D 27870 (Pages : 2) Name...

Reg. No...

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2007

Mathematics

Paper III—REAL ANALYSIS—I

(2003 admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

I. (a) Let A be the set of all sequences whose elements are the digits 0 and 1. Prove that A is uncountable.

(b) Prove that a finite set has no limit points.

(c) Prove that monotonic functions have no discontinuities of the second kind.

(d) Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

UNIT I

II. (a) Prove that every bounded infinite subset of R^k has a limit point in R^k.

(b) Define compact sets in a metric space. Give an example. Prove that closed subsets of compact sets are compact.

III. (a) Define closed balls in R^k. Prove that closed balls are convex sets.

(b) Prove that a mapping f of a metric space X into a metric space Y is continuous if and only if f^{-1} (C) is closed in X for every closed set C in Y.

IV. (a) If E is a non compact set in R^1, prove that there exists a continuous function on E which is not bounded.

(b) Suppose f is a continuous real function on a compact metric space X, and $M = \sup \{f(p), p \in X\}$ and $m = \inf \{f(p), p \in X\}$. Then prove that there exists points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

UNIT II

V. (a) If f is differentiable at a point $x \in |a, b|$, prove that f is continuous at x. Using an example establish that the converse is not true.

(b) State and prove the generalized mean value theorem.

Turn over

VI. (a) Suppose f is real differentible function on $[a, b]$ and suppose that $f'(a) < \lambda < f'(b)$. Then prove that there is point $x \in (a, b)$ such that $f'(x) = \lambda$.

(b) Prove that $f \in R(\alpha)$ if and only if for every $\epsilon > 0$, there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

VII. (a) Assume α increases monotonically and α' is Riemann integrable on $[a, b]$. Let f be a real bounded function on $[a, b]$. Then prove that $f \in R(\alpha)$ if and only if $f\alpha'$ is Riemann integrable.

(b) If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$ then prove that $fg \in R(\alpha)$.

UNIT III

VIII. (a) State and prove the fundamental theorem of calculus.

(b) If f maps $[a, b]$ into R^k and $f \in R(\alpha)$ for some monotonically increasing function α on $[a, b]$, then prove that $|f| \in R$ (2) $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha$.

IX. (a) Suppose F and G are differentiable functions on $[a, b]$. $F' = f$, $G' = g$ and f and g are Riemann integrable. Then prove that $\int_a^b F(x) g(x) dx = F(b) G(b) - F(a) G(a) - \int_a^b f(x) G(x) dx$.

(b) Let γ be a rectifiable arc. Then define the length of γ, denoted by $\wedge(\gamma)$. Prove that if γ' is continuous on $[a, b]$, then $\wedge(\gamma) = \int_a^b |\gamma'(t)| dt$.

X. (a) Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that $\lim_{t \to x} f_n(t) = A_n$ ($n = 1, 2, 3, \ldots$). Then prove that $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$.

(b) Prove that there exists a real continuous function on the real line which is nowhere differentiable.

D 27869 (Pages : 2) Name.....................................

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FIRST SEMESTER M.Sc. DEGREE EXAMINATION, JANUARY 2007

Mathematics

Paper II—LINEAR ALGEBRA

(2003 admission onward)

Time : Three Hours Maximum : 80 Marks

Answer **all** *questions in Part A.*
Part A carries 16 *marks.*
Answer **four** *question from Part B without omitting any unit.*
Each question carries 16 *marks.*

Part A

Answer **all** *questions.*

I. (a) Let $V = \mathbb{R}^3$ be a vector space over $\mathbb{R}$. Let $v_1 = (1, 2, 1)$, $v_2 = (1, 3, 1)$, $v_3 = (1, 4, 1)$. Verify which of the following are linear combinations of v_1, v_2, v_3.

(i) (2, 3, 2). (ii) (3, 2, 4).

(iii) (4, 1, 4). (iv) (3, 0, 2).

(b) Find the dimension of the subspace of $\mathbb{R}^3$ spanned by $S = \{(1, 1, 2), (1, 1, 3), (1, 1, 4)\}$.

(c) Find the rank of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + y, x - y, 0)$.

(d) Describe the T-cyclic subspace $Z(\alpha, T)$ where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x, y) = (0, y)$ and $\alpha = (1, 1)$.

(4 × 4 = 16 marks)

Part B

Unit I

II. (a) Let W_1, W_2 be subspaces of a vector space V. Show that $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$ is a subspace of V.

(b) Show that the space spanned by $W_1 \cup W_2$ is same as $W_1 + W_2$.

(c) Let $V = \mathbb{R}^3$, $W_1 = \{(x, x, x) : x \in \mathbb{R}\}$ and $W_2 = \{(x, x, 0) : x \in \mathbb{R}\}$. Show that $W_1 + W_2 = \{(x, x, y) : x, y \in \mathbb{R}\}$.

III. (a) Let V be a vector space of dimension *n*. Show that :

(i) any subset of V which contains more than *n* elements is linearly dependent.

(ii) If S is a subset of V containing less than *n* elements, then S can not span V.

(b) Give an example of a two dimensional space V and a basis of V.

IV. (a) Let $T : V \to W$ be a linear transformation. Show that the null space of T is a subspace of V.

(b) With the usual notations, prove that rank (T) + nullity (T) = dim V where T is a linear transformation on V.

Turn over

UNIT II

V. (a) Let V, W be finite dimensional vector spaces and T : V $\rightarrow$ W be a linear transformation. Describe the matrix of T w.r.t on ordered basis of V and an ordered basis of W.

(b) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by $T(x, y) = (x + y, y, 0)$. Fix an ordered basis of $\mathbb{R}^2$, and an ordered basis of $\mathbb{R}^3$ and give the matrix of T w.r.t. these bases.

VI. (a) Let V be a finite dimensional vector space over a field F and for each $v \in V$ let $L_v(f) = f(v)$ for each $f \in V^*$. Show that the map $v \rightarrow L_v$ is an isomorphism of V onto V^{**}.

(b) Let V be a finite dimensional space and L be a linear functional on V^*. Show that there exists a unique $v \in V$ such that $L(f) = f(v)$ for all $f \in v^*$.

VII. (a) Let T be an operator on a finite dimensional space V over F and let CE F. Show that the following are equivalent :—

(i) There exists $v \in V, v \neq 0$ such that $T(v) = cv$.

(ii) $(T - cI)$ is not invertible.

(iii) dot $(T - cI) = 0$.

(b) Find a characteristic value of $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x + y, y)$.

UNIT III

VIII. (a) Let $V = W_1 \oplus W_2 \oplus \ldots \oplus W_n$. Show that there exist projections $E_1, E_2, \ldots, E_n$ on V satisfying the following.

(i) $E_i E_j = 0$ if $i \neq j$.

(ii) $E_1 + E_2 + \ldots + E_n = I$, the identity on V.

(iii) range of $E_i = W_i$ for $i = 1, 2, \ldots, n$.

(b) Find projections E_1, E_2, on $\mathbb{R}^3$ such that range of $E_1 = \{(x, x, x) : x \in \mathbb{R}\}$ and range of $E_2 = \{(0, y, z) : y, z \in \mathbb{R}\}$.

IX. (a) Let T be a linear operator on a finite dimensional space V ; and let $\alpha \in V$. Define T-annhilator of α. Find the T-annihilator of $(0, 1) \in \mathbb{R}^2$ where $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $T(x, y) = (x, x + y)$.

(b) Let A be a 3 × 3 matrix whose characteristic polynomial is $(x - 1)^2 (x + 2)$. Describe the possible Jordan forms of A.

X. (a) Define orthogonal set of vectors in an inner product space. Show that any orthogonal set of vectors is linearly independent.

(b) Let V be finite dimensional and E be an orthogonal projection of V on a subspace W. Show that $W^{\perp}$ is the null space of E.

(4 × 16 = 64 marks)