PH.D THESIS MATHEMATICS

A FURTHER STUDY IN GENERALIZED TOPOLOGICAL SPACES

Thesis submitted to the

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by

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CERTIFICATE

I hereby certify that the thesis entitled A FURTHER STUDY IN GENERALIZED TOPOLOGICAL SPACES submitted by **Bushra Beevi K K** to St.Joseph's college(Autonomous), Devagiri, Kozhikode - 6730008 for the award of the degree of **Doctor of Philosophy** is a bonafide record of of the research work carried out by her under my supervision and guidance. The contents of this thesis, in full or parts, have not been submitted and will not be submitted to any other institute or university for the award of any degree or diploma.

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I, BUSHRA BEEVI K K, hereby declare that the thesis, entitled "**A FURTHER STUDY IN GENERALIZED TOPOLOGICAL SPACES**" submitted to the University of Calicut for the award of the degree of Doctor of Philosophy in Mathematics is a bonafide record of the work done by me under the guidance and supervision of Dr. Baby Chacko, Department of Mathematics, St. Joseph's College, Devagiri. This thesis contains no material which has been accepted for the award of any degree or diploma of any university or institution and to the best of my knowledge and belief, it contain no material previously published by any other person, except where due reference was made. The content of the thesis are undergone plagiarism check using iThenticate software at C.H.M.K.Library, University of Calicut, and the similarity index found within the permisible limit. I also declare that the thesis is free from AI generated content.

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July 2025 Bushra Beevi K K

ABSTRACT

This thesis investigates the theory of generalized topological spaces (GTS), extending classical topology by relaxing or modifying the axioms that define open sets. Motivated by limitations in standard topological frame works when applied to non-traditional structures in analysis. The thesis defines $\mathcal{G}$ - neghbourhood, $\mathcal{G}$ - neighbourhood system, $\mathcal{G}$ - separated sets, $\mathcal{G}$ - Lindeloff spaces, $\mathcal{G}$ - derived set, $\mathcal{G}$ - dense set, $\mathcal{G}$ - locally finite, $\sigma_{\mathcal{G}}$ - locally finite, $\mathcal{G}$ - paracompact spaces, $I_{\mathcal{G}}$ - convergent sequence, $I_{\mathcal{G}}^*$ - convergent sequence and presents fundamental properties, theorems and equivalent characterizations. The work explored the properties of the generalized closed sets, generalized connectedness and generalized local connectedness. $\mathcal{G}$ - locally finite collections and $\sigma_{\mathcal{G}}$ - locally collections of sets introduced and used to extend the definition of paracompactness into generalized topological spaces. The work investigate the role of separation axioms in generalized paracompactness. The thesis defines $I_{\mathcal{G}}$ limit points and $I_{\mathcal{G}}$ - cluster points of a sequence and found some relations between ordinary sequential convergence and $I_{\mathcal{G}}$ - convergence in generalized topological spaces. The study advances into the notions of $\mathcal{G}$ - regular open sets, $\mathcal{G}$ - regular closed sets and discuss about some properties of these type of concepts. Further introduced minimal and maximal $\mathcal{G}$ - regular open sets and minimal and maximal $\mathcal{G}$ - regular closed sets and investigate the role of these concepts in generalized topological spaces. Overall, the thesis provides a comprehensive and unified frame work that not only generalizes classical topological concepts but also enriches the theoretical landscape of topological spaces.

Key words

$\mathcal{G}$ - topology, $\mathcal{G}$ - open set, $\mathcal{G}$ closed set,$M_{\mathcal{G}}$, $\mathcal{G}$ closure, $\mathcal{G}$ - interior, $\mathcal{G}$ - dense, $\mathcal{G}$ - derived set, $\mathcal{G}$ - neighbourhood, $\mathcal{G}$ - accumulation point, $\mathcal{G}$ - boundary, $\mathcal{G}$ - connectedness, $\mathcal{G}$ - continuous, $\mathcal{G}$ - homeomorphism, $\mathcal{G}$ - open cover, $\mathcal{G}$ -open subcover, $\mathcal{G}$ -compact, $\mathcal{G}$ - Lindeloff, $\mathcal{G}$ - local connectedness, $\mathcal{G}$ - seperated sets. $\mathcal{G}$ - locally finite, $\sigma_{\mathcal{G}}$ - locally finite, $\mathcal{G}T_1$ - space, $\mathcal{G}T_2$ - space, $\mathcal{G}T_3$ - space, $\mathcal{G}T_3$ - space, $\mathcal{G}$ - normal space, $\mathcal{G}$ - paracompact, $I_{\mathcal{G}}$ - convergence, $I^*_{\mathcal{G}}$ -convergence, Filter, Ideal, admissible Ideal, $I_{\mathcal{G}}$ -limit point, $I_{\mathcal{G}}$ - cluster point, $I^*_{\mathcal{G}}$ - limit point, $I^*_{\mathcal{G}}$ - cluster point, $\mathcal{G}$ - regular open set, $\mathcal{G}$ - regular closed set, maximal $\mathcal{G}$ - regular open set, minimal $\mathcal{G}$ - regular open set, maximal $\mathcal{G}$ - regular closed set, minimal $\mathcal{G}$ - regular closed set, weakly $\mathcal{G}$ - regular open set, weakly $\mathcal{G}$ - regular closed set.

സംഗ്രഹം

ഈ പ്രബന്ധം ജനറലൈസ്ഡ് ടോപ്പോളജിക്കൽ സ്പേസുകളെ കുറിച്ചുള്ള ഒരു സമഗ്ര പഠനമാണ്. ഓപ്പൺ സെറ്റുകളെ നിർവചിക്കുന്ന അടിസ്ഥാന തത്വങ്ങളിൽ അയവ് വരുത്തുകയോ മാറ്റങ്ങൾ വരുത്തുകയോ ചെയ്തുകൊണ്ട് ക്ലാസിക്കൽ ടോപ്പോളജിയുടെ വ്യാപ്തി വർദ്ധിപ്പിക്കുകയാണ് ഇതിലൂടെ ലക്ഷ്യമിടുന്നത്. വിശകലനത്തിലെ പരമ്പരാഗതമല്ലാത്ത ഘടനകളിൽ സാധാരണ ടോപ്പോളജിക്കൽ ചട്ടക്കൂടുകൾക്ക് നേരിടുന്ന പരിമിതികളാണ് ഈ പഠനത്തിന് പ്രധാന പ്രചോദനം.

ഈ പ്രബന്ധത്തിൽ, ,$\mathcal{G}$ -നെയിബർഹുഡ്, G-നെയിബർഹുഡ് സിസ്റ്റം, $\mathcal{G}$ -സെപറേറ്റഡ് സെറ്റ്, ,$\mathcal{G}$ -ലിൻഡെലോഫ് സ്പേസുകൾ, ,$\mathcal{G}$ -ഡെറൈവ്ഡ് സെറ്റ്, $\mathcal{G}$ -ഡെൻസ് സെറ്റ്, $\mathcal{G}$ -ലോക്കലി ഫൈനൈറ്റ്, $\sigma_{\mathcal{G}}$ -ലോക്കലി ഫൈനൈറ്റ്, ,$\mathcal{G}$ -പാരാകോംപാക്റ്റ് സ്പേസ്, $I_{,\mathcal{G}}$-കൺവേർജന്റ് സ്വീകൻസ്, $I_{\mathcal{G}}^{*}$-കൺവേർജന്റ് സ്വീകൻസ് എന്നിവയെ നിർവചിക്കുകയും അവയുടെ അടിസ്ഥാനപരമായ സവിശേഷതകൾ, സിദ്ധാന്തങ്ങൾ, തുല്യമായ സ്വഭാവഗുണങ്ങൾ എന്നിവ വിശദീകരിക്കുകയും ചെയ്യുന്നു.

ജനറലൈസ്ഡ് ക്ലോസ്ഡ് സെറ്റുകൾ, ജനറലൈസ്ഡ് കണക്റ്റഡ്നെസ്സ്, ജനറലൈസ്ഡ് ലോക്കൽ കണക്റ്റഡ്നെസ്സ് എന്നിവയുടെ സവിശേഷതകൾ ഈ പഠനം ആഴത്തിൽ പരിശോധിക്കുന്നു. $\mathcal{G}$ - ലോക്കലി ഫൈനൈറ്റ് ശേഖരങ്ങളും $\sigma_{\mathcal{G}}$ - ലോക്കലി ശേഖരങ്ങളും അവതരിപ്പിക്കുകയും പാരാകോംപാക്റ്റ്നെസ്സിന്റെ നിർവചനത്തെ ജനറലൈസ്ഡ് ടോപ്പോളജിക്കൽ സ്പേസുകളിലേക്ക് വ്യാപിപ്പിക്കുന്നതിന് ഇവ ഉപയോഗിക്കുകയും ചെയ്യുന്നു.

ജനറലൈസ്ഡ് പാരാകോംപാക്റ്റ്നെസ്സിൽ സെപറേഷൻ തത്വങ്ങളുടെ പങ്ക് ഈ പഠനം അന്വേഷിക്കുന്നു. ഒരു ശ്രേണിയുടെ $I_{\mathcal{G}}$ ലിമിറ്റ് പോയിന്റുകളും IG ക്ലസ്റ്റർ പോയിന്റുകളും ഈ പ്രബന്ധം നിർവചിക്കുകയും സാധാരണ സ്വീക്വൻഷ്യൽ കൺവേർജൻസും ജനറലൈസ്ഡ് ടോപ്പോളജിക്കൽ സ്പേസുകളിലെ $I_{,\mathcal{G}}$കൺവേർജൻസും തമ്മിലുള്ള ചില ബന്ധങ്ങൾ കണ്ടെത്തുകയും ചെയ്യുന്നു.

$\mathcal{G}$ - റെഗുലർ ഓപ്പൺ സെറ്റുകൾ, $\mathcal{G}$ - റെഗുലർ ക്ലോസ്ഡ് സെറ്റുകൾ എന്നിവയെക്കുറിച്ചുള്ള ആശയങ്ങളിലേക്ക് ഈ പഠനം മുന്നേറുകയും ഈ ആശയങ്ങളുടെ വിവിധ സവിശേഷതകളെക്കുറിച്ച് ചർച്ച ചെയ്യുകയും ചെയ്യുന്നു. കൂടാതെ, മിനിമൽ, മാക്സിമൽ G-റെഗുലർ ഓപ്പൺ സെറ്റുകളും മിനിമൽ, മാക്സിമൽ G-റെഗുലർ ക്ലോസ്ഡ് സെറ്റുകളും അവതരിപ്പിക്കുകയും ജനറലൈസ്ഡ് ടോപ്പോളജിക്കൽ സ്പേസുകളിൽ ഈ ആശയങ്ങളുടെ പങ്ക് വിശദീകരിക്കുകയും ചെയ്യുന്നു.

മൊത്തത്തിൽ, ഈ പ്രബന്ധം ക്ലാസിക്കൽ ടോപ്പോളജിക്കൽ ആശയങ്ങളെ സാമാന്യവൽക്കരിക്കുക മാത്രമല്ല, ടോപ്പോളജിക്കൽ സ്പേസുകളുടെ സൈദ്ധാന്തിക ഭൂമികയെ സമ്പന്നമാക്കുകയും ചെയ്യുന്ന ഒരു സമഗ്രവും ഏകീകൃതവുമായ ചട്ടക്കൂട് പ്രദാനം ചെയ്യുന്നു.

പ്രധാന വാക്കുകൾ

$\mathcal{G}$ - ടോപ്പോളജി , $\mathcal{G}$-ഓപ്പൺ , $\mathcal{G}$ -ക്ലോസ്ഡ് , $\mathcal{G}$ - ക്ലോഷർ , $\mathcal{G}$ – ഇന്റീരിയർ , $\mathcal{G}$- ഡെൻസ് , $\mathcal{G}$ - ഡെറൈവ്ഡ് സെറ്റ് , G-നെയിബർഹുഡ് , $\mathcal{G}$ -അക്യുമുലേഷൻ പോയിന്റ് , $\mathcal{G}$ - ബൗണ്ടറി , $\mathcal{G}$ - കണക്റ്റഡ്നെസ്സ് , $\mathcal{G}$ - കണ്ടിന്യൂസ് , $\mathcal{G}$ - ഹോമിയോമോർഫിസം , $\mathcal{G}$ - ഓപ്പൺ കവർ , $\mathcal{G}$ - ഓപ്പൺ സബ്കവർ , $\mathcal{G}$ - കോംപാക്റ്റ് , $\mathcal{G}$ - ലിൻഡെലോഫ് , $\mathcal{G}$ - ലോക്കൽ കണക്റ്റഡ്നെസ്സ് , $\mathcal{G}$ - സെപറേറ്റഡ് സെറ്റുകൾ, $\mathcal{G}$ - ലോക്കലി ഫൈനൈറ്റ് , $\sigma_{\mathcal{G}}$ – ലോക്കലി ഫൈനൈറ്റ് , $\mathcal{G}\mathrm{T}_1$ സ്പേസ് , $\mathcal{G}\mathrm{T}_2$- സ്പേസ് , $\mathcal{G}\mathrm{T}_3$-സ്പേസ് , $\mathcal{G}$ -നോർമൽ സ്പേസ് , $\mathcal{G}$ -പാരാകോംപാക്റ്റ് , $I_{\mathcal{G}}$ - കൺവേർജൻസ് , $I_{\mathcal{G}}^*$ – കൺവേർജൻസ് . ഫിൽട്ടർ ,ഐഡിയൽ, അഡ്മിസിബിൾ ഐഡിയൽ , $I_{\mathcal{G}}$-ലിമിറ്റ് പോയിന്റ് , $I_{\mathcal{G}}$ -ക്ലസ്റ്റർ പോയിന്റ്, $I_{\mathcal{G}}^*$ -ലിമിറ്റ് പോയിന്റ് , $I_{\mathcal{G}}^*$ - ക്ലസ്റ്റർ പോയിന്റ്, $\mathcal{G}$ – റെഗുലർ ഓപ്പൺ സെറ്റ്, $\mathcal{G}$ - റെഗുലർ ക്ലോസ്ഡ് സെറ്റ് , മാക്സിമൽ $\mathcal{G}$ - റെഗുലർ ഓപ്പൺ സെറ്റ് , മിനിമൽ $\mathcal{G}$ - റെഗുലർ ഓപ്പൺ സെറ്റ് , മാക്സിമൽ $\mathcal{G}$ - റെഗുലർ ക്ലോസ്ഡ് സെറ്റ് , മിനിമൽ -$\mathcal{G}$ - റെഗുലർ ക്ലോസ്ഡ് സെറ്റ് , വീക്ലി $\mathcal{G}$ - റെഗുലർ ഓപ്പൺ സെറ്റ്, വീക്ലി $\mathcal{G}$ - റെഗുലർ ക്ലോസ്ഡ് സെറ്റ്

Contents

Chapter 0

Introduction

Topology is the study of qualitative questions about geometric structures. The motivation behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together.

In 1963, Levine N given a generalization of open sets in topological space. He also define semi open sets and discussed its properties. This sets are nothing but sets lie between an open set and its closure.

More recent research papers are devoted to the study of subsets of spaces, containing the family of open sets and possessing properties more or less similar to those open sets. This type of generalization given by Prof. $Cs'asz'ar$ in 1997[13] . Generalized topological space is an important generalization of topological spaces. $Cs'asz'ar$ and others are worked so many years on the generalized topological spaces and they develop a basic theory for them. Especially $Cs'asz'ar$ developed this theory using some basic operators. For this he consider all monotonic functions from power set of X to power set of X, where X is a nonempty set. The collection of all monotonic functions is denoted by $\Gamma(X)$. For $\gamma \in \Gamma$, let $\mathcal{G} = \{A \subset X : A \subset \gamma(A)\}$. $Cs'asz'ar$ found that $\phi \in \mathcal{G}$ and $\mathcal{G}$ is closed under arbitrary union. He named such families as generalized topology and the pair $(X, \mathcal{G})$ is called generalized topological spaces.

The generalized topological topic is of great importance in preserving some topological properties under conditions weaker than topology. $Cs'asz'ar$ introduced this notion (generalized topology) in his research article titled by 'Generalized topology, Generalized continuity'. He introduce this concept in connection with γ - open sets. In this paper he formulate some simple properties of generalized topologeis and to study, based on this concept, suitable generalizations of the concept of continuous maps. Here he also discussed about generalized continuity, θ - continuity, θ - irresolute map, weakly θ - irresolute maps and β - continuous map in terms of generalized topology.

Gh.Abbaspour Tabadkan and A. Taghavi deal with a concrete example of generalized topological space and they examine some topological properties such as the usual continuity, the net continuity, and the net closure property for it in the research article titled by ' A note on Generalized Topology' . In this article they found that a generalized topological space has net continuous property or net closure property if and only if it is a topological space. In the same article they also give examples of continuous real functions which their sum and product are not generalized continuous. A generalized topological space X is said to have a net- closure property if for each subset A of X and $x \in C_{\mathcal{G}}(A)$, there exist a net in A converges to x.

Connectedness is a main property in general topology. In 2003 $Cs'asz'ar$ introduced the concept of connectedness in generalized topological spaces. He introduced this concept in terms γ - open sets.

I. Basdouri, R. Messaoud, A. Missaoui[5] discussed about some specific properties of the notions of connectedness and hyperconnectedness in generalized topological spaces in the paper titled by ' connected and hyperconnected generalized topological spaces'. In this paper he introduce the concepts $\mathcal{G} - \alpha$ - connected, $\mathcal{G} - \sigma$ - connected, $\mathcal{G} - \pi$ - connected and $\mathcal{G} - \beta$ - connected properties in generalized topological spaces. A generalized topological space X is said to be $\mathcal{G} - \alpha -$ connected ($\mathcal{G} - \beta -$ connected) if there are no nonempty disjoint $\mathcal{G} - \alpha$ - open ($\mathcal{G} - \beta$ - open) subsets U, V of X such that $U \cup V = X$. Similarly a generalized topological space X is said to be $\mathcal{G} - \sigma -$ connected ($\mathcal{G} - \pi -$ connected) if there are no nonempty disjoint $\mathcal{G}$ - semi open ($\mathcal{G} -$ preopen) subsets U, V of X such that $U \cup V = X$. In this paper he found the following results:

1. Contra $\alpha-$ continuous image of a $\mathcal{G} - \alpha$ - connected set is $\mathcal{G}-$ connected.

2. The contra $\beta-$ continuous image of a $\mathcal{G}-\beta-$ connected set is $\mathcal{G}-$ connected.

In the last section he introduce the notion of hyperconnectedness in generalized topological spaces, it is nothing but every nonempty $\mathcal{G}-$ open set in a generalized topological space X is $\mathcal{G}-$ dense in X. Using the same technique in ordinary topological spaces he define concepts like $\mathcal{G} - \alpha-$ hyperconnected, $\mathcal{G} - \sigma-$ hyperconnected, $\mathcal{G} - \pi-$ hyperconnected and $\mathcal{G} - \beta-$ hyperconnected properties in generalized topological spaces.

In 2004, $Cs'asz'ar$ introduced the notion of separation axioms, in his research article titled ' Separation axioms for generalized topologies' . In this paper he found that their most important consequences remain true if we replace open sets by elements of a generalized topology. In this paper he introduced T_0, T_1, T_2, S_1, S_2 axioms in generalized topological spaces and he define the envelop operator and weak envelop operator.

Won Keun Min[33] is also introduce the concept separation axioms in some other way in his paper titled by ' Remarks on Separation Axiom on Generalized topological spaces'. Here he define the set $M_{\mathcal{G}} = \cup\{G : G \in \mathcal{G}\}$, where $(X, \mathcal{G})$ is a generalized topological space. In this paper defines GT_1, GT_2, G - regular and

G - normal spaces in generalized topological spaces. Finally he concluded with the relation $GT_4 \implies GT_3 \implies GT_2 \implies GT_1$.

Here also he found that GT_1, GT_2, G - regular and G - normal spaces in generalized topological spaces are coincide with T_1, T_2,, regular, normal spaces if we replace generalized topological spaces with ordinary topological spaces.

In 2011, R. Khayyeri[25] introduced the concept of base in generalized topological space. A collection $\mathcal{B} \subset P(X)$, is called a base for a generalized topology if $\mathcal{G} = \{\cup\mathcal{B}' : \mathcal{B}' \subset \mathcal{B}\}$. In ordinary topological space every subset of $P(X)$ form a sub base for some topology. But in this paper he claimed that every sub collection of $P(X)$ is a base for some generalized topology on X.

Thomas Jyothis[31] introduced the notion of $\mathcal{G}$ - compact spaces in his research article, titled by 'μ - compactness in generalized topological spaces'. In this paper he define concepts like $\mathcal{G}$ - cover, $\mathcal{G}$ - subcover and $\mathcal{G}$ - open cover. $\mathcal{G}$ - compactness is nothing but every $\mathcal{G}$ - open cover of a generalized topological space has a finite $\mathcal{G}$ - sub cover. In this paper he claimed that every finite generalized topological space is $\mathcal{G}$ - compact and every finite union of $\mathcal{G}$ - compact space is $\mathcal{G}$ - compact. He also claimed that $\mathcal{G}$ - continuous image of a $\mathcal{G}$ - compact space is $\mathcal{G}$ - compact and every $\mathcal{G}$ - closed subset of a $\mathcal{G}$ - compact set is $\mathcal{G}$ - compact. He also extended the concept $\mathcal{G}$ - compactness in subspace generalized topological spaces. In this section he proved that ' If E is $\mathcal{G}$ - compact and F is $\mathcal{G}$ - closed then $F \cap E$ is $\mathcal{G}$ - compact'. So almost all results regarding compactness in ordinary topological spaces is also hold in generalized topological spaces. Finally he discussed about the relation between separation axioms and compactness in generalized topological spaces.

Sequential convergence is a well developed concept in topological spaces. R. Baskaran introduced this concept in generalized topological spaces. In a topological space every point has a neighbourhood, because X is always open in X. The strength as well as weakness of generalized topological space is that not all points have neighbourhoods containing them. If X is a strong generalized topological space then every point in X has a generalized neighbourhood. So if there exist a point x in a generalized topological space X, such that x has no neighbourhood in X, then every sequence in X converges to x.

In this paper, *R. Baskaran* [26], also introduced the concept of attraction of a set. It is nothing but a point x in a generalized topological space X is said to be a point of attraction of $A \subset X$, if every sequence in A converges to x.

The concept of convergence of a sequence of real numbers has been extend to statistical convergence independently by *Fast.* Any convergent sequence of real numbers is statistically convergent, but the converse is not true. The concept of I - convergence of real sequences is a generalization of statistical convergence which is based on the structure of the ideal I of subsets of the set of natural numbers.

The concept of I^* - convergence of real sequences arises from the following result on statistical convergence; a real sequence $x = (x_n)$ is statistically convergent to ζ if and only if there exist a set $M = \{m_1 \leq m_2 \leq ... \leq m_k \leq ...\} \subset \mathbb{N}$ such that $d(M) = 1$ and $lim_{k\to\infty} x_{m_k} = \zeta$, where $d(M)$ is the density of M and defined by $d(M) = lim_{k\to\infty} \frac{|M_n|}{n}, M_n = \{k \in M : K \leq n\}$ and $|M_n|$ is the cardinality of M_n.

B . K. Lahari and Pratulananda Das[6] extended concepts I - convergence and I^* - convergence in topological spaces. In chapter 4, we extend these concepts in generalized topological spaces.

Murad Hussain and Moizul din khan and Cenap Ozel[23] are introduced the notion of generalized topological groups using generalized topological structure in their research article titled by 'On Generalized Topological Groups'. In this paper they also discussed some basic properties of this kind of structures. A $\mathcal{G}$ - topological group G is a group that is also a $\mathcal{G}T_2$ - space such that the multiplication map of $G \times G$ into G sending $x \times y$ into $x.y$, and the inverse map of G into G sending x into x^{-1}, are $\mathcal{G}$ - continuous. In this paper they also define morphisms of $\mathcal{G}$ - topological groups, it is nothing but a function $\phi : G \to G^{'}$, is both $\mathcal{G}$ - continuous and group homomorphism.

Sini P[28] studied heriditerily homogeneous generalized topological spaces in her research article titled by ' Heriditerly Homogeneous Generalized Topological

Spaces'. In this paper she also found a characterization for a generalized topological space which is heriditerily homogeneous, it is nothing but ' a generalized topological space is heriditerily homogenous if and only if every transposition of X is $\mathcal{G}$ homeomorphism in X.

Concept of regular open sets was introduced by M.H.Stone in 1937. It has been shown by him that the collection of all regular open sets form a complete lattice. He also gave some application of Boolean algebras in general Topology. Anuradha. N and Baby Chacko define regular open sets, regular closed sets, weakly regular open sets, weakly regular closed sets in topological spaces [2,3.4]. Here we extend the concept of regular open sets in generalized topological spaces.

The whole work is divided in to five chapters. Chapters divided in to sections . In chapter 1, we introduce concepts like genaralized closure operator, generalized dense, genaralized neighbourhood, generalized neighbourhood system, generalized accumulation point and genaralized derived set. This chapter contain 4 sections. Section 2 contains preliminary ideas on generalized topological spaces. In section 3, we discuss some more results regarding generalized closed sets. Here we found that some results in ordinary topological space need not be true in generalized topological spaces, for example ' in ordinary topological space finite union of closed sets is closed but in generalized topological space finite union of $\mathcal{G}$ - closed sets need not be $\mathcal{G}$ - closed. In this section also found that $C_{\mathcal{G}}(A \cup B) \neq C_{\mathcal{G}}(A) \cup C_{\mathcal{G}}(B)$, but in ordinary topological space *closure of* $(A \cup B)$ = *closure* (A) $\cup$ *closure* (B). In section 4, we introduce $\mathcal{G}$-neighbourhoods, $\mathcal{G}$-neighbouhood system, $\mathcal{G}$-dense etc. and discuss about some results relating these concepts. In this section we found that in ordinary topological space the neighbourhood system must be nonempty but in generalized topological space generalized neighbourhood system may be empty. If a generalized topological space is strong then generalized neighbourhood system must be nonempty.

Chapter 2 contains 5 sections and in this chapter we introduce the concepts like generalized interior points, generalized boundary, $\mathcal{G}$ - separated sets, $\mathcal{G}$ - component and $\mathcal{G}$ - Lindeloff spaces. Section 2 contains preliminary ideas on connectedness and compactness in generalized topological spaces. In section 3 we discuss some results regarding $\mathcal{G}$ - connectedness and introduce some concepts like $\mathcal{G}$ - int of a set etc. Here we found that the generalized closure of $\mathcal{G}$ connected set need not be $\mathcal{G}$ - connected, but in ordinary topological space the closure of a connected set must be connected. In this section also found that the $\mathcal{G}$ - components need not be $\mathcal{G}$ - closed but in ordinary topological space the components are closed. Section 4 explains local connectedness in generalized topological spaces. In section 5 , we introduce the concept $\mathcal{G}$ - Lindeloff spaces and explain some results regarding these type of spaces.

In chapter 3 we introduce the concepts like $\mathcal{G}$ - locally finite, $\sigma_{\mathcal{G}}$ - locally finite properties in generalized topological spaces. Then introduce the concept paracompactness in generalized topological spaces. This chapter contains 4 sections. Section 2 contains the preliminary ideas used in subsequent sections. In section 3 we discuss about $\mathcal{G}$ - locally finite and $\sigma_{\mathcal{G}}$ - locally finite properties in generalized topology. These two properties are coincides with locally finite and σ - locally finite properties in general topology if we replaced the generalized toplogical space with ordinary topological space. Here also discuss about some basic properties about $\mathcal{G}$ - locally finite and $\sigma_{\mathcal{G}}$ - locally finite properties. In this section we found that in ordinary topological space, the union of a locally finite collection of closed subsets of a topological space is itself closed. But in generalized topological space this result does not hold. In section 4 we introduce the main concept paracompactness in generalized topological space. Here also discuss about some relation between separation axioms and paracompactness in generalized topological spaces and claim that every $\mathcal{G}$ - paracompact space is $\mathcal{G}$ - normal.

In chapter 4, we extends the concepts I - Convergence and I^* - Convergence in generalized topological spaces. This chapter contain five sections. First section is the introduction. Second section contain preliminary ideas used for the subsequent sections. In third section we extend the concept I - convergence into generalized topological spaces. Mainly in this section we introduce the the concept $I_{\mathcal{G}}$ - convergence in generalized topological spaces and discuss about some properties. Here we found that the $I_{\mathcal{G}}$ convergence is preserved under $\mathcal{G}$ - continuous functions. In fourth section we introduce the concept of $I^*_{\mathcal{G}}$ - convergence in generalized topological spaces. Here we also discuss about the relation between $I_{\mathcal{G}}$ and $I^*_{\mathcal{G}}$ - convergence in Generalized topological spaces. Section 5 contain the definitions of $I_{\mathcal{G}}$ -limit point and $I_{\mathcal{G}}$ - cluster points of a sequence and discuss about some properties of these type of points.

In chapter 5, we introduce the concepts like of $\mathcal{G}$ - regular open sets, $\mathcal{G}$ - regular closed sets, minimal $\mathcal{G}$ - regular open sets, maximal $\mathcal{G}$ - regular open sets, minimal $\mathcal{G}$ - regular closed sets, maximal $\mathcal{G}$ - regular closed sets etc. Here we also discuss about some properties of these types of sets. In this chapter contain nine sections.

Chapter 1

On Generalized closed sets

1.1 Introduction

In this chapter we introduce concepts like genaralized closure operator, generalized dense, genaralized neighbourhood, generalized neighbourhood system, generalized accumulation point and genaralized derived set. Section 2 contains preliminary ideas on generalized topological spaces. In section 3, we discuss some more results regarding generalized closed sets. In section 4, we introduce $\mathcal{G}$-neighbourhoods, $\mathcal{G}$-neighbouhood system, $\mathcal{G}$-dense etc. and discuss about some results relating these concepts. In this chapter, we also found that some of the results in ordinary topological spaces does not hold in generalized topological spaces.

1.2 Preliminary Ideas on Generalized Topological Spaces

Definition 1.2.1. [13] *Let X be a set and $P(X)$ its power set. A subset $\mathcal{G}$ of $P(X)$ is called a generalized topology (GT) on X and $(X,\mathcal{G})$ is called a generalized topological space (GTS) if $\mathcal{G}$ has the following properties.*

1. *$\phi \in \mathcal{G}$*

2. *Any union of elements of $\mathcal{G}$ belongs to $\mathcal{G}$*

Here $\mathcal{G}$ is also called $\mathcal{G}$-topology on X.

Definition 1.2.2. [13] *A generalized topology $\mathcal{G}$ is called strong if $X \in \mathcal{G}$.*

In a generalized topological space $(X, \mathcal{G})$, define $M_{\mathcal{G}} = \cup\{U : U \in \mathcal{G}\}$

Definition 1.2.3. [13] *A subset A of a generalized topological space is called $\mathcal{G}$-open if $A \in \mathcal{G}$. A subset B is called $\mathcal{G}$-closed if $(X$-$B)$ is $\mathcal{G}$-open.*

Definition 1.2.4. [7] *The $\mathcal{G}$-closure of A is denoted by $C_{\mathcal{G}}(A)$ is the intersection of all $\mathcal{G}$-closed sets containing A.*

Theorem 1.2.5. [7] *Let $(X, \mathcal{G})$ be a generalized topological space and $A, B \subset X$. Then the following statements are hold.*

1. *$x \in C_{\mathcal{G}}(A)$ if and only if $x \in U \in \mathcal{G}$ implies $U \cap A \neq \phi$*

2. *If* $U, V \in \mathcal{G}$ *and* $U \cap V = \phi$ *then* $C_{\mathcal{G}}(U) \cap V = \phi$ *and* $U \cap C_{\mathcal{G}}(V) = \phi$

Definition 1.2.6. [34] *Let (X,$\mathcal{G}$) be a generalized topological space and $\mathcal{B}$ be a sub collection of $\mathcal{G}$ then $\mathcal{B}$ is called a base for $\mathcal{G}$ if every $\mathcal{G}$-open set can be expressed as union of some members of $\mathcal{B}$*

Theorem 1.2.7. [34] *Any subset of power set of X is a base for some generalized topology on X.*

Remark 1.2.8. *We can generate a generalized topology on X from any sub collection of power set of X (generate means taking possible unions).*

1.3 Some Results on Generalized Closed sets

Theorem 1.3.1. *Let $\mathcal{C}$ be the family of all generalized closed sets in a generalized topological space $(X, \mathcal{G})$. Then $\mathcal{C}$ has the following properties.*

1. $X \in \mathcal{C}$

2. *$\mathcal{C}$ is closed under arbitrary intersections.*

Conversely, given any set X and a family $\mathcal{C}$ of its subsets which satisfies these two properties, there exists a unique generalized topology $\mathcal{G}$ on X, such that $\mathcal{C}$ coincides with the family of all generalized closed subsets of $(X, \mathcal{G})$.

Proof. The first part follows from the definition of $\mathcal{G}$ -closed sets and De Morgan's laws. Define $\mathcal{G} = \{B \subset X : X - B \in \mathcal{C}\}$

Claim: $\mathcal{G}$ is a generalized topology on X.

1. $\phi \in \mathcal{G}$.

2. Let the collection (V_α) be an arbitrary collection of members of $\mathcal{G}$. Then for each $\alpha, (X - V_\alpha) \in \mathcal{C}$. To prove that $(\cup_\alpha V_\alpha)^c \in \mathcal{C}$. We have by De Morgan's law $(\cup_\alpha V_\alpha)^c = \cap_\alpha V_\alpha^c \in \mathcal{G}$. Thus $\mathcal{G}$ is closed under arbitrary union. Hence $\mathcal{G}$ is a GT on X. The $\mathcal{G}$-open subsets of X are precisely the complements of elements of $\mathcal{C}$. Also $\mathcal{G}$ is unique.

□

Remark 1.3.2. *In ordinary topological space finite union of closed sets is closed. But in generalized topological space finite union of $\mathcal{G}$-closed sets need not be $\mathcal{G}$-closed.*

Example 1.3.3. *Let X =$\{a, b, c, d\}$ and $\mathcal{G} = \{\phi, \{c, d\}, \{a, d\}, \{a, c, d\}\}$. Then $\mathcal{G}$ is a generalized topology on X. Note that $\{a, b\}$ and $\{b, c\}$ are $\mathcal{G}$ - closed but their union $\{a,b,c\}$ is not $\mathcal{G}$ - closed.*

Theorem 1.3.4. *Let A, B be subsets of a generalized topological space $(X, \mathcal{G})$. Then the following conditions hold.*

1. *$C_{\mathcal{G}}(A)$ is $\mathcal{G}$-closed in X. More over it is the smallest $\mathcal{G}$-closed subset of X containing A.*

2. *A is $\mathcal{G}$-closed in X if and only if $C_{\mathcal{G}}(A) = A$.*

3. $C_{\mathcal{G}}(C_{\mathcal{G}}(A)) = C_{\mathcal{G}}(A)$

4. $C_{\mathcal{G}}(A) \cup C_{\mathcal{G}}(B) \subset C_{\mathcal{G}}(A \cup B)$.

Proof. 1. Using theorem 1.3.1, $C_{\mathcal{G}}(A)$ is $\mathcal{G}$-closed in X. By the definition of generalized closure of a set $C_{\mathcal{G}}(A)$ is the smallest $\mathcal{G}$ - closed subset of X containing A.

2. Suppose A is $\mathcal{G}$ - closed. Since $C_{\mathcal{G}}(A)$ is the smallest $\mathcal{G}$ - closed subset of X containing A, we get $C_{\mathcal{G}}(A) = A$. Converse part is trivial from the definition of generalized closure of a set.

3. Since $C_{\mathcal{G}}(A)$ is $\mathcal{G}$ - closed in X, from 2, $C_{\mathcal{G}}(C_{\mathcal{G}}(A)) = C_{\mathcal{G}}(A)$

4. If $A_1 \subset A_2$ then $C_{\mathcal{G}}(A_1) \subset C_{\mathcal{G}}(A_2)$. Since $A \subset (A \cup B)$,
$C_{\mathcal{G}}(A) \subset C_{\mathcal{G}}(A \cup B)$. Similarly $C_{\mathcal{G}}(B) \subset C_{\mathcal{G}}(A \cup B)$.
Hence $C_{\mathcal{G}}(A) \cup C_{\mathcal{G}}(B) \subset C_{\mathcal{G}}(A \cup B)$.

□

Remark 1.3.5. *If A and B are subsets of a generalized topological space, then $C_{\mathcal{G}}(A \cup B)$ need not be contained in $C_{\mathcal{G}}(A) \cup C_{\mathcal{G}}(B)$.*

Example 1.3.6. *Let* $X = \{a,b,c,d\}$ *and* $\mathcal{G} = \{X, \phi, \{c,d\}, \{a,b\}, \{a,c,d\}\}$. *Then* $\mathcal{G}$ *is a generalized topology on* X. *Let* $A = \{a,b\}$ *and* $B = \{b,c\}$. *Then* $C_{\mathcal{G}}(A) = \{a,b\}$ *and* $C_{\mathcal{G}}(B) = \{b,c\}$. *But* $C_{\mathcal{G}}(A \cup B) = X$.
That is $C_{\mathcal{G}}(A \cup B) \neq C_{\mathcal{G}}(A) \cup C_{\mathcal{G}}(B)$.

Definition 1.3.7. *Let* $(X, \mathcal{G})$ *be a generalized topological space. The generalized closure operator associated with it is defined as the function* $c : P(X) \to P(X)$ *such that* $c(A) = C_{\mathcal{G}}(A)$, $A \in P(X)$.

Remark 1.3.8. *A generalized closure operator has the following properties.*

1. *The fixed points of* c *are precisely the* $\mathcal{G}$ *- closed subsets of* X.

2. c *is idempotent. That is* $c \circ c = c$ *or in other words for any* $A \in P(X)$, $c(c(A)) = c(A)$.

Theorem 1.3.9. *Let* X *be a set and* $\theta : P(X) \to P(X)$ *be a function such that*

1. *For any* $A \in P(X)$, $A \subset \theta(A)$.

2. θ *is idempotent.*

3. $\theta(A) \cup \theta(B) \subset \theta(A \cup B)$.

Then there exists a unique generalized topology on X *such that* θ *coincides with the generalized closure operator associated with it. Conversely any generalized closure operator satisfies these properties.*

Proof. Let $\mathcal{C} = \{A \subset X : \theta(A) = A\}$. Here we want to prove that $\mathcal{G}$ satisfies the the hypothesis of the theorem 1.3.1

1. For any $A \in P(X), A \subset \theta(A)$. So in particular $X \subset \theta(X)$.
   Hence $X = \theta(X)$ Therefore $X \in \mathcal{C}$.

2. Let $\{A_\alpha\}$ be any collection of members in $\mathcal{C}$. Note that θ is monotonic, because if $A \subset B$, then $B = A \cup (B - A)$
   implies $\theta(B) = \theta(A \cup (B - A)) \supset \theta(A) \cup \theta(B - A) \supset \theta(A)$
   implies $\theta(A) \subset \theta(B)$.
   Since $\cap_\alpha A_\alpha \subset A_\alpha$ for each α, $\theta(\cap_\alpha A_\alpha) \subset \theta(A_\alpha)$ for each α, because θ is monotonic. We want to prove that $\cap_\alpha A_\alpha \in \mathcal{C}$. For this we want to prove that $\theta(\cap_\alpha A_\alpha) = \cap_\alpha A_\alpha$.
   Since $\theta(\cap_\alpha A_\alpha) \subset \theta(A_\alpha)$ for each α, we see that $\theta(\cap_\alpha A_\alpha) \subset \cap_\alpha \theta(A_\alpha)$. But each A_α is a member of $\mathcal{C}$, so $\theta(A_\alpha) = A_\alpha$. Hence $\theta(\cap_\alpha A_\alpha) \subset \cap_\alpha(A_\alpha)$.
   Also using property 1, we get $\cap_\alpha(A_\alpha) \subset \theta(\cap_\alpha A_\alpha)$. Hence $\theta(\cap_\alpha A_\alpha) = \cap_\alpha(A_\alpha)$. Therefore $\cap_\alpha(A_\alpha) \in \mathcal{C}$.

So $\mathcal{C}$ satisfies all the conditions given in the theorem 1.3.1. So by theorem 1.3.1 we get, there is a unique generalized topology $\mathcal{G}$ on X such that $\mathcal{C}$ coincides with the family of all $\mathcal{G}$-closed subsets of $(X, \mathcal{G})$.
It remains to verify that the generalized closure operator associated with $\mathcal{G}$ is coincides with θ. Let $A \subset X$. To prove that $\theta(A) = C_{\mathcal{G}}(A)$. We have $C_{\mathcal{G}}(A)$ is

the intersection of all $\mathcal{G}$- closed subsets of X containing A. But $\mathcal{G}$-closed subsets of X with respect to $\mathcal{G}$ are members of $\mathcal{C}$.

Therefore $C_{\mathcal{G}}(A) = \cap\{B \subset X : A \subset B, \theta(B) = B\}$. Now whenever $A \subset B$, $\theta(A) \subset \theta(B)$. So if $A \subset B$ and $\theta(B) = B$ implies that $\theta(A) \subset B$. But $C_{\mathcal{G}}(A)$ is the intersection of such sets B's and so $C_{\mathcal{G}}(A) \supset \theta(A)$. Using condition 2, we see that $\theta(\theta(A)) = \theta(A)$ which implies $\theta(A) \in \mathcal{C}$. From property 1, $A \subset \theta(A)$. That is $\theta(A)$ is a $\mathcal{G}$-closed set containing A. But $C_{\mathcal{G}}(A)$ is the smallest $\mathcal{G}$ - closed set containing A. Hence $C_{\mathcal{G}}(A) \subset \theta(A)$. Therefore $C_{\mathcal{G}}(A) = \theta(A)$ □

Definition 1.3.10. *If A is a subset of a generalized topological topological space. Then A is said to be generalized dense (or $\mathcal{G}$ - dense) in X if $C_{\mathcal{G}}(A) = X$.*

Example 1.3.11. *Let $X = \{a, b, c, d\}$ and $\mathcal{G} = \{\phi, \{a, b\}\}$ and $A = \{b, c\}$ then $C_{\mathcal{G}}(A) = X$. So A is $\mathcal{G}$ - dense in X.*

Theorem 1.3.12. *A subset A of a generalized topological space X is $\mathcal{G}$ - dense in X if and only if A intersects every $\mathcal{G}$ - open subset of X.*

Proof. Suppose A is $\mathcal{G}$- dense in X and B is a non empty $\mathcal{G}$-open subset of X. If $A \cap B = \phi$. Then $A \subset (X - B)$. Since $(X - B)$ is $\mathcal{G}$ -closed and $C_{\mathcal{G}}(A)$ is the smallest $\mathcal{G}$ -closed set containing A, we get $C_{\mathcal{G}}(A) \subset (X - B)$. But $(X - B)$ is a proper nonempty subset of X, contradicting $C_{\mathcal{G}}(A) = X$. Hence $A \cap B \neq \phi$.

Conversely suppose that A intersects every nonempty $\mathcal{G}$- open subsets of X. Since ϕ is $\mathcal{G}$-open, the only $\mathcal{G}$- closed set containing A is X.

So $C_{\mathcal{G}}(A) = X$ and hence A is $\mathcal{G}$-dense in X. □

1.4 Generalized Neighbourhoods and Generalized Accumulation Points.

Definition 1.4.1. *Let $(X, \mathcal{G})$ be a generalized topological space, $x_0 \in X$ and $N \subset X$. Then N is said to be a generalized neighbourhood ($\mathcal{G}$-neighbourhood) of x_0, if there is a $\mathcal{G}$ - open set V such that $x_0 \in V$ and $V \subset N$.*

Example 1.4.2. *Let $X = \{a, b, c, d\}$ and $\mathcal{G} = \{\phi, \{a, b\}\}$. Let $N = \{a, b, c\}$. Then N is a generalized neighbouhood of both a and b. But N is not a $\mathcal{G}-$ neighbourhood of c. Also c has no $\mathcal{G}$ - neigbourhood in X.*

Remark 1.4.3. *In ordinary topological space every element has a neighbourhood because the whole set is a neighbourhood of each of its points. But in generalized topological space every point need not have a $\mathcal{G}$- neighbourhood. In a strong generalized topological space every point has a $\mathcal{G}$- neighbourhood (see definition 1.2.2).*

Theorem 1.4.4. *A subset A of a generalized topological space X is $\mathcal{G}$- open if and only if it is a $\mathcal{G}$- neigbourhood of each of its point.*

Proof. Let X be a generalized topological space and $G \subset X$. Suppose G is $\mathcal{G}$-open. By the definition of $\mathcal{G}$-neighbourhood, G is a $\mathcal{G}$-neighbourhood of each of its points.

Conversely suppose G is a $\mathcal{G}$- neighbourhood of each of its points. So for each $x \in G$ there exists a $\mathcal{G}$ - open set V_x such that $x \in V_x \subset G$. Hence $G = \cup_{x \in G} V_x$

implies G is $\mathcal{G}$- open. $\square$

Definition 1.4.5. *Let $(X, \mathcal{G})$ be a generalized topological space. Let η_x be the set of all $\mathcal{G}$ -neighbourhoods of x in X. Then the family η_x is called the generalized neighbourhood system at x.*

Example 1.4.6. *Let $X = \{a, b, c, d\}$ and $\mathcal{G} = \{\phi, \{a, b\}\}$. Here c has no $\mathcal{G}$-neighbourhood. So the generalized neighbourhood system at c, $\eta_c = \phi$.*
But $\eta_a = \{\{a, b\}, \{a, b, c\}, X\} = \eta_b$.

Remark 1.4.7. *In ordinary topological spaces the neighbourhood system must be nonempty. But the generalized neighbourhood system may be empty.*

Theorem 1.4.8. *Let X be a generalized topological space and for $x \in X$ such that the generalized neighbouhood system at x, η_x is nonempty. Then*

1. *If $U \in \eta_x$, then $x \in U$.*
2. *If $V \in \eta_x$ and $U \supset V$, then $U \in \eta_x$.*
3. *A set G is $\mathcal{G}$- open in X if and only if $G \in \eta_x$ for all $x \in G$.*
4. *If $U \in \eta_x$, then there exists $V \in \eta_x$ such that $V \subset U$ and $V \in \eta_y$ for all $y \in V$.*

Proof. 1. If $U \in \eta_x$ then U is a $\mathcal{G}$- neighbourhood of x. So $x \in U$.

2. $V \in \eta_x$ implies V is a $\mathcal{G}$- neighbourhood of x. So there exists a $\mathcal{G}$- open set W such that $x \in W \subset V$. $U \supset V$ then $x \in W \subset V \subset U$ implies $U \in \eta_x$.

3. A set G is $\mathcal{G}$- open if and only if G is a $\mathcal{G}$- neighbourhood of each of its points (by theorem 1.4.4) if and only if $G \in \eta_x$ for all $x \in G$.

4. If $U \in \eta_x$ then U is a $\mathcal{G}$- neighbourhood of x. So there exists $\mathcal{G}$- open set V such that $x \in V \subset U$. By theorem 1.4.4, V is a $\mathcal{G}$ - neighbourhood of each of its points, in particular it is a $\mathcal{G}$- neighbourhood of x. That is $V \in \eta_x$ such that $V \subset U$ and $V \in \eta_y$ for all $y \in V$.

$\square$

Theorem 1.4.9. *Let X be a set and suppose for each $x \in X$, η_x be a family of subsets of X satisfying the following properties.*

1. *If $U \in \eta_x$, then $x \in U$.*

2. *If $V \in \eta_x$, and $U \supset V$ then $U \in \eta_x$.*

3. *If $U \in \eta_x$, then there exists $V \in \eta_x$ such that $V \subset U$ and $V \in \eta_y$ for all $y \in V$.*

Then there exists a unique generalized topological space $\mathcal{G}$ on X such that for each $x \in X$, η_x coincides with the family of all $\mathcal{G}$ -neighbourhoods of x with respesct to $\mathcal{G}$.

Proof. Let $\mathcal{G} = \{U \subset X : U \in \eta_x, \text{ for every } x \in U\}$. First we want to prove that $\mathcal{G}$ is a generalized topology on X. Clearly $\phi \in \mathcal{G}$.

Let $\{A_\alpha\}$ be an arbitrary collection of members of $\mathcal{G}$. By definition of $\mathcal{G}$, each A_α is a member of η_x. By condition 2, we have $\cup_\alpha A_\alpha \in \eta_x$, for every $x \in \cup_\alpha A_\alpha$. Hence $\mathcal{G}$ is a generalized topology on X.

Next we want to prove that for any $x \in X$, the generalized neighbourhood system at x with respect to $\mathcal{G}$ coincides with η_x.

Let $U \in \eta_x$. By condition 3 there exists $V \in \eta_x$ such that $V \subset U$ and $V \in \eta_y$ for all $y \in V$. So $V \in \mathcal{G}$ implies U is a $\mathcal{G}$- neighbourhood of x.

Conversely, let U be a $\mathcal{G}$- neighbourhood of x. So there exists $V \in \mathcal{G}$ such that $x \in V \subset U$. Hence $V \in \eta_x$. From condition 2, $U \in \eta_x$.

□

Definition 1.4.10. *Let A be a subset of a generalized topological space X and $y \in X$. Then y is said to be a generalized accumulation point ($\mathcal{G}$- accumulation point) of A if every $\mathcal{G}$ - open set containing y contains at least one point of A other than y.*

Example 1.4.11. *Let $\mathcal{G}$ be a generalized topology on the set of real numbers generated by*

$\{(a, \infty), (-\infty, b) : a, b \in \mathbb{R}\}$. *Then every real number is a generalized accumulation point of $\mathbb{N}$, the set of natural numbers.*

Definition 1.4.12. *Let A be a subset of a generalized topological space X, then the generalized derived set of A is the set of all generalized accumulation points of A. It is denoted by $A'^{\mathcal{G}}$.*

Theorem 1.4.13. *If A is a subset of a generalized topological space X, then $C_{\mathcal{G}}(A) = A \cup A'^{\mathcal{G}}$.*

Proof. First we want to prove that $A \cup A'^{\mathcal{G}}$ is $\mathcal{G}$ - closed. It is enough to prove that $(X - (A \cup A'^{\mathcal{G}}))$ is $\mathcal{G}$- open.

Let $y \in (X - (A \cup A'^{\mathcal{G}}))$. Then y is not a $\mathcal{G}$ - accumulation point of A. So there is a $\mathcal{G}$ - open set V containing y such that V contains no point of A except possibly y. But $y \notin A$. So $A \cap V = \phi$. Next we claim that $(A'^{\mathcal{G}} \cap V) = \phi$. For, let $z \in (A'^{\mathcal{G}} \cap V)$. So z is a $\mathcal{G}$- accumulation point of A and V is a $\mathcal{G}$- open set containing y. Hence $A \cap V \neq \phi$. Which is a contradiction . So $(A'^{\mathcal{G}} \cap V) = \phi$ and $V \subset X - (A \cup A'^{\mathcal{G}})$. Hence $(A \cup A'^{\mathcal{G}})$ is $\mathcal{G}$ - closed. Therefore $C_{\mathcal{G}}(A) \subset (A \cup A'^{\mathcal{G}})$. For the reverse inclusion we claim that $A'^{\mathcal{G}} \subset C_{\mathcal{G}}(A)$. Let $y \in A'^{\mathcal{G}}$. If $y \notin C_{\mathcal{G}}(A)$ then $y \in (X - C_{\mathcal{G}}(A))$. So $(X - C_{\mathcal{G}}(A))$ is a $\mathcal{G}$ - open set containing y. But since y is a $\mathcal{G}$ - accumulation point of A, so $A \cap (X - (C_{\mathcal{G}}(A))) \neq \phi$. Which is a contradiction as $(X - (C_{\mathcal{G}}(A))) \subset X - A$. Hence $y \in C_{\mathcal{G}}(A)$. Therefore $A'^{\mathcal{G}} \subset C_{\mathcal{G}}(A)$. So $A \subset C_{\mathcal{G}}(A)$ and $A'^{\mathcal{G}} \subset C_{\mathcal{G}}(A)$ which implies $A \cup A'^{\mathcal{G}} \subset C_{\mathcal{G}}(A)$. Hence $C_{\mathcal{G}}(A) = A \cup A'^{\mathcal{G}}$. $\square$

Chapter 2

Connectedness and Lindeloff Property in Generalized Topological Spaces

2.1 Introduction

Here we introduce the concepts like generalized interior point, generalized boundary, $\mathcal{G}$ - separated sets, $\mathcal{G}$ - component, $\mathcal{G}$ - local connectedness and $\mathcal{G}$ - Lindeloff spaces. Section 2 contains preliminary ideas of connectedness and compactness in generalized topological spaces. In section 3, we discuss some results regarding $\mathcal{G}$ - connectedness and introduce some concepts like $\mathcal{G}$ - int of a set etc. In section 4, we introduce local connectedness in generalized topological spaces and discuss about some properties of this concept. In section 5, introduce the concept of $\mathcal{G}$ - Lindeloff spaces and explain some results regarding these type of spaces.

2.2 Preliminary Ideas Relating Connectedness and Compactness in generalized topological spaces.

Definition 2.2.1. [31] *Let $(X, \mathcal{G})$ be a generalized topological space. A collection $\mathcal{U}$ of subsets of X is said to be a $\mathcal{G}$ - cover of X if the union of elements of $\mathcal{U}$ equals X.*

Definition 2.2.2. [31] *Let $(X, \mathcal{G})$ be a generalized topological space. A $\mathcal{G}$ - sub cover of a $\mathcal{G}$ - cover $\mathcal{U}$ is a sub collection μ of $\mathcal{U}$ which itself a $\mathcal{G}$ - cover. If the elements of $\mathcal{U}$ are $\mathcal{G}$ - open then we say that $\mathcal{U}$ is a $\mathcal{G}$ - open cover.*

Definition 2.2.3. [31] *If every $\mathcal{G}$ - open cover of X has a finite $\mathcal{G}$ - sub cover then we say that X is $\mathcal{G}$ - compact (generalized compact) .*

Definition 2.2.4. [13] *Let $(X, \mathcal{G})$ and $(Y, \mathcal{G}')$ be two generalized topological spaces, thenthefunction$f : X \to Y$ is called $(\mathcal{G}, \mathcal{G}')$ continuous on X, if for any $\mathcal{G}'$ - open set O in Y, $f^{-1}(O)$ is $\mathcal{G}$ - open in X.*

Definition 2.2.5. [13] *A function $f : X \to Y$ is called a $\mathcal{G}$ - homeomorphism, if both f and f^{-1} are $\mathcal{G}$ - continuous. If we have a $\mathcal{G}$ - homeomorphism between X and Y, we say that they are $\mathcal{G}$ - homeomorphic.*

Theorem 2.2.6. [31] *$\mathcal{G}$ - continuous image of a $\mathcal{G}$ - compact set is $\mathcal{G}$ - compact.*

Theorem 2.2.7. [25] *$\mathcal{B} \subset P(X)$ is a base for a generalized topology $\mathcal{G}$ if and only if whenever U is a $\mathcal{G}$ - open set such that $x \in U$, then there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.*

Definition 2.2.8. [23] *Let $(X, \mathcal{G}_X)$ and $(Y, \mathcal{G}_Y)$ be two generalized topological spaces. The product $\mathcal{G}$ - topology on $X \times Y$ is the $\mathcal{G}$ - topology having as a basis the collection $\mathcal{B}$ of all sets of the form $U \times V$ where $U \in \mathcal{G}_X$ and $V \in \mathcal{G}_Y$.*

Definition 2.2.9. [5] *Let $(X, \mathcal{G})$ be a generalized topological space. X is called $\mathcal{G}$ - connected if there are no nonempty disjoint $\mathcal{G}$ - open subsets U, V of X such that $U \cup V = X$.*

Definition 2.2.10. [28] *Let $(X, \mathcal{G})$ be a generalized topological space and Y be a subset of X then $\mathcal{G}_Y = \{U \cap Y : U \in \mathcal{G}\}$ is a generalized topology on Y and it is called the subspace generalized topology on Y.*

2.3 Some Results on $\mathcal{G}$ - Connectedness

Theorem 2.3.1. *The $\mathcal{G}$ - connectedness is preserved under $\mathcal{G}$ - continuous functions.*

Proof. Let $f : X \to Y$ be a $\mathcal{G}$ continuous onto function. Assume that X is $\mathcal{G}$ - connected. To show that Y is $\mathcal{G}$ - connected. On the contrary we that there are two disjoint $\mathcal{G}$ - open subsets U, V of Y such that $U \cup V = Y$. Then

$X = f^{-1}(Y) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. Thus X can be expressed as union of two disjoint $\mathcal{G}$ - open subsets of X, which is a contradiction. Hence Y is $\mathcal{G}$ - connected. □

Theorem 2.3.2. *Suppose X and Y are $\mathcal{G}$ - homeomorphic generalized topological spaces. Then X and Y are either both $\mathcal{G}$ connected or both not $\mathcal{G}$ - connected.*

Proof. Given that $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ are $\mathcal{G}$ - homeomorphic. So there exists a mapping $f : X \to Y$, which is one-one, onto, $(\mathcal{G}_1, \mathcal{G}_2)$ - continuous and its inverse function is also $(\mathcal{G}_2, \mathcal{G}_1)$ continuous. If X is $\mathcal{G}$- connected, using theorem 2.3.1 we get Y is $\mathcal{G}$ - connected. If X is not $\mathcal{G}$ - connected then apply theorem 2.3.1 on the inverse function f^{-1}, we see that Y is not $\mathcal{G}$ - connected. □

Theorem 2.3.3. *A union of two intersecting $\mathcal{G}$ - connected subspaces is $\mathcal{G}$ - connected. That is, suppose $X = U \cup V$, where U, V are both $\mathcal{G}$ - connected, and $U \cap V \neq \phi$. Then X is $\mathcal{G}$- connected.*

Proof. Assume X is not $\mathcal{G}$- connected and $X = A \cup B$, where A and B are nonempty, disjoint and $\mathcal{G}$ - open subsets. Pick a point $v \in U \cap V$. We can assume $v \in A$. Now consider $U \cap A$ and $U \cap B$. They are $\mathcal{G}$ - open subsets in the subspace generalized topology $\mathcal{G}_U$. Also $U \cap A$ contains v and so is nonempty. So if $U \cap B \neq \phi$ then we get U is not $\mathcal{G}$ - connected. So we must have $U \cap B$ is empty. So $U \subset A$. Using the same arguments if $v \in B$. Now consider $V \cap A$ and $V \cap B$. They are $\mathcal{G}-$ open subsets in the subspace generalized topology $\mathcal{G}_v$.Also $V \cap A$ contains v and so is nonempty. So if $V \cap B \neq \phi$ then we get V

is not $\mathcal{G}$ - connected. So we must have $V \cap B$ is empty. Hence we get $V \subset A$. But then $X = U \cup V$ is contained in A and so B must be empty. This is a contradiction. □

Theorem 2.3.4. *Let $(X, \mathcal{G})$ be a generalized topological space, then X is $\mathcal{G}$ - connected if and only if there are no two nonempty subsets A, B of X such that $X = A \cup B$ and $C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = \phi$.*

Proof. Assume X is $\mathcal{G}$ - connected. If possible there exists two nonempty subsets A, B of X such that $X = A \cup B$ and $C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = \phi$. Taking complements and apply Demorgan's law on both sides of the equation $C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = \phi$, implies that $C_{\mathcal{G}}(A)^c \cup C_{\mathcal{G}}(B)^c = X$.

Also $C_{\mathcal{G}}(A)^c \cap C_{\mathcal{G}}(B)^c = \phi$, for if there exists $z \in X$ such that $z \in C_{\mathcal{G}}(A)^c \cap C_{\mathcal{G}}(B)^c$, which implies that $z \notin C_{\mathcal{G}}(A)$ and $z \notin C_{\mathcal{G}}(B)$ implies $z \notin C_{\mathcal{G}}(A) \cup C_{\mathcal{G}}(B) \supset A \cup B = X$ implies $z \notin X$. This become a contradiction. Hence X can be expressed as union of two disjoint nonempty $\mathcal{G}$ - open sets. This is a contradiction to the fact that X is $\mathcal{G}$ - connected.

Conversely, assume that there are no two nonempty subsets A, B of X such that $X = A \cup B$ and $C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = \phi$. If possible let X is not $\mathcal{G}$ - connected. Then there exist two nonempty disjoint $\mathcal{G}$ - open sets A, B of X such that $X = A \cup B$. So $A = X - B$ is $\mathcal{G}$ - closed. Similarly B is $\mathcal{G}$ - closed. Hence $C_{\mathcal{G}}(A) = A$ and $C_{\mathcal{G}}(B) = B$. So there are two nonempty subsets A, B of X such that $X = A \cup B$ and $C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = \phi$, which is a contradiction to our assumption. Hence X is $\mathcal{G}$ - connected. □

Definition 2.3.5. *Let $(X, \mathcal{G})$ be a generalized topological space, $x_0 \in X$ and $N \subset X$. Then x_0 is said to be a generalized interior point of N, if there is a $\mathcal{G}$ - open set V such that $x_0 \in V \subset N$. The set of all generalized interior points of N is called the generalized interior of N and is denoted by $int_{\mathcal{G}}(N)$.*

Example 2.3.6. *Consider $X = \mathbb{R}$. If $\mathcal{G} = \{U : U \subset \mathbb{R} - \mathbb{Z}\}$, then $\mathcal{G}$ is a generalized topology on X and $\frac{1}{2}$ is a generalized interior point of the set of rational numbers $\mathbb{Q}$, because $\{\dfrac{1}{2}\}$ is a $\mathcal{G}$ - open set contained in $\mathbb{Q}$. Using the same arguments, we can prove that any number belongs to $\mathbb{Q} - \mathbb{Z}$ is a generalized interior point of $\mathbb{Q}$*

Also $\sqrt{2}$ is not a generalized interior point of $\mathbb{Q}$, because there is no $\mathcal{G}$ - open set containing $\sqrt{2}$ and contained in $\mathbb{Q}$. By the similar arguments we can prove that any number belongs to $(\mathbb{Q} - \mathbb{Z})^c$ is not a generalized interior point of $\mathbb{Q}$. Hence the generalized interior of Q is $\mathbb{Q} - \mathbb{Z}$.

Theorem 2.3.7. *Let $(X, \mathcal{G})$ be a generalized topological space and $A \subset X$. Then $int_{\mathcal{G}}(A)$ is the union of all $\mathcal{G}$ - open sets contained in A. It is also the largest $\mathcal{G}$ - open subset of X contained in A.*

Proof. Let $\mathcal{M}$ be the family of all $\mathcal{G}$ - open subsets of X contained in A. Clearly $\mathcal{M}$ is nonempty because $\phi \in \mathcal{M}$. Let $V = \cup_{M \in \mathcal{M}} M$.
Claim: $V = int_{\mathcal{G}}(A)$.

Let $x \in V$. Then $x \in M$ for some $M \in \mathcal{M}$. That is $x \in M \subset A$ implies $x \in int_{\mathcal{G}}(A)$.

Conversely, let $x \in int_{\mathcal{G}}(A)$. Then there is a $\mathcal{G}$ - open set H such that

$x \in H \subset A$. Then $H \in \mathcal{M}$ and so $H \subset V$. So $x \in V$, that $V = int_{\mathcal{G}}(A)$.

Suppose G is any $\mathcal{G}$ - open set contained in A. Then $G \in \mathcal{M}$ and so $G \subset int_{\mathcal{G}}(A)$. Hence $int_{\mathcal{G}}(A)$ is the largest $\mathcal{G}$ - open set contained in A. □

Definition 2.3.8. *Let A be a subset of a generalized topological space X. Then its generalized boundary is the set $C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(X - A)$. It is denoted by $\partial_{\mathcal{G}}(A)$*

Theorem 2.3.9. *The generalized boundary of a set is always a $\mathcal{G}$ - closed set.*

Proof. Let $G = C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(X - A)$. To prove that G is $\mathcal{G}$ - closed. It is enough to prove that G^c is $\mathcal{G}$ - open.

But $G^c = (C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(X - A))^c = (C_{\mathcal{G}}(A))^c \cup (C_{\mathcal{G}}(X - A))^c$. That is G^c is the union of two $\mathcal{G}$ - open sets. So G^c is $\mathcal{G}$ - open. □

Remark 2.3.10. *The generalized boundary of a set is same as the generalized boundary of its complement.*

Theorem 2.3.11. *Let X be a generalized topological space and $A \subset X$. Then $int_{\mathcal{G}}(A) \cup (C_{\mathcal{G}}(X - A)) = X$.*

Proof. Claim: $X - int_{\mathcal{G}}(A) = C_{\mathcal{G}}(X - A)$.

Let O be any $\mathcal{G}$ - open subset of X such that $O \subset A$. Then $(X - A) \subset (X - O)$. Since O is $\mathcal{G}$ - open, we see that $X - O$ is $\mathcal{G}$ - closed. But $\mathcal{G}$ - closure of a set is the smallest $\mathcal{G}$ - closed set containing it. So by the minimality, $C_{\mathcal{G}}(X - A) \subset C_{\mathcal{G}}(X - O) = X - O$. Taking $O = int_{\mathcal{G}}(A)$. Then we get $C_{\mathcal{G}}(X - A) \subset (X - int_{\mathcal{G}}(A))$............(1).

For the reverse inclusion, let C be a $\mathcal{G}$ - closed set such that $(X-A) \subset C$. Since C is $\mathcal{G}$ - closed, we get $X-C$ is $\mathcal{G}$ - open and $(X-C) \subset A$. Since the generalized interior of a set is the largest $\mathcal{G}$ - open subset of that set, we get $(X-int_{\mathcal{G}}(A)) \subset C$. Now taking $C = C_{\mathcal{G}}(X-A)$, we get $(X-int_{\mathcal{G}}(A)) \subset C_{\mathcal{G}}(X-A)$...........(2)
From (1) and (2), we get $(X-int_{\mathcal{G}}(A)) = C_{\mathcal{G}}(X-A)$............(3)

Hence $int_{\mathcal{G}}(A) \cup (C_{\mathcal{G}}(X-A)) = int_{\mathcal{G}}(A) \cup (X-int_{\mathcal{G}}(A)) = X$.

□

Theorem 2.3.12. *Let A be a subset of a generalized topological space X. Then $C_{\mathcal{G}}(A)$ is the disjoint union of $int_{\mathcal{G}}(A)$ with the generalized boundary of A.*

Proof. Claim: $C_{\mathcal{G}}(A) = int_{\mathcal{G}}(A) \cup \partial_{\mathcal{G}}A$

We have $int_{\mathcal{G}}(A) \cup \partial_{\mathcal{G}}(A)$

$= int_{\mathcal{G}}(A) \cup (C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(X-A))$

$= [int_{\mathcal{G}}(A) \cup C_{\mathcal{G}}(A)] \cap [int_{\mathcal{G}}(A) \cup (C_{\mathcal{G}}(X-A))]$

$= (C_{\mathcal{G}}(A)) \cap X$ [from theorem 2.3.11]

$= C_{\mathcal{G}}(A)$ □

Theorem 2.3.13. *Let $(X, \mathcal{G})$ be a generalized topological space and $A \subset X$. Then A is $\mathcal{G}$ - closed if and only if it contains its generalized boundary and A is $\mathcal{G}$ - open if and only if it is disjoint from its generalized boundary.*

Proof. Assume A is $\mathcal{G}$ - closed. Then $C_{\mathcal{G}}(A) = A$. We have $\partial_{\mathcal{G}}(A) = C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(X-A) \subset C_{\mathcal{G}}(A) = A$.

Conversely suppose that $\partial_{\mathcal{G}}(A) \subset A$. It is enough to prove $C_{\mathcal{G}}(A) \subset A$. From theorem 2.3.11, $C_{\mathcal{G}}(A) = int_{\mathcal{G}}(A) \cup \partial_{\mathcal{G}}(A) \subset A$. Hence $C_{\mathcal{G}}(A) = A$. So A is $\mathcal{G}$ - closed.

Assume A is $\mathcal{G}$ - open. Then

$$\partial_{\mathcal{G}}(A) \cap A = [C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(X - A)] \cap A$$
$$= [C_{\mathcal{G}}(A) \cap (X - A)] \cap A = \phi.$$

Conversely assume that $\partial_{\mathcal{G}}(A) \cap A = \phi$. Then $\partial_{\mathcal{G}}(A) \subset X - A$. But the generalized boundary of A is same as the generalized boundary of $X - A$. So the generalized boundary of $X - A$ is contained in $(X - A)$. Hence by first part $(X - A)$ is $\mathcal{G}$ - closed. So A is $\mathcal{G}$ - open. □

Theorem 2.3.14. *Let $(X, \mathcal{G})$ be a generalized topological space and $A \subset X$. Then A is both $\mathcal{G}$ - closed and $\mathcal{G}$ - open (i.e $\mathcal{G}$ - clopen) if and only if $\partial_{\mathcal{G}} A = \phi$.*

Proof. A is both $\mathcal{G}$ - closed and $\mathcal{G}$ - open implies $\partial_{\mathcal{G}}(A) \subset A$ and $(\partial_{\mathcal{G}}(A)) \cap A = \phi$ (By theorem 2.3.12). Hence $\partial_{\mathcal{G}}(A) = \phi$. Conversely assume that $\partial_{\mathcal{G}}(A) = \phi$. Then $\partial_{\mathcal{G}}(A) \subset A$ and $(\partial_{\mathcal{G}} A) \cap A = \phi$. Hence by theorem 2.3.12, A is both $\mathcal{G}$ - closed and $\mathcal{G}$ - open. □

Example 2.3.15. *Consider $X = \mathbb{R}$. If $\mathcal{G} = \{U : U \subset \mathbb{R} - \mathbb{Z}\}$, then $\mathcal{G}$ is a generalized topology on X and the generalized boundary of $\mathbb{Z}$ is $C_{\mathcal{G}}(\mathbb{Z}) \cap C_{\mathcal{G}}(\mathbb{R} - \mathbb{Z}) = \mathbb{Z} \cap \mathbb{R} = \mathbb{Z}$*

Theorem 2.3.16. *Let $(X, \mathcal{G})$ be a generalized topological space and A, B are subsets of X. Then the given statements are equivalent:*

1. *$A \cup B = X$ and $C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = \phi$.*

2. *$A \cup B = X$ and $A \cap B = \phi$ and A, B are both $\mathcal{G}$ - closed in X.*

3. *$B = X - A$ and A, B are both $\mathcal{G}$ - open and $\mathcal{G}$ - closed in X.*

4. *$B = X - A$ and the generalized boundary of A is empty .*

5. *$A \cup B = X$ and $A \cap B = \phi$ and A, B are both $\mathcal{G}$ - open in X.*

Proof. **(1) implies (2)**.

$C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = \phi$ implies $A \cap B = \phi$ as $A \subset C_{\mathcal{G}}(A)$ and $B \subset C_{\mathcal{G}}(B)$. Also $C_{\mathcal{G}}(A) \subset X - C_{\mathcal{G}}(B) \subset X - B = A$. So $C_{\mathcal{G}}(A) = A$ as $A \subset C_{\mathcal{G}}(A)$. Hence A is $\mathcal{G}$ - closed in X.
Similarly $C_{\mathcal{G}}(B) \subset X - C_{\mathcal{G}}(A) \subset X - A = B$. So $C_{\mathcal{G}}(B) = B$ as $B \subset C_{\mathcal{G}}(B)$ and we get B is $\mathcal{G}$ - closed in X.

(2) implies (3).

By assumption A, B are both $\mathcal{G}$ - closed in X. So both $X - A$ and $X - B$ are $\mathcal{G}$ - open in X. But $X - A = B$ and $X - B = A$. So both A and B are $\mathcal{G}$ - open in X . Hence both A and B are $\mathcal{G}$ - open and $\mathcal{G}$ - closed in X.

(3) implies (4).

Since A is $\mathcal{G}$ - closed in X, we have $C_{\mathcal{G}}(A) = A$. Since $B = X - A$ is $\mathcal{G}$ - closed in X, we get $C_{\mathcal{G}}(X - A) = B$. So the generalized boundary of $A = \partial_{\mathcal{G}}(A) = C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(X - A) = A \cap B = \phi$.

(4) implies (5)

Assume $B = X - A$ and the generalized boundary of A is empty. So we see that $A \cup B = X$ and $A \cap B = \phi$. Since $\partial_{\mathcal{G}}(A) = \phi \subset A$, by theorem 2.3.13, A is $\mathcal{G}$ - closed in X. Also $A \cap \partial_{\mathcal{G}}(A) = \phi$. By theorem 2.3.13, we see that A is $\mathcal{G}$ - open. Since $B = X - A$ and A is both $\mathcal{G}$ - closed and $\mathcal{G}$ - open, we get B is both $\mathcal{G}$ - open and $\mathcal{G}$ - closed in X.

(5) implies (1)

Assume $A \cup B = X$ and $A \cap B = \phi$ and A, B are both $\mathcal{G}$ - open in X. Then $A = X - B$ and $B = X - A$. Hence A, B are both $\mathcal{G}$ - open and $\mathcal{G}$ - closed in X. So $C_{\mathcal{G}}(A) = A$ and $C_{\mathcal{G}}(B) = B$. Hence $C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = A \cap B = \phi$. □

Theorem 2.3.17. *If X is not a strong generalized topological space, then X must be $\mathcal{G}$ - connected.*

Proof. If X is not a strong generalized topological space, then X is not $\mathcal{G}$ - open. So X cannot be expressed as disjoint union of two nonempty $\mathcal{G}$ - open sets in X. □

Theorem 2.3.18. *Let $(X, \mathcal{G})$ be a generalized topological space. Then the following statements are equivalent:*

1. *X is $\mathcal{G}$ - connected.*

2. *X cannot be written as the disjoint union of two nonempty $\mathcal{G}$ - closed subsets of X.*

3. *Every nonempty proper subset of X has a nonempty generalized boundary.*

4. *X cannot be written as the disjoint union of two non empty $\mathcal{G}$ - open subsets of X.*

Proof. (1) implies (2)

Suppose X is $\mathcal{G}$ - connected. By the definition of $\mathcal{G}$ - connectedness, X cannot be written as the disjoint union of two non empty $\mathcal{G}$ - open sets. We want to prove that X cannot be written as the disjoint union two non empty $\mathcal{G}$ - closed subsets of X.

If possible let assume that there are two non empty disjoint $\mathcal{G}$ - closed sets A, B such that $X = A \cup B$. Since A, B are $\mathcal{G}$ - closed sets, we have $C_{\mathcal{G}}(A) = A$ and $C_{\mathcal{G}}(B) = B$. Since $A \cap B = \phi, C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = \phi$. So we get $X = A \cup B$ and $C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = \phi$. By theorem 2.3.16, we get A, B are both $\mathcal{G}$ - open in X. Hence X can be expressed as disjoint union of two non empty $\mathcal{G}$ - open sets, which is a contradiction to our assumption that X is $\mathcal{G}$ - connected. So X cannot be written as the disjoint union of two non empty $\mathcal{G}$ - closed subsets of X.

(2) implies (3)

Suppose X cannot be written as the disjoint union of two non empty $\mathcal{G}$ - closed subsets of X. To prove that every nonempty proper subsets of X has non empty generalized boundary. Let A be a non empty proper subset of X. We want to prove that $\partial_{\mathcal{G}}(A) \neq \phi$.

If possible let $\partial_{\mathcal{G}}(A) = \phi$. So A contains its generalized boundary. By theorem 2.3.13, we get A is $\mathcal{G}$ - closed in X. Also we know that A and A^c have same generalized boundary. So $\partial_{\mathcal{G}}(A^C) = \phi$. Again by theorem 2.3.13, we get A^c is $\mathcal{G}$ - closed. That is $X = A \cup A^c$ and both A and A^c are both proper non empty disjoint $\mathcal{G}$ - closed subsets of X. This is a contradiction to our assumption. Hence $\partial_{\mathcal{G}}(A) \neq \phi$.

(3) implies (4)

Suppose every nonempty proper subset of X has a nonempty generalized boundary. To prove that X cannot be written as the disjoint union of two non empty $\mathcal{G}$ - open subsets of X.

If possible, there are two non empty $\mathcal{G}$- open subsets A, B of X such that $X = A \cup B$. By theorem 2.3.16, the generalized boundary of A is empty. This is a contradiction to our assumption that every non empty proper subset of X has a non empty boundary.

(4) implies (1) .

Assume (4). That is X cannot be written as the disjoint union of two non empty $\mathcal{G}$ -open subsets of X. By the definition of $\mathcal{G}$ - connectedness, we get X is $\mathcal{G}$ - connected.

□

Definition 2.3.19. *Two subsets A and B of a generalized topological space X are said to generalized separated ($\mathcal{G}$ - separated) if $C_{\mathcal{G}}(A) \cap B = \phi$ and $A \cap C_{\mathcal{G}}(B) = \phi$*

Example 2.3.20. *Let $X = \{1, 2, 3\}$. Let $\mathcal{G} = \{X, \phi, \{1, 2\}, \{3\}\}$ be a generalized topology on X. Then $\{1, 2\}$ and $\{3\}$ are $\mathcal{G}$ - separated.*

Example 2.3.21. *Consider $X = \mathbb{R}$, then $\mathcal{G} = \{U : U \subset \mathbb{R} - \mathbb{Z}\}$ is a generalized topology on X. Then $\mathbb{Q} - \mathbb{Z}$ and $\mathbb{Z}$ are not $\mathcal{G}$ - separated subsets of X. Note that $\mathbb{Q}^c$ is $\mathcal{G}$ - open. So $\mathbb{Q}$ is $\mathcal{G}$ - closed. Hence the smallest $\mathcal{G}$ - closed set containing $\mathbb{Q} - \mathbb{Z}$ is $\mathbb{Q}$. But $\mathbb{Q} \cap \mathbb{Z} \neq \phi$.*

Theorem 2.3.22. *Let Y be a generalized subspace of X. Then a set C is $\mathcal{G}$ - closed in Y if and only if it equals the intersection of a $\mathcal{G}$ - closed set of X with Y.*

Proof. Let $A = C \cap Y$. Since C is $\mathcal{G}$ - closed in X, we have $X - C$ is $\mathcal{G}$ - open in X. So $(X - C) \cap Y = Y - A$. So $Y - A$ is $\mathcal{G}$ - open in Y. That is A is $\mathcal{G}$ - closed in Y. Hence $C \cap Y$ is $\mathcal{G}$ - closed in Y.

Conversely suppose that A is a $\mathcal{G}$ - closed subset of Y. To prove that $A = C \cap Y$, where C is $\mathcal{G}$ - closed in X. A is $\mathcal{G}$ - closed in Y implies that $Y - A$ is $\mathcal{G}$ - open in Y. So $Y - A = B \cap Y$, where B is $\mathcal{G}$ - open in X. But $A = (X - B) \cap Y$ and $(X - B)$ is $\mathcal{G}$ - closed in X. So we take $C = X - B$

□

Theorem 2.3.23. *Let X be a generalized topological space and A, B are subsets of X. Then A and B are $\mathcal{G}$ - separated in $A \cup B$ if and only if they are disjoint $\mathcal{G}$ - closed subsets of $A \cup B$ with subspace generalized topology in $A \cup B$.*

Proof. Suppose A, B are $\mathcal{G}$ - separated subsets of X. So $C_{\mathcal{G}}(A) \cap B = \phi$ and $A \cap C_{\mathcal{G}}(B) = \phi$. But $A = C_{\mathcal{G}}(A) \cap (A \cup B)$ and $B = C_{\mathcal{G}}(B) \cap (A \cup B)$. Since $C_{\mathcal{G}}(A)$ and $C_{\mathcal{G}}(B)$ are $\mathcal{G}$ - closed subsets of X and by using theorem 2.3.22, we get A and B are $\mathcal{G}$ - closed subsets of $A \cup B$.

Claim: $A \cap B = \phi$

We have $A \cap B = (C_{\mathcal{G}}(A) \cap (A \cup B)) \cap (C_{\mathcal{G}}(B) \cap (A \cup B))$

$= [(C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B)) \cap A] \cup [C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) \cap B] = \phi.$

Conversely suppose that A, B are two disjoint $\mathcal{G}$ - closed subsets of $A \cup B$. To prove that A, B are $\mathcal{G}$- separated subsets of $A \cup B$. Since A and B are $\mathcal{G}$ - closed subsets $A \cup B$, we have $C_{\mathcal{G}}(A) = A$ and $C_{\mathcal{G}}(B) = B$. So $C_{\mathcal{G}}(A) \cap B = A \cap B = \phi = A \cap C_{\mathcal{G}}(B)$. Therefore A and B are $\mathcal{G}$ - separated subsets $A \cup B$. □

Theorem 2.3.24. *A generalized topological space is $\mathcal{G}$ - connected if it is not the union of two nonempty $\mathcal{G}$ - separated subsets.*

Proof. Using theorem 2.3.17, a generalized topological space X is $\mathcal{G}$ - connected if and only if it cannot be written as the union of two disjoint $\mathcal{G}$ - closed subsets of X if and only if X is not the union of two nonempty $\mathcal{G}$ - separated subsets of X by theorem 2.3.23 .

□

Theorem 2.3.25. *Let X be a generalized topological space and C be a $\mathcal{G}$ - connected subset of X. Suppose $C \subset A \cup B$ where A and B are $\mathcal{G}$ - separated subsets of X. Then either $C \subset A$ or $C \subset B$.*

Proof. Let $G = C \cap A$ and $H = C \cap B$. From theorem 2.3.24 we see that both G and H are $\mathcal{G}$ - closed subsets of $G \cup H$. But $G \cup H = (C \cap A) \cup (C \cap B) = C \cap (A \cup B) = C$. Also $G \cap H = (C \cap A) \cap (C \cap B) = C \cap (A \cap B) = C \cap \phi = \phi$. So G and H are two disjoint subsets of C. Therefore G and H are $\mathcal{G}$ - closed in C with respect to the subspace generalized topology in C. Also $G \cap H = (C \cap A) \cap (C \cap B) = C \cap (A \cap B) = C \cap \phi = \phi$. Given that C is a $\mathcal{G}$ - connected subset of X. So C cannot be expressed as the union of two disjoint $\mathcal{G}$ - closed subsets of X. Hence we must have either $G = \phi$ or $H = \phi$. So in the first case $C \subset B$ while in the second case $C \subset A$. □

Theorem 2.3.26. *Let $\mathcal{C}$ be a collection of $\mathcal{G}$ - connected subsets of X such that no two members of $\mathcal{C}$ are $\mathcal{G}$ - separated. Then $\cup_{C \in \mathcal{C}} C$ is also $\mathcal{G}$ - connected.*

Proof. Let $M = \cup_{C \in \mathcal{C}} C$. If M is not $\mathcal{G}$ - connected, then by theorem 2.3.24 we can write M as $A \cup B$ where A, B are nonempty and $\mathcal{G}$ - separated subsets of X. By theorem 2.3.25, for each $C \in \mathcal{C}$ either $C \subset A$ or $C \subset B$.

Claim: $C \subset A$ for all $C \in \mathcal{C}$ or $C \subset B$ for all $C \in \mathcal{C}$.

If not, then there exists $C, D \in \mathcal{C}$ such that $C \subset A$ and $D \subset B$. But A, B are $\mathcal{G}$ - separated subsets of X. Hence their subsets are also $\mathcal{G}$ - separated subsets. This is a contradiction to the fact that no two members of $\mathcal{C}$ are $\mathcal{G}$ - separated. Thus either all members of $\mathcal{C}$ are contained in A or all members of $\mathcal{C}$ are contained in B. So either $M = A$ or $M = B$, contradicting that A, B are nonempty. So M cannot be expressed as union of two nonempty disjoint $\mathcal{G}$ - separated subsets of X. By theorem 2.3.24 we get M is $\mathcal{G}$ - connected. □

Theorem 2.3.27. *Let $\mathcal{C}$ be a collection of $\mathcal{G}$ - connected subsets of a generalized topological space X and K be a $\mathcal{G}$ - connected subset of X (not necessarily a member $\mathcal{C}$) such that $C \cap K \neq \phi$ for all $C \in \mathcal{C}$. Then $(\cup_{C\in\mathcal{C}} C) \cup K$ is $\mathcal{G}$ - connected.*

Proof. Let $M = (\cup_{C\in\mathcal{C}} C) \cup K$. Let $\mathcal{H} = \{K \cup C : C \in \mathcal{C}\}$. Clearly $M = \cup_{H\in\mathcal{H}} H$. By theorem 2.3.26 each member of $\mathcal{H}$ is $\mathcal{G}$ - connected, as it is a union of two $\mathcal{G}$ - connected sets in which their intersection is nonempty. Now any two members of $\mathcal{H}$ contains points of K. So any two members of $\mathcal{H}$ are $\mathcal{G}$ - separated. Hence by theorem 2.3.26 M is $\mathcal{G}$ - connected. $\square$

Theorem 2.3.28. *Let X_1, X_2 be generalized topological space and $X = X_1 \times X_2$ with $\mathcal{G}$ - product topology. Then X is $\mathcal{G}$ - connected.*

Proof. If either X_1 or X_2 is empty then so is X and the result hold trivially. So assume both X_1 and X_2 are non empty. Fix a point $y \in X_1$. Then the set $\{y\} \times X_2$ is $\mathcal{G}$ - homeomorphic X_2 and hence is $\mathcal{G}$ - connected. Let $K = \{y\} \times X_2$. Using the same arguments, we see that for each $x \in X_2$, the set $X_1 \times \{x\}$ is $\mathcal{G}$ - connected and its intersection with K is nonempty. Also note that $X_1 \times X_2 = (\cup_{x\in X_2}(X_1 \times \{x\})) \cup K$. By theorem 2.3.27 $X_1 \times X_2$ is $\mathcal{G}$ - connected. $\square$

Remark 2.3.29. *In an ordinary topological space closure of a connected set is connected. But in generalized topological space the $\mathcal{G}$ - closure of a $\mathcal{G}$ - connected set is need not be $\mathcal{G}$ - connected.*

Example 2.3.30. *Let* $X = \{a, b, c, d\}$

and $\mathcal{G} = \{X, \phi, \{a, b\}, \{b, c\}, \{a, b, c\}, \{c, d\}, \{b, c, d\}\}$*. Let* $A = \{a, b, c\}$*. Then* A *is* $\mathcal{G}$ *- connected and* $C_{\mathcal{G}}(A) = X$*. But* $X = \{a, b\} \cup \{c, d\}$ *and so* X *is not* $\mathcal{G}$ *- connected. That is* $C_{\mathcal{G}}(A)$ *is not* $\mathcal{G}$ *- connected.*

Definition 2.3.31. *A generalized component* ($\mathcal{G} - component$) *of a generalized topological space is a maximally* $\mathcal{G}$ *- connected subset which is not properly contained in any* $\mathcal{G}$ *- connected subset of that generalized topological space.*

Remark 2.3.32. *In ordinary topological spaces components are closed but in generalized topological spaces the* $\mathcal{G}$ *- components need not be* $\mathcal{G}$ *- closed.*

Example 2.3.33. *Let* $X = \{a, b, c, d\}$ *and* $\mathcal{G} = \{\phi, \{a, b\}, \{b, c\}, \{a, b, c\}\}$*.*
Then $(X, \mathcal{G})$ *is a generalized topological space.*

Let $A = \{a, b, c\}$*.*

Then A *is a* $\mathcal{G}$ *- connected subset of* X *and it is a* $\mathcal{G}$ *- component of* X*. But* A *is not* $\mathcal{G}$ *- closed.*

Theorem 2.3.34. *Any two distinct* $\mathcal{G}$ *- components are mutually disjoint.*

Proof. Let C, C' be two $\mathcal{G}$ - components of a generalized topological space X. If $C \cap C'$ is nonempty then by theorem 2.3.27, we see that $C \cup C'$ is $\mathcal{G}$ - connected. But $C \subset C \cup C'$ and $C' \subset C \cup C'$. Then by maximality of C and C'. So $C = C \cup C'$ and $C' = C \cup C'$. That is $C = C'$. Thus two distinct $\mathcal{G}$ - components are mutually disjoint. $\square$

Theorem 2.3.35. *Every nonempty $\mathcal{G}$ - connected subset is contained in a unique $\mathcal{G}$ - component.*

Proof. Let A be a nonempty $\mathcal{G}$ - connected subset of a generalized topological space X. Let $\mathcal{C}$ be the collection of all $\mathcal{G}$ - connected subsets of X containing A and let $M = \cup_{C \in \mathcal{C}} C$. Since each member of $\mathcal{C}$ containing A, any two members of $\mathcal{C}$ intersect. So by theorem 2.3.26, we get M is $\mathcal{G}$ - connected. Clearly $A \subset M$.

Claim: M is a $\mathcal{G}$ - component.

Suppose N is a $\mathcal{G}$ - connected subset of X containing M. Then $A \subset N$ and $N \in \mathcal{C}$. So $N \subset M$. So $A \subset N$ and hence $N \in \mathcal{C}$. Therefore $N \subset M$. Hence $M = N$ and M is a maximally $\mathcal{G}$ - connected subset of X. Thus every nonempty subset is contained in a $\mathcal{G}$ - component. Such a $\mathcal{G}$ - component is unique, since two distinct $\mathcal{G}$ - components are disjoint. $\square$

Theorem 2.3.36. *Let $\mathcal{B}$ be a base for a generalized topology $\mathcal{G}$ on a set X and also let $Y \subset X$ and $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$. Then $\mathcal{B}_Y$ is a base for the subspace generalized topology $\mathcal{G}_Y$ on Y.*

Proof. Let $y \in Y$ and U be a $\mathcal{G}$ - open set in Y containing y. Then $U = H \cap Y$ for some $\mathcal{G}$ - open set H in X. Since $y \in U$, we get $y \in H$. Since $\mathcal{B}$ is a bases for $\mathcal{G}$ and H is a $\mathcal{G}$ - open set containing y , by theorem 2.2.7 there exist $B \in \mathcal{B}$ such that $y \in B \subset H$. Then $y \in B \cap Y \subset H \cap Y = U$ and $B \cap Y \in \mathcal{B}_Y$. Hence again by theorem 2.2.7, $\mathcal{B}_Y$ is a base for subspace generalized topology $\mathcal{G}_Y$ on Y. $\square$

Theorem 2.3.37. *If A is a subset of a generalized topological space X, then*

$(X - \partial_{\mathcal{G}}(A)) = int_{\mathcal{G}}(A) \cup (int_{\mathcal{G}}A^c)$.

Proof. Let $x \in (X - \partial_{\mathcal{G}}(A))$. Then by definition, x cannot be both in $C_{\mathcal{G}}(A)$ and $C_{\mathcal{G}}(X - A)$. This means that x has a $\mathcal{G}$ - neighbourhood which entirely in A or x has a $\mathcal{G}$ - neighbourhood which entirely in $(X - A)$. Therefore either $x \in int_{\mathcal{G}}(A)$ or $x \in int_{\mathcal{G}}(X - A)$.
Thus $(X - \partial_{\mathcal{G}}(A)) \subset int_{\mathcal{G}}(A) \cup (int_{\mathcal{G}}A^c)$..................(1).

Let $y \in int_{\mathcal{G}}(A) \cup (int_{\mathcal{G}}A^c)$. Then y is a $\mathcal{G}$ - interior point of A or y is a $\mathcal{G}$ - interior point of $(X - A)$. So either y has a $\mathcal{G}$ - neighbourhood which lies entirely in A or y has a $\mathcal{G}$ - neghbourhood which lies entirely in $(X - A)$. Then by definition of generalized boundary of $A, y \notin \partial_{\mathcal{G}}(A)$.
Thus $int_{\mathcal{G}}(A) \cup (int_{\mathcal{G}}A^c) \subset (X - \partial_{\mathcal{G}}(A))$.................(2)

From (1) and (2) we get $(X - \partial_{\mathcal{G}}(A)) = int_{\mathcal{G}}(A) \cup (int_{\mathcal{G}}A^c)$. □

Theorem 2.3.38. *Let A be a subset of a generalized topological space X and C be a $\mathcal{G}$ - connected subspace of X that intersect both A and $X - A$. Then C intersects $\partial_{\mathcal{G}}(A)$.*

Proof. If possible $C \cap \partial_{\mathcal{G}}(A) = \phi$. Then $C \subset (X - \partial_{\mathcal{G}}(A))$. By theorem 2.3.37, $C \subset int_{\mathcal{G}}(A) \cup (int_{\mathcal{G}}A^c)$. By theorem 2.3.25, since C is a $\mathcal{G}$ - connected subspace of X either $C \subset int_{\mathcal{G}}(A)$ or $C \subset int_{\mathcal{G}}(X - A)$.

If $C \subset int_{\mathcal{G}}(A)$, then $C \cap (X - A) = \phi$. This is a contradiction, because C intersects $(X - A)$.

If $C \subset int_{\mathcal{G}}(X - A)$, then $C \cap A = \phi$. This is a contradiction.

Hence C must intersect $\partial_{\mathcal{G}}(A)$. $\square$

Theorem 2.3.39. *If X is a strong generalized topological space, then X is $\mathcal{G}$ - connected if and only if the only subsets of X which are both $\mathcal{G}$ - open and $\mathcal{G}$ - closed are X and the empty set.*

Proof. Suppose that X is a strong $\mathcal{G}$ - connected space. Let A be a non empty proper subset of X, which is both $\mathcal{G}$ - open and $\mathcal{G}$ - closed in X. Then X can be expressed as a union of two disjoint nonempty $\mathcal{G}$ - open sets A and $X - A$. This is a contradiction to our assumption that X is $\mathcal{G}$ - connected. Therefore there is no such non empty proper subset A exists. So the only subsets of X which are both $\mathcal{G}$ - open and $\mathcal{G}$ - closed are X and ϕ only.

Conversely suppose that the only subsets of X which are both $\mathcal{G}$ - open and $\mathcal{G}$ - closed are X and the empty set. To prove that X is $\mathcal{G}$ - connected. If not, there are two proper non empty disjoint $\mathcal{G}$ - open sets U and V such that $X = U \cup V$. Then $U = X - V$ is $\mathcal{G}$ -closed as V is $\mathcal{G}$ - open. Thus U is a proper non empty subset of X, which is both $\mathcal{G}$ - open and $\mathcal{G}$ - closed, which is a contradiction to our assumption. Hence X is $\mathcal{G}$ - connected. $\square$

Remark 2.3.40. *In theorem 2.3.39, X must be a strong generalized topological space. Otherwise X is not $\mathcal{G}$ - open and ϕ is not $\mathcal{G}$ - closed.*

Theorem 2.3.41. *Let X be a strong generalized topological space and let Z be a subset of X. If Z is $\mathcal{G}$ - connected and $\mathcal{G}$ - dense in X, then X is $\mathcal{G}$ - connected.*

Proof. To prove this theorem we use theorem 2.3.39. Let A be a nonempty subset of X which is both $\mathcal{G}$ - open and $\mathcal{G}$ - closed in X. Since Z is $\mathcal{G}$ - dense in X, by theorem 1.3.12 Z must intersect every nonempty $\mathcal{G}$ - open subsets of X. Therefore $A \cap Z$ is nonempty. Since X is a strong generalized topological space, we see that Z is strong generalized sub space of X. Now $A \cap Z$ is both $\mathcal{G}$ -open and $\mathcal{G}$ - closed subset of Z. Since Z is $\mathcal{G}$ - connected, by theorem 2.3.39 we get $A \cap Z = Z$. Therefore $Z \subset A$. Hence $X = C_{\mathcal{G}}(Z) \subset C_{\mathcal{G}}(A) = A$. This implies $X = A$. By theorem 2.3.39 we get X is $\mathcal{G}$ - connected. □

Theorem 2.3.42. *If A is a proper non empty subset of a $\mathcal{G}$ - connected space, then $\partial_{\mathcal{G}}(A) \neq \phi$.*

Proof. If possible assume that $\partial_{\mathcal{G}}(A) = \phi$. Then A contains its generalized boundary. So by theorem 2.3.13 A is $\mathcal{G}$ -closed. Also note that A is disjoint from its generalized boundary. Again by theorem 2.3.13 A is $\mathcal{G}$ - open. That is, A is a proper subset of X which is both $\mathcal{G}$ - open and $\mathcal{G}$ - closed. This is a contradiction for the theorem 2.3.39. □

Theorem 2.3.43. *Let A and B be non empty subsets of a generalized topological space X. If $(C_{\mathcal{G}}(A) \cap B) \cup (A \cap C_{\mathcal{G}}(B)) = \phi$. Then $A \cup B$ is not $\mathcal{G}$ - connected*

Proof. Since the union of two sets are empty implies the individual sets are empty and $(C_{\mathcal{G}}(A) \cap B) \cup (A \cap C_{\mathcal{G}}(B)) = \phi$, it follows that $(C_{\mathcal{G}}(A) \cap B) = \phi$ and $(A \cap C_{\mathcal{G}}(B)) = \phi$. This implies that $B \subset X - C_{\mathcal{G}}(A)$ and $A \subset X - C_{\mathcal{G}}(B)$. Therefore $A \cup B \subset (X - C_{\mathcal{G}}(A)) \cup (X - C_{\mathcal{G}}(B))$. Hence $A \cup B = [(X - C_{\mathcal{G}}(A)) \cap (A \cup B)] \cup [(X - C_{\mathcal{G}}(B)) \cap (A \cup B)]$. Thus $A \cup B$ can be expressed as union of two $\mathcal{G}$ -

open sets in $A\cup B$. More over $[(X-C_{\mathcal{G}}(A))\cap(A\cup B)]$ and $[(X-C_{\mathcal{G}}(B))\cap(A\cup B)]$ are disjoint, for if, $y \in [(X - C_{\mathcal{G}}(A)) \cap (A \cup B)] \cap [(X - C_{\mathcal{G}}(B)) \cap (A \cup B)]$, then $y \in (X - C_{\mathcal{G}}(A))$ and $y \in A \cup B$ and $y \in (X - C_{\mathcal{G}}(B))$. Then $y \in A \cup B$ implies $y \in A$ or $y \in B$. If $y \in A$, then $y \in (X - C_{\mathcal{G}}(A))$ which is impossible. Similarly, if $y \in B$ we get a contradiction. Therefore $[(X - C_{\mathcal{G}}(A)) \cap (A \cup B)]$ and $[(X - C_{\mathcal{G}}(B)) \cap (A \cup B)]$ are disjoint. Hence $A \cup B$ is not $\mathcal{G}$ - connected. $\square$

Theorem 2.3.44. *Let X be a generalized topological space and $A \subset X$. Then A is $\mathcal{G}$ - disconnected if and only if there exist $\mathcal{G}$ - open sets O_1 and O_2, such that:*

1. $A \subset O_1 \cup O_2$.
2. $O_1 \cap O_2 \subset (X - A)$, *and*
3. $A \cap O_1 \neq \phi$ *and* $A \cap O_2 \neq \phi$.

Proof. Suppose that A is $\mathcal{G}$ - disconnected. By the definition of $\mathcal{G}$ - connectedness there are two non empty disjoint $\mathcal{G}$ - open sets O_1 and O_2 such that $A = O_1 \cup O_2$. Also note that $O_1 \cap O_2 = \phi \subset (X - A)$ and $O_1 \cap A \neq \phi, O_2 \cap A \neq \phi$, otherwise either O_1 or O_2 become empty.

Conversely suppose that the given three conditions are hold. To prove that A is $\mathcal{G}$ -disconnected. Note that $A \cap O_1$ and $A \cap O_2$ are $\mathcal{G}$ - open in A and $A = (A \cap O_1) \cup (A \cap O_2)$. Also note that $A \cap O_1 and A \cap O_2$ are disjoint for if, $z \in (A \cap O_1) \cap (A \cap O_2)$, then $z \in A$ and $z \in O_1 \cap O_2$. But $O_1 \cap O_2 \subset (X - A)$. Therefore $z \in A$ and $z \in (X - A)$ which is impossible. So A can be expressed as union of two disjoint nonempty $\mathcal{G}$ - open subsets of A. Hence A is a $\mathcal{G}$ - disconnected subset of X. $\square$

Theorem 2.3.45. *Let X and Y be generalized topological spaces. Then $f : X \to Y$ is $\mathcal{G}$ - continuous if and only if the inverse image of each $\mathcal{G}$ - closed set in Y is $\mathcal{G}$ - closed in X.*

Proof. Suppose that f is $\mathcal{G}$ - continuous. Let B be a $\mathcal{G}$ - closed subset of Y. To prove that $f^{-1}(B)$ is $\mathcal{G}$ - closed in X. Let $A = f^{-1}(B)$. We want to prove that $C_{\mathcal{G}}(A) = A$. By elementary set theory we have $f(A) = f(f^{-1}(B)) \subset B$. Clearly $A \subset C_{\mathcal{G}}(A)$. For the reverse inclusion, let $x \in C_{\mathcal{G}}(A)$. Then $f(x) \in f(C_{\mathcal{G}}(A)) \subset C_{\mathcal{G}}(f(A)) \subset C_{\mathcal{G}}(B) = B$. This implies that $x \in f^{-1}(B) = A$. Thus $C_{\mathcal{G}}(A) \subset A$. So that $A = C_{\mathcal{G}}(A)$.

Conversely suppose that the inverse image of a $\mathcal{G}$ - closed set in Y is $\mathcal{G}$ - closed in X. To prove that f is $\mathcal{G}$ - continuous. Let V be a $\mathcal{G}$ - open set in Y. Let $B = Y - V$. Then B is $\mathcal{G}$ - closed in Y. By hypothesis $f^{-1}(B)$ is $\mathcal{G}$ - closed in X. But $f^{-1}(B) = F^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$. Since $f^{-1}(B)$ is $\mathcal{G}$ - closed in X, we get $f^{-1}(V)$ is $\mathcal{G}$ - open in X. $\square$

2.4 Local Connectedness in Generalized Topological Spaces.

In this section we localize the concept of $\mathcal{G}$ - connectedness.

Definition 2.4.1. *A generalized topological space X is said to be $\mathcal{G}$ - locally connected at a point x in it if for every $\mathcal{G}$ - neighbourhood N of x in X there exists a $\mathcal{G}$ - connected neighbourhood M of x such that $M \subset N$. A generalized topological space X is said to be $\mathcal{G}$ - locally connected if it is $\mathcal{G}$ - locally connected at each of its points.*

Theorem 2.4.2. *A generalized topological space X is $\mathcal{G}$ - locally connected if and only if for every $\mathcal{G}$ - open set U of X, each $\mathcal{G}$ - component of U is $\mathcal{G}$ - open in X.*

Proof. Suppose that X is $\mathcal{G}$ - connected. Let U be a $\mathcal{G}$ - open set in X and C be a $\mathcal{G}$ - component of U. Let $x \in C$, we can choose a $\mathcal{G}$ - connected neighbourhood V_x of X such that $V_x \subset U$. Since V_x is $\mathcal{G}$ - connected and C is a $\mathcal{G}$ - component of U, we must have $V \subset C$. Therefore $C = \cup_{x \in C} V_x$. Hence C is $\mathcal{G}$ - open in X.

Conversely suppose that the $\mathcal{G}$ - components of $\mathcal{G}$ - open sets in X are $\mathcal{G}$ -open. Given a point x of X and a $\mathcal{G}$ - neighbourhood U of x, let C be the $\mathcal{G}$ - component of U containing x. By assumption C is $\mathcal{G}$ - open in X. Therefore for each $x \in X$ and each $\mathcal{G}$ - neghbourhood U of x, there is $\mathcal{G}$ - connected neighbourhood U of x such that $x \in C \subset U$. Hence X is $\mathcal{G}$ - locally connected. $\square$

2.5 $\mathcal{G}$ - Lindeloff Spaces

Definition 2.5.1. *Let $(X, \mathcal{G})$ be a generalized topological space. Then X is said to be $\mathcal{G}$ - Lindeloff space if and only if each $\mathcal{G}$ - open cover of X has a countable*

$\mathcal{G}$ - open sub cover.

Example 2.5.2. *Let $X = \mathbb{N}$ and a be a fixed element in $\mathbb{N}$. Define $\mathcal{G} = \{U \subset \mathbb{N} : a \in U\} \cup \{\phi\}$. Then $\mathcal{G}$ - is a generalized topology on X. Since X is countable, it must be $\mathcal{G}$ - Lindeloff.*

Example 2.5.3. *Every countable generalized topological spaces are $\mathcal{G}$ - Lindeloff.*

Remark 2.5.4. *Every $\mathcal{G}$ - compact space is $\mathcal{G}$ - Lindeloff.*

Theorem 2.5.5. *If $(X, \mathcal{G})$ is a countable generalized topological space, then X is $\mathcal{G}$ - Lindeloff.*

Proof. Let $X = \{x_1, x_2, ..., x_n, ...\}$. Let $\mathcal{F}$ be a $\mathcal{G}$ - open cover of X. Then each element of X belongs to at least one of the element of $\mathcal{F}$, say $x_1 \in F_1, x_2 \in F_2, ..., x_n \in F_n, etc.$, where each $F_i \in \mathcal{F}$ for $i = 1, 2, ..., n, ...$. Then $\{F_1, F_2, etc.\}$ is a countable sub collection of $\mathcal{F}$ and which is a $\mathcal{G}$ - open cover of X. Hence X is $\mathcal{G}$ - Lindeloff. □

Theorem 2.5.6. *If $(X, \mathcal{G})$ is a generalized topological space, where $\mathcal{G} = \{U \subset X : (X - U)$ is either countable or all of $X\}$, then X is $\mathcal{G}$ - Lindeloff.*

Proof. Let $\mathcal{F}$ be a $\mathcal{G}$ - open cover of X. Since each member of $\mathcal{F}$ is $\mathcal{G}$ - open, by definition of $\mathcal{G}$, the complement of each member of $\mathcal{F}$ is either countable or all of X. Let F be a nonempty member of $\mathcal{F}$, then $X - F$ is countable. Let $X - F = \{x_1, x_2, ..., x_n, ...\}$. Since $\mathcal{F}$ is a $\mathcal{G}$ - open cover of X, each $x_i, i = 1, 2, ..., n, etc.$, belongs to at least one member of $\mathcal{F}$, say $x_1 \in F_1, x_2 \in F_2, ..., x_n \in F_n, etc.$. Then the collection $\{U, F_1, F_2, F_3, ..., F_n, ...\}$is a countable subcollection of $\mathcal{F}$, which is a $\mathcal{G}$ - open cover of X. Hence X is $\mathcal{G}$ - Lindeloff. $\square$

Theorem 2.5.7. *Let $(X, \mathcal{G})$ be a generalized topological space. If $F_1, F_2, ..., F_n, ...$ are $\mathcal{G}$ - Lindeloff subsets of X, then $\cup_{n=1}^{\infty} F_n$ is $\mathcal{G}$ - Lindeloff. That is, countable union of $\mathcal{G}$ - Lindeloff sets is $\mathcal{G}$ - Lindeloff.*

Proof. Let U and V be any two $\mathcal{G}$ - Lindeloff subsets of X. Let $\mathcal{F}$ be a $\mathcal{G}$ - open cover of $U \cup V$. Then $\mathcal{F}$ is also a $\mathcal{G}$ - open cover of both U and V. So by hypothesis there exists a countable subcollection of $\mathcal{F}$ of $\mathcal{G}$ - open sets, say, $\{U_1, U_2, ..., U_n, ...\}$ and $\{V_1, V_2, ..., V_n, ...\}$ covering U and V respectively. Clearly the collection $\{U_1, U_2, ..., U_n, ..., V_1, V_2, ..., V_n, ...\}$ is a countable collection of $\mathcal{G}$ - open sets covering $U \cup V$. Also it is a subcollection of $\mathcal{F}$. Hence $U \cup V$ is $\mathcal{G}$ - Lindeloff. Using mathematical induction, we see that countable union of $\mathcal{G}$ - Lindeloff sets is $\mathcal{G}$ - Lindeloff. $\square$

Theorem 2.5.8. *Let $(X, \mathcal{G})$ be a generalized topological space and $A \subset X$. Then A is a $\mathcal{G}$ - Lindeloff subset of X if and only if $(A, \mathcal{G}_A)$ is $\mathcal{G}$ - Lindeloff.*

Proof. Assume that A is a $\mathcal{G}$ - Lindeloff subset of X. Let $\mathcal{F}$ be a $\mathcal{G}$ - open cover of $(A, \mathcal{G}_A)$. Then each member F of $\mathcal{F}$ is of the form $H \cap A$, for some $H \in \mathcal{G}$. For each $F \in \mathcal{F}$, fix $D(F) \in \mathcal{G}$ such that $F = D(F) \cap A$. Then the family $\{D(F) : F \in \mathcal{F}\}$ is a $\mathcal{G}$ - open cover of A by members of $\mathcal{G}$. Since A is a $\mathcal{G}$ - Lindeloff subset of X, this open cover has a countable $\mathcal{G}$ - open sub cover, say $\{D(F_i) : i = 1, 2, 3, ..., n, ...\}$, where $F_i \in \mathcal{F}$ for $i = 1, 2, ..., n, ...$ Then

$\{F_1, F_2, ..., F_n, ...\}$ is countable subcover of $\mathcal{F}$ which cover A. Hence $(A, \mathcal{G}_A)$ is $\mathcal{G}$ - Lindeloff.

Conversely, assume that $(A, \mathcal{G}_A)$ is a $\mathcal{G}$ - Lindeloff space. To prove that A is a $\mathcal{G}$ - Lindeloff subset of X. Let $\mathcal{F}$ be a $\mathcal{G}$ - open cover of A by members of $\mathcal{G}$. Then $\{G \cap A : G \in \mathcal{F}\}$ is a $\mathcal{G}$ - open cover of A by members of $\mathcal{G}_A$. By assumption this collection has a countable $\mathcal{G}$ - open subcover, say, $\{G_i \cap A : i = 1, 2, 3, ..., n, ...\}$ where $G_i \in \mathcal{F}$ for $i = 1, 2, ..., n, ...$ Clearly $\{G_1, G_2, ..., G_n, ...\}$ is countable subcollection of $\mathcal{F}$, which cover A by members of $\mathcal{G}$. Thus A is a $\mathcal{G}$ - Lindeloff subset of X. □

Theorem 2.5.9. *Let $(X, \mathcal{G})$ be a generalized topological space and X is $\mathcal{G}$ - Lindeloff. If $A \subset X$ is $\mathcal{G}$ - closed in X, then $(A, \mathcal{G}_A)$ is $\mathcal{G}$ - Lindeloff.*

Proof. Assume that X is $\mathcal{G}$ - Lindeloff and $A \subset X$ is $\mathcal{G}$ - closed in X. Let $\mathcal{F}$ be a $\mathcal{G}$ - open cover of A by members of $\mathcal{G}_A$. For each $U \in \mathcal{F}$, fix a $\mathcal{G}$ - open set $V(U)$ such that $U = V(U) \cap A$. Note that the collection $\{V(U) : U \in \mathcal{F}\}$ cover A and $X - A$ is a $\mathcal{G}$ - open set. Then the family $\mathcal{L} = \{V(U) : U \in \mathcal{F}\} \cup \{X - A\}$ is a $\mathcal{G}$ - open cover of X. By assumption this collection has a countable $\mathcal{G}$ - open cover consisting of $\{V(U_1), V(U_2), ..., V(U_n), ...\}$ and possibly $X - A$. But the collection $\{U_1, U_2, ..., U_n, ...\}$ covers A and is a countable subcollection of $\mathcal{F}$. Hence $(A, \mathcal{G}_A)$ is $\mathcal{G}$ - Lindeloff. □

Theorem 2.5.10. *Every uncountable subset of a $\mathcal{G}$ - Lindeloff space X has at least one $\mathcal{G}$ - cluster point in X.*

Proof. Assume X is a $\mathcal{G}$ - Lindeloff space and A is an uncountable subset of X. If possible assume that A has no $\mathcal{G}$ cluster point in X. Then for each $x \in X$, $U_x \in \mathcal{G}$ with $x \in U_x$ and $U_x \cap A = \{x\}$ or ϕ. Now the collection $\{U_x : x \in X\}$ is a $\mathcal{G}$ - open cover of X. Since X is $\mathcal{G}$ - Lindeloff, this collection has a countable subcover , say, $U_{x_1}, U_{x_2}, ..., U_{x_n}, ...$ So the collection $\{(U_{x_i} \cap A) : i = 1, 2, 3, ..., n, ...\}$ cover A. Therefore $\cup_{i=1}^{\infty}(U_{x_i} \cap A) = A$. But $(U_{x_1} \cap A) \cup (U_{x_2} \cap A) \cup .... \cup (U_{x_n} \cap A) \cup ... = \{x_1\} \cup \{x_2\} \cup .... \cup \{x_n\} \cup ...$ or ϕ. This implies that $A = \{x_1, x_2, ..., x_n, ...\}$ or ϕ, Which is a contradiction to the fact that A is uncountable. So A has at least one $\mathcal{G}$ - cluster point in X. □

Theorem 2.5.11. *The $\mathcal{G}$ - Lindeloff property is preserved under $\mathcal{G}$ - continuous functions.*

Proof. Let $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ be two generalized topological spaces. Let $f : (X, \mathcal{G}_1) \to (Y, \mathcal{G}_2)$ be $(\mathcal{G}_1, \mathcal{G}_2)$ continuous functions from X onto Y. Let X be $\mathcal{G}_1$ - Lindeloff. To prove that Y is $\mathcal{G}_2$ - Lindeloff. Let $\mathcal{F}$ be any $\mathcal{G}_2$ open cover of Y. Then the collection $\{f^{-1}(G) : G \in \mathcal{F}\}$ is a $\mathcal{G}_1$ - open cover of X. Since X is $\mathcal{G}_1$ - Lindeloff, there exist a countable subcollection $\{f^{-1}(G_1), f^{-1}(G_2), ..., f^{-1}(G_n), ...\}$ each $G_i \in \mathcal{F}, i = 1, 2, ..., n, ...,$ which is also a $\mathcal{G}_1$ - open cover of X. Since the mapping is onto , the collection $\{G_1, G_2, ..., G_n, ...\}$ is a countable subcollection of $\mathcal{F}$, which cover Y. Hence Y is $\mathcal{G}_2$ - Lindeloff. □

Theorem 2.5.12. *Let Y be a subset of the generalized topological space X and A be a subset of Y. Let $C_{\mathcal{G}}(A)$ denote the generalized closure of A in X. Then the generalized closure of A in Y equals $C_{\mathcal{G}}(A) \cap Y$.*

Proof. Let B denote the generalized closure of A in Y. The set $C_{\mathcal{G}}(A)$ is $\mathcal{G}$ - closed in X. By theorem 2.3.22, $C_{\mathcal{G}}(A) \cap Y$ is $\mathcal{G}$ - closed in Y. Since $A \subset C_{\mathcal{G}}(A)$ and $A \subset Y$, $A \subset C_{\mathcal{G}}(A) \cap Y$. By the definition of generalized closure B is the smallest $\mathcal{G}$ -closed set containing A. So we must have $B \subset C_{\mathcal{G}}(A) \cap Y$.

On the other hand B is $\mathcal{G}$ - closed in Y. Hence by theorem 2.3.22, $B = C \cap Y$ for some $\mathcal{G}$ - closed set C in X. Since $A \subset B$ we have C is a $\mathcal{G}$ - closed subset of X containing A. But $C_{\mathcal{G}}(A)$ is the smallest $\mathcal{G}$ - closed subset of X containing A. Therefore $C_{\mathcal{G}}(A) \subset C$. Then $C_{\mathcal{G}}(A) \cap Y \subset C \cap Y = B$. Hence $B = C_{\mathcal{G}}(A) \cap Y$. □

Theorem 2.5.13. *Let A be a subset of the generalized topological space X. Then $x \in C_{\mathcal{G}}(A)$ if and only if every $\mathcal{G}$ -open set U containing x intersects A.*

Proof. It is enough to prove that $x \notin C_{\mathcal{G}}(A)$ if and only if there exists a $\mathcal{G}$ - open set U containing x that doesn't intersect A.

Suppose that $x \notin C_{\mathcal{G}}(A)$. Then the set $U = X - C_{\mathcal{G}}(A)$ is a $\mathcal{G}$ - open set containing x that does not intersect A.

Conversely suppose that if there exists a $\mathcal{G}$ - open set U containing x that does not intersect A. Then $X - U$ is a $\mathcal{G}$ -closed set containing A. But $C_{\mathcal{G}}(A)$ - is the smallest $\mathcal{G}$ - closed set containing A. Therefore $C_{\mathcal{G}}(A) \subset X - U$. Hence $x \notin C_{\mathcal{G}}(A)$. □

Theorem 2.5.14. *Let A be a subset of a generalized topological space X and let $\mathcal{B}$ be a basis for X. Then $x \in C_{\mathcal{G}}(A)$ if and only if every basis element B containing x must intersects A.*

Proof. Suppose that $x \in C_{\mathcal{G}}(A)$. We know that every basis element is also a $\mathcal{G}$ - open set. So by theorem 2.5.13, every basis element must intersects A.

Conversely suppose every basis element intersects A. Since every $\mathcal{G}$ - open set contain a basis element, we must have every $\mathcal{G}$ - open set containing x intersects A. By theorem 7, we get $x \in C_{\mathcal{G}}(A)$. □

Theorem 2.5.15. *Let X be a generalized topological space and X has a countable basis for its generalized topology. Then every $\mathcal{G}$ - open covering of X contains a countable sub collection covering X.*

Proof. Let $\mathcal{B} = \{B_n\}$ be a countable basis for the generalized topology $\mathcal{G}$. Let $\mathcal{A}$ be a $\mathcal{G}$ - open cover of X. To prove that $\mathcal{A}$ has a countable sub collection which cover X. By the definition of base for generalized topological space for each $A \in \mathcal{A}$, there is a $B \in \mathcal{B}$ such that $B \subset A$. So for each positive integer n for which it is possible, choose an element A_n of $\mathcal{A}$ containing the basis element B_n. Let $\mathcal{A}^{'}$ be the collection of all such $A_n^{\prime}s$. Since $\mathcal{B}$ is countable we get $\mathcal{A}^{'}$ is countable.

Claim: $\mathcal{A}^{'}$ cover X.

Let $x \in X$. Since $\mathcal{A}$ cover X, there is an $A \in \mathcal{A}$ such that $x \in A$. Since $\mathcal{B}$ is a basis for the generalized topology, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset A$. By the definition $\mathcal{A}^{'}$, we get $A \in \mathcal{A}^{'}$. Hence $\mathcal{A}^{'}$ cover X. Thus $\mathcal{A}^{'}$ is a countable sub collection $\mathcal{A}$ which cover X. □

Theorem 2.5.16. *Let X be a generalized topological space and X has a countable basis for its generalized topology. Then there exists a countable subset of X that* *is $\mathcal{G}$ - dense in X.*

Proof. Let $\mathcal{B} = \{B_n\}$ be a countable basis for $\mathcal{G}$. For each non empty basis element B_n choose a point x_n. Let D be the set consisting of all points x_n.

Claim: D is $\mathcal{G}$ - dense in X.

It is enough to prove that $C_{\mathcal{G}}(D) = X$. Clearly $C_{\mathcal{G}}(D) \subset X$. For the reverse inclusion let $x \in X$. By the definition of D every basis element containing x must intersect D. Therefore $x \in C_{\mathcal{G}}(D)$. □

Theorem 2.5.17. *Let Y be a $\mathcal{G}$ - subspace of X. Then Y is $\mathcal{G}$ - Lindeloff if and only if every covering of Y by $\mathcal{G}$ - open sets in X contains a countable sub collection covering Y.*

Proof. Suppose Y is a $\mathcal{G}$ - Lindeloff subspace of X. Let $\mathcal{A} = \{A_\alpha\}$ be a $\mathcal{G}$ - open cover of Y by $\mathcal{G}$ - open sets in X. Then the collection $\{(A_\alpha \cap Y) : A_\alpha \in \mathcal{A}\}$ is $\mathcal{G}$ - open covering of Y by $\mathcal{G}$ -open sets in Y. Since Y is $\mathcal{G}$ - Lindeloff, a countable sub collection of $\{(A_\alpha \cap Y) : A_\alpha \in \mathcal{A}\}$ Say, $(A_{\alpha_1} \cap Y), (A_{\alpha_2} \cap Y), ...$ cover Y. Then $\{A_{\alpha_1}, A_{\alpha 2}, ...\}$ is a sub collection of $\mathcal{A}$ that covers Y.

Conversely suppose that the given condition holds. Let $\mathcal{A}' = \{A'_\alpha\}$ be an open covering of Y by $\mathcal{G}$ - open sets in Y. For each α, choose a set $A_\alpha, \mathcal{G}$ - open in X such that $A'_\alpha = A_\alpha \cap Y$. The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by $\mathcal{G}$ - open sets in X. By hypothesis there is a countable sub collection $\{A_{\alpha_1}, A_{\alpha_2}, etc.\}$ cover Y. Then $\{A'_{\alpha_1}, A'_{\alpha_2}, etc.\}$ is a countable sub collection of $\mathcal{A}'$ that cover Y. □

Chapter 3

Paracompactness in Generalized Topological Spaces

3.1 Introduction

In this chapter we introduce the concepts like $\mathcal{G}$ - locally finite, $\sigma_{\mathcal{G}}$ - locally finite properties in generalized topological spaces and introduce the concept of paracompactness in generalized topological spaces. This chapter contains 4 sections. Section 2 contains the preliminary ideas used in subsequent sections. In section 3, we discuss about $\mathcal{G}$ - locally finite and $\sigma_{\mathcal{G}}$ - locally finite properties in generalized topology. These two properties of generalized topological spaces coincides with locally finite and σ - locally finite properties of ordinary topology, if we replace the generalized topological space with ordinary topological space.

Here, also we discuss about some basic properties about $\mathcal{G}$ - locally finite and $\sigma_{\mathcal{G}}$ - locally finite properties. Also we found that in ordinary topological space, the union of a locally finite collection of closed sets is itself closed. But in generalized topological space this result does not hold. In section 4, we introduce the main concept paracompactness in generalized topological space. Also we discuss about some relation between separation axioms and paracompactness in generalized topological spaces and claim that every $\mathcal{G}$ - paracompact space is $\mathcal{G}$ - normal.

3.2 Preliminary Ideas Regarding Paracompactness in Generalized Topological Spaces

Definition 3.2.1. [33] *Let $(X, \mathcal{G})$ be a generalized topological space. Then X is called a $\mathcal{G}_{T_1}$ - space, if for every distinct points x_1, x_2 in $M_{\mathcal{G}}$, there exist $U, V \in \mathcal{G}$ such that $x_1 \in U, x_2 \notin U$ and $x_2 \in V, x_1 \notin V$*

Definition 3.2.2. [33] *A generalized topological space is said to be $\mathcal{G}_{T_2}$, if x and y are two distinct points in $M_{\mathcal{G}}$ then there exist two disjoint $\mathcal{G}$ - open sets U and V containing x and y respectively.*

Definition 3.2.3. [33] *Let $(X, \mathcal{G})$ be a generalized topological space. Then X is said to be $\mathcal{G}$ - regular if for each $x \in M_{\mathcal{G}}$ and a $\mathcal{G}$ - closed set F such that $x \notin F$, there are disjoint $\mathcal{G}$ - open sets U and V such that $x \in U$ and $F \cap M_{\mathcal{G}} \subset V$. If X is $\mathcal{G}_{T_1}$ and $\mathcal{G}$ - regular, then we say that X is $\mathcal{G}_{T_3}$.*

Definition 3.2.4. [33] *A generalized topological space X is said to be $\mathcal{G}$ - normal if for any two $\mathcal{G}$ - closed sets A and B such that $A \cap B \cap M_{\mathcal{G}} = \phi$, there exist disjoint $\mathcal{G}$ - open sets U and V such that $A \cap M_{\mathcal{G}} \subset U$ and $B \cap M_{\mathcal{G}} \subset V$. If X is $\mathcal{G}_{T_2}$ and $\mathcal{G}$ - normal, we say that X is $\mathcal{G}_{T_4}$.*

3.3 $\mathcal{G}$ - locally finite and $\sigma_{\mathcal{G}}$ - locally finite.

Definition 3.3.1. *Let X be a generalized topological space. Then a family $\mathcal{U}$ of subsets of X is said to be $\mathcal{G}$ - locally finite if for each $x \in M_{\mathcal{G}}$, there exist a $\mathcal{G}$ - neighbourhood N of x which intersects only finitely many members of $\mathcal{U}$.*

Definition 3.3.2. *Let X be a generalized topological space. Then a family $\mathcal{V}$ of subsets of X is said to be $\sigma_{\mathcal{G}}$ - locally finite, if it can be written as the union of countably many subfamilies each of which is $\sigma_{\mathcal{G}}$ - locally finite.*

Example 3.3.3.

1. *Every finite family of subsets of a generalized topological space X is $\mathcal{G}$ - locally finite.*
2. *Every countable family of subsets of a generalized topological space X is $\sigma_{\mathcal{G}}$ - locally finite.*

Example 3.3.4. *Let $X = \mathbb{R}$, the set of real numbers . Let $\mathcal{G}$ be the generalized topology on $\mathbb{R}$ which is generated by the set $\{\{n\} : n \in \mathbb{N}\}$.*
Then $\mathcal{U} = \{\{1, 2, 3, ...\}, \{2, 3, 4, ...\}, \{3, 4, 5, ...\}, ..., \{k, k+1, k+2, ...\}, ...\}$, which is a countable collection of subsets of $\mathbb{R}$. Note that $\mathcal{G}$ is the power set of $\mathbb{N}$. If

$x \in M_{\mathcal{G}}$, then x must be a member of N. Therefore $\{x\}$ is a $\mathcal{G}$ - neighbourhood of x, which intersect only finitely many members of $\mathcal{U}$. So $\mathcal{U}$ is $\mathcal{G}$ - locally finite. Since $\mathcal{U}$ is a countable collection, U is $\sigma_{\mathcal{G}}$ - locally finite.

Theorem 3.3.5. *Every $\mathcal{G}$ - locally finite collection of subsets of $\mathcal{G}$ - compact strong generalized topological space must be finite.*

Proof. Let $(X, \mathcal{G})$ be a generalized topological space and let $\mathcal{F} = \{F_a : a \in \Lambda\}$ be a $\mathcal{G}$ - locally finite family of subsets of X. For each point $x \in X$, choose a $\mathcal{G}$ - open neighbourhood U_x that intersect a finite number of subsets in $\mathcal{F}$. Clearly the family of subsets $\{U_x : x \in X\}$ is a $\mathcal{G}$ - open cover of X. Since X is $\mathcal{G}$ - compact, this cover has a finite subcover, say $\{U_{k_n} : n = 1, 2, 3, ..., r\}$. Since each $U_{k_i}, i = 1, 2, 3, ..., r$ intersects only a finite number of subsets of $\mathcal{F}$, suppose U_{k_1} intersect $F_{k_{1_1}}, F_{k_{1_2}}, ..., F_{k_{1_{n_1}}}$

U_{k_2} intersect $F_{k_{2_1}}, F_{k_{2_2}}, ..., F_{k_{2_{n_2}}}$

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U_{k_r} intersect $F_{k_{r_1}}, F_{k_{r_2}}, ..., F_{k_{r_{n_l}}}$.

If possible assume that $\mathcal{F}$ is infinite. Then there exists $G \in \mathcal{F}$ such that $G \neq F_{k_{i_m}}$ for $i = 1, 2, 3, ..., r$, $m = 1, 2, 3, ..., l$ and $U_{k_i}, i = 1, 2, 3, ..., r$ does not intersect G. So there exists $g \in G$ such that $g \notin \cup_{i=1}^{r} U_{k_i}$. But $\cup_{i=1}^{r} U_{k_i} = X$. Thus $g \in G$ but not belongs to X, which is a contradiction. So $\mathcal{F}$ is infinite which is wrong. □

Remark 3.3.6. *In ordinary topological space the union of a locally finite collection of closed subsets is closed. But in generalized topological space this result need not be hold.*

That is, the union of $\mathcal{G}$ - locally finite collection of $\mathcal{G}$ - closed subsets need not be $\mathcal{G}$ - closed.

Example 3.3.7. *Let $X = \{a, b, c, d\}$ and $\mathcal{G} = \{\phi, \{c, d\}, \{a, d\}, \{a, c, d\}\}$. Then $\mathcal{G}$ is a generalized topology on X and $(X, \mathcal{G})$ is a generalized topological space.*

Let $\mathcal{F} = \{\{a, b\}, \{b, c\}\}$. Since $\mathcal{F}$ is a finite collection, it is a $\mathcal{G}$ - locally finite collection.

But $\{a, b\} \cup \{b, c\} = \{a, b, c\}$ is not $\mathcal{G}$ - closed.

Lemma 3.3.8. *For a subset A of a generalized topological space X, $C_{\mathcal{G}}(A) = \{y \in X$: every $\mathcal{G}$ - neighbourhood of y meets A non vacuously. $\}$*

Proof. Let η_y be the generalized neighbourhood system of y. That is η_y be the collection of all $\mathcal{G}$ - neighbourhoods of y.

Let $B = \{y \in X : U \in \eta_y$ implies $U \cap A \neq \phi\}$.

Claim : $C_{\mathcal{G}}(A) = B$.

We have $C_{\mathcal{G}}(A) = A \cup A'^{\mathcal{G}}$. So we want to claim that $A \cup A'^{\mathcal{G}} = B$.

Let $y \in A \cup A'^{\mathcal{G}}$. Then $y \in A$ or $y \in A'^{\mathcal{G}}$.

If $y \in A$, then every $\mathcal{G}$ - neighbourhood of y intersects A, because every $\mathcal{G}$ - neighbourhood of y contains y and $y \in A$. So in this case $y \in B$.

Next we consider the case $y \in A'^{\mathcal{G}}$. Then y is a generalized accumulation point of A. By the definition of generalized accumulation point, we see that every $\mathcal{G}$ - neighbourhood of y meets A non vacuously. So $y \in B$. Hence $A \cup A'^{\mathcal{G}} \subset B$.

Conversely, let $y \in B$. If $y \notin A \cup A'^{\mathcal{G}}$, then $y \notin C_{\mathcal{G}}(A)$, because, $C_{\mathcal{G}}(A) = A \cup A'^{\mathcal{G}}$. Since $C_{\mathcal{G}}(A)$ is the smallest $\mathcal{G}$ - closed set containing A, we see that $(X - C_{\mathcal{G}}(A))$ is a $\mathcal{G}$ - open set and so it is a $\mathcal{G}$ - neighbourhood of y which does not meet A. This is a contradiction to the fact that $y \in B$. So $B \subset A \cup A'^{\mathcal{G}}$ and hence $B = C_{\mathcal{G}}(A)$. □

Theorem 3.3.9. *If $\mathcal{C}$ is a $\mathcal{G}$ - locally finite family of subsets of a generalized topological space X, then the collection $\{C_{\mathcal{G}}(C) : C \in \mathcal{C}\}$ is $\mathcal{G}$ - locally finite.*

Proof. Assume the contrary that $\{C_{\mathcal{G}}(C) : C \in \mathcal{C}\}$ is not $\mathcal{G}$ - locally finite. Then there exists $x \in M_{\mathcal{G}}$ such that every $\mathcal{G}$ - neighbourhood of x intersect infinitely many members of $\{C_{\mathcal{G}}(C) : C \in \mathcal{C}\}$. By lemma 3.3.8, $x \in C_{\mathcal{G}}(C_{\mathcal{G}}(C))$ for infinitely many $C \in \mathcal{C}$. But we have $C_{\mathcal{G}}(C_{\mathcal{G}}(C)) = C_{\mathcal{G}}(C)$, and therefore $x \in C_{\mathcal{G}}(C)$ for infinitely many $C \in \mathcal{C}$. Again by lemma 3.3.8, every $\mathcal{G}$ - neighbourhood of x intersect infinitely many $C \in \mathcal{C}$. Since $x \in M_{\mathcal{G}}$ and $\mathcal{C}$ is $\mathcal{G}$ - locally finite, so there exists a $\mathcal{G}$ - neighbourhood N_x of x which intersects only finitely many members of $\mathcal{C}$, which is a contradiction, since we just claim that every $\mathcal{G}$ - neighbourhood of x intersects infinitely many members of $\mathcal{C}$. Hence $\{C_{\mathcal{G}}(C) : C \in \mathcal{C}\}$ is $\mathcal{G}$ - locally finite. □

3.4 Paracompactness in Generalized Topological Spaces

Definition 3.4.1. *A collection $\mathcal{F}$ of subsets of a set X is called a refinement of a collection $\mathcal{H}$ if every $F \in \mathcal{F}$ there is a $H \in \mathcal{H}$ such that $F \subset H$.*

Definition 3.4.2. *A generalized topological space is called $\mathcal{G}$ - paracompact if it is $\mathcal{G}$ - regular and if every $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$ has $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement which is also a cover of $M_{\mathcal{G}}$.*

Lemma 3.4.3. *If A, B be nonempty subsets of a generalized topological space X, then $A'^{\mathcal{G}} \cup B'^{\mathcal{G}} = (A \cup B)'^{\mathcal{G}}$.*

Proof. Let $x \in A'^{\mathcal{G}} \cup B'^{\mathcal{G}}$. Then x is a generalized accumulation point of A or that of B.

If x is a generalized accumulation point of A, then every $\mathcal{G}$ - neighbourhood of x intersects A. So every $\mathcal{G}$ - neighbourhood of x intersect $A \cup B$. That is, x is a generalized accumulation point of $A \cup B$, and this implies $x \in (A \cup B)'^{\mathcal{G}}$.

Similarly, if x is a generalized accumulation point of B we can prove that $x \in (A \cup B)'^{\mathcal{G}}$.

Therefor $A'^{\mathcal{G}} \cup B'^{\mathcal{G}} \subset (A \cup B)'^{\mathcal{G}}$ ....(1).

Let $y \in (A \cup B)'^{\mathcal{G}}$. Then y is a generalized accumulation point of $(A \cup B)$. So every $\mathcal{G}$ - neighbourhood of y intersect $A \cup B$. That is every $\mathcal{G}$ - neighbourhood of y intersect A or B or both. In any case we get $y \in A'^{\mathcal{G}} \cup B'^{\mathcal{G}}$.

$(A \cup B)'^{\mathcal{G}} \subset A'^{\mathcal{G}} \cup B'^{\mathcal{G}}$................................(2).

From (1) and (2), we get $A'^{\mathcal{G}} \cup B'^{\mathcal{G}} = (A \cup B)'^{\mathcal{G}}$. □

Theorem 3.4.4. *A $\mathcal{G}$ - closed subspace of a $\mathcal{G}$ - paracompact space is $\mathcal{G}$ - paracompact.*

Proof. Let X be a $\mathcal{G}$ -paracompact space and A be a $\mathcal{G}$ - closed subspace of X. Since X is $\mathcal{G}$ - paracompact it is $\mathcal{G}$ - regular. So A is $\mathcal{G}$ - regular. Let $M_{\mathcal{G}_{\mathcal{A}}}$ be the union of all $\mathcal{G}$ - open sets in A. We want to prove that every $\mathcal{G}$ - open cover of $M_{\mathcal{G}_{\mathcal{A}}}$ has a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement which also cover $M_{\mathcal{G}_{\mathcal{A}}}$.

Let $\mathcal{F}$ be a $\mathcal{G}$ - open cover of $M_{\mathcal{G}_A}$. Note that each member F in $\mathcal{F}$ is of the form $O \cap A$ where O is $\mathcal{G}$ - open in X. Let $\mathcal{V}$ be the collection of all such $\mathcal{G}$ - open sets. Then $\mathcal{V} = \{O : F = O \cap A, F \in \mathcal{F}\}$. Note that $\mathcal{V}$ is a $\mathcal{G}$ - open cover of A by $\mathcal{G}$ - open sets in X. Hence $\mathcal{V} \cup \{X - A\}$ form a $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$. Since X is $\mathcal{G}$ - paracompact, $\mathcal{V}$ has a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement $\mathcal{U}$ which also cover $M_{\mathcal{G}}$. If $\mathcal{U}$ contain $X - A$ we remove it and consider all other elements and form another collection $\mathcal{W}$ of $\mathcal{G}$ - open sets in A. Let $\mathcal{W} = \{U \cap A : U \in \mathcal{U}, U \neq X - A\}$. Since $\mathcal{U}$ is a $\mathcal{G}$ open cover of $M_{\mathcal{G}}$, we get $\mathcal{W}$ is a $\mathcal{G}$ - open cover of $M_{\mathcal{G}_A}$. Since $\mathcal{G}$ - locally finite, we see that $\mathcal{W}$ is $\mathcal{G}$ locally finite. Also since $\mathcal{W}$ is a refinement $\mathcal{V}$ we get $\mathcal{W}$ is a refinement of $\mathcal{F}$. Hence A is $\mathcal{G}$ - paracompact. □

Theorem 3.4.5. *Let X be a $\mathcal{G}_{T_2}$ space. Then X is $\mathcal{G}$ - paracompact if and only if it has the property that every $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$ has a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement which is also a cover of $M_{\mathcal{G}}$.*

Proof. Suppose X is$\mathcal{G}$ paracompact. Then by the definition 3.4.2 of $\mathcal{G}$ paracompactness, we get X is $\mathcal{G}$ - regular and every $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$ has $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement which is also a cover of $M_{\mathcal{G}}$.

Conversely suppose that X is $\mathcal{G}_{T_2}$ and the given property holds. We want to prove that X is $\mathcal{G}$ - paracompact. We need only to show that X is $\mathcal{G}$ - regular.

Let $x \in X$ and C is a $\mathcal{G}$ - closed subset of X not containing x. We want to find two disjoint $\mathcal{G}$ - open sets U and V such that $x \in U$ and $C \cap M_{\mathcal{G}} \subset V$. Since C is a $\mathcal{G}$ - closed set not containing x, we get $X - C$ is a $\mathcal{G}$ - open set containing x . So $x \in X - C \subset M_{\mathcal{G}}$.

Let $y \in C \cap M_{\mathcal{G}}$. Then $y \in C$ and $y \in M_{\mathcal{G}}$. Which implies x and y are two distinct elements in $M_{\mathcal{G}}$. Since X is $\mathcal{G}_{T_2}$, there exist disjoint $\mathcal{G}$ - open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$. So $X - U_x$ is $\mathcal{G}$ - closed in X and $U_y \subset X - U_x$. That is $X - U_x$ is a $\mathcal{G}$ - closed set containing U_y, but $C_{\mathcal{G}}(U_y)$ is the smallest $\mathcal{G}$ - closed set containing U_y. Therefore $C_{\mathcal{G}}(U_y) \subset X - U_x$. So $x \notin C_{\mathcal{G}}(U_y)$ as $x \notin X - U_x$.

Let $\mathcal{U} = \{U_y : y \in C \cap M_{\mathcal{G}}\} \cup \{X - C\}$. Then $\mathcal{U}$ is a $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$. Let $\mathcal{V}$ be a $\mathcal{G}$ - open, $\mathcal{G}$- locally finite refinement of $\mathcal{U}$ which is also a cover of $M_{\mathcal{G}}$. So by the definition of refinement, every member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$ or in $X - C$. In the second case it cannot intersect C. Let $\mathcal{W} = \{V \in \mathcal{V} : V \cap C \neq \phi\}$. Then every member of $\mathcal{W}$ is contained in some U_y. Let $G = \cup_{W \in \mathcal{W}} W$. Since each W is $\mathcal{G}$ - open, we see that G is $\mathcal{G}$ - open.

Claim: $(C \cap M_{\mathcal{G}}) \subset G$

Let $a \in (C \cap M_{\mathcal{G}})$. Then $a \in C$ and $a \in M_{\mathcal{G}}$. Since $a \in M_{\mathcal{G}}$ and $\mathcal{V}$ is a $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$, there exists $V \in \mathcal{V}$ such that $a \in V$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, there exists $U_y \in \mathcal{U}$ such that $a \in V \subset U_y$. Since $a \in V$ and $a \in C$, this implies $V \cap C \neq \phi$. Therefore $V \in \mathcal{W}$ which implies $a \in G$. Hence $(C \cap M_{\mathcal{G}}) \subset G$.

Claim: $x \notin C_{\mathcal{G}}(G)$.

If possible let $x \in C_{\mathcal{G}}(G)$. Then $x \in C_{\mathcal{G}}(\cup_{W \in \mathcal{W}} W) = (\cup_{W \in \mathcal{W}} W) \cup (\cup_{W \in \mathcal{W}} W)'^{\mathcal{G}}$. So either $x \in (\cup_{W \in \mathcal{W}} W)$ or $x \in (\cup_{W \in \mathcal{W}} W)'^{\mathcal{G}}$. That is either $x \in W$ for some $W \in \mathcal{W}$ or x is a generalized accumulation point of W for some $W \in \mathcal{W}$ by lemma 3.4.3. Since $x \in U_x$ and each U_y does not intersect U_x, we see that $x \notin V$, for all $V \in \mathcal{V}$. Therefore $x \notin W$, for all $W \in \mathcal{W}$. Then x must be a generalized accumulation point of W for some $W \in \mathcal{W}$. Hence every $\mathcal{G}$ - neighbourhood of x intersect W. But this is impossible, because the $\mathcal{G}$ - open set U_x containing x, does not intersect W.

Therefore $x \notin C_{\mathcal{G}}(G)$ which implies $x \in X - C_{\mathcal{G}}(G)$. Also note that $X - C_{\mathcal{G}}(G)$ is $\mathcal{G}$ - open. So we take $U = X - C_{\mathcal{G}}(G)$ and $V = G$. Then $x \in U$ and $C \cap M_{\mathcal{G}} \subset V$ and $U \cap V = \phi$. □

Lemma 3.4.6. *Let X be a generalized topological space. If X is $\mathcal{G}$ - regular, then for any $x \in X$ and any $\mathcal{G}$ - open set G containing x, there exists a $\mathcal{G}$ - open set H containing x such that $C_{\mathcal{G}}(H) \subset G \cup M_{\mathcal{G}}^c$.*

Proof. Suppose X is a $\mathcal{G}$ - regular space. Let $x \in X$ and G be any $\mathcal{G}$ - open subset of X containing x.

So $X - G$ is a $\mathcal{G}$ - closed set not containing x. So there exist disjoint $\mathcal{G}$ - open sets U and V such that $x \in U$ and $(X - G) \cap M_{\mathcal{G}} \subset V$. This implies

$(G^c \cap M_{\mathcal{G}})^c \supset V^c$ (taking complements on both sides) . By De Morgan's law $(X - V) \subset (G \cup M_{\mathcal{G}}^c)$.

Since $U \subset (X-V)$ and $(X-V)$ is $\mathcal{G}$ - closed, we get $C_{\mathcal{G}}(U) \subset (X-V)$. Hence $C_{\mathcal{G}}(U) \subset G \cup M_{\mathcal{G}}^c$. Take $U = H$, we see that $x \in H$ and $C_{\mathcal{G}}(H) \subset G \cup M_{\mathcal{G}}^c$. $\square$

Theorem 3.4.7. *Every $\mathcal{G}$ - paracompact space is $\mathcal{G}$ - normal.*

Proof. Let X be a $\mathcal{G}$ - paracompact space. Let A and B be two $\mathcal{G}$ - closed subsets of X such that $A \cap B \cap M_{\mathcal{G}} = \phi$. From the definition of $\mathcal{G}$ - paracompactness, X is $\mathcal{G}$ - regular and every $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$ has a $\mathcal{G}$ - open , $\mathcal{G}$ - locally finite refinement which is also cover $M_{\mathcal{G}}$.

Since X is $\mathcal{G}$ - regular, for each $x \in A \cap M_{\mathcal{G}}$ and $x \notin B \cap M_{\mathcal{G}}$, there are disjoint $\mathcal{G}$ - open sets U_x and V_x such that $x \in U_x$ and $B \cap M_{\mathcal{G}} \subset V_x$. Since X is $\mathcal{G}$ - regular, using lemma 3.4.4, there exists $\mathcal{G}$ - open set H_x such that $C_{\mathcal{G}}(H_x) \subset U_x \cup M_{\mathcal{G}}^c$, $x \in H_x$.

Let $\mathcal{U} = \{H_x : x \in A \cap M_{\mathcal{G}}\} \cup \{X - A\}$. Then $\mathcal{U}$ is a $\mathcal{G}$ - open cover of $M_{\mathcal{G}}$. Since X is $\mathcal{G}$ - paracompact , by the definition of $\mathcal{G}$ - paracompactness $\mathcal{U}$ has a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement which is also a cover of $M_{\mathcal{G}}$.

Let $\mathcal{V}$ be a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite refinement of $\mathcal{U}$ which is also a cover of $M_{\mathcal{G}}$. So by the definition of refinement, every member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. Let $\mathcal{W} = \{V \in \mathcal{V} : V \cap A \neq \phi\}$. Now let $G = \cup_{W \in \mathcal{W}} W$. Then G is $\mathcal{G}$ - open.

Claim 1: $A \cap M_{\mathcal{G}} \subset G$.

Let $z \in A \cap M_{\mathcal{G}}$. This implies $z \in H_z$, where $H_z \in \mathcal{U}$. So $z \in W$, for some

$W \in \mathcal{W}$. Hence $z \in G$.

Claim 2: $B \cap M_{\mathcal{G}} \subset (X - C_{\mathcal{G}}(G))$.

Let $z \in B \cap M_{\mathcal{G}}$. This implies $z \in B$ and $z \in M_{\mathcal{G}}$. To prove that $z \notin C_{\mathcal{G}}(G)$.

If possible let $z \in C_{\mathcal{G}}(G)$. Using theorem 1.4.13, we have $C_{\mathcal{G}}(G) = G \cup G'^{\mathcal{G}}$. Hence $z \in G$ or $z \in G'^{\mathcal{G}}$.

Case(i) : $z \in G$

If $z \in G \implies z \in (\cup_{W \in \mathcal{W}} W)$.

$\implies z \in W$ for some $W \in \mathcal{W}$.

$\implies z \in V$ for some $V \in \mathcal{V}$ and $V \cap A \neq \phi$.

Since every member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$, So there exists $U \in \mathcal{U}$ such that $V \subset U$. Then $U = H_x$, for some $x \in A \cap M_{\mathcal{G}}$ or $U = (X - A)$. Since $(X - A)$ does not intersect A, U is of the form $H_x, x \in A \cap M_{\mathcal{G}}$.

So by the definition of $H_x, x \in A \cap M_{\mathcal{G}}$, there exist disjoint $\mathcal{G}$ - open sets such that $x \in U_x$ and $B \cap M_{\mathcal{G}} \subset V_x$. Also $C_{\mathcal{G}}(H_x) \subset U_x \cup M_{\mathcal{G}}^c$. So $z \in V \subset H_x \subset C_{\mathcal{G}}(H_x) \subset U_x \cup M_{\mathcal{G}}^c$. Since $z \in M_{\mathcal{G}}$, we get $z \in U_x$. Since Also $B \cap M_{\mathcal{G}} \subset V_x$, we get $z \in V_x$, which is a contradiction to the fact that $U_x \cap V_x = \phi$. Therefore $z \notin G$.

Case(ii) : $z \in G'^{\mathcal{G}}$**.**

If $z \in G'^{\mathcal{G}}$, by theorem 3.4.2 we have $G'^{\mathcal{G}} = (\cup_{W \in \mathcal{W}} W)'^{\mathcal{G}} = \cup_{W \in \mathcal{W}} (W'^{\mathcal{G}})$. Therefore $z \in \cup_{W \in \mathcal{W}} (W'^{\mathcal{G}}) \implies z \in W'^{\mathcal{G}}$ for some $W \in \mathcal{W}$ which implies z is a $\mathcal{G}$ - accumulation point of W. By the definition of $\mathcal{W}, W = V$ for some $V \in \mathcal{V}$ such that $V \cap A \neq \phi$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, by the definition of refinement, there exists $U \in \mathcal{U}$ such that $V \subset U$. By the definition of $\mathcal{U}, U$

is of the form H_x for some $x \in A$. That is, there exists $H_x \in \mathcal{U}$ such that $W \subset H_x, x \in A$. So there exist disjoint $\mathcal{G}$ - open sets U_x and V_x such that $x \in U_x, B \cap M_{\mathcal{G}} \subset V_x$ and $C_{\mathcal{G}}(H_x) \subset U_x \cup (M_{\mathcal{G}})^c$. Since $W \subset H_x$ and $H_x \subset C_{\mathcal{G}}(H_x)$ implies $W \subset C_{\mathcal{G}}(H_x) \subset U_x \cup M_{\mathcal{G}}^c$.
This implies $W \subset U_x$ since $W \subset M_{\mathcal{G}}$.

Also, $z \in B \cap M_{\mathcal{G}}$ and $B \cap M_{\mathcal{G}} \subset V_x \implies z \in V_x$. Since V_x is $\mathcal{G}$ - open and $z \in V_x$, it is a $\mathcal{G}$ - neighbourhood of z. Since z is a $\mathcal{G}$ - accumulation point of W, we see that $V_x \cap W \neq \phi$.

Since $W \subset U_x$ and $W \cap V_x \neq \phi$, we get $U_x \cap V_x \neq \phi$, which is a contradiction since U_x and V_x are disjoint. Hence $z \notin G'^{\mathcal{G}}$ Therefore$z \notin C_{\mathcal{G}}(G) i.e, z \in B \cap M_{\mathcal{G}}$ implies $z \in (X - C_{\mathcal{G}}(G))$. Therefore $B \cap M_{\mathcal{G}} \subset (X - C_{\mathcal{G}}(G))$.

That is, there are two $\mathcal{G}$ - open sets G and $(X - C_{\mathcal{G}}(G))$ such that $A \cap M_{\mathcal{G}} \subset G$ and $B \cap M_{\mathcal{G}} \subset (X - C_{\mathcal{G}}(G))$. Hence X is $\mathcal{G}$ - normal. □

Lemma 3.4.8. *Let X be a $\mathcal{G}$ - normal space. Then for any $\mathcal{G}$ - closed set C and any $\mathcal{G}$ - open set G containing C, there exist $\mathcal{G}$ - open set H such that $C \cap M_{\mathcal{G}} \subset H$ and $C_{\mathcal{G}}(H) \subset G \cup M_{\mathcal{G}}^c$.*

Proof. Suppose X is a $\mathcal{G}$ - normal space. Let C be a $\mathcal{G}$ - closed subset of X and G be a $\mathcal{G}$ - open subset of X containing C. Then $X - G$ is a $\mathcal{G}$ - closed subset of X not containing C. That is, C and $(X - G)$ are two disjoint $\mathcal{G}$ - closed subsets of X. Since X is $\mathcal{G}$ - normal, by the definition of $\mathcal{G}$ - normality, there are disjoint $\mathcal{G}$ - open sets U and V such that $C \cap M_{\mathcal{G}} \subset U$ and $(X - G) \cap M_{\mathcal{G}} \subset V$. Since $U \cap V = \phi$, we see that $U \subset (X - V)$ and $X - V$ is a $\mathcal{G}$ - closed subset of X .

But we know that $C_{\mathcal{G}}(U)$ is the smallest $\mathcal{G}$ - closed set containing U. Therefore $C_{\mathcal{G}}(U) \subset (X - V)$.

Also $(X - G) \cap M_{\mathcal{G}} \subset V \implies ((X - G) \cap M_{\mathcal{G}})^c \supset (X - V)$.

$\implies (X - V) \subset G \cup M_{\mathcal{G}}^c$ (by De Morgan's law).

But $C_{\mathcal{G}}(U) \subset X - V \subset G \cup M_{\mathcal{G}}^c \implies C_{\mathcal{G}}(U) \subset G \cup M_{\mathcal{G}}^c$. If we take $U = H$, then we get $C \cap M_{\mathcal{G}} \subset H$ and $C_{\mathcal{G}}(H) \subset G \cup M_{\mathcal{G}}^c$. □

Theorem 3.4.9. *If $\mathcal{U}$ is a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite cover of a $\mathcal{G}$ - normal space X,then for each $U \in \mathcal{U}$ there exists a $\mathcal{G}$ - open set $G(U)$ such that $C_{\mathcal{G}}(G(U)) \subset U \cup M_{\mathcal{G}}^c$ and the family $\{G(U) : U \in \mathcal{U}\}$ covers X.*

Proof. Let $(X, \mathcal{G})$ be a generalized topological space. Given that X is a $\mathcal{G}$ - normal space and let $\mathcal{U}$ be a $\mathcal{G}$ - open, $\mathcal{G}$ - locally finite cover of X. Since $\mathcal{U}$ is a $\mathcal{G}$ - open cover of X, we see that $M_{\mathcal{G}} = X$.

We want to find a function $G : \mathcal{U} \to \mathcal{G}$ with the property that for each $U \in \mathcal{U}, G(U)$ is $\mathcal{G}$ - open, $C_{\mathcal{G}}(G(U)) \subset U \cup M_{\mathcal{G}}^c$ and the family $\{G(U) : U \in \mathcal{U}\}$ covers X. Call such a function G as a total shrinking function for $\mathcal{U}$.

Next define a partial shrinking function for $\mathcal{U}$ to be the function $F : \mathcal{V} \to \mathcal{G}$, where $\mathcal{V} \subset \mathcal{U}, C_{\mathcal{G}}(F(V)) \subset V \cup M_{\mathcal{G}}^c$, for all $V \in \mathcal{V}$ and the family $(\mathcal{U} - \mathcal{V}) \cup \{F(V) : V \in \mathcal{V}\}$ is a cover of X. Let $\mathcal{F}$ be a family of all partial shrinking functions for $\mathcal{U}$. Note that $\mathcal{F}$ is nonempty because the function with empty domain belong to it.

Define an order $\leq$ in $\mathcal{F}$ as follows. Given two partial shrinking functions $F, H \in \mathcal{F}$, we say that $F \leq H$ if,

1. domain of F is contained in that of H.

2. $F(V) = H(V)$ for all $V \in$ domain of F.

Claim 1: $\leq$ is a partial order on $\mathcal{F}$.

1. *Reflexivity :*

   Let $F \in \mathcal{F}$, then clearly $F \leq F$.

2. *Antisymmetric:*

   Let $F, H \in \mathcal{F}$ such that $F \leq H$ and $H \leq F$. To prove that $F = H$.

   $F \leq H \implies$ the domain of F is contained in the domain of H.

   $H \leq F \implies$ the domain of H is contained in the domain F.

   Therefore domain of $F =$ domain of H.

   Let domain of $F =$ domain of $H = \mathcal{V}$. Then

   $F \leq H$ and $H \leq F \implies F(V) = H(V)$ for all $V \in \mathcal{V}$. Hence $F = H$.

3. *Transitivity:*

   Let F, G, H be three members in $\mathcal{F}$ such that $F \leq G$ and $G \leq H$. To prove that $F \leq H$.

   $F \leq G \implies$ domain of F is contained in the domain of G and

   $G \leq H \implies$ domain of G is contained in the domain of H.

   Hence domain of F is contained in the domain of H.

   Let $\mathcal{D}(F), \mathcal{D}(G), \mathcal{D}(H)$ denotes the domain of F, G and H respectively. Then

$F \leq G \implies F(V) = G(V)$ for all $V \in \mathcal{D}(F)$.

$G \leq H \implies G(V) = H(V)$ for all $V \in \mathcal{D}(G)$.

$\mathcal{D}(F) \subset \mathcal{D}(G) \subset \mathcal{D}(H) \implies \mathcal{D}(F) \subset \mathcal{D}(H)$ and hence $F(V) = H(V)$ for all $V \in \mathcal{D}(F)$.

Therefore $F \leq H$. Hence $\leq$ is a partial order on $\mathcal{F}$.

Claim 2: $\mathcal{F}$ has a maximal element.

Let $\{F_i : i \in I\}$ is a chain in $\mathcal{F}$. We want to find an upper bound for this chain. First we construct a partial shrinking function F from this chain as follows:

Let $\mathcal{D}(F)$ be the domain of F then $\mathcal{D}(F) = \cup_{i \in I} \mathcal{D}(F_i)$ where $\mathcal{D}(F_i)$ is the domain of F_i for $i \in I$. If $V \in \mathcal{D}(F)$, so $V \in \mathcal{D}(F_i)$ for some $i \in I$ and set $F(V) = F_i(V)$. This is well defined because if $V \in \mathcal{D}(F_i) \cap \mathcal{D}(F_j)$ for $i \neq j$ in I, then $V \in \mathcal{D}(F_i)$ and $\mathcal{D}(F_j)$. If $F_i \neq F_j$, then we may assume that $F_i \leq F_j$. So $\mathcal{D}(F_i) \subset \mathcal{D}(F_j) implies \mathcal{D}(F_i) \cap \mathcal{D}(F_j) = \mathcal{D}(F_i)$. Also $F_i = F_j$ on $\mathcal{D}(F_i) \implies F_i(V) = F_j(V)$ since $V \in \mathcal{D}(F_i)$.

Claim 3: $C_{\mathcal{G}}(F(V)) \subset V \cup M_{\mathcal{G}}^c, for V \in \mathcal{D}(F)$.

Let $V \in \mathcal{D}(F)$.Then $V \in \mathcal{D}(F_i)$ for some $i \in I$. By the definition F, we have $F(V) = F_i(V)$ for some $i \in I$. Since for each $i \in I, F_i$ is a partial shrinking on $\mathcal{U}$, by the definition of partial shrinking $C_{\mathcal{G}}(F_i(V)) \subset V \cup M_{\mathcal{G}}^c \implies C_{\mathcal{G}}(F(V)) \subset V \cup M_{\mathcal{G}}^c$ for all $V \in \mathcal{D}(F)$.

In order to show that F is a partial shrinking for $\mathcal{U}$, it only remains to show that the family $(\mathcal{U} - \mathcal{D}(F)) \cup \{F(V) : V \in \mathcal{D}(F)\}$ is a cover of X.

Let $x \in X$. Then $M_{\mathcal{G}} = X \implies x \in M_{\mathcal{G}}$. Since $\mathcal{U}$ is locally finite, by the definition of locally finite property, there is a $\mathcal{G}$ - neighbourhood N of x which intersect only finitely many members of $\mathcal{U}$. So x contains only finitely many members of $\mathcal{U}$. Suppose that $U_1, U_2, ..., U_n$ are the only members in $\mathcal{U}$ that contains x. If at least one of the U_i is $(\mathcal{U} - \mathcal{D}(F))$ we are done. Otherwise, since $\mathcal{F}$ is the collection of all partial shrinking for $\mathcal{U}$, there exist $i_1, i_2, i_3, ..., i_n$ in I, such that $U_r \in \mathcal{D}(F_{i_r})$ for $r = 1, 2, 3, ..., n$. Since $\{F_i : i \in I\}$ is a chain, without loss of generality we may assume that $F_{i_1} \leq F_{i_2} \leq .... \leq F_{i_n}$. By the definition of $\leq$, we get $\mathcal{D}(F_{i_1}) \subset \mathcal{D}(F_{1_2}) \subset .... \subset \mathcal{D}(F_{i_n})$. Since $U_1 \in \mathcal{D}(F_{i_1}), U_2 \in \mathcal{D}(F_{i_2}), ..., U_n \in \mathcal{D}(F_{i_n})$, we get $U_r \in \mathcal{D}(F_{i_n})$ for all $r = 1, 2, ..., n$. Since these are the only members of $\mathcal{U}$ containing x and by the definition of F_{i_n}, we see that $(\mathcal{U} - \mathcal{D}(F_{i_n})) \cup \{F_{i_n}(V) : V \in \mathcal{D}(F_{i_n})\}$ is a cover of X. So there is a $V \in \mathcal{D}(F_{i_n})$ such that $x \in V$. But we have $\mathcal{D}(F) = \cup_{i \in I} \mathcal{D}(F_i)$ and $F = F_i$ on $\mathcal{D}(F_i)$. Therefore $F(V) = F_{i_n}(V)$. Thus we get $(\mathcal{U} - \mathcal{D}(F)) \cup \{F(V) : V \in \mathcal{D}(F)\}$ is a cover of X. So F is a partial shrinking function for $\mathcal{U}$ and by its construction it is an upper bound for the chain $\{F_i : i \in I\}$.

Thus every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. Then by Zorn's lemma, $\mathcal{F}$ has a maximal element. That is, we get a partial shrinking function, say G for $\mathcal{U}$ which is maximal with respect to the ordering $\leq$.

To conclude the proof, we assert that G is a total shrinking for $\mathcal{U}$. That is to show that the domain of $G, \mathcal{D}(G) = \mathcal{U}$. Clearly $\mathcal{D}(G) \subset \mathcal{U}$ as G is a partial shrinking function on $\mathcal{U}$. So we want to prove that $\mathcal{U} \subset \mathcal{D}(G)$

If not there exists $U \in \mathcal{U}$ such that $U \notin \mathcal{D}(G)$. That is, $U \in \mathcal{U} - \mathcal{D}(G)$. Let Wbe the union of the sets $\cup\{G(V) : V \in \mathcal{D}(G)\}$ and the set $(\cup(\mathcal{U} - \mathcal{D}(G) - U))$.Since each $G(v)$ is $\mathcal{G}$ - open, $\cup\{G(V) : V \in D(g)\}$ is $\mathcal{G}$ - open. The second collection in the definition of W is also $\mathcal{G}$ - open, because it is a subcollection of $\mathcal{U}$. So their union must be $\mathcal{G}$ - open. Hence W is $\mathcal{G}$ - open. Therefore $X - W$ is $\mathcal{G}$ - closed. Since $(\mathcal{U} - \mathcal{D}(G)) \cup \{G(V) : V \in \mathcal{D}(G)\}$ is a cover of X, we get $X - W \subset U$. Now by $\mathcal{G}$ - normality and using lemma 3.4.7, we see that there exists a $\mathcal{G}$ - open set Q such that $(X - W) \cap M_{\mathcal{G}} \subset Q$ and $C_{\mathcal{G}}(Q) \subset U \cup M_{\mathcal{G}}^c$.

Define $H : \mathcal{D}(G) \cup \{U\} \to \mathcal{G}$ by $H(V) = G(V)$ for $V \in \mathcal{D}(G)$ and $H(U) = Q$. Since G is a partial shrinking function for $\mathcal{U}$, we get H is a partial shrinking function for $\mathcal{U}$. Also note that H is strictly greater than G. Which contradict the maximality of G. Thus $\mathcal{D}(G) = \mathcal{U}$.

$\square$

Chapter 4

I - Convergence and I^* - Convergence in Generalized Topological Spaces

4.1 Introduction

This chapter contains five sections. First section is the introduction and second section contains preliminary ideas used for the subsequent sections. In third section we extend the concept of I - convergence to generalized topological spaces. Mainly in this section we introduce the concept $I_{\mathcal{G}}$ - convergence in generalized topological spaces and discuss about its some properties. Here we found that the $I_{\mathcal{G}}$ convergence is preserved under $\mathcal{G}$ - continuous functions. In fourth section we introduce the concept of $I^*_{\mathcal{G}}$ - convergence in Generalized topological spaces. Here we also discuss about the relation between $I_{\mathcal{G}}$ and $I^*_{\mathcal{G}}$ - convergence in

Generalized topological spaces. Section 5 contains definitions of $I_{\mathcal{G}}$ -limit point and $I_{\mathcal{G}}$ - cluster points of a sequence and discuss about some properties of these type of points.

4.2 Preliminaries

Definition 4.2.1. [19] *A sequence (x_n) in a set X is said to be eventually in $U \subset X$, if there exists $k \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq k$.*

Definition 4.2.2. [26] *Let $(X, \mathcal{G})$ be a generalized topological space. A sequence (x_n) in X does not converge to a point x if there exists a $\mathcal{G}$ - open set U containing x such that (x_n) is not eventually in U. Otherwise we say that (x_n) converges to x.*

Theorem 4.2.3. [26] *If $(X, \mathcal{G})$ and $(Y, \mathcal{G}')$ are two generalized topological spaces and $f : X \to Y$ be $(\mathcal{G}, \mathcal{G}')$ continuous, then $x_n \to x$ in X implies that $f(x_n) \to f(x)$ in Y.*

Definition 4.2.4. [5] *If X is a nonempty set, then a family of sets $I \subset 2^X$ is an ideal if*

1. $\phi \in I$.

2. $A, B \in I \implies A \cup B \in I$ *and*

3. $A \in I, B \subset A \implies B \in I$.

An ideal is called nontrivial, if $I \neq \{\phi\}$ and $X \notin I$.

Definition 4.2.5. [6] *A nonempty family $\mathcal{F}$ of subsets of a nonempty set X is called a filter if*

1. $\phi \notin \mathcal{F}$
2. $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$ *and*
3. $A \in \mathcal{F}, A \subset B \implies B \in \mathcal{F}$.

If I is a nontrivial ideal on X then $F = F(I) = \{A \subset X : X - A \in I\}$ is clearly a filter on X .

A nontrivial ideal I is called admissible if it contains all the singleton sets.

Remark 4.2.6. *Throughout this chapter $(X, \mathcal{G})$ will stands for a generalized topological spaces and I for a non trivial ideal of $\mathbb{N}$, the set of all natural numbers.*

4.3 $I_{\mathcal{G}}$ -Convergence In Generalized Topological Spaces.

Definition 4.3.1. *A sequence (x_n) in X does not $I_{\mathcal{G}}$ - convergent to $x_0 \in X$, if there exists a nonempty $\mathcal{G}$ - open set U containing $x_0, \{n \in \mathbb{N}; x_n \notin U\} \notin I$, otherwise we say that (x_n) is $I_{\mathcal{G}}$ - convergent to x_0. In this case we write $I_{\mathcal{G}}$ - $lim x_n = x_0$ and x_0 is called the $I_{\mathcal{G}}$ - limit of (x_n).*

Example 4.3.2. *Let* $X = \{1, 2, 3, 4\}$ *and* $\mathcal{G} = \{\phi, \{1, 2, 3\}, \{1\}, \{1, 2\}\}$. *Then* $(X, \mathcal{G})$ *is a Generalized topological space.*

1. *Consider the constant sequence* (1). *If we take any* $\mathcal{G}$ *- open set* U *containing* 1, *we see that* $\{n \in \mathbb{N} : x_n \notin U\} = \phi \in I$. *Therefore the sequence* (1) *is* $I_{\mathcal{G}}$ *- convergent to* 1.

2. *Clearly the sequence* (1) *is not* $I_{\mathcal{G}}$ *- convergent to* 2.

3. *Sequence* (1) *is* $I_{\mathcal{G}}$ *- convergent to* 3, *because the only* $\mathcal{G}$ *- open set containing* 3 *is* $\{1, 2, 3\}$.

Remark 4.3.3. *So in general the* $I_{\mathcal{G}}$ *- limit of a sequence is not unique.*

Theorem 4.3.4. *If* I *is admissible, then ordinary sequential convergence in generalized topological space implies* $I_{\mathcal{G}}$ *convergence.*

Proof. Let (x_n) be a sequence in a generalized topological space X and suppose $x_n \to x_0$ in X. If possible (x_n) does not $I_{\mathcal{G}}$ convergent to x_0. Then there exists a $\mathcal{G}$ - open set U containing x_0 such that $\{n \in \mathbb{N} : x_n \notin U\} \notin I$.

Since $x_n \to x_0$, by definition of sequential convergence in generalized topological space, (x_n) is eventually in U. So there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$, for all $n \geq n_0$. Since I is admissible, it contains all singletons and since I is closed under finite union, $\{1, 2, 3, ..., n_0\} \in I$. If $n \geq n_0$, then $x_n \in U$. Therefore $\{n \in \mathbb{N} : x_n \notin U\} \subset \{1, 2, ..., n_0\}$.

Since $\{1, 2, 3, ..., n_0\} \in I$ and $\{n \in \mathbb{N} : x_n \notin U\} \subset \{1, 2, ..., n_0\}$, by definition of I we get $\{n \in \mathbb{N} : x_n \notin U\} \in I$. This is a contradiction and hence (x_n) is $I_{\mathcal{G}}$ convergent to x_0. $\square$

Theorem 4.3.5. *If $I$is admissible and I does not contain any infinite set then the sequential convergence and $I_{\mathcal{G}}$ - convergence of a sequence in a generalized topological space X coincide.*

Proof. Let (x_n) be a sequence in X and $x \in X$.

Suppose $x_n \to x$. By theorem 4.3.3, (x_n) is $I_{\mathcal{G}}$ - convergent to x.

Conversely suppose that (x_n) is $I_{\mathcal{G}}$ - convergent to x. To show that $(x_n) \to x$. If not, there exists a $\mathcal{G}$ - open set U containing x such that (x_n) is not eventually in U. By the definition of $I_{\mathcal{G}}$ - limit, $\{n \in \mathbb{N} : x_n \notin U\} \in I$. Since (x_n) is not eventually in U for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \geq n$ and $x_k \notin U$. Therefore $\{k : x_k \notin U\}$ is an infinite set. But $\{k : x_k \notin U\} \subset \{n \in \mathbb{N} : x_n \notin U\} \in I$. By the definition of $I, \{k : x_k \notin U\} \in I$, which is a contradiction to the fact that I does not contain any infinite set. Therefore the sequence (x_n) converges to x. $\square$

Theorem 4.3.6. *If X is a strong $\mathcal{G}_{T2}$ - space, then $I_{\mathcal{G}}$ - convergent sequence has a unique $I_{\mathcal{G}}$ - limit.*

Proof. If possible suppose that an $I_{\mathcal{G}}$ convergent sequence (x_n) has two distinct $I_{\mathcal{G}}$ - limit say x_0 and y_0. Since X is a strong generalized topological space, there exists $U, V \in \mathcal{G}$ such that $x_0 \in U$ and $y_0 \in V, U \cap V = \phi$.

Since (x_n) is $I_{\mathcal{G}}$ - convergent to x_0 and $x_0 \in U$ it follows that $\{k \in \mathbb{N} : x_k \notin$

$U\} \in I$. Also since (x_n) is $I_{\mathcal{G}}$ - convergent to y_0 and $y_0 \in V$, $\{k \in \mathbb{N} : x_k \notin V\} \in I$. We have $\{k : x_k \in (U \cap V)^c\} \subset \{k : x_k \in U^c\} \cup \{k : x_k \in V^c\} \in I$. Since I is nontrivial, there exist $k_0 \in \mathbb{N}$ such that $k_0 \notin \{k : x_k \in (U \cap V)^c\}$ which implies $x_{k0} \notin (U \cap V)^c$. This implies $x_{k0} \in U \cap V$, which is a contradiction to the fact that $U \cap V = \phi$.

□

Theorem 4.3.7. *If I is an admissible ideal and if there exists a sequence (x_n) of distinct elements in a set $E \subset X$ which is $I_{\mathcal{G}}$ - convergent to $x_0 \in X$, then x_0 is a generalized limit point of E.*

Proof. Let U be a generalized neighbourhood of x_0. Since $I_{\mathcal{G}}$ - $\lim x_n = x_0$ and by the definition of $I_{\mathcal{G}}$ - limit, $\{n : x_n \notin U\} \in I$ and so $\{n : x_n \in U\} \notin I$. Since I is admissible, it contain all singletons and hence all finite subsets of $\mathbb{N}$. So $\{n : x_n \in U\}$ is an infinite subset of $\mathbb{N}$. Choose $k_0 \in \{n : x_n \in U\}$ such that $x_{k0} \neq x_0$. Then $x_{k0} \in U \cap (E - \{x_0\})$. Thus x_0 is a generalized limit point of E. □

Theorem 4.3.8. *A $\mathcal{G}$ - continuous function $g : X \to X$ preserves $I_{\mathcal{G}}$ convergence.*

Proof. Let (x_n) be a sequence in X such that $I_{\mathcal{G}}$ - limit $(x_n) = x$. To prove that $I_{\mathcal{G}}$ - limit $(g(x_n)) = g(x)$.

If possible let $I_{\mathcal{G}}$ - limit $(g(x_n)) \neq g(x)$. Then there exists a $\mathcal{G}$ - open set U such that $\{n \in \mathbb{N} : g(x_n) \notin U\} \notin I$. Since g is $\mathcal{G}$ - continuous , $g^{-1}(U)$ is a $\mathcal{G}$ -

open set such that $x \in g^{-1}(U)$. Also note that $\{n \in \mathbb{N} : x_n \notin g^{-1}(U)\} \notin I$. This implies $I_{\mathcal{G}} - limit(x_n) \neq x$, a contradiction. $\square$

Remark 4.3.9. *The converse of the theorem* 4.3.7 *is need not be true.*

Example 4.3.10. *Let* $X = \{1, 2, 3, 4\}$ *and* $\mathcal{G} = \{\phi, \{1, 2, 3\}, \{1\}, \{1, 2\}\}$. *Then* $(X, \mathcal{G})$ *is a Generalized topological space. Define* $g : X \to X$ *by* $g(x) = 1$. *Let* (x_n) *be an* $I_{\mathcal{G}}$ *convergent sequence in* X. *Then sequence* $(g(x_n))$ *is the constant sequence* (1) *which is* $I_{\mathcal{G}}$ *- convergent in* X. *But the function* g *is not* $\mathcal{G}$ *- continuous, because* $g^{-1}(\{1\}) = X$ *is not* $\mathcal{G}$ *- open in* X.

4.4 $I^*_{\mathcal{G}}$ - convergence in generalized topological Spaces.

Definition 4.4.1. *A sequence* (x_n) *in a generalized topological space* X *is* $I^*_{\mathcal{G}}$ *convergent to* $x \in X$ *if and only if there exist a set* $M \in F(I)$, $M = \{m_1 \leq m_2 \leq ... \leq m_k \leq ...\}$, *such that* $(x_{m_k}) \to x$ *as* $k \to \infty$. *In this case we write* $I^*_{\mathcal{G}}$ *- lim*$x_n = x$ *and* x *is called the* $I^*_{\mathcal{G}}$ *- limit of* (x_n).

Example 4.4.2. *Let* $X = \{1, 2, 3, 4\}$ *and* $\mathcal{G} = \{\phi, \{1, 2, 3\}, \{1\}, \{1, 2\}\}$. *Then* $(X, \mathcal{G})$ *is a Generalized topological space. The constant sequence* (1) *is* $I^*_{\mathcal{G}}$ *- convergent to* 1. *In this case we take* $M = \mathbb{N}$.

Theorem 4.4.3. *If* I *is admissible then* $I^*_{\mathcal{G}}$ *- limit*$x_n = x$ *implies* $I_{\mathcal{G}}$ *- limit*$x_n = x$ *and so in addition if* X *is a* $\mathcal{G}_{T_2}$ *- space then* $I^*_{\mathcal{G}}$ *- limit*x_n *is unique.*

Proof. Given I is admissible. Suppose $I^*_{\mathcal{G}} - limitx_n = x$. To prove that $I_{\mathcal{G}} - limitx_n = x$. Since $I^*_{\mathcal{G}} - limitx_n$ implies there exists $K \in I$ such that for $M = \mathbb{N} - K = \{m_1 \leq m_2 \leq ... \leq m_k \leq ...\}$, we have $limx_{m_k} = x$. By the definition of sequential convergence in generalized topological space, for any $\mathcal{G}$ - open set U containing x sequence x_{m_k} is eventually in U. That is there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $x_{m_k} \in U$. Clearly $\{n : x_n \notin U\} \subset \{m_1, m_2, ..., m_{k0}\} \cup K \in I$ as I is admissible. Therefor $I_{\mathcal{G}} - limitx_n = x$. □

Theorem 4.4.4. *If I is admissible, X is a strong $\mathcal{G}_{\mathcal{T}_\in}$ - space and (x_n) be an $I^*_{\mathcal{G}}$ convergent sequence in X, then $I^*_{\mathcal{G}} - limitx_n$ is unique.*

Proof. If possible suppose that (x_n) has two distinct $I^*_{\mathcal{G}}$ limit say x_0 and y_0. Then by theorem 4.4.3 (x_n) has two distinct $I_{\mathcal{G}}$ - limit. By theorem 4.3.5 the $I_{\mathcal{G}}$ limit of a sequence is unique. So $x_0 = y_0$. This is a contradiction. Therefore $I^*_{\mathcal{G}}$ - limit of a sequence is unique. □

Theorem 4.4.5. *If X has no generalized accumulation point then $I_{\mathcal{G}}$ and $I^*_{\mathcal{G}}$ - convergence are coincide for every admissible ideal I.*

Proof. Let (x_n) be a sequence in X and let $I_{\mathcal{G}}$ - $limx_n = x_0$. Since X has no generalized accumulation point, $U = \{x_0\}$ is $\mathcal{G}$ - open. Since $I_{\mathcal{G}}$ - limit $x_n = x_0$, we have $\{n : x_n \notin U\} \in I$. This implies $\{n : x_n \notin \{x_0\}\} \in I$ and hence $\{n : x_n \neq x_0\} \in I$. By the definition of $F(I)$, the complement of $\{n : x_n \neq x_0\}$ belongs to $F(I)$.

i.e, $\{n : x_n = x_0\} \in F(I)$. Thus $I^*_{\mathcal{G}} - limx_n = x_0$. □

4.5 $I_{\mathcal{G}}$ - Limit points and $I_{\mathcal{G}}$ - cluster points.

Definition 4.5.1. *Let $x = (x_n)$ be a sequence in X. An element $y \in X$ is called $I_{\mathcal{G}}$- limit point of x, if there exists a set $M = \{m_1 \leq m_2 \leq ...\} \subset \mathbb{N}$ such that $M \notin I$ and $lim_{k\to\infty} x_{m_k} = y$. Denote $I_{\mathcal{G}}(L_x)$ be the collection of all $I_{\mathcal{G}}$ - limit points of the sequence x.*

Example 4.5.2. *Let $X = \{1, 2, 3, 4\}, \mathcal{G} = \{\phi, \{1, 2\}, \{1, 2, 3\}\}$.*

1. *Consider the sequence $(x_n) = (2)$ in X. Let $M = \mathbb{N}$, so $M \notin I$ and $lim_{k\to\infty} x_k = 2$. So 2 is a $I_{\mathcal{G}}$ - limit point of (x_n).*
2. *Consider the sequence 1, 2, 1, 2, ..., 1, 2,... in X. Note that 1, 2, 3, 4 are $I_{\mathcal{G}}$- limit point of this sequence.*

Example 4.5.3. *Consider $X = \mathbb{R}$, the set of real numbers and $\mathbb{Z}$ be the set of all integers. Let $\mathcal{G} = \{G : G \subset \mathbb{R} - \mathbb{Z}\}$. Then $(X, \mathcal{G})$ is a Generalized topological space. Note that every sequence in X converges to every $z \in \mathbb{Z}$. So every $z \in Z$ is a $I_{\mathcal{G}}$ - limit point of every sequence in $\mathbb{R}$.*

Definition 4.5.4. *Let $x = (x_n)$ be a sequence of elements of X and let $y \in X$ is said to be an $I_{\mathcal{G}}$ - cluster point of x, if there exists a $\mathcal{G}$ - open set U containing y, and $\{n : x_n \in U\} \notin I$. Denote $I_{\mathcal{G}}(C_x)$ be the collection of all $I_{\mathcal{G}}$ - cluster points of the sequence x.*

Example 4.5.5. *Let $X = \{1, 2, 3, 4\}, \mathcal{G} = \{\phi, \{1, 2\}, \{1, 2, 3\}\}$. Consider the sequence (x_n) as the constant sequence (4). Note that 4 is an $I_{\mathcal{G}}$ - cluster point* *of* (x_n) *and* 3 *is an* $I_{\mathcal{G}}$ *- cluster point of* (x_n)*, because the only* $\mathcal{G}$ *- open set containing* 3 *is* $U = \{1, 2, 3\}$*, and* $\{n \in \mathbb{N} : x_n \in U\} = \phi \notin I$ *as* I *is nontrivial.*

Theorem 4.5.6. *If* I *is an admissible ideal, then* $I_{\mathcal{G}}(L_x) \subset I_{\mathcal{G}}(C_x)$*, where* $x = (x_n)$ *be a sequence in* X*.*

Proof. Let $y \in I_{\mathcal{G}}(L_x)$. So there exists a set $M = \{m_1 \leq m_2 \leq ....\} \subset \mathbb{N}$ such that $M \notin I$ and $lim_{k\to\infty} x_{m_k} = y$.

Claim: $y \in I_{\mathcal{G}}(C_x)$**.**

If possible $y \notin I_{\mathcal{G}}(C_x)$. Then there exists a $\mathcal{G}$ - open set U containing y, such that $\{n : x_n \in U\} \in I$. Since $y \in U$ and $x_{m_k} \to y$ implies (x_{m_k}) is eventually in U, there exist $k_0 \in \mathbb{N}$ such that $x_{m_k} \in U$ for all $k \geq k_0$. Note that $\{n : x_n \in U\} \supset M - \{m_1, m_2, ..., m_{k_0}\}$. That is, $M - \{m_1, m_2, ..., m_{k_0}\} \subset \{n : x_n \in U\} \in I$..

Since I is admissible, it contain all singletons and hence I contain all finite subsets of $\mathbb{N}$. So in particular $\{m_1, m_2, ..., m_{k_0}\} \in I$. Since I is closed under finite union, we get $M = (M - \{m_1, m_2, ..., m_{k_0}\}) \cup \{m_1, m_2, ..., m_{k_0}\} \in I$. This is a contradiction to the fact that $M \notin I$. Therefore $y \in I_{\mathcal{G}}(C_x)$. □

Remark 4.5.7. *If* I *is admissible, then* $I_{\mathcal{G}}(L_x) \neq I_{\mathcal{G}}(C_x)$*.*

Theorem 4.5.8. *If* I *is an admissible ideal, then* $I_{\mathcal{G}}(C_x)$ *is* $\mathcal{G}$ *- closed for each sequence* $x = (x_n)$ *in* X*.*

Proof. We want to prove that $C_{\mathcal{G}}(I_{\mathcal{G}}(C_x)) = I_{\mathcal{G}}(C_x)$. For this, to prove that $C_{\mathcal{G}}(I_{\mathcal{G}}(C_x)) \subset I_{\mathcal{G}}(C_x)$.

Let $y \in C_{\mathcal{G}}(I_{\mathcal{G}}(C_x))$. Then $y \in (I_{\mathcal{G}}(C_x) \cup (I_{\mathcal{G}}(C_x)'^{\mathcal{G}}$. If $y \in (I_{\mathcal{G}}(C_x)'^{\mathcal{G}}$. Let

U be a $\mathcal{G}$ - open set containing y. Since $y \in (I_{\mathcal{G}}(C_x)'^{\mathcal{G}}$, by the definition of $\mathcal{G}$ - limit point, there exists $z \in I_{\mathcal{G}}(C_x)$ such that $z \neq y$ and $z \in U$. So by the definition of $I_{\mathcal{G}}(C_x), z \in I_{\mathcal{G}}(C_x)$ for $z \in U$. Therefore $\{n : x_n \in U\} \notin I$, which implies $y \in I_{\mathcal{G}}(C_x)$. $\square$

Chapter 5

Regular open sets in Generalized Topological Spaces.

5.1 Introduction

In this chapter we extend the concepts regular open sets, regular closed sets, weakly regular open sets, weakly regular closed sets, into generalized topological spaces. This chapter contain nine sections. In section 2, we introduce the concept $\mathcal{G}$ - regular open sets and discuss about some of its properties. In the remaining sections we introduce concepts like minimal $\mathcal{G}$ - regular open sets, maximal $\mathcal{G}$ - regular open sets, $\mathcal{G}$ - regular closed sets, minimal $\mathcal{G}$ - regular closed sets, maximal $\mathcal{G}$ - regular closed sets, weakly $\mathcal{G}$ - regular open sets, weakly $\mathcal{G}$ - regular closed sets. Here also we discuss about some properties of these concepts. Here we found that finite intersection of $\mathcal{G}$ - regular open sets need not be $\mathcal{G}$ - regular open, but in ordinary topological space finite intersection of regular open sets is regular open.

5.2 $\mathcal{G}$ - Regular Open Sets.

Definition 5.2.1. *A subset A of a generalized topological space X is said to be $\mathcal{G}$ - regular open, if $A = int_{\mathcal{G}}(c_{\mathcal{G}}(A))$.*

Example 5.2.2. *Let $X = \{a, b, c, d\}$ and $\mathcal{G} = \{\phi, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a\}.\}$ Clearly $\mathcal{G}$ is a generalized topology on X. Let $A = \{a\}$. Then $C_{\mathcal{G}}(A) = \{a, d\}$ and $int_{\mathcal{G}}(C_{\mathcal{G}}(A)) = int_{\mathcal{G}}(\{a, d\}) = \{a\} = A$. Therefore A is a $\mathcal{G}$ - regular open subset of X.*

Definition 5.2.3. *A subset of a generalized topological space is said to be $\mathcal{G}$ - clopen if it is both $\mathcal{G}$ - open and $\mathcal{G}$ - closed.*

Theorem 5.2.4. *Every $\mathcal{G}$ - clopen set in a generalized topological space is $\mathcal{G}$ - regular open.*

Theorem 5.2.5. *Every $\mathcal{G}$ - regular open set in a generalized topological space is $\mathcal{G}$ - open.*

Proof. Proof follow from the fact that the generalized interior of a set is always $\mathcal{G}$- open. □

Theorem 5.2.6. *If A and B are subsets of generalized topological space X, then $C_{\mathcal{G}}(A \cap B) \subset C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B)$.*

Proof. Since $(A \cap B) \subset A$ and $(A \cap B) \subset B$, then $C_{\mathcal{G}}(A \cap B) \subset C_{\mathcal{G}}(A)$ and $C_{\mathcal{G}}(A \cap B) \subset C_{\mathcal{G}}(B)$. Hence $C_{\mathcal{G}}(A \cap B) \subset C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B)$. □

Remark 5.2.7. *In ordinary topological spaces $closure(A \cap B) = closure(A) \cap closure(B)$ but in generalized topological space $C_{\mathcal{G}}(A \cap B) \neq C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B)$.*

Example 5.2.8. *Let $X = \{a, b, c, d\}$ and $\mathcal{G} = \{X, \phi, \{a, b\}, \{c, d\}, \{a, c, d\}\}$. Then $\mathcal{G}$ is a generalized topology on X. If $A = \{a, b\}$ and $B = \{b, d\}$, then $C_{\mathcal{G}}(A) = \{a, b\}$ and $C_{\mathcal{G}}(B) = X$. Therefore $C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B) = \{a, b\}$. Also $A \cap B = \{b\}$ and $C_{\mathcal{G}}(A \cap B) = \{b\}$. Hence $C_{\mathcal{G}}(A \cap B) \neq C_{\mathcal{G}}(A) \cap C_{\mathcal{G}}(B)$.*

Remark 5.2.9. *In ordinary topological space finite intersection of regular open sets is regular open, but in generalized topological space finite intersection of $\mathcal{G}$ - regular open set need not be $\mathcal{G}$ - regular open.*

Example 5.2.10. *Let $X = \{a, b, c, d\}$ and $\mathcal{G} = \{X, \phi, \{c, d\}, \{a, b\}, \{a, c, d\}, \{b\}, \{b, c, d\}\}$. Then $\mathcal{G}$ is a generalized topology on X. Let $A = \{a, b\}$ and $B = \{a, c, d\}$. Then $int_{\mathcal{G}}(C_{\mathcal{G}}(A)) = A$ and $int_{\mathcal{G}}(C_{\mathcal{G}}(B)) = B$. Therefore A and B are $\mathcal{G}$ - regular open sets in X. But $A \cap B = \{a\}$ and $int_{\mathcal{G}}(C_{\mathcal{G}}(\{a\})) = \phi \neq \{a\}$.*

5.3 Minimal $\mathcal{G}$ - regular open sets and Maximal $\mathcal{G}$ - regular open sets.

Definition 5.3.1. *A proper non empty $\mathcal{G}$ - regular open set U of a generalized topological space X is said to be a minimal $\mathcal{G}$ - regular open set, if any $\mathcal{G}$ - regular open set which is contained in U is U or ϕ.*

Example 5.3.2. *Let $X = \{a, b, c, d\}$ and $\mathcal{G} = \{X, \phi, \{a, b\}, \{c, d\}, \{a, c, d\}\}$. Then $\mathcal{G}$ is a generalized topology on X. Here $\{a\}$ is a minimal $\mathcal{G}$ - regular open set in X.*

Definition 5.3.3. *A proper non empty $\mathcal{G}$ - regular open subset U of a generalized topological space X is said to be a maximal $\mathcal{G}$ - regular open set if any $\mathcal{G}$ - regular open set contains U is U or X.*

Example 5.3.4. *Let $X = \{a, b, c\}$ and $\mathcal{G} = \{\phi, \{a\}, \{a, b\}, \{b\}\}$. Then $\{a, b\}$ is a maximal $\mathcal{G}$ - regular open subset of X.*

5.4 $\mathcal{G}$ - regular Closed Sets in a Generalized Topological Space.

Definition 5.4.1. *A subset A of a generalized topological space X is said to be $\mathcal{G}$ - regular closed, if $A = C_{\mathcal{G}}(int_{\mathcal{G}}(A))$.*

Example 5.4.2. *Let* $X = \{a, b, c\}$ *and* $\mathcal{G} = \{\phi, \{a\}, \{a, b\}, \{b\}\}$. *Then* $\{c\}$, $\{b, c\}$*and* $\{a, c\}$ *are* $\mathcal{G}$ *- regular closed subsets of* X.

Theorem 5.4.3. *Every* $\mathcal{G}$ *- clopen set in a generalized topological space is* $\mathcal{G}$ *- regular closed.*

Theorem 5.4.4. *Every* $\mathcal{G}$ *- regular closed set in a generalized topological space is* $\mathcal{G}$ *- closed.*

Proof. Let A be a $\mathcal{G}$ - regular closed subset of a generalized topological space X. Then $A = C_{\mathcal{G}}(int_{\mathcal{G}}(A))$. Since generalized closure of a set is $\mathcal{G}$ - closed. Therefore A is $\mathcal{G}$ - closed. □

5.5 Minimal $\mathcal{G}$ - regular Closed Sets and Maximal $\mathcal{G}$ - regular Closed sets.

Definition 5.5.1. *A proper non empty* $\mathcal{G}$ *- regular closed subset* F *of a generalized topological space* X *is said to be minimal* $\mathcal{G}$ *- regular closed set, if any* $\mathcal{G}$ *- regular closed set which is contained in* F *is* F *or* ϕ.

Example 5.5.2. *Let* $X = \{a, b, c\}$ *and* $\mathcal{G} = \{\phi, \{a\}, \{a, b\}, \{b\}\}$. *Then* $\{c\}$ *is a minimal* $\mathcal{G}$ *- regular closed subset of* X.

Definition 5.5.3. *A proper non empty* $\mathcal{G}$ *- regular closed subset* F *of a generalized topological space* X *is said to be maximal* $\mathcal{G}$ *- regular closed set, if any* $\mathcal{G}$ *- regular closed set which contains* F *is* F *or* X.

Example 5.5.4. *Let $X = \{a, b, c\}$ and $\mathcal{G} = \{\phi, \{a\}, \{a, b\}, \{b\}\}$. Then $\{b, c\}$and $\{a, c\}$ are maximal $\mathcal{G}$ - regular closed subsets of X.*

5.6 Properties of Minimal $\mathcal{G}$ - regular Open Sets and Maximal $\mathcal{G}$ - regular Open Sets.

Theorem 5.6.1. *Let X be a generalized topological space and $F \subset X$. Then F is a minimal $\mathcal{G}$ - regular open set if and only if $X - F$ is a maximal $\mathcal{G}$ -regular closed set and F is a maximal $\mathcal{G}$ - regular open set if and only if $X - F$ is a minimal $\mathcal{G}$ - regular closed set.*

Proof. Suppose that F is a minimal $\mathcal{G}$ - regular open set. Then $int_{\mathcal{G}}(C_{\mathcal{G}}(F)) = F$ and any $\mathcal{G}$ - regular open set contained in F is F or ϕ.

Claim 1 : $X - F$ **is** $\mathcal{G}$ **- regular closed.**

$C_{\mathcal{G}}(int_{\mathcal{G}}(X - F)) = C_{\mathcal{G}}((X - F) - \partial_{\mathcal{G}}(X - F)) = X - F$. Therefore $X - F$ is a $\mathcal{G}$ - regular closed subset of X.

Claim 2 : $X - F$ **is a maximal** $\mathcal{G}$ **- regular closed subset of** X.

If H is a $\mathcal{G}$ - regular closed set contains $X - F$, then using claim 1, $X - H$ is a $\mathcal{G}$ - regular closed set contained in F. Since F is a minimal $\mathcal{G}$ - regular open set, we get $X - H = F$ or $X - H = \phi$. Therefore $H = X - F$ or $H = X$. Hence $X - F$ is a maximal $\mathcal{G}$ - regular closed subset of X. Similarly we can prove the second part.

□

Theorem 5.6.2. *Let U be a minimal $\mathcal{G}$ - regular open set and W be a $\mathcal{G}$ - regular open set and $U \cap W$ is a $\mathcal{G}$ -regular open set. Then either $U \cap W = \phi$ or $U \subset W$.*

Proof. Given that U is a minimal $\mathcal{G}$ - regular open set and $U \cap W$ is a $\mathcal{G}$ - regular open set. Since $U \cap W \subset U$, we must have either $U \cap W = \phi$ or $U \cap W = U$. Therefore either $U \cap W = \phi$ or $U \subset W$. □

Theorem 5.6.3. *If U and W are two minimal $\mathcal{G}$ - regular open sets and their intersection is also a $\mathcal{G}$ - regular open set, then either $U \cap W = \phi$ or $U = W$.*

Proof. By theorem 5.6.2, we get either $U \cap W = \phi$ or $U \subset W$ and $W \subset U$. Therefore either $U \cap W = \phi$ or $U = W$. □

Theorem 5.6.4. *Let U and W be $\mathcal{G}$ - regular open subsets of a generalized topological space X and U be a maximal $\mathcal{G}$ - regular open and their union is also a $\mathcal{G}$ - regular open set. Then either $U \cup W = X$ or $W \subset U$.*

Proof. Given that U is a maximal $\mathcal{G}$ - regular open set and $U \cup W$ is also a $\mathcal{G}$ - regular open set. Then by the definition of maximal $\mathcal{G}$ - regular open set. We must have either $U \cup W = X$ or $W \subset U$. □

Theorem 5.6.5. *Let U and W be maximal $\mathcal{G}$ - regular open subsets of a generalized topological space X and their union is also a $\mathcal{G}$ - regular open set. Then either $U \cup W = X$ or $U = W$.*

Proof. By theorem 5.6.4, we have either $U \cup W = X$ or $U \subset W$ and $W \subset U$. Therefore either $U \cup W = X$ or $U = W$. □

5.7 Properties of Minimal $\mathcal{G}$ - regular Closed Sets and Maximal $\mathcal{G}$ - regular closed sets.

Theorem 5.7.1. *Let X be a generalized topological space and $F \subset X$. Then F is a minimal $\mathcal{G}$ - regular closed set if and only if $X - F$ is a maximal $\mathcal{G}$ -regular open set and F is a maximal $\mathcal{G}$ - regular closed set if and only if $X - F$ is a minimal $\mathcal{G}$ - regular open set.*

Proof. Suppose that F is a minimal $\mathcal{G}$ - regular closed subset of X. Then $C_{\mathcal{G}}(int_{\mathcal{G}}(F)) = F$ and any $\mathcal{G}$ - regular closed set contained in F is F or ϕ.

Claim 1: $(X - F)$ **is** $\mathcal{G}$ **- regular open**.

$int_{\mathcal{G}}(C_{\mathcal{G}}(X - F)) = int_{\mathcal{G}}((X - F) \cup \partial_{\mathcal{G}}(X - F)) = X - F$. Therefore $X - F$ is a $\mathcal{G}$ - regular open subset of X.

Claim 2 : (X - F) is a maximal $\mathcal{G}$ - regular open subset of X.

If H is a $\mathcal{G}$ - regular closed set contains $X - F$, then using claim 1, $X - H$ is a $\mathcal{G}$ -regular open set contained in F. Since F is a minimal $\mathcal{G}$ - regular closed set, we $X - H = F$ or $X - H = \phi$. Therefore $H = X - F$ or $H = X$. Hence $X - F$ is a maximal $\mathcal{G}$ - regular open subset of X.

Conversely suppose that $X - F$ is a maximal $\mathcal{G}$ -regular open set. To prove that F is a minimal $\mathcal{G}$ - regular closed set. By theorem 5.6.1 $[X - (X - F)] = F$ is a minimal $\mathcal{G}$ - regular closed set.

□

Theorem 5.7.2. *Let U be a minimal $\mathcal{G}$ - regular closed set and W be a $\mathcal{G}$ - regular closed set and their intersection is also a $\mathcal{G}$ - regular closed set. Then either $U \cap W = \phi$ or $U \subset W$.*

Proof. Given that U is a minimal $\mathcal{G}$ - regular closed set and $U \cap W \subset U$, we must have either $U \cap W = \phi$ or $U \cap W = U$. Therefore either $U \cap W = \phi$ or $U \subset W$. □

Theorem 5.7.3. *Let U be a maximal $\mathcal{G}$ - regular closed set and W be a $\mathcal{G}$ - regular closed set and their union is also a $\mathcal{G}$ - regular closed set. Then either $U \cup W = X$ or $W \subset U$.*

Proof. Given that U is a maximal $\mathcal{G}$ - regular closed set and $U \subset U \cup W$ is a $\mathcal{G}$ - regular closed set. Therefore either $U \cup W = X$ or $W \subset U$. □

Theorem 5.7.4. *Let U and W are maximal $\mathcal{G}$ - regular closed sets and if their union is also a $\mathcal{G}$ - regular closed set, then either $U \cup W = X$ or $U = W$.*

Proof. By theorem 5.7.3, either $U \cup W = X$ or $U \subset W$ and $W \subset U$. Hence either $U \cup W = X$ or $U = W$. □

5.8 Weakly $\mathcal{G}$ - regular Open Sets and Weakly $\mathcal{G}$ - regular Closed Sets.

In this section we consider X as a strong generalized topological space.

Definition 5.8.1. *A proper subset A of a generalized topological space X is said to be weakly $\mathcal{G}$ - regular open, the only $\mathcal{G}$ - regular open set containing A is X.*

Example 5.8.2. *Let $X = \{a, b, c\}$ and $\mathcal{G} = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $\{a, b\}$ is a weakly $\mathcal{G}$ - regular open subset of X.*

Definition 5.8.3. *A proper subset A of a generalized topological space X is said to be weakly $\mathcal{G}$ - regular closed, if its complement is weakly $\mathcal{G}$ - regular open.*

Example 5.8.4. *Let $X = \{a, b, c\}$ and $\mathcal{G} = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $\{c\}$ is a weakly $\mathcal{G}$ - regular closed subset of X.*

Theorem 5.8.5. *The union of two maximal $\mathcal{G}$ - regular open sets of a generalized topological space X is either a weakly $\mathcal{G}$ - regular open set or the whole set X.*

Proof. Let U and W are two maximal $\mathcal{G}$ - regular open set. If their union is a $\mathcal{G}$ - regular open set, then by theorem 5.6.5, either $U \cup W = X$ or $U = W$.

If $U \cup W$ is not a $\mathcal{G}$ - regular open set, then the only $\mathcal{G}$ - regular open set containing $U \cup W$ is X because, U and W are maximal $\mathcal{G}$ - regular open set.

Hence $U \cup W$ is either a weakly $\mathcal{G}$ - regular open set or the whole set X. □

Theorem 5.8.6. *The intersection of two minimal $\mathcal{G}$ - regular closed sets in a generalized topological space is either a weakly $\mathcal{G}$ - regular closed set or the empty set.*

Proof. Let F and G be two minimal $\mathcal{G}$ - regular closed subsets of a generalized topological space X. By theorem 5.5.1, we get $X - F$ and $X - G$ are maximal $\mathcal{G}$ - regular closed sets. By theorem 5.7.1, $(X - F) \cup (X - G)$ is either weakly $\mathcal{G}$ - regular open or the whole set X. Using De Morgan's law, we get $(X - (F \cap G))$ is either weakly $\mathcal{G}$ - regular open or the whole set X. Therefore $F \cap G$ is either a weakly $\mathcal{G}$ - regular closed set or the empty set. □

Theorem 5.8.7. *The intersection of two weakly $\mathcal{G}$ - regular closed sets in a generalized topological space is again a weakly $\mathcal{G}$ - regular closed set.*

Proof. Let F and G are two weakly $\mathcal{G}$ - regular closed sets. Then $(X - F)$ and $(X - G)$ are two weakly $\mathcal{G}$ - regular open sets. So their union is also a weakly $\mathcal{G}$ - regular open set. That is $((X - F) \cup (X - G))$ is weakly $\mathcal{G}$ - regular open. Hence by De Morgan's law $(F \cap G)$ is weakly $\mathcal{G}$ - regular closed. □

CONCLUSION

Through this thesis we were able to extend some topological concepts to generalized topological spaces.

Chapter 1, named On generalized closed sets. Here we proved some fundamental results about generalized closed sets. Also found that some results in ordinary topological spaces need not be hold in generalized topological spaces. Some of them are given below.

1. In ordinary topological space finite union of closed sets is closed. But in generalized topological space finite union of $\mathcal{G}$ - closed sets need not be $\mathcal{G}$ - closed.

2. In ordinary topological space $\overline{A \cup B} = \overline{A} \cup \overline{B}$. But in generalized topological space $C_{\mathcal{G}}(A \cup B) \neq C_{\mathcal{G}}(A) \cup C_{\mathcal{G}}(B)$.

3. In ordinary topological space every element has a neighbourhood. But in generalized topological space, every element need not have a $\mathcal{G}$ - neighbourhood.

4. In ordinary topological space the neighbourhood system at a point must be non empty. But in a generalized topological space the generalized neighbourhood system may be empty.

In this chapter we also proved some theorems about generalized neighbourhoods, generalized accumulation points, generalized closure operator, etc. Some of them are given below.

1. Generalized closure of a set is the smallest $\mathcal{G}$ - closed set containing that set.

2. A is $\mathcal{G}$ - closed in a generalized topological space if and only if $C_{\mathcal{G}}(A) = A$.

3. The fixed points of a generalized closure operator are precisely the $\mathcal{G}$ - closed subsets of X.

4. The generalized closure operator is idempotent.

5. A subset of a generalized topological space is $\mathcal{G}$ - open if and only if it is a $\mathcal{G}$ - neighbourhood of each of its point.

In chapter 2 we extend topological properties like connectedness and Lindeloffness in generalized topological spaces. Here we prove some more results on $\mathcal{G}$ - connectedness and $\mathcal{G}$ - Lindeloff properties. Also found that some results in ordinary topological space need not be hold in generalized topological space. Some of them are given below.

1. In ordinary topological space closure of a connected set is connected. But in generalized topological space the $\mathcal{G}$ - closure of a $\mathcal{G}$ - connected set need not be $\mathcal{G}$ - connected.

2. In ordinary topological space components are closed. But in generalized topological space the $\mathcal{G}$ - components need not be $\mathcal{G}$ - closed.

Here we also found some important results about $\mathcal{G}$ -connectedness and $\mathcal{G}$ - Lindeloff property in generalized topological space. Some of them are given below.

1. A generalized topological space is $\mathcal{G}$ - connected if it is not the union of two non empty $\mathcal{G}$ - separated sets.
2. Let $\mathcal{C}$ be a collection of $\mathcal{G}$ -connected subsets of X such that no two members of $\mathcal{C}$ are $\mathcal{G}$ - separated. Then $\cup_{C\in\mathcal{C}}C$ is also $\mathcal{G}$ - connected.
3. Finite products of $\mathcal{G}$ - connected spaces is $\mathcal{G}$ -connected.
4. Arbitrary union of $\mathcal{G}$ - Lindeloff spaces is $\mathcal{G}$ - Lindeloff.
5. The $\mathcal{G}$ - Lindeloff property is preserved under $\mathcal{G}$ - continuous functions.

The third chapter is titled by Paracompactness in Generalized topological Spaces. Here we found some topological results about locally finite property and paracompactness are not true in generalized topological spaces. Some important results in this chapter are given below.

1. In ordinary topological space the union of a locally finite collection of closed subsets is closed. But in generalized topological space the union of $\mathcal{G}$ - locally finite collection of $\mathcal{G}$ - closed subsets need not be $\mathcal{G}$ - closed.
2. If $\mathcal{C}$ is a $\mathcal{G}$ - locally finite family of subsets of generalized topological space X, then the collection $\{C_{\mathcal{G}}(C) : C \in \mathcal{C}\}$ is $\mathcal{G}$ - locally finite.

3. Every $\mathcal{G}$ - paracompact space is $\mathcal{G}$ - normal.

In fourth chapter we extend the notion of I - convergence and I^* - convergence into generalized topological spaces. Some of important results in this chapter given below,

1. If I is admissible, then sequential convergence in generalized topological space implies $I_{\mathcal{G}}$ - convergence.
2. If X is a strong $\mathcal{G}_{T_2}$ - space, then $I_{\mathcal{G}}$ convergent sequence has unique $I_{\mathcal{G}}$ - limit.
3. A $\mathcal{G}$ -continuous function preserves $I_{\mathcal{G}}$ convergence.
4. If X has no generalized accumulation points, then $I_{\mathcal{G}}$ and $I^*_{\mathcal{G}}$ - convergence are coincide for every admissible ideal I.
5. If I is an admissible ideal, then $I_{\mathcal{G}}(C_x)$ is $\mathcal{G}$ - closed for each sequence $x = (x_n)$ in X.

Chapter 5 was named as Regular open sets in generalized topological space. Here we found that some results in ordinary topological space does not hold in generalized topological space. Some of them are listed below.

1. In ordinary topological space finite intersection of regular open sets is regular open, but in generalized topological space finite intersection of $\mathcal{G}$ - regular open sets need not be $\mathcal{G}$ - regular open.

Here we also proved some results about $\mathcal{G}$ - regular open sets.

1. Every $\mathcal{G}$ - regular closed set is is $\mathcal{G}$ - closed.

2. Let U be a minimal $\mathcal{G}$ - regular open set and W be a $\mathcal{G}$ - regular open set and $U \cap W$ is a $\mathcal{G}$ -regular open set. Then either $U \cap W = \phi$ or $U \subset W$.

3. Let U be a minimal $\mathcal{G}$ - regular closed set and W be $\mathcal{G}$ - regular closed set and their intersection is also a $\mathcal{G}$ - regular closed set. Then either $U \cap W = \phi$ or $U \subset W$.

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APPENDIX

List of Publications

1) *K K Bushra Beevi and Baby Chacko,* , More Results On Generalized Closed Sets, Malaya Journal of Matematik , vol.8, No.3, 1158 - 1161, 2020

2) *K K Bushra Beevi and Baby Chacko,* On $\mathcal{G}$ - connected and $\mathcal{G}$ - Lindeloff spaces in Generalized topology., Malaya Journal of Matematik, Vol.9, No.1,480 - 485, 2021.

3) *K K Bushra Beevi and Baby Chacko,* $I_{\mathcal{G}}$ -Convergence and $I_{\mathcal{G}^*}$ - Convergence in GTS, Zeichen Journal,vol.8,Issue 5,(330 - 336),2022.

4) *K K Bushra Beevi and Baby Chacko,* Paracompactness in Generalized Topological spaces, South East Asian Journal Mathematics and Mathematical Sciences, Vol.19, No.1(2023), pp- 287 - 300.

5) *K K Bushra Beevi and Baby Chacko,* Regular open sets in Generalized topological spaces, South east Asian Journal of Mathematics and Mathematical Sciences. (Communicated.)