D 111775 **(Pages : 4)** **Name..**

Reg. No......................................

THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2024

(CCSS)

Mathematics

MAT 314—COMPLEX ANALYSIS

(2022 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 mark.

1. Prove that an absolutely convergent series is convergent.

2. Find the fixed points of a dilation on $\mathbb{C}_\infty$.

3. Let $\gamma : [0,2\pi] \to \mathbb{C}$ be defined by $\gamma(t) = e^{it}$ and let $f : \mathbb{C} - \{0\} \to \mathbb{C}$ be defined by $f(z) = \frac{1}{z}$. Evaluate $\int_\gamma f$.

4. Let f be analytic in $B(a;R)$ and suppose that $|f(z)| \le M$ for all z in $B(a;R)$. Prove that $\left|f^{(n)}(a)\right| \le \frac{n!\,M}{R^n}$.

5. Evaluate $\int_\gamma \frac{2z+1}{z^2+z+1}\,dz$ where γ is the circle $|z| = 2$.

6. Let G be an open set which is a-star shaped. If γ_0 is the curve which is constantly equal to a, then show that every closed rectifiable curve in G is homotopic to γ_0.

Turn over

7. Determine the nature of the isolated singularity at $z = 0$ for the function $f(z) = \frac{\cos z - 1}{z}$.

8. Find the number of zeros of the polynomial $p(z) = z^{10} - 6z^9 - 3z + 1$ inside the circle $|z| = 1$.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} . z^n$.

10. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence R > 0. Prove that the series $\sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$ also has radius of convergence R.

11. Prove that a branch of the logarithm function is analytic and its derivative is z^{-1}.

12. Let $\gamma : [a, b] \to \mathbb{C}$ be a rectifiable path and let $\varphi : [c, d] \to [a, b]$ be a continuous non-decreasing function with $\varphi(c) = a, \varphi(d) = b$. For any function f continuous on $\{\gamma\}$, prove that

$$\int_{\gamma} f = \int_{\gamma \circ \varphi} f .$$

13. If $p(z)$ is a non constant polynomial then prove that there is a complex number a with $p(a) = 0$.

14. Let G be a region and suppose that f is a non constant analytic function on G. For every open set U in G, prove that $f(\mathrm{U})$ is open.

15. If $f : \mathrm{G} \to \mathbb{C}$ is analytic and one-one, then show that $f'(z) \neq 0$ for any z in G.

16. If f has an essential singularity at $z = a$, then prove that $\{f[ann:(a;0,\gamma)]\}^{-} = \mathbb{C}$ for every $\delta > 0$.

17. Find all poles of the function $f(z) = \dfrac{z^2}{1+z^4}$ and calculate the residue of $f(z)$ at any two poles.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. (a) If $\sum a_n (z-a)^n$ is a given power series with radius of convergence R, then prove that

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right|,$$

if this limit exists.

(b) Briefly describe the extended plane and its spherical representation.

19. (a) If G is open and connected and $f: G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all z in G, then prove that f is a constant.

(b) Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Prove that the cross ratio (z_1, z_2, z_3, z_4) is a real number if and only if all four points lie on a circle.

20. (a) Let $f: G \to \mathbb{C}$ be analytic and suppose $\bar{B}(a;r) \subset G$ where $r > 0$. If $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$, then prove that

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw$$

for $|z-a| < r$.

(b) If f and g are analytic on a region G, then prove that $f \equiv g$ if and only if $\{z \in G : f(z) = g(z)\}$ has a limit point in G.

Turn over

21. (a) Let γ be a rectifiable curve and suppose ϕ is continuous on $\{\gamma\}$. For each $m \geq 1$, let

$$F_m(z) = \int_\gamma \frac{\phi(w)}{(w-z)^m} \, dw$$

for $z \notin \{\gamma\}$. Prove that each F_m is analytic on $\mathbb{C} \setminus \{\gamma\}$.

(b) Let G be an open set and let $f : G \to \mathbb{C}$ be a differentiable function. Prove that f is analytic on G.

22. (a) State and prove Residue Theorem.

(b) Let G be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on G^- which is analytic in G. Prove that

$$\max\{|f(z)| : z \in G^-\} = \max\{|f(z)| : z \in \delta G\}.$$

(8 × 3 = 24 marks)

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THIRD SEMESTER P.G. (CCSS) DEGREE EXAMINATION NOVEMBER 2024

Mathematics

MAT 3 15—FUNCTIONAL ANALYSIS

(2022 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

*Answer **all** questions.*

Each question is of 1 mark.

1. Show that a Cauchy sequence in a metric space X need not be convergent in X.
2. Show that in a normed space addition and scalar multiplication are continuous.
3. Show that a linear map on a linear space X may be continuous with respect to some norm on X and discontinuous with respect to another norm on X.
4. Show that for normed spaces X and Y, BL (X, Y) = {0} if and only if Y = {0}.
5. Show that c_{00} with norm $\| \|_\infty$ is not a Banach space.
6. State uniform boundedness principle.
7. Prove that a closed map need not be continuous.
8. Let X, Y and Z be normed spaces. Show that if $F : X \to Y$ is continuous and $G : Y \to Z$ is closed then $G \circ F : X \to Z$ is closed.

(8 × 1 = 8 marks)

Part B

*Answer any **six** questions.*

Each question is of 3 marks.

9. Show that if $L^\infty(|a, b|)$ is not separable.

Turn over

10. Let $X = K^3$. For $(x_1, x_2, x_3) \in X$ let

$$\| x \| = \left(\left(|x_1|^2 + |x_2|^2 \right)^{3/2} + |x_3|^3 \right)^{1/3}.$$

Show that $\| \ \|$ is a norm on K^3.

11. Show that the linear functional f on c defined by $f(x) = \lim_{j \to \infty} x(j), x \in c$ is continuous and $\| f \| = 1$.

12. Let Y be a subspace of a normed space X and $a \in X$ and $a \notin \bar{Y}$. Show that there is some $f \in X'$ such that $f | Y = 0$, $f(a) = dist\,(a, \bar{Y})$ and $\| f \| = 1$.

13. Show that a Banach space can not have a denumerable basis.

14. Let X and Y be normed spaces and $F : X \to Y$ be linear. Show that F is continuous if and only if $g \circ f$ is continuous for every $g \in Y'$.

15. Let $F : X \to Y$ be a linear map such that the subspace Z (F) is closed in X. Let $\tilde{F} : X | Z(F) \to Y$ be defined by $\tilde{F}\ (x + Z(F)) = F(x)$ for $x \in X$. Show that F is an open map if and only if F is an open map.

16. Let X and Y be Banach spaces and $F \in BL(X, Y)$. Show that R (F) is linearly homeo-morphic to X/Z (F) if and only if R (F) is closed in Y.

17. Show that any *two* comparable complete norms on a linear space are equivalent.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question is of 8 marks.

18. (a) Let X be a metric space. Show that the intersection of a finite number of dense open subsets of X is dense in X.

(b) Let X, Y be normed spaces and $F : X \to Y$ be a linear map. Show that F is continuous on X if and only if $\| F(x) \| \le \alpha \| x \|$ for some $\alpha > 0$ and all $x \in X$.

19. (a) Let X be a normed space. Show that every closed and bounded subset of X is compact if and only if X is finite dimensional.

(b) Show that a normed space X is a Banach space if and only if every absolutely summable series is summable in X.

20. (a) State and prove Hahn-Banach separation theorem.

(b) Let X, Y be Banach spaces and let $F \in BL(X, Y)$. Show that if F is bijective then $F^{-1} \in BL(Y, X)$.

21. (a) Let X and Y be Banach spaces and $F : X \to Y$ be a closed linear map. Show that F is continuous.

(b) Let T be a metric space. Show that the subspace $C_0(T) =$

$\{x \in C(T) :$ for every $\in > 0$ there is a compact set $E \subset T$ with $|x(t)\backslash < \in$ for all $t \notin E\}$ is not a Banach space.

22. State Riemann-Lebesgue Lemma and show that its converse does not hold.

(3 × 8 = 24 marks)

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THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2024

(CCSS)

Mathematics

MAT 3 16—PDE AND INTEGRAL EQUATIONS

(2022 Admission onwards)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*

Each question carries 1 *mark.*

1. State Cauchy problem for quasi linear equations.

2. Consider the equation

$$u_x + u_y = 1$$

with initial condition $u(x, 0) = f(x)$. What are the projections of characteristic curves on the (x, y)-plane.

3. Consider the equation

$$xu_{xx} - yu_{yy} + \frac{1}{2}\left(u_x - u_y\right) = 0.$$

Find the domain where the equation is elliptic and the domain where it is hyperbolic.

4. Show that the Cauchy problem :

$$u_{tt} - c^2 u_{xx} = \mathrm{F}(x, t) \;\; ; -\infty < x < \infty, t > 0$$

with $u(x, 0) = f(x), u_t(x, 0) = g(x)$ admits at most one solution.

5. State Dirichlet problem.

6. State the weak maximum principle.

Turn over

7. Show that if $y'' = F(x)$ and y satisfies the initial conditions $y(0) = y_0$ and $y'(0) = y'_0$ then

$$y(x) = \int_0^x (x - \xi) F(\xi) \, d\xi + y'_0 \, x + y_0.$$

8. Determine the resolvent kernel associated with $k(x, \xi) = x\xi$ in the interval (0, 1) in the form of a power series in λ, obtaining the first two terms.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries of 3 *marks.*

9. Solve the equation $u_x + 3y^{2/3} u_y = 2$ subject to the initial condition $u(x, 1) = 1 + x$.

10. Compute the function $u(x, y)$ satisfying the eikonal equation $u_x^2 + u_y^2 = n^2$ and the initial condition $u(x, 1) = n\sqrt{1 + x^2}$ where n is a constant.

11. Find the canonical form of the equation $u_{xx} + 4u_{xy} + u_x = 0$.

12. Let $u(x, t)$ be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ which is defined in the whole plane. Assume that u is constant on the line $x = 2 + ct$. Prove that $u_t + cu_x = 0$.

13. Let u be a function in $C^2(D)$ satisfying the mean value property at every point in D. Show that u is harmonic in D.

14. Solve the equation $u_t = 17u_{xx}$, $0 < x < \pi$, $t > 0$ with boundary condition $u(0, t) = u(\pi, t) = 0$, $t \geq 0$ and initial condition :

$$u(x, 0) = \begin{cases} 0 \text{ if } 0 \leq x \leq \pi/2 \\ 2 \text{ if } \pi/2 < x \leq \pi. \end{cases}$$

15. Transform the problem :

$$\frac{d^2 y}{dx^2} + y = x, \; y(0) = 1, \; y'(1) = 0$$

to a Fredholm integral equation.

16. Determine the characteristic values λ and the corresponding characteristic functions of the equation

$$y(x) = F(x) + \lambda \int_0^{2\pi} \cos(x+\xi)\, y(\xi)\, d\xi.$$

17. Consider the integral equation $y(x) = \lambda \int_0^1 x\xi y(\xi)\, d\xi + 1$. Show that the iterative procedure leads to the expression

$$y(x) = 1 + x\left(\frac{\lambda}{2} + \frac{\lambda^2}{6} + \frac{\lambda^3}{18} + \ldots\right).$$

(6 × 3 = 18 marks)

Part C

Answrr any **three** *questions.*
Each question carries 8 *marks.*

18. (a) Use the Langrange method to find a function $u(x, y)$ that satisfies the problem :

$$uu_x + u_y = 1, u(3x, 0) = -x, -\infty < x < \infty.$$

(b) Solve the problem :

$$u_{tt} - u_{xx} = t^7, -\infty < x < \infty, t > 0$$

$$u(x, 0) = 2x + \sin x, -\infty < x < \infty$$

$$u_t\ (x, 0) = 0, -\infty < x < \infty.$$

19. (a) Solve the problem using Mongo's method.

$$p^2 + q^2 - n_0^2 = 0$$

with initial conditions $x(0, s) = s, y(0, s) = 2s, u(0, s) = 1.$

(b) Write short, notes on :

(i) Domain of dependence.

(ii) Region of influence.

Turn over

20. (a) Find the general solution of the one - dimensional wave equation :

$$u_{tt} - c^2 u_{xx} = 0, -\infty < x < \infty, t > 0.$$

(b) Using separation of variables method find a solution of the vibrating string with fixed ends given by :

$$u_{tt} - c^2 u_{xx} = 0, 0 < x < L, \ t > 0$$

$$u(0, t) = u(L, t) = 0, t \geq 0$$

$$u(x, 0) = f(x), u_t(x, 0) = g(x), 0 \leq x \leq L.$$

21. (a) Find a co-ordinate system $S = S(x, y)$, $t = t(x, y)$ that transforms the equation

$$u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0$$

into its canonical form and find the general solution of the equation.

(b) Obtain Poisson formula for the solution of the Laplace equation in a disc of radius a with Dirichlel boundary conditions.

22. (a) Show that any solution of the integral equation :

$$y(x) = \lambda \int_0^1 (1 - 3x\xi)\, y(\xi)\, d\xi + F(x)$$

can be expressed as the sum of F (x) and some linear combination of the characteristic functions.

(b) Show that the kernel $K(x, \xi) = \sin x \cos \xi$ has no characteristic numbers associated with (0.2π).

(3 × 8 = 24 marks)