D 121955 (Pages : 2) Name.............................

Reg. No...............................

FOURTH SEMESTER P.G. (CCSS) DEGREE EXAMINATION, APRIL 2025

Mathematics

MAT4E22—ALGEBRAIC GRAPH THEORY

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *mark.*

1. Show that $K_{1,3}$ is not an induced subgraph of a line graph.
2. Define proper coloring of a graph.
3. If X is vertex of the graph X and g is an automorphism of X then prove that the vertex $y = x^2$ has the same valency as x.
4. Prove the Cayley graph X(G, c) is connected if and only if C is a generating set for G.
5. Is the Petersen graph a Cayley graph ? Justify your answer.
6. Define vertex transitivity and edge transitivity of a graph with an example.
7. Show that any two disjoint edge atoms are vertex disjoint.
8. Prove that the eigenvalues of a real symmetric matrix A are real numbers.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Let G be a permutation graph acting on V and let x be a point in V. Prove that $|G_x||{}_{x}{}^{G}| = |G|$.
10. Prove that the size of the isomorphism class containing X is $\dfrac{n!}{|\text{Aut}(X)|}$.
11. Prove that chromatic number of a graph X is the least integer such that there is a homomorphism from X to K_r.
12. State and prove Orbit-stabilizer Lemma.

Turn over

13. Let G be a transitive permutation group on V. Show that G is primitive if and only if each non-diagonal orbit is connected.
14. A graph X is a line graph if and only if each induced subgraph of X on at most six vertices is a line graph.
15. Show that the Cayley graph X(G, C) is vertex transitive.
16. Let X be vertex and edge transitive, but not are transitive. Show that its valency is even.
17. Prove two properties of fragments.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. (a) Prove that any two paths of maximum length in a connected graph must have at least one vertex in common.
    (b) Prove that a transitive abelian permutation group is regular.
19. (a) Prove that almost all graphs are asymmetric.
    (b) Let X be a connected vertex-transitive graph. Then prove that X has a matching that misses at most one vertex and each edge is contained a maximum matching.
20. Let D be a directed graph such that in-valency and out-valency of any two vertices are equal. Then show that D is strongly connected if and only if D is weakly connected.
21. (a) Show that X and $\overline{X}$ have the same automorphism group for any graph X.
    (b) Let u and v be distinct vertices in the graph X. Then prove that the maximum number of openly disjoint paths from u to v equals the minimum size of a set of vertices S such that u and v lie in distinct components of X\S.
22. (a) Define adjaceny matrix of a directed graph.
    (b) Let X be a directed graph with adjacency matrix A. Prove that the number of walks from the vertices u to v in X with length r is $(A^r)_{uv}$.

(3 × 8 = 24 marks)

D 121956 **(Pages : 2)** **Name..............................**

Reg. No...............................

FOURTH SEMESTER P.G. (CCSS) DEGREE EXAMINATION, APRIL 2025

Mathematics

MAT4E23—ADVANCED FUNCTIONAL ANALYSIS

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *the questions.*
Each question carries 1 *mark.*

1. Let X be a normed space and A ∈ BL(X), the space of bounded operators on X. Show that A is invertible if and only if A is bounded below and surjective.
2. Distinguish between weak and weak* convergences.
3. Prove that if X is a reflexive normed space, then its dual X′ is reflexive.
4. Let F, G∈CL(X,Y), the space of compact linear maps from a normed space X to a normed space Y, then show that F + G∈CL(X,Y).
5. State the Bessel's inequality.
6. State the Riesz-Fischer theorem.
7. State and prove the Paralleogram law.
8. Let H be a Hilbert space, A and B be self-adjoint operators. Prove that AB is self-adjoint if and only if A and B commute.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Give an example to show that $\sigma_a(A) \not\subset \sigma_e(A)$ is general, where $\sigma_a(A)$ and $\sigma_e(A)$ represents the approximate eigenspectrum and eigenspectrum respectively.
10. Let X be a normed space. Show that if the dual space X′ is separable, then so is X.
11. Let X be a non-zero reflexive space. Prove that every non-empty closed convex subset of X contains an element of minimal norm.

Turn over

12. Let X and Y be normed spaces and $F : X \to Y$ be linear. Prove that F is a compact map if and only if for every bounded sequence (x_n) in X, $(F(x_n))$ has a sub sequence which converges in Y.
13. Let H be a Hilbert space and $H \neq 0$. Prove that H contains an orthonormal set.
14. Prove that every separable Hilbert space has a Schauder basis.
15. Let X be an inner product space and $f \in X'$. Let $\{u_1, u_2, \ldots\}$ be an orthonormal set in X. Then prove that $\sum |f(u_n)|^2 \leq \|f\|^2$.
16. State and prove the Schwarz inequality.
17. Let $A \in BL(H)$, then show that there are unique self-adjoint operators B and C on H, such that $A = B + iC$.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 marks.

18. Let X be a nonzero Banach space over C and $A \in BL(X)$. Prove that $\sigma(A)$, the spectrum of A is non-empty.
19. Let X and Y be normed spaces and $F \in BL(X, Y)$. Prove that if $F \in CL(X, Y)$, then $F' \in CL(Y', X')$. Also, prove that the converse holds if Y is a Banach space.
20. Let H be a non-zero Hilbert space over K. Then show that the following conditions are equivalent.
    - (a) H has a countable orthonormal basis.
    - (b) H is linearly isometric to K^n for some n or to l^2.
    - (c) H is separable.
21. State and prove the Projection theorem.
22. State and prove the Generalized Schwarz inequality.

(3 × 8 = 24 marks)

D 121957 **(Pages : 3)** **Name..............................**

Reg. No...............................

FOURTH SEMESTER P.G. (CCSS) DEGREE EXAMINATION, APRIL 2025

Mathematics

MAT4E24—ADVANCED COMPLEX ANALYSIS

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. If a set $\mathcal{F} \subset H(G)$ is locally bounded then prove it is uniformly bounded on every compact subsets of G.
2. True or false : "The subspace M(G) of $C(G, \mathbb{C}_\infty)$ is complete." Justify.
3. Let V and U be open subsets of $\mathbb{C}$ with $V \subset U$ and $\partial V \cap U = \phi$. Show that if H is a component of U and $H \cap V \neq \phi$ then $H \subset V$.
4. Define "homeomorphism". Show that $\mathbb{C}$ and $D = \{z : |z| < 1\}$ are homeomorphic.
5. Prove that if f is an entire function of order λ then f' also has order λ.
6. Find the order of the entire function $\sin(z)$.
7. Define the term function element, germ and analytic continuation of a function element along a path.
8. State Monodromy Theorem.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Suppose $\mathcal{F} \subset C(G, \Omega)$ is equicontinuous at each point of G. Prove that $\mathcal{F}$ is equicontinuous over each compact subsets of G.
10. Let $\{f_n\}$ be a sequence in M(G) and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Show that either f is meromorphic or $f \equiv \infty$.

Turn over

11. Let $Re z_n > 0$ for all $n \geq 1$. Show that $\prod_{n=1}^{\infty} z_n$ converages to a non-zero number iff the series $\sum_{n=1}^{\infty} \log(z_n)$ converges.

12. Let γ be a rectifiable curve and let K be a compact set such that $K \cap \{\gamma\} = \phi$. Show that for given $\varepsilon > 0$ and a f is a continuous function defined on $\{\gamma\}$ there exists a rational function R(z) having all its poles on $\{\gamma\}$, such that $\left|\int_\gamma \frac{f(w)}{w-z}\,dw - R(z)\right| < \varepsilon$ for all z in K.

13. The set $G = \left\{re^{it} : -\infty < t < 0 \text{ and } 1 + e^t < r < 1 + 2e^t\right\}$ is called a cornucopia. Show that G is simply connected.

14. State and prove Harnack's Theorem.

15. State and prove Jensen's formula.

16. State and prove Little Picard Theorem.

17. Let $\gamma[0,1] \to \mathbb{C}$ be a path and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. Show that or $0 \leq t \leq 1$, if R(t) be the radius of convergence of the power series expansion of f_t about $z = \gamma_t$ then either $R(t) \equiv \infty$ or $R : [0,1] \to (0, \infty)$ is continuous.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. Let G be a region and let $\{a_j\}$ be a sequence of distinct points in G with no limit point in G; and let $\{m_j\}$ be a sequence of integers. Show that there is an analytic function f defined on G whose only zeros are at the point a_j; furthermore, a_j is a zero of f of multiplicity m_j.

19. Let K be a compact subset of the region G; then show that there are straight line segments $\gamma_1, \gamma_2, \dots \gamma_n$ in G – K such that for every function f in H(G), $f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z}\,dw$ for all z in K. The line segments form a finite number of polygons.

20. Let $D = \{z : |z| < 1\}$. Show that if $f : \partial D \mapsto \mathbb{R}$ is a continuous function then there is a continuous function $u = \bar{D} \mapsto \mathbb{R}$ such that

    (a) $u(z) = f(z)$ for z in ∂D;

    (b) u is harmonic in D.

    Moreover u is unique and is defined by the formula

$$u\left(re^{i\theta}\right) = \frac{1}{2}\int_{-\pi}^{\pi} P_r(\theta - t) f\left(e^{it}\right) dt$$

for $0 \le r < 1, 0 \le \theta \le 2\pi$. Here P_r denotes the Poisson kernel.

21. Let f be an entire function of genus μ. Show that for each positive number α there is a number r_0 such that for $|z| > r_0, |f(z)| < \exp\left(\alpha |z|^{\mu+1}\right)$.

22. State and prove Schwarz reflection principle.

(3 × 8 = 24 marks)

D 121958 (Pages : 2) Name..............................

Reg. No..............................

FOURTH SEMESTER P.G. (CCSS) DEGREE EXAMINATION, APRIL 2025

Mathematics

MAT4E25—COMMUTATIVE ALGEBRA

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Verify whether $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ is a subring of the ring $\mathbb{Q}$ or rationals.
2. Verify whether the ideal generated by $x^2 + 1$ in $\mathbb{Q}[x]$ is a prime ideal.
3. Let $A = a_1 \oplus a_2$ be a direct sum of rings a_1 and a_2. Show that a_1 is isomorphic to A/a_2.
4. Verify whether the sequence $M \xrightarrow{f} M/N \to 0$ is exact where N is a submodule of M and f is the quotient map.
5. Verify whether the ideal generated by 4 in the ring $\mathbb{Z}$ of integers is a primary ideal.
6. Verify whether $\sqrt{2}$ is integral over $\mathbb{Z}$.
7. Verify whether the polynomial ring $k[x]$ over a field k satisfies a.c.c. on ideals.
8. Verify whether the ring $\mathbb{Z}[i]$ of Gaussian integers is noetherian.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Let A be a ring such that 0 and A are the only ideals of A. Show that A is a field.
10. Let m be an ideal of a ring A such that every $x \in A - m$ is a unit. Show that m is a maximal ideal of A.
11. Let $L \supseteq M \supseteq N$ be an A-modules. Let θ : L/N → L/M be defined by $\theta(x + N) = x + M$. Show that θ is an A-module homomorphism.

Turn over

12. Let M be a finitely generated A-module. Show that M is isomorphic to a quotient of A^n for some positive integer n.
13. Let M, N be A-modules. Show that the tensor product $M \otimes N$ is isomorphic to $N \otimes M$.
14. Show that if q is a primary ideal of a ring A then the radical $r(q)$ is a prime ideal.
15. Show that if $x \in B$ is integral over A then $A[x]$ is a finitely generated A-module.
16. Show that every Noetherian A-module is finitely generated.
17. Show that every ideal in a Noetherian ring is a finite intersection of irreducible ideals.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. (a) Define local ring and give an example.
    (b) Let A be a ring and m be a maximal ideal of A. Show that if every element of $1 + m$ is a unit in A then A is a local ring.
19. (a) Define Jocobson radical of a ring.
    (b) Let $\mathcal{R}$ be the Jacobson radical of a ring A and $x \in A$. Show that $x \in \mathcal{R}$ if and only if $1 - xy$ is a unit in A for all $y \in A$.
20. (a) Define tensor product $M \otimes N$ of A-modules M and N.
    (b) Show that if $M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is an exact sequence of A-modules then $M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \to 0$ is exact for any A-module N.
21. (a) Define integrally closed domain.
    (b) Show that if A is an integrally closed domain then A_p is integrally closed for all prime ideals P.
22. (a) Define Artinian rign and given an example of a ring which is not Artinian.
    (b) Show that in an Artinian ring the nilradical is nilpotent.

(3 × 8 = 24 marks)

D 121959 (Pages : 2) Name..............................

Reg. No..............................

FOURTH SEMESTER P.G. (CCSS) DEGREE EXAMINATION, APRIL 2025

Mathematics

MAT4E26—GRAPH THEORY

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Section A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Draw a bipartite graph with six vertices.
2. Draw an edge deleted subgraph of Petersen graph.
3. Explain unilateral digraph.
4. Explain Cayley's formula.
5. State the Fan lemma.
6. Determine the edge connectivity of the Kneser graph $KG_{m,n}$.
7. State the Jordan curve theorem.
8. Show that the dual of any plane graph is connected.

(8 × 1 = 8 marks)

Section B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. If a graph G has m edges and n vertices $m \geq n$, then show that G contains a cycle.
10. Show that any two longest paths in a connected graph have a vertex in common.
11. Prove that a connected digraph is eulerian if and only if it is even.
12. Show that $K_{3,3}$ has no algebraic dual.
13. Let G be a k-connected graph, and let X and Y be subsets of V of cardinality at least k. Then show that there exists in G a family of k pairwise disjoint (X, Y)-paths.
14. Let G be a 3-connected graph, let v be a vertex of G of degree at least four, and let H be an expansion of G at v. Then prove that H is 3-connected.

Turn over

15. If G is a connected chordal graph which is not complete, and S be a minimal vertex cut of G. Then show that S is a clique cut of G.

16. Prove that a graph G is embeddable on the plane if and only if it is embeddable on the sphere.

17. Prove that the dual of a nonseparable plane graph is nonseparable.

(6 × 3 = 18 marks)

Section C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. (a) Prove that, in a connected graph G, a non-empty edge cut $\delta(X)$ is a bond if and only if both G[X] and G[V\X] are connected.

    (b) Show that an arc of a digraph is contained either in a directed cycle, or in a directed bond, but not both.

19. Prove that the number of labelled trees on n vertices in $n^{(n-2)}$.

20. (a) Let G be a k-connected graph and let H be a graph obtained from G by adding a new vertex y and joinint it to at least k vertices of G. Then show that H is also k-connected.

    (5 marks)

    (b) Let G be a 3-connected graph on at least five vertices, and let $e = xy$ be an edge of G such that $G \backslash e$ is not 3-connected. Then show that there exists a vertex z such that $\{x, y, z\}$ is a 3-vertex cut of G.

    (3 marks)

21. Let G be a graph which is locally $2k$-edge-connected modulo v, where v is a vertex of even degree in G. Given any link uv incident with v, then show that there exists a second link vw incident with v such that the graph G′ obtained by splitting off uv and vw at v is also locally 2k-edge-connected modulo v.

22. (a) Prove that, in a nonseparable plane graph other than K_1 or K_2, each face is bounded by a cycle.

    (5 marks)

    (b) Prove that, in a loopless 3-connected plane graph, the neighbours of any vertex lie on a common cycle.

    (3 marks)

[3 × 8 = 24 marks]

D 121966 **(Pages : 3)** **Name..............................**

Reg. No...............................

FOURTH SEMESTER P.G. (CCSS) DEGREE EXAMINATION, APRIL 2025

Mathematics

MAT4E33—TOPOLOGICAL GROUPS

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Verify whether the set $\mathbb{R}$ of reals with usual topology is a topological group with addition as the group operation.
2. Give a fundamental neighbourhood system at 0 of symmetric neighbourhoods for the topological group ($\mathbb{R}$, +) with usual topology.
3. Let G be a topological group and A be a subset of G. Show that $\overline{a\mathrm{A}a^{-1}} = a\bar{\mathrm{A}}a^{-1}$.
4. Let the open interval U = (–1, 1) be a neighbourhood of 0 in the topological group ($\mathbb{R}$, +) with usual topology. Find $\cup_{n\geq 1}\mathrm{U}^n$.
5. Verify whether the topological group $(\mathbb{R}-\{0\},\cdot)$ is locally compact.
6. Let E be a locally compact topological group and F be any topological group. Show that if there is a continuous open homomorphism from E to F then F is locally compact.
7. Let G be an additive abelian topological group and G′ be the dual group. For x, $y \in$ G and $x' \in$ G′ show that $\langle x+y, x'\rangle = \langle x, x'\rangle + \langle y, x'\rangle$.
8. Define point open topology on the dual G′ of a topological group G.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Let G be a topological group. Show that the inner automorphism $x \mapsto axa^{-1}$ for all $a \in$ G is a homeomorphism of G onto itself.

Turn over

10. Let G be a topological group and e be the identity of G. Show that there is a fundamental system of symmetric neighbourhoods of e in G.
11. Show that every T_0 topological group is a T_1 space.
12. Let ($\mathbb{R}$+) be a topological group with usual topology. Let A = (0, 1) and B = (1, 2) be open intervals. Show that $\overline{A} + \overline{(-B)} = \overline{A - B}$.
13. Show that the center of a Hausdorff topological group is closed.
14. Let G be a topological group having a compact neighbourhood of the identity. Show that G is locally compact.
15. Show that the set Gn(K) of regular $n \times n$ matrices over a field K is an open set in Mn(K).
16. Let $\mathcal{S}$ denote the family of all finite subsets of a topological group and $\mathcal{V}$ be a fundamental system of open symmetric neighbourhoods of 0 in the one dimensional circle group T. Show that $\{T(M, V) : M \in \mathcal{S}, V \in \mathcal{V}\}$ is a basis of neighbourhoods of the identity 0′ in the dual group G′.
17. Let $\mathcal{S}$ denote the family of all compact subsets of a topological group and $\mathcal{V}$ be a fundamental system of open symmetric neighbourhoods of 0 in the one dimensional circle group T. Let $M \in \mathcal{S}$ and U, V$\in \mathcal{V}$. Show that if $U + U \subseteq V$ then $T(M, U) + T(M, U) \subseteq T(M, V)$.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. (a) Define neighbourhood of a point in a topological group.
    (b) Let $\mathcal{U}_e$ be the system of all neighbourhoods of the identity e in a topological group G and A be a subset of G. Show that $\overline{A} = \cap\{AU : U \in \mathcal{U}_e\}$.
    (c) Show that for every neighbourhood U of the identity e there is a neighbourhood V of e such that $\overline{V} \subseteq U$.
19. (a) Define fundamental system of neighbourhoods of a point in a topological group.
    (b) Let F be a closed subset and C be a compact subset of a topological group G. Show that FC is closed.
    (c) Let F = [0, 1] and B = (0, 1) in the topological group ($\mathbb{R}$, +) with usual topology. Show that F + B is not closed.

20. Let $G = \Pi\{G_\alpha : \alpha \in A\}$ be a product of topological groups with product topology. Show that :

    (a) G is a topological group.

    (b) For each $\alpha \in A$ the projection $p_\alpha : G \to G_\alpha$ is an open map.

21. (a) Define locally Euclidean topological group.

    (b) Show that $M_n(K)$ is a locally Euclidean group.

    (c) Find a neighbourhood U of the zero matrix in $M_2(\mathbb{R})$ such that U is homeomorphic to $\mathbb{R}^4$.

22. (a) Define the dual group G' of a topological group G.

    (b) Show that G' with compact open topology is a topological group.

(3 × 8 = 24 marks)