C 63531 **(Pages : 3)** Name....................................

Reg. No...................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2014

(CCSS)

Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 marks.

1. Describe all inner products on R, where R is vector space of all real numbers over R.
2. Show that an orthogonal set of non-zero vectors is linearly independent.
3. State Picard's theorem. For what points (x_0, y_0) does Picard's theorem imply that the initial value problem $y' = |y|$, $y(x_0) = y_0$ has a unique solution on some interval $|x - x_0| \le h$.
4. Verify that $y = c_1 x^{-1} + c_2 x^5$ is the general solution of the equation $x^2 y'' - 3xy' - 5y = 0$ on any interval $[a, b]$ that does not contain zero.
5. Define radius of convergence of a power series. If p is not zero or a positive integer, then find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{p(p-1)\ldots(p-n+1)}{n!} x^n$.
6. Locate and classify the singular points on the x-axis of the equation $x^2 (x^2 - 1)^2 y'' - x(1 - x) y' + 2y = 0$.
7. Show that $F'(a, b, c, x) = \frac{ab}{c} F(a+1, b+1, c+1, x)$.
8. Let $f(x)$ be a function defined on the interval $-1 \le x \le 1$. Find a polynomial $p(x)$ of degree n such that $I = \int_{-1}^{1} [f(x) - p(x)]^2 \, dx$ is minimum.
9. Define the Gamma function $\Gamma(p)$ and show that $\Gamma(n+1) = n!$ for any integer $n \ge 0$.

Turn over

10. Describe the phase portrait of the system :

$$\frac{dx}{dt} = -x, \frac{dy}{dt} = -y.$$

11. Show that a function of the form $ax^3 + bx^2 y + cxy^2 + dy^3$ cannot be either positive definite or negative definite.

12. Show that (0, 0) is an asymptotically stable critical point of the system :

$$\frac{dx}{dt} = -y - x^3, \quad \frac{dy}{dt} = x - y^3.$$

(12 × 4 = 48 marks)

Part B

Answer A **or** *B of each question.*
Each question carries 8 *marks.*

13. (A) Let W be the subspace of R^2 spanned by the vector (3, 4). Using the standard inner product, let E be the orthogonal projection of R^2 onto W. Find

(i) a formula for E (x_1, x_2).

(ii) the matrix of E in the standard ordered basis.

(iii) $W^{\perp}$.

(iv) an orthonormal basis in which E is represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(B) (a) Explain Picard's method of successive approximation of solving the initial value problem $y' = f(x, y), y(x_0) = y_0$; where $f(x, y)$ is an arbitrary function defined and continuous in some neighbourhood of (x_0, y_0).

(5 marks)

(b) Find the exact solution of the problem $y' = x + y, \quad y(0) = 1$. Starting with $y_0(x) = 1$, apply Picard's method to calculate $y_1(x)$, $y_2(x)$ and $y_3(x)$.

(3 marks)

14. (A) (a) Find the general solution of the equation $(x^2 - 1) y'' - 2xy' + 2y = (x^2 - 1)^2$. (5 marks)

(b) Find the indicial equation and its roots of the differential equation

$$x^3 y'' + (\cos 2x - 1) y' + 2xy = 0.$$

(3 marks)

(B) (a) Find a series solution $y(x)$ of the equation $y'' + y' - xy = 0$ such that $y(0) = 1$ and $y'(0) = 0$.

(4 marks)

(b) Express $\sin^{-1} x$ in the form of a power series of the form $\sum a_n x^n$ by solving $y' = (1-x^2)^{-1/2}$; $y(0) = 0$ in two ways.

(4 marks)

15. (A) (a) Find the general solution of the Gauss's hypergeometric equation :

$$x(1-x)\,y'' + [c-(a+b+1)\,x]\,y' - aby = 0$$ near its singular point $x = 0$.

(6 marks)

(b) Show that, for a positive integer $m \geq 0$, $J_m(x)$ and $J_{-m}(x)$ are linearly independent.

(2 marks)

(B) (a) Derive Rodrigue's formula for Legendre polynomials. (5 marks)

(b) Show that $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$. (3 marks)

16. (A) (a) Find the general solution of the system :

$$\frac{dx}{dt} = -4x - y, \quad \frac{dy}{dt} = x - 2y.$$

(5 marks)

(b) Determine the nature and stability properties of the critical point (0, 0) for the system :

$$\frac{dx}{dt} = 2x, \quad \frac{dy}{dt} = 3y.$$ (3 marks)

(B) For the non-linear system, $\frac{dx}{dt} = y(x^2 + 1)$, $\frac{dy}{dt} = -x(x^2 + 1)$.

(i) Find the critical points.

(ii) Find the differential equation of the paths.

(iii) Solve this equation and find the paths.

(iv) Sketch the few of the paths and show the direction of increasing t.

($4 \times 2 = 8$ marks)

[$4 \times 8 = 32$ marks]

C 63530 (Pages : 3) Name..

Reg. No......................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2014

(CCSS)

Mathematics

MAT 2C 07—REAL ANALYSIS—II

Time : Three Hours | Maximum : 80 Marks

Part A

*Answer **all** questions.*
Each question carries 4 marks.

I 1. Prove that a linear operator A on a finite dimensional vector space X is one to one if and only if the range of A is all of X.

2. Let Ω be the set of invertible operators on $\mathbb{R}^n$ and let $A \in \Omega$. Prove that Ω is an open subset of $\mathbb{R}^n$ and the mapping $A \to A^{-1}$ is continuous on Ω.

3. Let C be a collection of subsets of a set X. Prove that there exists a smallest algebra containing C.

4. If $m(E) = 0$, then prove that E is a measurable set.

5. Prove that every Borel set is measurable.

6. Let E be a measurable set. Prove that for every $\epsilon > 0$, there is an open set O containing E such that $m^*(O \sim E) < \epsilon$.

7. If f is a measurable function and $f = g$ *a.e.*, then prove that g is measurable.

8. Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then prove that f measurable and

$$R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx.$$

9. Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions and let $f = \lim f_n$ *a.e.*. Prove that $\int f = \lim \int f_n$.

10. Show that if f is integrable over a measurable set E, then so is $|f|$ and

$$\left|\int_E f\right| \leq \int_E |f|.$$

Turn over

11. If f is of bounded variation on $[a, b]$, then prove that $T_a{}^b = P_a{}^b + N_a{}^b$.

12. If f is absolute continuous on $[a, b]$, then prove that f is of bounded variation on $[a, b]$.

(12 × 4 = 48 marks)

Part B

Answer **A** *or* **B** *of each question.*
Each question carries 8 marks.

II A (a) Let E be an open subset of $\mathbb{R}^n$ and f maps E into $\mathbb{R}^m$. If f is differentiable at a point $\mathbf{x} \in E$, then prove that the partial derivatives $(D_j f_i)(\mathbf{x})$ exist.

(b) If X is complete metric space and if φ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\varphi(x) = x$

B State and prove inverse function theorem.

III A (a) Prove that intervals are measurable.

(b) Prove that there exists a non-measurable set.

B (a) Prove that outer measure of an interval is its length.

(b) Let $\{f_n\}$ be a sequence of measurable functions with the same domain of definition. Prove that $\sup_n f_n$ is measurable.

IV A (a) Let $E_1, E_2, \ldots, E_n$ be disjoint measurable sets and let $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$. Prove that $\int \varphi = \sum_{i=1}^{n} a_i m(E_i)$.

(b) Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure and suppose that there is a real M such that $|f_n(x)| \leq M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each x in E, then prove that

$$\int_E f = \lim \int_E f_n.$$

B (a) Let f and g be integrable over E. If A and B are disjoint measurable sets contained in E, then prove that

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

(b) Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all x in E we have $f(x) = \lim f_n(x)$. Prove that

$$\int_E f = \lim \int_E f_n.$$

V A (a) If f is of bounded variation on $[a, b]$, then prove that $f'(x)$ exists for almost all x.

(b) Let f be an integrable function on $[a, b]$ and let

$$F(x) = F(a) + \int_a^x f(t)\, dt.$$

Prove that $F'(x) = f(x)$ for almost all x in $[a, b]$.

B (a) Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure. Prove that $\{f_n\}$ converges to f in measure if and only if every subsequence of $\{f_n\}$ has in turn a subsequence that converges almost everywhere to f.

(b) If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ *a.e.*, then prove that f is a constant.

(4 × 8 = 32 marks)

C 63529 (Pages : 3) Name..

Reg. No......................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2014

(CCSS)

Mathematics

MAT 2C 06—ALGEBRA–II

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Prove that $R[x]/<x^2+1>$ is isomorphic to the field c of complex numbers.

2. Find irr (α, Q) where Q is the field of rationals and $\alpha = \sqrt{2}+\sqrt{3}$.

3. Prove that $\sqrt[3]{2}$ is not constructible using straight edge and compass.

4. Describe all primitive 8^{th} roots of 1.

5. Verify whether $i+\sqrt{2}$ and $i-\sqrt{2}$ are conjugates over Q.

6. Let E be a field of characteristic 2. Show that $\psi : E \to E$ defined by $\psi(\alpha)\alpha^2$ for $\alpha \in E$ is an automorphism of E.

7. Let E be the splitting field of x^3-2 over Q. Find [E : Q].

8. Let K be a finite normal extension of F and $F \le E \le K$. Let $\sigma, \tau \in G(K/F)$ and $\sigma(a) = \tau(a)$ for all $a \in E$. Show that $\sigma = \tau\mu$ some $\mu \in G(K/E)$.

9. Let K be a finite normal extension of F. Show that the Galois group $G(K|F)$ is finite.

10. Verify whether $xy + xz + yz + xyz$ in a symmetric function on x, y, z.

11. Let K be a the splitting field of x^4+1 over Q. Show that G (K/Q) is isomorphic to the Klein four group.

12. Show that the n^{th} cyclotomic polynomial $\Phi_n(x)$ over Q belongs to Q [x].

(12 × 4 = 48 marks)

Turn over

Part B

Answer **either** *A or B of each question.*
Each question carries 8 *marks.*

13. (A) (a) Let F (α) an algebraic extension of F and let deg irr $(\alpha, F) = n > 1$. Show that every $\beta \in E$ can be uniquely expressed as $\beta = b_0 + b_1 \alpha + \ldots + \beta_{n-1}{}^{\alpha^{n-1}}$ where bi$\in$ F.

(b) Describe all elements of the field $Z_2(\alpha)$ where α is a zero of $x^2 = x + 1 \in Z_2[x]$.

(B) (a) Prove that every finite extension of a field F is an algebraic extension of F.

(b) Prove by an example that an algebraic extension need not be a finite extension.

14. (A) (a) Let E be a finite field. Show that the number of elements in E is p^n for a prime p and a natural number n.

(b) Let E be a field of p^n elements and let $E \subseteq \overline{Z}_p$ where $\overline{Z}_p$ is the algebraic closure of Z_p. Prove that elements of E are precisely the zeros of $x^q - x$ in $\overline{Z}p$ where $q = pn$.

(B) (a) Let F be a field and α,β be conjugates over F. Show that there exists an isomorphism of F(α) on to F (β) which maps α to β.

(b) Let α be algebraic over F and $\psi : F(\alpha) \to \overline{F}$ be an isomorphism such that $\psi(a) = a$ for all $a \in F$. Prove that $\psi(\alpha)$ is a conjugate of α over F.

15. (A) (a) Let $F \leq E \leq \overline{F}$ and let E be a splitting field over F. Prove that every automorphism of $\overline{F}$ leaving F fixed maps E on E.

(b) Let E be a splitting field over F and $f(x)$ be an irreducible polynomial in F $[x]$ having a zero in E. Show that F $[x]$ splits in E$[x]$.

(B) (a) Show that if $f(x) \in f[x]$ is irreducible then all zeros of $f(x)$ in $\overline{F}$ have the same multiplicity.

(b) Let $F \leq E \leq K$ and let K be a finite extension of F. Show that if E is separable over F and K is separable over E then K is separable over F.

Define elementary symmetric functions in $y_1, y_2, \ldots, y_n$ over a field F.

Show that if K is the field of all symmetric functions in $y_1, y_2, \ldots, y_n$ over F then $K = F(s_1, s_2, \ldots, s_n)$ where $s_1, s_2, \ldots, s_n$ are elementary symmetric functions over F.

Define the n^{th} cyclotomic polynomial $\Phi_n(x)$ over a field F.

Show that the Galois group of $\Phi_n(x)$ over Q is isomorphic to the multiplicative group of positive integers and less than n and relatively prime to n under multiplication mode n.

(4 × 8 = 32 marks)

C 27854 (Pages 2) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JULY 2012

(CCSS)

Mathematics

MAT 2C 09—TOPOLOGY—I

(2010 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

1. Give an example of a normal topology that is not discrete.
2. Prove that closed balls in a metric space are closed sets.
3. Define a topology in the set of real numbers which is stronger than the usual topology but strictly weaker than the discrete topology.
4. Define the scattering topology.
5. Write a sub base for the set of real numbers with usual topology.
6. Is the real line with usual topology separable? Justify your answer.
7. Find the derived set of the set of integers in the real line with usual topology.
8. Prove that every closed surjective map is a quotient map.
9. Give an example of a topology that satisfy the countable chain condition.
10. Define extension problem and lifting problem in topological spaces.
11. Distinguish between connectedness and locally connectedness in topological spaces.
12. Justify the terms 'box' and 'wall' geometrically for products of copies of the real line.

Part B

Answer **either** *A* **or** *B part of each question.*
Each question carries 8 *marks.*

13. A (a) Let $\{x_n\}$ be a sequence in a metric space $(X; d)$. Then prove that $\{x_n\}$ converges to y in X if and only if for every open set U containing y, there exists a positive integer N such that for every integer $n \geq N$, $x_n \in U$.

 (b) Determine the topology induced by a discrete metric on a set.

 B (a) Prove that in a co-countable topology, the only convergent sequences are those which are eventually constant.

Turn over

(b) Prove that a space is second countable if and only if it has a countable sub-base.

14. A (a) Define dense subset of a topological space. State and prove necessary and sufficient condition for a set to be dense.

(b) Prove that a subset of a topological space is open if and only if it is the neighbourhood of each of its points.

B (a) For any three spaces X_1, X_2, X_3, prove that $X_1 \times (X_2 \times X_3)$ is homeomorphic to $(X_1 \times X_2) \times X_3$.

(b) Prove that the product topology is the weak topology determined by the projection functions.

15. A (a) Prove that every second countable space is Lindeloff.

(b) Prove that every second countable space is first countable.

B (a) Prove that a subset of the set of real numbers is connected if and only if it is an interval.

(b) Prove that regularity is a hereditary property.

16. A (a) Prove that every Tychnoff space is T_3.

(b) Prove that normality is a weak hereditary property.

B (a) Prove that a subset of X is box if and only if it is the intersection of a family of walls.

(b) Prove that arbitrary product of T_1 spaces is T_1.

C 27853 (Pages : 3) Name.......................................

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JULY 2012

(CCSS)

Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

(2010 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

1. For $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ in R^2, let $(\alpha \mid \beta) = x_1 y_1 - x_2 y_1 - x_1 y_2 + 4x_2 y_2$. Show that $(\,|\,)$ is an inner product on R^2.
2. Show that an orthogonal set of non-zero vectors in an inner product space is linearly independent.
3. Show that $f(x, y) = y^{1/2}$ does not satisfy a Lipschitz condition on the rectangle $|x| \le 1$ and $0 \le y \le 1$.
4. Find the general solution of $y'' - f(x)y' + [f(x) - 1]y = 0$.
5. Show that $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$ by solving the equation $y' = 1 + y^2,\ y(0) = 0$ in two ways.
6. Locate and classify the singular points on the x–axis of the equation $(3x+1)xy'' - (x+1)y' + 2y = 0$.
7. Find the general solution of the differential equation :

   $(x^2 - x - 6)y'' + (5 + 3x)y' + y = 0$ near its singular point $x = 3$.
8. Let $f(x)$ be a function defined on the interval $-1 \le x \le 1$. Find a polynomial $p(x)$ of degree n such that

   $I = \int_{-1}^{1} [f(x) - p(x)]^2\, dx$ is minimum.
9. Define the gamma function $\Gamma(p)$ and show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
10. Describe the phase portrait of the system :

    $$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 2.$$

Turn over

11. Determine the nature and stability properties of the critical point (0, 0) for the system :

$$\frac{dx}{dt} = -4x - y, \quad \frac{dy}{dt} = x - 2y.$$

12. Show that (0, 0) is an asymptotically stable critical point of the system :

$$\frac{dx}{dt} = -2x + xy^3 \; ; \; \frac{dy}{dt} = -x^2y^2 - y^3.$$

Part B

Answer (a) **or** *(b) of each question.*
Each question carries 8 marks.

13. (a) (i) Let W be a subspace of an inner product space V and let β be a vector in V. Show that the vector α in W is a best approximation to β by vectors in W iff $\beta - \alpha$ is orthogonal to every vector in W.

(ii) Consider C^3, with the standard inner product. Find an orthonormal basis for the subspace spanned by

$\beta_1 = (1, 0, i)$ and $\beta_2 = (2, 1, 1 + i)$.

(b) (i) Explain Picard's method of successive approximation of solving the initial value problem $y' = f(x, y)$; $y(x_0) = y_0$, where $f(x, y)$ is an arbitrary function defined and continuous in some neighbourhood of (x_0, y_0).

(ii) Starting with $y_0(x) = 0$, apply Picard's method to calculate $y_1(x), y_2(x), y_3(x)$ and $y_4(x)$ of the initial value problem $y' = 2x(1+y), \; y(0) = 0$.

14. (a) (i) Describe the method of variation of parameters for determining a particular solution of the equation $y'' + P(x)y' + Q(x)y = R(x)$, provided the general solution of the homogeneous equation $y'' + P(x)y' + Q(x)y = 0$ is already known.

(ii) Find a particular solution of $y'' + 2y' + 5y = e^{-x} \sec 2x$.

(b) (i) Find the general solution of the equation $y'' - 2xy' + 2py = 0$, where p is a constant.

(ii) Show that the equation $x^2y'' + xy' + (x^2 - 1)y = 0$ has only one Frobenius series solution.

15. (a) (i) Show that the Gauss's hyper geometric equation $x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$ has three regular singular points 0, 1 and ∞ with corresponding exponents 0 and $1 - c$, 0 and $c - a - b$ and a and b.

(ii) Show that, for a positive integer $m \geq 0$, $J_m(x)$ and $J_{-m}(x)$ are linearly independent.

(b) (i) State and prove the orthogonality property of Legendre polynomials.

(ii) If $f(x) = x^p$ for the interval $0 \le x < 1$, show that its Bessel series in the functions $J_p(\lambda_n x)$, where the λ_n's are the positive zeros of $J_p(x)$, is $x^p = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_{p+1}(\lambda_n)} J_p(\lambda_n x)$.

16. (a) Find the general solution of the system :

$$\frac{dx}{dt} = x - 2y \ ; \ \frac{dy}{dt} = 4x + 5y.$$

(b) (i) State and prove Liapunov's stability theorem for an isolated critical point of the autonomous system :

$$\frac{dx}{dt} = F(x, y) \ ; \ \frac{dy}{dt} = G(x, y).$$

(ii) Show that the linear system, $\frac{dx}{dt} = 4y$; $\frac{dy}{dt} = -x$; has the origin as an isolated critical point and find the differential equation of the paths.

2, 3, 3

C 27852 (Pages : 3) Name.................................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2012

(CCSS)

Mathematics

MAT 2C 07—REAL ANALYSIS—II

(2010 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

I. 1 Let Ω be the set of all invertible linear operators in $\mathbb{R}^n$. Prove that Ω is an open subset of $L(\mathbb{R}^n)$ and the mapping $A \to A^{-1}$ is continuous on Ω.

2 Let f be a differentiable real-valued function defined in an open set $E \subseteq \mathbb{R}^n$ and let f has a local maximum at a point $x \in E$. Prove that $f'(x) = 0$.

3 Prove that the Lebesgue outer measure m* is translation invariant.

4 Prove that intersection of two measurable sets is measurable.

5 Let f and g be measurable functions. prove that max $\{f, g\}$ is a measurable function.

6 Prove that the characteristic function χ_E of a set E is measurable if and only if E is measurable.

7 Let ϕ and ψ be simple functions which vanishes outside a set of measure zero. Prove that $\int(\phi + \psi) = \int\phi + \int\psi$.

8 Let f be a nonnegative measurable function. Show that $\int f = 0$ implies $f = 0$ a.e.

9 Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f. Prove that there is a subsequence $\{f_{n_k}\}$ that converges to f almost everywhere.

10 Let f be a function. Show that $D^+[-f(x)] = -D_+ f(x)$.

Turn over

11 Show that if $a \le c \le b$, then $T_a^b = T_a^c + T_C^b$.

12 Let f be integrable on $[a, b]$. Prove that the function F on $[a, b]$ defined by $F(x) = \int_a^x f(t)\,dt$ is continuous and is of bounded variation on $[a, b]$.

(12 × 4 = 48 marks)

Part B

Answer A **or** B *of each question.*
Each question carries 8 *marks.*

II. A (a) Let $E \subset \mathbb{R}^n$ be an open set and the map $f: E \to \mathbb{R}^n$ be differentiable at $x_0 \in E$. if g maps an open set containing $f(E)$ into $\mathbb{R}^m$ and g is differentiable at $f(x_0)$, then prove that the map $F: E \to \mathbb{R}^m$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0)) f'(x_0)$.

(b) Let f map an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Prove that f is continuously differentiable in E if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m, 1 \le j \le n$.

B State and rove implicit function theorem.

III. A (a) Let C be any collection of subsets of X. Prove that there exists a smallest σ-algebra containing C.

(b) For any $a \in \mathbb{R}$, prove that (a, ∞) is measurable.

B (a) Let $\{E_n\}$ be a sequence of pairwise disjoint measurable sets. prove that

$$m\left(\bigcup_i E_i\right) = \sum_i m(E_i).$$

(b) Let E be a measurable set. Prove that for every $\in > 0$. there exists and open set O containing E such that $m^*(O - E) < \in$.

(c) Prove that sum of two measurable functions is measurable.

IV. A (a) Let f be a bounded function define on $[a, b]$. If f is Reimann integrable on $[a, b]$, then prove that f is measurabled and

$$R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx.$$

(b) State and prove bounded convergence theorem.

B (a) Let f be a bounded function defined on a measurable set of finite measure. Prove that f is measurable if and only if :

$$\inf_{f\leq\psi}\int_E \psi(x)\,dx = \sup_{f\leq\varphi}\int_E \varphi(x)\,dx$$

for all measurable simple function φ and ψ.

(b) Let f and g be integrable over E. Prove that $f + g$ is integrable over E and $\int_E (f+g) = \int_E f + \int_E g$.

V. A Let f be an increasing real valued function on the interval $[a, b]$. Prove that f is differentiable almost everywhere, the derivative f' is measurable and

$$\int_a^b f'(x)\,dx \leq f(b) - f(a).$$

B (a) Prove that a function f is of bounded variation on $[a, b]$ if an only if it is the difference of two monotone real valued functions on $[a, b]$.

(b) If f is absolutely continuous, then prove that f has a derivative almost everywhere.

(4 × 8 = 32 marks)

C 27851 (Pages : 2) Name..............................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JULY 2012

(CCSS)

Mathematics

MAT 2C 06—ALGEBRA—II

(2010 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

1. Show that there exists a finite field of 27 elements.
2. Show that $Q\left(2^{1/2}, 2^{1/3}\right) = Q\left(2^{1/6}\right)$.
3. Prove that the field C of complex numbers is an algebraically closed field.
4. Let E be a field of p^n elements (p – a prime) contained in an algebraic closure $\overline{Z}_p$ of Z_p. Show that the elements of E are precisely the zeros in $\overline{Z}_p$ of the polynomial $x^{p^n} - x$ in $Z_p[x]$.
5. Find the number of primitive 10^{th} roots of unity in $GF(23)$.
6. Prove that if E is an algebraic extension of a field F, then two algebraic closures $\overline{F}$ and $\overline{E}$ of F and E, respectively, are isomorphic.
7. Find the degree over Q of the splitting field over Q of the polynomial $(x^2-2)(x^3-2)$ in $Q[x]$.
8. What is $G\left(Q\left(\sqrt[3]{2}\right)/Q\right)$?
9. State the main theorem of Galois theory.
10. Find $\phi_8(x)$ over Z_3.
11. Show that the Galois group of the p^{th} cyclotomic extension of Q for a prime p is cyclic of order $p-1$.
12. Is it true that the splitting field of $x^{17}-5$ over Q has a solvable Galois group ? Justify your answer.

(12 × 4 = 48 marks)

Turn over

Part B

Answer A **or** B *of each questions.*
Each question carries 8 *marks.*

13. A. (i) Let F be a field and let $f(x)$ be a nonconstant polynomial in $F[X]$. Show that there exists an extension field E of F and an $\alpha \varepsilon E$ such that $f(\alpha)=0$.

(ii) Show that $\sqrt{\sqrt[3]{2}-i}$ is algebraic over Q by finding $f(x) \varepsilon Q[x]$ such that $f(\alpha)=0$.

B. (i) Let E be an algebraic extension of a field F. Show that there exist a finite number of elements $\alpha_1, \alpha_2, ..., \alpha_n$ in E such that $E = F(\alpha_1, \alpha_2, ..., \alpha_n)$ iff E is finite-dimensional vector space over F.

(ii) Show that trisecting the angle is impossible.

14. A. (i) Show that if F is any field, then for every positive integer n, there is an irreducible polynomial in $F[X]$ of degree n.

(ii) State Isomorphism extension theorem.

B. (i) Let F be a field, and let α and β be algebraic over F with $\deg(\alpha, F) = n$. Show that the map $\Psi_{\alpha,\beta} : F(\alpha) \rightarrow F(\beta)$ defined by $\Psi_{\alpha,\beta}(c_0 + c_1\alpha + ... + c_{n-1}\alpha^{n-1}) = c_0 + c_1\beta + ... + \beta_{n-1}\beta^{n-1}$ for $c_i \varepsilon F$ is an isomorphism of $F(\alpha)$ onto $F(\beta)$ iff α and β are conjugate over F.

(ii) Find all conjugates of the number $\sqrt{2}+i$ over R.

15. A. (i) Define splitting field. Show that if $E \leq \bar{F}$ is a splitting field over F, then every irreducible polynomial in $F[x]$ having a zero in E splits in E.

(ii) Show that if E is a finite extension of F, then $\{E:F\}$ divides $[E:F]$.

B. (i) Show that every finite field is perfect.

(ii) Verify that $Q(\sqrt{2}, \sqrt{3})$ is separable over Q.

16. A. Let K be the splitting field in C of x^4-2 over Q. Describe the group $G(K/Q)$ and give the lattice diagram for the subgroups of $G(K/Q)$.

B. Let F be a field of characteristics 0, and let $a \varepsilon F$. Show that if K is the splitting field of $x^n - a$ over F, then $G(K/F)$ is a solvable group.

(4 × 8 = 32 marks)

C 16428 (Pages : 2)

Name.....Nashida P.P

Reg. No...............................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JULY 2011

CCSS

Mathematics

MAT 2C 06—ALGEBRA—II

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

1. Let E be an extension field of a finite field F, where F has q elements. Let $\alpha \in E$ be algebraic over F of degree n. Prove that $F(\alpha)$ has q^n elements.
2. Prove that the field C of complex numbers is an algebraically closed field.
3. Find the degree and a basis for the field $Q(\sqrt{2}, \sqrt{6})$ over $Q(\sqrt{3})$.
4. Find all primitive 18^{th} roots of unity in GF (19).
5. Prove that a finite field GF (P^n) of P^n elements exists for every prime power P^n.
6. Let $f(x) \in R[x]$. Show that if $f(a+ib) = 0$ for $(a+bi) \in C$, where $a, b, \in R$, then $f(a-ib) = 0$.
7. Let E be the splitting field of $x^3 - 2$ over Q. What is G (E/Q) ?
8. Prove that if E is an algebraic extension of a perfect field F, then E is perfect.
9. Is it true that two different subgroups of a Galois group may have the same fixed field ? Justify your answer.
10. Let ξ be a primitive 7^{th} root of unity in C. Show that Q (ξ) in the splitting field of x^7-1 over Q.
11. Find $\phi_3(x)$ and Z_2.
12. Show that the polynomial $x^5 - 1$ is solvable by radicals over Q.

(12 × 4 = 48 marks)

Part B

Answer A or B of each question.
Each question carries 8 *marks.*

Unit I

13. A (i) Let E be a simple extension $F(\alpha)$ of a field F, and let α be algebraic over F. Let the degree of irr (α, F) be $n \geq 1$. Show that every element β of $E = F(\alpha)$ can be uniquely expressed in the form

$$\beta = b_0 + b_1\alpha + \ldots + b_{n-1}\alpha^{n-1}$$

where the b_i are in F.

(ii) Show that a finite extension field E of a field F is an algebraic extension of F.

Turn over

B (i) Prove in detail that $Q\left(\sqrt{3}+\sqrt{7}\right) = Q\left(\sqrt{3}+\sqrt{7}\right)$.

(ii) Show that trisecting the angle is impossible.

Unit II

14. A (i) Let E be a field of P^n elements contained in an algebraic closure $\overline{Z}_p$ of Z_p. Show that the elements E are precisely the zeros in $\overline{Z}_p$ of the polynomial $x^{p^n} - x$ in $Z_p[x]$.

(ii) Let $\{\sigma_i : i \varepsilon I\}$ be a collection of automorphisms of a field E. Show that the set $E_{\{\sigma_i\}}$ of all $a \varepsilon E$ left fixed by every σ_i for $i \varepsilon I$ forms a subfield of E.

B (i) Let F be a finite field of characteristic p. Show that the map $\sigma_p : F \rightarrow F$ defined by $\sigma_p(a) = a^p$ for $a \varepsilon F$ is an automorphism and that $F_{\{\sigma_p\}} = Z_p$.

(ii) Give an example to show that the map $\sigma_p : F \rightarrow F$ given by $\sigma_p(a) = a^p$ for $a \varepsilon F$ need not be automorphism in the case that F is an infinite field of characteristic $p \neq 0$.

Unit III

15. A (i) Show that if $E \leq \overline{F}$ is a splitting field over F, then every irreducible polynomial in F [x] having a zero in E splits in E.

(ii) Show that if K is a finite extension of E and E is a finite extension of F, then K is separable over F iff K is separable over E and E is separable over F.

B (i) Show that every finite field is perfect.

(ii) State the main theorem of Galois theory.

Unit IV

16. A (i) Let K be the splitting field of $(x^4 + 1)$ over Q. Describe the group G (K /Q). Give the lattice diagrams for the subfields of K and for the subgroup of G (K /Q).

(ii) Show that the Galois group of the p^{th} cyclotomic extension of Q for a prime p is cyclic of order $p - 1$.

B (i) Let F be a field of characteristic zero, and let $F \leq E \leq K \leq \overline{F}$, where E is a normal extension of F and K is an extension F by radicals. Show that G (E/F) is a solvable group.

(ii) Is it true that the Galois group of a finite extension of a finite field is solvable ? Justify your answer.