

D 1541



(Pages : 2)

Name.....

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2009

Mathematics

Paper XIV—DIFFERENTIAL GEOMETRY

(2002 admissions)

Time : Three Hours

Maximum : 80 Marks

Part A

Answer all questions.

Each question carries 4 marks.

1. (a) Find the integral curve through the point (1, 1) of the vector field X on \mathbb{R}^2 with associated function $X : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $X(x_1, x_2) = (x_2, x_1)$.
- (b) Let S be an n -surface in \mathbb{R}^{n+1} . Define : tangent vector field S , normal vector field on S . Give an example of a tangent vector field and normal vector field on the 1-sphere.
- (c) Let X and Y be smooth vector fields tangent to an n -surface S along a parametrised curve $\alpha : I \rightarrow S$. Verify : $(X \cdot Y)' = X' \cdot Y + X \cdot Y'$.
- (d) Find a formula for a chart whose domain in the unit sphere S^2 with one point deleted.

(4 × 4 = 16 marks)

Part B

Answer any four questions without omitting any unit.

Each question carries 16 marks.

UNIT I

2. (a) Let X be a smooth vector field on an open set $U \subseteq \mathbb{R}^{n+1}$ and let $p \in U$. Then prove that there exists an open interval I containing 0 and are integral curve $\alpha : I \rightarrow U$ of such that (i) $\alpha(0) = p$, (ii) $\beta : \tilde{I} \rightarrow U$ is any other integral curve of X with $\beta(0) = p$, then $\tilde{I} \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.
- (b) Verify that a surface of revolution is a 2-surface.
3. (a) Let U be an open set in \mathbb{R}^{n+1} and let $f : U \rightarrow \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^\perp$.
- (b) Let $S \subseteq \mathbb{R}^{n+1}$ be a connected n -surface in \mathbb{R}^{n+1} . Prove that there exist on S exactly two smooth unit normal vector fields N_1 and N_2 and $N_2(p) = -N_1(p)$ for all $p \in S$.

Turn over

4. (a) Let S be a compact connected oriented n -surface in \mathbb{R}^{n+1} . Prove that the Gauss map maps S onto the unit sphere S^n .
- (b) Describe the spherical image of the cylinder $x_2^2 + x_3^2 = 1$ oriented by $\nabla f / \|\nabla f\|$, where $f(x_1, x_2, x_3) = x_2^2 + x_3^2$.

UNIT II

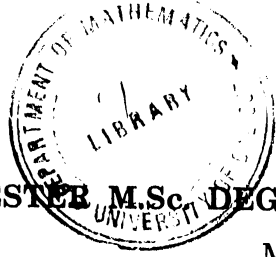
5. (a) Let S be an n -surface in \mathbb{R}^{n+1} , let $p, q \in S$, and let α be a piecewise smooth parametrized curve from p to q . Prove that the parallel transport $P_\alpha : S_p \rightarrow S_q$ along α is a vector space isomorphism which preserves products.
- (b) Show that if $\alpha : I \rightarrow \mathbb{R}^{n+1}$ is a parametrized curve with constant speed, the $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$.
6. (a) Prove that the Weingarten map L_p is self-adjoint.
- (b) Find the global parametrization of the plane curve $x_1^2 + \frac{x_2^2}{4} = 1$, oriented by $\nabla f / \|\nabla f\|$, where $f(x_1, x_2) = x_1^2 + (x_2^2/4)$.
7. (a) Let C be a connected, oriented plane curve and let $\beta : I \rightarrow C$ be a unit speed global parametrization of C . Then prove that β is either one-to-one or periodic. Further, prove that β is periodic if and only if C is compact.
- (b) Let C be a connected, oriented plane curve and let \tilde{C} be the same curve with opposite orientation. Prove that $l(\tilde{c}) = -l(c)$.

UNIT III

8. (a) Let S be a compact, oriented n -surface in \mathbb{R}^{n+1} . Prove that there exists a point $p \in S$ such that the second fundamental form at p is definite.
- (b) Find the Gaussian curvature $K : S \rightarrow \mathbb{R}$ where S is the surface $x_1^2 + x_2^2 - x_3^2 = 0, x_3 > 0$.
9. (a) Let $\varphi : U \rightarrow \mathbb{R}^{n+1}$ be a parametrized n -surface in \mathbb{R}^{n+1} and let $p \in U$. Prove that there exists an open set $U_1 \subseteq U$ about p such that $\varphi(U_1)$ is an n -surface in \mathbb{R}^{n+1} .
- (b) Let $\varphi : U_1 \rightarrow U_2$ and $\psi : U_2 \rightarrow \mathbb{R}^k$ be smooth, where $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^m$. Verify the chain rule $d(\psi \circ \varphi) = d\psi \circ d\varphi$.
10. (a) State and prove the inverse function theorem for n -surfaces.
- (b) Let S be a compact connected oriented n -surface in \mathbb{R}^{n+1} whose Gauss-Kronecker curvature is nowhere zero. Then, prove that the Gauss-map $N : S \rightarrow S^n$ is a diffeomorphism.

(4 × 16 = 64 marks)

D 1540



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Name.....

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THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2009

Mathematics

Paper XIII—TOPOLOGY—II

(2003 admissions)

Time : Three Hours

Maximum : 80 Marks

Part A

Answer all questions.

Each question carries 4 marks.

- I. (a) Let $f: (0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = \sin(y_n)$. Give two sequences (x_n) and (y_n) in $(0, 1]$ such that

$$x_n \rightarrow 0, y_n \rightarrow 0 \text{ and } f(x_n) \rightarrow 0, f(y_n) \rightarrow 1.$$

- (b) Let $\{X_i : i \in I\}$ be a collection of topological spaces and $X = \pi X_i$ be the product space. Prove that each projection $\pi_i : X \rightarrow X_i$ is open.
- (c) Let $f_1 : X \rightarrow Y$ and $f_2 : X \rightarrow Z$ be functions. Describe the evaluation map $e : X \rightarrow Y \times Z$ of the family $\{f_1, f_2\}$.
- (d) Let D be the set of all partitions of the closed interval $[0, 1]$. For $P, Q \in D$ let $P \geq Q$ mean P is a refinement of Q . Verify whether (D, \geq) is a directed set.

(4 × 4 = 16 marks)

Part B

Answer any four questions without omitting any unit.

Each question carries 16 marks.

UNIT I

- II. (a) Let X be a topological space and $A \subseteq X$. Let $f : A \rightarrow \mathbb{R}$ be continuous. Show that any two continuous extensions of f to X agree on \bar{A} .
- (b) Let X be a topological space such that for any closed subset A of X , any continuous function $f : A \rightarrow \mathbb{R}$ has a continuous extension to X . Show that X is a normal space.
- III. (a) Prove that a subset of $X = \pi X_i$ is a box if and only if it is the intersection of a family of walls.
- (b) Prove that the intersection of finitely many walls is a large box.
- IV. (a) Let $X = \pi X_i$ be a product space. Prove that if each X_i is T_2 then X is also T_2 .
- (b) Prove that if each X_i is completely regular then X is completely regular.

Turn over

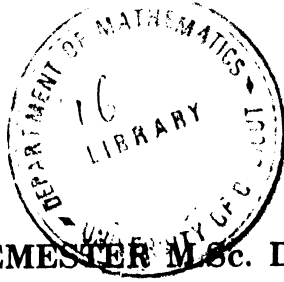
- V. (a) Describe the evaluation map of a family of functions $\{f_i : X \rightarrow Y_i : i \in I\}$.
(b) Show that the evaluation map is one-to-one if and only if the family distinguishes points.
- VI. (a) Prove that every second countable T_3 space is embeddable in the Hilbert cube.
(b) Show that every second countable T_3 space is metrizable.
- VII. (a) Let τ_1, τ_2 be topologies on a set X such that a net in X converges to a point x w.r.t. τ_1 if and only if it converges to x w.r.t. τ_2 . Show that $\tau_1 = \tau_2$.
(b) Show that a topological space X is Hausdorff if and only if limits of all nets in X are unique.

UNIT III

- VIII. (a) Show that every countably compact metric space is second countable.
(b) Deduce that every countably compact metric space is compact.
- IX. (a) Define locally compact space.
(b) Prove that every locally compact Hausdorff space is regular.
- X. (a) Show that every compact metric space is complete.
(b) Show that every compact metric space is totally bounded.

(4 × 16 = 64 marks)

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(Pages : 2)

Name.....

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THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2009

Mathematics

Paper XI—COMPLEX ANALYSIS—I

(2002 admissions)

Time : Three Hours

Maximum : 80 Marks

Part A

Answer all questions.

Each question carries 4 marks.

1. (a) Find the linear transformation which carries $0, -i, i$ into $1, -1, 0$ respectively.
- (b) Compute $\int_{|z|=2} \frac{dz}{z^2+1}$ by decomposition of the integrand in partial fractions.
- (c) Develop $\frac{1}{1+z^2}$ in powers of $(z-1)$.
- (d) State and prove Morera's theorem.

(4 × 4 = 16 marks)

Part B

Answer any four questions without omitting any unit.

Each question carries 16 marks.

UNIT I

2. (a) Prove that the set of fractional linear transformations $w = sz = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ form a group under composition.
- (b) Suppose that $f(z)$ is analytic and satisfies the condition $|f(z)^2 - 1| < 1$ in a region Ω . Show that either $\operatorname{Re} f(z) > 0$ or $\operatorname{Re} f(z) < 0$ throughout the region Ω .
3. (a) Let z_1, z_2, z_3, z_4 be distinct points in the extended complex plane and let T be any linear transformation. Prove that $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.
- (b) Find the linear transformation which carries the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin and the origin into i .

Turn over

4. (a) Let $f(z)$ be analytic on a set R' obtained from a rectangle R by omitting a finite number of interior points z_j . Suppose further that

$$\lim_{z \rightarrow z_j} (z - z_j) f(z) = 0 \text{ for all } j.$$

Then prove that $\int_{\partial R} f(z) dz = 0$.

- (b) Map the complement of the line segment $[-1, 1]$ onto the unit open disc $|z| < 1$.

UNIT II

5. (a) State and prove Cauchy's integral formula.
 (b) Prove that an entire function which satisfies an inequality $|f(z)| < |z|^n$ for some n and for all sufficiently large $|z|$ reduces to a polynomial.
6. (a) Suppose that $f(z)$ is analytic at $z_0, f(z_0) = w_0$ and that $f(z) - w_0$ has a zero of order n of z_0 . Given $\varepsilon > 0$, prove that there exists a corresponding $\delta > 0$ such that for all a with $|a - w_0| < \delta$, the equation $f(z) = a$ has exactly n roots in the disc $|z - z_0| < \varepsilon$.
 (b) Deduce the open mapping theorem.
7. (a) State and prove the residue theorem.
 (b) Evaluate the integral

$$\int_0^{\infty} \frac{x^2 dx}{x^4 + 5x^2 + 6} \text{ by the method of residues.}$$

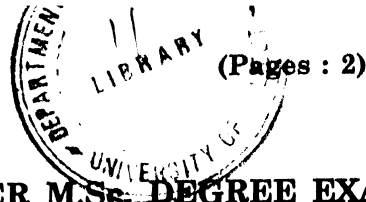
UNIT III

8. (a) Prove the maximum principle for harmonic functions.
 (b) State and prove Hurwitz's theorem.
9. (a) With the usual notation prove that the function $P_u(z)$ is harmonic for $|z| < 1$ and $\lim_{z \rightarrow e^{i\theta_0}} P_u(z) = u(\theta_0)$, provided that u is continuous at θ_0 .
 (b) Prove that the Laurent development is unique.
10. (a) Prove that an elliptic function without poles is constant.
 (b) Prove that a non-constant elliptic function has equally many poles as it has zeroes.
 (c) Prove :

$$\mathcal{P}(2z) = \frac{1}{4} \left(\frac{\mathcal{P}''(z)}{\mathcal{P}'(z)} \right)^2 - 2\mathcal{P}(z).$$

(4 × 16 = 64 marks)

D 41642



Name.....

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, FEBRUARY 2008

Mathematics

Paper XIII—TOPOLOGY—II

(2003 Admissions)

Time : Three Hours

Maximum : 80 Marks

Part A

Answer all questions.

Each question carries 4 marks.

- I. 1 Let $A = [0, 1]$; and $f : A \rightarrow \mathbb{R}$ be defined by $f(x) = \sin(Y_x)$. Is f continuous? Does there exist a continuous extension of f to $[0, 1]$? Justify your answer.
- 2 Prove that the projection $\pi_j : X \rightarrow X_j$ is onto where X is the product $\prod X_j$.
- 3 Let τ be the indiscrete topology on \mathbb{R} . Describe the convergence of nets in (\mathbb{R}, τ) .
- 4 Prove that if A is totally bounded then every subset of A is totally bounded.

(4 × 4 = 16 marks)

Part B

Answer any four questions from this part without omitting any unit.

Each question carries 16 marks.

UNIT I

- II. (a) Let X be a topological space and Y be a metric space. Let $f_n : X \rightarrow Y$ be continuous for each n . Show that if f_n converges to f uniformly then f is continuous.
- (b) Let (f_n) be a sequence of real valued functions on a topological space X . Let $\sum M_n$ be a convergent series of reals such that $|f_n(x)| \leq M_n$ for all x . Show that $\sum f_n$ converges uniformly on X .
- III. Let (X_i, τ_i) be a family of topological spaces and $X = \prod_{i \in I} X_i$ be the product space. Prove that the following are equivalent :—
- (i) τ is the smallest topology on X such that each projection $\pi_i : X \rightarrow X_i$ is continuous.
- (ii) $\{\pi_i^{-1}(V_i) : V_i \text{ is open in } X_i, i \in I\}$ is a sub-base for τ .

Turn over

UNIT II

- V. (a) Give an example of a family of continuous functions $f_i : X \rightarrow Y_2$ which distinguishes points from closed sets.
- (b) Let $\{f_i : X \rightarrow Y_i, i \in I\}$ be a family of continuous functions. Show that if this family distinguishes points from closed sets then the evaluation map $e : X \rightarrow \prod Y_i$ is an open map from X to $e(X)$.
- VI. (a) Define directed set. Give an example.
- (b) Show that a topological space X is Hausdorff if and only if limits of nets in X are unique.
- VII. (a) Define the net associated with a filter.
- (b) Let \mathcal{F} be a filter in a topological space X ; and S be the associated net. Show that \mathcal{F} converges to x in X if and only if S converges to x in X .

UNIT III

- VIII. (a) Prove that every countably compact space is second countable.
- (b) Prove that every continuous real valued function on a countably compact space is bounded.
- IX. (a) Define the one-point compactification X^+ of a space X .
- (b) Show that if X is locally compact and Hausdorff then the one-point compactification X^+ of X is Hausdorff.
- X. (a) Prove that every non-empty complete metric space is of second category.
- (b) Prove that every contraction of a complete metric space into itself has a unique fixed point.

(4 × 16 = 64 marks)