

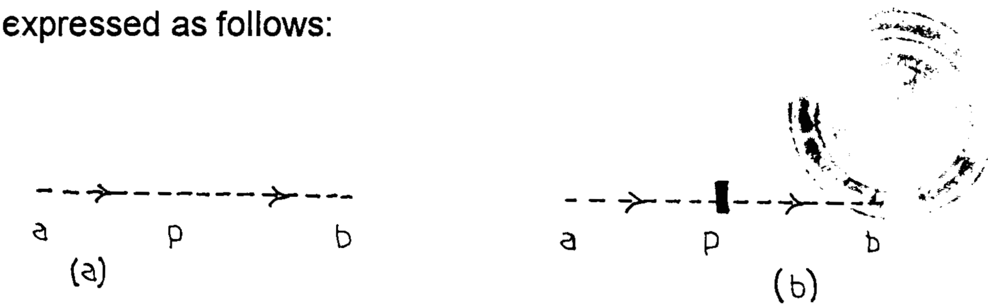
Fig. 4.2

They are expressed as follows:

$$- \frac{i\delta_{ab}\eta^{\alpha\beta}}{p^2-i\epsilon} \exp\left(-\frac{p^2}{\Lambda^2}\right) = -i\delta_{ab}\eta^{\alpha\beta} \int_1^{\infty} \frac{d\tau}{\Lambda^2} \exp\left(-\tau \frac{p^2}{\Lambda^2}\right) \tag{4.3.3a}$$

$$- \frac{i\delta_{ab}\eta^{\alpha\beta}}{p^2-i\epsilon}\left[1-\exp\left(-\frac{p^2}{\Lambda^2}\right)\right] = -i\delta_{ab}\eta^{\alpha\beta} \int_0^{1} \frac{d\tau}{\Lambda^2} \exp\left(-\tau \frac{p^2}{\Lambda^2}\right) \tag{4.3.3b}$$

The ghosts only occur in the internal parts of Feynman diagrams, and have the wrong spin–statistics relation. The ghost propagates like a scalar particle but has Fermi statistics. The barred and unbarred ghost propagators are shown in fig.4.3 and can be expressed as follows:



The smeared and shadow ghost propagators
of non local Yang – Mills
Fig. 4.3

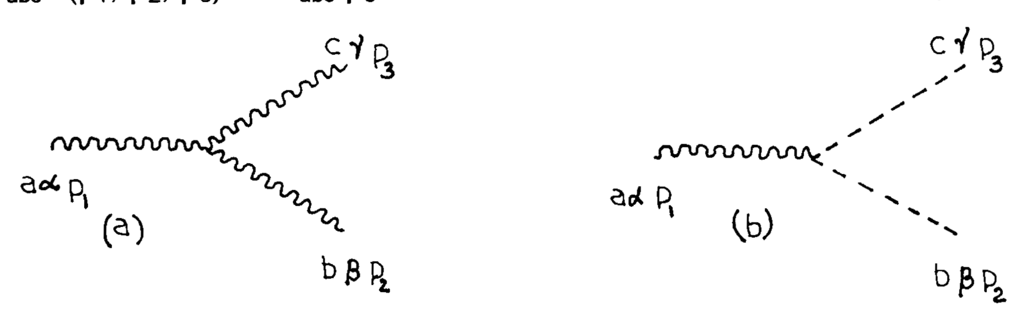
$$-\frac{i\delta_{ab}}{p^2-i\epsilon}\exp\left(-\frac{p^2}{\Lambda^2}\right) = -i\delta_{ab}\int_1^{\infty}\frac{d\tau}{\Lambda^2}\exp\left(-\tau\frac{p^2}{\Lambda^2}\right) \quad (4.3.4a)$$

$$-\frac{i\delta_{ab}}{p^2-i\epsilon}\left[1-\exp\left(-\frac{p^2}{\Lambda^2}\right)\right] = -i\delta_{ab}\int_0^{1}\frac{d\tau}{\Lambda^2}\exp\left(-\tau\frac{p^2}{\Lambda^2}\right) \quad (4.3.4b)$$

The local 3–point vertices of Yang–Mills are given in fig. 4.4 and can be expressed as follows [1,80]:

$$I_{abc}^{\alpha\beta\gamma}(p_1, p_2, p_3) \equiv -i f_{abc}\{\eta^{\alpha\beta}(p_2-p_1)^{\gamma} + \eta^{\beta\gamma}(p_3-p_2)^{\alpha} + \eta^{\gamma\alpha}(p_1-p_3)^{\beta}\} \quad (4.3.5a)$$

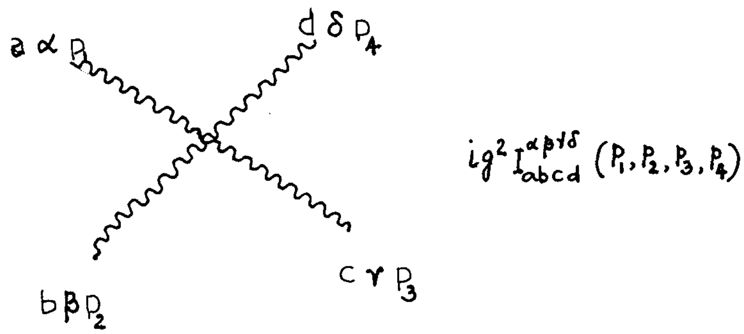
$$I_{abc}^{\alpha}(p_1, p_2, p_3) \equiv i f_{abc}\, p_3^{\alpha} \quad (4.3.5b)$$



The 3–point vertices of local Yang–Mills.
Fig. 4.4

The local 4–point vertex is depicted in fig.4.5 and has the form:

$$I_{abcd}^{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) \equiv -\{ f_{abe} f_{cde} (\eta^{\alpha\gamma}\eta^{\beta\delta} - \eta^{\gamma\beta}\eta^{\delta\alpha})$$
$$+ f_{ace} f_{dbe} (\eta^{\alpha\delta}\eta^{\gamma\beta} - \eta^{\delta\gamma}\eta^{\alpha\beta})$$
$$+ f_{ade} f_{bce} (\eta^{\alpha\beta}\eta^{\delta\gamma} - \eta^{\beta\delta}\eta^{\gamma\alpha})\} \qquad (4.3.6)$$



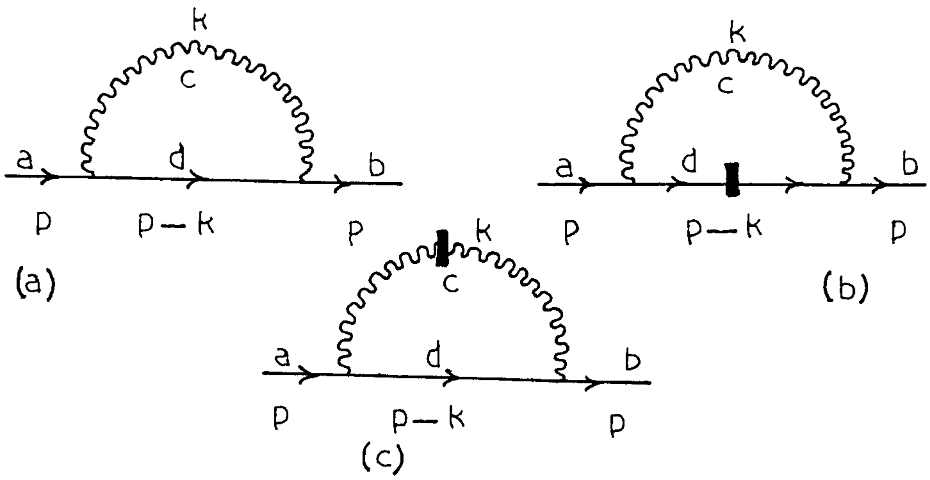
The 4-point vertex of local Yang-Mills.
Fig. 4.5

With the help of these rules, one can proceed to evaluate the quark self-energy and the quark-gluon vertex in nonlocal QCD.The vacuum polarization for the pure nonlocal Yang-Mills (involving only gluons and ghosts) has been evaluated by Kleppe and Woodard [51].

4.4 Quark Self-energy in nonlocal regularization

The contribution to the quark self-energy at one loop comes from the three diagrams given in fig.4.6. In the diagrams, a,b,c, and d are SU(3) labels. Applying Feynman rules, the contribution to the quark self-energy from fig.4.6(a) is:

$$-i\Sigma_1^{ab}(p) \equiv \int \frac{d^4k}{(2\pi)^4} \{ig\gamma_\mu (T^c)_{ad}\} \left\{ \frac{-i\delta_{ab}}{\not p - \not k + m - i\epsilon} \right\} \{ig\gamma_\nu (T^c)_{db}\}$$
$$\times \left\{ \frac{-i\delta_{ab}\eta^{\mu\nu}}{k^2 - i\epsilon} \right\} \exp\left[-\frac{p^2+m^2}{\Lambda^2} - \frac{(p-k)^2+m^2}{\Lambda^2} - \frac{k^2}{\Lambda^2} \right] \qquad (4.4.1a)$$



Quark self-energy at one loop in nonlocal QCD.

Fig.4.6

Promoting the propagators to Schwinger integrals,

$$-i\Sigma_1^{ab}(p) = ig^2 \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \delta_{ab}\,(T^c)_{ad}\,(T^c)_{da} \int \frac{d^4k}{(2\pi)^4}\, \gamma_\mu\,(\not{p}-\not{k}+m)\,\gamma^\mu$$

$$\times \int_1^\infty \frac{d\tau_1}{\Lambda^2} \int_1^\infty \frac{d\tau_2}{\Lambda^2} \exp\left[-\tau_1 \frac{(p-k)^2+m^2}{\Lambda^2} - \tau_2 \frac{k^2}{\Lambda^2}\right] \tag{4.4.1b}$$

The factor of i comes from the rotation to the Euclidean space. Performing the momentum integral,

$$-\,i\Sigma_1^{ab}(p) = -\frac{ig^2}{8\pi^2}\,(T^c)_{ad}\,(T^c)_{da}\,\delta_{ab}\,\exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)$$

$$\times \int_1^\infty d\tau_1 \int_1^\infty d\tau_2 \left[\frac{\tau_2\,\not{p}}{(\tau_1+\tau_2)^3} + \frac{2m}{(\tau_1+\tau_2)^2}\right]$$

$$\times \exp\left[-\frac{\tau_1\,\tau_2}{\tau_1+\tau_2}\,\frac{p^2}{\Lambda^2} - \tau_1\,\frac{m^2}{\Lambda^2}\right] \tag{4.4.2}$$

Now $(T^c)_{ad}\,(T^c)_{da}$ is just the group theoretic factor. Also

$$T^c \cdot T^c = C_2(F). \qquad (4.4.3)$$

For SU(3), $C_2(F) = \dfrac{4}{3}$. So we have,

$$-i\Sigma_1^{ab}(p) = -\frac{ig^2}{8\pi^2} C_2(F)\,\delta_{ab} \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \int_1^\infty d\tau_1 \int_1^\infty d\tau_2$$

$$\times \left[\frac{\tau_2 \not{p}}{(\tau_1+\tau_2)^3} + \frac{2m}{(\tau_1+\tau_2)^2}\right] \exp\left[-\frac{\tau_1\tau_2}{\tau_1+\tau_2}\frac{p^2}{\Lambda^2} - \tau_1 \frac{m^2}{\Lambda^2}\right] \qquad (4.4.4)$$

Now make the change of variables

$$\tau_1 = \lambda\,(1-x) \quad \text{and} \quad \tau_2 = \lambda\, x \qquad (4.4.5)$$

Here x is the usual Feynman parameter. The result of the completely unbarred loop is

$$\Sigma_1^{ab}(p) = \frac{g^2}{8\pi^2} C_2(F)\,\delta_{ab} \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \left[\int_{1/2}^{1} dx \int_{1/1-x}^{\infty} d\lambda + \int_0^{1/2} dx \int_{1/x}^{\infty} d\lambda\right]$$

$$\times \frac{1}{\lambda}(x\not{p} + 2m) \exp\left[-\lambda x\,(1-x)\frac{p^2}{\Lambda^2} - \lambda\,(1-x)\frac{m^2}{\Lambda^2}\right] \qquad (4.4.6)$$

Including the barred graphs just extends the range of parameter integration to the entire positive quadrant excepting the unit square $0 < \tau_i < 1$.

$$\Sigma_2^{ab}(p) = \frac{g^2}{8\pi^2} C_2(F)\,\delta_{ab} \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \int_{1/2}^{1} dx \int_{1/x}^{1/1-x} \frac{d\lambda}{\lambda}$$

$$\times (x\not{p} + 2m) \exp\left[-\lambda x\,(1-x)\frac{p^2}{\Lambda^2} - \lambda\,(1-x)\frac{m^2}{\Lambda^2}\right] \qquad (4.4.7)$$

and

$$\Sigma_3^{ab}(p) = \frac{g^2}{8\pi^2} C_2(F)\,\delta_{ab} \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \int_0^{1/2} dx \int_{1/1-x}^{1/x} \frac{d\lambda}{\lambda}$$

$$\times\ (x\not{p} + 2m) \quad \exp\left[-\lambda x(1-x)\frac{p^2}{\Lambda^2} - \lambda(1-x)\frac{m^2}{\Lambda^2}\right] \qquad (4.4.8)$$

The sum of the barred and unbarred graphs is therefore

$$\Sigma^{ab}(p) = \frac{g^2}{8\pi^2} C_2(F)\,\delta_{ab} \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right) \left[\int_{1/2}^{1} dx \int_{1/x}^{\infty} d\lambda + \int_0^{1/2} dx \int_{1/1-x}^{\infty} d\lambda\right]$$

$$\times\ \frac{1}{\lambda}(x\not{p} + 2m) \quad \exp\left[-\lambda x(1-x)\frac{p^2}{\Lambda^2} - \lambda(1-x)\frac{m^2}{\Lambda^2}\right] \qquad (4.4.9a)$$

$$\Sigma^{ab}(p) = \frac{g^2}{8\pi^2} C_2(F)\,\delta_{ab} \exp\left(-\frac{p^2+m^2}{\Lambda^2}\right)$$

$$\times \left[\int_{1/2}^{1} dx\, E_1\left\{\frac{1}{x}\left[x(1-x)\frac{p^2}{\Lambda^2} + (1-x)\frac{m^2}{\Lambda^2}\right]\right\}\right.$$

$$\left. + \int_0^{1/2} dx\, E_1\left\{\frac{1}{1-x}\left[x(1-x)\frac{p^2}{\Lambda^2} + (1-x)\frac{m^2}{\Lambda^2}\right]\right\}\right] \qquad (4.4.9b)$$

where $E_1(z)$ is the exponential integral :

$$E_1(z) \equiv \int_z^{\infty} dt \frac{\exp(-t)}{t} = -\ln z - \gamma - \sum_{n=1}^{\infty} \frac{(-z)^n}{nn!} \qquad (4.4.9c)$$

One can develop an asymptotic expansion in Λ by expanding the exponential integral:

$$\Sigma^{ab}(p) = \frac{g^2}{8\pi^2} C_2(F)\delta_{ab} \left[\left(\frac{1}{2}\not{p}+2m\right) \ln(\Lambda^2) - \left(\frac{1}{2}\not{p}+2m\right)\gamma + \frac{1}{4}\not{p} + \left(\frac{1}{2}\not{p}+2m\right)\ln(2) - \int_0^1 dx\,(x\not{p}+2m)\ln(xp^2+m^2) + O\left(\frac{\ln(\Lambda^2)}{\Lambda^2}\right)\right] \qquad (4.4.10)$$

The result is the same as that obtained from dimensional regularization [1,92] provided one suggests the correspondence

$$\frac{2}{4-D} \sim \ln(\Lambda^2) \qquad (4.4.11)$$

for the co-efficients of logarithmic divergences in the two methods. Also, the divergent parts of the electron and the quark self-energies are related by,

$$\Sigma_{div}{}^{ab}(p) = C_2(F)\,\delta_{ab}\,\Sigma_{div}(QED) \qquad (4.4.12)$$

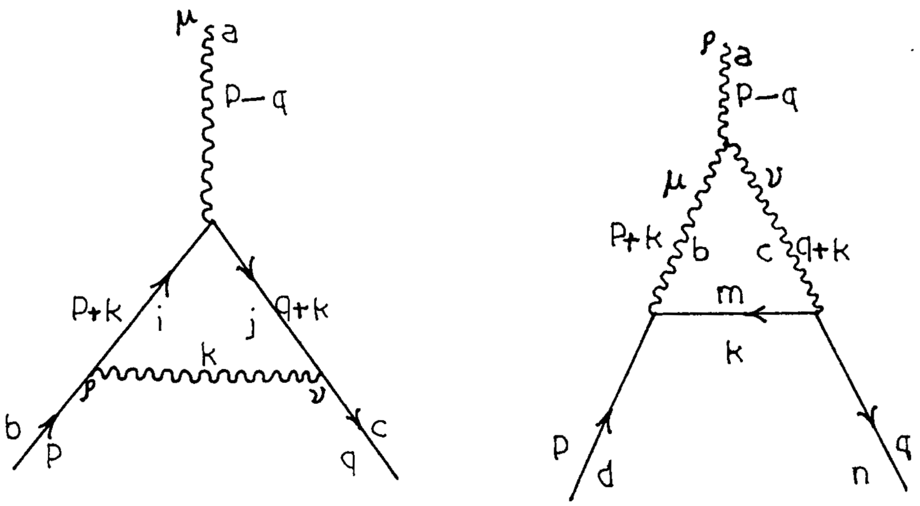
In D-dimensions, the exponential integral is replaced by the incomplete γ function just as in QED. The incomplete γ function $\Gamma(n, z)$ is

$$\Gamma(n, z) \equiv \int_z^\infty dt\, t^{n-1} \exp(-t) \qquad (4.4.13a)$$

$$= (n-1)\,\Gamma(n-1, z) + z^{n-1}\exp(-z) \qquad (4.4.13b)$$

4.5 QCD Vertex part in nonlocal regularization

In this section, the nonlocal quark-gluon vertex function is calculated. Two distinct Feynman diagrams contribute to this in QCD [86,94]. They are given in fig. 4.7.



Correction to quark-gluon vertex

Correction to quark-gluon vertex involving the 3-gluon vertex

Fig.4.7

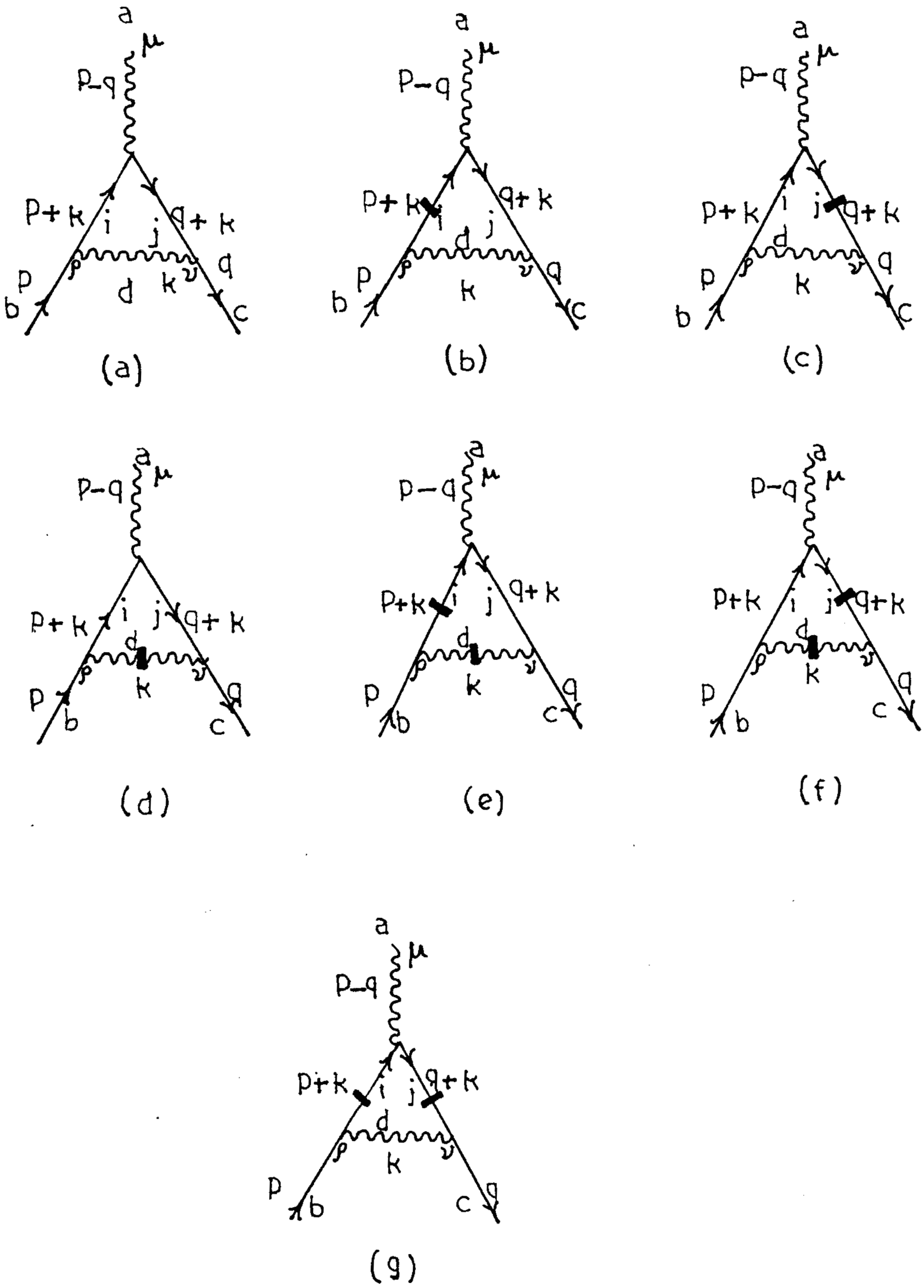
Fig.4.7(a) is analogous to the QED vertex part. But fig.4.7(b) is certainly non–Abelian in character, with its 3-gluon coupling. The nonlocal diagrams corresponding to fig.4.7(a) are given in fig.4.8. Applying Feynman rules, the contribution to the quark–gluon vertex from fig 4.8 (a) is :

$$
-ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(a) \equiv \int \frac{d^4k}{(2\pi)^4}\,\{ig\gamma_\rho\,(T^d)_{bi}\}\left\{\frac{-i}{\not p+\not k+m-i\epsilon}\right\}\{ig\ \gamma_\mu\,(T^a)_{ij}\}
$$

$$
\times\left\{\frac{-i}{\not q+\not k+m-i\epsilon}\right\}\{ig\ \gamma_\nu\,(T^d)_{jc}\}\left\{\frac{-i\delta_{bc}\eta^{\rho\nu}}{k^2-i\epsilon}\right\}
$$

$$
\times\ \exp\left[-\frac{(p+k)^2+m^2}{\Lambda^2}-\frac{(q+k)^2+m^2}{\Lambda^2}-\frac{k^2}{\Lambda^2}\right]
$$

$$
\times\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right] \qquad (4.5.1a)
$$



Nonlocal vertex graphs corresponding to
fig.4.7(a)

Fig.4.8.

Promoting the propagators to Schwinger integrals

$$- ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(a) \equiv ig^3\,(T^dT^aT^d)\,\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int\frac{d^4k}{(2\pi)^4}\,\gamma_\nu\,(\not p+\not k-m)\,\gamma_\mu\,(\not q+\not k-m)\,\gamma^\nu\int_1^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}$$

$$\times\int_1^\infty\frac{d\tau_3}{\Lambda^2}\,\exp\left[-\tau_1\frac{k^2}{\Lambda^2}-\tau_2\frac{(p+k)^2+m^2}{\Lambda^2}-\tau_3\frac{(q+k)^2+m^2}{\Lambda^2}\right]$$

(4.5.1b)

Introduce the new integration variable

$$k' = k + \frac{\tau_2 p + \tau_3 q}{\tau_1+\tau_2+\tau_3} \qquad (4.5.2)$$

Now perform the momentum integral over k'. The term which is quadratic in k' is divergent whereas the term linear in k' integrate to zero. The rest terms are convergent. The divergent part is :

$$- ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(a) = -\frac{ig^3}{8\pi^2}\gamma_\mu(T^dT^aT^d)\,\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int_1^\infty d\tau_1\int_1^\infty d\tau_2\int_1^\infty d\tau_3\,\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left.-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\,\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-(\tau_2+\tau_3)\frac{m^2}{\Lambda^2}\right]$$

(4.5.3)

The barred contributions are :

$$- ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(b) = -\frac{ig^3}{8\pi^2}\gamma_\mu(T^dT^aT^d)\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times \int_1^\infty d\tau_1 \int_0^1 d\tau_2 \int_1^\infty d\tau_3 \exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left.-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\,\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-(\tau_2+\tau_3)\frac{m^2}{\Lambda^2}\right]$$

(4.5.4)

$$- ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(c) = -\frac{ig^3}{8\pi^2}\gamma_\mu(T^dT^aT^d)\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times \int_1^\infty d\tau_1 \int_1^\infty d\tau_2 \int_0^1 d\tau_3 \exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left.-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\,\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-(\tau_2+\tau_3)\frac{m^2}{\Lambda^2}\right]$$

(4.5.5)

$$- ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(d) = -\frac{ig^3}{8\pi^2}\gamma_\mu(T^dT^aT^d)\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times \int_0^1 d\tau_1 \int_1^\infty d\tau_2 \int_1^\infty d\tau_3 \exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left.-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\,\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-(\tau_2+\tau_3)\frac{m^2}{\Lambda^2}\right]$$

(4.5.6)

$$- ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(e) = -\frac{ig^3}{8\pi^2}\gamma_\mu(T^dT^aT^d)\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int_0^1 d\tau_1\int_0^1 d\tau_2\int_1^\infty d\tau_3\,\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left.-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\,\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-(\tau_2+\tau_3)\frac{m^2}{\Lambda^2}\right]$$

(4.5.7)

$$- ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(f) = -\frac{ig^3}{8\pi^2}\gamma_\mu(T^dT^aT^d)\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int_0^1 d\tau_1\int_1^\infty d\tau_2\int_0^1 d\tau_3\,\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left.-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\,\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-(\tau_2+\tau_3)\frac{m^2}{\Lambda^2}\right]$$

(4.5.8)

$$- ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(g) = -\frac{ig^3}{8\pi^2}\gamma_\mu(T^dT^aT^d)\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int_1^\infty d\tau_1\int_0^1 d\tau_2\int_0^1 d\tau_3\,\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left.-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\,\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-(\tau_2+\tau_3)\frac{m^2}{\Lambda^2}\right]$$

(4.5.9)

The various barred contributions merely serve to extend the range of integration. Since we include all diagrams except the completely barred one, the final range is characterized by $\tau_i > 0$, excepting the region for which all the τ_i's are between zero and one. Make a change of variables :

$$\tau \equiv \sum_{i=1}^{3} \tau_i \tag{4.5.10a}$$

$$x_i \equiv \frac{1}{\tau} \sum_{j=i+1}^{3} \tau_j \tag{4.5.10b}$$

The x_i's are the usual Feynman parameters. The total contribution from the entire class of diagrams [50] in fig.4.8 is :

$$- ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(A) = -\frac{ig^3}{8\pi^2}\gamma_\mu(T^dT^aT^d)\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times \int_0^1 dx \int_0^x dy \sum_{k=1}^{3} \Gamma\left(0, \frac{1}{\xi_k}\left[x(1-x)\frac{p^2}{\Lambda^2} + y(1-y)\frac{q^2}{\Lambda^2}\right.\right.$$

$$\left.\left. - 2xy\frac{p.q}{\Lambda^2} + (x+y)\frac{m^2}{\Lambda^2}\right]\right)\prod_{i\neq k}\theta(\xi_k - \xi_i)\ \theta(\xi_i) \tag{4.5.11}$$

where we have put $x_2 = y$. Also $\xi_i \equiv x_{i-1} - x_i$, $\xi_{i0} \equiv 0$ and $\xi_{iN+1} \equiv \infty$, where N is the number of internal lines. The argument of the incomplete gamma function is just the bracketed term of the parameter integrand divided by $\xi_k \Lambda^2$. Γ (0, z) is nothing but the exponential integral $E_1(z)$. An asymptotic expansion in Λ gives for the divergent part :

$$- ig\,(\Lambda_\mu{}^a)_{cd}(p,p-q,q)(A_D) = -\frac{ig^3}{8\pi^2}\gamma_\mu(T^dT^aT^d)\int_0^1 dx \int_0^x dy\,\ln\left(\Lambda^2\right) \quad (4.5.12a)$$

$$= -\frac{ig^3}{8\pi^2}\gamma_\mu(T^dT^aT^d)\left[\frac{1}{2}\ln\,(\Lambda^2)\right] \quad (4.5.12b)$$

The group theoretic factor $(T^dT^aT^d)$ is easily evaluated:

$$T^dT^aT^d = T^d\,[T^a,T^d] + T^dT^dT^a = i\,f^{adc}\,T^d\,T^c + C_2\,(F)\,T^a \quad (4.5.13a)$$

$$= -\frac{1}{2}f^{adc}f^{dcb}\,T^b + C_2(F)\,T^a = \left[-\frac{1}{2}C_2(G) + C_2(F)\right]T^a \quad (4.5.13b)$$

Therefore, (4.5.12) becomes,

$$(\Lambda_\mu{}^a)_{cd}(p,p-q,q)\,(A_D) = \frac{g^2}{8\pi^2}\gamma_\mu\left[-\frac{1}{2}C_2(G) + C_2(F)\right]T^a\left[\frac{1}{2}\ln(\Lambda^2)\right] \quad (4.5.14)$$

This suggests the correspondence,

$$\ln\left(\Lambda^2\right) \sim \frac{2}{4-D} \quad \text{just as in the nonlocal QED.}$$

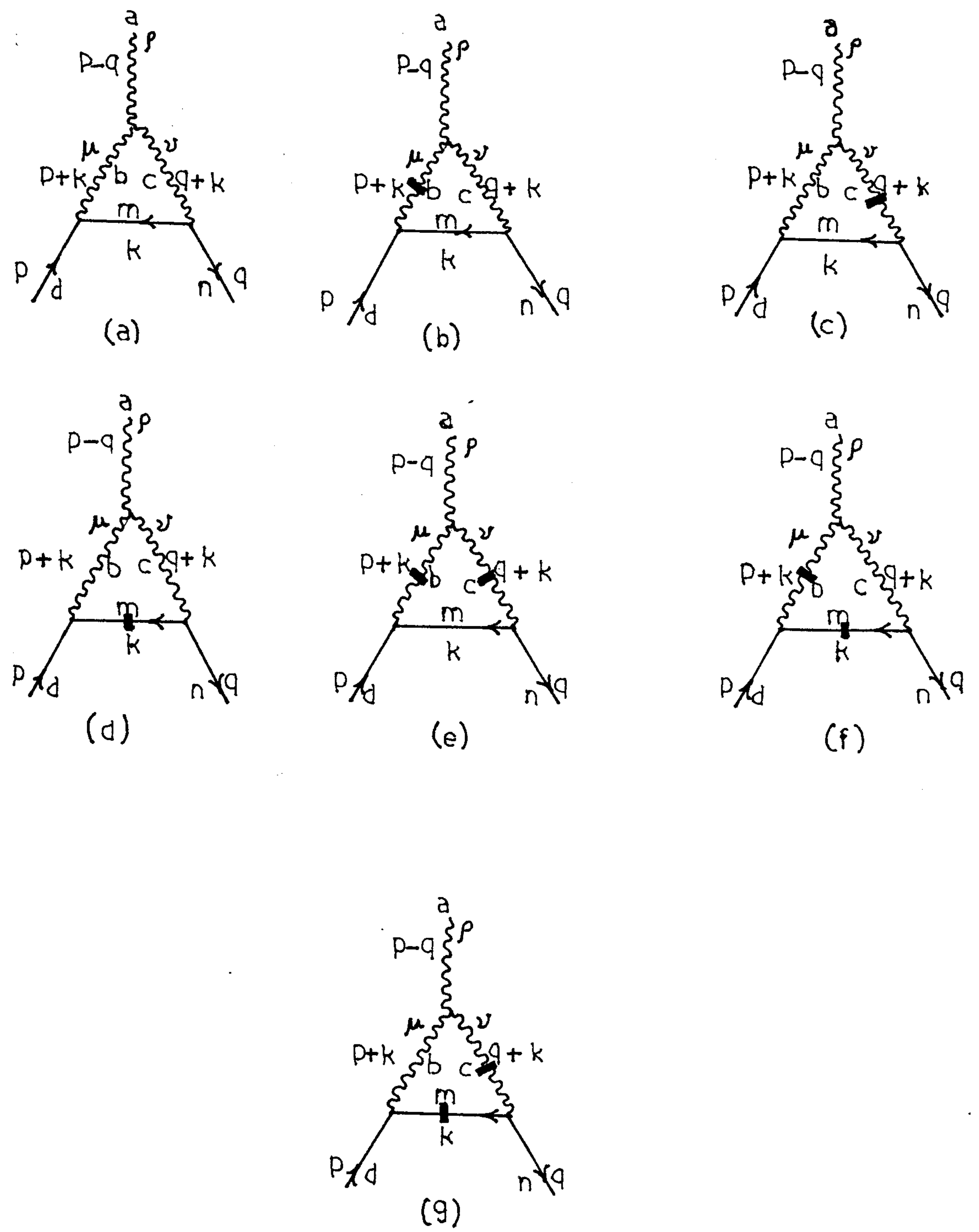
The second vertex diagram [fig 4.7(b)] has a 3-gluon coupling. The corresponding Feynman diagrams in nonlocal QCD are given in fig. 4.9. The contribution from fig. 4.9 (a) to the nonlocal QCD vertex part is :

$$- ig\,\Lambda_\rho{}^a\,(a) \equiv \int \frac{d^4k}{(2\pi)^4}\{ig\gamma_\mu(T^b)_{dm}\}\left\{\frac{-i}{(p+k)^2}\exp\left(-\frac{(p+k)^2}{\Lambda^2}\right)\right\}$$

$$\times\left\{ig\,I_{abc}{}^{\rho\mu\nu}\,[-(p-q),(p+k),-(q+k)]\right\}\left\{\frac{-i\,\delta_{nd}}{\not{k}-m+i\epsilon}\right\}$$

$$\times\exp\left(-\frac{k^2+m^2}{\Lambda^2}\right)\{ig\,\gamma_\nu(T^c)_{mn}\}\left\{\frac{-i}{(q+k)^2}\exp\left(-\frac{(q+k)^2}{\Lambda^2}\right)\right\}$$

$$\times\exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2}\right] \quad (4.5.15)$$



Nonlocal vertex graphs corresponding to fig.4.7(b)

Fig.4.9.

where $I_{abc}^{\alpha\beta\gamma}(p_1, p_2, p_3)$ is as given in equation (4.3.5a). So,

$$-ig\Lambda_\rho^a(a) = -g^3 f_{abc}(T^b)_{dm}(T^c)_{mn}\delta_{nd}\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int\frac{d^4k}{(2\pi)^4}\,\gamma_\mu(\not{k}-m)\gamma_\nu\int_1^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}\int_1^\infty\frac{d\tau_3}{\Lambda^2}$$

$$\times\exp\left[-\tau_1\frac{k^2+m^2}{\Lambda^2}-\tau_2\frac{(p+k)^2}{\Lambda^2}-\tau_3\frac{(q+k)^2}{\Lambda^2}\right]$$

$$\times\{\eta^{\rho\mu}(2p+k-q)^\nu+\eta^{\mu\nu}(-2k-p-q)^\rho+\eta^{\nu\rho}(2q+k-p)^\mu\} \qquad (4.5.17)$$

Shift the integration variable k to k′:

$$k' \rightarrow k+\frac{\tau_2 p+\tau_3 q}{\tau_1+\tau_2+\tau_3} \qquad (4.5.18)$$

$$-ig\Lambda_\rho^a(a) = -g^3 f_{abc}(T^b)_{dm}(T^c)_{mn}\delta_{nd}\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times\int\frac{d^4k'}{(2\pi)^4}\gamma_\mu\left[k'-\frac{\tau_2 p+\tau_3 q}{\tau_1+\tau_2+\tau_3}-m\right]\gamma_\nu\int_1^\infty\frac{d\tau_1}{\Lambda^2}\int_1^\infty\frac{d\tau_2}{\Lambda^2}\int_1^\infty\frac{d\tau_3}{\Lambda^2}$$

$$\times\exp\left[-(\tau_1+\tau_2+\tau_3)\frac{k'^2}{\Lambda^2}\right]\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}\right.$$

$$\left.+\frac{2\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-\tau_1\frac{m^2}{\Lambda^2}\right]\left\{\eta^{\rho\mu}\left[k'+\frac{2\tau_1+\tau_2+2\tau_3}{\tau_1+\tau_2+\tau_3}p\right.\right.$$

$$\left.-\frac{\tau_1+\tau_2+2\tau_3}{\tau_1+\tau_2+\tau_3}q\right]^\nu+\eta^{\mu\nu}\left[-2k'+\frac{\tau_2-\tau_1-\tau_3}{\tau_1+\tau_2+\tau_3}p+\frac{\tau_3-\tau_1-\tau_2}{\tau_1+\tau_2+\tau_3}q\right]^\rho$$

$$+\eta^{\nu\rho}\left[k'-\frac{\tau_1+2\tau_2+\tau_3}{\tau_1+\tau_2+\tau_3}p+\frac{2\tau_1+2\tau_2+\tau_3}{\tau_1+\tau_2+\tau_3}q\right]^\mu \qquad (4.5.19)$$

Only terms quadratic in k' are divergent. The terms linear in k' integrate to zero and those with no k' in the numerator are finite (convergent), so may be ignored. The divergent part is,

$$N_\rho = \gamma_\mu \not{k}' \gamma_\nu \{\eta^{\rho\mu} k'^\nu - \eta^{\mu\nu} 2k'^\rho + \eta^{\nu\rho} k'^\mu\} \qquad (4.5.20)$$

By a simple bit of 'Diracology', this becomes

$$N_\rho = -2\gamma^\rho k'^2 - 4\not{k}' k'^\rho \qquad (4.5.21)$$

Hence the divergent term is :

$$-ig\Lambda_\rho^a(a_D) = -g^3 f_{abc}(T^b)_{dm}(T^c)_{mn}\delta_{nd} \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2}\right]$$

$$\times \int_1^\infty \frac{d\tau_1}{\Lambda^2} \int_1^\infty \frac{d\tau_2}{\Lambda^2} \int_1^\infty \frac{d\tau_3}{\Lambda^2} \int \frac{d^4k'}{(2\pi)^4} \{-2\gamma^\rho k'^2 - 4\not{k}' k'^\rho\}$$

$$\times \exp\left[-(\tau_1+\tau_2+\tau_3)\frac{k'^2}{\Lambda^2}\right] \exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left. - \frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2} + \frac{2\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2} - \tau_1\frac{m^2}{\Lambda^2}\right] \qquad (4.5.22)$$

Performing the momentum integration,

$$-ig\Lambda_\rho^a(a_D) = -\frac{3g^3}{8\pi^2}\gamma_\rho f_{abc}(T^b)_{dm}(T^c)_{mn}\delta_{nd} \exp\left[-\frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2}\right.$$

$$\left. - \frac{(p-q)^2}{2\Lambda^2}\right] \int_1^\infty d\tau_1 \int_1^\infty d\tau_2 \int_1^\infty d\tau_3 \exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left. - \frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2} + \frac{2\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2} - \tau_1\frac{m^2}{\Lambda^2}\right] \qquad (4.5.23)$$

The other contributions are :

$$-ig\Lambda_\rho^a\,(b_D) = -\frac{3g^3}{8\pi^2}\gamma_\rho f_{abc}(T^b)_{dm}(T^c)_{mn}\delta_{nd}\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]\int_1^\infty d\tau_1\int_0^1 d\tau_2\int_1^\infty d\tau_3\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-\tau_1\frac{m^2}{\Lambda^2}\right] \quad (4.5.24)$$

$$-ig\Lambda_\rho^a\,(c_D) = -\frac{3g^3}{8\pi^2}\gamma_\rho f_{abc}(T^b)_{dm}(T^c)_{mn}\delta_{nd}\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]\int_1^\infty d\tau_1\int_1^\infty d\tau_2\int_0^1 d\tau_3\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-\tau_1\frac{m^2}{\Lambda^2}\right] \quad (4.5.25)$$

$$-ig\Lambda_\rho^a\,(d_D) = -\frac{3g^3}{8\pi^2}\gamma_\rho f_{abc}(T^b)_{dm}(T^c)_{mn}\delta_{nd}\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}-\frac{(p-q)^2}{2\Lambda^2}\right]\int_0^1 d\tau_1\int_1^\infty d\tau_2\int_1^\infty d\tau_3\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-\tau_1\frac{m^2}{\Lambda^2}\right] \quad (4.5.26)$$

$$-ig\Lambda_\rho{}^a\,(e_D) = -\frac{3g^3}{8\pi^2}\,\gamma_\rho f_{abc}\,(T^b)_{dm}\,(T^c)_{mn}\delta_{nd}\,\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}\right.$$

$$\left.-\frac{(p-q)^2}{2\Lambda^2}\right]\int_1^\infty d\tau_1\int_0^1 d\tau_2\int_0^1 d\tau_3\,\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left.-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-\tau_1\frac{m^2}{\Lambda^2}\right] \quad (4.5.27)$$

$$-ig\Lambda_\rho{}^a\,(f_D) = -\frac{3g^3}{8\pi^2}\,\gamma_\rho f_{abc}\,(T^b)_{dm}\,(T^c)_{mn}\delta_{nd}\,\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}\right.$$

$$\left.-\frac{(p-q)^2}{2\Lambda^2}\right]\int_0^1 d\tau_1\int_0^1 d\tau_2\int_1^\infty d\tau_3\,\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left.-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-\tau_1\frac{m^2}{\Lambda^2}\right] \quad (4.5.28)$$

$$-ig\Lambda_\rho{}^a\,(g_D) = -\frac{3g^3}{8\pi^2}\,\gamma_\rho f_{abc}\,(T^b)_{dm}\,(T^c)_{mn}\delta_{nd}\,\exp\left[-\frac{p^2+m^2}{2\Lambda^2}-\frac{q^2+m^2}{2\Lambda^2}\right.$$

$$\left.-\frac{(p-q)^2}{2\Lambda^2}\right]\int_0^1 d\tau_1\int_1^\infty d\tau_2\int_0^1 d\tau_3\,\exp\left[-\frac{\tau_2(\tau_1+\tau_3)}{\tau_1+\tau_2+\tau_3}\frac{p^2}{\Lambda^2}\right.$$

$$\left.-\frac{\tau_3(\tau_1+\tau_2)}{\tau_1+\tau_2+\tau_3}\frac{q^2}{\Lambda^2}+\frac{2\tau_2\tau_3}{\tau_1+\tau_2+\tau_3}\frac{p.q}{\Lambda^2}-\tau_1\frac{m^2}{\Lambda^2}\right] \quad (4.5.29)$$

The total contribution to the nonlocal three gluon vertex is the sum of all the terms from (4.5.23) to (4.5.29). Converting the Schwinger parameters to the Feynman parameters as before one obtains the following result for the entire class [50] :

$$- ig\, \Lambda_\rho^{\ a}(B_D) = - \frac{3g^3}{8\pi^2}\, \gamma_\rho\, f_{abc}(T^b)_{dm}(T^c)_{mn}\delta_{nd}\, \exp\left[- \frac{p^2+m^2}{2\Lambda^2} - \frac{q^2+m^2}{2\Lambda^2} - \frac{(p-q)^2}{2\Lambda^2} \right] \int_0^1 dx \int_0^x dy \sum_{k=1}^{3} \Gamma\left(0, \frac{1}{\xi_k}\left[x(1-x)\frac{p^2}{\Lambda^2} + y(1-y)\frac{q^2}{\Lambda^2} - 2xy\frac{p.q}{\Lambda^2} + x\frac{m^2}{\Lambda^2} \right] \right) \prod_{i\neq k} \theta(\xi_k - \xi_i)\ \theta(\xi_i) \tag{4.5.30}$$

Γ (0, z) is the exponential integral $E_1(z)$. For asymptotically large values of Λ, $E_1(z)$ can be approximated as :

$$E_1(z) \equiv - \ln z - \gamma - \sum_{n=1}^{\infty} \frac{(-z)^n}{nn!} \tag{4.5.31}$$

Expression (4.5.30) has a pole part $I_\rho^{\ a}$, which is given as follows :

$$-igI_\rho^{\ a} = - \frac{3g^3}{8\pi^2}\, \gamma_\rho\, f_{abc}\, (T^b)_{dm}\, (T^c)_{mn}\delta_{nd} \left\{ \frac{1}{2} \ln(\Lambda^2) \right\} \tag{4.5.32}$$

The group theoretic factor $f_{abc}\,(T^b)_{dm}\,(T^c)_{mn}\delta_{nd}$ can be expressed as :

$$f_{abc}\,(T^b)_{dm}\,(T^c)_{mn}\delta_{nd} = f_{abc}\, T^b . T^c \tag{4.5.33}$$

We have $[T^a, T^b] = if^{abc}\, T^c$ (4.5.34)

Using (4.5.34), (4.5.33) can be written as,

$$f_{abc}\, T^b . T^c = \frac{i}{2}\, f_{abc}\, f^{bcd}\, T^d \qquad (4.5.35a)$$

$$= \frac{i}{2}\, C_2(G)\, T^a \qquad (4.5.35b)$$

Therefore, the divergent part is,

$$-igI_\rho{}^a = -\frac{3g^3}{8\pi^2}\,\gamma_\rho \cdot \frac{i}{2}\, C_2(G)T^a . \left\{\frac{1}{2}\ \ln(\Lambda^2)\right\} \qquad (4.5.36a)$$

Putting $\rho \to \mu$,

$$I_\mu{}^a = \frac{g^2}{8\pi^2}\left[\frac{3C_2(G)}{2}\right]\gamma_\mu\ (T^a) . \left\{\frac{1}{2}\ \ln(\Lambda^2)\right\} \qquad (4.5.36b)$$

Again comparison with the dimensional regularization method suggests the correspondence,

$$\ln(\Lambda^2) \sim \frac{2}{4-D} \qquad (4.5.37)$$

Adding together the two vertex contributions, the total nonlocal divergent vertex part in QCD is :

$$\Lambda_\mu{}^a = \Lambda_\mu{}^a\ (A_D) + I_\mu{}^a .$$

$$= \frac{g^2}{8\pi^2}\,\gamma_\mu\left[-\frac{1}{2}\ C_2(G) + C_2(F)\right](T^a)\left\{\frac{1}{2}\ \ln(\Lambda^2)\right\}$$

$$+ \frac{g^2}{8\pi^2}\,\gamma_\mu\left[\frac{3}{2}C_2(G)\right](T^a)\left\{\frac{1}{2}\ \ln(\Lambda^2)\right\} \qquad (4.5.38a)$$

$$= \frac{g^2}{8\pi^2}\,\gamma_\mu\left\{\frac{1}{2}\ \ln(\Lambda^2)\right\} T^a\ [C_2(G) + C_2(F)] \qquad (4.5.38b)$$

For SU(3), $C_2(F) = 4/3$ and $C_2(G) = 3$.

Hence the total nonlocal divergent part becomes,

$$\Lambda_\mu^{a} = \frac{g^2}{8\pi^2} \left\{ \frac{1}{2} \ln(\Lambda^2) \right\} \cdot \frac{13}{3} \gamma_\mu T^a \qquad (4.5.39)$$

This result is yet to be published [95]. It can be seen that the result obtained for the divergent part in both quark self-energy and the quark-gluon vertex by the method of nonlocal regularization is in perfect agreement with the corresponding results of dimensional regularization, provided one suggests the correspondence [28].

$$\ln \Lambda^2 \sim \frac{2}{4-D} .$$

4.6 Discussion

The method of nonlocal regularization is extended to Yang-Mills theories also, which is a non-abelian one. We are emphasizing particularly on the SU(3) gauge symmetry. The vacuum polarization for the pure Yang-Mills (involving gluons and ghosts) is calculated by Kleppe and Woodard [51]. In this chapter the quark self-energy and the QCD vertex part in nonlocal regularization are evaluated, with the help of the two kinds of propagators–smeared and shadow . The results obtained by this method and the former conventional method are the same. The divergent parts of the quark and electron self–energies are related as

$$\Sigma_{div}{}^{ab}(p) \qquad = \qquad C_2(F)\, \delta_{ab}\, \Sigma_{div}\, QED$$

here also. $C_2(F)$ is the group theoretic factor. Unlike in QED, there are two types of vertices in QCD–the quark–gluon and the gluon–gluon. The total vertex contribution to QCD, which is the sum of the two types of vertices is the same as that obtained from the dimensional regularization method. Again one can suggest the correspondence

$$\ln(\Lambda^2) \sim \frac{2}{4-D},$$

for the co-efficients of the logarithmic divergences in the two methods, for the two physical processes in QCD also.

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