
Ph.D. THESIS

MATHEMATICS

ORDER THEORETIC AND TOPOLOGICAL ASPECTS OF
GENERALIZATIONS OF WELL ORDERED SETS

Thesis submitted to the

University of Calicut

for the award of the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

ANGELA SUNNY

under the Guidance of

Dr. SINI P.



Department of Mathematics, University of Calicut
Kerala, India 673 635.

FEBRUARY 2026



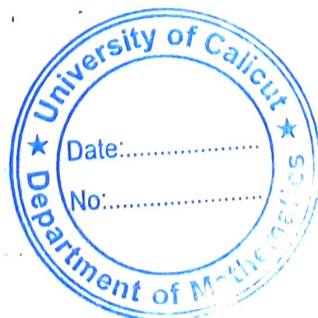
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALICUT

Dr. Sini P.
Associate Professor

University of Calicut
03 February 2026

CERTIFICATE

I hereby certify that the thesis entitled "Order Theoretic and Topological Aspects of Generalizations of Well Ordered Sets" is a bona fide work carried out by **Ms. ANGELA SUNNY**, under my guidance for the award of Degree of Ph.D. in Mathematics from the University of Calicut. This work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.



Dr. Sini P.
Research Supervisor

Dr. SINI. P.
Associate Professor & Head
Department of Mathematics
University of Calicut
Malappuram-673635

DECLARATION

I hereby declare that the work presented in the thesis entitled “**Order Theoretic and Topological Aspects of Generalizations of Well Ordered Sets**” is based on the original work done by me under the guidance of **Dr. Sini P.**, Associate Professor, Department of Mathematics, University of Calicut and has not been included in any other thesis submitted previously for the award of any degree. The contents of the thesis are undergone plagiarism check using iThenticate software at C.H.M.K. Library, University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.



Angela Sunny
Research Scholar



Dr. Sini P.
Research Supervisor

University of Calicut

03 February 2026.

To my dearest parents,

Ligi & Sunny,

for their unconditional love and support.

ACKNOWLEDGEMENT

Words are inadequate to express my gratitude and indebtedness to Dr. Sini P., my supervising teacher and Head of the Department, Department of Mathematics, University of Calicut, for her continuous guidance, support and encouragement throughout my research. Her patience and willingness to spend time on valuable discussions were truly instrumental to my progress. Whatever little mathematics I managed to do during my Ph.D. would not have been possible without her unwavering support and encouragement.

I extend my heartfelt gratitude to Late Prof. P. T. Ramachandran, former faculty and Head of the Department of Mathematics, University of Calicut. His suggestions, mentorship and support during the initial stages of my research were immensely helpful. The weekend meetings with him remain unforgettable memories.

My sincere thanks to Prof. Raji Pilakkat, Prof. Anil Kumar V., Dr. Preethi Kuttipulakkal, Dr. Prasad T. and Dr. Mubeena T. for their suggestions, support and for providing all necessary facilities for my research. I also thank all the office staff and librarians of the department for their kind assistance. Special thanks to Mr. Praveen K. V., a former office staff member, for his dedicated support.

I also thank the staff of the C.H. Mohammed Koya Library and the Directorate of Research of the University for their valuable and kind assistance. I further thank the Hostel warden, office staff and matrons of the University Ladies Hostel for their cooperation and assistance.

I am grateful to Dr. Dhanya and Dr. Kavita, students of Prof. P.T. Ramachandran, for their support during my research. I also extend a special thanks to Mr. Nitheesh Unnikrishnan.

My heartfelt thanks to Dr. Abhinand, Mr. Abhinav, Mr. Ajeesh, Ms. Ameena, Dr. Anusha, Ms. Archana, Dr. Darsana, Ms. Dhanya, Mr. Fazil, Mr. Midhun, Ms. Naheeda, Ms. Nithya, Ms. Nivisha, Dr. Noushida, Dr. Priya, Dr. Rafiya, Dr. Safeera, Ms. Saira, Ms. Saji, Mr. Saleel, Dr. Simi, Ms. Sophiya and Dr. Sruthi for their wonderful company and the valuable discussions-both in mathematics and beyond.

Sincere thanks to my dear hostel mates: Dr. Anjitha, Ms. Archana, Dr. Asha, Dr. Drishya, Dr. Eirin, Dr. Greeshma, Dr. Mable Rose, Dr. Neethu, Dr. Prasily, Ms. Reeha, Ms. Reshma, Ms. Shobina, Ms. Smisha, Ms. Sreelegha, Ms. Surya Lisamma and Dr. Tasleema. I also extend my gratitude to the members of Deepthi Cultural Center and Christian Chair and my dear friends Ms. Anila, Dr. Anjima, Mr. Babu, Ms. Chinju, Ms. Gigi, Ms. Linu, Ms. Neethu and Dr. Nisha.

Heartfelt thanks to Ms. Angel, Ms. Aneeta, Ms. Hima, Ms. Josphine, Ms. Melbin, Ms. Nikky, Ms. Rafflesia and Ms. Riya for their friendship and support. I also thank Mr. Abhijith, a research scholar at the Central University of Kerala, Kasaragod, for his pleasant company.

I thank all the postgraduate and M.Phil. students of the department for their support and friendship. I would also like to mention the postgraduate students who helped me academically during this journey: Ms. Abhisha, Ms. Akshara, Ms. Vandana and Ms. Sreeleksmi.

I am deeply grateful to all my teachers who have guided and inspired me over

the years.

I am thankful to UGC, Government of India, for providing financial assistance through the JRF and SRF fellowships.

Most importantly, I express my heartfelt gratitude to my family - without their love and support, this achievement would not have been possible. I am especially indebted to my parents, Ligi and Sunny, for their unconditional love, encouragement and constant support throughout my academic journey. I am truly blessed to have my siblings, Anjaly, Anitta and Albit, whose support has been a source of strength and motivation for me. I also extend my sincere thanks to my cousins and relatives for their encouragement and blessings.

Finally and above all, I bow in gratitude to the Almighty, whose grace made everything possible.

Angela Sunny

University of Calicut

03 February 2026.

ABSTRACT

Set theory, order theory and topology are closely interconnected disciplines that provide a framework for the study of mathematical structures. The interactions among these areas have proved instrumental in a wide range of applications, including computer algorithms, formal languages and automata theory, combinatorics, mathematical modelling, algebra, logic and related fields. Motivated by the study of anti-homogeneous topological spaces, the notion of semi-well ordered sets was introduced to characterize such spaces. This thesis undertakes a detailed investigation of the set-theoretic and order-theoretic properties of semi-well ordered sets. Illustrative examples are presented, additional structural properties are examined and order ideals are studied. Furthermore, the concept of semi-ordinals is introduced and their arithmetic and fundamental properties are explored. We further study the order topology on semi-well ordered sets and show that these spaces are zero-dimensional and strongly locally compact. The isolated points of semi-well ordered sets are identified and characterizations of infinite homogeneous and compact semi-well ordered sets are obtained. In addition, continuous functions between semi-well ordered sets are characterized and it is shown that any function between two such sets is the point-wise limit of a net of continuous functions. Further topological properties are examined and the dynamics of semi-ordinals are also investigated.

We introduce partially semi-well ordered sets as a generalization of both semi-well ordered sets and partially well ordered sets. The main motivation for this generalization is to verify the Aharoni–Korman conjecture ($k = 1$) for this broader class of partially ordered sets. We investigate the order-theoretic and structural properties of partially semi-well ordered sets, the properties of their order ideals and their reversibility. Using a specific structural characterization, we verify the Aharoni–Korman conjecture ($k = 1$) for partially semi-well ordered sets. The non-existence of an infinite antichain in a partially semi-well ordered set ensures that the interval topology and the Dedekind topology coincide on these sets; we refer to this common topology as the intrinsic topology. Conditions for Dedekind completeness, compactness and connectedness of this space are examined. We characterize the closure of subsets and establish a relationship between the relative topology and the intrinsic topology on subsets of partially semi-well ordered sets. We further show that this topological space is anti-rigid and we also discuss additional properties of this topological space. Finally, we characterize continuous functions between partially semi-well ordered sets and derive a result concerning functions between such sets based on this characterization.

Key Words: well ordered set, semi-well ordered set, partially semi-well ordered set, interval topology, order topology, Aharoni-Korman conjecture.

സംഗ്രഹം

ഗണിത ഘടനകളുടെ പഠനത്തിന് അടിസ്ഥാനപരമായ ഒരു ചട്ടക്കൂട് നൽകുന്ന പരസ്പരബന്ധിതമായ ഗണിതശാസ്ത്ര ശാഖകളാണ് സെറ്റ് തിയറി, ഓർഡർ തിയറി, ടോപ്പോളജി എന്നിവ. ഇവയുടെ പരസ്പര ഇടപെടൽ കമ്പ്യൂട്ടർ ആൽഗോരിതങ്ങൾ, ഔപചാരിക ഭാഷകളും ഓട്ടോമാറ്റ തിയറിയും, കോംബിനേറ്റോറിക്സ്, ഗണിത മോഡലിംഗ്, ആൽജിബ്ര, തർക്കശാസ്ത്രം തുടങ്ങിയ വിവിധ മേഖലകളിൽ വ്യാപകമായ പ്രയോഗങ്ങൾക്ക് വഴിയൊരുക്കിയിട്ടുണ്ട്.

ആന്റി-ഹോമജീനിയസ് ടോപ്പോളജിക്കൽ സ്പേസുകളുടെ പഠനമാണ് സെമി-വെൽ ഓർഡേർഡ് ഗണം എന്ന ആശയത്തിന്റെ വികസനത്തിന് പ്രചോദനമായത്. ഇത്തരം സ്പേസുകളെ ക്യാരക്ടറൈസ് ചെയ്യുന്നതിനായി സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങൾ എന്ന ആശയം അവതരിപ്പിക്കപ്പെട്ടു. ഈ ഗവേഷണപ്രബന്ധത്തിൽ സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങളുടെ സെറ്റ്-സിദ്ധാന്തപരവും ക്രമസിദ്ധാന്തപരവുമായ പ്രത്യേകതകൾ വിശദമായി പഠിച്ചിട്ടുണ്ട്. ഉദാഹരണങ്ങളിലൂടെ ഈ ആശയം വിശദീകരിക്കുകയും, അവയുടെ ഘടനാപരമായ സവിശേഷതകളും ഓർഡർ ഐഡിയലുകളും പരിശോധിക്കുകയും ചെയ്യുന്നു. കൂടാതെ, സെമി-ഓർഡിനൽസ് എന്ന ആശയം അവതരിപ്പിച്ച് അവയുടെ അടിസ്ഥാനസവിശേഷതകളും വിശകലനം ചെയ്യുന്നു. തുടർന്ന് സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങളിലെ ഓർഡർ ടോപ്പോളജി കൂടുതൽ വിശദമായി പഠിക്കുന്നു. ഈ സ്പേസുകൾ സീറോ-ഡൈമെൻഷണലും സ്ക്രോംഗ്ലി ലോക്കലി കോംപാക്റ്റും ആണെന്ന് തെളിയിക്കുന്നു. സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങളിലെ ഐസൊലേറ്റഡ് പോയിന്റുകൾ തിരിച്ചറിയുകയും, അനന്തമായ ഹോമജീനിയസ് സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങളും കോംപാക്റ്റ് സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങളും സംബന്ധിച്ച ക്യാരക്ടറൈസേഷനുകൾ അവതരിപ്പിക്കുകയും ചെയ്യുന്നു. കൂടാതെ, സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങൾ തമ്മിലുള്ള കൺസിന്റസ് ഫങ്ഷനുകൾ ക്യാരക്ടറൈസ് ചെയ്യുകയും, ഇത്തരം രണ്ട് ഗണങ്ങൾക്കിടയിലെ ഏതൊരു ഫങ്ഷനും കൺസിന്റസ് ഫങ്ഷനുകളുടെ, നെറ്റിന്റെ പോയിന്റ്-വൈസ് ലിമിറ്റായി രേഖപ്പെടുത്താൻ കഴിയുമെന്ന് തെളിയിക്കുകയും ചെയ്യുന്നു. ഇതിന് പുറമെ, മറ്റ് ടോപ്പോളജിക്കൽ സവിശേഷതകൾ പരിശോധിക്കുകയും സെമി-ഓർഡിനലുകളുടെ ഡൈനാമിക്സ് പഠനവിധേയമാക്കുകയും ചെയ്തിട്ടുണ്ട്.

സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങളുടെയും പാർഷ്യലി വെൽ ഓർഡേർഡ് ഗണങ്ങളുടെയും പൊതുവായ വിപുലീകരണമായി പാർഷ്യലി സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങൾ എന്ന ആശയം നിർവ്വചിക്കുന്നു. ലീനിയർ ഓർഡേർഡ് ഗണം ഒരു പാർഷ്യലി വെൽ ഓർഡേർഡ് ഗണം ആകുന്നത് അത് വെൽ ഓർഡേർഡ് ആണെങ്കിൽ മാത്രമാണ്. അതിനാൽ പാർഷ്യലി വെൽ ഓർഡേർഡ് ഗണങ്ങളും സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങളും വെൽ ഓർഡേർഡ് ഗണത്തിന്റെ രണ്ട് വ്യത്യസ്ത വിപുലീകരണങ്ങളാണ്. ഇതിന്റെ അടിസ്ഥാനത്തിൽ കൂടുതൽ പൊതുവായ പാർഷ്യൽ ഓർഡേർഡ് ഗണങ്ങളുടെ ശേഖരം നിർവ്വചിക്കപ്പെടുന്നു. അഹറോണി-കോർമാൻ അനുമാനം ($k = 1$) അടിസ്ഥാനമാക്കിയാണ് പാർഷ്യലി സെമി-വെൽ ഓർഡേർഡ് ഗണം എന്ന ആശയം വികസിപ്പിച്ചെടുത്തത്. പാർഷ്യലി സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങളുടെ ക്രമസിദ്ധാന്തപരവും ഘടനാപരവുമായ പ്രത്യേകതകൾ പരിശോധിക്കുകയും, അവയുടെ ഓർഡർ ഐഡിയലുകളുടെ സവിശേഷതകളും റിവേഴ്സിബിലിറ്റിയും പഠിക്കുകയും ചെയ്യുന്നു. ഈ ഗണങ്ങളുടെ ഒരു പ്രത്യേക ഘടനാപരമായ ക്യാരക്ടറൈസേഷൻ ഉപയോഗിച്ച് ഇവയിൽ അഹറോണി-കോർമാൻ അനുമാനം ($k = 1$) ശരിയാണെന്ന് സ്ഥാപിക്കുന്നു.

അനന്തമായ ആന്റിചെയിൻ ഒരു പാർഷ്യലി സെമി-വെൽ ഓർഡേർഡ് ഗണത്തിൽ ഉണ്ടാകാത്തത് ഈ ഗണങ്ങളിലെ ഇൻറർവൽ ടോപ്പോളജിയും ഡെഡിക്കിൻഡ് ടോപ്പോളജിയും സമാനമാണെന്ന് ഉറപ്പാക്കുന്നു; ഈ പൊതുവായ ടോപ്പോളജിയെ ഇൻട്രിൻസിക് ടോപ്പോളജി എന്ന് വിളിക്കുന്നു. ഈ സ്പേസിന്റെ ഡെഡിക്കിൻഡ് കമ്പ്ലിറ്റ്നെസ്, കോംപാക്റ്റ്നെസ്, കണക്റ്റഡ്നെസ് എന്നിവ പരിശോധിക്കുന്നു. ഒരു പാർഷ്യലി സെമി-വെൽ ഓർഡേർഡ് ഗണത്തിന്റെ ഉപസെറ്റുകളുടെ ക്ലോഷറുകൾ ക്യാരക്ടറൈസ് ചെയ്യുകയും, ഇവയിലെ റിലേറ്റീവ് ടോപ്പോളജിയും ഇൻട്രിൻസിക് ടോപ്പോളജിയും തമ്മിലുള്ള ബന്ധം സ്ഥാപിക്കുകയും ചെയ്യുന്നു. കൂടാതെ, ഒരു പാർഷ്യലി സെമി-വെൽ ഓർഡേർഡ് ഗണത്തിൽ നിർവ്വചിക്കപ്പെട്ട ഇൻട്രിൻസിക് ടോപ്പോളജി ആന്റി-റിജിഡ് ആണെന്ന് തെളിയിക്കുകയും, ഇതിന്റെ മറ്റ് പ്രത്യേകതകൾ പരിശോധിക്കുകയും ചെയ്യുന്നു. അവസാനമായി, പാർഷ്യലി സെമി-വെൽ ഓർഡേർഡ് ഗണങ്ങൾ തമ്മിലുള്ള കൺസിന്റസ് ഫങ്ഷനുകൾ ക്യാരക്ടറൈസ് ചെയ്യുകയും, ഈ ക്യാരക്ടറൈസേഷൻ അടിസ്ഥാനമാക്കി ഇത്തരം സെറ്റുകൾക്കിടയിലെ ഫങ്ഷനുകളെ സംബന്ധിച്ച ഒരു റിസൾട്ട് അവതരിപ്പിക്കുകയും ചെയ്യുന്നു.

സൂചകപദങ്ങൾ: വെൽ ഓർഡേർഡ് ഗണം, സെമി-വെൽ ഓർഡേർഡ് ഗണം, പാർഷ്യലി സെമി-വെൽ ഓർഡേർഡ് ഗണം, ഇൻറർവൽ ടോപ്പോളജി, ഓർഡർ ടോപ്പോളജി, അഹറോണി-കോർമാൻ അനുമാനം.

Contents

0	Introduction	1
0.1	Motivation and Survey of Literature	1
0.2	Organization of the Thesis	5
1	Preliminaries	8
1.1	Sets with Additional Structures	9
1.1.1	Well Ordered Sets and Ordinal Numbers	12
1.1.2	Arithmetic of Ordinals	14
1.2	Topology	16
1.3	Dynamical Systems	19
2	Semi-Well Ordered Sets and Semi-Ordinals	21
2.1	Introduction	22
2.2	Semi-Well Ordered Sets	23
2.3	Order Ideals of Semi-Well Ordered Sets	29
2.4	Semi-Ordinals	34
2.4.1	Arithmetic of Semi-Ordinals	36
3	Topology on Semi-Well Ordered Sets	44
3.1	Introduction	45
3.2	Topology on Semi-Well Ordered Sets	45
3.3	Dynamics on Semi-Ordinals	63
4	Partially Semi-Well Ordered Sets: Characterizations and Structural Properties	70

Contents

4.1	Introduction	71
4.2	Partially Semi-Well Ordered Sets and its Properties	71
4.3	Order Ideals of Partially Semi-Well Ordered Sets	83
4.4	Reversibility of Partially Semi-Well Ordered Sets	85
5	Partial Semi-Well Ordering and the Aharoni-Korman Conjecture	89
5.1	Introduction	90
5.2	The Aharoni-Korman Conjecture	91
5.3	Some Advancements in the Aharoni-Korman Conjecture	95
5.3.1	Partial Semi-Well Ordering and the Aharoni-Korman Conjecture	97
6	The Topology of a Partially Semi-Well Ordered Set	102
6.1	Introduction	103
6.2	The Topology of a Partially Semi-Well Ordered Set	103
6.2.1	Partially Ordered Sets with No Infinite Antichain	106
6.2.2	Partially Semi-Well Ordered Sets	111
6.2.3	Continuous Functions	115
7	Conclusion and Recommendations	119
7.1	Conclusion	119
7.2	Recommendations	121
	Bibliography	122
	Appendix	129
	Index	131

Introduction

Set theory, order theory and topology are closely connected disciplines that provide a framework for studying mathematical structures. Their interactions also enable a wide range of applications in combinatorics, computer algorithms, mathematical modelling, algebra, logic and related fields. In this introductory chapter, we first present the motivation for the study and provide a review of the relevant literature. This is followed by an outline of the organization of the thesis.

0.1 Motivation and Survey of Literature

In 1979, Paul Bankston [11] introduced a method for finding a new topological property from a given property. The central theme of his work is the uniform generation of new topological properties from existing ones. Given a class of topological spaces S , he defined the class $Anti(S)$. The spectrum of S , denoted by $Spec(S)$, is defined as the class of all cardinal numbers α such that any

topology on a set of cardinality α belongs to S . For example, every finite set is compact, whereas every infinite set admits a non-compact topology. Hence, $Spec(\{\text{compact spaces}\}) = \mathbb{N} \cup \{0\}$. Also, $Spec(\{\text{Lindelöff spaces}\}) = \mathbb{N} \cup \{0, \aleph_0\}$, $Spec(\{\text{connected spaces}\}) = \{0, 1\}$ and $Spec(\{\text{perfect spaces}\}) = \{0\}$.

Now $Anti(S)$ is defined as the class of all spaces X such that, for every subset $A \subset X$, one has $A \in S$ if and only if the cardinality of A belongs to $Spec(S)$. It can be easily seen that $Anti(\{\text{connected spaces}\})$ consists of the totally disconnected spaces, $Anti(\{\text{perfect spaces}\})$ consists of the scattered spaces and $Anti(\{\text{compact spaces}\})$ consists of the pseudofinite spaces.

In his paper [11] Bankston presented several general properties of the operation $Anti(\cdot)$ and investigated anti-compactness, anti-connectedness and anti-Lindelöfness. He also examined the preservation of anti-compactness under various topological operations.

Building on this work, Reily, Vamanamurthy and their collaborators undertook an extensive investigation into various interesting anti-properties [23, 63, 64]. Their research focused on separation axioms and provided several notable results. In fact, they completely determined all the anti-properties of $T_0, T_1, T_2, T_3, T_{3\frac{1}{2}}, T_4$ and T_5 , as well as discreteness, indiscreteness and metrizability. They also studied anti-compactness and anti-Lindelöfness. In [23], they proved that any topological space with at least four elements contains a normal three element subspace, implying that there exists only one non-trivial anti-normal space.

McMaster [46] and his collaborators investigated the total negations of three classical countability conditions in general topology – first countability, second countability and separability. Their work also examined the iterative application of Bankston’s total negation operator, $Anti(\cdot)$, to arbitrary classes of topologi-

cal spaces [45, 47]. In addition, a survey article by Reilly [62] consolidates the research carried out by his team over the past two decades and highlights subsequent developments in the study of anti-properties.

The study of anti-homogeneity and anti-rigidity was initiated by Ramachandran [60, 61], who established that anti-homogeneity is equivalent to hereditary rigidity. Further, Ramachandran and Kannan [35] characterized hereditarily homogeneous spaces in their subsequent work. In this context, Ramachandran introduced the notion of semi-well ordered sets to characterize anti-homogeneous topological spaces. A semi-well ordered set is defined as a partially ordered set in which every non-empty subset of it has either a least element or a greatest element. He also developed several characterizations of these sets and showed that each semi-well ordered structure admits a unique representation as the sum of a well ordered set and co-well ordered set. Moreover, he proved that a partially ordered set is semi-well ordered if and only if it is hereditarily rigid. In 2013, Sreeja [66] further investigated anti-rigid topological spaces and demonstrated that a semi-well ordered set, when equipped with order topology, is both scattered and anti-rigid.

In the initial chapters of this thesis, we undertake a detailed investigation of the set-theoretic, order-theoretic and topological properties of semi-well ordered sets and semi-ordinals. For foundational concepts in set theory and order theory, we refer to the textbooks [22, 28, 36, 43, 51, 68], while the basics of topology are covered in [31, 37, 52, 67]. Moreover, for topologies on linearly ordered sets, we refer to [9, 38].

We also discuss the dynamics of semi-ordinals. For the foundations of dynamical systems we refer the textbooks [2, 13]. We also refer to the research articles [10, 15, 16, 32–34].

Another notable extension of the classical concepts of well ordered sets and ordinal numbers is the theory of partially well ordered sets (also known as well partially ordered sets) and partial ordinals. This theory finds wide applications in computer algorithms, combinatorics, algebra, formal languages, order theory and graph theory. These sets have been extensively studied by Higman [30], Kruskal [39, 40], Jongh and Parikh [18] and Wolk [72]. Some of these authors considered quasi orderings—relations that do not require anti-symmetry—instead of partial orderings [58, 70]. However, except for technical details, the two concepts are the same. The notion of well founded partially ordered sets [44, 50, 65] further generalizes partially well ordered sets and consequently, well ordered sets.

It can be noted that a semi-well ordered set is partially well ordered if and only if it is well ordered. This motivates us to define a more general class of partially ordered sets, which we call partially semi-well ordered sets. We discuss in detail the structural and order-theoretic properties of partially semi-well ordered sets. We also discuss their reversibility and verify the Aharoni-Korman conjecture for $k = 1$ for partially semi-well ordered sets. To study reversibility, we refer to [41, 42, 59] and a detailed literature review on the Aharoni-Korman conjecture is provided in chapter five of this thesis.

We further discuss the properties of the interval topology on partially semi-well ordered sets. Since partially semi-well ordered sets contain no infinite antichain, both the interval topology and the Dedekind topology coincide on them and we refer to this topology as the intrinsic topology. The properties of the topology on partially well ordered sets are discussed in [73]. We also refer to [17, 29, 48, 53, 71] to study the topologies on partially ordered sets.

0.2 Organization of the Thesis

The research presented in the thesis, in addition to the introduction, is organized into seven chapters.

The **first chapter** outlines the essential background needed for the later chapters. It introduces the basic ideas of set theory, topology and topological dynamical systems.

In the **second chapter**, the concept of semi-well ordered sets is illustrated through examples and additional properties of these sets are examined in detail. We study the order ideals of a semi-well ordered set with an aim of determining all of its order ideals. It is shown that the collection of all order ideals of a semi-well ordered set is itself semi-well ordered under the usual set inclusion. Furthermore, we identify the range of surjective order isomorphisms from a semi-well ordered set onto its order ideal. We further introduce the notion of the order type of a semi-well ordered set, called a semi-ordinal, denoted by (α, β^*) , where α is an ordinal number and β is a limit ordinal. We proceed to develop the arithmetic of semi-ordinals and examine its properties. Although not all aspects of classical ordinal arithmetic carry over to this broader concept, several significant properties remain valid.

In **chapter three**, we explore the properties of semi-well ordered set equipped with the order topology. As every semi-well ordered set is linearly ordered, such spaces are Hausdorff, Sober [38] and completely normal [8]. Furthermore, [66] Sreeja established that semi-well ordered sets are scattered and anti-rigid. Building on these foundational results, the second chapter establishes additional properties of this topological space. We show that a semi-well ordered set with the order topology is zero dimensional, totally disconnected and strongly locally com-

pact. We also identify and analyze the isolated points in such spaces. Furthermore, we provide a characterization of infinite homogeneous semi-well ordered sets, compact semi-well ordered sets and continuous functions defined between semi-well ordered sets. Several other properties of these topological spaces are also explored in detail. In the last section of this chapter, we investigate the dynamics of semi-ordinals equipped with the order topology.

In the **fourth chapter**, we define partially semi-well ordered sets. Both semi-well ordered sets and partially well ordered sets can be viewed as extensions of well ordered sets. However, it is important to note that a linearly ordered set is partially well ordered if and only if it is well ordered, highlighting the distinct nature of these two concepts as separate extensions of well ordered sets. This distinction provides the motivation for the introduction of a more general class of ordered structures: partially semi-well ordered sets. The primary objective behind the introduction of partially semi-well ordered set was to prove the Aharoni-Korman conjecture ($k = 1$) for these sets, with a significant characterization of them was utilized in achieving this result. We explore the order-theoretic properties of partially semi-well ordered sets and provide various characterizations of them. Additionally, we investigate their structural properties, the properties of their order ideals and their reversibility.

The **fifth chapter** deals with the Aharoni-Korman conjecture. Here, we provide an introduction to the Aharoni-Korman conjecture, including a brief overview of its development and the motivation behind its proposal. We outline the key advancements related to the conjecture, with a focus on the class of partially ordered sets for which the conjecture has been verified. We verify the Aharoni-Korman conjecture ($k = 1$) for partially semi-well ordered sets using a specific characterization of these partially ordered sets as the base for our proof.

In **chapter six**, we discuss the properties of the intrinsic topology on a partially semi-well ordered set. The non-existence of an infinite antichain in a partially semi-well ordered set ensures that the interval topology and the Dedekind topology coincide on this set. We refer to this common topology as the intrinsic topology. In 1958, Wolk [71] proved that any poset with no infinite antichain is Hausdorff; hence, this topological space is Hausdorff. We then examine the conditions under which this space is Dedekind complete, compact and connected. In addition, we characterize the elements of the closure and establish a relationship between the relative topology and the intrinsic topology on a subset of a partially ordered set with no infinite antichain. Furthermore, we show that the partially semi-well ordered set equipped with intrinsic topology is anti-rigid. Moreover, we verify that it is neither normal nor topologically reversible. More properties of this topological space are explored through examples and counter examples. We also present a characterization for continuous functions between two partially semi-well ordered sets. Using this characterization, we establish a result concerning mapping between such sets.

The **final chapter** presents the conclusion along with several recommendations. A bibliography, a list of research publications and presentations and an index are also included thereafter.

Chapter 1

Preliminaries

This chapter provides a brief overview of the preliminary concepts that will be used in the subsequent chapters. It includes the basic definitions and results that are foundational for later discussions. Specifically, it includes the fundamentals of set theory, topology and topological dynamical systems. The chapter is organized into three sections: the first section covers the preliminary concepts of set theory, the second section outlines the fundamental results in topology and the final section discusses the basic concepts of topological dynamical systems.

1.1 Sets with Additional Structures

Let X be a set and let $\leq$ be a binary relation on X . The relation $\leq$ is said to be a *partial order* [31] on X if it satisfies the following properties for all $x, y, z \in X$:

- (i) *Reflexivity*: $x \leq x$.
- (ii) *Anti-symmetry*: If $x \leq y$ and $y \leq x$, then $x = y$.
- (iii) *Transitivity*: If $x \leq y$ and $y \leq z$, then $x \leq z$.

A set X equipped with a partial order $\leq$ is called a *partially ordered set* (or *poset*) [31] and is denoted by the pair $(X, \leq)$.

A partial order $\leq$ on a set X is said to be *linear* (or *total* or *simple*) [31] if, for all $x, y \in X$, either $x \leq y$ or $y \leq x$ holds. In this case, the pair $(X, \leq)$ is called a *chain* (or *linearly ordered set* or *totally ordered set*) [31].

Let $(X, \leq)$ and $(Y, \leq')$ be two posets. A function $f : X \rightarrow Y$ is said to be *order preserving* [31] if, for all $x, y \in X$, $x \leq y$ implies $f(x) \leq' f(y)$. Furthermore, f is said to be *strictly order preserving* [31] if, for all $x, y \in X$, $x < y$ implies $f(x) <' f(y)$. If f is a bijection and both f and f^{-1} are order preserving, then f is called an *order isomorphism* [31] and the two posets X and Y are said to be *order isomorphic* [31] or to have the same order type. The *order type* [31] of a poset is defined as the class of all posets that are order isomorphic to it.

Let $(X, \leq)$ be a poset and let $A \subseteq X$. An element $x \in X$ is said to be an *upper bound* [31] of A if, for all $a \in A$, $a \leq x$. A set that has at least one upper bound is said to be *bounded above* [31]. The notions of *lower bound* and *bounded*

below are defined analogously. A set that is both bounded above and bounded below is said to be *bounded* [31].

Let $(X, \leq)$ be a poset and let $A \subseteq X$. An element $x \in A$ is called a *minimal element* [31] of A if there does not exist any $a \in A$ such that $a < x$. A set may have more than one minimal element. The concept of *maximal elements* is defined in a similar manner.

Let $(X, \leq)$ be a poset and let $A \subseteq X$. A subset $M \subseteq A$ is said to be a *minimal subset* [72] of A if: (i) each $m \in M$ is a minimal element of A ; and (ii) for every $a \in A$, there exists $m \in M$ such that $m \leq a$. A finite minimal subset refers to a non-empty minimal subset containing a finite number of elements. The notion of a *maximal subset* is defined analogously.

Let $(X, \leq)$ be a poset and let $A \subseteq X$. An element $x \in A$ is said to be the *greatest element* [31] of A if $a \leq x$ for all $a \in A$. It is evident that a set can have at most one greatest element. The concept of the *least element* of a set can be defined similarly.

Let $(X, \leq)$ be a poset and let $A \subseteq X$. The least element of the set of all upper bounds of A (if it exists) is called the *least upper bound* [31] (*l.u.b.*) of A . The concept of the *greatest lower bound* (*g.l.b.*) is defined analogously.

A poset is said to be *complete* [31] if every non-empty subset that is bounded above has a least upper bound. This condition is equivalent to the requirement that every non-empty subset that is bounded below has a greatest lower bound.

Let $(X, \leq)$ be a poset. For elements $x, y \in X$, we say that x and y are *incomparable* [72] if and only if $x \not\leq y$ and $y \not\leq x$. In this case, their incomparability is denoted by $x \parallel y$.

Let $(X, \leq)$ be a poset. A subset $A \subseteq X$ is called *totally unordered* (or an *antichain*) [72] if and only if every pair of distinct elements of A is incomparable.

Let $\leq_1$ and $\leq_2$ be two partial order relations on the same set X . If $\leq_1 \subseteq \leq_2$, then $\leq_2$ is called an *extension* [72] of $\leq_1$.

Theorem 1.1.1. [69] *Any partial order on a set X has a linear extension.*

Let $\{X_a : a \in I\}$ be an arbitrary family of sets. Define $\prod\{X_a : a \in I\} = \{f : I \rightarrow \bigcup_{a \in I} X_a \text{ such that } f(a) \in X_a \text{ for all } a \in I\}$. The elements of $\prod X_a$ are called *choice functions* [72] for the family $\{X_a : a \in I\}$.

Suppose that $\leq_a$ is a partial order defined on each X_a . Then a partial order $\leq$ can be defined on $\prod X_a$ by $f \leq g$ if and only if $f(a) \leq_a g(a)$, for all $a \in I$. The poset $(\prod X_a, \leq)$ is called the *cardinal product* [72] of the family of posets $(X_a, \leq_a)$.

The *dimension* [21] of the poset $(X, \leq)$ is defined as the smallest cardinal k such that the partial order $\leq$ can be expressed as the intersection of k linear orders on X . An equivalent characterization [55] states that the dimension of $(X, \leq)$ is the smallest cardinal k for which the poset $(X, \leq)$ can be isomorphically embedded into the cardinal product of k linearly ordered sets.

A poset $(X, \leq)$ is said to be a *lattice* [31] if every two-element subset of X possesses both a least upper bound (called the join of these elements) and a greatest lower bound (called the meet of these elements). A *complete lattice* [12] is a poset in which every subset has both a least upper bound and a greatest lower bound.

A poset $(X, \leq)$ is called *partially well ordered* [72] if, for every sequence $\{x_n\}$ in X , there exist $i < j$ such that $x_i \leq x_j$. Clearly, every well ordered set is

partially well ordered.

A *well founded* [74] poset is a poset that contains no strictly decreasing sequence. A *dually well founded* [74] poset is a poset that contains no strictly increasing sequence. A well founded poset with no infinite antichain is partially well ordered [72].

For two disjoint chains $(X, \leq)$ and $(Y, \leq')$, the notation $X + Y$ denotes the set $X \cup Y$ equipped with the linear order $\leq \cup \leq' \cup \{(x, x') : x \in X, x' \in Y\}$. Also, for two disjoint posets $(X, \leq)$ and $(Y, \leq')$, the symbol $X \sqcup Y$ denotes the poset $X \cup Y$ equipped with the order $\leq \cup \leq'$.

1.1.1 Well Ordered Sets and Ordinal Numbers

In this subsection, we present the fundamental concepts of well ordered sets and ordinal numbers.

A *well ordered set* [68] is a partially ordered set in which every non-empty subset has a least element. The dual of a well ordered set is referred to as a *co-well ordered set* [61]. It follows immediately that every well ordered set is linearly ordered [68]. In a well ordered set $(X, \leq)$, each element a —except possibly the greatest element, if one exists—has an immediate successor, defined as the least element of the set $\{x \in X : x > a\}$. However, an element in a well ordered set need not have an immediate predecessor. For instance, in the set $\{1 - 1/n : n \in \mathbb{N}\} \cup \{2 - 1/n : n \in \mathbb{N}\} \cup \{2\}$, neither 1 nor 2 has an immediate predecessor.

In a linearly ordered set $(X, \leq)$ with $x \in X$, the sets $Xx = \{y \in X : y < x\}$ and $xX = \{y \in X : x < y\}$ are called the *initial* [68] and *terminal segments*

determined by x , respectively.

A well ordered set is not order isomorphic to any of its initial segments [68]. Moreover, if $(X, \leq)$ and $(Y, \leq')$ be two well ordered sets. Then exactly one of the following holds [68] [43]: (i) X is order isomorphic to Y ; (ii) X is order isomorphic to an initial segment of Y ; or (iii) Y is order isomorphic to an initial segment of X .

Theorem 1.1.2. [68] *The following statements are equivalent:*

- (i) *Zermelo's Axiom of Choice: For every set X , there exists a function f defined on the collection of non-empty subsets of X such that for each non-empty subset $A \subseteq X$, we have $f(A) \in A$.*
- (ii) *Hausdorff's Maximal Principle: Every partially ordered set contains a maximal chain, that is, a chain which is not properly contained in any other chain.*
- (iii) *Zorn's Lemma: Every non-empty partially ordered set in which every chain has an upper bound contains at least one maximal element.*
- (iv) *Well Ordering Principle: Every set can be well ordered.*

The order type of a well ordered set is called an *ordinal number* or simply an *ordinal* [68].

Let α and β be two ordinals. Then $\alpha < \beta$ [68] if and only if there exist representatives $X \in \alpha$ and $Y \in \beta$ such that X is order isomorphic to an initial segment of Y .

Let Δ be a set of ordinals. Then $\alpha \leq \beta$ if $\alpha < \beta$ or $\alpha = \beta$ makes $(\Delta, \leq)$ a partially ordered set, which is in fact well ordered [68]. Furthermore, there exists an ordinal greater than every element of Δ [68] and there exists a least such ordinal.

Theorem 1.1.3. [68] *Let α be an ordinal. Then the set $S(\alpha) = \{\beta : \beta < \alpha\}$ of all ordinals less than α is a well ordered set whose order type is α .*

This result provides a standard representation of an ordinal number. That is, whenever permissible, a set X whose order type is α may be replaced by the set $S(\alpha) = \{\beta : \beta < \alpha\}$, which is order isomorphic to X .

An ordinal other than 0 is referred to as a *limit ordinal* [68] if it lacks an immediate predecessor; otherwise, it is called a *non-limit ordinal*.

1.1.2 Arithmetic of Ordinals

In this section, we discuss the arithmetic operations defined on ordinal numbers and explore their fundamental properties.

Let α, β be two ordinal numbers. Suppose that $(X, \leq)$ and $(Y, \leq')$ are representatives of α and β , respectively.

- (i) Sum of Ordinals [68]: If $X \cap Y = \emptyset$, then the sum of ordinals α and β , denoted $\alpha + \beta$, is defined as the order type of $X + Y$.
- (ii) Product of Ordinals [68]: If either $\alpha = 0$ or $\beta = 0$, then the product of α and β , denoted $\alpha \cdot \beta$, is defined to be 0. If $\alpha > 0$ and $\beta > 0$, then $\alpha \cdot \beta$ is defined as the order type of the set $X \times Y$ equipped with the relation $(a, b) < (a', b')$ if and only if $b < b'$ or $b = b'$ and $a < a'$.

Theorem 1.1.4. [68] *Let α and β be ordinals.*

(i) *If $\beta > 0$, then $\alpha + \beta > \alpha$.*

(ii) *If $\alpha < \beta$, then there exists a unique ordinal $\gamma > 0$ such that $\alpha + \gamma = \beta$.*

(iii) *If $\beta > 0$, then α has a unique representation of the form $\alpha = \beta \cdot \xi + \rho$, where $0 \leq \rho < \beta$.*

(iv) *The ordinal $\alpha + 1$ is the successor of α .*

Theorem 1.1.5. [68] *For ordinals α , β and γ the following statements holds:*

(i) $\alpha < \beta \Leftrightarrow \gamma + \alpha < \gamma + \beta$.

(ii) $\alpha < \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$ and conversely, $\alpha + \gamma < \beta + \gamma \Rightarrow \alpha < \beta$.

(iii) $\alpha < \beta$ and $\gamma > 0 \Rightarrow \gamma \cdot \alpha < \gamma \cdot \beta$; conversely, $\gamma \cdot \alpha < \gamma \cdot \beta \Rightarrow \alpha < \beta$.

(iv) $\gamma + \alpha = \gamma + \beta \Rightarrow \alpha = \beta$.

(v) $\gamma \cdot \alpha = \gamma \cdot \beta$ and $\gamma > 0 \Rightarrow \alpha = \beta$.

We now present the hierarchy of ordinal numbers arranged according to their order [43]. The ordering begins with finite ordinals, followed by the first limit ordinal ω and its successors. This is then followed by the second limit ordinal ω^2 and its successors and so on. This is then followed by the powers of ω :

$0, 1, 2, 3, \dots, \omega, \omega + 1, \omega + 2, \omega + 3, \dots, \omega^2, \omega^2 + 1, \omega^2 + 2, \omega^2 + 3, \dots, \omega^3, \omega^3 + 1, \dots, \omega^4, \dots, \omega^5, \dots, \omega\omega = \omega^2, \omega^2 + 1, \omega^2 + 2, \dots, \omega^2 + \omega, \omega^2 + \omega + 1, \dots, \omega^2 + \omega^2, \dots, \omega^2 + \omega^3, \dots, \omega^2 + \omega^2 = \omega^2 \cdot 2, \omega^2 \cdot 2 + 1, \dots, \omega^2 \cdot 3, \dots, \omega^2 \cdot 4, \dots, \omega^2 \omega = \omega^3, \omega^3 + 1, \dots, \omega^4, \omega^5, \dots, \omega^\omega, \omega^\omega + 1, \dots, \omega^{\omega^\omega}, \dots, \omega^{\omega^{\omega^\omega}}, \dots, \dots$

After all this ordinals, we have ϵ_0 , the smallest ordinal satisfying $\epsilon_0 = \omega^{\epsilon_0}$. Then comes $\epsilon_0 + 1$, $\epsilon_0 + 2$ and so on.

It is important to note that each ordinal enumerated thus far corresponds to the order type of a countable set. The smallest uncountable ordinal is denoted by Ω_0 and Ω' denotes the ordinal $\Omega_0 + 1$ [31].

1.2 Topology

In this section, we present some fundamental concepts and results in topology that are essential for the discussions in the following chapters.

A topological space is said to be *completely normal* [8] if every pair of separated sets A and B are contained in disjoint open sets. The sets A and B are said to be *separated* [8] if neither intersects the closure of the other; that is, $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. It is known [8] that complete normality is equivalent to the condition that all subspaces of the space are normal with respect to their relative topologies.

A topological space is said to be *sober* [38] if every non-empty closed subset is either the closure of a unique point or the union of two proper closed subsets.

Let X be a topological space. An element $x \in X$ is called an *isolated point* [52] if the singleton set $\{x\}$ is open in X . A topological space X is said to be *scattered* [66, 67] if every non-empty subset of X contains an isolated point with respect to the relative topology.

A topological space X is defined as *rigid* [61] if the only homeomorphism from X onto itself is the identity map. A topological space X is called *anti-rigid* [66]

if it contains no rigid subspace with more than one element.

A topological space is said to be *zero dimensional* [67] if it has a basis consisting of clopen subsets.

If every point in a topological space is contained in an open set whose closure is compact, the space is said to be *strongly locally compact* [67].

A topological space X is said to be locally compact at a point $x \in X$ if there exists a compact neighborhood of x . The space X is called *locally compact* [31] if it is locally compact at every point in X . Clearly, every strongly locally compact space is locally compact and a Hausdorff locally compact space is strongly locally compact [67].

A topological space X is said to be *homogeneous* [61] if, for any $x, y \in X$, there exists a homeomorphism f on X such that $f(x) = y$.

A topological space is said to be *paracompact* [31] if it is regular and if every open cover has an open, locally finite refinement that is also a cover.

For a poset $(X, \leq)$, the *interval topology* [72] on X is the topology $\mathcal{T}$ having $\{x \in X : x \leq a\}$ or $\{x \in X : x \geq b\}$ as the sub-base for closed sets.

Let $(X, \leq)$ be a linearly ordered set. Let B denote the collection of all sets of the following types: (i) all open intervals (a, b) in X ; (ii) all intervals of the form $[a_0, b)$, where a_0 is the least element (if it exists) of X ; and (iii) all intervals of the form $(a, b_0]$, where b_0 is the greatest element (if it exists) of X . Then the collection B forms a basis for a topology on X , referred to as the *order topology* [52].

In a linearly ordered set, the interval topology and the order topology coincide.

A subset C of a directed set $(D, \leq)$ is said to be a *cofinal* [31] if, for every $d \in D$, there exists a $c \in C$ such that $d \leq c$.

Let $(D, \leq)$ be a directed set and $(X, \leq)$ be a poset. (Note that the same symbol “ $\leq$ ” is used to denote the order relation in both D and X .) A net $\{S_n : n \in D, \leq\}$ is said to be *monotonically increasing (respectively, decreasing)* [73] if, for all $m, n \in D$ with $m \leq n$, it follows that $S_m \leq S_n$ (respectively, $S_m \geq S_n$). A net is called *monotone* [73] if it is either monotonically increasing or monotonically decreasing.

Let X be a topological space. A net $S : D \rightarrow X$ is said to *converge* [31] to a point $x \in X$ if, for every open set U containing x , there exists an element $d \in D$ such that $S_n \in U$ for all $n \in D$ with $n \geq d$.

Let $\{f_n : n \in D\}$ be a net of functions from a topological space X to a topological space Y . The function $f : X \rightarrow Y$ is said to be the pointwise limit of the net $\{f_n\}$ if, for every $x \in X$, the net $\{f_n(x)\}$ converges to $f(x)$.

Theorem 1.2.1. [31] *Let X and Y be two topological spaces and suppose that X is first countable at a point $x \in X$. Then a function $f : X \rightarrow Y$ is continuous at x if and only if, for every sequence $\{x_n\}$ converging to x , the sequence $\{f(x_n)\}$ converges to $f(x)$.*

Theorem 1.2.2. (Urysohn’s Metrization Theorem) [52] *A second countable topological space is metrizable if and only if it is T_3 .*

Theorem 1.2.3. [31] *A topological space X is compact if and only if every net in X has a subnet that converges to a point in X .*

A subset Y of a topological space T is called a *retract* [52] of T if there exists a

continuous function $f : T \rightarrow Y$ such that the restriction of f to Y is the identity function; that is, $f(y) = y$ for all $y \in Y$.

Proposition 1.2.4. [36] *Let T be a separable metrizable space. Then T is zero-dimensional if and only if every closed subset of T is a retract of T .*

1.3 Dynamical Systems

In this section, we present some preliminary concepts related to topological dynamical systems.

A *dynamical system* [10] is defined as a pair (X, f) , where X is a topological space and $f : X \rightarrow X$ is a continuous function. For a point $x \in X$, the sequence $(x, f(x), f^2(x), \dots)$ is referred to as the *trajectory* of x and the set $\{f^n(x) : n \in \mathbb{N} \cup \{0\}\}$ is called the *orbit* of x . An element $x \in X$ is called a *periodic point* [13] if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$. The smallest such n is referred to as the *period* [13] of x . The set of all periodic points of X under f is denoted by $P(f)$. A subset $A \subseteq X$ is said to be forward f -invariant or simply *forward invariant* [10], if $f(A) \subseteq A$. We now present some examples of dynamical system:

Example 1.3.1. (Circular rotations) [13]. Consider the unit circle $S^1 = [0, 1]/\sim$, where the equivalence relation $\sim$ identifies the points 0 and 1. The usual metric on $[0, 1]$ induces a metric on S^1 , given by $d(x, y) = \min(|x - y|, 1 - |x - y|)$. For $\alpha \in \mathbb{R}$, let R_α be the rotation of S^1 by angle $2\pi\alpha$, that is, $R_\alpha(x) = x + \alpha \pmod{1}$. Then (S^1, R_α) is a dynamical system for each $\alpha \in \mathbb{R}$.

If $\alpha = p/q$ is rational, then R_α^q is the identity map on S^1 and thus every orbit

is periodic. Such rotations are referred to as rational rotations. On the other hand, if α is irrational, then the dynamical system has no periodic points and every orbit is dense in S^1 . These rotations are known as irrational rotations.

Example 1.3.2. (Expanding endomorphisms) [13]. For $m \in \mathbb{Z}$ with $|m| > 1$, define the *times $-m$* map $E_m : S^1 \rightarrow S^1$ by $E_m(x) = mx \pmod{1}$. Then (S^1, E_m) is a dynamical system.

Example 1.3.3. (Tent map) [2] Define the map $T : [0, 1] \rightarrow [0, 1]$ by

$$T(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}], \\ 2 - 2x & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

This map is known as the tent map. An interesting property of the tent map is that it has periodic points of every period. That is, for every $n \in \mathbb{N}$, there exists an $x \in [0, 1]$ such that $T^n(x) = x$ and n is the smallest such positive integer. Moreover, the set of periodic points of T is dense in $[0, 1]$.

Example 1.3.4. [10] Let $X = \{\frac{2m}{2n+1} : 0 \leq m \leq n\} \cup \{1\}$ and define a function $t : X \rightarrow X$ by $t(x) = 1 - |1 - 2x|$ for all $x \in X$. Then (X, t) is a dynamical system. Here, X is a Hausdorff space that is not locally compact and $\overline{P(t)} \setminus P(t) = \{1\}$. Note also that the continuous extension of t to the interval $[0, 1]$ coincides with the tent map.

Chapter 2

Semi-Well Ordered Sets and Semi-Ordinals

A semi-well ordered set is a partially ordered set in which every non-empty subset possesses either a least element or a greatest element. The aim of this work is to investigate the set-theoretic and order-theoretic properties of semi-well ordered sets in detail. In addition, we examine the characteristics of order ideals within a semi-well ordered set. We introduce the notion of semi-ordinals and provide a representation for them. Furthermore, we explore the arithmetic of semi-ordinals and analyze their fundamental properties.

2.1 Introduction

This concept of a semi-well ordered set was introduced by Ramachandran [60,61], who defined such sets as partially ordered sets in which every non-empty subset possesses either a least element or a greatest element. The primary objective behind this introduction was to characterize anti-homogeneous topological spaces using the concept of semi-well ordered sets. He further showed that such sets admit a unique representation as a linear sum of a well ordered set and the dual of a well ordered set. Additionally, he established that a partially ordered set $(X, \leq)$ is semi-well ordered if and only if it is hereditarily rigid. Later, in 2013, Sreeja [66] demonstrated that when equipped with the order topology, semi-well ordered sets are both scattered and anti-rigid spaces.

In this chapter, we undertake a comprehensive study of semi-well ordered sets. Our objective is to extend the classical theory of well ordered sets and ordinal numbers to a broader framework involving semi-well ordered sets and semi-ordinals. This investigation focuses on exploring the order-theoretical properties of these structures.

In the second section, the concept of semi-well ordered sets is illustrated through examples and their additional properties are examined in detail. The third section explores the characteristics of order ideals of a semi-well ordered set, with the aim of determining all of its order ideals. Furthermore, we identify the range of surjective order isomorphisms from a semi-well ordered set onto its order ideal. In the final section, we introduce the notion of the order type of a semi-well ordered set, termed a semi-ordinal, denoted by (α, β^*) , where α is an ordinal number and β is a limit ordinal. We then develop the arithmetic of semi-ordinals and investigate their properties. Although not all aspects of classical

ordinal arithmetic extend to this broader concept, several significant properties remain valid.

In the sequel, we use the abbreviation ‘*swo*’ for semi-well ordered.

2.2 Semi-Well Ordered Sets

Here, we investigate the fundamental properties of *swo* sets and discuss some of their characterizations.

Definition 2.2.1. [60, 61] A poset $(X, \leq)$ is said to be semi-well ordered if every non-empty subset of X has either a least element or a greatest element.

Both well ordered and dually (or co-) well ordered sets are *swo* sets. The following two examples, however, illustrate *swo* sets that are neither well ordered nor co-well ordered.

Example 2.2.2. Let $(\mathbb{N}_0 \times \mathbb{N}, <)$ be a poset where $<$ denote the lexicographic (dictionary) ordering and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N} \setminus \{2^m : m \in \mathbb{N}_0\}$, let $s(n)$ denote the unique pair $(r, k) \in \mathbb{N}_0 \times \mathbb{N}$ such that k is odd and $n = 2^r k$. Now, consider the ordering $\prec$ on the set of positive integers $\mathbb{N}$ as follows: for $a, b \in \mathbb{N}$, define $a \prec b$ if one of the following conditions holds:

- (i) $a \in \mathbb{N} \setminus \{2^m : m \in \mathbb{N}_0\}$ and $b \in \{2^m : m \in \mathbb{N}_0\}$.
- (ii) $a, b \in \mathbb{N} \setminus \{2^m : m \in \mathbb{N}_0\}$ with $s(a) < s(b)$.
- (iii) $a, b \in \{2^m : m \in \mathbb{N}_0\}$ with $a > b$.

2.2. Semi-Well Ordered Sets

That is,

$$\begin{aligned}
 &3 \prec 5 \prec 7 \prec 9 \prec \dots \\
 &\prec 2 \times 3 \prec 2 \times 5 \prec 2 \times 7 \prec \dots \\
 &\quad \vdots \\
 &\prec 2^n \times 3 \prec 2^n \times 5 \prec 2^n \times 7 \prec \dots \\
 &\quad \vdots \\
 &\dots \prec 2^n \prec \dots \prec 2^3 \prec 2^2 \prec 2 \prec 1
 \end{aligned}$$

With this ordering, the set of all positive integers forms a *swo* set. This ordering plays an important role in the study of discrete dynamical systems and is known as Sharkovsky's ordering [14, 16].

Example 2.2.3. The ordering on the set of all integers $\mathbb{Z}$ defined as $0 < 1 < 2 < 3 < \dots < \dots < -3 < -2 < -1$ is a semi-well ordering.

For a *swo* set $(X, \leq)$ with $x, y \in X$ such that $x \neq y$, the subset $\{x, y\}$ has either a least element or a greatest element. Therefore, either $x \leq y$ or $y \leq x$. Hence, $(X, \leq)$ is a chain. Moreover, any subset of X is itself a *swo* set under the order inherited from X .

Remark A *swo* set is partially well ordered if and only if it is well ordered. Thus, the classes of partially well ordered sets and *swo* sets are two distinct extensions of the class of well ordered sets, whose intersection coincides with the class of well ordered sets.

It is straightforward to verify that a linearly ordered set that is order-isomorphic

to a *swo* set is itself *swo*. Moreover, the order isomorphism between two order isomorphic *swo* sets is unique.

The following two theorems, due to Ramachandran [60,61], provide two characterizations of *swo* sets.

Theorem 2.2.5. [60] *Let $(X, \leq)$ be a poset. Then X is *swo* if and only if it can be expressed in the form $\mathcal{W} + \mathcal{C}$, where $(\mathcal{W}, \leq)$ is a well ordered set and $(\mathcal{C}, \leq)$ is a co-well ordered set.*

The set $\mathcal{W}$ is defined as $\mathcal{W} = \{x \in X : x \text{ has no immediate predecessor or } x \text{ is the } n^{\text{th}} \text{ successor of an element having no immediate predecessor}\}$ and $\mathcal{C}$ is defined as its complement, $\mathcal{C} = X \setminus \mathcal{W}$.

Throughout this chapter, a *swo* set will be denoted by $X = \mathcal{W} + \mathcal{C}$.

Definition 2.2.6. A poset $(X, \leq)$ is said to be rigid [60] if identity map is the only order isomorphism from X onto itself. Furthermore, it is said to be *hereditarily rigid* if every subset of X is rigid.

Theorem 2.2.7. [60, 61] *A poset is *swo* if and only if it is hereditarily rigid.*

Remark While a well ordered set is not order isomorphic to any of its initial segment and a co-well ordered set is not order isomorphic to any of its terminal segments, a *swo* set can be order isomorphic to one of its initial or terminal segments. For example, let $X = \{\dots, -3, -2, -1, 0\}$ and $X' = \{0, 1, 2, \dots\}$. Then the initial segment $X_0 = \{\dots, -3, -2, -1\}$ is order isomorphic to X and the terminal segment ${}_0X' = \{1, 2, 3, \dots\}$ is order isomorphic to X' .

2.2. Semi-Well Ordered Sets

The following theorem provides an additional characterization for *swo* sets.

Theorem 2.2.9. *A linearly ordered set is swo if and only if it does not embed $\omega^* + \omega$, where ω is the first infinite ordinal and ω^* denotes the dual of ω .*

Proof. Let $(X, \leq)$ be chain and suppose that X is *swo*. Then $X = \mathcal{W} + \mathcal{C}$. Now, assume the contrary that X embeds $\omega^* + \omega$; that is, X contains a subset of the form $C + D$, where C is a strictly decreasing sequence and D is a strictly increasing sequence. Since C has no least element, it must be entirely contained in $\mathcal{C}$. Therefore, $C + D$ is contained in $\mathcal{C}$. Consequently, $\mathcal{C}$ contains a strictly increasing sequence D , which contradicts the definition of a co-well ordered set. Hence, no such subset exists.

Conversely, suppose that X does not embed $\omega^* + \omega$; that is, it contains no subset of the form $C + D$. Assume that X is not *swo*. Then there exists a non-empty subset $A \subseteq X$ that has neither a least element nor a greatest element. Let $x_0 \in A$. Since A has no least element, there exists $x_{-1} \in A$ with $x_{-1} < x_0$. Similarly, as x_{-1} is not the least element, there exists $x_{-2} \in A$ with $x_{-2} < x_{-1}$. Continuing in this way, we obtain a strictly decreasing sequence $C' = \{\dots, x_{-2}, x_{-1}, x_0\} \subseteq A$. Likewise, since x_0 is not the greatest element of A , there exists $x_1 \in A$ such that $x_0 < x_1$ and again x_1 is not the greatest element, so there exists $x_2 \in A$ with $x_1 < x_2$ and so on. This yields a strictly increasing sequence $D' = \{x_1, x_2, x_3, \dots\} \subseteq A$. Thus, the set $C' + D'$ is a subset of X of the form $C + D$, with C strictly decreasing and D strictly increasing-contradicting the assumption. Hence, X is semi-well ordered. □

Theorem 2.2.10. *A swo set $(X = \mathcal{W} + \mathcal{C}, \leq)$ is complete if and only if $\mathcal{W}$ has a greatest element.*

Proof. Suppose that $(X = \mathcal{W} + \mathcal{C}, \leq)$ is complete. If $\mathcal{C}$ is empty, then X is well ordered and therefore has a greatest element. Now, suppose $\mathcal{C}$ is non-empty. If $\mathcal{W}$ is empty, then $\mathcal{C} = (S(\alpha))^*$ for some limit ordinal α , where $(S(\alpha))^*$ denotes the dual of $(S(\alpha))$. In this case, $X = (S(\alpha))^*$, which contradicts the completeness of X . Therefore, $\mathcal{W}$ must be non-empty whenever $\mathcal{C}$ is non-empty. Hence, $\mathcal{C}$ has lower bounds by elements of $\mathcal{W}$. If these lower bounds have a supremum, then this supremum must belong to $\mathcal{W}$ and is the greatest element of $\mathcal{W}$.

Conversely, suppose that $\mathcal{W}$ has a greatest element. If $\mathcal{C}$ is empty, then $\mathcal{W}$ is well ordered and hence X is complete. Now suppose that $\mathcal{C}$ is non-empty. Let Y be a non-empty subset of X that is bounded above. Then $Y = A + B$, where $A \subseteq \mathcal{W}$ and $B \subseteq \mathcal{C}$. If B is empty, then $A \subseteq \mathcal{W}$ has a least upper bound in $\mathcal{W}$, since $\mathcal{W}$ is well ordered with a greatest element. If B is non-empty, then B has a greatest element, which is the least upper bound of $Y = A + B$. Therefore, X is complete. $\square$

Theorem 2.2.11. *Consider the following two statements:*

- (i) *Every set can be endowed with a semi-well ordering.*
- (ii) *Every set can be well ordered.*

These two statements are equivalent.

Proof. Suppose that (i) holds. Let X be any set. Then by (i), there exists a semi-well ordering $\leq$ on X . By Theorem 2.2.5, X can be expressed as $X = \mathcal{W} + \mathcal{C}$, where $(\mathcal{W}, \leq)$ is well ordered and $(\mathcal{C}, \leq)$ is co-well ordered. If $\mathcal{C}$ is empty, then X is well ordered. Otherwise, since $(\mathcal{C}, \leq)$ is co-well ordered, the reverse ordering $(\mathcal{C}, \geq)$ is well ordered. Define a relation on X by $\leq \cup \geq \cup \{(x, y): x \in \mathcal{W}, y \in \mathcal{C}\}$. This relation defines a well ordering on X . Hence, any set can be well ordered.

Since every well ordered set is *swo*, the converse follows. $\square$

Remark Statement (ii) is the well known Well ordering theorem. Accordingly, we refer to statement (i) as the Semi-Well ordering theorem. Hence, the Axiom of Choice and the Semi-Well ordering theorem are equivalent.

Let X_1 and X_2 be two disjoint *swo* sets, where $X_1 = \mathcal{W}_1 + \mathcal{C}_1$ and $X_2 = \mathcal{W}_2 + \mathcal{C}_2$. Since $\mathcal{W}_1 + \mathcal{W}_2$ is well ordered and $\mathcal{C}_2 + \mathcal{C}_1$ is co-well ordered, we define the relation on $X_1 \cup X_2$ as $(\mathcal{W}_1 + \mathcal{W}_2) + (\mathcal{C}_2 + \mathcal{C}_1)$. Equipped with this relation, $X_1 \cup X_2$ is clearly *swo*. As a generalization, let I be a well ordered indexed set and $\{X_i = \mathcal{W}_i + \mathcal{C}_i: i \in I\}$ be a disjoint collection of *swo* sets. Consider the relation $\leq^*$ on $\bigcup_{i \in I} X_i$ defined by $(\mathcal{W}_i + \mathcal{W}_j) + (\mathcal{C}_j + \mathcal{C}_i)$ for $i < j$.

For example, if I is the set of positive integers with the usual ordering, then the relation on $\bigcup_{i=1}^{\infty} X_i$ can be depicted as $(\mathcal{W}_1 + \mathcal{W}_2 + \cdots) + (\cdots + \mathcal{C}_2 + \mathcal{C}_1)$.

Theorem 2.2.13. *Let I be a well ordered indexed set. If $\{X_i: i \in I\}$ is a disjoint collection of *swo* sets, then the union $\bigcup_{i \in I} X_i$ is *swo* under the relation $\leq^*$.*

2.3 Order Ideals of Semi-Well Ordered Sets

In this section, we examine the order ideals of a *swo* set. A subset A of a poset $(X, \leq)$ is called an *order ideal* (or an initial segment) [72] of X if, for every $y \in A$, $x \leq y$ implies $x \in A$. Let $I(X)$ denote the set of all order ideals of X . Clearly, $I(X)$ contains both the empty set $\emptyset$ and the entire set X . Moreover, $I(X)$ forms a complete lattice under set inclusion.

It is known that if X is well ordered, then $I(X)$ contains no strictly decreasing sequence [72]. We now establish the dual result:

Proposition 2.3.1. *If $(X, \leq)$ is co-well ordered set, then $I(X)$ contains no strictly increasing sequence.*

Proof. Suppose, for contradiction, that $I(X)$ contains a strictly increasing sequence $\{A_n : n \in \mathbb{N}\}$. Since the sequence is strictly increasing, for each n , we can choose an element $a_n \in A_{n+1} \setminus A_n$. Consider the set $A = \{a_n : n \in \mathbb{N}\}$. Clearly, A is a non-empty subset of X and since X is co-well ordered, A must have a greatest element, say a_j . Thus, $a_n \leq a_j$ for all n . In particular, $a_{j+1} \leq a_j$, so $a_{j+1} \in A_{j+1}$ because A_{j+1} is an order ideal. However, according to the construction, $a_{j+1} \in A_{j+2} \setminus A_{j+1}$, which is a contradiction. Hence, $I(X)$ cannot contain a strictly increasing sequence. $\square$

Remark The collection $I(X)$ of order ideals of a semi-well ordered set X can contain a strictly increasing sequence or a strictly decreasing sequence. For example, consider the set of positive integers $\mathbb{N}$ under the usual ordering. Then the sequence of order ideals $\{\{1\}, \{1, 2\}, \{1, 2, 3\}, \dots\}$ is a strictly increasing.

Similarly, consider the set of negative integers under the usual ordering. The sequence $\{\{\dots, -3, -2, -1\}, \{\dots, -3, -2\}, \{\dots, -4, -3\}, \dots\}$ is a strictly decreasing sequence of order ideals.

Lemma 2.3.3. *Suppose that $(X, \leq)$ is a well ordered set. Then the order ideal of X are precisely $\emptyset$, X and the initial segments of the form $Xx = \{y \in X : y < x\}$, where $x \in X$.*

Proof. It is clear that both $\emptyset$ and X are order ideals of X . Let W be a non-empty, non-trivial order ideal of X . Then the complement $X \setminus W$ is a non-empty subset of X and since X is well ordered, $X \setminus W$ has a least element, say a . We claim that $W = Xa$. To verify this, let $y \in W$. If $y \notin Xa$, then $a \leq y$ and since W is an order ideal, it would follow that $a \in W$, which is a contradiction. Hence, $y \in Xa$ and thus $W \subseteq Xa$. Again, let $x \in Xa$. Then $x < a$, which implies $x \notin X \setminus W$, so $x \in W$. Therefore, $Xa \subseteq W$. It follows that $W = Xa$. Thus, every non-empty, non-trivial order ideals of X is of the form Xx , for some $x \in X$. $\square$

Lemma 2.3.4. *The order ideals of a co-well ordered set $(X, \leq)$ are precisely $\emptyset$, X and the sets of the form $Xx \cup \{x\}$, where $x \in X$ and $Xx = \{y \in X : y < x\}$.*

Proof. Clearly, both $\emptyset$ and X are order ideals of X . Let W be a non-empty, proper order ideal of X . Since X is co-well ordered, W has a greatest element, say a . We claim that $W = Xa \cup \{a\}$. If $y \in Xa$, then $y < a$ and since W is an order ideal and $a \in W$, it follows that $y \in W$. Hence, $Xa \cup \{a\} \subseteq W$. Now,

let $x \in W$. Since a is the greatest element of W , we have $x \leq a$. Therefore, $x \in Xa \cup \{a\}$ and so $W \subseteq Xa \cup \{a\}$. Thus, $W = Xa \cup \{a\}$, completing the proof. $\square$

Theorem 2.3.5. *Let $(X, \leq)$ be a swo set such that $X = \mathcal{W} + \mathcal{C}$, where $\mathcal{C}$ is non-empty. Then the order ideals of X are precisely: $\emptyset, X, \mathcal{W}, Xa$ for $a \in \mathcal{W}$ and $Xb \cup \{b\}$ for $b \in \mathcal{C}$.*

Proof. Let A be a non-empty, proper order ideal of X . If $A \subseteq \mathcal{W}$, then by Lemma 2.3.3, either $A = Xa$ for some $a \in \mathcal{W}$ or $A = \mathcal{W}$. Suppose now that A is not entirely contained in $\mathcal{W}$. Then $A = \mathcal{W} \cup A'$, where $A' \subseteq \mathcal{C}$ and A' is an order ideal of $\mathcal{C}$. Since $\mathcal{C}$ is co-well ordered and non-empty, Lemma 2.3.4 implies that either $A' = \mathcal{C}$ or $A' = \mathcal{C}b \cup \{b\}$ for $b \in \mathcal{C}$. Then $A = X$ or $A = Xb \cup \{b\}$ for some $b \in \mathcal{C}$. $\square$

Theorem 2.3.6. *If $(X, \leq)$ is a swo set, then $I(X)$, the collection of order ideals of X , is itself swo under the usual set inclusion.*

Proof. Let $(X, \leq)$ be a non-empty swo set, where $X = \mathcal{W} + \mathcal{C}$. We consider the following three cases:

(i) $\mathcal{W}$ is non-empty and $\mathcal{C}$ is empty.

In this case, X is well ordered and by Lemma 2.3.3, $I(X) = \{X, \emptyset\} \cup \{Xx; x \in X\}$. Let A be a non-empty subset of $I(X)$ with more than one element.

If $\emptyset \in A$, then $\emptyset$ is the least element of A . If $\emptyset \notin A$, then Xa , where a is the

least element of $\{x \in X: Xx \in A\}$, is the least element of A . Hence, every non-empty subset of $I(X)$ has a least element, which implies that $I(X)$ is well ordered.

(ii) $\mathcal{C}$ is non-empty and $\mathcal{W}$ is empty.

Then X is an infinite co-well ordered set and by Lemma 2.3.4, $I(X) = \{\emptyset, X\} \cup \{Xx \cup \{x\}; x \in X\}$. Let A be a non-empty subset of $I(X)$ containing more than one element. If $X \in A$, then X is the greatest element of A . If $X \notin A$, then $Xa \cup \{a\}$, where a is the greatest element of $\{x \in X: Xx \cup \{x\} \in A\}$, is the greatest element of A . Hence, $I(X)$ is co-well ordered.

(iii) Both $\mathcal{C}$ and $\mathcal{W}$ are non-empty.

In this case, using Theorem 2.3.5, we have: $I(X) = \{X, \emptyset, \mathcal{W}\} \cup \{Xx: x \in \mathcal{W}\} \cup \{Xx \cup \{x\}: x \in \mathcal{C}\}$. Let A be a non-empty subset of $I(X)$. If $\emptyset$ or X belongs to A , then they are the least and the greatest element of A , respectively. Suppose A contains neither $\emptyset$ nor X but contains only elements of the form Xx or $\mathcal{W}$, where $x \in \mathcal{W}$. Then, Xa is the least element of A , where a is the least element of $\{x \in X: Xx \in A\}$. On the other hand, if A does not contain $\emptyset$ and X but contains elements of the form $Xb \cup \{b\}$ for $b \in \mathcal{C}$, then $Xy \cup \{y\}$, where y is the greatest element of $\{b \in X: Xb \cup \{b\} \in A\}$, is the greatest element of A . Therefore, A has either a least element or a greatest element and so $I(X)$ is *swo*.

In all the cases, every non-empty subset of $I(X)$ has either a least element or a greatest element. Therefore, $I(X)$ is *swo* under the usual set inclusion. $\square$

Theorem 2.3.7. *Let $(X, \leq)$ be a swo set such that $X = \mathcal{W} + \mathcal{C}$. If $f: X \rightarrow Y$ is an order isomorphism and $Y \in I(X)$, then either $f(X) = X$ or $f(X) = Xc \cup \{c\}$ for some $c \in \mathcal{C}$.*

Proof. Clearly, if $X = \emptyset$, then $f(\emptyset) = \emptyset$ and the result holds trivially. Now, suppose X is non-empty. We consider the following three cases:

(i) $X = \mathcal{W}$.

In this case, X is well ordered and $I(X) = \{X, \emptyset\} \cup \{Xx: x \in X\}$. It is known that a well ordered set is not order isomorphic to any of its initial segments. Hence, $f(X) = X$.

(ii) $X = \mathcal{C}$.

Here, X is an infinite co-well ordered set and $I(X) = \{X, \emptyset\} \cup \{Xx \cup \{x\}: x \in X\}$. Then clearly, $f(X) = X$ or $f(X) = Xc \cup \{c\}; c \in \mathcal{C}$.

(iii) $X = \mathcal{W} + \mathcal{C}$, with both $\mathcal{W}$ and $\mathcal{C}$ non-empty.

In this case, $I(X) = \{X, \emptyset, \mathcal{W}\} \cup \{Xx: x \in \mathcal{W}\} \cup \{Xy \cup \{y\}: y \in \mathcal{C}\}$. We now show that $f(X) = \mathcal{W}$ and $f(X) = Xx$ for any $x \in \mathcal{W}$ are not possible. For it, suppose, for contradiction, that $f(X) = \mathcal{W}$. Now consider $f(\mathcal{W})$. Let $x \in f(\mathcal{W})$ and $y \leq x$, since $\mathcal{W}$ is an order ideal of X and f^{-1} is order preserving, $y \in f(\mathcal{W})$. Therefore, $f(\mathcal{W})$ is an order ideal of X and

$f(\mathcal{W}) \subseteq \mathcal{W}$. Thus, $f(\mathcal{W}) = \mathcal{W}$. However, this contradicts the fact that $\mathcal{C} \neq \emptyset$.

Suppose now that $f(X) = Xx$ for some $x \in \mathcal{W}$. Let $x \in f(\mathcal{W})$ and $y \leq x$. Since f is an order isomorphism and $\mathcal{W}$ is an order ideal, $y \in f(\mathcal{W})$. So, $f(\mathcal{W})$ is an order ideal of X and $f(\mathcal{W}) \subseteq Xx$. Therefore, $f(\mathcal{W}) = Xx_1$ for some $x_1 \leq x$ and $x_1 \in \mathcal{W}$, a contradiction. Hence, the only possibilities of $f(X)$ are: $f(X) = X$ or $f(X) = Xc \cup \{c\}$ for some $c \in \mathcal{C}$.

□

2.4 Semi-Ordinals

In this section, we introduce the concept of semi-ordinals, which generalize both ordinals and co-ordinals. The order type of a co-well ordered set is called a *co-ordinal number* or simply a *co-ordinal*. Recall that if $(X, \leq)$ is a co-well ordered set, then its dual $(X, \geq)$ is well ordered. Accordingly, we denote a co-ordinal as β^* , where β is an ordinal.

The order type of a *swo* set is referred to as a *semi-ordinal number* or simply a *semi-ordinal*. Since any chain that is order isomorphic to a *swo* set is itself *swo*, every representative of a semi-ordinal is a *swo* set. Thus, the class of semi-ordinals properly extends both the class of ordinals and the class of co-ordinals.

We now present a canonical representation of semi-ordinals. For each *swo* set $(X = \mathcal{W} + \mathcal{C}, \leq)$ we associate the ordered pair (α, β^*) , where α is the order type of $\mathcal{W}$ and β^* is the order type of $\mathcal{C}$. In this representation, β is either 0 or a limit ordinal and this representation of a semi-ordinal is unique. Moreover, the class of all semi-ordinals can be viewed as the Cartesian product of the class of

all ordinals with the class of all limit co-ordinals.

For ordinals α_1 and α_2 , $\alpha_1 < \alpha_2$ [68] if and only if there exist well ordered sets A_1 and A_2 , having order types α_1 and α_2 respectively, such that A_1 is order isomorphic to an initial segment of A_2 . Similarly, for co-ordinals β_1^* and β_2^* , we define $\beta_1^* < \beta_2^*$ if and only if there exist co-well ordered sets B_1 and B_2 , of order type β_1^* and β_2^* respectively, such that B_1 is order isomorphic to a terminal segment of B_2 . It follows directly that, for any two ordinals α_1 and α_2 , we have $\alpha_1 < \alpha_2$ if and only if $\alpha_1^* < \alpha_2^*$. Based on this, we define an ordering on semi-ordinals as follows: for semi-ordinals (α_1, β_1^*) and (α_2, β_2^*) , we write $(\alpha_1, \beta_1^*) < (\alpha_2, \beta_2^*)$ if and only if $\beta_1 < \beta_2$ or $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$.

Now let Δ be a set of semi-ordinals. Then $(\Delta, \leq)$ forms a poset, where for any $x, y \in \Delta$, we define $x \leq y$ if and only if $x < y$ or $x = y$. Since any two ordinals are comparable, $(\Delta, \leq)$ is a chain.

Theorem 2.4.1. *Let Δ be any set of semi-ordinals. Then $(\Delta, \leq)$ is well ordered.*

Proof. Let Δ' be a non-empty subset of Δ . Then $\Delta' = \{(\alpha_i, \beta_i^*) : i \in I\}$, where I is some indexed set. Now consider the set $\{\beta_i : i \in I\}$. Since any set ordinals is well ordered, this set has a least element, say β . Let α be the least element of the set $\{\alpha_i : (\alpha_i, \beta) \in \Delta'\}$. Then, (α, β^*) is the least element of Δ' . Hence, $(\Delta, \leq)$ is well ordered. $\square$

Proposition 2.4.2. *For any given semi-ordinal, there exists a semi-ordinal that is strictly greater than the given one.*

Proof. Let (α, β^*) be the given semi-ordinal. Since β is a limit ordinal, there

exists a limit ordinal β' such that $\beta < \beta'$. Thus $(\alpha, \beta^*) < (\alpha, (\beta')^*)$. $\square$

A semi-ordinal other than $(0, 0)$ is said to be a *limit* semi-ordinal if it has no immediate predecessor under the semi-ordinal ordering. All other semi-ordinals are called a *non-limit* semi-ordinal.

2.4.1 Arithmetic of Semi-Ordinals

In this section, we define the arithmetic operations on semi-ordinals and explore their properties.

1. Sum of two semi-ordinals: Let (α_1, β_1^*) and (α_2, β_2^*) be two semi-ordinals.

The sum of these two semi-ordinals is defined as: $(\alpha_1, \beta_1^*) + (\alpha_2, \beta_2^*) = (\alpha_1 + \alpha_2, (\beta_1 + \beta_2)^*)$.

2. Product of two semi-ordinals: The product of two semi-ordinals (α_1, β_1^*) and (α_2, β_2^*) is defined as: $(\alpha_1, \beta_1^*) \cdot (\alpha_2, \beta_2^*) = (\alpha_1 \cdot \alpha_2, (\beta_1 \cdot \beta_2)^*)$.

Remark The sum and product of two semi-ordinals are not commutative, since those of ordinals are not.

Let I be a well ordered indexed set and let $\{(\alpha_i, \beta_i^*) : i \in I\}$ be a collection of semi-ordinals. Suppose $(X_i, \leq_i)$ is a *swo* set of order type (α_i, β_i^*) for each $i \in I$. Then $\{X_i \times \{i\} : i \in I\}$ is a well ordered collection of disjoint *swo* sets. By Theorem 2.2.13, the union $\bigcup_{i \in I} X_i \times \{i\}$ is itself a *swo* set. This motivates the following definition.

Definition 2.4.4. Let I be a well ordered indexed set and let $\{(\alpha_i, \beta_i^*) : i \in I\}$ be a collection of semi-ordinals, where (α_i, β_i^*) denotes the order type of

2.4. Semi-Ordinals

a *swo* set X_i . Then the sum $\sum_{i \in I} (\alpha_i, \beta_i^*)$ is defined as the order type of the *swo* set $\bigcup_{i \in I} X_i \times \{i\}$.

Examples:

$$(a) \sum_{i \in \mathbb{N}} (\omega, \omega^*) = (\omega^2, (\omega^2)^*) \text{ and } \sum_{i \in \mathbb{N}} (1, 0) = (\omega, 0).$$

$$(b) \sum_{i \in \mathbb{N}} (0, \omega^*) = (0, (\omega^2)^*) \text{ and } \sum_{i \in \mathbb{N}} (\omega, 0) = (\omega^2, 0).$$

Proposition 2.4.5. *The following properties hold for semi-ordinals:*

(i) *Associative law for addition.*

(ii) *Associative law for multiplication.*

(iii) *Left distributive law.*

Proof. Let (α_1, β_1^*) , (α_2, β_2^*) and (α_3, β_3^*) be three semi-ordinals.

(i) We know that associative law for addition holds for ordinals. Now,

$$\begin{aligned} ((\alpha_1, \beta_1^*) + (\alpha_2, \beta_2^*)) + (\alpha_3, \beta_3^*) &= ((\alpha_1 + \alpha_2), (\beta_1 + \beta_2)^*) + (\alpha_3, \beta_3^*) \\ &= ((\alpha_1 + \alpha_2) + \alpha_3, ((\beta_1 + \beta_2) + \beta_3)^*) \\ &= (\alpha_1 + (\alpha_2 + \alpha_3), (\beta_1 + (\beta_2 + \beta_3))^*) \\ &= (\alpha_1, \beta_1^*) + ((\alpha_2 + \alpha_3), (\beta_2 + \beta_3)^*) \\ &= (\alpha_1, \beta_1^*) + ((\alpha_2, \beta_2^*) + (\alpha_3, \beta_3^*)) \end{aligned}$$

Thus, addition of semi-ordinals is associative.

2.4. Semi-Ordinals

(ii) We know ordinal multiplication is associative. So,

$$\begin{aligned}
 ((\alpha_1, \beta_1^*) \cdot (\alpha_2, \beta_2^*)) \cdot (\alpha_3, \beta_3^*) &= ((\alpha_1 \cdot \alpha_2), (\beta_1 \cdot \beta_2)^*) \cdot (\alpha_3, \beta_3^*) \\
 &= ((\alpha_1 \cdot \alpha_2) \cdot \alpha_3, ((\beta_1 \cdot \beta_2) \cdot \beta_3)^*) \\
 &= (\alpha_1 \cdot (\alpha_2 \cdot \alpha_3), (\beta_1 \cdot (\beta_2 \cdot \beta_3))^*) \\
 &= (\alpha_1, \beta_1^*) \cdot ((\alpha_2 \cdot \alpha_3), (\beta_2 \cdot \beta_3)^*) \\
 &= (\alpha_1, \beta_1^*) \cdot ((\alpha_2, \beta_2^*) \cdot (\alpha_3, \beta_3^*))
 \end{aligned}$$

Thus semi-ordinal multiplication is associative.

(iii) The distribution law from the left holds for ordinals. So,

$$\begin{aligned}
 (\alpha_1, \beta_1^*) \cdot ((\alpha_2, \beta_2^*) + (\alpha_3, \beta_3^*)) &= (\alpha_1, \beta_1^*) \cdot ((\alpha_2 + \alpha_3), (\beta_2 + \beta_3)^*) \\
 &= (\alpha_1 \cdot (\alpha_2 + \alpha_3), (\beta_1 \cdot (\beta_2 + \beta_3))^*) \\
 &= (\alpha_1 \cdot \alpha_2 + \alpha_1 \cdot \alpha_3, (\beta_1 \cdot \beta_2 + \beta_1 \cdot \beta_3)^*) \\
 &= (\alpha_1 \cdot \alpha_2, (\beta_1 \cdot \beta_2)^*) + (\alpha_1 \cdot \alpha_3, (\beta_1 \cdot \beta_3)^*) \\
 &= ((\alpha_1, \beta_1^*) \cdot (\alpha_2, \beta_2^*)) + ((\alpha_1, \beta_1^*) \cdot (\alpha_3, \beta_3^*))
 \end{aligned}$$

Hence, left distributive law holds for semi-ordinals.

□

Remark It is known that the right distributive law, as well as right cancellation for both addition and multiplication, do not hold for ordinals. Therefore, these properties are also not satisfied by semi-ordinals.

Proposition 2.4.7. *Let (α_1, β_1^*) and (α_2, β_2^*) be semi-ordinals with $(\alpha_2, \beta_2^*) > (0, 0)$. Then $(\alpha_1, \beta_1^*) + (\alpha_2, \beta_2^*) > (\alpha_1, \beta_1^*)$.*

Proof. The condition $(\alpha_2, \beta_2^*) > (0, 0)$ implies that either $\beta_2 > 0$ or $\beta_2 = 0$ and $\alpha_2 > 0$. If $\beta_2 > 0$, then $\beta_1 + \beta_2 > \beta_1$ and hence $(\alpha_1, \beta_1^*) + (\alpha_2, \beta_2^*) = (\alpha_1 + \alpha_2, (\beta_1 + \beta_2)^*) > (\alpha_1, \beta_1^*)$. If $\beta_2 = 0$ and $\alpha_2 > 0$, then $\alpha_1 + \alpha_2 > \alpha_1$ and hence $(\alpha_1, \beta_1^*) + (\alpha_2, \beta_2^*) = (\alpha_1 + \alpha_2, \beta_1^*) > (\alpha_1, \beta_1^*)$. $\square$

Theorem 2.4.8. *Let (α_1, β_1^*) , (α_2, β_2^*) and (α_3, β_3^*) be three semi-ordinals. Then:*

- (i) $(\alpha_1, \beta_1^*) < (\alpha_2, \beta_2^*)$ if and only if $(\alpha_3, \beta_3^*) + (\alpha_1, \beta_1^*) < (\alpha_3, \beta_3^*) + (\alpha_2, \beta_2^*)$.
- (ii) If $(\alpha_3, \beta_3^*) + (\alpha_1, \beta_1^*) = (\alpha_3, \beta_3^*) + (\alpha_2, \beta_2^*)$, then $(\alpha_1, \beta_1^*) = (\alpha_2, \beta_2^*)$. [Left cancellation law for addition]

Proof. (i) Suppose that $(\alpha_1, \beta_1^*) < (\alpha_2, \beta_2^*)$. Then $\beta_1 < \beta_2$ or $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$. If $\beta_1 < \beta_2$, then $\beta_3 + \beta_1 < \beta_3 + \beta_2$ and hence $(\alpha_3, \beta_3^*) + (\alpha_1, \beta_1^*) < (\alpha_3, \beta_3^*) + (\alpha_2, \beta_2^*)$. If $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$, then $\beta_3 + \beta_1 = \beta_3 + \beta_2$ and $\alpha_3 + \alpha_1 < \alpha_3 + \alpha_2$. Therefore, in this case as well, $(\alpha_3, \beta_3^*) + (\alpha_1, \beta_1^*) < (\alpha_3, \beta_3^*) + (\alpha_2, \beta_2^*)$.

For the converse part, assume that $(\alpha_3, \beta_3^*) + (\alpha_1, \beta_1^*) < (\alpha_3, \beta_3^*) + (\alpha_2, \beta_2^*)$. This implies $(\alpha_3 + \alpha_1, (\beta_3 + \beta_1)^*) < (\alpha_3 + \alpha_2, (\beta_3 + \beta_2)^*)$. Therefore, $\beta_3 + \beta_1 < \beta_3 + \beta_2$ or $\beta_3 + \beta_1 = \beta_3 + \beta_2$ and $\alpha_3 + \alpha_1 < \alpha_3 + \alpha_2$. If

2.4. Semi-Ordinals

$\beta_3 + \beta_1 < \beta_3 + \beta_2$, then $\beta_1 < \beta_2$ and $(\alpha_1, \beta_1^*) < (\alpha_2, \beta_2^*)$. If $\beta_3 + \beta_1 = \beta_3 + \beta_2$ and $\alpha_3 + \alpha_1 < \alpha_3 + \alpha_2$, then $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$. So, $(\alpha_1, \beta_1^*) < (\alpha_2, \beta_2^*)$.

(ii) Suppose $(\alpha_3, \beta_3^*) + (\alpha_1, \beta_1^*) = (\alpha_3, \beta_3^*) + (\alpha_2, \beta_2^*)$. Then, this implies $(\alpha_3 + \alpha_1, (\beta_3 + \beta_1)^*) = (\alpha_3 + \alpha_2, (\beta_3 + \beta_2)^*)$. Therefore, $\alpha_3 + \alpha_1 = \alpha_3 + \alpha_2$ and $\beta_3 + \beta_1 = \beta_3 + \beta_2$. Since the left cancellation law holds for ordinal addition, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Thus, $(\alpha_1, \beta_1^*) = (\alpha_2, \beta_2^*)$. □

Remark The following are some properties satisfied by ordinals but do not hold in general for semi-ordinals:

(i) For ordinals α_1, α_2 and α_3 , with $\alpha_1 < \alpha_2$ and $\alpha_3 > 0$, it holds that $\alpha_3 \cdot \alpha_1 < \alpha_3 \cdot \alpha_2$. However, this property does not necessarily hold for semi-ordinals. For example, consider the semi-ordinals $(1, \omega^*)$, $(2, \omega^*)$ and $(0, \omega^*)$. Here, $(1, \omega^*) < (2, \omega^*)$ and $(0, \omega^*) > (0, 0)$, but $(0, \omega^*) \cdot (1, \omega^*) \not< (0, \omega^*) \cdot (2, \omega^*)$. Similarly, although $(2, \omega^*) < (1, \omega 2^*)$ and $(\omega, 0) > (0, 0)$, we have $(\omega, 0) \cdot (2, \omega^*) \not< (\omega, 0) \cdot (1, \omega 2^*)$.

(ii) For ordinals α_1, α_2 and α_3 , if $\alpha_3 \cdot \alpha_1 < \alpha_3 \cdot \alpha_2$, then $\alpha_1 < \alpha_2$. However, this is not necessarily true for semi-ordinals. For example, $(2, 0) \cdot (\omega, (\omega 2)^*) < (2, 0) \cdot (\omega 2, \omega^*)$, but $(\omega, (\omega 2)^*) \not< (\omega 2, \omega^*)$.

(iii) For ordinals α_1, α_2 and α_3 , if $\alpha_3 \cdot \alpha_1 = \alpha_3 \cdot \alpha_2$ and $\alpha_3 > 0$, then $\alpha_1 = \alpha_2$. However, this property does not hold for semi-ordinals. For example, consider the semi-ordinals $(0, \omega^*)$, $(2, \omega^*)$, $(2, (\omega 2)^*)$ and $(2, 0)$. We have

$(0, \omega^*) \cdot (2, \omega^*) = (0, \omega^*) \cdot (0, \omega^*)$ with $(0, \omega^*) > (0, 0)$, yet $(2, \omega^*) \neq (0, \omega^*)$. Similarly, $(2, 0) \cdot (2, \omega^*) = (2, 0) \cdot (2, (\omega 2)^*)$ and $(2, 0) > (0, 0)$, but $(2, \omega^*) \neq (2, (\omega 2)^*)$.

(iv) For ordinals α_1, α_2 and α_3 , if $\alpha_1 \cdot \alpha_3 = \alpha_2 \cdot \alpha_3$ and α_3 is a non-limit ordinal, then $\alpha_1 = \alpha_2$. However, this property does not hold for semi-ordinals. For example, $(\omega, (\omega 2)^*) \cdot (1, 0) = (\omega, \omega^*) \cdot (1, 0)$, where $(1, 0)$ is a non-limit semi-ordinal, yet $(\omega, (\omega 2)^*) \neq (\omega, \omega^*)$. Similarly, $(\omega, (\omega 2)^*) \cdot (1, \omega^*) = (\omega, \omega^*) \cdot (1, \omega^*)$, where $(1, \omega^*)$ is also a non-limit semi-ordinal, but $(\omega, (\omega 2)^*) \neq (\omega, \omega^*)$.

Theorem 2.4.10. *Let (α_1, β_1^*) , (α_2, β_2^*) and (α_3, β_3^*) be semi-ordinals. Then the following hold:*

- (i) *If $(\alpha_1, \beta_1^*) < (\alpha_2, \beta_2^*)$ and $\alpha_3 > 0, \beta_3 > 0$, then $(\alpha_3, \beta_3^*) \cdot (\alpha_1, \beta_1^*) < (\alpha_3, \beta_3^*) \cdot (\alpha_2, \beta_2^*)$. Also, if $(\alpha_3, \beta_3^*) \cdot (\alpha_1, \beta_1^*) < (\alpha_3, \beta_3^*) \cdot (\alpha_2, \beta_2^*)$ and $\beta_3 > 0$, then $(\alpha_1, \beta_1^*) < (\alpha_2, \beta_2^*)$.*
- (ii) *If $(\alpha_3, \beta_3^*) \cdot (\alpha_1, \beta_1^*) = (\alpha_3, \beta_3^*) \cdot (\alpha_2, \beta_2^*)$ and $\alpha_3 > 0, \beta_3 > 0$, then $(\alpha_1, \beta_1^*) = (\alpha_2, \beta_2^*)$.*
- (iii) *If $(\alpha_1, \beta_2^*) \cdot (\alpha_3, 0) = (\alpha_2, \beta_2^*) \cdot (\alpha_3, 0)$ and α_3 is a non-limit ordinal, then $(\alpha_1, \beta_2^*) = (\alpha_2, \beta_2^*)$.*

Proof. (i) Suppose that $(\alpha_1, \beta_1^*) < (\alpha_2, \beta_2^*)$ and $\alpha_3 > 0, \beta_3 > 0$. Then either $\beta_1 < \beta_2$ or $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$. If $\beta_1 < \beta_2$, then $\beta_3 \cdot \beta_1 < \beta_3 \cdot \beta_2$.

2.4. Semi-Ordinals

Therefore, $(\alpha_3, \beta_3^*) \cdot (\alpha_1, \beta_1^*) < (\alpha_3, \beta_3^*) \cdot (\alpha_2, \beta_2^*)$. If $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$, then $\beta_3 \cdot \beta_1 = \beta_3 \cdot \beta_2$ and $\alpha_3 \cdot \alpha_1 < \alpha_3 \cdot \alpha_2$. Thus, again, $(\alpha_3, \beta_3^*) \cdot (\alpha_1, \beta_1^*) = (\alpha_3 \cdot \alpha_1, (\beta_3 \cdot \beta_1)^*) < (\alpha_3 \cdot \alpha_2, (\beta_3 \cdot \beta_2)^*) = (\alpha_3, \beta_3^*) \cdot (\alpha_2, \beta_2^*)$.

Now, suppose that $(\alpha_3, \beta_3^*) \cdot (\alpha_1, \beta_1^*) < (\alpha_3, \beta_3^*) \cdot (\alpha_2, \beta_2^*)$ and $\beta_3 > 0$. Then either $\beta_3 \cdot \beta_1 < \beta_3 \cdot \beta_2$ or $\beta_3 \cdot \beta_1 = \beta_3 \cdot \beta_2$ and $\alpha_3 \cdot \alpha_1 < \alpha_3 \cdot \alpha_2$. If $\beta_3 \cdot \beta_1 < \beta_3 \cdot \beta_2$, then $\beta_1 < \beta_2$. Therefore, $(\alpha_1, \beta_1^*) < (\alpha_2, \beta_2^*)$. If $\beta_3 \cdot \beta_1 = \beta_3 \cdot \beta_2$ and $\alpha_3 \cdot \alpha_1 < \alpha_3 \cdot \alpha_2$, then $\beta_1 = \beta_2$ and $\alpha_1 < \alpha_2$, hence again $(\alpha_1, \beta_1^*) < (\alpha_2, \beta_2^*)$.

(ii) Suppose $(\alpha_3, \beta_3^*) \cdot (\alpha_1, \beta_1^*) = (\alpha_3, \beta_3^*) \cdot (\alpha_2, \beta_2^*)$ with $\alpha_3 > 0$ and $\beta_3 > 0$. Then, by the definition of multiplication on semi-ordinals, $(\alpha_3 \cdot \alpha_1, (\beta_3 \cdot \beta_1)^*) = (\alpha_3 \cdot \alpha_2, (\beta_3 \cdot \beta_2)^*)$. This equality implies that $\alpha_3 \cdot \alpha_1 = \alpha_3 \cdot \alpha_2$ and $\beta_3 \cdot \beta_1 = \beta_3 \cdot \beta_2$. Since $\alpha_3 > 0$ and $\beta_3 > 0$, by the left cancellation law for ordinals, it follows that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Hence, $(\alpha_1, \beta_1^*) = (\alpha_2, \beta_2^*)$.

(iii) Suppose $(\alpha_1, \beta_2^*) \cdot (\alpha_3, 0) = (\alpha_2, \beta_2^*) \cdot (\alpha_3, 0)$. By the definition of multiplication on semi-ordinals, this implies $(\alpha_1 \cdot \alpha_3, 0) = (\alpha_2 \cdot \alpha_3, 0)$. Thus, $\alpha_1 \cdot \alpha_3 = \alpha_2 \cdot \alpha_3$, where α_3 is a non-limit ordinal. By the property of ordinals, it follows that $\alpha_1 = \alpha_2$. Therefore, $(\alpha_1, \beta_2^*) = (\alpha_2, \beta_2^*)$.

□

Remark It is known that for any ordinals α_1 and α_2 with $\alpha_1 < \alpha_2$, there exists a unique ordinal $\alpha_3 > 0$ such that $\alpha_1 + \alpha_3 = \alpha_2$. However, this property does not hold in the case of semi-ordinals. For example, consider the semi-ordinals (ω, ω^*) and $(1, (\omega 2)^*)$. Clearly, $(\omega, \omega^*) < (1, (\omega 2)^*)$, but there exists no semi-ordinal (α, β^*) such that $(\omega, \omega^*) + (\alpha, \beta^*) = (1, (\omega 2)^*)$. The following theorem

presents a specific case in which this property holds for semi-ordinals.

Theorem 2.4.12. *Let (α_1, β_1^*) and (α_2, β_2^*) be two semi-ordinals such that $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. Then, there exists a unique semi-ordinal (α_3, β_3^*) with $\alpha_3 > 0$ and $\beta_3 > 0$ such that $(\alpha_1, \beta_1^*) + (\alpha_3, \beta_3^*) = (\alpha_2, \beta_2^*)$.*

Proof. Since $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, there exist unique ordinals $\alpha_3 > 0$ and $\beta_3 > 0$ such that $\alpha_1 + \alpha_3 = \alpha_2$ and $\beta_1 + \beta_3 = \beta_2$. Hence, $(\alpha_1, \beta_1^*) + (\alpha_3, \beta_3^*) = (\alpha_2, \beta_2^*)$. Since α_3 and β_3 are uniquely determined, the semi-ordinal (α_3, β_3^*) is uniquely defined and satisfies the required condition. $\square$

Theorem 2.4.13. *In the class of all semi-ordinals, every set of semi-ordinals has a least upper bound.*

Proof. Let Γ denote the class of all semi-ordinals and let $X = \{(\alpha_i, \beta_i^*) : i \in I\}$ be a set of semi-ordinals, where I is an index set. Define $Y = \{\beta_i : i \in I\}$. Then Y is a set of ordinals and hence it has a least upper bound, say β . Now, this β may or may not belong to Y . If $\beta \in Y$, then consider the set $Z = \{\alpha_i : (\alpha_i, \beta_i^*) \in X\}$. Since Z is a set of ordinals, it has a least upper bound, say α . Then, (α, β^*) is the least upper bound of X . On the other hand, if $\beta \notin Y$, then β is a limit ordinal and $(0, \beta^*)$ is the least upper bound of X . $\square$

Chapter 3

Topology on Semi-Well Ordered Sets

In this chapter, we investigate the properties of order topology on semi-well ordered sets. We demonstrate that these topological spaces are zero dimensional, strongly locally compact and totally disconnected. The isolated points of a semi-well ordered set are identified and a characterization of compact semi-well ordered sets, infinite homogeneous semi-well ordered sets and continuous functions defined between semi-well ordered sets are provided. Several additional properties of these topological spaces are also examined. Finally, we investigate the dynamics of semi-ordinals.

3.1 Introduction

In this chapter, we explore the properties of semi-well ordered set equipped with the order topology. As every semi-well ordered set is linearly ordered, such spaces are Hausdorff, completely normal [8] and Sober [38]. Furthermore, Sreeja [66] established that semi-well ordered sets together with order topology are both scattered and anti-rigid.

Building on these foundational results, the second section establishes additional properties of this topological space. We show that a semi-well ordered set with the order topology is zero dimensional, totally disconnected and strongly locally compact. We also identify and analyze the isolated points in such spaces. Furthermore, we provide a detailed characterization of homogeneous and compact semi-well ordered sets. Further properties of these topological spaces are also explored in detail.

In the last section, we investigate the dynamics of semi-ordinals, in particular, the dynamics of semi-well ordered sets of order type $(\omega^2, (\omega^2)^*)$. The motivation for considering $(\omega^2, (\omega^2)^*)$, along with the key observations derived from our findings, is discussed in detail.

Throughout this chapter, we often denote a semi-well ordered set by $X = \mathcal{W} + \mathcal{C}$.

3.2 Topology on Semi-Well Ordered Sets

Here, we examine the properties of order topology on semi-well ordered sets.

Theorem 3.2.1. *Let $(X, \leq)$ be a swo set and let $Y \subseteq X$. Then the order topology on Y is a subset of the relative topology on Y inherited from X .*

3.2. Topology on Semi-Well Ordered Sets

Proof. Let $\mathcal{T}$ and $\mathcal{D}$ denote the order topology and the relative topology on Y , respectively. If X is well ordered or co-well ordered, then clearly $\mathcal{T} \subseteq \mathcal{D}$. Now suppose that X is neither well ordered nor co-well ordered. Then X has both a least element and a greatest element, denoted by x_0 and y_0 , respectively. Given $Y \subseteq X$, suppose Y has a least element, say a . Then any interval of the form $[a, y)$, where $y \in Y$ and $y \neq a$ is a basis element of $\mathcal{T}$. Now, $[a, y) = [x_0, y) \cap Y$, so $[a, y) \in \mathcal{D}$. Also, if $(x, y) \in \mathcal{T}$ for $x, y \in Y$, then $(x, y) = (x, y) \cap Y$, so $(x, y) \in \mathcal{D}$. Again, if Y has a greatest element, say b , then the intervals of the form $(x, b]$, where $x \in Y$ and $x \neq b$ are basis elements of $\mathcal{T}$. Also, $(x, b] = (x, y_0] \cap Y$, so $(x, b] \in \mathcal{D}$. Since all the basis elements of $\mathcal{D}$ are elements of $\mathcal{T}$, it follows that $\mathcal{T} \subseteq \mathcal{D}$. $\square$

Remark By Theorem 3.2.1, the order topology on Y is always a subset of the subspace topology, but the two topologies need not coincide. For example, consider the *swo* set with ordering $0 < 1 < 2 < \dots < \omega < \omega + 1 < \omega + 2 < \dots < \omega^2 < \omega^2 + 1 < \dots < \omega^3 < \dots < \omega^4 < \dots < \omega^5 < \dots < \dots < \dots < (\omega^3)^* < \dots < (\omega^2)^* < \dots < (\omega + 1)^* < \omega^* < \dots < 2^* < 1^* < 0^*$. The order type of this *swo* set is $(\omega^2, (\omega^2)^*)$. Let $Y = \{\omega + 1\} + \{\dots < 2^* < 1^* < 0^*\}$, with the order inherited from X . Here, the set $\{\omega + 1\} = (\omega, \omega + 2) \cap Y$ is open in the subspace topology on Y , but it is not open in the order topology on Y .

Since every *swo* set is linearly ordered, the following result follows directly from Theorem 4 in [9].

Theorem 3.2.3. *Let $(X, \leq)$ be a swo set and let $Y \subset X$. Then the order*

topology on Y is equivalent to the relative topology on Y (inherited from X) if and only if, for any $y \in Y$ and $x \in X \setminus Y$, there exists an element $a \in Y$ between x and y or if $y < x$, then the set $\{k \in Y : x < k\}$ has a least element or if $x < y$, then the set $\{k \in Y : k < x\}$ has a greatest element.

Corollary 3.2.4. *Let $X = \mathcal{W} + \mathcal{C}$ be a swo set, where $\mathcal{W}$ has no greatest element. Let Y be a non-empty swo set. Then $X + Y$ is a swo set and the subspace topology on X , viewed as a subspace of $X + Y$, is equivalent to the order topology on X .*

Proof. Let $Y = \mathcal{W}_1 + \mathcal{C}_1$, where $\mathcal{W}_1$ is well ordered and $\mathcal{C}_1$ is co-well ordered. Then $X + Y = (\mathcal{W} + \mathcal{W}_1) + (\mathcal{C}_1 + \mathcal{C})$. Now, for any $x \in X$ and $y \in (X + Y) \setminus X$ there exists an element between x and y , since $\mathcal{W}$ has no greatest element. Hence, by Theorem 3.2.3, the corollary follows. $\square$

The following lemma is used in establishing subsequent theorems.

Lemma 3.2.5. *Let $(X, \leq)$ be a swo set, where $X = \mathcal{W} + \mathcal{C}$ and $\mathcal{W}$ is non-empty. Then:*

- (i) *If $x \in \mathcal{W}$, where x is neither the least nor the greatest element of $\mathcal{W}$ and I is an open set containing x , then there exists an open set in X of the form $(y, x]$ such that $y \in A$ and $x \in (y, x] \subseteq I$.*
- (ii) *If $y \in \mathcal{C}$, where y is not the greatest element of $\mathcal{C}$ and I is an open set containing y , then there exists an open set in X of the form $[y, z)$ such that $z \in C$ and $y \in [y, z) \subseteq I$.*

Proof. (i) Let x_0 be the least element of X and let $x \in \mathcal{W}$. Then, we have the following:

- (a) If $x \in [x_0, y)$, then $x \in (x_0, x] \subseteq [x_0, y)$.
- (b) If $x \in (y_1, y_2)$, then $x \in (y_1, x] \subseteq (y_1, y_2)$.
- (c) If X has a greatest element, say y_0 and $x \in (y, y_0]$, then $x \in (y, x] \subseteq (y, y_0]$.

(ii) Let y_0 be the greatest element of X . Then:

- (a) If $y \in (x, y_0]$, then $y \in [y, y_0) \subseteq (x, y_0]$.
- (b) If $y \in (x_1, x_2)$, then $y \in [y, x_2) \subseteq (x_1, x_2)$.
- (c) If $y \in [x_0, x)$, then $x \in [y, x) \subseteq [x_0, x)$.

□

The last element of a well ordered set, if it exists, may be either isolated or non-isolated. Moreover, this last element is isolated if and only if it has an immediate predecessor. For example, in the well ordered set $S(\omega + 1)$, the last element ω is non-isolated, whereas in the well ordered set $S(\omega + 2)$, the last element $\omega + 1$ is an isolated point. The following theorem provides a characterization of the isolated points of a semi-well ordered set.

Theorem 3.2.6. *Let $(X = \mathcal{W} + \mathcal{C}, \leq)$ be a swo set, where $\mathcal{W}$ has no greatest element. Then the isolated points of X are precisely those elements that include the least element of $\mathcal{W}$ (if it exists), the greatest element of $\mathcal{C}$ (if it exists), the elements of $\mathcal{W}$ that have an immediate predecessor and the elements of $\mathcal{C}$ that have an immediate successor.*

Proof. We prove the theorem by considering different cases separately.

(i) Suppose $\mathcal{W}$ is non-empty and $\mathcal{C}$ is empty.

Let x_0 be the least element of X . Since $\{x_0\}$ is open in X , x_0 is an isolated point. Let $x \neq x_0$ be an element of X with an immediate predecessor, say x_{-1} . Then, $\{x\} = (x_{-1}, x_1)$ is an open set in X , where x_1 is the immediate successor of x . Hence, x is also an isolated point. On the other hand, if $y \neq x_0$ is an element of X with no immediate predecessor, then by Lemma 3.2.5 any open set containing y contains subsets of the form $(a, y]$, where $a < y$. Therefore, y is not an isolated point.

(ii) Suppose $\mathcal{W}$ is empty and $\mathcal{C}$ is non-empty,

Let y_0 be the greatest element of X . Clearly, $\{y_0\}$ is open in X , so y_0 is an isolated point. If $y \neq y_0$ is an element of X with an immediate successor, then $\{y\} = (y_{-1}, y_1)$, where y_{-1} and y_1 are the immediate predecessor and immediate successor of y , respectively. Thus, y is an isolated point. If $x \neq y_0$ is an element of X with no immediate successor, then by Lemma 3.2.5 every open set containing x contains a subset of the form $[x, a)$, where $x < a$. Hence, x is not an isolated point.

(iii) Suppose both $\mathcal{W}$ and $\mathcal{C}$ are non-empty.

In this case, X has both a greatest element and a least element. Let these be y_0 and x_0 , respectively. Clearly, $\{x_0\}$ and $\{y_0\}$ are open in X , so both x_0 and y_0 are isolated points. Let $x \neq x_0$ be an element of $\mathcal{W}$ with no

immediate predecessor. Then, by Lemma 3.2.5, x is not an isolated point. However, if x has an immediate predecessor, then x is an isolated point. Similarly, if $y \neq y_0$ is an element of $\mathcal{C}$ with an immediate successor, then y is an isolated point. If y has no immediate successor, then by Lemma 3.2.5, y is not an isolated point of X . $\square$

Remark If $(X = \mathcal{W} + \mathbb{C}, \leq)$ is a *swo* set in which $\mathcal{W}$ has a greatest element, say w and $\mathcal{C}$ is non-empty, then w is not an isolated point of X .

Proposition 3.2.8. *If $(X = \mathcal{W} + \mathcal{C}, \leq)$ is a swo set where $\mathcal{W}$ has no greatest element, then both $\mathcal{W}$ and $\mathcal{C}$ are open in X .*

Proof. Let $x \in \mathcal{W}$. If x is the least element of $\mathcal{W}$, then $\{x\}$ is an open set in X and is contained in $\mathcal{W}$. If x is not the least element of $\mathcal{W}$, then $(y, x]$, where $y \in \mathcal{W}$ is an open set in X containing x contained in $\mathcal{W}$. Therefore, $\mathcal{W}$ is open in X . Similarly, we can prove that $\mathcal{C}$ is open in X . $\square$

If $\mathcal{W}$ has the greatest element, then $\mathcal{W}$ need not be open in X . As an illustration, consider the following example: Let X be a *swo* set with ordering given by $1 < 2 < 3 < \dots < 0 < \dots < -3 < -2 < -1$. Then $\mathcal{W} = \{1, 2, 3, \dots, 0\}$ and the complement of $\mathcal{W}$ is not closed in X . Thus, $\mathcal{W}$ is not open in X .

Recall that every linearly ordered set, together with the order topology, is completely normal [8]. So, we have the following result.

Theorem 3.2.9. *A semi-well ordered set, together with order topology, is completely normal.*

Theorem 3.2.10. *Let $(X = \mathcal{W} + \mathcal{C}, \leq)$ be a non-empty swo set. Then X is compact if and only if $\mathcal{W}$ has a greatest element.*

Proof. Assume that X is compact. Suppose, for contradiction, that both $\mathcal{W}$ and $\mathcal{C}$ are non-empty and that $\mathcal{W}$ has no greatest element. Let x_0 and y_0 denote the least and greatest elements of X , respectively. Let $U = \{[x_0, x) : x \in \mathcal{W}\}$ and $V = \{(y, y_0] : y \in \mathcal{C}\}$. Then $U \cup V$ is an open cover of X , where U covers $\mathcal{W}$ and V covers $\mathcal{C}$ by open subsets of X . Since X is compact, this cover has a finite sub-cover. Note that U and V consists of nested intervals. Therefore, there exist $[x_0, x') \in U$ and $(y', y_0] \in V$ such that $\mathcal{W} = [x_0, x')$ and $\mathcal{C} = (y', y_0]$. Hence, $X = [x_0, x') + (y', y_0]$ and $x', y' \notin X$, which is a contradiction. This contradiction implies that $\mathcal{W}$ must have a greatest element.

For the converse, assume that $\mathcal{W}$ has a greatest element, say w . If $\mathcal{C}$ is empty, then $X = \mathcal{W}$, which is a well ordered set with a greatest element. Hence, X is compact. Now suppose that $\mathcal{C}$ is non-empty. Let x_0 and y_0 denote the least and greatest elements of X , respectively. Let $\mathcal{U}$ be an open cover of X consisting of basic open sets. We then consider the following cases:

- (i) The open set in $\mathcal{U}$ that contains w is a basic open set of the form (x, y) , where $x \in \mathcal{W}$ and $y \in \mathcal{C}$.

Without loss of generality, assume that all elements of $\mathcal{U}$, other than (x, y) , is of the form $\{x_0\}$, $\{y_0\}$, $(x, x'] \subset \mathcal{W}$ or $[y, y') \subset \mathcal{C}$. Clearly, $\{\{x_0\}, \{y_0\}, (x, y)\}$ is a subset of $\mathcal{U}$. If $x \neq x_0$, let $b_{-1} = x$. Since $x \neq x_0$, there exists

$a_{-1} \in \mathcal{W}$ such that $(a_{-1}, b_{-1}] \in \mathcal{U}$. If $a_{-1} = x_0$, then the collection $\{\{x_0\}, (a_{-1}, b_{-1}], (x, y)\}$ will cover $\mathcal{W}$. If $a_{-1} \neq x_0$, set $a_{-1} = b_{-2}$. Then, there exists $a_{-2} \in \mathcal{W}$ such that $(a_{-2}, b_{-2}] \in \mathcal{U}$ and $a_{-1} \in (a_{-2}, b_{-2}]$. If $a_{-2} = x_0$, then the argument is complete. If not, the process continues. Eventually, for some n , $a_{-n} = x_0$; otherwise, we would obtain a strictly decreasing sequence $\{a_{-1}, a_{-2}, a_{-3}, \dots\}$ in $\mathcal{W}$, which is not possible since $\mathcal{W}$ is well ordered.

Similarly, if $y \neq y_0$, let $b_1 = y$. Then there exists $a_1 \in \mathcal{C}$ with $[b_1, a_1) \in \mathcal{U}$. If $a_1 = y_0$, then the collection $\{\{x_0\}, (x_{-1}, y_{-1}], \dots, (x_{-n}, y_{-n}], (x, y), [y, y_0), \{y_0\}\}$ is a finite subcover of $\mathcal{U}$. If $a_1 \neq y_0$, set $a_1 = b_2$. Then there exists $[b_2, a_2) \in \mathcal{U}$ such that $a_1 \in [b_2, a_2)$ and $a_1 < a_2$. If $a_2 = y_0$, we are done. If not, the process continues. Eventually, for some m , $a_m = y_0$; otherwise, the sequence $\{a_1, a_2, \dots\}$ would be a strictly increasing sequence in $\mathcal{C}$, which is not possible since $\mathcal{C}$ is co-well ordered. Therefore, $\{\{x_0\}, (x_{-1}, y_{-1}], \dots, (x_{-n}, y_{-n}], (x, y), [b_1, a_1), \dots, [b_m, a_m), \{y_0\}\}$ is a finite subcover of $\mathcal{U}$.

(ii) The open set in $\mathcal{U}$ that contains w is a basic open set of the form $(x, y_0]$, where $x \in \mathcal{W}$.

(iii) The open set in $\mathcal{U}$ that contains w is a basic open set of the form $[x_0, y)$, where $y \in \mathcal{C}$.

The cases (ii) and (iii) can be proved similarly. Therefore, we obtain a finite subcover of $\mathcal{U}$ and hence X is compact. $\square$

Proposition 3.2.11. *If $(X = \mathcal{W} + \mathcal{C}, \leq)$ is a swo set, then X is zero dimensional.*

Proof. We consider the following cases:

(i) $\mathcal{W}$ is non-empty and $\mathcal{C}$ is empty.

Let x_0 be the least element of X . Then the collection $\{\{x_0\}\} \cup \{(x, w] : x, w \in \mathcal{W}, w \neq x_0\}$ forms a clopen basis for X .

(ii) $\mathcal{W}$ is empty and $\mathcal{C}$ is non-empty.

Let y_0 be the greatest element of $\mathcal{C}$. Then the collection $\{\{y_0\}\} \cup \{[y, c) : y, c \in \mathcal{C}, y \neq y_0\}$ forms a clopen basis for X .

(iii) $\mathcal{W}$ is a non-empty set with no greatest element and $\mathcal{C}$ is non-empty.

Let x_0 and y_0 be the least and greatest elements of X , respectively. Then the collection $\{\{x_0\}\} \cup \{(x_1, x_2] : x_1, x_2 \in \mathcal{W}, x_2 \neq x_0\} \cup \{\{y_0\}\} \cup \{[y_1, y_2) : y_1, y_2 \in \mathcal{C}, y_1 \neq y_0\}$ forms a clopen basis for X .

(iv) $\mathcal{W}$ is a non-empty set with a greatest element, say a and $\mathcal{C}$ is non-empty.

If $x_0 = a$, then the collection $\{[a, c) : c \in \mathcal{C}\} \cup \{\{y_0\}\} \cup \{[y_1, y_2) : y_1, y_2 \in \mathcal{C}, y_1 \neq y_0\}$ forms a clopen basis for X . If $x_0 \neq a$, then $\{\{x_0\}\} \cup \{(x_1, x_2] : x_1, x_2 \in \mathcal{W}, x_2 \neq x_0, x_2 \neq a\} \cup \{\{y_0\}\} \cup \{[y_1, y_2) : y_1, y_2 \in \mathcal{C}, y_1 \neq y_0\} \cup \{(w_1, c_1) : w_1 \in \mathcal{W}, c_1 \in \mathcal{C}, w_1 < a\}$ forms a clopen basis for X .

In all cases, X admits a clopen basis. Therefore, X is zero dimensional. $\square$

Proposition 3.2.12. *Every swo set is strongly locally compact.*

Proof. Let $(X = \mathcal{W} + \mathcal{C}, \leq)$ be a swo set. If $\mathcal{C}$ is empty, then X is a well ordered set and hence it is strongly locally compact. Now suppose that $\mathcal{C}$ is non-empty. If $\mathcal{W}$ has a greatest element, then by Theorem 3.2.10, X is compact. Therefore, X is strongly locally compact in this case as well. Assume now that $\mathcal{W}$ has no greatest element. Let x_0 be the least element of $\mathcal{W}$ and let y_0 be the greatest element of $\mathcal{C}$. For the element x_0 , the singleton $\{x_0\}$ is an open set in X whose closure is $\{x_0\}$, which is compact. For any $x \in \mathcal{W}$ with $x \neq x_0$, the set $(w, x]$, where $w \in \mathcal{W}$, is a clopen neighbourhood of x that is compact. Similarly, if $y \in \mathcal{C}$ and $y \neq y_0$, then $[y, c)$, for some $c \in \mathcal{C}$, is a clopen neighbourhood of y that is compact. Hence, in all the cases, X is strongly locally compact. $\square$

Remark Every strongly compact space is locally compact. Therefore, every swo set is locally compact.

Theorem 3.2.14. *Let $(X, \leq)$ and $(Y, \leq')$ be two swo sets. Let $\mathcal{T}$ denote the interval topology on $X \times Y$ and let $\mathcal{D}$ be the product of the interval topologies on X and Y . Then $\mathcal{T} \subseteq \mathcal{D}$.*

Proof. The sub-base elements of closed subsets of $(X \times Y, \mathcal{T})$ are of the form, $K = \{(x, y) : x \leq a, y \leq' b\}$ or $K' = \{(x, y) : a \leq x, b \leq' y\}$. The projection of K onto X is $\pi_1(K) = \{x \in X : x \leq a\}$ and the projection onto Y is $\pi_2(K) = \{y \in Y : y \leq' b\}$. Clearly, $\pi_1(K)$ and $\pi_2(K)$ are closed in X and Y , respectively. Moreover, $K = \pi_1^{-1}(\pi_1(K)) \cap \pi_2^{-1}(\pi_2(K))$ and hence K is closed in $(X \times Y, \mathcal{D})$,

the product topology. Similarly, K' is also closed in $(X \times Y, \mathcal{D})$. Therefore, $\mathcal{T} \subseteq \mathcal{D}$. $\square$

Remark Consider $-\mathbb{N}$ and $\mathbb{N}$ together with their usual ordering. Then $-\mathbb{N} \times \mathbb{N}$, together with the cardinal product order, forms a lattice. Moreover, under this ordering, $-\mathbb{N} \times \mathbb{N}$ cannot be written as the union of two closed rays. Therefore, on $-\mathbb{N} \times \mathbb{N}$, the interval topology and the product of the interval topologies are not equal [9]. Consequently, $\mathcal{T} \neq \mathcal{D}$.

Theorem 3.2.16. *Let X and Y be two sets, each having both a least and a greatest element. Then $\mathcal{T} = \mathcal{D}$.*

Proof. If X and Y each have a least and a greatest element, then their cardinal product $X \times Y$ also has both a least and a greatest element. Consequently, $\mathcal{T} = \mathcal{D}$ [9]. $\square$

We state the following theorem for an arbitrary poset:

Theorem 3.2.17. [73] *Let X be a poset and $\mathcal{T}$ be the interval topology on it. For a net $\{S_n : n \in D\}$ in X that converges to y , the following holds:*

(i) *If $S_n \geq y$ for all $n \in D$, then $y = g.l.b.(range(S_n))$.*

(ii) *If $S_n \leq y$ for all $n \in D$, then $y = l.u.b.(range(S_n))$.*

Proof. (i) Let A be the range set of $\{S_n : n \in D\}$. Suppose that $y \neq g.l.b.(range(S_n))$. Let $P = \{x \in X : x \text{ is an upper bound of } A\}$. Since

$y \neq g.l.b.(range(S_n))$, y is not the least element of P . Hence, there exists such that $z \not\leq y$. Define $Q = \{x \in X : z \leq x\}$. Clearly, $y \notin Q$ and $A \subseteq Q$. Now, Q is closed and $X \setminus Q$ is an open set containing y . Therefore, $\{S_n\}$ is eventually in $X \setminus Q$, which is a contradiction, since $A \subseteq Q$.

(ii) This can be proved similarly. □

Theorem 3.2.18. *Let $(X, \leq)$ be a swo set where $X = \mathcal{W} + \mathcal{C}$ and $\mathcal{C} \neq \phi$. If $\{S_n\}$ is a net in X that converges to $y \in X$, then:*

(i) *If $y \in \mathcal{W}$ and y is not the greatest element of $\mathcal{W}$, then $y = l.u.b.\{range(S_n) : m \leq n\}$ for some $m \in \mathbb{N}$.*

(ii) *If $y \in \mathcal{C}$, then $y = g.l.b.\{range(S_n) : m \leq n\}$ for some $m \in \mathbb{N}$.*

Proof. (i) Let $K = \{x \in X : x \leq y\}$. Since y is not the greatest element of $\mathcal{W}$, K is an open set containing y . Hence, there exists $m \in \mathbb{N}$ such that $S_n \in K$ for all $n \geq m$, that is, $S_n \leq y$ for all $n \geq m$. By Theorem 3.2.17, it follows that $y = l.u.b.\{range(S_n) : n \geq m\}$.

(ii) Let $Q = \{x \in X : x \geq y\}$. Clearly, Q is open and contains y , so the net $\{S_n\}$ is eventually in Q . Therefore, by Theorem 3.2.17, there exists $m \in \mathbb{N}$ such that $y = g.l.b.\{range(S_n) : n \geq m\}$. □

Proposition 3.2.19. *Let $(X, \leq)$ be an infinite swo set where $X = \mathcal{W} + \mathcal{C}$. Then X is homogeneous if and only if the least element of X (if it exists) has an*

immediate successor, the greatest element of X (if it exists) has an immediate predecessor and every other element has both an immediate predecessor and an immediate successor.

Proof. The sufficient part follows trivially, since under the given conditions, the order topology on X coincides with the discrete topology.

For the necessary part, suppose that X has a least element a_0 with no immediate successor. Then clearly, $\mathcal{C}$ is non-empty and contains an element b that has both an immediate successor and an immediate predecessor. Consider a homeomorphism $f : X \rightarrow X$ such that $f(a_0) = b$, which exists as X is homogeneous. Since $\{b\}$ is open in X , its preimage $f^{-1}(\{b\}) = \{a_0\}$ must also be open. However, $\{a_0\}$ is not open in X , which is a contradiction. Therefore, if X has a least element, then it must have an immediate successor.

Again, suppose that X has a greatest element b_0 with no immediate predecessor. Then clearly $b_0 \in \mathcal{W}$ and $\mathcal{C}$ is empty. Since $\mathcal{W}$ is infinite, it contains an element a that has both an immediate predecessor and an immediate successor. Consider a homeomorphism $g : X \rightarrow X$ such that $g(b_0) = a$. Since $\{a\}$ is open in X , its preimage $g^{-1}(\{a\}) = \{b_0\}$ must also be open. However, $\{b_0\}$ is not open in X , leading to a contradiction.

Now, suppose that X contains an element x with neither an immediate predecessor nor an immediate successor. Then $x \in \mathcal{W}$ and $\mathcal{C}$ is non-empty. Consequently $\mathcal{C}$ contains an element y that has both an immediate predecessor and an immediate successor. Consider a homeomorphism $h : X \rightarrow X$ such that

$h(x) = y$. Here, $\{y\}$ is open in X and $h^{-1}(\{y\}) = \{x\}$ is not open in X , which is a contradiction. Hence, the proposition follows. $\square$

Hence, an infinite *swo* set is homogeneous if and only if the order topology on it coincides with the discrete topology.

Proposition 3.2.20. *Let $(X, \leq)$ be a swo set. Then X is connected if and only if it is a singleton or the empty set.*

Proof. Let $X = \mathcal{W} + \mathcal{C}$, where $\mathcal{W}$ is well ordered and $\mathcal{C}$ is co-well ordered. Clearly, if X is singleton or the empty set, then it is connected. Conversely, suppose that X is connected. Assume, for contradiction, that X has more than one element. Then $\mathcal{C}$ must be non-empty, since a well ordered set with more than one element is totally disconnected. Let c be the largest element of $\mathcal{C}$ and consider the singleton set $\{c\}$. Note that, $\{c\} = (c - 1, c]$, so it is both open and closed in X . This contradicts the assumption that X is connected. $\square$

Remark A *swo* set with more than one element is totally disconnected.

Note For a chain $(X, \leq)$, a subset $Y \subseteq X$ is closed if and only if for every subset A of Y with $\text{l.u.b.}(A) = k$ or $\text{g.l.b.}(A) = m$, then $k, m \in Y$.

Lemma 3.2.23. *Let X and X' be two linearly ordered sets equipped with the order topology. A function $f : X \rightarrow X'$ is continuous if and only if, for every subset $Y \subseteq X$ with $\text{l.u.b.}(Y) = y$ (or $\text{g.l.b.}(Y) = y$), there exists a subset $K \subseteq f(Y)$ such that $\text{l.u.b.}(K) = f(y)$ or $\text{g.l.b.}(K) = f(y)$.*

Proof. Suppose that f is continuous. Let $Y \subseteq X$ with $\text{l.u.b.}(Y) = y$. Assume, for the sake of contradiction, that $f(y)$ is neither the l.u.b. nor the g.l.b. of any subset of $f(Y)$. Define the subsets $C = \{f(x) \in f(Y) : f(x) < f(y)\}$ and $D = \{f(x) \in f(Y) : f(x) > f(y)\}$. Since $f(y)$ is not the least upper bound of C , there exists an element $c \in X'$ such that $f(x) < c < f(y)$ for all $f(x) \in C$. Similarly, since $f(y)$ is not the greatest lower bound of D , there exists an element $d \in X'$ such that $f(x) > d > f(y)$ for all $f(x) \in D$. The interval (c, d) is an open set in X' containing $f(y)$, but by construction, it contains no elements of $f(Y)$. Since f is continuous, the preimage $f^{-1}((c, d))$ is an open set in X containing y , but it contains no elements of Y , which contradicts the fact that $\text{l.u.b.}(Y) = y$.

Conversely, let C be a closed subset of X' . We aim to show that $A = f^{-1}(C)$ is closed in X . Suppose $Y \subseteq A$ and $y = \text{l.u.b.}(Y)$ (or $y = \text{g.l.b.}(Y)$). By assumption, there exists a subset $E \subseteq C$ such that either $\text{l.u.b.}(E) = f(y)$ or $\text{g.l.b.}(E) = f(y)$. Since C is closed, we have $f(y) \in C$. Therefore, $y \in f^{-1}(C) = A$. This shows that A is closed in X . Consequently, we conclude that f is continuous. $\square$

Theorem 3.2.24. *Let X and X' be two linearly ordered sets equipped with the order topology. Then every function $f : X \rightarrow X'$ is the point-wise limit of a net of continuous functions from X to X' .*

Proof. Let $\mathcal{F}(X)$ denote the set of all finite subsets of X . Then, $\mathcal{F}(X)$ forms a directed set under set inclusion. For an element $x \in X$ and $a \in A$, where

$A \in \mathcal{F}(X)$, we say that a is the A – *middle* of x if:

- (i) $x = a$, or
- (ii) $x \notin A$ and $a < x$ and there is no $b \in A$ such that $a < b < x$, or
- (iii) $x \notin A$ and $a > x$ and there is no $b \in A$ such that $a > b > x$.

Since X is a chain and A is finite, each element of X has an A – *middle*. Let $c \in X'$ be fixed. For each $A \in \mathcal{F}(X)$, we define a function $f_A: X \rightarrow X'$ as follows:

$$f_A(x) = \begin{cases} f(a), & \text{if } x \text{ has precisely one } A\text{-middle, namely } a, \\ c, & \text{if } x \text{ has more than one } A\text{-middle.} \end{cases}$$

Let $x \in X$ and let A be any finite set containing x . Then, for all $A_i \in \mathcal{F}(X)$ with $A_i \supseteq A$, we have $f_{A_i}(x) = f(x)$. Therefore, $\lim\{f_A(x), A \in \mathcal{F}(X)\} = f(x)$. Hence, f is the pointwise limit of the net of functions f_A .

It now remains to prove that each f_A is continuous. For this, we use Lemma 3.2.23. To apply the lemma, let $Y \subseteq X$ be a subset such that $l.u.b.(Y) = y$.

Case (i): Suppose that y has more than one A – *middle*. Since X is a chain and $Y \subseteq X$, the set Y is directed. Let $K = \{x \in Y: x \text{ has precisely one } A\text{ – middle}\}$. Then K is not cofinal in Y . Suppose, for the sake of contradiction, that K is cofinal in Y . Then K itself is a directed set. Define

$K_a = \{x \in K: a \text{ is an } A\text{-middle of } x\}$. Then $\bigcup_{a \in A} K_a = K$ and since A is finite, K_a is cofinal in K for some $a \in A$. Consequently, K_a is cofinal in Y and $l.u.b.(K_a) = y$. Therefore, a is an $A\text{-middle}$ of y . Now let b be another $A\text{-middle}$ of y . Then b should be incomparable with a , a contradiction. So, K is not cofinal in Y . So, there exists an element $x' \in Y$ with $k < x'$ for all $k \in K$. Hence, for all $x \in Y$ with $x' \leq x$, $f_A(x) = c = f_A(y)$.

Case (ii): Suppose that y has precisely one $A\text{-middle}$, say x' . Let $L = \{x \in Y: x \text{ has an } A\text{-middle other than } x'\}$. Then L is not cofinal in Y . Suppose, for the sake of contradiction, that it is. Let $a \neq x'$ and define $L_a = \{x \in L: a \text{ is an } A\text{-middle of } x\}$. Then L_a is cofinal in L for some $a \in A$ and $a \neq x'$. Hence, L_a is cofinal in Y and so $l.u.b.(L_a) = y$. This implies that a is an $A\text{-middle}$ of y , contradicting the assumption that y has a unique $A\text{-middle}$ x' . Therefore, L is not cofinal in Y . Thus, there exists an element $z \in Y$ with $l < z$ for all $l \in L$. Hence, for all $x \in Y$ with $x \geq z$, we have $f_A(x) = f(a) = f_A(y)$.

In a similar way, we can consider Y with $y = g.l.b.(Y)$. Hence, f_A is continuous for all $A \in \mathcal{F}(X)$. □

A poset is referred to as *order reversible* [42] if every order preserving self bijection on it is also an order isomorphism. Similarly, a topological space is defined as *topologically reversible* [59] if every bijective continuous function on it is a homeomorphism. Further details on reversibility can be found in [41, 42, 59].

Since every linearly ordered set is order reversible, every *swo* set is hereditar-

ily order reversible. However, a *swo* need not be topologically reversible when endowed with the order topology. The following example illustrates this:

Example 3.2.25. Let $X = S(\omega 2)$, where $S(\omega 2)$ denotes the collection of all ordinals less than $\omega 2$ under the usual ordinal ordering. Clearly, X is well ordered and therefore *swo*. Now consider X endowed with the order topology. Define a function $f : X \rightarrow X$ by: $f(0) = 0$, $f(\omega) = \omega$, $f(n) = 2n$, $f(\omega + (2n - 1)) = 2n - 1$, $f(\omega + 2n) = \omega + n$ for all $n \in \mathbb{N}$. It is straightforward to verify that f is a bijection and continuous function. However, consider $(0, \omega + 1) \subseteq X$; while this set is open in the order topology on X , its image under f is $f((0, \omega + 1)) = \{2, 4, 6, \dots\} \cup \{\omega\}$, which is not open in X . Therefore, f is not a homeomorphism and X is not topologically reversible.

Remark It can be shown that any well ordered set whose order type is strictly less than $\omega 2$, when equipped with the order topology, is topologically reversible.

Let $(X, \mathcal{T})$ be a topological space. Then the relation on X defined by $x <_{\mathcal{T}} y$ if and only if $x \in \overline{\{y\}}$ is a preorder on X . Now, we have the following theorem:

Theorem 3.2.27. *Let $(X, \mathcal{T})$ be a topological space such that $(X, \leq_{\mathcal{T}})$ is a *swo* set. Then $(X, \mathcal{T})$ is hereditarily reversible.*

Proof. Let f be a continuous bijective function on $(X, \mathcal{T})$. Then f is a bijective order preserving function on $(X, \leq_{\mathcal{T}})$. Since every chain is reversible, it follows that f is an order isomorphism. Moreover, since $(X, \leq_{\mathcal{T}})$ is a *swo* set by Theorem 2.2.7, f must be the identity function. Therefore, f is a homeo-

morphism. Because every subset of a *swo* set is also *swo*, $(X, \mathcal{T})$ is hereditarily reversible. □

3.3 Dynamics on Semi-Ordinals

In this section, we study the dynamics of semi-well ordered sets equipped with the order topology. Throughout, by a semi-ordinal equipped with the order topology we mean any semi-well ordered set of the given order type, endowed with the order topology.

We first identify semi-ordinals for which there exists a homeomorphism f whose $P(f)$ is empty. Clearly, for every finite semi-ordinal, $P(f)$ is never empty. However, for $(\omega, 0)$, we can easily construct a homeomorphism with $P(f) = \emptyset$. For instance, let $k : \mathbb{Z} \rightarrow (\omega, 0)$ be a bijection, which exists since $(\omega, 0)$ is countable. Thus, $(\omega, 0) = \{k(i) : i \in \mathbb{Z}\}$. Define $g : (\omega, 0) \rightarrow (\omega, 0)$ by $g(x) = k(i + 1)$ whenever $x = k(i)$. Then $P(g) = \emptyset$. We now have the following proposition.

Proposition 3.3.1. *There exists a self homeomorphism g on $(\omega^n, (\omega^m)^*)$, for $n, m \in \mathbb{N}$, such that $P(g)$ is empty.*

Proof. We will first prove that there exists a self homeomorphism h on $(\omega^n, 0)$, for $n > 1$, such that $P(h) = \emptyset$. Consider the semi-well ordered set $S = \{1, 2, 3, \dots, \omega, \omega+1, \dots, \omega 2, \dots, \omega 3, \dots, \omega^2, \dots, \omega^{n-1}, \omega^{n-1}+1, \dots, \omega^{n-1} 2, \dots, \omega^{n-1} 3, \dots\}$. Clearly, the order type of S is $(\omega^n, 0)$.

Consider the subset $K = \{\omega^{n-1}, \omega^{n-1} 2, \omega^{n-1} 3, \dots\}$ of S and define a function

3.3. Dynamics on Semi-Ordinals

$k : \mathbb{Z} \rightarrow K$ to be a bijection, which exists since K is countable. Thus, $K = \{k(i) : i \in \mathbb{Z}\}$.

For each $k(i) = \omega^{n-1}j$, define the set $\{k_{il}\} = \{\omega^{n-1}(j-1) + l, 1 \leq l < \omega^{n-1}\}$. Clearly, $\bigcup_{i \in \mathbb{Z}} \{K_{il}\} = S \setminus K$. Now define $h : (\omega^n, 0) \rightarrow (\omega^n, 0)$ by $h(k(i)) = k(i+1)$ and $h(k_{il}) = k_{(i+1)l}$. Then h is a homeomorphism and $P(h) = \emptyset$.

Now by gluing lemma, we can find a homeomorphism on g on $(\omega^n, (\omega^m)^*)$ such that $P(g)$ is empty for all $n, m \in \mathbb{N}$. $\square$

We now characterize the set of periodic points of continuous self-maps on a semi-well ordered set of order type $(\omega^2, (\omega^2)^*)$, equipped with the order topology. In [10, 16], it was shown that a subset Y is the set of periodic points of a continuous self-map on ω^2 if and only if $\overline{Y} \setminus Y$ is either empty or infinite. We can extend this result to the space $(\omega^2, (\omega^2)^*)$.

From here onwards in this section, let X denote a semi-well ordered set of order type $(\omega^2, (\omega^2)^*)$, equipped with the order topology. For any topological space Y , let Y_0 denote the set of all isolated points of Y and let $Y_1 = Y \setminus Y_0$ denote the set of all non-isolated points of Y . Note that no sequence of distinct elements in X_1 converges in X_1 and that $(\omega^2, (\omega^2)^*)$ is the largest semi-ordinal with this property.

In [10, 16], the following proposition is established for ω^2 ; by an application of the gluing lemma, this result extends to $(\omega^2, (\omega^2)^*)$. For completeness, we present a detailed proof for this extension.

Proposition 3.3.2. *Let $K \subseteq \mathbb{N}$. Then there exists a self-homeomorphism f on X such that the set of periodic points $P(f)$ equals X_0 and the set of periods of f*

is K if and only if K is infinite.

Proof. Suppose there exists a self-homeomorphism $f : X \rightarrow X$ such that $P(f) = X_0$ and the set of periods of f is K . Assume, for the sake of contradiction, that K is finite. Let $m = \text{l.c.m}\{n \in K\}$. Then $f^m(x) = x$ for all $x \in X_0$. Suppose $y \in X_1$ and let $\{y_i\}$ be a sequence in X_0 converging to y . Since f^m is a homeomorphism, the sequence $\{f^m(y_i)\}$ converges to $f^m(y)$. But $f^m(y_i) = y_i$ for all $i \in \mathbb{N}$. Therefore, $f^m(y) = y$, which contradicts the assumption that $y \in X_1$ and $P(f) = X_0$.

Conversely, suppose K is infinite and let $K = \{n_1, n_2, n_3, \dots\}$ with $n_i < n_j$ for all $i < j$. Let $X = \mathcal{W} + \mathcal{C}$. We now define a homeomorphism $f_1 : \mathcal{W} \rightarrow \mathcal{W}$ as follows:

Let $t : K \rightarrow \mathbb{Z}$ be the function defined by $t(n_l) = (-1)^{n_l} \lfloor \frac{n_l}{2} \rfloor$. Also, let $\mathcal{W}_1 \subset \mathcal{W}$ be the set of all non-isolated points in $\mathcal{W}$, that is, $\mathcal{W}_1 = \{\omega, \omega 2, \omega 3, \dots\}$. Since $\mathcal{W}$ is countably infinite, we index $\mathcal{W}_1$ as $\{w_i : i \in \mathbb{Z}\}$. For each w_i , where $w_i = \omega n$ for some $n \in \mathbb{N}$, define the set $\{w_{ij} : j \in \mathbb{N}\} = \{\omega(n-1) + j : j \in \mathbb{N}\}$. Clearly, $\bigcup_{i \in \mathbb{Z}} \{w_{ij} : j \in \mathbb{N}\} = \mathcal{W}_0$.

We now define the sets S_{n_l} for all $n_l \in K$ inductively as follows:

Define

$$S_{n_1} = \{w_{01}, w_{11}, w_{-11}, w_{21}, w_{-21}, w_{31}, w_{-31}, w_{41}, \dots, w_{t(n_1)1}\}.$$

For $k \geq 1$, define the set $A(n_k)$ by,

3.3. Dynamics on Semi-Ordinals

$$A(n_k) = \begin{cases} \{w_{-t(n_k)1}, w_{(t(n_k)+1)1}, w_{(-t(n_k)-1)1}, w_{(t(n_k)+2)1}, \dots, w_{t(n_{k+1})1}\}, & \text{if } n_k \text{ is even,} \\ \{w_{(-t(n_k)+1)1}, w_{(t(n_k)-1)1}, w_{(-t(n_k)+2)1}, w_{(t(n_k)-2)1}, \dots, w_{t(n_{k+1})1}\}, & \text{if } n_k \text{ is odd.} \end{cases}$$

Then define

$$S_{n_{k+1}} = \{w_{i(j+1)} : w_{ij} \in S_{n_k}\} \cup A(n_k).$$

Clearly, $\bigcup_{n_l \in K} S_{n_l} = \mathcal{W}_0$. Define a function f_l on S_{n_l} by

$$f_l(w_{ij}) = \begin{cases} w_{(i+1)m}, & \text{if } w_{(i+1)m} \in S_{n_l} \text{ for some } m, \\ w_{(-t(n_l)+1)1}, & \text{if } t(n_l) > 0 \text{ and } w_{(i+1)m} \notin S_{n_l}, \\ w_{t(n_l)1}, & \text{if } t(n_l) < 0 \text{ and } w_{(i+1)m} \notin S_{n_l}. \end{cases}$$

Clearly, f_l is a homeomorphism on each S_{n_l} . Next, define a function f' on $\mathcal{W}_1$ by $f'(w_i) = w_{i+1}$. Now define a function $f_1 : \mathcal{W} \rightarrow \mathcal{W}$ by:

$$f_1(x) = \begin{cases} f_l(x), & \text{if } x \in \mathcal{W}_0 \text{ and } x \in S_{n_l}, \\ f'(x), & \text{if } x \in \mathcal{W}_1. \end{cases}$$

Clearly, f_1 is a bijective open function. Furthermore, if a sequence $\{a_i\}$ is a sequence in $\mathcal{W}$ that converges to an element a , then the sequence $\{f_1(a_i)\}$ converges to $f_1(a)$. Thus, f_1 is continuous. Therefore, f_1 is a homeomorphism

on $\mathcal{W}$ with $P(f_1) = \mathcal{W}_0$ and the set of periods of f_1 is K . In a similar way, we can define a homeomorphism on $\mathcal{C}$. Then, by the Gluing Lemma, there exists a homeomorphism f on X such that $P(f) = X_0$ and the set of periods of f is K . $\square$

It was shown in the last section that every semi-well ordered set is zero-dimensional. In addition, such spaces are Hausdorff and normal. Since X is countable, it is also second countable. Therefore, by Urysohn's Metrization Theorem, X is metrizable. We now state the following proposition.

Proposition 3.3.3. *If g is a continuous self-map on a closed subset Y of X , then it can be extended to a continuous map f on X such that $P(f) = P(g)$.*

Proof. Since X is a zero-dimensional separable metric space, it follows from Proposition 1.2.4 that there exists a continuous function $f_1 : X \rightarrow Y$ such that $f_1(y) = y$ for all $y \in Y$. Define $f = g \circ f_1$. Then f is a continuous self-map on X and for all $n \in \mathbb{N}$ and $x \in X$, we have $f^n(x) = g^n(f_1(x))$. Hence, $P(f) = P(g)$. $\square$

Theorem 3.3.4. *A subset $Y \subseteq X$ is the set of periodic points of a continuous self-map on X if and only if $\overline{Y} \setminus Y$ is either empty or infinite.*

Proof. Suppose $f : X \rightarrow X$ is a continuous function such that $P(f) = Y$. We first claim that if $\overline{Y} \setminus Y$ is non-empty, then it is forward invariant. Let $x \in \overline{Y} \setminus Y$. Since f is continuous and $x \in \overline{Y}$, it follows that $f(x) \in \overline{f(Y)} \subseteq \overline{Y}$. Assume, for contradiction, that $f(x) \in Y$. Note that $x \in \overline{Y} \setminus Y$ implies that $x \in X_1$. Let

(x_n) be a sequence of distinct elements in Y converging to x . By the continuity of f , we have $f(x) \in X_1$ and the sequence $(f(x_n))$ converges to $f(x)$. Since $f(x) \in Y$, let l denote the period of $f(x)$, so that $f^l(f(x)) = f(x)$. Since $x \notin Y$, it follows that $f^n(f(x)) \neq x$ for all $n \leq l$. Furthermore, as f is continuous, each $f^n(f(x)) \in X_1$ and the sequence $(f^n(f(x_n)))$ converges to $f^n(f(x))$. Hence, for sufficiently large n , elements x_n are not periodic, contradicting the assumption that $x_n \in Y$. Therefore, $f(x) \in \bar{Y} \setminus Y$ and $\bar{Y} \setminus Y$ is forward invariant. In particular, this implies that $\bar{Y} \setminus Y$ must be infinite.

Conversely, suppose that $\bar{Y} \setminus Y$ is either empty or infinite. If $\bar{Y} \setminus Y$ is empty, then Y is a closed subset of X . Define $g : Y \rightarrow Y$ by $g(x) = x$ for all $x \in Y$. Clearly, g is a continuous function on Y with $P(g) = Y$. By Proposition 3.3.3, there exists a continuous function f on X extending g such that $P(f) = P(g) = Y$.

Suppose that $\bar{Y} \setminus Y$ is infinite. We first define a continuous function on $\bar{Y}$ such that $P(f) = Y$. Clearly, $\bar{Y} \setminus Y \subseteq X_1$. For any element $x \in X_1 \cap \omega^2$ that is, $x = \omega k$, by a sequence (x_n) converging to x here we mean $\omega(k-1) \leq x_n \leq \omega k$ for all $n \in \mathbb{N}$. A similar argument applies when $x \in X_1 \cap (\omega^2)^*$. Now, for any $x \in \bar{Y} \setminus Y$, there exists a sequence in Y converging to x . Let T denote the union of the ranges of all sequences in Y that converge to elements of $\bar{Y} \setminus Y$. Define $Z = T \cup (\bar{Y} \setminus Y)$. Then, as in the proof of Proposition 3.3.3, there exists a homeomorphism h on Z such that $P(h) = T$. Now define $g : \bar{Y} \rightarrow \bar{Y}$ by,

$$g(x) = \begin{cases} h(x), & \text{if } x \in Z, \\ x, & \text{if } x \notin Z. \end{cases}$$

Clearly, g is a continuous function on $\overline{Y}$ with $P(g) = Y$. Then by Proposition 3.3.3, there exists a continuous function $f : X \rightarrow X$ that extends g such that $P(f) = Y$. $\square$

An important consequence of Theorem 3.3.4 is the following: if $K \subset L \subset [0, 1]$, where L is homeomorphic to $(\omega^2, (\omega^2)^*)$ and $\overline{L} \setminus L$ is finite, then if K cannot occur as the set of periodic points of any continuous self-map on L , then K also cannot occur as the set of periodic points of any continuous self-map on $[0, 1]$.

It was shown in [15, 16] that if Y denotes the set of periodic points of a compact Hausdorff space K , then the set $\overline{Y} \setminus Y$ is either empty or infinite. This conclusion, however, does not necessarily hold when K is not locally compact [10], see Example 1.3.4. It was further established that the result remains valid when K is the ordinal space ω^2 [10] endowed with the order topology and the converse also holds in this case. Whether this result holds for arbitrary locally compact Hausdorff spaces remains an open problem [10].

Here, we extend these results to semi-well ordered sets of order type $(\omega^2, (\omega^2)^*)$ endowed with the order topology which is again a locally compact Hausdorff space. An interesting open question is whether analogous results holds for semi-well ordered sets of order type $(\omega^n, (\omega^m)^*)$ for all $n, m \in \mathbb{N}$.

Partially Semi-Well Ordered Sets: Characterizations and Structural Properties

This chapter introduces partially semi-well ordered sets, providing the motivation behind their definition, their relation to semi-well ordered sets and partially well ordered sets and the primary objective of their introduction. We explore their order-theoretic and structural properties and present several characterizations. In addition, we study the properties of order ideals in partially semi-well ordered sets. Moreover, we investigate the reversibility of partially semi-well ordered sets.

4.1 Introduction

Both semi-well ordered sets and partially well ordered sets can be viewed as extensions of well ordered sets. However, it is important to note that a linearly ordered set is partially well ordered if and only if it is well ordered, highlighting the distinct nature of these two concepts as separate extensions of well ordered sets. This distinction provides the motivation for the introduction of a more general class of ordered structures: partially semi-well ordered sets. The primary objective behind the introduction of partially semi-well ordered set is to verify the Aharoni-Korman conjecture for the case $k = 1$ within this class. A key characterization of these sets plays a significant role in establishing this result.

In this chapter, we explore the order-theoretic and structural properties of partially semi-well ordered sets. The second section provides a formal definition of partially semi-well ordered sets, examines their fundamental properties and presents several characterizations. In the third section, we analyze the properties of their order ideals. The final section is devoted to investigating the reversibility of partially semi-well ordered sets.

4.2 Partially Semi-Well Ordered Sets and its Properties

A partially semi-well ordered set is defined as follows:

Definition 4.2.1. A poset $(X, \leq)$ is said to be partially semi-well ordered (*pswo*) if every non-empty subset of X has either a finite minimal subset or a

finite maximal subset. The relation $\leq$ is referred to as the partial semi-well ordering on X .

In a partially well ordered set (*pwo*) set, every non-empty subset has a finite minimal subset [72]. Therefore, every *pwo* set is also *pswo*. Similarly, every non-empty subset of a *swo* set has either a least element or a greatest element, so every *swo* set is also *pswo*. Hence, the class of *pswo* sets extends both the class of *pwo* sets and the class of *swo* sets. The following example illustrates that this extension is, in fact, a proper extension.

Example 4.2.2. Let $\mathbb{N}$ denote the set of positive integers and $-\mathbb{N}$ the set of negative integers, each equipped with the usual total order. Consider the sets $X = \mathbb{N} \times \mathbb{N}$ and $Y = -\mathbb{N} \times -\mathbb{N}$, each endowed with the cardinal product order, denoted by $\leq'$ and $\leq''$, respectively. Define a partial order $\leq$ on $X \cup Y$ by taking $\leq = \leq' \cup \leq'' \cup X \times Y$. One can easily check that the poset $(X \cup Y, \leq)$ is *pswo*. However, it is neither *pwo* nor *swo*.

The following proposition provides additional examples of *pswo* sets.

Proposition 4.2.3. (i) A linearly ordered set $(X, \leq)$ is *pswo* if and only if it is *swo*.

(ii) The dual of a *pswo* set is *pswo*.

Proof. (i) If X is *pswo*, then every non-empty subset of X has either a finite minimal subset or a finite maximal subset. This minimal or maximal subset will be a singleton, as X is linearly ordered. The converse is trivial.

(ii) Let $(X, \leq)$ be a *pswo* set. If A is a non-empty subset of X , then A has a finite minimal or a finite maximal subset in $(X, \leq)$. Clearly, a minimal (maximal) subset of a poset is a maximal (minimal) subset in its dual. Therefore, A has a finite minimal subset or a finite maximal subset in $(X, \geq)$.

□

The following theorem is proved using the subsequent lemma:

Lemma 4.2.4. [49] [König's Infinity Lemma] Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of non-empty, pairwise disjoint finite sets and let $f_n: I_{n+1} \rightarrow I_n$ be a function for each n . Then, for each n , there exists an $a_n \in I_n$ such that $f_n(a_{n+1}) = a_n$.

The following theorem provides several characterizations of a *pswo* set.

Theorem 4.2.5. For a poset $(X, \leq)$, the following properties are equivalent:

- (i) X is *pswo*.
- (ii) X has no subsets of the form $C \sqcup D$, where C is a strictly decreasing sequence, D is a strictly increasing sequence and no element of D is strictly less than any element of C and X has no infinite antichain.
- (iii) If $\{x_n\}$ and $\{y_n\}$ are two sequences in X with $y_m \not\leq x_n$ for all n, m , then either there exists $i < j$ with $x_i \leq x_j$ or $k < h$ with $y_k \geq y_h$.
- (iv) If $\{x_n\}$ and $\{y_n\}$ are two sequences with distinct elements of X , such that $y_m \not\leq x_n$ for all n, m , then either $\{x_n\}$ has a strictly increasing subsequence or $\{y_n\}$ has a strictly decreasing subsequence.

Proof. (i) $\Leftrightarrow$ (ii): Suppose that X is a *pswo* set. Clearly, X contains no infinite antichain. Now, assume that X has a subset of the form $C \sqcup D$, as described in (ii). Let $C = \{\dots, x_{-2}, x_{-1}, x_0\}$ and $D = \{y_0, y_1, y_2, \dots\}$. If any element $x_j \in C$ is less than any element $y_k \in D$, then $C \sqcup D$ contains a subset of the form $A + B$, where A is a strictly decreasing sequence and B is a strictly increasing sequence. This leads to a contradiction, since such a subset $A + B$ contains no finite minimal or maximal subsets. Next, suppose that for all $x_j \in C$ and $y_k \in D$, the set $\{x_j, y_k\}$ forms an antichain. In this case, the only minimal element of $C \sqcup D$ is y_0 , but $\{y_0\}$ is not a minimal subset. Similarly, the only maximal element of $C \sqcup D$ is x_0 , but $\{x_0\}$ is not a maximal subset of $C \sqcup D$. Thus, $C \sqcup D$ has neither a finite minimal subset nor a finite maximal subset, leading to a contradiction.

Suppose that (ii) holds. Let A be an infinite subset of X . Assume that A has neither a finite minimal subset nor a finite maximal subset. We now aim to prove that A contains either an infinite antichain or has a subset of the form $C \sqcup D$. Suppose that A has neither minimal nor maximal elements. Let $x \in A$. Since x is not a minimal element, there exists an $x_1 \in A$ with $x_1 < x$ and x_1 is not a minimal element. Therefore, there exists $x_2 \in A$ with $x_2 < x_1 < x$. Continuing this process, we get $\dots < x_2 < x_1 < x$. Similarly, since x is not a maximal element, there exists $y_1 \in A$ with $x < y_1$ and y_1 is not a maximal element. Hence, there exists $y_2 \in A$ with $x < y_1 < y_2$ and continuing this process, we get $x < y_1 < y_2 < \dots$. Let $C = \{\dots, x_2, x_1, x\}$ and $D = \{y_1, y_2, y_3, \dots\}$. Then C is a strictly decreasing sequence, D is a strictly increasing sequence and no

element of D is strictly less than any element of C .

Suppose that A has minimal elements and no maximal elements. Let P denote the set of all minimal elements of A . If P is infinite, then it is an infinite antichain. If P is finite, since A has no finite minimal subset, P is not a minimal subset of A . Thus, there exists an $x_0 \in A$ with $m \not\leq x_0$ for all $m \in P$. Since $x_0 \notin P$, there exists an $x_{-1} \in A$ with $x_{-1} < x_0$. Now, if there exists an $m_1 \in P$ with $m_1 \leq x_{-1}$, then by transitivity, $m_1 < x_0$, which is not possible. Therefore, $m \not\leq x_{-1}$ for all $m \in P$ and there exists $x_{-2} \in A$ with $x_{-2} < x_{-1} < x_0$. Again, $m \not\leq x_{-2}$ for all $m \in P$. Continuing this process, we obtain $\dots < x_{-3} < x_{-2} < x_{-1} < x_0$. Since A has no maximal elements, there exist $x_1, x_2, \dots \in A$ with $\dots < x_{-3} < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \dots$. Let $C = \{\dots, x_{-3}, x_{-2}, x_{-1}, x_0\}$ and $D = \{x_1, x_2, \dots\}$. Then, C is a strictly decreasing sequence, D is a strictly increasing sequence and no element of D is strictly less than any element of C . A similar argument holds if A has maximal elements and no minimal elements.

Suppose that A has both minimal and maximal elements. Let P be the set of all minimal elements of A and let Q be the set of all maximal elements of A . If either P or Q is infinite, then it forms an infinite antichain in A . Suppose that both P and Q are finite. Since P is not a minimal subset of A , there exists $x_1 \in A$ such that $m \not\leq x_1$ for all $m \in P$. Clearly, $x_1 \notin P$, so there exists $x_2 \in A$ with $x_2 < x_1$ and $m \not\leq x_2$ for all $m \in P$. Continuing this process, we get a strictly decreasing sequence $C = \{\dots, x_3, x_2, x_1\}$. Again, since Q is not a maximal subset of A , there exists $y_1 \in A$ such that $y_1 \not\leq m$ for all $m \in Q$. Now, $y_1 \notin Q$, so

there exists $y_2 \in A$ with $y_1 < y_2$ and $y_2 \not\leq m$ for all $m \in Q$. Continuing this, we obtain a strictly increasing sequence $D = \{y_1, y_2, y_3, \dots\}$. Without loss of generality, assume that $C \cap D = \emptyset$ (if they intersect, their intersection must be finite). Suppose that there exist $y_i \in D$ and $x_j \in C$ such that $y_i < x_j$. If there exists $m \in P$ with $m \leq y_i$, then by transitivity, $m < x_j$, which is not possible by the choice of x_j . Therefore, $m \not\leq y_i$ for all $m \in P$. Also, since $y_i \notin P$, there exists $z_1 \in A$ with $z_1 < y_i$, and again $m \not\leq z_1$ for all $m \in P$. Hence, there exists z_2 with $z_2 < z_1 < y_i$. Continuing this process, we obtain $\dots < z_2 < z_1 < y_i < y_{i+1} < y_{i+2} < \dots$. Let $C' = \{z_1, z_2, z_3, \dots\}$ and $D' = \{y_i, y_{i+1}, \dots\}$. Then C' is a strictly decreasing sequence and D' is a strictly increasing sequence, with no element of D' is strictly less than any element of C' .

(ii) $\Leftrightarrow$ (iii): Assume (ii). Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X with distinct elements and suppose that $y_n \not\leq x_m$ for all n, m . Assume that for all $i < j$, $x_i \not\leq x_j$ and $y_i \not\leq y_j$. Let X_1 and Y_1 denote the range sets of $\{x_n\}$ and $\{y_n\}$, respectively. Since X contains no infinite antichain, X_1 must contain only a finite number of minimal elements. Now suppose X_1 has a finite minimal subset, say M_1 . Let $i' = \max\{i \in \mathbb{N} : x_i \in M_1\}$. Consider $x_{i'+1}$, then there exists $j < i' + 1$ such that $x_j \in M_1$ and $x_j < x_{i'+1}$, which contradicts our assumption. Therefore, X_1 cannot have a finite minimal subset, implying that X_1 contains a strictly decreasing sequence, say C . Similarly, Y_1 cannot have an infinite number of maximal elements and no finite maximal subset. Hence, Y_1 contains a strictly increasing sequence, say D . Thus, $C \sqcup D$ is a subset of X , where C is a strictly

decreasing sequence and D is a strictly increasing sequence, with no element of D being strictly less than any element of C , which leads to a contradiction. Consequently, there must exist $i < j$ with $x_i \leq x_j$ or $k < h$ such that $y_k \geq y_h$.

Suppose that (iii) holds. Then, it follows trivially that X cannot have a subset of the form $C \sqcup D$, where C is a strictly decreasing sequence, D is a strictly increasing sequence and no element of D is strictly less than any element of C . Now, let $\{q_1, q_2, q_3, \dots\}$ be an infinite denumerable antichain in X . By defining $\{x_n\} = \{q_1, q_3, q_5, \dots\}$ and $\{y_n\} = \{q_2, q_4, q_6, \dots\}$, we arrive at a contradiction.

(ii) $\Leftrightarrow$ (iv): Assume that (ii) holds. Suppose, for contradiction, that $\{x_n\}$ has no strictly increasing subsequence and $\{y_n\}$ has no strictly decreasing subsequence. Since $\{x_n\}$ has no strictly increasing subsequence and no infinite antichain, each subsequence of $\{x_n\}$ must have a finite maximal subset. First, choose I_1 such that $\{x_i : i \in I_1\}$ is a maximal subset of $\{x_n\}$. Continuing this process, for $n \geq 1$, select I_{n+1} such that $\{x_i : i \in I_{n+1}\}$ is a maximal subset of $\{x_m : m > k\}$, where $k = \max(\bigcup_{i=1}^n I_i)$. For each n , define a function $f_n: I_{n+1} \rightarrow I_n$ such that $x_i < x_{f_n(i)}$ for all $i \in I_{n+1}$. By König's Infinity Lemma (Lemma 4.2.4), we can select an element x_{i_n} such that $x_{i_n} = x_{f_n(i_{n+1})}$ and thus $x_{i_n} > x_{i_{n+1}}$ for all n . Let $C = \{x_{i_n}\}_{n=1}^{\infty}$. Then, C is a strictly decreasing sequence of $\{x_n\}$. Similarly, we can find a strictly increasing subsequence D of $\{y_n\}$. Thus, $C \sqcup D$ is a subset of X , where no element of D is strictly less than any element of C , leading to a contradiction. Therefore, $\{x_n\}$ must contain a strictly increasing subsequence or

$\{y_n\}$ must contain a strictly decreasing subsequence. The converse is trivial. $\square$

Proposition 4.2.6. *Let $(X, \leq)$ be a pswo set and let $(Y, \leq')$ be an arbitrary poset.*

(i) *If $f : X \rightarrow Y$ is an onto order preserving map, then $(Y, \leq')$ is also a pswo set.*

(ii) *If $f : X \rightarrow Y$ is a strict order-preserving map, then for any $y \in Y$, the preimage $f^{-1}(y)$ is finite.*

Proof. (i) Let $\{x_n\}$ and $\{y_n\}$ be two disjoint sequences in Y , with $y_n \not\leq' x_n$ for all n . Then, the sequences $\{f^{-1}(x_n)\}$ and $\{f^{-1}(y_n)\}$ contain two disjoint sequences $\{a_n\}$ and $\{b_n\}$, respectively, such that $b_n \not\leq a_n$ for all n . Therefore, there must exist indices $i < j$ such that $a_i \leq a_j$ or $k < h$ such that $b_k \geq b_h$. Since f is order preserving, this implies $x_i \leq' x_j$ or $y_h \leq' y_k$, where $x_i = f(a_i)$, $x_j = f(a_j)$, $y_h = f(b_h)$ and $y_k = f(b_k)$. Thus, $(Y, \leq')$ is a pswo set.

(ii) Let $y \in Y$. Then, $f^{-1}(y) = \{x \in X : f(x) = y\}$. Since f is strict, $f^{-1}(y)$ forms an antichain in X . As X is pswo, it follows that $f^{-1}(y)$ must be finite.

$\square$

We now present the following lemma, which will be used to prove the subsequent theorem.

Lemma 4.2.7. *If $(X, \leq_0)$ is a poset that contains an infinite antichain, then there exists a linear extension of $\leq_0$ that is not a semi-well ordering on X .*

Proof. Let $\{t_1, t_2, t_3, \dots\}$ be an infinite denumerable antichain of X . Define a relation $\leq_1$ on X by $x \leq_1 y$ if and only if (i) $x \leq_0 y$ or (ii) $x \leq_0 t_2$ and $t_1 \leq_0 y$. Clearly, $\leq_1$ is a partial ordering on X and is an extension of $\leq_0$ because $t_2 \leq_1 t_1$. Next, define $\leq_2$ on $(X, \leq_1)$ by $x \leq_2 y$ if and only if (i) $x \leq_1 y$ or (ii) $t_3 \leq_1 y$ and $x \leq_1 t_1$. Again, $\leq_2$ is a partial ordering on X and an extension of $\leq_1$, as $t_1 \leq_2 t_3$. Continuing inductively for $n = 1, 2, 3, \dots$, define $\leq_{2n+1}$ by $x \leq_{2n+1} y$ if and only if (i) $x \leq_{2n} y$ or (ii) $x \leq_{2n} t_{2n+2}$ and $t_{2n} \leq_{2n} y$. Also, define $\leq_{2n+2}$ by $x \leq_{2n+2} y$ if and only if (i) $x \leq_{2n+1} y$ or (ii) $x \leq_{2n+1} t_{2n+1}$ and $t_{2n+3} \leq_{2n+1} y$.

Thus, each $\leq_n$ (for $n = 0, 1, 2, \dots$) is an extension of $\leq_0$ and $\leq_n$ is a subset of $\leq_{n+1}$ for all n . Let $\leq = \cup\{\leq_n: n = 0, 1, 2, \dots\}$. Then, $\leq$ is a partial ordering on X and in the poset $(X, \leq)$, we have the sequence $\dots < t_4 < t_2 < t_1 < t_3 < t_5 < \dots$. Thus, the sequence $\{t_n\}$ is of the form $C + D$, where $C = \{\dots, t_4, t_2\}$ is a strictly decreasing sequence and $D = \{t_1, t_3, t_5, \dots\}$ is a strictly increasing sequence. Let $(X, \leq^*)$ be a linear extension of $\leq$. Since the sequence $\{t_n\}$ is of the form $C + D$, it follows that $(X, \leq^*)$ is not *swo* set. Moreover, $\leq^*$ is an extension of $\leq_0$, completing the proof. $\square$

We now present another characterization of a *pswo* set.

Theorem 4.2.8. *Let $(X, \leq)$ be a poset. A necessary and sufficient condition for X to be *pswo* is that every linear extension of $\leq$ is a semi-well ordering on*

X .

Proof. Suppose X is *pswo*. Consider a linear extension $\leq'$ of $\leq$. Define a map $f: (X, \leq) \rightarrow (X, \leq')$ by $f(x) = x$ for all $x \in X$. Clearly, f is an order preserving map from $(X, \leq)$ onto $(X, \leq')$. Then, by Proposition 4.2.6, $(X, \leq')$ is *pswo* and since it is also a chain, $(X, \leq')$ is *swo*.

Conversely, suppose that every linear extension of $\leq$ is *swo*. Then $(X, \leq)$ cannot have a subset of the form $C \sqcup D$, where C is a strictly decreasing sequence, D is a strictly increasing sequence and no element of D is strictly less than any element of C . This is because such a subset $C \sqcup D$ will be $C + D$ in some linear extension of $\leq$, which is not possible. Furthermore, by Lemma 4.2.7, X cannot contain an infinite antichain. Hence, by Theorem 4.2.5, $(X, \leq)$ is *pswo*. $\square$

Theorem 4.2.9. *If $(X, \leq)$ is a finite dimensional *pswo* set, then X is isomorphic to a subset of the cardinal product of a finite number of *swo* sets.*

Proof. Given that $(X, \leq)$ is a finite dimensional *pswo* set. Therefore, there exists a finite number of linear extensions of $\leq$, such that $\leq = \bigcap_{i=1}^n \leq_i$. Also, by Theorem 4.2.8, each $(X, \leq_i)$ is a *swo* set. Let $(K, \leq')$ denote the cardinal product of $(X, \leq_i)$ for $i = 1, 2, \dots, n$. Define a function g from $(X, \leq)$ to $(K, \leq')$ by $g(x) = f_x$, where $f_x(i) = x$ for all $i = 1, 2, \dots, n$. Clearly, g is a one-to-one order preserving function. Therefore, X is isomorphic to $\{f_x: x \in X\}$, which is a subset of K . $\square$

Remark The converse of the above theorem is not true. Specifically, $(\mathbb{Z}, \leq)$ is a

poset of dimension one and it is isomorphic to a subset of the cardinal product of $(\mathbb{N}, \leq)$ and $(-\mathbb{N}, \leq)$. Both $(\mathbb{N}, \leq)$ and $(-\mathbb{N}, \leq)$ are *swo* sets, but $(\mathbb{Z}, \leq)$ is not a *pswo* set.

The following theorem establishes a structural property of *pswo* sets:

Theorem 4.2.11. *Let $(X, \leq)$ be a *pswo* set. Then, $X = P \sqcup P'$, where $(P, \leq)$ is a *pwo* set, $(P', \leq)$ is a dually *pwo* set and no element of P' is strictly less than any element of P .*

Proof. Let $A' = \{x \in X : x \text{ has no immediate predecessor or } x \text{ is the } n^{\text{th}} \text{ successor of an element having no immediate predecessor}\}$. Define P as the set $\{x \in A' : \text{all predecessors of } x \text{ belong to } A'\}$ and let P' be the complement of P in X . Then, for any $p \in P'$, either p has no immediate successor or p is the n^{th} predecessor of an element having no immediate successor. To prove this, suppose, for contradiction, that there exists a $p_0 \in P'$ with an immediate successor and p_0 is not the n^{th} predecessor of an element having no immediate successor. In this case, p_0 must have an n^{th} successor for all positive integers n , denoted p_n . Now, consider the sequence $D = \{p_0, p_1, p_2, \dots\}$ which is clearly strictly increasing. Clearly, $p_0 \notin P$. If $p_0 \notin A'$, then p_0 must have an n^{th} predecessor for all n , leading to the existence of a strictly decreasing sequence. Alternatively, if $p_0 \in A'$, then p_0 must have a predecessor that does not belong to A' , which again results in a strictly decreasing sequence. In both cases, we obtain a strictly decreasing sequence. Let $C = \{q_1, q_2, \dots\}$ be this strictly decreasing sequence. Therefore,

$C \cup D = \{\dots < q_2 < q_1 < p_0 < p_1 < p_2 < \dots\}$ contains no finite minimal or maximal subset, leading to a contradiction.

Next, we prove that no element of P' is strictly less than any element of P . Suppose, for contradiction, that there exists $p \in P$ and $q \in P'$ such that $q < p$. Since $p \in P$, it follows that q must belong to A' . Furthermore, since $q \notin P$, there must exist a predecessor c of q , such that $c \notin A'$ and $c < q < p$. However, this leads to a contradiction, as it implies that p has a predecessor c that does not belong to A' , which contradicts the assumption that $p \in P$.

To show that P is *pwo*, let M be an infinite subset of P . Suppose, for the sake of contradiction, that M has no finite minimal subset. Since X is a *pswo* set, it follows that M must have a finite maximal subset, say C . Without loss of generality, assume that each element of C has no immediate predecessor. Then, $M \setminus C$ cannot have a finite minimal or maximal subset, which leads to a contradiction.

Next we show that P' is dually *pwo*. It is clear that P' has no infinite antichain. Suppose, for contradiction, that P' contains a strictly increasing sequence $\{x_1, x_2, \dots\}$ such that $x_1 < x_2 < x_3 < \dots$. Since $x_1 \notin P$, there must exist a strictly decreasing sequence in X , say $\{x_{-1}, x_{-2}, \dots\}$. Considering the sequences $D = \{x_1, x_2, \dots\}$ and $C = \{x_{-1}, x_{-2}, \dots\}$, we obtain $C \cup D = \{\dots < x_{-2} < x_{-1} < x_1 < x_2 < \dots\}$, which has no finite minimal or maximal subset. This leads to a contradiction and therefore, P' is dually *pwo*. □

The converse of Theorem 4.2.11 is not true. Specifically, the disjoint union of a *pwo* set and a dually *pwo* does not necessarily form a *pswo* set. To demonstrate this, consider the following example.

Example 4.2.12. Consider the set of all positive integers $\mathbb{N}$ and the set of all negative integers $-\mathbb{N}$, each equipped with their usual ordering. The disjoint union $\mathbb{N} \sqcup -\mathbb{N}$ does not form a *pswo* set.

The cardinal product of finitely many *pwo* is known to be *pwo* [72]. However, the cardinal product of two *swo* sets need not be *swo*. This rises the question of whether the cardinal product of two *swo* sets is *pswo*. The answer is negative, as illustrated by the following example.

Example 4.2.13. Consider the set of all non-negative integers $\mathbb{N} \cup \{0\}$ and the set of all negative integers $-\mathbb{N}$, both equipped with their usual orderings. The cartesian product $(\mathbb{N} \cup \{0\}) \times -\mathbb{N}$, with the cardinal product order, is not a *pswo* set. This is because the set $\{(0, -1), (1, -2), (2, -3), (3, -4), \dots\}$ forms an infinite antichain within this product set.

4.3 Order Ideals of Partially Semi-Well Ordered Sets

In this section, we examine the order ideals of a *pswo* set.

Definition 4.3.1. If $(X, \leq)$ is a *pswo* set and $I(X)$, denoting the set of all order ideals of X with set inclusion, is also *pswo*, then X is said to be normal .

By Theorem 2.3.6, every *swo* set is normal. The following provides an example of a non-normal *pswo* set.

Example 4.3.2. Let $X = \mathbb{N} \times \mathbb{N}$. For elements $n = (n_1, n_2)$ and $m = (m_1, m_2)$, define a relation $\leq$ on X as follows: $n \leq m$ if and only if $n_2 = m_2$ and $n_1 \leq m_1$ or $n_2 < m_2$ and $n_1 + n_2 \leq m_2$. Clearly, $(X, \leq)$ is a *pswo* set. Now, let $K_i = \{(x, y) : y = i\}$ for all $i \in \mathbb{N}$. Let O_i denote the order ideal of X generated by K_i , that is, $O_i = \{x \in X : x \leq y \text{ for some } y \in K_i\}$. The collection $\{O_i : i \in \mathbb{N}\}$ forms an infinite antichain in the set of all order ideals of X under set inclusion. Therefore, $I(X)$ is not *pswo* and hence X is not normal.

Proposition 4.3.3. *If $(X, \leq)$ is a normal *pswo* set and $Y \subseteq X$, then $(Y, \leq_Y)$ is also normal.*

Proof. Let $\{J_n\}$ and $\{K_n\}$ be two sequences of order ideals of Y with $K_n \not\subseteq J_m$ for all n, m . Define: $C_n = \{x \in X : x \leq y \text{ for some } y \in J_n\}$, $B_n = \{x \in X : x \leq y \text{ for some } y \in K_n\}$. Suppose $B_n \subseteq C_m$ for some n and m . Then, we would have $K_n \subseteq C_m$, but by assumption, $K_n \not\subseteq J_m$. Therefore, there exists $y \in K_n$ such that $y \notin J_m$ and $y \in C_m$. Since $y \in C_m$, there exists $z \in J_m$ with $y < z$. However, $y \in Y$ and $y \notin J_m$, this implies that J_m is not an order ideal of Y , which leads to a contradiction. Hence, $B_n \not\subseteq C_m$ for all n, m . Therefore, $\{C_n\}$ and $\{B_n\}$ are two sequences of order ideals of X such that $B_n \not\subseteq C_m$ for all n, m . Since $I(X)$ is *pswo*, it follows that either there exists $i < j$ with $C_i \subseteq C_j$ or $h < k$ with $B_h \supseteq B_k$. In either case, we have: $J_i = C_i \cap Y \subseteq C_j \cap Y = J_j$ or

$K_h = B_h \cap Y \supseteq B_k \cap Y = K_k$. Thus, $I(Y)$ is *pswo* and Y is normal. $\square$

Theorem 4.3.4. *Let $(X, \leq)$ be a *pswo* set. Then $I(X)$ has no subset of the form $C \sqcup D$, where $C = \{W_n\}$ is a strictly decreasing sequence, $D = \{K_n\}$ is a strictly increasing sequence and $W_n \cap K_m = \emptyset$, for all m, n .*

Proof. Suppose that X contains a subset of the form $C \sqcup D$. For each n , select $x_n \in W_n \setminus W_{n+1}$ and $y_n \in K_{n+1} \setminus K_n$. Then, $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $y_n \not\leq x_m$ for all n, m . Since X is *pswo*, there must exist $i < j$ with $x_i \leq x_j$ or $h < k$ with $y_h \geq y_k$. However, $x_i \leq x_j$ implies $x_i \in W_j$ and $y_h \geq y_k$ implies $y_k \in K_h$, but neither of these cases is possible, which leads to a contradiction. $\square$

Remark For a *swo* set X , $I(X)$ can contain a strictly increasing or decreasing sequence. Thus, it is clear that if X is *pswo*, $I(X)$ can also contain a strictly increasing or decreasing sequence.

4.4 Reversibility of Partially Semi-Well Ordered Sets

In this section, we discuss the reversibility of *pswo* sets. It was proved in [42] that every *pwo* set is hereditarily reversible. We also verify that every *swo* set is hereditarily reversible. We now state the following theorem.

Theorem 4.4.1. *Every pswo is hereditarily order reversible.*

Proof. Let $(X, \leq)$ be a pswo set and let $f: X \rightarrow X$ be a bijective order preserving map. By Theorem 4.2.11, we can express $X = P \sqcup P'$, where $(P, \leq)$ is a pswo set, $(P', \leq)$ is a dually pswo set and no element of P' is strictly less any element of P . Let the levels of P be denoted by P_α 's, where $P_\alpha = \min(P \setminus \bigcup_{\beta < \alpha} P_\beta)$ represents the α^{th} level of $(P, \leq)$. Similarly, let the levels of P' be denoted by P'_α 's, where $P'_\alpha = \max(P' \setminus \bigcup_{\beta < \alpha} P'_\beta)$ represents the α^{th} level of $(P', \leq)$. We will now prove by transfinite induction that $f(P_\alpha) = P_\alpha$ for all α . Suppose there exists an element $x \in P_0$ such that $x \notin f(P_0)$. By the definition of P' , every element of P' has an immediate predecessor, so no element of P' can be mapped to x . Suppose that $x = f(y)$ for some $y \in P \setminus P_0$. If $y \in P \setminus P_0$, then there must exist an element $a \in P_0$ such that $a < y$. Since f is an order preserving bijection, we must have $f(a) < f(y) = x$, which leads to a contradiction. Therefore, $f(P_0) = P_0$, since P_0 is finite.

Now, suppose that for all $\alpha < \beta$, $f(P_\alpha) = P_\alpha$. Let $Q = P \setminus \bigcup_{\alpha < \beta} P_\alpha$. Then $f|_{Q \cup P'}$ is an order preserving bijection and the first level of Q is P_β . By the same argument as above, we have $f(P_\beta) = P_\beta$. Therefore, $f(P) = P$ and $f|_P : P \rightarrow P$ is an order isomorphism. Similarly, $f|_{P'} : P' \rightarrow P'$ is also an order isomorphism.

Finally, let $p \in P$ and $q \in P'$ with $p \parallel q$. Then, $p \in P_\alpha$ and $q \in P'_\beta$ for some ordinals α and β . Thus, $f|_{P_\alpha \cup P'_\beta} : P_\alpha \cup P'_\beta \rightarrow P_\alpha \cup P'_\beta$ is an order preserving bijection on a finite set and hence it is an order isomorphism. Therefore, we

conclude that $f(p) \parallel f(q)$. □

Let $(X, \mathcal{T})$ be a topological space. The ordering on X , defined by $x <_{\mathcal{T}} y$ if and only if $x \in \overline{\{y\}}$, is a preorder [41]. Here, $x <_{\mathcal{T}} y$ if and only if every open set containing x contains y .

For a topological space $(X, \mathcal{T})$, the fact that $(X, \leq_{\mathcal{T}})$ is reversible does not necessarily imply that $(X, \mathcal{T})$ is reversible. Consider, for instance, the set of all real numbers $\mathbb{R}$ and let $\mathcal{T}_u$ denote the usual topology on $\mathbb{R}$, while $\mathcal{T}_d$ denotes the discrete topology on the set of all positive real numbers. Define $\mathcal{T}$ as the topology on $\mathbb{R}$ generated by $\mathcal{T}_u \cup \mathcal{T}_d$. Then the topological space $(\mathbb{R}, \mathcal{T})$ is not reversible, since the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x - 2$ is a continuous bijection that is not a homeomorphism. However, the poset $(\mathbb{R}, \leq_{\mathcal{T}})$ is totally unordered and therefore hereditarily reversible. We now state the following theorem:

Theorem 4.4.2. *If $(X, \mathcal{T})$ is a topological space such that $(X, \leq_{\mathcal{T}})$ is a pswo set, then $(X, \mathcal{T})$ is hereditarily reversible.*

Proof. Let $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T})$ be a continuous bijection and let $A \in \mathcal{T}$. By Theorem 4.4.1, $f : (X, \leq_{\mathcal{T}}) \rightarrow (X, \leq_{\mathcal{T}})$ is an order isomorphism. Additionally, since $(X, \leq_{\mathcal{T}})$ is pswo, its dual $(X, \geq_{\mathcal{T}})$ is also pswo and thus, $f : (X, \geq_{\mathcal{T}}) \rightarrow (X, \geq_{\mathcal{T}})$ is an order isomorphism.

Now, since A is an open set, we have $A = A \uparrow$, where $A \uparrow = \{x \in X : x \geq u \text{ for some } u \in A\}$. Additionally, A is a non-empty subset of X , so it must have either a finite minimal subset or a finite maximal subset. Suppose A has a finite minimal subset. Then, we have $A = \min(A) \uparrow$.

4.4. Reversibility of Partially Semi-Well Ordered Sets

Since $(X, \leq_{\mathcal{T}})$ is *pswo*, it can be represented as the disjoint union of a *pwo* set and dual of a *pwo* set, which partitions X into disjoint levels. Since $\min(A)$ is finite, it belongs to a finite number of levels of X , say $X_{\alpha 1}, X_{\alpha 2}, \dots, X_{\alpha n}$. Thus, $\min(A) \subset X_{\alpha 1} \cup X_{\alpha 2} \cup \dots \cup X_{\alpha n} = K$ and for each i , we have $f(X_{\alpha i}) = X_{\alpha i}$. Now, $f|_K : K \rightarrow K$ is an order isomorphism and since K is finite, $f|_K^n = I_K$, the identity on K . Hence, $f^n(\min(A)) = \min(A)$, which implies

$$f^n(A) = f^n(\min(A \uparrow)) = (f^n(\min(A)) \uparrow = (\min(A)) \uparrow = A.$$

Therefore, we have $f(A) = f^{-(n-1)}(A)$ and since f is continuous, $f(A)$ is open. Thus, f is a homeomorphism.

If A has a finite maximal subset, we can prove that f is a homeomorphism by considering $(X, \geq_{\mathcal{T}})$ instead of $(X, \leq_{\mathcal{T}})$. Hence, X is reversible. Since any subset of a *pswo* set is *pswo*, $(X, \mathcal{T})$ is hereditarily reversible. $\square$

Chapter 5

Partial Semi-Well Ordering and the Aharoni-Korman Conjecture

This chapter is devoted to the study of the Aharoni–Korman conjecture. We provide an introduction to the conjecture, including a brief overview of its development and the motivation behind its proposal. We also outline the key advancements related to the conjecture, with a focus on the class of posets for which the conjecture has been verified. We verify the conjecture for partially semi-well ordered sets using a specific characterization of these posets as the base for our proof.

5.1 Introduction

Dilworth's theorem [19] states that the maximum size of an antichain in a finite poset is equal to the minimum number of chains required to cover it. This theorem was later extended to show that any poset of finite width k can be partitioned into at most k chains. Peles [56] noted that the strongest form of an “infinite Dilworth's theorem” one might hope for – namely, that any poset of width at most λ can be partitioned into at most λ -many chains – is false. For example, consider the least ordinal α with cardinality $\aleph_1$; the poset $\alpha \times \alpha$, under the cardinal product order, contains no infinite antichain (and thus has width $\aleph_0$), yet it cannot be partitioned into less than $\aleph_1$ -many chains. However, Dilworth's theorem can be generalized to certain infinite posets that contain no infinite antichain, provided there exists some partition into finitely many chains. This result was proved by Abraham [1] in 1987.

Oellrich and Steffens [54] established a stronger version of Dilworth's theorem, showing that if $(X, \leq)$ is a countable poset without an infinite chain, then there exists a partition of X into chains C_i and there is an antichain that intersects each C_i . Using the theorem from [3] instead of the countable version, the proof can be extended to the general (infinite) case. Additionally, Greene and Kleitman [25] extended the finite case of Dilworth's theorem by considering the maximal union of k antichains, rather than a single maximal antichain.

If we replace the word “chain” by “antichain” and vice versa in Dilworth's theorem, we obtain a statement that holds for finite posets. Specifically, for every finite poset, there exists a partition into antichains A_i and a chain C that intersects each A_i . This can be proven by taking A_i as the levels of the poset and C as the chain of maximum length.

Capturing the spirit of Dilworth’s theorem, Greene-Kleitman’s theorem for infinite posets was proved in 1992 [6]. Aharoni and Korman [6] described this theorem as the “ ‘correct’ infinite version” of Greene and Kleitman’s well-known theorem concerning Sperner k -families. The Aharoni-Korman conjecture [6] is the dual of Greene-Kleitman’s theorem.

5.2 The Aharoni-Korman Conjecture

In this section, we present a concise review of the existing literature related to the Aharoni-Korman conjecture.

The Aharoni-Korman conjecture emerged naturally from and is closely connected to the study of covers and matchings in infinite hypergraphs. Often referred to as “the fishbone conjecture” (due to the resemblance of the structure of chains and antichains to the skeleton of a fish), there have been many attempts to prove the conjecture; indeed, it has been described as “one of the most attractive” conjectures in the area [4]. However, despite significant attention, progress on resolving the conjecture has been limited. In this section, we explore the key advancements made regarding the Aharoni-Korman conjecture.

Theorem 5.2.1. *[6](Greene-Kleitman’s Theorem) Let $(X, \leq)$ be a poset with no infinite chains and let k be a positive integer. Then, there exists a partition $\{C_i: i \in I\}$ of X into disjoint chains and k disjoint antichains $A_j (1 \leq j \leq k)$ such that each chain C_i meets $\min(k, |C_i|)$ antichains A_j .*

Remark In Greene-Kleitman’s theorem, the condition of “no infinite chain” is necessary. To demonstrate this [5], consider the poset $(X, \leq)$, where $X = \{x_i :$

5.2. The Aharoni-Korman Conjecture

$0 \leq i < \omega\} \cup \{y_i : 0 < i < \omega\} \cup \{z_i : 0 < i < \omega\}$, and the only relations in X are: $x_i > z_i$ for all $0 < i < \omega$; $x_0 > z_i$ for all $i \geq 1$; $x_i > y_j$ for all $0 \leq j \leq i < \omega$ (with $j \neq 0$); and $y_i > y_j$ for all $0 < j < i < \omega$. It is clear that X contains an infinite chain and it can be proven that Greene-Kleitman's theorem for $k = 1$ does not hold for this poset $(X, \leq)$.

By swapping the words “chain” and “antichain” in Greene-Kleitman's theorem, we obtain the following conjecture proposed by Aharoni and Korman:

Conjecture 5.2.1. [6] (Aharoni-Korman Conjecture) *If a poset $(X, \leq)$ contains no infinite antichain, then for every positive integer k , there exist k disjoint chains $C_1, C_2, \dots, C_k$ and a partition of X into disjoint antichains $\{A_i : i \in I\}$ such that each A_i meets $\min(k, |A_i|)$ chains C_j .*

The disjoint union of chains $[6] X_i$ of lengths $i = 1, 2, \dots$ illustrates that the condition of having ‘no infinite antichain’ is necessary.

The Aharoni-Korman conjecture holds for finite posets, as established in a theorem by Greene [24]. Furthermore, it was shown in [6] that the conjecture holds for any poset of width two. This proof relies on the König duality theorem for infinite bipartite graphs [3]. An alternative proof is presented in the next section, based on the Greene-Kleitman theorem.

A graph is said to be *chordal* [7] if each cycle of length four or more contains a chord. The Aharoni-Korman conjecture for $k = 1$ has been verified for posets whose comparability graph is a chordal graph or the complement of a chordal graph. Specifically, if X is a poset with no infinite antichain and its comparability graph is either a chordal graph or the complement of a chordal graph [7], then

there exists a partition of X into antichains and there is a chain in X that intersects every member of the partition.

Following the Compactness Theorem of First Order Logic, the Aharoni-Korman conjecture for $k = 1$ holds for well founded posets in which every level is finite [6]. Duffus and Goddard subsequently provided a constructive proof of this result [20].

Theorem 5.2.3. *[20] If X is a well founded poset with every level finite, then there exists a chain in X that intersects every level of X .*

In addition, Duffus and Goddard [20] established the Aharoni-Korman conjecture for $k = 1$ for posets of the form $C \times X$, where C is a chain and X is finite, as well as for posets that contain neither infinite antichains nor infinite intervals.

A graph is said to be *locally finite* [74] if each of its vertices is adjacent to only finitely many other vertices. Zaguia [74] verified the Aharoni-Korman conjecture for the case $k = 1$ for posets whose incomparability graph is locally finite.

A poset X is said to be $(3+1)$ -free [26] if X does not have an induced subposet isomorphic to the disjoint union of a 3-element chain and a 1-element chain. The collection of posets whose incomparability graphs are locally finite contains all $(3 + 1)$ -free posets that do not contain infinite antichains. Consequently, the conjecture holds for all $(3+1)$ -free posets with no infinite antichains.

Let $(X, \leq)$ be a poset. A 4-tuple (x, y, z, w) of distinct elements of X is an N in $(X, \leq)$ if $y > x$ and $y > z$, $w > z$ and if these are the only strict relations between the elements x, y, z, w . A poset is N -free [74] if it does not contain an N . Imed [74] further verified the Aharoni-Korman conjecture for $k = 1$ for N -free posets that do not contain infinite antichains.

5.2. The Aharoni-Korman Conjecture

Recently, Hallom [27] provided a counterexample to the conjecture in its full generality. He demonstrated that the Aharoni–Korman conjecture does not hold for the following poset.

Example 5.2.4. [27] Consider the set $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Define a relation $\leq$ on this set by $(n_1, n_2, k) \leq (m_1, m_2, l)$ if and only if of the following conditions holds: (i) $k \geq l + 2$ or (ii) $k = l$ and $n_1 \leq m_1$ and $n_2 \leq m_2$ or (iii) $k = l + 1$ and $\min\{n_1, n_2\} + 1 \leq \min\{m_1, m_2\}$ or (iv) $k = l + 1$ and $n_1 + n_2 \leq 2(m_1 + m_2)$.

A chain C of a poset $(X, \leq)$ is said to be *contiguous* [27], if there does not exist an element $x \in X \setminus C$ and elements $y, z \in C$ such that $y < x < z$ and $C \cup \{x\}$ forms a chain. The *linear sum* $\bigoplus_{i \in I} X_i$ of a collection of posets $(X_i : i \in I)$, indexed by a linear order I , is defined as the poset formed by the disjoint union $\bigsqcup_{i \in I} X_i$, where every element of X_j is greater than every element of X_i whenever $j > i$ in I .

Hallom also verified that the Aharoni–Korman conjecture for the case $k = 1$ for a countable poset X with no infinite antichain, such that for any contiguous chain $C \subseteq X$, neither C nor its dual C^* can be expressed as a linear sum $\bigoplus_{i \in \omega} C_i$, where each C_i is infinite and co-well founded.

Although the Aharoni–Korman conjecture does not hold in full generality, it remains valid for a broad and significant class of posets.

5.3 Some Advancements in the Aharoni-Korman Conjecture

We now present a simple alternating proof for the Aharoni-Korman conjecture for the case $k = 1$ for a poset of width two.

Theorem 5.3.1. *If $(X, \leq)$ is a poset of width two (that is, it contains no antichain of size three), then the Aharoni-Korman conjecture holds for all k .*

Proof. Given that $(X, \leq)$ is a poset of width two, it can be covered by two disjoint chains, C_1 and C_2 . Define a relation $\leq'$ on X such that $x <' y$ if and only if $x \in C_1$, $y \in C_2$ and $x \parallel y$. Then, $(X, \leq')$ forms a poset that does not contain an infinite chain. By the Greene-Kleitman theorem, for any positive integer k , there exists a partition of X into disjoint chains $(C_i: i \in I)$ and there exist antichains $A_1, A_2, \dots, A_k$ such that each chain C_i meets $\min(k, |C_i|)$ antichains A_j . A chain in $(X, \leq')$ is an antichain in $(X, \leq)$ and an antichain in $(X, \leq')$ is a chain in $(X, \leq)$. Therefore, for each positive integer k , the Aharoni-Korman conjecture holds. $\square$

Remark For posets of width l , the Aharoni-Korman conjecture is true for all $k \geq l$. In this case, take $C_1, C_2, \dots, C_l$ as the disjoint chains that cover the poset and set $C_i = \emptyset$ for $l < i \leq k$. Then, the decomposition of the poset into singleton antichains satisfies the desired conditions.

The following is a more general theorem.

Theorem 5.3.3. *If $(X, \leq)$ is a poset of finite width and the incomparability relation on X is transitive (i.e, $x||y, y||z$ implies $x||z$ for all $x, y, z \in X$), then the Aharoni-Korman conjecture is true for all k .*

Proof. Since $(X, \leq)$ is a poset of finite width, X can be covered by n disjoint chains, say $C_1, C_2, C_3, \dots, C_n$. Now define a relation $\leq'$ on X by $x < y$ if and only if $x \in C_i, y \in C_j, i < j$ and $x||y$. Then $(X, \leq')$ is a poset with no infinite chain. So, by Greene-Kleitman theorem, this follows. $\square$

Theorem 5.3.4. *Let $(X, \leq)$ be a pwo set and $(A_i: i \in I)$ be a partition of X into disjoint levels. Suppose C_1 is a chain that meets each A_i exactly once and the set $\overline{C_1} = \{x \in X \setminus C_1: x < c \text{ for some } c \in C_1\}$ is empty. Then the Aharoni-Korman conjecture holds for $k = 2$.*

Proof. Let $K = X \setminus C_1$ equipped with the induced order. Clearly, K is a pwo set and can be partitioned into disjoint levels, say $B_1, B_2, B_3, \dots$ and there is a chain C_2 that meets each B_i exactly once. Let $C_1 = \{c_1, c_2, \dots\}$. For each B_i , let A_{n_i} be the level of X containing an element of B_i such that no element of B_i is contained in A_j for $j > n_i$. Now define a new partition of X as follows: $A'_1 = \{c_1\}, A'_2 = \{c_2\}, \dots, A'_{n_1-1} = \{c_{n_1-1}\}, A'_{n_1} = \{c_{n_1}\} \cup B_1, A'_{n_1+1} = \{c_{n_1+1}\}, A'_{n_1+2} = \{c_{n_1+2}\}, \dots, A'_{n_2} = \{c_{n_2}\} \cup B_2, \dots$. Then, $(A'_i: i \in I)$ is a partition of X into disjoint antichains, and C_1 and C_2 are disjoint chains in X satisfying the Aharoni-Korman conjecture for $k = 2$. $\square$

5.3.1 Partial Semi-Well Ordering and the Aharoni-Korman Conjecture

In this section, we verify the Aharoni-Korman conjecture for partially semi-well ordered sets. To do so, we utilized a particular characterization of these sets, which plays a key role in the verification of the conjecture.

Theorem 5.3.5. *A poset $(X, \leq)$ is psw if and only if it has no infinite antichain and it embeds neither $\omega^* + \omega$ nor $\omega^* \sqcup \omega$.*

Proof. Suppose that X is psw. Both an infinite antichain in X and $\omega^* + \omega$ do not contain a finite minimal or maximal subset. Consider the set $\omega^* \sqcup \omega$. The only minimal element of $\omega^* \sqcup \omega$ is $0 \in \omega$ and $\{0\}$ is not a minimal subset of $\omega^* \sqcup \omega$. Similarly, the only maximal element of $\omega^* \sqcup \omega$ is $0 \in \omega^*$, but $\{0\}$ is not a maximal subset of $\omega^* \sqcup \omega$. Therefore, X contains no infinite antichain and it neither embeds $\omega^* + \omega$ nor $\omega^* \sqcup \omega$.

For the converse, suppose that X does not embed $\omega^* + \omega$ or $\omega^* \sqcup \omega$. Assume, for contradiction, that there exists an infinite subset A of X that has no finite minimal or maximal subset. We first prove that $(A, \leq)$ is neither well founded nor anti-well founded and we will argue to a contradiction.

Indeed, if $(A, \leq)$ is well founded, then the set of all minimal elements of A forms a minimal subset of A . To prove this, let B denote the set of all minimal elements of A . Clearly, B is non-empty and since it is an antichain, B is finite. Let $a_1 \in A \setminus B$. Since $a_1 \notin B$, there exists $a_2 \in A$ such that $a_2 < a_1$. If $a_2 \notin B$,

then there exists $a_3 \in A$ such that $a_3 < a_2 < a_1$. Since A is well founded, there exists an $a_k \in B$ with $a_k < \cdots < a_3 < a_2 < a_1$. Therefore, for each $a \in A$, there exists $b \in B$ such that $b \leq a$. This implies that B is a finite minimal subset of A . Similarly, if $(A, \leq)$ is dually well founded, then the set of all maximal elements of A forms a finite maximal subset of A .

Hence, $(A, \leq)$ embeds both ω and ω^* . Since A cannot embed $\omega^* + \omega$ and $\omega^* \sqcup \omega$, it follows that A must embed $\omega + \omega^*$. So A must have a minimal and maximal element. Let P and Q denote the collections of all minimal elements and all maximal elements of A , respectively. If either P or Q is infinite, then it forms an infinite antichain in A . Suppose that, both P and Q are finite. Since P is not a minimal subset of A , there exists an $x_1 \in A$ with $m \not\leq x_1$ for all $m \in P$. Clearly, $x_1 \notin P$, so there exists $x_2 \in A$ with $x_2 < x_1$ and $m \not\leq x_2$ for all $m \in P$. Continuing, we get a strictly decreasing sequence $C = \{\cdots, x_3, x_2, x_1\}$. Again, Q is not a maximal subset of A , so there exists $y_1 \in A$ with $y_1 \not\leq m$ for all $m \in Q$. Now, $y_1 \notin Q$, so there exists $y_2 \in A$ with $y_1 < y_2$ and $y_2 \not\leq m$ for all $m \in Q$. Continuing, we get a strictly increasing sequence $D = \{y_1, y_2, y_3, \cdots\}$. Without loss of generality, take $C \cap D$ as the empty set (if they intersect, then the intersection is finite). If no element of D is strictly less than any element of C , then A embeds either $\omega^* \sqcup \omega$ or $\omega^* + \omega$. If not, suppose that there exists $y_i \in D$ and $x_j \in C$ with $y_i < x_j$. If there exists an $m \in P$ with $m \leq y_i$, then by transitivity $m < x_j$, which is not possible by the choice of x_j . So $m \not\leq y_i$ for all $m \in P$. Also, $y_i \notin P$, therefore there exists $z_1 \in A$ with $z_1 < y_i$, again $m \not\leq z_1$

for all $m \in P$. So, there exists z_2 with $z_2 < z_1 < y_i$. Continuing this, we get $\cdots < z_2 < z_1 < y_i < y_{i+1} < y_{i+2} < \cdots$, that is, A embeds $\omega^* + \omega$.

Therefore, every non-empty subset of X must have either a finite minimal or maximal subset. Thus, $(X, \leq)$ is *pswo*. $\square$

A subset Y of a poset $(X, \leq)$ is said to be an *initial segment* of X if $x \in Y$ whenever $y \in Y$ and $x \leq y$. The definition of a *final segment* is analogous to that of an initial segment.

Theorem 5.3.6. *The posets that do not embed $\omega^* + \omega$ are unions of an initial segment that is well founded and a final segment that is dually well founded.*

Proof. Let $(X, \leq)$ be a poset that does not embed $\omega^* + \omega$. Without loss of generality, assume that X is infinite. If X embeds ω , then the initial segments generated by a chain of order type ω are well-founded. To show this, let C be a chain in X of order type ω and let $Xc = \{x \in X : x \leq c\}$ be an initial segment of C , where $c \in C$. If Xc is not well-founded, it must contain an infinite decreasing sequence. Since every element of this sequence is less than c , it follows that X would embed $\omega^* + \omega$, which contradicts the assumption.

Now, consider the set of all well founded initial segments of X , say $\mathcal{J}$. Clearly, $\mathcal{J}$ is a non-empty poset under set inclusion. Let $\{A_i : i \in I\}$ be a chain in $\mathcal{J}$ and $A = \bigcup_{i \in I} A_i$. If $x \in A$ and $y \leq x$, then $x \in A_i$ for some i . Since A_i is an initial segment, we have $y \in A_i \subseteq A$. Additionally, if $\{a_1 > a_2 > \cdots\}$ is an infinite strictly decreasing sequence in A , then $a_1 \in A_j$ for some j . Since A_j is

an initial segment, the sequence $\{a_1 > a_2 > \dots\}$ is contained in A_j , which is not possible because A_j is well founded. Thus, A cannot contain a strictly decreasing sequence. Therefore, A is a well founded initial segment in X such that $A_i \subseteq A$ for all $i \in I$. By Zorn's lemma, a maximal well founded initial segment exists in X ; let it be K .

Now, consider $X \setminus K$. If $X \setminus K$ is non-empty, then it must be an infinite final segment. Additionally, no element of $X \setminus K$ is strictly less than any element of K , since K is an initial segment. Suppose that $X \setminus K$ contains a strictly increasing sequence, say S . Let B be the union of S and all the initial segments of S . Then, B is a well founded initial segment and so $K \cup B$ is also a well founded initial segment. This contradicts the assumption that K is the maximal well founded initial segment of X . Therefore, $X \setminus K$ is dually well founded. If X does not embed ω but embeds ω^* , apply the previous argument to the dual of $(X, \leq)$. $\square$

Using the above two theorems and Theorem 5.2.3, the following theorem verifies the Aharoni-Korman conjecture for $k = 1$ for partially semi-well ordered sets.

Theorem 5.3.7. *If $(X, \leq)$ is a partially semi-well ordered set, then there is a partition of X into antichains and there is a chain that meets each member of the partition exactly once.*

Proof. By Theorems 5.3.5 and 5.3.6, X is a disjoint union of an initial segment that is well-founded, say P and a final segment that is dually well founded, say

5.3. Some Advancements in the Aharoni-Korman Conjecture

P' , with every level of P and P' finite. Also, note that no element of P' is strictly less than any element of P .

Since the Aharoni-Korman conjecture ($k = 1$), is true for well founded sets with every level finite, the finite levels form partitions of P and P' into disjoint antichains and there exist chains C_1 and C_2 in P and P' respectively, which meets each member of the partition exactly once. Since X does not embed $\omega \sqcup \omega^*$, every element of C_1 is strictly less than every element of C_2 . So, the antichain partitions of P and P' together forms the antichain partition of $(X, \leq)$ and the union of chains C_1 and C_2 is the required chain. $\square$

Chapter 6

The Topology of a Partially Semi-Well Ordered Set

A partially semi-well ordered set is a partially ordered set in which every non-empty subset has either a finite minimal subset or a finite maximal subset. In this chapter, we examine in detail the properties of the interval topology on partially ordered sets with no infinite antichain and in particular on partially semi-well ordered sets. We further investigate the continuous functions on partially semi-well ordered sets and establish a result concerning mappings between such sets.

6.1 Introduction

In 1958, Wolk [71] established that a partially ordered set with no infinite antichain is Hausdorff with respect to its interval topology. Since partially semi-well ordered sets do not admit infinite antichains, this conclusion also holds for such partially ordered sets. Furthermore, since the interval topology and the Dedekind topology coincide in a partially ordered set with no infinite antichain, we refer to this common topology as the intrinsic topology.

In this chapter, the next section is organized into three subsections. The first subsection discusses the properties of the intrinsic topology on partially ordered sets with no infinite antichain. The second subsection examines the intrinsic topology on partially semi-well ordered sets. In the final subsection, we present a characterization of continuous functions between two partially semi-well ordered sets and using this characterization, we establish a result on functions between such sets.

6.2 The Topology of a Partially Semi-Well Ordered Set

In this section, we examine the properties of the interval topology defined on a partially semi-well ordered set. Throughout, for a poset $(X, \leq)$, we define $(\downarrow x) = \{a \in X : a \leq x\}$ and $(\uparrow x) = \{a \in X : x \leq a\}$.

Definition 6.2.1. Let $(X, \leq)$ be a poset. The *interval topology* [73] on X

is the topology $\mathcal{T}$ for which the family of sets $\{(\downarrow x), (\uparrow y) : x, y \in X\}$ forms a sub-basis for the closed sets.

Definition 6.2.2. A subset A of a poset $(X, \leq)$ is said to be *up-directed* (respectively, *down-directed*) [73] if, for any $x, y \in A$, there exists $z \in A$ such that $x \leq z$ and $y \leq z$ (respectively, $z \leq x$ and $z \leq y$).

Definition 6.2.3. A subset A of a poset $(X, \leq)$ is said to be *Dedekind closed* [48] if, for any up-directed subset $U \subseteq A$ with $l.u.b.(U) = k$ or any down-directed subset $U \subseteq A$ with $g.l.b.(U) = k$, it follows that $k \in A$.

The union of two Dedekind closed subsets of a poset is also Dedekind closed. To show this, let $(X, \leq)$ be a poset and let A and B be two Dedekind closed subsets of X . Let U be an up-directed subset of $A \cup B$ such that $g = l.u.b.(U)$. We have to show that $g \in A \cup B$. If $U \subseteq A$ or $U \subseteq B$ then the result follows immediately from the Dedekind closedness of A or B , respectively. Now, suppose that both $U \cap A$ and $U \cap B$ are non-empty.

We first claim that, g is the least upper bound of either $A \cap U$ or $B \cap U$. Suppose, for the sake of contradiction, that g is not the least upper bound of $A \cap U$ and $B \cap U$. Then there exist upper bounds x and y of $A \cap U$ and $B \cap U$, respectively, such that $g \not\leq x$ and $g \not\leq y$. Now, x cannot be an upper bound of $B \cap U$ and y cannot be an upper bound of $A \cap U$. Therefore, there exists $b \in B \cap U$ such that $b \not\leq x$ and $a \in A \cap U$ such that $a \not\leq y$. Since $a, b \in U$ and U is up-directed, there exists $c \in U$ such that $a \leq c$ and $b \leq c$. But $c \in U \subseteq A \cup B$, so $c \in A \cap U$ or $c \in B \cap U$, which is a contradiction. Hence, the claim follows.

It is clear that at least one of the sets $A \cap U$ or $B \cap U$ is up-directed. We

now proceed to prove the following: If $g = l.u.b.(B \cap U)$, $A \cap U$ is up-directed and $B \cap U$ is not up-directed, then $g = l.u.b.(A \cap U)$. Suppose, for contradiction, that $g \neq l.u.b.(A \cap U)$. Then there exists an upper bound x of $A \cap U$ such that $x \not\leq g$. Now, x cannot be an upper bound of $B \cap U$. Hence, there exists $b \in B \cap U$ such that $b \not\leq x$. Let $b_1, b_2 \in B \cap U$ be arbitrary. Since U is up-directed, there exists $u \in U$ such that $b_1 \leq u$, $b_2 \leq u$ and $b \leq u$. If $u \in A$, then $b \leq u \leq x$, contradicting $b \not\leq x$. Therefore, $u \in B \cap U$ and so $B \cap U$ is up-directed, a contradiction. Hence, $g = l.u.b.(A \cap U)$. Thus, $g \in A \cup B$ and therefore $A \cup B$ is Dedekind closed.

Hence, the collection of all Dedekind closed subsets of a poset $(X, \leq)$, denoted by $\mathcal{D}$, forms a topology on X , called the *Dedekind topology* [73].

Clearly, $\mathcal{T} \subseteq \mathcal{D}$; however, they are not necessarily equal. For example, consider the following:

Example 6.2.4. Let $(X, \leq)$ be the disjoint union of chains $C_1, C_2, \dots$, where each chain C_i has cardinality $|C_i| = i$. Clearly, X is a poset. For each $i \in \mathbb{N}$, let c_i denote the least element of C_i . Then the set $\{c_i: i \in \mathbb{N}\}$ is Dedekind closed; however, it is not closed in the interval topology.

A topological space $(X, \mathcal{U})$ is said to be *order-compatible* [71] if $\mathcal{T} \subseteq \mathcal{U} \subseteq \mathcal{D}$, where $\mathcal{T}$ is the interval topology and $\mathcal{D}$ is the Dedekind topology on X . In 1958, Wolk [71] proved that if $(X, \leq)$ is a poset of finite width, then it admits a unique order-compatible topology. Later, Naito [53] extended this result by showing that if every antichain in a poset $(X, \leq)$ has finite cardinality, then X possesses a unique order-compatible topology. He further proved that if $(X, \leq)$ is a complete lattice, then every antichain in X has finite cardinality if and only if it admits a

unique order-compatible topology.

Since every totally unordered subset of a *pswo* set is finite, the interval topology and the Dedekind topology coincide on such a set. Following [73], we refer to this topology as the *intrinsic topology* on a *pswo* set.

6.2.1 Partially Ordered Sets with No Infinite Antichain

In this section, we discuss some properties of the intrinsic topology on partially ordered sets with no infinite antichain. We now state the following theorem for general posets:

Theorem 6.2.5. [73] *If a net $\{S_n : n \in D\}$ in a poset $(X, \leq)$ converges to y in the interval topology on X and $S_n \leq y$ for all $n \in D$, then $y = l.u.b.(range(S_n))$.*

Using this theorem, we prove the following theorem for posets with no infinite antichain.

Theorem 6.2.6. *Let $(X, \leq)$ be a poset with no infinite antichain and let $\{S_n : n \in D\}$ be a monotonically increasing (respectively, decreasing) net in X . Then $y = l.u.b.(range(\{S_n\}))$ (respectively, $y = g.l.b.(range(\{S_n\}))$) if and only if $\{S_n\}$ converges to y in the intrinsic topology on X .*

Proof. Suppose that $y = l.u.b.(range(S_n))$, but $\{S_n\}$ does not converge to y in the intrinsic topology on X . Then there exists a closed set $C \subseteq X$ such that $y \notin C$ and, for all $n \in D$, there exists $m_n \in D$ with $n \leq m_n$ and $S_n \leq S_{m_n}$ and $S_{m_n} \in C$. Consider the subnet $\{S_{m_n} : n \in D\}$. Since $\{S_n\}$ is monotonically increasing, $range(\{S_{m_n}\})$ is an up-directed subset of C .

We now show that y is the least upper bound of $\text{range}(\{S_{m_n}\})$. First, it is clear that y is an upper bound of this set. Suppose, for contradiction, that some $x < y$ is also an upper bound. Then, there exists $S_k \in \{S_n : n \in D\}$ such that $x < S_k$ or $x \parallel S_k$. Then there exist $k' \in D$ and $S_{k'} \in \{S_{m_n}\}$ with $x < S_k \leq S_{k'}$ or $x \parallel S_k \leq S_{k'}$, which contradicts the assumption that x is an upper bound. Therefore, no element less than y can be an upper bound of $\text{range}(\{S_{m_n}\})$. Next, suppose that there exists an upper bound l of $\text{range}(\{S_{m_n}\})$ such that $l \parallel y$. Then l cannot be an upper bound of $\text{range}(\{S_n\})$, so there exists $S_k \in \{S_n\}$ such that either $l < S_k$ or $l \parallel S_k$. Since l is assumed to be an upper bound of $\text{range}(\{S_{m_n}\})$ and there exists $k'' \geq k$ such that $S_{k''} \in \{S_{m_n}\}$ with $S_k \leq S_{k''}$, this leads to a contradiction.

Thus, $\text{range}(\{S_{m_n}\})$ is an up-directed subset of C with y as its least upper bound. Since C is closed, it must contain y , which contradicts the assumption that $y \notin C$. Therefore, $\{S_n\}$ converges to y .

Conversely, suppose that $\{S_n\}$ converges to y . We will show that $S_n \leq y$ for all n . Assume, for the sake of contradiction, that there exists some $m \in D$ such that $S_m \not\leq y$. Then $y \leq S_m$ or $y \parallel S_m$. In either case, the complement of the set $(\uparrow S_m)$ is an open set containing y . Since $\{S_n\}$ converges to y , there exists $k \in D$ such that for all $j \geq k$, $S_j \in (\uparrow S_m)^c$. In particular, $S_k \in (\uparrow S_m)^c$. Because D is a directed set, there exists $a \in D$ such that $k \leq a$ and $m \leq a$. Since $\{S_n\}$ is monotonically increasing, we have $S_k \leq S_a$ and $S_m \leq S_a$. Therefore, $S_a \in (\uparrow S_m)^c \cap (\uparrow S_m)$, which is a contradiction. Thus, $S_n \leq y$ for all n . By

Theorem 6.2.5, it follows that $y = l.u.b.(range(S_n))$. $\square$

In 1967, R. W. Hansell [29] proved the following theorem:

Theorem 6.2.7. [29] *If $(X, \leq)$ is a poset with no infinite antichain, then every net in X has a monotone subnet.*

Now we have the following corollary:

Corollary 6.2.8. *Let $(X, \leq)$ be a poset with no infinite antichain and let $S = \{S_n : n \in D\}$ be a net in X . If S converges to y with respect to the interval topology $\mathcal{T}$, then for every subnet T of S , there exists an up-directed or down-directed subset of $range(T)$ whose $l.u.b.$ or $g.l.b.$, respectively, is y .*

Proof. Suppose that S converges to y and let T be an arbitrary subnet of S . Since every net contains a monotone subnet, T contains a monotone subnet, say R . Clearly, R also converges to y . By Theorem 6.2.6, if R is monotonically increasing, then $y = l.u.b.(range(R))$ and $range(R)$ is up-directed. Similarly, if R is monotonically decreasing, then $y = g.l.b.(range(R))$ and $range(R)$ is down-directed. $\square$

A poset X is said to be *Dedekind complete* [73] if every up-directed subset of X has a least upper bound in X and every down-directed subset of X has a greatest lower bound in X . The following theorem presents characterizations of compact sets in a poset with no infinite antichain.

Theorem 6.2.9. *Let $(X, \leq)$ be a poset with no infinite antichain then the following are equivalent:*

(i) X is compact.

(ii) $(X, \leq)$ is Dedekind complete.

(iii) Every linearly ordered subset of X has both a least upper bound and a greatest lower bound in X .

Proof. (i) $\Leftrightarrow$ (ii): Suppose that $(X, \leq)$ is Dedekind complete. Let $S = \{S_n\}$ be a net in X . Since X contains no infinite antichain, by Theorem 6.2.7, S has a monotone subnet, say $\{A_n\}$. If $\{A_n\}$ is monotonically increasing, then its range set is up-directed and therefore has a least upper bound, say a . Similarly, if $\{A_n\}$ is monotonically decreasing, then its range set contains a greatest lower bound, say b . In either case, by Theorem 6.2.6, the subnet $\{A_n\}$ converges to a or b , respectively. Hence, every net in X has a convergent subnet, which implies that X is compact.

For the converse, it is known that if a poset is compact in its interval topology, then it is Dedekind complete [72].

(i) $\Leftrightarrow$ (iii): Suppose that every linearly ordered subset of X has a least upper bound and a greatest lower bound in X . In [17], it is proved that if every linearly ordered subset of a poset has a least upper bound (greatest lower bound), then every up-directed (down-directed) subset of X has a least upper bound (greatest lower bound) in X . So, X is Dedekind complete. The converse part is trivial, since every linearly ordered set is both up-directed and down-directed. $\square$

Theorem 6.2.10. (*Kelley's Theorem on Iterated Limits*) [37]: Let D be a di-

rected set and for each $m \in D$, let E_m be a directed set. Define $F = D \times \prod\{E_m : m \in D\}$. For each $(m, f) \in F$, define $R(m, f) = (m, f(m))$. Suppose $S(m, n)$ is an element of a topological space for each $m \in D$ and $n \in E_m$. Then the net $S \circ R$ converges to $\lim_m \lim_n S(m, n)$ whenever this iterated limit exists.

Theorem 6.2.11. *Let $(X, \leq)$ be a poset with no infinite antichain and let $A \subseteq X$. Then $\bar{A} = \{y \in X : y = l.u.b.(K) \text{ for some up-directed set } K \subseteq A \text{ or } y = g.l.b.(L) \text{ for some down-directed set } L \subseteq A\}$.*

Proof. Let $\tilde{A}$ denote the set $\{y \in X : y = l.u.b.(K) \text{ for some up-directed set } K \subseteq A \text{ or } y = g.l.b.(L) \text{ for some down-directed set } L \subseteq A\}$. We have to show that $\bar{A} = \tilde{A}$. Clearly, $A \subseteq \tilde{A}$. Now, if $y \in \tilde{A}$ and C is a Dedekind closed subset of X that contains A , then $y \in C$. This implies $\tilde{A} \subseteq \bar{A}$. We now prove that $\tilde{A}$ is Dedekind closed.

Let D be an up-directed subset of $\tilde{A}$ such that $l.u.b.(D) = y$. By the definition of $\tilde{A}$, for each $m \in D$, there exists an up-directed set or a down-directed set $E_m \subseteq A$ satisfying $l.u.b.(E_m) = m$ or $g.l.b.(E_m) = m$. If $(E_m, \leq)$ is down-directed, here we consider $(E_m, \geq)$ as the directed set. For each fixed m , define the net $S(m, n)$ by $S(m, n) = n$ for all $n \in E_m$. Then $\lim_{m \in D} [\lim_{n \in E_m} S(m, n)] = y$.

We now apply Kelly's Theorem on Iterated Limits. Let $F = D \times \prod\{E_m : m \in D\}$, where F is equipped with the cardinal product order. For each $(m, f) \in F$, define $R(m, f) = (m, f(m)) \in D \times E_m$. Then the composition $S \circ R$, given by $S(R(m, f)) = f(m)$, defines a net that converges to y . Moreover, $range(S \circ R) \subseteq$

$\cup\{E_m : m \in D\} \subseteq A$. Since $S \circ R$ is a net, it possesses a monotone subnet [29], denoted by T , which also converges to y . By Theorem 6.2.6, it follows that $y = l.u.b.(range(T))$ or $y = g.l.b.(range(T))$. Hence, the range of T forms either an upward-directed or a downward-directed subset of A , with y as its least upper bound or greatest lower bound, respectively. Therefore, $y \in \tilde{A}$. Thus, $\tilde{A}$ is a Dedekind closed subset of X and consequently $\tilde{A} = \overline{A}$. $\square$

Theorem 6.2.12. *Let $(X, \leq)$ be a poset with no infinite antichain and let $\mathcal{T}$ be its intrinsic topology. For any subset $A \subseteq X$, let $\mathcal{S}$ denote the subspace topology on A induced by $\mathcal{T}$ and let $\mathcal{C}$ be the intrinsic topology on A , then $\mathcal{C} \subseteq \mathcal{S}$.*

Proof. Let B be a $\mathcal{C}$ -closed subset of A . We aim to show that $B = \overline{B}^{\mathcal{T}} \cap A$, where $\overline{B}^{\mathcal{T}}$ denotes the closure of B in $(X, \mathcal{T})$. The inclusion $B \subseteq \overline{B}^{\mathcal{T}} \cap A$ is immediate. To show the reverse inclusion, let $y \in \overline{B}^{\mathcal{T}} \cap A$. Because $y \in \overline{B}^{\mathcal{T}}$, Theorem 6.2.11 guarantees the existence of either an up-directed subset $D \subseteq B$ such that $l.u.b.(D) = y$ or a down-directed set $E \subseteq B$ such that $g.l.b.(E) = y$. Since $y \in A$ and B is $\mathcal{C}$ -closed in A , it follows that $y \in B$. Hence, $B = \overline{B}^{\mathcal{T}} \cap A$, implying that B is closed in $\mathcal{S}$. Therefore, every $\mathcal{C}$ -closed subset of A is also $\mathcal{S}$ -closed and thus $\mathcal{C} \subseteq \mathcal{S}$. $\square$

6.2.2 Partially Semi-Well Ordered Sets

In this section, we discuss the properties of the intrinsic topology on partially semi-well ordered sets. We already showed in the previous chapter that every partially semi-well ordered set can be expressed as the disjoint union of a partially

well ordered set and dually partially well ordered set. We now state the following lemma:

Lemma 6.2.13. *Let $(X = P \sqcup P', \leq)$ be a pswo set equipped with the intrinsic topology such that P is infinite. Then the minimal subset of P is the minimal subset of X and every subset of the finite minimal subset of X is clopen.*

Proof. If P' is empty, then X is a *pwo* set. Let $M = \{m_1, m_2, \dots, m_n\}$ be the finite minimal subset of X . Consider the set $\{m_1\}$. Clearly, $X \setminus \{m_1\}$ has a finite minimal subset in X , say $\{k_1, k_2, \dots, k_r\}$. Moreover, $X \setminus \{m_1\} = \bigcup_{i=1}^r (\uparrow k_r)$, which is closed in X . Since X is Hausdorff, $\{m_1\}$ is a clopen subset of X . Similarly, $\{m_i\}$ is a clopen subset of X for all $i = 2, 3, \dots, n$.

Now, suppose that P' is non-empty. Then X is neither *pwo* nor dually *pwo* and both P and P' must be infinite. Let M be the finite minimal subset of P . We will now show that M is the minimal subset of X . Clearly, no element of P' is strictly less than any element of M , since P' is a final segment. Suppose, for contradiction, that there exists an element $p \in P'$ such that $p \parallel m$ for all $m \in M$. Since $p \in P'$, there exists a strictly decreasing sequence in P' with $\dots p_2 < p_1 < p$. Clearly, $p_i \parallel m$ for all $m \in M$. Since P is infinite, there exists a strictly increasing sequence in P whose first element belongs to M . This implies that X embeds $\omega \sqcup \omega^*$, contradicting the assumption that X is *pswo*. Therefore, M is a minimal subset of X . Additionally, since X is Hausdorff, the singleton set $\{m\}$ is closed for all $m \in M$. Moreover, since $X \setminus \{m\}$ has a finite minimal subset, this implies that $\{m\}$ is open in X . □

If $(X, \leq)$ is a dually *pwo* set equipped with the intrinsic topology, then it can be similarly shown that every subset of the finite maximal subset of X is clopen. Thus, every *pswo* set contains a clopen subset. The following theorem is now trivial.

Theorem 6.2.14. *A pswo set is connected if and only if it is singleton or empty.*

As discussed in chapter 4, every order preserving bijection on a *pswo* set $(X, \leq)$ is an order isomorphism. In other words, every *pswo* set is order reversible. Furthermore, since every subset of a *pswo* set is itself *pswo*, it follows that the set is hereditarily order reversible. However, a *pswo* set endowed with its intrinsic topology need not be topologically reversible; that is, a bijective continuous map on such a set is not necessarily a homeomorphism. This is illustrated in Example 3.2.25.

The structural property of *pswo* sets is utilized to establish the subsequent theorem.

Theorem 6.2.15. *Let $(X, \leq)$ be a pswo set such that $X = P \sqcup P'$, where P is the maximal pwo initial segment of X and P' is the dually pwo final segment of X . Then P is a closed subset of X .*

Proof. We now show that P is Dedekind closed in X . Let A be an infinite up-directed subset of P and let $a = l.u.b.(A)$. Suppose, for contradiction, that $a \notin P$. Then every element of A has a predecessor in A . Now, $a \in P'$ implies that there exists a strictly decreasing sequence $\cdots < x_2 < x_1 < a$ in P' . Since a is the least upper bound of A , there exists an element in A that is not less than some x_i . In this case, the poset X would embed $\omega \sqcup \omega^*$, contradicting the assumption

that X is *pswo*. Hence, $a \in P$. Moreover, every down-directed subset of P is finite and therefore P is Dedekind closed in X . $\square$

Theorem 6.2.16. *Every pswo set, when endowed with its intrinsic topology, is anti-rigid.*

Proof. Let $(X, \leq)$ be a *pswo* set and let A be a non-empty subset of X . If $(A, \leq)$ is *pwo*, then it contains a finite minimal subset, say $\{a_1, a_2, \dots, a_n\}$. The singleton $\{a_1\}$ is open in the subspace topology on A , since it is open in the intrinsic topology on A . Thus, a_1 is an isolated point of A . Similarly, if $(A, \leq)$ is dually *pwo*, then it contains a finite maximal subset and again, we can find an isolated point in A by the same reasoning. Now suppose that $(A, \leq)$ is a *pswo* subset of X that is neither *pwo* nor dually *pwo*. Then, by Lemma 6.2.13, A contains an element a such that $\{a\}$ is open in the intrinsic topology on A ; hence a is an isolated point of A .

Therefore, every non-empty subset of X contains an isolated point. Consequently, X is a scattered Hausdorff space and hence X is anti-rigid. $\square$

Remark It is known that the cardinal product of two *pwo* sets is again *pwo* [72] and that the product topology coincides with the intrinsic topology of the product [73]. For the well ordered sets Ω_0 and Ω' , the product $\Omega_0 \times \Omega'$ is *pwo* but not normal in the product topology [31]. Therefore, the intrinsic topology on $\Omega_0 \times \Omega'$ is also not normal [73]. Consequently, a *pswo* set equipped with the intrinsic topology need not be normal.

Again, since Ω' [31] is neither first countable, separable nor perfectly normal and does not satisfy the countable chain condition, these properties need not hold for *pswo* sets. Moreover, Ω_0 [31, 67] is not metacompact, paracompact, Lindelöf or σ -compact; therefore, *pswo* sets do not necessarily possess these properties. In addition, *pswo* sets need not be pseudocompact, weakly countably compact, countably compact, sequentially compact or homogeneous, since one can easily find well ordered sets that do not possess these properties. Furthermore, since every *pswo* set is Hausdorff, it is also Sober [38].

6.2.3 Continuous Functions

In this section, X and X' denote *pswo* sets equipped with the intrinsic topology. The following is a characterization for continuous functions on *pswo* sets.

Theorem 6.2.18. *Let X and X' be two *pswo* sets. Then a function $f : X \rightarrow X'$ is continuous if and only if for every up-directed (or down-directed) subset $D \subseteq X$ with $\text{l.u.b.}(D) = y$ (or $\text{g.l.b.}(D) = y$), the image $f(D)$ contains an up-directed or a down-directed subset whose *l.u.b.* or *g.l.b.*, respectively, is $f(y)$.*

Proof. Suppose that f is continuous and let D be an up-directed subset of X with $\text{l.u.b.}(D) = y$. Define a net $S : D \rightarrow X$ by $S_n = n$ for all $n \in D$. Since f is continuous, the net $\{f(S_n) : n \in D\}$ converges to $f(y)$. By Corollary 6.2.8, it follows that the range of $\{f(S_n) : n \in D\}$ contains either an up-directed set E with $\text{l.u.b.}(E) = f(y)$ or a down-directed set F with $\text{g.l.b.}(F) = f(y)$. If $(D, \leq)$

is down-directed, consider $(D, \geq)$ as the directed set when defining the net S_n ; in this case, the net S_n is monotonically decreasing.

Conversely, let $C \subseteq X'$ be a closed set and define $A = f^{-1}(C)$. To show that A is closed in X , consider an up-directed subset $D \subseteq A$ with $\text{l.u.b.}(D) = y$. By assumption, $f(D)$ contains either an up-directed subset $E \subseteq C$ with $\text{l.u.b.}(E) = f(y)$ or a down-directed subset $F \subseteq C$ with $\text{g.l.b.}(F) = f(y)$. Since C is closed, $f(y) \in C$ and hence $y \in A$. Therefore, A is closed in X , which implies that f is continuous. $\square$

Theorem 6.2.19. *Let $(X = P \sqcup P', \leq)$ and $(X', \leq')$ be two pswo sets, where P has no maximal elements. Then every function $f : X \rightarrow X'$ is the point-wise limit of a net of continuous functions from X to X' .*

Proof. Let $\mathbb{D}$ denote the collection of all finite subsets of X . Clearly $\mathbb{D}$ is a directed set under the usual set inclusion. For each $x \in X$ and $D \in \mathbb{D}$, we say that x has a D -cover if one of the following holds: (i) $x \in D$; or (ii) $x \notin D$, $x \in P$ and there exists $d \in D$ such that $x < d$ and there is no $e \in D$ with $x < e < d$; or (iii) $x \notin D$, $x \in P'$ and there exists $d \in D$ such that $x > d$ and there is no $e \in D$ with $x > e > d$.

Let $k \in X'$ be fixed. For each $D \in \mathbb{D}$, define a function $f_D : X \rightarrow X'$ as follows:

$$f_D(x) = \begin{cases} k, & \text{if } x \text{ has no } D\text{-cover,} \\ k, & \text{if } x \text{ has more than one } D\text{-cover,} \\ f(c), & \text{if } x \text{ has precisely one } D\text{-cover } c. \end{cases}$$

Clearly, f_D is a well defined function from X to X' .

Let $x \in X$ and let $D \in \mathbb{D}$ such that $x \in D$. Then, for any $D' \in \mathbb{D}$ with $D' \supseteq D$, we have $f_{D'}(x) = f(x)$. Hence, the net of functions $\{f_D : D \in \mathbb{D}\}$ converges to f point-wise.

It remains to show that each $f_D : X \rightarrow X'$ is continuous. For this, let U be an up-directed subset of X with $\text{l.u.b.}(U) = u$.

If $U \cap P'$ is non-empty, then clearly it is a finite subset of P' . Since U is an up-directed set, the maximal subset of $U \cap P'$ is a singleton set, say $\{a\}$. Moreover, as P has no maximal elements, we have $d \leq a$ for all $d \in U$. Since u is the $\text{l.u.b.}(U)$, it follows that $u = a$ and hence $u \in U$. Therefore, $f_D(U)$ contains an up-directed singleton set $E = \{f_D(u)\}$ such that $\text{l.u.b.}(E) = f_D(u)$.

Now, suppose $U \cap P' = \emptyset$. Since P has no maximal element, it follows that $u \in P$.

Case (i): u has no D -cover. Let M denote the set of all elements of U that have at least one D -cover. Then M is not cofinal in U . Suppose, to the contrary, that M is cofinal in U . Clearly, M is an up-directed subset of U . For each $a \in D$, define $Ma = \{m \in M : m \leq a\}$. It is clear that $\bigcup_{a \in D} Ma = M$. Since D is finite, there exists some $a \in D$ such that Ma is cofinal in M and hence cofinal

in U . Consequently, $l.u.b.(Ma) = u$ and $u \leq a$, which is a contradiction, since u has no D -cover and D is finite. Therefore, M is not cofinal in U . It follows that there exists an element $d \in U$ such that $d \not\leq m$ for all $m \in M$. Hence, $f_D(d) = f_D(u) = k$.

Case (ii): u has more than one D -cover. Let N denote the set of all elements of U that have precisely one D -cover. Suppose, to the contrary, that N is cofinal in U . Clearly, N is an up-directed subset of U . For each $a \in D$, define $Na = \{n \in N : a \text{ is a } D\text{-cover of } n\}$. It is evident that $\bigcup_{a \in D} Na = N$. Since D is finite, there exists some $a \in D$ such that Na is cofinal in N and hence cofinal in U . Consequently, $l.u.b.(Na) = u$ and therefore $u \leq a$. Thus, a is a D -cover of u . Let b be another D -cover of u . Then a and b must be incomparable, which contradicts the fact that every element in Na has exactly one D -cover, namely a . Thus, N is not cofinal in U . It follows that there exists an element $d \in U$ with $d \not\leq n$ for all $n \in N$. Hence, $f_D(d) = f_D(u) = k$.

Case (iii): u has precisely one D -cover, say c . Let O denote the set of all elements of U for which c is the unique D -cover. It can be shown that $U \setminus O$ is not cofinal in U . If $U \setminus O$ is cofinal in U , then the set $P = \{x \in U : x \text{ has no } D\text{-cover}\}$ would be cofinal in U and u would be the $l.u.b.$ of P . Since D is finite, this is not possible. Therefore, there exists an element $d \in O$ such that $f_D(d) = f_D(u) = f(c)$.

Analogous cases arise when U is down-directed and $u = g.l.b.(U)$. Hence, by Theorem 6.2.18, each f_D is continuous. $\square$

Conclusion and Recommendations

This chapter presents the conclusion of the study by summarizing the main findings. It also provides recommendations and outlines the scope for future research.

7.1 Conclusion

In this thesis, the concept of semi-well ordered sets is illustrated through examples and additional properties of these sets are examined in detail. We investigated the order ideals and introduced the concept of semi-ordinals. The arithmetic of semi-ordinals and its properties are also explored.

We further studied the order topology on semi-well ordered sets and showed that these spaces are zero dimensional, totally disconnected and strongly locally compact. The isolated points were identified and characterizations of infinite

homogeneous semi-well ordered sets and compact semi-well ordered sets were provided. Additional properties of these topological spaces were also explored in detail. We further investigated the dynamics of semi-ordinals.

We defined partially semi-well ordered sets as an extension of both semi-well ordered sets and partially well ordered sets. We explored the order-theoretic properties of partially semi-well ordered sets and provided various characterizations of them. Additionally, we investigated their structural properties, the properties of their order ideals and the reversibility of these sets.

We studied the Aharoni-Korman conjecture and presented the motivation behind its proposal. We also outlined the key advancements related to the conjecture, with a focus on the class of partially ordered sets for which the conjecture has been verified. Furthermore, we verified the Aharoni-Korman conjecture ($k = 1$) for partially semi-well ordered sets.

The non-existence of an infinite antichain in a partially semi-well ordered set ensures that the interval topology and the Dedekind topology coincide; we referred to this common topology as the intrinsic topology. We examined the conditions under which this space is Dedekind complete, compact and connected. In addition, we characterized the elements of the closure of a subset in a partially ordered set with no infinite antichain and established a relationship between the relative topology and the intrinsic topology on its subsets. Furthermore, we showed that the partially semi-well ordered set with intrinsic topology is anti-rigid. Moreover, we verified that it is neither normal nor topologically reversible. Additional properties of these topological spaces were also explored. We also presented a characterization of continuous functions between two partially semi-well ordered sets. Using this characterization, we established a result concerning mappings between such sets.

7.2 Recommendations

In this section, we address some unsolved problems that may be considered for future research.

One possible direction for future research on semi-well ordered sets is the classification of semi-well ordered sets that are homeomorphic with respect to the order topology. It is already known from the literature that ordinals are topologically reconstructible [57]; therefore, the reconstruction of semi-ordinals can be investigated. In addition, the dynamics of semi-ordinals can be studied in greater depth.

Although the Aharoni–Korman conjecture does not hold in full generality, it remains valid for a broad and significant class of partially ordered sets. It is therefore an interesting open problem to investigate whether the conjecture holds for partially ordered sets of finite width, partially well ordered sets (for $k \geq 2$) and partially semi-well ordered sets (for $k \geq 2$). Furthermore, it is possible to identify more classes of partially ordered sets for which the conjecture is true.

Another direction of future research is the investigation of additional properties of partially well ordered sets and partially semi-well ordered sets when equipped with interval topology. For instance, one can verify whether a partially well ordered set or a partially semi-well ordered set is locally compact. If such a space is locally compact, then, since it is Hausdorff, it would also be completely regular. Furthermore, it remains open question whether a partially semi-well ordered set is zero dimensional.

Additionally, one can ask whether every function between two partially semi-well ordered sets can be expressed as the point-wise limit of continuous functions between them. A special case of this problem is addressed, whereas the general case remains open.

Bibliography

- [1] U. Abraham, A note on Dilworth's theorem in the infinite case, *Order*, **4** (1987), 107–125.
- [2] C. Adams and R. Franzosa, Introduction to topology pure and applied, *Pearson Education, Inc.*, (2008).
- [3] R. Aharoni, König's duality theorem for infinite bipartite graphs, *J. Lond. Math. Soc.*, **29** (1984), 1–12.
- [4] R. Aharoni and E. Berger, Menger's theorem for infinite graphs, *Invent. Math.*, **176**(1) (2009), 1–62.
- [5] R. Aharoni and E. Berger, Strongly maximal antichains in posets, *Discrete Math.*, **311** (2011), 1518–1522.
- [6] R. Aharoni and V. Korman, Greene-Kleitman's theorem for infinite posets, *Order*, **9** (1992), 245–253.
- [7] R. Aharoni and M. Loebl, Strongly perfect infinite graphs, *Israel J. Math.*, **90** (1995), 81–91.
- [8] R. A. Alo, A proof of the complete normality of chains, *Acta Math. Acad. Sci. Hung.*, **22**(3-4) (1971), 393–395.

Bibliography

- [9] R. A. Alo and O. Frink, Topologies of chains, *Math. Ann.*, **171** (1967), 239-246.
- [10] M. Archana, P. Chiranjeevi and V. Kannan, Dynamics on locally compact Hausdorff spaces, *Topol. Proc.*, **48** (2016), 309-325.
- [11] P. Bankston, The total negation of a topological property, *Illinois J. Math.*, **23** No. 2 (1979), 241-252.
- [12] G. Birkhoff, Lattice theory, *American Math. Soc. Colloquium Publications*, Vol. XXV (3rd ed.), (1948).
- [13] M. Brin and G. Stuck, Introduction to dynamical systems, *Cambridge University Press*, (2002).
- [14] K. Burns and B. Hasselblatt, The Sharkovsky theorem: a natural direct proof, *Amer. Math. Monthly*, **118**(3) (2011), 229-244.
- [15] P. Chiranjeevi, Periodic points of continuous self-maps on locally compact Hausdorff spaces, *Proceeding of Andra Pradesh Akademi of Sciences*, **14**(9) (2012), 4233-4237.
- [16] P. Chiranjeevi, Some results on periodicity and topological conjugacy in dynamical systems, Ph.D. Thesis, Submitted to University of Hyderabad, (2013).
- [17] P.M. Cohn, Universal algebra, *Harper and Row*, (1965).
- [18] D.H.J. de Jongh and R. Parikh, Well-partial ordering and hierarchies, *Indag. Math.*, **39** (1979), 195-207.
- [19] R. P. Dilworth, A decomposition theorem for partially ordered sets, *Ann. of Math.*, **51**(1) (1950), 161-166.
- [20] D. Duffus and T. Goddard, Some progress on the Aharoni-Korman conjecture, *Discrete Math.*, **250** (2002), 79-91.

Bibliography

- [21] B. Dushnik and E. W. Miller, Partially ordered sets, *Amer. J. Math.*, **63** (1941), 600-610.
- [22] R. C. Freiwald, An introduction to set theory and topology, *Washington university in St. Louis*, (2014).
- [23] D. B. Gauld, I. L. Reilly and M. K. Vamanamurthy, On normal subspaces, *Bull. Austral. Math. Soc.*, **23** (1981), 1-4.
- [24] C. Greene, Some partitions associated with a partially ordered set, *J. Combin. Theory Ser. A*, **20** (1976), 69-79.
- [25] C. Greene and D. J. Kleitman, The structure of Sperner K -families, *J. Combin. Theory Ser. A*, **20** (1976), 41-68.
- [26] M. Guay-Paquet, A. H. Morales and E. Rowland, Structure and enumeration of $(3 + 1)$ -free posets, *Ann. Comb.*, **18** (2014), 645–674.
- [27] L. Hallom, A resolution of the Aharoni-Korman conjecture, *arXived*, (2025).
- [28] P. R. Halmos, Naive set theory, *Springer*, (1960).
- [29] R. W. Hansell, Monotone subnets in partially ordered sets, *Proc. Amer. Math. Soc.*, **18** (1967), 854-858.
- [30] G. Higman, Ordering divisibility in abstract algebras, *Proc. Lond. Math. Soc.*, **2**(3) (1952), 326-336.
- [31] K. D. Joshi, Introduction to general topology, *New Age International (P) Limited, Third Edition*, (2023).
- [32] V. Kannan, A note on countable compact spaces, *Publicationes Mathematicae Debrecen*, **21** (1974), 113–114.

Bibliography

- [33] V. Kannan, Sets of periods of dynamical systems, *Indian J. Pure Appl. Math.*, **41**(1) (2010), 225-240.
- [34] V. Kannan and S. Gopal, Homeomorphisms on compact metric spaces with finite derived length, *Appl. Gen. Topol.*, **17** no. 2 (2016), 129-137.
- [35] V. Kannan and P. T. Ramachandran, Hereditarily homogeneous spaces, *Proceedings, International Seminar on Recent Trends in Topology and its Applications, St. Joseph's College, Irinjalakuda*, (2009), 48-51.
- [36] A. S. Kechris, Classical descriptive set theory, Graduate Texts in Mathematics, *New York: Springer-Verlag*, (1995).
- [37] J. L. Kelly, General topology, *Van Nostrand*, (1967).
- [38] R. D. Kopperman, E. H. Kronheimer and R. G. Wilson, Topologies on totally ordered sets, *Topology Appl.*, **90** (1998), 165-185.
- [39] J. B. Kruskal, The theory of well-quasi-ordering: a frequently discovered concept, *J. Combin. Theory Ser. A*, **13** (1972), 297-305.
- [40] J. B. Kruskal, Well-quasi-ordering, the tree theorem and Vazsonyi's conjecture, *Trans. Amer. Math. Soc.*, **95** (1960), 210-225.
- [41] M. Kukiela, Characterization of hereditarily reversible posets, *Math. Slovaca*, **66**(3) (2016), 539-544.
- [42] M. Kukiela, Reversible and bijectively-related posets, *Order*, **26** (2009), 119-124.
- [43] S. Lipschutz, Schaum's Outline of theory and problems of set theory and related topics, *McGraw-Hill*, (1964).
- [44] M. Malicki and A. Rutkowski, On operations and linear extensions of well partially ordered sets, *Order*, **21**(1) (2004), 7-17.

Bibliography

- [45] J. Matier and T. B. M. McMaster, Iteration of Bankston's 'Anti'-operation, *J. Inst. Math. Comput. Sci.*, (Mathematics Series **3**) No. 1 (1990), 31-35.
- [46] J. Matier and T. B. M. McMaster, Total negation of separability and related properties, *Proc. R. Ir. Acad.*, **90 A** No. 2 (1990), 131-137.
- [47] T. B. M. McMaster and C.R. Turner, Iteration of total negation in constrained environments, *Math. Proc. R. Ir. Acad.*, **101A**(1) (2001), 21-26.
- [48] E. J. McShane, Order preserving maps and integration processes, *Annals of Math. Studies* no. 31, Princeton, (1953).
- [49] E. Michael, A class of partially ordered sets, *Amer. Math. Monthly*, **67** (1960), 448-449.
- [50] E. C. Milner and N. Sauer, On chains and antichains in well-founded partially ordered sets, *J. Lond. Math. Soc.*, **24**(2) (1981), 15-33.
- [51] Y. Moschovakis, Notes on set theory second edition, *Springer publications*, (2005).
- [52] J. R. Munkres, Topology, *Pearson India Education Services Pvt. Ltd.*, Updated Second Edition, (2022).
- [53] T. Naito, On a problem of Wolk in interval topologies, *Proc. Amer. Math. Soc.* **11** (1960), 156-158.
- [54] H. Oellrich and K. Steffens, On Dilworth's decomposition theorem, *Discrete Math.*, **15** (1976), 301-304.
- [55] O. Ore, Theory of graphs, *American Math. Soc. Colloquium Publications*, **38** (1962).
- [56] M. A. Peles, On Dilworth's theorem in the infinite case, *Israel J. Math.*, **1** (1963), 108-109.

Bibliography

- [57] M. F. Pitz, Topological reconstruction and compactification theory, *Ph.D. Thesis, Submitted to University of Oxford*, (2015).
- [58] M. Pouzet, Well-quasi-ordering and embeddability of relational structures, *Order*, **41** (2024), 183–278.
- [59] M. Rajagopalan and A. Wilansky, Reversible topological spaces, *J. Aust. Math. Soc.*, **6**(2) (1966), 129-138.
- [60] P. T. Ramachandran, Anti homogeneity is equivalent to hereditary rigidity, *Ganita*, **35**(1) (1984), 1-9.
- [61] P. T. Ramachandran, Some problems in set topology relating group of homeomorphisms and order, *Ph.D. Thesis, Submitted to University of Cochin*, (1985), 21-28.
- [62] I. L. Reilly, Four topics and an example, *Hacettepe J. Math. Stat.*, **34 S** (2005), 27-34.
- [63] I. L. Reilly and M. K. Vamanamurthy, On spaces in which every denumerable subspace is discrete, *Mat. Vesnik*, **38** (1986), 97-102.
- [64] I. L. Reilly and M. K. Vamanamurthy, Some topological anti-properties, *Illinois J. Math.*, **24** No. 3 (1980), 382-389.
- [65] D. Schmidt, The relation between the height of a well-founded partial ordering and the order types of its chains and antichains, *J. Combin. Theory Ser. B*, **31** (1981), 183–189.
- [66] M. Sreeja, On the total negation of rigidity, *Int. J. Pure Appl. Math.*, **82**(3) (2013), 391-398.
- [67] L. A. Steen and J. A. Seebach, Counterexamples in topology, *Springer-Verlag New York Inc., Second Edition*, (1978).
- [68] R. R. Stoll, Set theory and logic, *Dover publications, Inc. New York*, (1963).

Bibliography

- [69] E. Szpilrajn, Sur l'extension de l'ordre partiel, *Fund. Math.*, **16** (1930), 386-389.
- [70] I. Vialard, On the width of the cartesian product of ordinals, *Order*, **42** (2025), 37–58.
- [71] E. S. Wolk, Order-compatible topologies on a partially ordered set, *Proc. Amer. Math. Soc.*, **9** (1958), 524-529.
- [72] E. S. Wolk, Partially well ordered sets and partial ordinals, *Fund. Math.*, **60** (1967), 175-186.
- [73] E. S. Wolk, Topology on a partially well ordered set, *Fund. Math.*, **62**(1968), 255-264.
- [74] I. Zaguia, Some progress on the Aharoni–Korman conjecture, *Discrete Math.*, **347** (2024), Article 114139.

Appendix

List of Publications

1. Angela Sunny, P. Sini: *Topology on semi-well ordered sets*, Kyungpook Mathematical Journal, 2024; 64(1), 161-169, <https://doi.org/10.5666/KMJ.2024.64.1.161>.
2. Angela Sunny, P Sini: *In-depth Analysis of Semi-Well Ordered Sets and Semi-Ordinals*, Accepted for publication in the Palestine Journal of Mathematics.
3. Angela Sunny, P Sini: *On Some Characterizations and Structural Properties of Partially Semi-Well Ordered Sets*, Accepted for publication in the Jordan Journal of Mathematics and Statistics.

List of Presentations

1. Angela Sunny: *Structural and topological properties of partially semi-well ordered sets*, National Seminar on Glimpses of Analysis and Geometry-2025, Department of Mathematics, University of Calicut, Malappuram, Kerala, India, 13-14 March 2025.
2. Angela Sunny: *Some advancements in the Aharoni-Korman Conjecture*, International Conference on Algebra and Discrete Mathematics (ICADM

2024), Department of Mathematics, Govt. College, Kattappana, Kerala, India in association with IMRT, 20-22 February 2024.

3. Angela Sunny: *Semi-well ordered sets and Semi-ordinals*, 2nd International Conference on “Recent Advances in Mathematical Science and Interdisciplinary Areas (RAMSIA 2023)”, GLA University, Mathura, India, 23 June 2023.
4. Angela Sunny: *Properties of Order Topology on a Semi-Well Ordered Set*, National Webinar on “Applications of Mathematics in Recent Times”, Ghatal Rabindra Satabarsiki Mahavidyalaya, West Bengal, India, 19 June 2023.
5. Angela Sunny: *On an extension of well ordered sets*, 35th Kerala Science Congress, Mar Basellious Christian College of Engineering and Technology Kuttikanam, Kerala, India, 12-14 February 2023.

Index

- Aharoni-Korman Conjecture, 92
- antichain, 11
- antirigid, 16
- bounded, 10
- bounded above, 9
- bounded below, 10
- cardinal product of posets, 11
- choice function, 11
- chordal graph, 92
- circular rotations, 19
- co-ordinal number, 34
- cofinal, 18
- complete, 10
- complete lattice, 11
- completely normal, 16
- contiguous chain, 94
- convergence of net, 18
- Dedekind closed, 104
- Dedekind complete, 108
- Dedekind topology, 105
- Dilworth's theorem, 90
- dimension, 11
- down-directed set, 104
- dually well founded poset, 12
- dynamical system, 19
- expanding endomorphisms, 20
- extension, 11
- fishbone conjecture, 91
- forward invariant, 19
- greatest element, 10
- greatest lower bound, 10
- Greene-Kleitman's theorem, 91
- Hausdorff's maximal principle, 13
- homogeneous, 17
- incomparable, 10
- initial segment, 12
- interval topology, 17
- intrinsic topology, 106
- isolated point, 16
- König duality theorem, 92

- König's Infinity Lemma, 73
- lattice, 11
- least element, 10
- least upper bound, 10
- limit ordinal, 14
- linearly ordered set, 9
- locally compact, 17
- locally finite graph, 93
- lower bound, 9
- maximal element, 10
- maximal subset, 10
- minimal element, 10
- minimal subset, 10
- N- free poset, 93
- non-limit ordinal, 14
- normal poset, 83
- orbit, 19
- order ideal, 29
- order isomorphism, 9
- order preserving, 9
- order topology, 17
- order type, 9
- order-compatible, 105
- ordinal, 13
- paracompact, 17
- partial order, 9
- partially semi-well ordered set, 71
- partially well ordered set, 11
- period, 19
- periodic point, 19
- pointwise limit of net of functions, 18
- poset, 9
- product of semi-ordinals, 36
- retract, 18
- reversible poset, 61
- reversible topological space, 61
- rigid, 16, 25
- scattered, 16
- semi-ordinal, 34
- semi-well ordered set, 23
- separated, 16
- Sharkovsky's ordering, 24
- sober, 16
- strictly order preserving, 9
- strongly locally compact, 17
- sum of semi-ordinals, 36
- tent map, 20
- terminal segment, 12
- totally unordered, 11
- trajectory, 19
- up-directed set, 104
- upper bound, 9
- Urysohn's metrization theorem, 18
- well founded poset, 12
- well ordered set, 12
- well ordering principle, 13
- Zermelo's Axiom of choice, 13
- zero dimensional, 17
- Zorn's lemma, 13
- $3 + 1$ - free poset, 93