D 43899 (Pages : 2) Name..................................

Reg. No...............................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2018

(CCSS)

Mathematics

MAT 2C 09 – TOPOLOGY

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Define topology on a set. Give an example of a topology on a particular set.
2. Define d on $\mathbb{R}^2$ by the rule $d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Describe the open balls in this space.
3. Give an example of an open set, which is not an open interval in the set of real numbers with usual topology. Justify your claim.
4. Prove that in a topological space composition of two continuous functions is a continuous function.
5. Prove that every closed surjective map is a quotient map.
6. Prove that the property of being a discrete space is divisible.
7. Prove that in a Hausdorff space, limits of sequences are unique.
8. Prove that the intersection of any family of boxes is a box.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by the rule $d(x, y) = |x - y| + |x^2 - y^2|$. Prove that d is a metric on $\mathbb{R}$ that is not translation invariant.
10. Find the boundary of the set N of natural numbers in the real line with usual topology.
11. Distinguish between continuous maps and open maps in topological spaces.
12. Prove that regularity is a hereditary property.
13. Justify the terms 'box' and 'wall' geometrically for product of copies of real line.
14. Prove that if a product is non-empty then each projection function is onto.

(4 × 4 = 16 marks)

Turn over

Part C

Answer A **or** *B of the following questions.*
Each question carries 12 *marks.*

15. A. (a) Prove that the usual topology on the Euclidean plane R^2 is strictly weaker than the topology induced by lexicographic ordering.

(b) Determine the topology induced by a discrete metric on a set.

B. (a) If X is any set, prove that the collection of all one-point subsets of X is a base for the discrete topology on X.

(b) Prove that metrisability is a hereditary property.

16. A. (a) Prove that a subset of a topological space is open if and only if it is the neighborhood of each of its points.

(b) Prove that a subset A of a space X is dense in X if and only if for every non-empty open subset B of X, $A \cap B \neq \phi$.

B. (a) For any subset A of a space X, with usual notations prove that $\bar{A} = A \cup A'$.

(b) For a subset A of a space X, prove that

$\bar{A} = \{y \in X$: every neighborhood of y meets A non-vacuously$\}$

17. A. (a) Let X have the weak topology determined by a family $\{f_i : X \to Y_i \mid i \in I\}$ of functions where each Y_i is a topological space, I being an index set. Then for any space Z, prove that a function $g : Z \to X$ is continuous if and only if for each $i \in I$, the composite $f_i \circ g : Z \to Y_i$ is continuous.

(b) Prove that the composite of two quotient maps is a quotient map.

B. (a) Prove that every continuous real-valued function on a compact space is bounded and attains its extrema.

(b) Prove that a subset of the set of real numbers is connected if and only if it is an interval.

18. A. (a) Prove that all metric spaces are T_3 spaces.

(b) Define large box in a topology. Prove that intersection of a finite number of large boxes is a large box.

B. (a) Prove that the projection functions are open.

(b) If the product is non-empty, then prove that each co-ordinate is embeddable in it.

(4 × 12 = 48 marks)

C 5889 **(Pages : 2)**

Name.....................................

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2016

(CCSS)

Mathematics

MAT 2C 09—TOPOLOGY—I

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 marks.

1. Give examples of two metrics on the same set that yield the same topology on this set.
2. Define the usual topology on the set of real numbers.
3. Define sub-base for a topology on a set. Give an example for a sub-base.
4. Define subspace of a topological space. Give an example.
5. Show that a projection map need not sent closed sets to closed sets.
6. Define nearness relation on a set X. Prove that there is a one-to-one correspondence between the set of topologies on a set and the set of nearness relations on that set.
7. Define T_1 space and write an example.
8. State and prove a relation between second countability and separability in topological spaces.
9. Prove that regularity is a hereditary property.
10. Define normal space. Give an example.
11. Define extension problem and lifting problem in topological spaces.
12. Distinguish between connectedness and locally connectedness in topological spaces.

(12 × 4 = 48 marks)

Part B

Answer either A **or** *B of each question.*
Each question carries 8 marks.

13. A. (a) Prove that the open balls in a metric space are open sets.

(b) Determine the topology induced by a discrete metric on a set.

B. (a) Define the co-countable topology. Prove that in a co-countable topology, the only convergent sequences are those which are eventually constant.

(b) Prove that a space is second countable if and only if it has a countable sub-base.

Turn over

14. A. (a) Prove that every T_3 space is T_2 Using an example prove that the converse is not true.

(b) Prove that a subset A of a space X is dense in X if and only if for every non-empty open subset B of X, $A \cap B \neq \phi$.

B. (a) Prove that in topological spaces, composition of continuous functions is continuous.

(b) For any three spaces X_1, X_2, X_3 prove that $X_1 \times (X_2 \times X_3)$ is homeomorphic to $(X_1 \times X_2) \times X_3$.

15. A. (a) Prove that every second countable space is first countable.

(b) State and prove the Lebesgue covering lemma.

B. (a) Define countable chain condition in a topological space. Prove that second countable space satisfies the countable chain condition.

(b) Prove that every closed and bounded interval is compact.

16. A. (a) Prove that metric spaces are T_4 spaces.

(b) Define completely regular space. Prove that every completely regular space is regular.

B. (a) Prove that normality is a weakly hereditary property.

(b) Prove that every Tychnoff space is T_3.

C.5888 (Pages : 4) Name.................................

Reg. No...............................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2016

(CCSS)

Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 marks.

1. Let A be a 2×2 matrix with real entries. For X, Y in $R^{2 \times 1}$, let $f_A(X, Y) = Y^t AX$. Show that if $A = A^t$, $A_{11} > 0, A_{22} > 0$ and det $A > 0$, then f_A is an inner product on $R^{2 \times 1}$.

2. Let V be an inner product space, W a finite dimensional subspace, and E the orthogonal projection of V on W. Show that the mapping $\beta \mapsto \beta - E\beta$ is the orthogonal projection of V on W.

3. Find the exact solution of the initial value problem $y' = 2x(1+y),\ y(0) = 0$. Starting with $y_0(x) = 0$, apply Picard's method to calculate $y_1(x), y_2(x)$ and $y_3(x)$.

4. Find the general solution of the equation $xy'' - (2x+1)y' + (x+1)y = 0$, given that $y = e^x$ is a solution.

5. Find a series solution of the equation $y'' + y' - xy = 0$ such that $y(0) = 1$ and $y'(0) = 0$.

6. Locate and classify the singular points on the x-axis of the equation :

$$x^3(x-1)y'' - 2(x-1)y' + 3xy = 0.$$

7. Find the general solution of the equation

$$\left(x^2 - x - 6\right)y'' + (5+3x)y' + y = 0 \text{ near its singular point } x = 3.$$

8. Show that $p_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}\left(x^2 - 1\right)^n$ satisfies the Legendre's equation

$$\left(1 - x^2\right)y'' - 2xy' + n(n+1)y = 0, \text{ where } n \text{ is a non-negative integer.}$$

Turn over

9. Express $J_2(x)$, $J_3(x)$ and $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$.

10. Show that the system $\frac{dx}{dt} = -x$, $\frac{dy}{dt} = -2y$ has the origin as an isolated critical point and find the differential equation of the paths.

11. Determine the nature and stability properties of the critical point (0, 0) for the system :

$$\frac{dx}{dt} = 4x - 3y, \frac{dy}{dt} = 8x - 6y.$$

12. Show that (0, 0) is an asymptotically stable critical point for the system :

$$\frac{dx}{dt} = -2x + xy^3, \frac{dy}{dt} = -x^2 y^2 - y^3.$$

(12 × 4 = 48 marks)

Part B

Answer (A) **or** *(B) of each question.*
Each question carries 8 marks.

13. (A) Let V be the subspace of $R[x]$ of polynomials of degree at most 3. Equip V with the inner product $(f/g) = \int_0^1 f(t)\,g(t)\,dt$ for $f, g \in V$.

(i) Find the orthogonal complement of the subspace of scalar polynomials.

(2 marks)

(ii) Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$. (6 marks)

(B) Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R with sides parallel to the axes. If (x_0, y_0) is any interior point of R, then show that there exists a number $h > 0$ with the property that the initial value problem

$$y' = f(x, y), y(x_0) = y_0$$

has a solution on the interval $|x - x_0| \le h$.

(8 marks)

14. (A) (a) Find a particular solution of the equation $y'' + 2y' + 5y = e^{-x} \sec 2x$.

(6 marks)

(b) Find a power series solution of the equation $y' + y = 1$. (2 marks)

(B) (a) Define regular singular point of the differential equation $y'' + p(x) y' + Q(x) y = 0$.

(1 mark)

(b) Show that the origin is a regular singular point and calculate two independent Frobenius series solutions of $2x^2 y'' + xy' - (x+1) y = 0$.

(7 marks)

15. (A) (a) Show that $\int_{-1}^{1} p_m(x)\, p_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}$. (6 marks)

(b) Determine the nature of the point $x = \infty$ for Legendre's equation

$(1 - x^2) y'' - 2xy' + p(p+1) y = 0$.

(2 marks)

(B) (a) Obtain $J_p(x)$ as a solution of Bessel's equation. (5 marks)

(b) Show that between any two zeros of $J_0(x)$ there is a zero of $J_1(x)$ and between any two zeros of $J_1(x)$ there is a zero of $J_0(x)$.

(3 marks)

16. (A) (a) Find the general solution of the system $\dfrac{dx}{dt} = 5x + 4y, \dfrac{dy}{dt} = -x + y$. (6 marks)

(b) Describe the phase portrait of the system :

$\dfrac{dx}{dt} = 1, \dfrac{dy}{dt} = 2.$

(2 marks)

(B) (a) State and prove Liapunov's stability theorem for an isolated critical point of the autonomous system :

$$\frac{dx}{dt} = F(x, y), \frac{dy}{dt} = G(x, y).$$

(6 marks)

(b) Show that a function of the form $ax^3 + bx^2y + cxy^2 + dy^3$ cannot be either positive definite or negative definite.

(2 marks)

[4 × 8 = 32 marks]

C 5887 (Pages : 3) Name.......................................

Reg. No...................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2016

(CCSS)

Mathematics

MAT 2C 07—REAL ANALYSIS—II

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

I. 1 Let $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$. Prove that $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space under the metric $d(A, B) = \| A - B \|$.

2 Let f be a differentiable real function in an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. If f has a local maximum at a point $x \in E$, then prove that $f'(x) = 0$.

3 Prove that the outer measure is translation invariant.

4 If E_1 and E_2 are measurable sets, then prove that $E_1 \cup E_2$ is measurable.

5 Prove that sum of two measurable functions defined on a same measurable set is measurable.

6 Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Prove that $\limsup f_n$ is a measurable function.

7 Let f be a non-negative measurable function. Show that $\int f = 0$ if and only if $f = 0$ a.e...

8 Let E be a measurable set. If f and g are integrable over E, then prove that $f + g$ is integrable over E and $\int_E f + g = \int_E f + \int_E g$.

Turn over

9. Define convergence in measure. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure. If $f_n \to f$ *a.e.*, then prove that $\{f_n\}$ converges to f in measure.

10. Let f and g be non-negative functions. If f and g are continuous at c, then prove that $D^+ (f \cdot g)(c) \leq f(c) D^+ g(c) + g(c) D^+ f(c)$.

11. If f is of bounded variation on $[a, b]$, then prove that $T_a^b = P_a^b + N_a^b$.

12. If f is absolute continuous on $[a, b]$, then prove that f has a derivative almost everywhere.

(12 × 4 = 48 marks)

Part B

Answer (A) **or** *(B) of each question.*
Each question carries 8 *marks.*

II. (A) (a) Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then prove that $\dim X \leq r$.

(b) Let E be an open set in $\mathbb{R}^n$, f be a mapping from E into $\mathbb{R}^m$ and f be differentiable at $x_0 \in E$. If g maps an open set containing $f(E)$ into $\mathbb{R}^k$ and g is differentiable at $f(x_0)$, then prove that the mapping F of E into $\mathbb{R}^k$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0)) f'(x_0)$.

(B) (a) Let E be an open subset of $\mathbb{R}^n$ and f maps E into $\mathbb{R}^m$. If f is differentiable at a point $x \in E$, then prove that the partial derivatives $(D_j f_i)(x)$ exist.

(b) If X is a complete metric space and if φ is a contraction of X into X, then prove that there exists one and only one $x \in E$ such that $\varphi(x) = x$.

III. (A) (a) Let $\mathcal{A}$ be an algebra of subsets and $\{A_i\}$ be a sequence of sets in $\mathcal{A}$. Prove that there is a sequence $\{B_i\}$ of sets in $\mathcal{A}$ such that $B_n \cap B_m = \varnothing$ for $n \neq m$ and $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$.

(b) If A is countable, then prove that $m * (A) = 0$.

(B) (a) Prove that outer measure of an interval is its length.

(b) Prove that there exists a non-measurable set.

IV. (A) Let f be defined and bounded on a measurable set E with finite measure. Prove that f is measurable if and only if

$$\inf_{f \le \psi} \int \psi(x)\, dx = \sup_{f \ge \varphi} \int_E \varphi(x)\, dx,$$

where ψ and φ are simple functions.

(B) (a) Let f be a non-negative function which is integrable over a set E. Prove that for a given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$ we have $\int_A f < \varepsilon$.

(b) State and prove Lebesgue convergence theorem.

V. (A) Let f be an increasing real valued function on the interval $[a, b]$. Prove that f is differentiable almost everywhere, the derivative f' is measurable and

$$\int_a^b f'(x) \le f(b) - f(a).$$

(B) (a) Let f be an integrable function on $[a, b]$ and let

$$F(x) = F(a) + \int_a^x f(t)\, dt.$$

Prove that $F'(x) = f(x)$ for almost all x in $[a, b]$.

(b) Prove that every absolutely continuous function is the indefinite integral of its derivative.

(4 × 8 = 32 marks)

C 5886 (Pages : 3) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. (CCSS) DEGREE EXAMINATION, JUNE 2016

Mathematics

MAT 2C 06—ALGEBRA–II

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Prove that $Q(\pi)$ is isomorphic to the field of rational functions over Q.
2. Find irr (α, Q) where $\alpha = i + \sqrt{2}$.
3. Prove that $\sqrt{\pi}$ is not constructible by straight edge and compass.
4. Let α be a zero of $x^3 + x^2 + 2 \in Z_3[x]$ and β be a zero of $x^3 + 2x + 1 \in Z_3[x]$. Prove that $Z_3(\alpha) = Z_3(\beta)$.
5. Let $E = Q(\sqrt{2}, \sqrt{3})$. Verify whether there is an automorphism Ψ of E with $\Psi(\sqrt{2} + \sqrt{3}) = \sqrt{6}$.
6. Show that in a field F of characteristic p the equation $(a + b)^p = a^p + b^p$ holds for all $a, b \in F$.
7. Verify whether $Q(\sqrt{2}\, i) = Q(\sqrt{2} + i)$.
8. Let K be a finite normal extension of F and $F \leq E \leq K$. Show that K is a normal extension of E.
9. Let K be a splitting field over F. Let $\alpha \in K$ and $\alpha \notin F$. Show that there exists $\sigma \in G(K/F)$ such that $\sigma(\alpha) \neq \alpha$.
10. Let K be the field of all symmetric rational functions in x, y over Q. Let $s_1 = x + y$ and $s_2 = xy$. Prove that $K = Q(s_1, s_2)$.
11. Let K be the splitting field of $x^4 - 2$ over Q. Find [K : Q].
12. Describe all primitive 8^{th} roots of units.

($12 \times 4 = 48$ marks)

Part B

Answer **either** *(A)* **or** *(B) of each question.*
Each question carries 8 *marks.*

13. (A) Let F be a field and $p(x)$ be an irreducible polynomial of degree n in F[x]. Prove that

    (i) $E = F[x]/\langle p(x) \rangle$ is a field.

    (ii) $\alpha = x + \langle p(x) \rangle \in E$ is a zero of $p(x)$.

    (iii) $[E : F] = n$.

    (B) (a) Let K be an extension of a field F. Show that $E = \{\alpha \in K : \alpha$ is algebraic over F$\}$ is a subfield of K.

    (b) Prove that the following are equivalent for a field F :—

    (i) Every non-constant polynomial in F[x] has a zero in F.

    (ii) F has no algebraic extension E with F < E.

14. (A) (a) Prove that if E is a finite extension of a field F of degree n and if $|F| = ?$ then $|E| = q^n$.

    (b) Prove that every finite extension of a finite field is a simple extension.

    (B) (a) Let E be a finite extension of F. Define the index {E : F}. Find the index {C : R} where C is the field of complex numbers and R is the field of reals.

    (b) Let E be a finite extension of F and σ be an automorphism of F. Show that the number of extensions of σ to an isomorphism of E to $\bar{F}$ is finite and independent of σ.

15. (A) (a) Let E be the splitting field of a polynomial $p(x) \in F[x]$ over F. Show that $E = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_1, \alpha_2 \ldots, \alpha_n$ are the zeros pf $p(x)$ in $\bar{F}$.

    (b) Let $F \leq E \leq \bar{F}$ and E be a splitting field over F. Show that if σ is an automorphism of $\bar{F}$ leaving F fixed then $\sigma(E) = E$.

    (B) (a) Prove that every finite field is perfect.

    (b) Let $E = F(\beta, \gamma)$ be a finite separable extension of P an infinite field F. Show that there is an $\alpha \in E$ such that $E = F(\alpha)$.

16. (A) Let F $(y_1, y_2, \ldots, y_n)$ be the field of an rational functions in $y_1, y_2 \ldots, y_n$ over F and K be the subfield of all symmetric rational functions in $y_1, y_2 \ldots, y_n$. Show that

(i) $f(x) = \pi_{i=1}^{n} (x - y_i)$ is a polynomial over K.

(ii) $[F(y_1, y_2, \ldots, y_n) : K] = n!$

(iii) The Galois group G $(F(y_1, y_2, \ldots, y_n) \mid K)$ is isomorphic to the symmetric group S_n.

(B) (a) Define Fermat primes. Show that a regular n-gon is constructible by straight edge and compass if and only if all odd primes dividing n are Fermat primes whose squares do not divide n.

(b) Show that the regular 60-gon is constructible.

(4 × 8 = 32 marks)

C 84222 (Pages : 2) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CCSS)

Mathematics

MAT 2C 09—TOPOLOGY—I

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

1. Write the discrete topology on the set $S = \{a, b, c\}$.
2. Give examples of *two* topologies on a finite set X such that one is weaker than the other.
3. Find the boundary of the set N of natural numbers in the real line.
4. Define subspace of a topological space. Give an example of a non-trivial subspace of the set of real numbers with usual topology.
5. Define quotient map from *one* topological space to another. Give an example.
6. Show that a projection map need not send closed sets to closed sets.
7. Give an example of a topological space that is T_0 but not T_1.
8. Prove that a metric space is a T_3 space.
9. Define second countability and separability in a topological space. Write a relation between the two.
10. Prove that the projection functions are open.
11. Is the union of any two connected sets connected ? Why ?
12. Justify the terms 'box' and 'wall' geometrically for products of copies of the real-line.

(12 × 4 = 48 marks)

Part B

Answer **either** *A* **or** *B of each question.*
Each question carries 8 *marks.*

13. (A) (a) Define a metric on R^2 other than the usual metric on R^2.

(b) Prove that the usual topology on the Euclidean plane R^2 is strictly weaker than the topology induced by the lexicographic ordering.

(B) (a) Define the Sierpinski space. Prove that this topology is not induced by a metric.

(b) Prove that metrisability is a hereditary property.

Turn over

14. (A) (a) If a space is second countable, prove that every open cover of it has a countable subcover.

(b) Prove that if each space (X_i, T_i) is second countable, for $i = 1, 2, ..., n$, then so is their topological product.

(B) (a) Prove that the real line with the semi-open interval topology is not metrisable.

(b) For any subset A of a space X, with usual notations prove that $\bar{A} = A \cup A'$.

15. (A) (a) Prove that the composite of two continuous functions is continuous.

(b) Prove that closed subset of a compact space is compact.

(B) (a) Prove that every closed surjective map is a quotient map.

(b) Prove that every continuous real-valued function on a compact space is bounded and attains its extrema.

16. (A) (a) Prove that normality is a weakly hereditary property.

(b) Prove that every closed and bounded interval is compact.

(B) Let $\{X_i : i \in I\}$ be a family of topological spaces and $X = \pi X_i$ be the product space. Prove that a sequence (x_n) in X converges to $x \in E$ if and only if $\pi_i(x_n)$ converges to $\pi_i(x)$ for each i.

(4 × 8 = 32 marks)

C 84221 (Pages : 4) Name..................................

Reg. No...............................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CCSS)

Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

(2008 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Let V be a vector space over a field F. Show that the sum of two inner products on V is an inner product on V.

2. Consider R^4 with the standard inner product. Let W be the subspace of R^4 consisting of all vectors which are orthogonal to both $\alpha = (1, 0, -1, 1)$ and $\beta = (2, 3, -1, 2)$. Find a basis for W.

3. Prove or disprove : The function $f(x, y) = y^{1/2}$ does not satisfy a Lipschitz condition on the rectangle $|x| \leq 1$ and $0 \leq y \leq 1$.

4. Show that one solution of the equation $xy'' - (2x+1)y' + (x+1) = y = 0$ is $y_1 = e^x$, and find the general solution.

5. Define radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$. If p is not zero or a positive integer, then show that the radius of convergence of $\sum_{n=1}^{\infty} \frac{p(p-1)\dots(p-n+1)}{n!} x^n$ is 1.

6. Find the indicial equation and its roots of the differential equation :

$$4x^2 y'' + \left(2x^4 - 5x\right) y' + \left(3x^2 + 2\right) y = 0.$$

Turn over

7. Show that the equation :

$(x-A)(x-B)y'' + (C+Dx)y' + Ey = 0$, where A, B, C, D and E are constants with $A \neq B$ can be transformed into a hypergeometric equation.

8. Find the first three terms of the Legendre series of $f(x) = e^x$.

9. Obtain $J_p(x)$, the Bessel function of the first kind of order p.

10. Describe the phase portrait of the system :

$$\frac{dx}{dt} = 1, \frac{dy}{dx} = 2.$$

11. Show that a function of the form $ax^3 + bx^2y + cxy^2 + dy^3$ cannot be either positive definite or negative definite.

12. Determine the nature and stability properties of the critical point (0, 0) for the system :

$$\frac{dx}{dt} = -3x + 4y, \frac{dy}{dt} = -2x + 3y.$$

(12 × 4 = 48 marks)

Part B

Answer A **or** *B of each question.*
Each question carries 8 marks.

13. (A) (i) Let W be a subspace of an inner product space V and let β be a vector in V. Show that the vector α in W is a best approximation to β by vectors in W iff $\beta - \alpha$ is orthogonal to every vector in W.

(ii) Let V be the subspace of $R[x]$ of polynomials with real coefficients of degree at most 3 equiped with the inner product $(f/g) = \int_0^1 f(t)\, g(t)\, dt$.

Find an orthonormal basis for V.

(B) (i) State Picard's theorem.

(ii) Solve the initial value problem by Picard's method :

$$\begin{cases} \dfrac{dy}{dx} = z, y(0) = 1 \\ \dfrac{dz}{dx} = -y, z(0) = 0 \end{cases}.$$

14. (A) (i) Discuss the general solution of the homogeneous equation $y'' + py' + qy = 0$, where p and q are constants.

(ii) Find the general solution of

$$y'' + 2y' + 5y = e^{-x} \sec 2x.$$

(B) (i) Find the general solution of Hermite's equation

$y'' - 2xy' + 2py = 0$, where p is a constant.

(ii) Show that the equation $x^2 y'' - 3xy' + (4x + 4) y = 0$

has only one Frobenius series solution. Find this solution.

15. (A) (i) Obtain the recursion formula :

$$(n + 1) p_{n+1}(x) = (2n + 1) x p_n(x) - n p_{n-1}(x).$$

Use this formula to calculate $p_2(x), p_3(x)$ and $p_4(x)$, assuming that $p_0(x) = 1$ and $p_1(x) = x$.

(ii) State and prove the orthogonality property for Legendre polynomials.

(B) (i) Show that $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$.

(ii) Prove that the positive zeros of $J_p(x)$ and $J_{p+1}(x)$ occur alternately, in the sense that between each pair of consecutive positive zeros of either there is exactly one zero of the other.

16. (A) Find the general solution of the system :

$$\frac{dx}{dt} = -3x + 4y, \frac{dy}{dt} = -2x + 3y.$$

(B) (i) Find the differential equation of the paths of the system

$$\begin{cases} \frac{dx}{dt} = 4y \\ \frac{dy}{dt} = -x \end{cases}$$

and solve it. Sketch a few of the paths, showing the direction of increasing t.

(ii) Verify that $(0,0)$ is a simple critical point for the following system, and determine its nature and stability properties :

$$\frac{dx}{dt} = x + y - 2xy$$

$$\frac{dy}{dt} = -2x + y + 3y^2.$$

(4 × 8 = 32 marks)

C 84220 (Pages : 3) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CCSS)

Mathematics

REAL ANALYSIS—II

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

I. (1) Let X be a vector space and let dim $X = n$. Prove that every basis of X has n elements.

(2) Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$. Prove that $\|A\| < \infty$ and A is a uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$.

(3) Let f map a convex open subset E of $\mathbb{R}^n$ into $\mathbb{R}^m$ and let f be differentiable in E. If $f'(x) = 0$ for all $x \in E$, then prove that f is a constant.

(4) If A is a countable subset of $\mathbb{R}$, then prove that $m^*(A) = 0$.

(5) If E_1 and E_2 are measurable sets, then prove that $E_1 \cap E_2$ and $E_1 - E_2$ are measurable sets.

(6) Prove that countable sets has measure zero.

(7) Prove that the characteristic function χ_E of a set E is measurable if and only if E is measurable.

(8) Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Prove that $\sup_n f_n$ is a measurable function.

(9) Let f and g be bounded measurable functions defined on a set E of finite measure and let $f = g$ *a.e.* Prove that $\int f = \int g$.

(10) Define convergence in measure. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure such that $f_n \to f$ *a.e.* Prove that $\{f_n\}$ converges to f in measure.

Turn over

(11) Let f be a function defined by

$$f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}.$$

Find $D^{+}f(x), D_{+}f(x), D^{-}f(x), D_{-}f(x)$ at $x=0$.

(12) If f is of bounded variation on $[a,b]$, then prove that $T_a^b = P_a^b + N_a^b$.

(12 × 4 = 48 marks)

Part B

Answer A **or** *B of each question.*
Each question carries 8 *marks.*

II. A (a) Let Ω be the set of all invertible linear operators in $\mathbb{R}^n$. Prove that Ω is an open subset of $L(\mathbb{R}^n)$ and the mapping $A\to A^{-1}$ is continuous on Ω.

(b) If $A\in L(\mathbb{R}^n,\mathbb{R}^m)$, and $B\in L(\mathbb{R}^m,\mathbb{R}^k)$, then prove that $\|A+B\|\leq\|A\|+\|B\|$.

(c) Let f be a differentiable real-valued function defined in an open set $E\subseteq\mathbb{R}^n$ and let f has a local maximum at a point $x\in E$. Prove that $f'(x)=0$.

B (a) Let $E\subset\mathbb{R}^n$ be an open set and let $f:E\to\mathbb{R}^m$ be a mapping differentiable at a point $x\in E$. Prove that the partial derivatives $(D_j f_i)(x)$ exist and

$$f'(x)\,e_j=\sum_{i=1}^{m}(D_j f_i)(x)\,u_i,$$

where $1\leq j\leq n$.

(b) If X is a complete metric space and if ϕ is a contraction of X into X, then prove that there exists one and only one $x\in E$ such that $\phi(x)=x$.

III. A (a) Let $\mathcal{A}$ be an algebra of subsets of a set X and $\{A_i\}$ a sequence of sets in $\mathcal{A}$. Prove that there exists a disjoint sequence $\{B_i\}$ of sets in $\mathcal{A}$ such that

$$\bigcup_{i=1}^{\infty}B_i=\bigcup_{i=1}^{\infty}A_i.$$

(b) Let $\{E_n\}$ be an infinite sequence of measurable sets such that $E_1 \subset E_2 \subset .....$ If $m(E_1)$ is finite, then prove that

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} m(E_n).$$

B (a) Prove that the outer measure of an interval is its length.

(b) Prove that Cantor set is of measure zero.

IV. A (a) Let f be a bounded function defined on $[a, b]$. If f is Reimann integrable on $[a, b]$, then prove that f is measurable and

$$R \int_a^b f(x)\, dx = \int_a^b f(x)\, dx.$$

(b) State and prove Fatou's lemma.

B (a) Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all $x \in E$ we have $f(x) = \lim f_n(x)$. Prove that

$$\int_E f = \lim \int_E f_n.$$

(b) Let f and g be integrable over a measurable set E. Prove that $f + g$ is integrable over E and $\int_E (f+g) = \int_E f + \int_E g.$

V. A (a) Prove that a function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotonic increasing functions.

(b) Define absolute continuity and give an example of it. If f is absolutely continuous, then prove that f has derivative almost everywhere.

B (a) If f is of bounded variation on $[a, b]$, then prove that $f'(x)$ exists for almost all x in $[a, b]$.

(b) Let f be a bounded measurable function on $[a, b]$ and let $F(x) = \int_a^x f(t)\, dt + F(a)$. Prove that $F'(x) = f(x)$ for almost all x in $[a, b]$.

(4 × 8 = 32 marks)

C 84219 **(Pages : 3)** **Name..**

Reg. No......................................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CCSS)

Mathematics

MAT 2C 06—ALGEBRA – II

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Construct a field having 16 elements.

2. Find a real number α such that $Q\left(\sqrt{3}, \sqrt{7}\right) = Q(\alpha)$.

3. Show that the field C of complex numbers is algebraically closed.

4. Find the degree and a basis for $Q\left(\sqrt{2}, \sqrt{6}\right)$ over $Q\left(\sqrt{3}\right)$.

5. Let E be a field of p^n elements contained in an algebraic closure $\overline{Z}_p$ of Z_p. Show that the elements of E are precisely the zeros in $\overline{Z}_p$ of the polynomial $x^{p^n} - x$ in $Z_p[x]$.

6. Find all primitive 8^{th} roots of unity in GF (9).

7. Find all conjugates of $\sqrt{2+i}$ over R.

8. Find the degree over Q of the splitting field over Q of the polynomial $x^3 - 1$ in Q $[x]$.

9. Show that if $[E : F] = 2$, then E is a splitting field over F.

10. State the Main theorem of Galois theory.

11. Show that $\Phi_8(x) = x^4 + 1$.

12. Prove that the polynomial $x^5 - 1$ is solvable by radicals.

(12 × 4 = 48 marks)

Turn over

Part B

Answer **either** *A* **or** *B of each question.*
Each question carries 8 *marks.*

13. A (i) Let F be a field and let $f(x)$ be a non-constant polynomial in F [x]. Show that there exists an extension field E of F and an $\alpha \varepsilon E$ such that $f(\alpha) = 0$.

(ii) Show that a field F is algebraically closed iff every non-constant polynomial in F [x] factors in F [x] into linear factors.

B (i) Let E be an algebraic extension of a field F. Show that there exists a finite number of elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ in E such that $E : F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ iff E is finite-dimensional vector space over F.

(ii) Show that trisecting the angle is impossible.

14. A (i) Show that if F is any field, then for every positive integer n, there is an irreducible polynomial in F [x] of degree n.

(ii) Let $\{\sigma_i : i \varepsilon I\}$ be a collection of automorphism of a field E. Show that the set $E_{\{\sigma_i\}}$ of all $a \varepsilon E$ left fixed by every σ_i for $i \varepsilon I$ form a subfield of E.

B (i) Let F be a finite field of characteristic p. Show that the map $\sigma_p : F \to F$ defined by $\sigma_p(a) = a^p$ for $a \varepsilon F$ is an automorphism of F. Also, show that $F\{\sigma_p\} \cong Z_p$.

(ii) Describe all extensions of the identity map of Q to an isomorphism mapping $Q(\sqrt[3]{2})$ onto a subfield of $\bar{Q}$.

15. A (i) Define splitting field. Show that if $E \leq \bar{F}$ is a splitting field over F, then every irreducible polynomial in F [x] having a zero in E splits in E.

(ii) Let K be a finite extension of degree n of a finite field F of p^r elements. Show that $G(K/F)$ is in cyclic of order n, and is generated by σ_{p^r}, where for $\alpha \varepsilon K, \sigma_{p^r}(\alpha) = \alpha^{p^r}$.

B (i) Show that every field of characteristic zero is perfect.

(ii) Show that a finite separable extension of a field is a simple extension.

16. A Let K be the splitting field of $x^4 + 1$ over Q. Compute G (K / Q). Give the lattice diagrams for subgroups of G (K / Q) and for the intermediate fields.

B (i) Prove that the regular n-gon is constructible iff all the odd primes dividing n are fermat primes whose squares do not divide n.

(ii) Is it true that the splitting field of $x^{17} - 5$ over Q has a solvable Galois group ? Justify your answer.

($4 \times 8 = 32$ marks)