488270

D 101795 (Pages : 3) Name.......................................

Reg. No..................................

FOURTH SEMESTER P.G. (CCSS) DEGREE EXAMINATION, APRIL 2024

Mathematics

MAT4E02—ADVANCED FUNCTIONAL ANALYSIS

(2019–2021 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 2 *marks.*

(1) Define the resolvent set of a bounded linear operator.

(2) Is the dual of a separable normed space separable? Justify your answer.

(3) Show that the dual of a reflexive normed space is reflexive.

(4) Distinguish between weak and weak* convergence.

(5) Let $\{x_1, x_2, \dots, x_n\}$ be an orthogonal set in an inner product space. Show that $\|x_1 + x_2 + \dots + x_n\|^2 = \|x_1\|^2 + \dots + \|x_1\|^2 + \dots + \|x_n\|^2$.

(6) State the Riesz representation theorem.

(7) Give an example to show that the Riesz representation theorem does not hold for incomplete inner product spaces.

(8) Let H be a Hilbert space over $\mathbb{K}$ and A, B are bounded linear operators on H. Show that $(A+B)^* = A^* + B^*$, where A* denotes the adjoint of A.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

(9) Show that $x_n \xrightarrow{w} x$ if and only if $x_n \to x$ in ℓ^1.

(10) Show that a finite dimensional normed space is reflexive.

(11) Let A be a self adjoint operator on a finite dimensional Hilbert space H. Show that every root of the characteristic polynomial of A is real.

Turn over

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(12) Let X be a normed space and $A \in CL(X)$, the space of compact operators on X. Show that the eigen spectrum and spectrum of A are countable sets and have 0 as the only possible limit point.

(13) Let $H = L^2([-\pi, \pi])$ and for $n = 0, \pm 1, \pm 2, \pm 3, \ldots, u_n(t) = \frac{e^{int}}{\sqrt{2\pi}}, t \in [-\pi, \pi]$

Show that $\{u_n : n = 0, \pm 1, \pm 2, \pm 3, \ldots, \}$ is an orthonormal set in H.

(14) Let H be Hilbert space and $A \in BL(H)$ the space of bounded linear operators on H. Show that there is a unique $B \in BL(H)$ such that for all $x, y \in H, \langle A(x), y \rangle = \langle x, B(y) \rangle$.

(4 × 4 = 16 marks)

Part C

Answer either part (a) or (b) of each of the following questions.
Each question carries **12** *marks.*

(15) (a) Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. Show that the dual of $\mathbb{K}^n$ with the norm $\| \|_p$ is linearly isometric to $\mathbb{K}^n$ with the norm $\| \|_q$.

Or

(b) Let X and Y be normed spaces and $F \in BL(X, Y)$, the space of bounded linear maps from X to Y. If $F \in CL(X, Y)$ the space of compact linear maps from X to Y, then show that $F' \in CL(Y', X')$ where F' is the transpose of F. Also, show that the converse holds if Y is a Banach space.

(16) (a) Let X and Y be normed spaces and $F : X \to Y$ be linear. Prove the following results :

(i) F is a compact map if and only if for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

(ii) If F is a compact map, then F (U) is a totally bounded subset of Y. Conversely if Y is Banach and F (U) is a totally bounded subset of Y, then F is a compact map.

Or

(b) (i) Let A be a compact operator on $H \neq \{0\}$. Show that if A is self adjoint, then $\|A\|$ or $-\|A\|$ is an eigen value of A.

(ii) Let A be a compact operator on $H \neq \{0\}$. Show that every nonzero approximate eigenvalue of A is an eigenvalue of A and the corresponding eigen space is finite dimensional.

(17) (a) Let H be a Hilbert space. Prove the following results :

(i) If A and B are self adjoint, then A + B is self adjoint. Also, AB is self adjoint if and only if A and B commute.

(ii) If A and B are unitary, then AB is unitary. Also A + B is unitary if and only if it is surjective and $\mathrm{Re}\langle A(x), B(x)\rangle == -1/2$ for every $x \in H$ with $\|x\| = 1$.

(iii) If A and B are normal and if A commutes with B^* and B commutes with A^*, then A + B and AB are normal.

Or

(b) Let H be a Hilbert space and $A \in BL(H)$. Then prove the following results :

(i) If R (A), the range space of A is finite dimensional, then A is compact.

(ii) If each A_n is a compact operator on H and $\|A_n - A\| \to 0$, then A is compact.

(iii) If A is compact, then so is A^*.

(18) (a) State and prove the Generalized Schwartz inequality.

Or

(b) Let H be a Hilbert space over $\mathbb{K}$ and $A \in BL(H)$.

(i) Show that $\mathbb{K} \in \sigma(A)$ if and only if $\bar{k} \in \sigma(A^*)$, where $k \in \mathbb{K}$ and $\sigma(A)$ represents the spectrum of A.

(ii) Show that $\sigma_e(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \{k : \bar{k} \in \sigma_e(A^*)\}$, where $\sigma_e(A)$ and $\sigma_a(A)$ represents the eigen spectrum and approximate eigen spectrum of A respectively.

(iii) Give an example to show that $k \in \sigma_e(A)$ need not imply that $\bar{k} \in \sigma_e(A^*)$.

(4 × 12 = 48 marks)

488006

D 101803 **(Pages : 3)** **Name.......................................**

Reg. No..................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT 4E 22—ALGEBRAIC GRAPH THEORY

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Prove that if X is a bipartite graph G with at least one edge, then $chi(\mathrm{X}) = 2$.
2. Prove that $Aut(\mathrm{C}_n)$ contains the cyclic subgroup $\mathrm{R} = \{g^m : 0 \le m \le n-1\}$.
3. Define the term isomorphism between two graphs and illustrate the same with an example.
4. If x and y are vertices of a graph X, and g is an automorphism of X, then prove that the vertex $y = x^2$ has the same valency as x.
5. Show that the chromatic number of graph X is the least integer r such that there is a homomorphism from X to K_r.
6. Define vertex connectivity and edge connectivity of a graph with an example.
7. Define primitivity of permutation groups.
8. Prove that the k-cube Q_k is vertex transitive.

(8 × 1 = 8 marks)

Turn over

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Part B

Answer any **six** *questions.*
Each question carries 3 marks.

9. Prove that the automorphism group of a group is equal to the automorphism group of its complement.

10. Characterize a line graph in terms of its induced subgraph.

11. If $v \geq k \geq i$, then prove that $Aut\left(\mathrm{J}\left(v, k, i\right)\right)$ contains a subgroup isomorphism to $Sym\left(v\right)$.

12. If X is regular graph with valency *k*, then prove that $\mathrm{L}\left(\mathrm{X}\right)$ is regular graph with valency $2k - 2$.

13. Prove that $\left| Aut\left(\mathrm{C}_n\right) \right| = 2n$.

14. Prove that if the line graph of a connected graph X is regular then, X is regular or bipartite and semi-regular.

15. State and prove Orbit-stabilizer lemma.

16. Prove or disprove. The Petersen graph is a Cayley graph.

17. Prove that if $v \geq k \geq i$, then $\mathrm{J}\left(v, k, i\right) \cong \mathrm{J}\left(v, v - k, v - 2k + i\right)$.

(6 × 3 = 18 marks)

Part C

Answer any **three** *of the following questions.*
Each question carries 8 marks.

18. (a) Prove that the *k*-cube Q_k is vertex transitive.

    (b) State and prove Euler's polyhedron formula for planar graphs.

19. If G is a transitive permutation group on V and $x \in \mathrm{V}$, then prove that G is primitive if and only if G_x is a maximal subgroup of G.

20. (a) Prove that almost all graphs are asymmetric.

    (b) Let X be a connected vertex-transitive graph. Then prove that X has a matching that misses at most one vertex and each edge is contained a maximum matching.

21. (i) Prove that chromatic number of a graph X is the least integer r such that there is a homomorphism from X to K_r.

    (ii) Show that X and $\bar{X}$ have the same automorphism group for any graph X.

22 If the graph X is vertex transitive and edge transitive, but not arc transitive, then prove that its valency is even.

(3 × 8 = 24 marks)

488010

D 101811 (Pages : 2) Name.................................

Reg. No...............................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT4E36—ALGEBRAIC TOPOLOGY

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *the questions.*
Each question carries 1 *mark.*

1. Explain the term barycentric co-ordinates.
2. Is the 2-sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sum x_i^2 = 1\}$ triangulable ? Explain.
3. Define the Euler characteristic of a complex.
4. Illustrate with an example the simplicial approximation of a function.
5. What is an n–dimensional torus ? Compute its fundamental group.
6. Give an example of a simply connected space which is not contractible.
7. State the Borsuk-Ulam Theorem.
8. Let $g : S^2 \to \mathbb{R}^2$ be a continuous map. Prove that there is at least one pair $x, -x$ of antipodal points for which $g(x) = g(-x)$.

(8 × 1 = 8)

Part B

Answer **six** *questions.*
Each question carries 3 *marks.*

9. If K is an oriented complex, then prove that $B_p(K) \subset Z_p(K)$ for each integer p such that $0 \leq p \leq n$, where n is the dimension of K, $B_p(K)$ denotes the p dimensional boundary group of K and $Z_p(K)$ the p dimensional cycle group of K.
10. Illustrate the homology groups with an example.

Turn over

11. Prove that there are only *five* regular, simple polyhedra.

12. If $f : S^n \to S^n$ and $g : S^n \to S^n$ are continuous maps, then prove that *deg* (*gf*) = *deg* (*g*).*deg* (*f*).

13. Prove that every contractible space is simply connected.

14. Let X and Y be spaces with x_0 in X and y_0 in Y. Prove that

$$\pi_1 \left(\mathrm{X} \times \mathrm{Y}\right), \left(x_0, y_0\right)\right) \cong \pi_1 \left(\mathrm{X}, x_0\right)\right) \oplus \ \pi_1 \left(\mathrm{Y}, y_0\right).$$

15. For any positive integer n, let $q_n : S^1 \to S^1$ be the map defined by $q_n(z) = z^n$, $z \in S^1$, where z^n is the n^{th} power of the complex number z. Show that (S^1, q_n) is a covering space of S^1.

16. Prove that if (E, p) is a covering space of B, then all the sets $p^{-1}(b)$, $b \in B$ have the same cardinal number.

17. Prove that any *two* universal covering spaces of a base space B are lsomorphic.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. Let K be a-complex with r combinatorial components. Prove that H_0(K) isomorphic to the direct sum of r copies of the group $\mathbb{Z}$ of integers.

19. Prove that an n–pseudomanifold K is orientable if and only if the n^{th} homology group H_n(K) is not the trivial group.

20. Prove that if a space X is path connected and x_0, x_1 are points in X, then the fundamental groups $\pi_1 \left(\mathrm{X}, x_0\right)$ and $\pi_1 \left(\mathrm{X}, x_1\right)$ are isomorphic.

21. What do you understand by degree of a map ? State and prove the Brouwer's degree theorem.

22. Prove that there is no continuous map $f : S^n \to S^{n-1}$ for which $f(-x) = -f(x)$ for all $x \in S^n$, $n \geq 1$.

(3 × 8 = 24 marks)

D 101807 **(Pages : 2)** **Name.......................................**

Reg. No...................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT 4E 26—GRAPH THEORY

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Prove that an edge e of a graph G is a tree if and only if every edge of G is a cut edge of G.
2. Show that if a graph G is a block, then every two points of G lies on a common cycle.
3. Define a Tree. Prove that every non-trivial tree has at least two end points.
4. Define diameter of a connected graph. Illustrate with an example.
5. Define chromatic number $\chi(G)$ of a graph G. Prove that $\chi(K_n) = n$.
6. Define point covering number $\alpha_0(G)$ of a graph G. Prove that $\alpha_0(K_p) = p-1.$.
7. Prove that $\alpha + \beta = v$, where α is the independence number and β is the covering number of a graph G of order v.
8. If G is k-critical, then prove that $\delta \geq k-1$.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Prove that every non trivial tree has at least two vertices of degree one.
10. If e is link of a graph G, then prove that $\tau(G) = \tau(G-e) + \tau(G \cdot e)$.

Turn over

11. Prove that a vertex v of a graph G is a cut vertex of G if and only if $d(v) > 1$.

12. Prove that in a bipartite graph, the number of maximum matching is equal to the number of vertices in a minimum covering.

13. If G is a bipartite graph then prove that $\chi' = \Delta$, where χ' is the edge chromatic number of G.

14. Prove that every tournament has a directed Hamilton path.

15. If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

16. Prove that $K_{3,3} - \{e\}$ is planar for any edge e of $K_{3,3}$.

17. If a graph G is planar, then prove that every subgraph of G is planar.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 marks.

18. (a) Let G be a, k-connected graph, $k \geq 2$. Prove that given any set of k vertices in G, there is a cycle in G containing all these vertices.

    (b) Prove that a graph is bipartite if and only if it contains no odd cycle.

19. If a simple graph with $v \geq 3$ and $\delta \geq \frac{v}{2}$, then prove that G is Hamiltonian.

20. Explain the Traveling Salesman problem.

21. State and prove Brooke's Theorem.

22. For any positive integer k, then prove that there exists a k-chromatic graph containing no triangle.

(3 × 8 = 24 marks)

D 101899 **(Pages : 3)** **Name.......................................**

Reg. No..................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT 4E 31—SPECTRAL THEORY

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question is of 1 *mark.*

1. State the parallelogram equality for norms arising from an inner product.
2. Find the orthogonal complement of the subspace $W = \text{span}\{(1,1)\}$ in $\mathbb{R}^2$ with usual inner product.
3. Verify whether $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.
4. Show that for operators T on a finite dimensional space the continuous spectrum $\sigma_c(T)$ is empty.
5. Show that the zero operator on any normed linear space is compact.
6. Find the adjoint S^* of the right shift operator S on l^2.
7. Show that if P is the projection onto a subspace Y of a Hilbert space X then $(I - P)$ is the projection onto $Y^{\perp}$.
8. Describe the positive part T^+ of an operator T on a Hilbert space.

(8 × 1 = 8 marks)

Turn over

Part B

Answer any **six** *questions.*
Each question is of 3 marks

9. Show that the sequence space l^1 is not a Hilbert space.

10. Let Y be a closed subspace of a Hilbert space X. Show that $X = Y \oplus Y^{\perp}$.

11. Let (e_k) be a total orthonormal set in a Hilbert space H. Show that $x = \sum \langle x, e_k \rangle e_k$ for every $x \in H$.

12. Let T_1, T_2 be linear operators on a finite dimensional normed linear space. Show that if $T_2 = C^{-1}T_1C$ for an invertible operator C then T_1 and T_2 have the same eigen values.

13. Let S be the right shift operator on l^2. Show that $\lambda = 0$ is a spectral value of S which is not an eigen value of S.

14. Show that every compact linear operator is bounded.

15. Let $T : H \to H$ be a bounded self adjoint operator on a complex Hilbert space H. Show that all eigen values of T are real.

16. Let P_1 be a projection onto a subspace Y_1 and P_2 be a projection onto a subspace Y_2 of a Hilbert space H. Show that if $P_1P_2 = P_2P_1$ then P_1P_2 is a projection. Find the image of P_1P_2 in this case.

17. Let P_1 be a projection onto a subspace Y_1 and P_2 be a projection onto a subspace Y_2 of a Hilbert space H. Show that if $Y_1 \subset Y_2$ then $P_2 - P_1$ is a projection.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions*
Each question is of 8 marks

18. Let Y be a subspace of a Hilbert space H. Prove that
    - (a) If Y is closed in H then Y is complete.
    - (b) If Y is finite dimensional then Y is complete.
    - (c) If Y is complete then Y is closed in H.

19. (a) Define total orthonormal set in a Hilbert space H.

    (b) Let M be a subspace of H such that M is total in H. Show that $M^{\perp} = \{0\}$.

    (c) Let (e_k) be an orthonormal set in H. Show that (e_k) is total in H if and only if $\sum \left| \langle x, e_k \rangle \right|^2 = \| x \|^2$ for all $x \in H$.

20. Let A be a complex Banach algebra with identity e. Show that

    (a) If $x \in A$ with $\| x \| \leq 1$ then $(e - x)$ is invertible and $(e - x)^{-1} = e + \sum_{i=1}^{\infty} x^i$.

    (b) The set G of all invertible elements of A is an open subset of A.

21. Let T be a bounded self adjoint operator on a complex Hilbert sapce H and $\rho\ (T)$ be the resolvent set of T. Show that

    (a) If $\lambda \in \rho\ (T)$ then there exists $c > 0$ such that $\| T_{\lambda} x \| \geq c \| x \|$ for all $x \in H$ where $T_{\lambda} = T - \lambda I$.

    (b) If there exists $c > 0$ such that $\| T_{\lambda} x \| \geq c \| x \|$ for all $x \in H$ then T_{λ} is a bijection.

22. (a) Define spectral family on a Hilbert space H in an interval $[a, b]$.

    (b) Let T be a linear operator on a Hilbert space H and T^{+} be the positive part of T. Show that if S is a linear operator such that $ST = TS$ then $ST^{+} = T^{+}S$.

(3 × 8 = 24 marks)

D 101900 **(Pages : 3)** **Name.................................**

Reg. No...............................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

History

MAT4E33—TOPOLOGICAL GROUPS

(2019 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question is of 1 *mark.*

1. Let G be any group. Show that with indiscrete topology G is a topological group.
2. Let $(\mathbb{R}, +)$ be the topological group with usual topology on $\mathbb{R}$. Give a fundamental neighbourhood system at 0.
3. Let G be a topological group and H be a subgroup of G. show that H is a topological group with respect to the subspace topology.
4. Let C be the connected component of the identity e of a topological group G. Show that C is a subgroup of G.
5. Give an example of a locally compact topological group.
6. Let H be an invariant subgroup of a locally compact topological group G. Show that the quotient group G/H is localy compact.
7. Let G be an additive abelian topological group and G' be the dual group, show that $\langle -x, x' \rangle = -\langle x, x' \rangle$ for all $x \in G$ and $x' \in G'$.
8. Define the topology of simple convergence in the dual G' of a topological group G.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*

Each question is of 3 *marks.*

9. Let F be a closed subset of a topological group G. Show that aF is closed for all $a \in G$.

Turn over

10. Let G be a topological group with identity e and $\mathcal{U}_e$ be the system of all neighbourhoods of e. Show that $\bar{A} = \cap\{AU : U \in \mathcal{U}_e\}$ for any $A \subseteq G$.

11. Let G be a T_2-topological group and let $\mathcal{U}_e$ be a fundamental system of neighbourhoods of the identity e. Show that $\bigcap\{U : U \in \mathcal{U}_e\} = \{e\}$.

12. Let H be an invariant subgroup of a topological group G. Show that $\bar{H}$ is also an invariant subgroup of G.

13. Let U be a symmetric neighbourhood of the identity e in a topological group G. Show that $H = U_{n \geq 1} U^n$ is a subgroup of G.

14. Let G be a topological group and H be an open and closed subgroup of G. Show that if H is normal then G is also normal.

15. Let Gn(K) be the group of regular $n \times n$ matrices over a field K. Show that the map $A \to A'$ is a homeomorphism of Gn(K) onto itself.

16. Let S denote the family of all finite subsets of additive abelian topological group and ν be a fundamental system of open symmetric neighbourhoods of 0 in the one dimenional circle group T. Show that for each $M \in S$ and $V \in \nu$, T (M, V) is a symmetric neighbourhood of the identity 0′ in the dual group G'.

17. Let G' be the dual of an additive abelian topological group G. Show that the map $x' \mapsto -x'$ is continuous from $G' \to G'$.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question is of 8 *marks.*

18. Let F be a closed subset and C be a compact subset of a topological group such that $F \cap C = \emptyset$. Prove the following.

    (a) There exists a neighbourhood U of C such that $FU \cap CU = \emptyset$.

    (b) FC is closed.

19. (a) Give an example of a fundamental system of symmetric neighbourhoods of the identity e in a topological group.

(b) Show that a topological group is a T_0,-space if and only if $\cap \{U : U \in \mathcal{U}_e\} = \{e\}$ where $\mathcal{U}_e$ is a fundamental system of neighbourhoods of the identity e.

20. Let H be an invariant subgroup of a topological group G. Show that :

(a) If H is compact then the canonical mapping $\phi : G \to G/H$ is closed.

(b) If H is closed and G is compact then G/H is compact.

21. (a) Define locally compact topological group.

(b) Show that a topological group G is locally compact if and only if there exists a compact neighbourhood of the identity in G.

(c) Show that every locally compact Hausdorff topological group is complete.

22. (a) Define character of a topological group and give an example.

(b) Let G be a locally compact Hausdorff abelian topological group and G' be the dual with compact open topology. Show that if G is compact then G' is discrete.

(3 × 8 = 24 marks)

D 101806 **(Pages : 3)** **Name.......................................**

Reg. No..................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2024

(CCSS)

Mathematics

MAT 4E 25—COMMUTATIVE ALGEBRA

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question is of 1 *mark.*

1. Verify whether $S = \{2n - 1 : n \in \mathbb{N}\}$ is a subring of the ring $\mathbb{Z}$ of integers.

2. Give an example of a prime ideal in $\mathbb{Q}[x]$.

3. Let $A = a_1 \oplus a_2$ be a direct sum of rings a_1 and a_2. Show that the ideal in A generated by the identity e_1 of a_1 is isomorphic to a_1.

4. Let $f : M \to M'$ be an A-module homomorphism. Verify whether the sequence $0 \to M/\ker f \to M'$ is exact.

5. Verify whether the ideal generated by 6 in the ring $\mathbb{Z}$ of integers is a primary ideal.

6. Verify whether $\frac{1}{2}$ is integral over $\mathbb{Z}$.

7. Verify whether the polynomial ring $k[x]$ over a field k satisfies d.c.c. on ideals.

8. Let A be a Noetherian ring. Show that A/a is Noetherian for every ideal a of A.

(8 × 1 = 8 marks)

Turn over

Part B

Answer any **six** *questions.*
Each question is of 3 *marks.*

9. Let A be a ring such that every homomorphism from A to a non zero ring B is injective. Show that A is a field.

10. Show that the set of all nilpotent elements of a ring A is an ideal of A.

11. Let $L \supseteq M \supseteq N$ be A-modules. Let $\theta : L/N \to L/M$ be defined by $\theta(x + N) = x + M$. Show that her $\ker \theta = M/N$.

12. Let M be a quotient of a free module A^n where $n > 0$. Show that M is a finitely generated A-module.

13. Show that if M is an A-module then the tensor product $A \otimes M$ is isomorphic to M.

14. Show that if q is a primary ideal of a ring A then every zero divisor in A/q is nilpotent.

15. Show that if $A[x]$ is a finitely generated A-module then x is integral over A.

16. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Show that M is Noetherian if and only if M' and M'' are Noetherian.

17. Show that in a Noetherian ring every irreducible ideal is primary.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question is of 8 *marks.*

18. (a) Define maximal ideal of a ring and give an example.

    (b) Show that every ideal of a ring A is contained in a maximal ideal of A.

    (c) Show that every non unit of A is contained in a maximal ideal of A.

19. (a) Define the radical $r(\alpha)$ of an ideal α of a ring A.

    (b) Show that if $\phi : A \to A/\alpha$ is the quotient map then $r(\alpha) = \phi^{-1}\left(\mathcal{R}_{A/\alpha}\right)$ where $\mathcal{R}_{A/\alpha}$ is the radical of A/α.

20. (a) Describe the ring of fractions $S^{-1}A$ of a ring A with respect to a multiplicative subset S of A.

(b) Let A, B be rings and $g : A \to B$ be a ring homomorphism. Let S be a multiplicative subset of A such that $g(s)$ is a unit in B for every $s \in S$. Show that there is a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$ where $f : A \to S^{-1}A$ is given by $f(x) = x/1$.

21. Let A be a subring of B :

(a) Define the integral closure of A in B.

(b) Let S be a multiplicatively closed set in A and let C be the integral closure of A in B. Show that $S^{-1}C$ is the integral closure of $S^{-1}A$ and $S^{-1}B$.

22. (a) Define composition series of an A-module M.

(b) Let M be an A-module having a composition series of length n. Show that

i) Every composition series of M has length n.

ii) Every chain in M can be extended to a composition series.

(3 × 8 = 24 marks)