D 52789 (Pages 4) Name..................................

Reg. No..............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2008

Mathematics

MAT IC 02—REAL ANALYSIS—I

(CCSS)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

I. 1 Let A be the set of all sequences whose elements are the digits 0 and 1. Prove that A is uncountable.

2 Let E be an infinite subset of a compact set K of a metric space X. Show that E lies a limit point in K.

3 Define neighbourhood of a point in a metric space. Prove that neighbourhood of a point is an open set.

4 Let f be a real function defined on $[a, b]$. If f has a local maximum at a point x in (a, b) and if $f'(x)$ exists, then prove that $f'(x) = 0$.

5 Let f be a continuous mapping of a compact metric space X into a metric space Y. Prove that $f(X)$ is compact.

6 Let E be a bounded non-compact subset of the real time R. Show that there exists a continuous function on E which is not uniformly continuous.

7 Let $\alpha(x) = I(x \geq S)$, where I is the unit step function and $a < s < b$. If f bounded on $[a, b]$ and if f is continuous at s, prove that $\int_a^b f\, d\alpha = f(s)$.

8 For all real x, define f by :

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$$

Prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Turn over

9 Let α be a monotonic increasing function on $[a, b]$. If $f, g \in \mathscr{R}(\alpha)$ on $[a, b]$, prove that $fg \in \mathscr{R}(\alpha)$ on $[a, b]$.

10 Let $f_n(x) = n^2 x \left(1 - x^2\right)^n$ $(0 \le x \le 1, n = 1, 2, ...)$ show that :

$$\int_0^2 \left[\lim_{n \to \infty} f_n(x)\right] dx \ne \lim_{n \to \infty} \int_0^1 f_n(x)\, dx.$$

11 Examine whether the sequence of functions $\{f_n\}$ is equi continuous on [0, 1] where :

$$f_n(x) = x^2 \big/ \left(x^2 + (1 - nx)^2\right) (0 \le x \le 1, n = 1, 3 \ldots .)$$

12 Let $\mathscr{C}$ (X) denote the set of all complex valued, continuous bounded functions on a metric space X. Show that a sequence $\{f_n\}$ in $\mathscr{C}$ (X), converges to a function f in $\mathscr{C}$ (X) with respect to the usual metrics of $\mathscr{C}$ (X) if and only if $f_n \to f$ uniformly on X.

Part B

Answer A **or** *B of each question.*
Each question carries 8 *marks.*

II. A (a) Let X be a metric space and K $\subset$ Y $\subset$ X. Prove that K is compact relative to X if and only if K is compact relative to Y.

(4 marks)

(b) Let Y be a subset of a metric space X. Show that a subset E of Y is open relative to Y if and only if E = Y $\cap$ G for some open subset G of X.

(4 marks)

B (a) State and prove a necessary and sufficient condition for a subset of the real line $\mathbb{R}$ to be connected.

(4 marks)

(b) Prove that every non empty perfect set in $\mathbb{R}^k$ is uncountable. (4 marks)

III. A (a) Let f be a continuous mapping of a metric space X into a metric space Y. If E is a connected of subset of X, prove that f(E) is connected.

(4 marks)

(b) Let f be a monotonic increasing function on (a, b). Prove that the set of discontinuities of f on (a, b) is atmost countable.

(4 marks)

B (a) Let f be a continuous real function on $[a, b]$. If $f(a) < f(b)$, and if c is a number such that $f(a) < c < f(b)$, then prove that there exists a point x in (a, b) such that $f(x) = 0$.

(4 marks)

(b) Let f be defined by :

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$$

Is f differentiable at all points ? Justify your answer.

(4 marks)

IV. A (a) Let α be a monotonic increasing function on $[a, b]$. (a) If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$, prove that $f_1 + f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$ and $\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1 \, d\alpha \mathrm{T} \int_a^b f_2 \, d\alpha$.

(4 marks)

(b) If f is monotonic increasing on $[a, b]$ and α is continuous on $[a, b]$, show that :

$f \in \mathcal{R}(\alpha)$ on $[a, b]$.

(4 marks)

B (a) Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$. For $a \leq x \leq b$, let $\mathrm{F}(x) = \int_a^x f(t)\, dt$. Prove that if f is continuous at a point x_0 in $[a, b]$, then F is differentiable at x_0 and $\mathrm{F}'(x) = f(x_0)$.

(b) State and prove Cauchy's criterion for uniform convergence of a sequence of real functions defined on a set E.

(4 marks)

V. A (a) Let α be a monotonic increasing function on $[a, b]$, such that $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ for $n = 1, 2 \ldots$ If $\{f_n\}$ converges uniformly on $[a, b]$ to a function f, prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and that :

$$\int_a^b f d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha.$$

(b) Let K be a compact metric space. If $\{f_n\}$ is a sequence of continuous functions on k and if $\{f_n\}$ converges uniformly on k, then prove that $\{f_n\}$ is equicontinuous on k.

(4 marks)

B (a) Suppose $\{f_n\}$ is a sequence of functions differentiable on $[a, b]$ such that $\{f_n(x)\}$ converges for some point x_0 in $[a, b]$. If $\{f'_n\}$ converges uniformly in $[a, b]$, then prove that $\{f_n\}$ converges uniformly on $[a, b]$ to a function f and :

$$f'(x) = \lim_{n \to \infty} f'_n(x), a \leq x \leq b.$$

(4 marks)

(b) If $\{f_n\}$ is a sequence of pointwise bounded and equicontinuous functions on a compact set k, prove that $\{f_n\}$ is uniformly bounded on K.

(4 marks)

[4 × 8 = 32 marks]

D 41301 (Pages : 2) Name..................................

Reg. No..............................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2008

Mathematics

Paper I—ALGEBRA—I

(2003 admissions)

Time : Three Hours Maximum : 80 Marks

Part A (Compulsory)

I. (a) Show that elements of B^n are in the same coset of C iff they have the same syndrome.

(b) Let $\phi : Z_{12} \to Z_3$ be the homomosphism such that $\phi(1) = 2$. Find the kernel K of ϕ and list the cosets in Z_{12} / K, showing the elements in each coset.

(c) Factorize $x^4 + 4$ in $Z_5 [x]$.

(d) Show that the ring $M_2 (R)$ of all 2×2 matrices with entries in R, the field of real numbers, are not isomorphic to the field C of complex numbers.

Part B

Answer any **four** *questions without omitting any unit.*

UNIT I

II. (a) Show that the group $Z_m \times Z_n$ is isomosphic to Z_{mn} iff m and n are relatively prime.

(b) Find all abelian groups, upto isomorphism of order 1089.

III. (a) Show that any two composition series of a group G are isomorphic.

(b) Find all composition series of $Z_5 \times Z_5$ and show that they are isomorphic.

(c) Show that Z has no composition series.

IV. (a) Let H be a subgroup of a group G. Show that left coset multiplication is well defined by the equation $(aH)(bH) = (ab)H$ iff $aH = Ha$ for all $a \in G$.

(b) Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G.

(c) Compute the factor group $\dfrac{Z_4 \times Z_6}{<(2,3)>}$.

UNIT II

V. (a) Let X be a G-set and let $x \in X$. Show that $|G_x| = (G : G_x)$.

(b) Find the number of orbits in {1, 2, 3, 4, 5, 6, 7, 8} under the subgroup of S_8 generated by (1, 3) and (2, 4, 7).

VI. (a) Let H and K be normal subgroups of a group G with K is a normal subgroup of H. Show that $\dfrac{G}{H} \simeq \dfrac{G/K}{H/K}$.

(b) Let K and L be normal subgroups of a group G with $K \vee L = G$, and $K \cap L = \{e\}$. Show that $G/K \cong L$ and $G/L \cong K$.

VII. (a) Let G be a finite group and let a prime p divide $|G|$. Show that G has a subgroup of order p.

(b) Show that no group of order 30 is simple.

(c) Write the class equation of S_3.

Unit III

VIII. (a) Show that $(x, y : y^2x = y, yx^2y = x)$ is a presentation of the trivial group of one element.

(b) Determine all groups of order 8 upto isomorphism.

IX. (a) Show that the multiplicative group of all non-zero elements of a finite field is cyclic.

(b) State and prove Eisenstein criteria for irreducibility.

X. (a) Give the addition and multiplication tables for the group algebra Z_2 (G), where $G = \{e, a\}$ is a cyclic group of order 2.

(b) Let R be a commentative ring with unity of prime characteristic p. Show that the map $\phi_p : R \to R$ given by $\phi(a) = a^p$ is a homomorphism.

(c) Give an example to show that a factor ring of an integral domain may be a field.

(4 × 16 = 64 marks)