

**A STUDY ON SOME OPEN SETS VIA REGULAR
OPEN SETS IN TOPOLOGICAL SPACE AND
BITOPOLOGICAL SPACE**

*Thesis submitted to the
University of Calicut for the award of the degree of*

DOCTOR OF PHILOSOPHY IN MATHEMATICS
under the Faculty of Science

by
MUHSINA V

under the guidance of
DR. BABY CHACKO



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AUGUST 2025

DECLARATION

I, MUHSINA V, hereby declare that this thesis entitled “A STUDY ON SOME OPEN SETS VIA REGULAR OPEN SETS IN TOPOLOGICAL SPACE AND BITOPOLOGICAL SPACE” submitted to the University of Calicut for the award of the degree of Doctor of Philosophy in Mathematics is a bonafide record of the work done by me under the guidance and supervision of Dr. Baby Chacko, Department of Mathematics, St. Joseph’s College, Devagiri. This thesis contains no material which has been accepted for the award of any degree or diploma of any university or institution and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference was made. The content of the thesis are undergone plagiarism check using iThenticate software at C.H.M.K Library, University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.

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CERTIFICATE

I hereby certify that, this thesis entitled **“A STUDY ON SOME OPEN SETS VIA REGULAR OPEN SETS IN TOPOLOGICAL SPACE AND BITOPOLOGICAL SPACE”** is a bonafide record of research work carried out by **Ms. MUHSINA V**, Research Scholar, Department of Mathematics, St. Joseph's College (Autonomous), Devagiri, Kozhikode-673008 under my supervision and guidance for the award of the Degree of Doctor of Philosophy in Mathematics, of the University of Calicut. The work reported herein does not form part of any other thesis or dissertation submitted previously for the award of any degree or diploma of any other university or institution. Also certified that the contents of the thesis have been checked using anti-plagiarism data base and no unacceptable similarity was found through the software check. The thesis is revised as per the modifications and recommendations reported by the adjudicators. Soft copy attached is the same as that of the revised copy.

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ABSTRACT

This thesis undertakes an in-depth investigation of some open sets in both topological and bitopological spaces by using regular open sets. The limitations of regular open set that the arbitrary intersections of regular open sets need not be regular open motivated to introduces the concept of I_r -open sets, defined as intersections of regular open sets. The thesis define I_r -closure, I_r -interior, and I_r -cluster points and presents fundamental properties, theorems, and equivalent characterizations. The work systematically introduced and explored the generalized separation axioms (I_r - T_0 , I_r - T_1 , I_r - T_2), I_r -regularity, and I_r -normality, along with their starred counterparts, I_r -connectedness, and I_r -compactness. This thesis extends the framework to define I_r^* -closed sets and investigates their properties, characterizations, and behavior under set operations. The I_r -cover of a set is defined and used to construct a supra-topology τ_{I_r} , within which several characterizations of separation axioms are studied. It further examines function classes including I_r -continuous, I_r -irresolute, and I_r -open functions, along with highlighting their role in characterizing topological properties. The study advances into the notions of minimal and maximal regular open sets and introduces somewhat mr-continuous and somewhat mr-open functions with their associated topologies. In the context of bitopological spaces, the thesis defines pairwise minimal and maximal regular open sets, proposes new separation axioms, spaces and investigates various continuity classes such as (i, j) -almost continuous, (i, j) -almost completely continuous, etc. Further, p -Min and p -Max r -continuous maps are defined, and their interrelations with existing function classes are studied, enhancing the structural understanding of bitopological continuity. Overall, this thesis provides a comprehensive and unified framework that not only generalizes classical topological concepts but also enriches the theoretical landscape of topological and bitopological spaces.

Keywords

I_r -open sets, I_r -Alexandroff spaces, I_r -cluster points, I_r -closure, I_r -interior, I_r -regular spaces, I_r -normal spaces, almost I_r -normal spaces, completely I_r -normal spaces, I_r^* -regular spaces, I_r^* -normal spaces, almost I_r^* -normal spaces, completely I_r^* -normal spaces, I_r -compactness, I_r -connectedness, I_r -cover, I_r^* -closed sets, I_r^* -open sets, I_r -continuous functions, I_r -irresolute functions, I_r -open functions, minimal regular open set, maximal regular open set, somewhat mr -continuous functions, somewhat mr -open functions, pairwise minimal regular open sets, pairwise maximal regular open sets, Pairwise- $rT_{\min}$ spaces, Pairwise- $rT_{\max}$ spaces, (i, j) - r -door spaces, (i, j) - $rT_{\frac{1}{2}}$ spaces, (i, j) -almost continuous functions, (i, j) -almost completely continuous functions, (i, j) -almost perfectly continuous functions, p -Min r -Continuous maps, p -Max r -Continuous maps.

സംഗ്രഹം

ഈ പ്രബന്ധം റെഗുലർ ഓപ്പൺ സെറ്റുകൾ ഉപയോഗിച്ച് ടോപ്പോളജിക്കൽ, ബൈടോപ്പോളജിക്കൽ സ്പേസുകളിലെ ചില ഓപ്പൺ സെറ്റുകളെക്കുറിച്ച് ആഴത്തിലുള്ള പഠനം നടത്തുന്നു. റെഗുലർ ഓപ്പൺ സെറ്റുകളുടെ ഇന്റർസെക്ഷൻ റെഗുലർ ഓപ്പൺ ആയിരിക്കണമെന്നില്ല എന്ന പരിമിതി റെഗുലർ ഓപ്പൺ സെറ്റുകളുടെ ഇന്റർസെക്ഷനായി നിർവ്വചിച്ചിരിക്കുന്ന I_r -ഓപ്പൺ സെറ്റുകളുടെ സങ്കല്പം അവതരിപ്പിക്കാൻ പ്രേരിപ്പിച്ചു.

ഈ പ്രബന്ധം I_r -ക്ലോഷർ, I_r -ഇന്റീരിയർ, I_r -ക്ലസ്റ്റർ പോയിന്റുകൾ എന്നിവ നിർവ്വചിക്കുകയും അടിസ്ഥാന ഗുണധർമ്മങ്ങൾ, സിദ്ധാന്തങ്ങൾ, തുല്യ സവിശേഷതകൾ എന്നിവ അവതരിപ്പിക്കുകയും ചെയ്യുന്നു. ഈ പ്രബന്ധം സെപ്പറേഷൻ ആക്സിയംസ് (I_r - T_0 , I_r - T_1 , I_r - T_2), I_r -റെഗുലാരിറ്റി, I_r -നോർമാലിറ്റി എന്നിവയും അവയുടെ സ്റ്റാർഡ് കൗണ്ടർപാർട്ടുകൾ, I_r -കണക്ടഡ്നെസ്, I_r -കോമ്പാക്റ്റ്നെസ് എന്നിവയും വ്യവസ്ഥിതമായി അവതരിപ്പിക്കുകയും പര്യവേക്ഷണം ചെയ്യുകയും ചെയ്യുന്നു.

ഈ പ്രബന്ധം I_r^* -ക്ലോസ്ഡ് സെറ്റുകൾ നിർവ്വചിക്കുന്നതിനായി ചട്ടക്കൂട് വിപുലീകരിക്കുകയും അവയുടെ ഗുണധർമ്മങ്ങൾ, സവിശേഷതകൾ, സെറ്റ് ഓപ്പറേഷനുകൾക്ക് കീഴിലുള്ള പെരുമാറ്റം എന്നിവ അന്വേഷിക്കുകയും ചെയ്യുന്നു. ഒരു സെറ്റിന്റെ I_r -കവർ നിർവ്വചിച്ച് സുപ്ര-ടോപ്പോളജി τ_{I_r} നിർമ്മിക്കാൻ ഉപയോഗിക്കുന്നു, അതിനുള്ളിൽ നിരവധി സവിശേഷതകൾ പഠിക്കുന്നു.

I_r -കണ്ടിന്യൂസ്, I_r -ഇറിസൊല്യൂട്ട്, I_r -ഓപ്പൺ ഫംഗ്ഷനുകൾ ഉൾപ്പെടുന്ന ഫംഗ്ഷൻ ക്ലാസുകളെ ഇത് കൂടുതൽ പരിശോധിക്കുകയും ടോപ്പോളജിക്കൽ ഗുണധർമ്മങ്ങളെ സവിശേഷമാക്കുന്നതിലുള്ള അവയുടെ പങ്ക് എടുത്തുകാട്ടുകയും ചെയ്യുന്നു. മിനിമൽ, മാക്സിമൽ റെഗുലർ ഓപ്പൺ സെറ്റുകളുടെ ആശയങ്ങളിലേക്ക് പഠനം മുന്നേറുകയും അവയുമായി ബന്ധപ്പെട്ട ടോപ്പോളജികളോടൊപ്പം സംവാദ് mr -കണ്ടിന്യൂസ്, സംവാദ് mr -ഓപ്പൺ ഫംഗ്ഷനുകൾ അവതരിപ്പിക്കുകയും ചെയ്യുന്നു.

ബൈടോപ്പോളജിക്കൽ സ്പേസുകളുടെ സന്ദർഭത്തിൽ, പ്രബന്ധം പെയർവൈസ് മിനിമൽ, മാക്സിമൽ റെഗുലർ ഓപ്പൺ സെറ്റുകൾ നിർവ്വചിക്കുകയും പുതിയ സെപ്പറേഷൻ ആക്സിയംസ് നിർദ്ദേശിക്കുകയും, (i, j) -ഓൾമോസ്റ്റ് കണ്ടിന്യൂസ്, (i, j) - ഓൾമോസ്റ്റ് കംപ്ലീറ്റ്ലി കണ്ടിന്യൂസ് തുടങ്ങിയ വിവിധ ഫംഗ്ഷൻ ക്ലാസുകൾ അന്വേഷിക്കുകയും ചെയ്യുന്നു. കൂടാതെ, p -Min, p -Max r -കണ്ടിന്യൂസ് മാപ്പുകൾ നിർവ്വചിക്കുകയും നിലവിലുള്ള ഫംഗ്ഷൻ ക്ലാസുകളുമായുള്ള

അവയുടെ പരസ്പര ബന്ധങ്ങൾ പഠിക്കുകയും ചെയ്യുന്നു, ഇത് ബൈടോപ്പോളജിയിൽ തുടർച്ചയുടെ ഘടനാപരമായ ധാരണ വർദ്ധിപ്പിക്കുന്നു.

മൊത്തത്തിൽ, ഈ പ്രബന്ധം ക്ലാസിക്കൽ ടോപ്പോളജിയിൽ സങ്കല്പങ്ങളെ സാമാന്യവൽക്കരിക്കുക മാത്രമല്ല, ടോപ്പോളജിക്കൽ, ബൈടോപ്പോളജിക്കൽ സ്പേസുകളുടെ സൈദ്ധാന്തികപ്രകൃതി സമ്പന്നമാക്കുകയും ചെയ്യുന്ന ഒരു സമഗ്രവും ഏകീകൃതവുമായ ചട്ടക്കൂട് നൽകുന്നു.

സൂചകപദങ്ങൾ: I_r -ഓപ്പൺ സെറ്റുകൾ, I_r -അലക്സാണ്ട്രോഫ് സ്പെയ്സുകൾ, I_r -ക്ലസ്റ്റർ പോയിന്റുകൾ, I_r -ക്ലോഷർ, I_r -ഇന്റീരിയർ, I_r -റെഗുലർ സ്പെയ്സുകൾ, I_r -നോർമൽ സ്പെയ്സുകൾ, ഓൾമോസ്റ്റ് I_r -നോർമൽ സ്പെയ്സുകൾ, കംപ്ലീറ്റ്ലി I_r -നോർമൽ സ്പെയ്സുകൾ, I_r^* -റെഗുലർ സ്പെയ്സുകൾ, I_r^* -നോർമൽ സ്പെയ്സുകൾ, ഓൾമോസ്റ്റ് I_r^* -നോർമൽ സ്പെയ്സുകൾ, കംപ്ലീറ്റ്ലി I_r^* -നോർമൽ സ്പെയ്സുകൾ, I_r -കോമ്പാക്റ്റ്നെസ്, I_r -കണക്ടനെസ്, I_r -കവർ, I_r^* -ക്ലോസ്ഡ് സെറ്റുകൾ, I_r^* -ഓപ്പൺ സെറ്റുകൾ, I_r -കണ്ടിന്യൂസ് ഫംഗ്ഷനുകൾ, I_r -ഇൻസൊല്യൂട്ട് ഫംഗ്ഷനുകൾ, I_r -ഓപ്പൺ ഫംഗ്ഷനുകൾ, മിനിമൽ റെഗുലർ ഓപ്പൺ സെറ്റ്, മാക്സിമൽ റെഗുലർ ഓപ്പൺ സെറ്റ്, സംവാട്ട് mr -കണ്ടിന്യൂസ് ഫംഗ്ഷനുകൾ, സംവാട്ട് mr -ഓപ്പൺ ഫംഗ്ഷനുകൾ, പെയർവൈസ് മിനിമൽ റെഗുലർ ഓപ്പൺ സെറ്റുകൾ, പെയർവൈസ് മാക്സിമൽ റെഗുലർ ഓപ്പൺ സെറ്റുകൾ, പെയർവൈസ്- $rT_{\min}$ സ്പെയ്സുകൾ, പെയർവൈസ്- $rT_{\max}$ സ്പെയ്സുകൾ, (i, j) - r -ഡോർ സ്പെയ്സുകൾ, (i, j) - $rT_{\frac{1}{2}}$ സ്പെയ്സുകൾ, (i, j) -ഓൾമോസ്റ്റ് കണ്ടിന്യൂസ് ഫംഗ്ഷനുകൾ, (i, j) -ഓൾമോസ്റ്റ് കംപ്ലീറ്റ്ലി കണ്ടിന്യൂസ് ഫംഗ്ഷനുകൾ, (i, j) -ഓൾമോസ്റ്റ് പെർഫെക്ട്ലി കണ്ടിന്യൂസ് ഫംഗ്ഷനുകൾ, p -Min r -കണ്ടിന്യൂസ് മാപ്പുകൾ, p -Max r -കണ്ടിന്യൂസ് മാപ്പുകൾ.

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Introduction

Concept of the regular open set was introduced by M. H. Stone in 1937. It has been shown by him that the collection of all regular open sets in a topological space form a complete lattice. M. H. Stone further gave results on application of theory of Boolean algebras to general topology. In 1980, R. C. Jain worked on role of regularly open sets in topology on his thesis. C. Ronse in 1990 studied generalizations of regular open sets in a complete lattice and discussed on relevance of such concepts for representing objects in continuous and digital spaces. Thus introduction of regular open sets raised many topological questions which have led to a productive study in which many new notions have been defined and examined. As a result of which many new properties and characterizations have been introduced. Purpose of this thesis is to investigate more on regular open sets, its types, properties, characterizations and also introduce new open sets via regular open sets and study properties.

In general topology, it is well-known that the arbitrary intersection of regular open sets in a topological space need not be regular open. This observation

motivates the formulation of a new class of open sets that facilitates the study of generalized closure and interior operations beyond the limitations of regular open sets. **Chapter 1** introduces the concept of I_r -open sets, defined as the subsets of a topological space that can be expressed as the intersection of regular open sets. This notion allows for the extension of classical topological properties and the study of new structures derived from regular open sets.

Building upon this foundation, we explore various properties of I_r -open sets and investigate associated topological spaces such as I_r -Alexandroff spaces, where the collection of I_r -open sets forms a base. Further, we define the concept of I_r -cluster points, a point $x \in X$ is an I_r -cluster point of a subset A of a topological space X if every I_r -open set of X containing x intersects A non-trivially. The collection of all I_r -cluster points forms the I_r -closure of A , denoted $I_r-cl(A)$. We provide new characterizations and theorems involving the behavior of I_r -closures under union and intersection, along with illustrative examples and equivalent formulations. For a topological space (X, τ) , the closure of a set A of X with respect to τ is denoted by $cl(A)$.

Similarly, the concept of I_r -interior is developed using I_r -open sets, a point $x \in X$ is said to be an I_r -interior point of A of a topological space X if there exists an I_r -open set U of X such that $x \in U \subset A$. The set of all such points constitutes the I_r -interior of A , denoted by $I_r-int(A)$. We introduce various theorems and properties related to this notion, including its interaction with unions and intersections, and provide the results for its determination. Finally, the relationship between I_r -closure and I_r -interior is examined, along with their

interplay with other topological concepts.

Chapter 2 is devoted to the study of generalized separation axioms and other fundamental topological properties within the framework of I_r -open sets. We begin by introducing and investigating the I_r - T_i separation axioms for $i = 0, 1, 2, 3, 4$, which serve as natural extensions of classical separation axioms through the lens of I_r -open sets. In addition, we define and explore I_r -regular, I_r -normal, almost I_r -normal, and completely I_r -normal spaces. These notions are further defined to new concepts, namely I_r^* -regular, I_r^* -normal, almost I_r^* -normal, and completely I_r^* -normal spaces. A detailed analysis of the structural properties, interrelations, and hereditary characteristics of these spaces is undertaken, supported by appropriate examples and counterexamples.

Subsequently, we introduce the study of I_r -compactness, which extends the classical notion of compactness. The final part of this chapter is dedicated to introduce the concept of I_r -connectedness, which generalizes the classical notion of connectedness by using I_r -open sets. A space X is defined to be I_r -connected if it cannot be expressed as the union of two non-empty, I_r -separated subsets. Similarly, a subset $C \subset X$ is said to be I_r -connected if it is I_r -connected in the subspace topology. If a space is not I_r -connected, it is termed the space is I_r -disconnected. Also introduce the concept of I_r -components, various characterizations and examples are provided to facilitate a deeper understanding of I_r -connected spaces. To support the investigations presented in this chapter, we introduce and explore the concept of the I_r -cover of a set. Given a subset A of a topological space (X, τ) , the I_r -cover of A , denoted

by $I_r(A)$, is defined as the intersection of all regular open sets containing A , $I_r(A) = \bigcap \{V : V \text{ is a regular open set and } A \subset V\}$. This construction plays a fundamental role in understanding generalized separation axioms and their characterizations. Several remarks and results are obtained linking the I_r -cover to I_r -separation axioms. A significant result in this concept establishes that the structure (X, τ_{I_r}) , where $\tau_{I_r} = \{A^c : I_r(A) = A, A \subset X\}$, forms a supra topological space. Moreover, the I_r -cover $I_r(A)$ of a subset A in X coincides with the supra-closure of A with respect to the supra topology τ_{I_r} . This provides a foundational framework for further topological investigations using regular open sets.

The study of generalized closed sets forms a crucial part of modern topology, particularly in the exploration of structures beyond classical open and closed sets. Building upon the foundational notions of I_r -open and I_r -closed sets, in **Chapter 3**, introduces and investigates the concept of I_r^* -closed sets, which serves as a natural extension enabling the formulation of finer generalizations of topological properties. A subset A of a topological space X is said to be I_r^* -closed if there exists an I_r -closed set F such that $A \subset F \subset U$ whenever U is an I_r -open set containing A . This chapter develops several characterizations and structural properties of I_r^* -closed sets and I_r^* -open sets, examining their behavior under arbitrary and finite unions, hereditary nature, and their connection with the I_r -closure operator. Also includes the theorems like every I_r -closed set is I_r^* -closed, while the converse need not hold.

In addition, we study the relationship between I_r^* -closed sets and I_r -compactness,

highlighting the role of I_r^* -closedness in refining classical compactness concepts. The behavior of I_r^* -closed sets under the general set-theoretic operations is thoroughly analyzed. Special attention is given to the class of I_r -Alexandroff spaces, within which alternate and simplified characterizations of I_r^* -closed sets are established.

In **Chapter 4**, we develop the concept of continuity using the framework of I_r -open sets. In classical topology, several well-established relationships exist among continuity, openness, closedness, and invertibility of functions. These connections have been widely introduced and studied. The primary aim of this chapter is to investigate whether similar relationships hold among the various function types introduced in this chapter within the context of I_r -open sets. Let (X, τ) and (Y, σ) be topological spaces, a function $f : X \rightarrow Y$ is said to be *I_r -continuous at a point $x \in X$* if, for every I_r -open set V containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subset V$. If this condition holds for every point in X , then f is called *I_r -continuous*.

In addition to I_r -continuity, we introduce and examine other related function classes, a function $f : X \rightarrow Y$ is called *I_r -irresolute* if the preimage of every I_r -open set in Y is I_r -open in X , a function is said to be *I_r -open* if the image of every open set in X is I_r -open in Y . The chapter further establishes several theorems concerning these function types and various topological properties such as I_r -connectedness and I_r -compactness. Special attention is given to the behavior of these functions in spaces satisfying particular conditions.

In 2001 and 2003, F. Nakaoka and N. Oda introduced and studied *minimal*

open (respectively, *minimal closed*) sets and *maximal open* (respectively, *maximal closed*) sets, which form subclasses of open (respectively, closed) sets. The complement of a minimal open set is a maximal closed set, and the complement of a maximal open set is a minimal closed set. These concepts enrich the classical notions of openness and closedness by introducing minimal and maximal conditions. S. S. Benchali, Basavaraj Ittanagi and R. S. Wali studied on minimal open sets and functions in topological spaces. Subsequently in 1937, Stone introduced the notion of regular open sets, which provided a foundation for the development of the semi-regularization of a topological space. Building on these ideas, in 2015 N. Anuradha and Baby Chacko [5] defined the concepts of minimal regular open sets and maximal regular closed sets, which extended the study of open and closed sets within the topological space. **Chapter 5** aims to investigate and establish certain properties of these sets. we provide examples and theorems concerning the minimal and maximal regular open sets.

Gentre and Hoyle [27] explored the concepts of somewhat continuous functions and somewhat open functions. In **Chapter 6**, we introduce a new class of functions, namely somewhat *mr*-continuous and somewhat *mr*-open functions, explores their properties and provide characterizations. The relationship between somewhat *mr*-continuous functions and other types of functions are studied, along with a discussion on *mr*-weakly stronger topologies.

Earlier, in 1963, J. C. Kelly [23] defined a *bitopological space* (X, τ_1, τ_2) as a set X equipped with two topologies τ_1 and τ_2 . Bitopological spaces provide studies on core topological concepts such as closure, interior, continuity, and convergence

by examining how these notions behave and interact under two different topologies defined on the same set. In 2012, S. S. Benchalli, Basavaraj M. Ittanagi, and R. S. Wali [10] introduced the concepts of *pairwise minimal open sets* (respectively (i, j) -minimal open sets) and *pairwise maximal open sets* (respectively (i, j) -maximal open sets) in bitopological spaces. These notions further generalized the idea of minimality and maximality of sets relative to each topology in the pair. In **Chapter 7**, we introduce and investigate *pairwise minimal regular open sets* (respectively (i, j) -minimal regular open sets) and *pairwise maximal regular open sets* (respectively (i, j) -maximal regular open sets) in bitopological spaces. We explore their properties and analyze how they relate to other types of open sets within the bitopological space. We also introduce and examine new separation axioms, namely $Pairwise-rT_{\min}$ and $Pairwise-rT_{\max}$ spaces, as well as the concepts of (i, j) -*r-door spaces* and (i, j) - $rT_{\frac{1}{2}}$ -spaces. These spaces provide further insights into the structure and behavior of bitopological spaces under regular open and closed conditions. In addition, we define and study several types of continuous functions such as (i, j) -almost continuous, (i, j) -almost completely continuous, and (i, j) -almost perfectly continuous functions. We also consider functions defined with respect to a single topology, such as i -completely continuous, i -strongly continuous, i -almost continuous, i -almost completely continuous, i -almost perfectly continuous, and i -totally continuous functions. The relationships among these classes of functions are thoroughly explored. Finally, we introduce two new classes of functions p -Min r -Continuous Maps and p -Max r -Continuous Maps and study their properties and interconnections with the existing classes of functions in bitopological spaces. These investigations aim

to enhance the understanding of continuity in bitopological space. Throughout this chapter a subset A of a bitopological space (X, τ_1, τ_2) is said to be τ_i -open (regular open, minimal open, minimal regular open respectively) if it open (regular open, minimal open, minimal regular open respectively) with respect to the topology τ_i for $i \in \{1, 2\}$. Also interior(closure respectively) of A with respect to the topology $\tau_i, i \in \{1, 2\}$ denoted by $i\text{-int}(A)$ ($i\text{-cl}(A)$ respectively).

In the final chapter, chapter 8, the conclusion of the work is stated. This chapter also put forward certain proposals for further research. Throughout this thesis $\mathcal{P}(A)$ denote the power set of a set A .

Chapter 1

I_r -Open Sets and Derived Concepts in Topological Space

1.1 Introduction

In general topology, the arbitrary intersection of regular open sets need not be regular open. This motivates the introduction of I_r -open sets, defined as subsets of a topological space that can be expressed as intersections of regular open sets. In this chapter we examine properties of I_r -open sets and define associated spaces such as I_r -Alexandroff spaces, where the collection of I_r -open sets forms a base. The notion of I_r -cluster points leads to the definition of the I_r -closure, denoted $I_r-cl(A)$, while the dual concept of I_r -interior, denoted $I_r-int(A)$, is also introduced. We present theorems and characterizations related to both operations, including their behavior under unions and intersections.

Some preliminary definitions are introduced to facilitate understanding of the main topic:

Definition 1.1.1. [48] Let (X, τ) be a topological space. Then a subset A of (X, τ) is called *regular open* if $A = \text{int}(\text{cl}(A))$ and *regular closed* if $A = \text{cl}(\text{int}(A))$.

Definition 1.1.2. [31] If (X, τ) is a topological space, then the *semi-regularization topology* τ_s on X of τ is the coarser topology on X generated by the family of all open domains of (X, τ) where a subset U is called an *open domain* if $U = \text{int}(\overline{U})$. Then (X, τ_s) is called *semi-regularization space of a space* (X, τ) .

Definition 1.1.3. [46] A topological space (X, τ) is said to be an *Alexandroff space* if arbitrary intersection of open sets is open.

1.2 I_r -Open Sets and Related Topological Spaces

Definition 1.2.1. Let (X, τ) be a topological space. A set $A \subset X$ is called *I_r -open* if A can be expressed as the intersection of a family of regular open sets. The complement of *I_r -open* set is called *I_r -closed*.

Example 1.2.1. In the space $\mathbb{R}$ with usual topology, every singleton set is *I_r -open*, since for any $x \in \mathbb{R}$, $\{x\} = \bigcap_{i=1}^{\infty} (x - 1/n, x + 1/n)$. Also for any $a, b \in \mathbb{R}$, $[a, b] = \bigcap_{i=1}^{\infty} (a - 1/n, b + 1/n)$ is *I_r -open*.

Definition 1.2.2. A subset $A \subset X$ is said to be *I_r -clopen* if it is both *I_r -open* and *I_r -closed*.

Example 1.2.2. In the space $\mathbb{R}$ with usual topology, since $[a, b] = \bigcap_{i=1}^{\infty} (a - 1/n, b + 1/n)$ for any $a, b \in \mathbb{R}$, I_r -open. Also since $[a, b]$ is a regular closed set, it is I_r -closed. Therefore $[a, b]$ is I_r -clopen.

Theorem 1.2.1. *Arbitrary intersection of I_r -open sets in a topological space is I_r -open.*

Proof. Let (X, τ) be a topological space and $\{A_i : i \in I\}$ be the collection of I_r -open sets in X . Then by the definition of I_r -open sets, for $i \in I$, there exists family $\{B_{i,j} : J_i\}$ of regular open sets such that $A_i = \bigcap_{j \in J_i} B_{i,j}$. Since $\bigcap_{i \in I} A_i = \bigcap_{(i,j) \in (I, J_i)} B_{i,j}$, $\bigcap_{i \in I} A_i$ is I_r -open in X .

Corollary 1.2.1.1. *Arbitrary union of I_r -closed sets in a topological space is I_r -closed set*

Proof. Proof follows by using 1.2.1 and De Morgan's Law.

Remark. Let (X, τ) be a topological space. Then

1. ϕ and X are I_r -open.
2. Arbitrary intersection of I_r -open sets is I_r -open.
3. Arbitrary union of I_r -closed sets is I_r -closed set.
4. Arbitrary union of regular closed sets is I_r -closed, since every regular closed set is I_r -closed.

5. Union of two I_r -open sets need not be I_r -open, since union of two regular open sets need not be regular open.
6. Every regular open set is I_r -open.

Corresponding to every topological space (X, τ) , there exists a topology τ_r on X with the collection of all I_r -open sets is a base for τ_r . This topology is finer than the semi regularisation topology τ_s .

Definition 1.2.3. Let (X, τ) be a topological space. Then the space (X, τ_r) is called *I_r -Alexandroff space corresponding to τ* . A topological space (X, τ) is said to be *I_r -Alexandroff* if $(X, \tau) = (X, \tau_r)$.

Example 1.2.3. I_r -Alexandroff space corresponding to $\mathbb{R}$ with usual topology is the discrete space.

Lemma 1.2.2. *Any discrete topological space is an I_r -Alexandroff space.*

Proof. Proof follows from the fact that any subset of a discrete space is I_r -open.

Theorem 1.2.3. *Let (X, τ) be metric space, then X is I_r -Alexandroff if and only if X has the discrete topology.*

Proof. Suppose X is an I_r -Alexandroff space. For each $x \in X$, $\{x\} = \bigcap_{n \in \mathbb{N}} B(x, \frac{1}{n})$, where B is open ball with centre x and radius $\frac{1}{n}$, regular open, then $\{x\}$ is I_r -open. Since X is an I_r -Alexandroff space, τ is a discrete topology. Converse follows from the above lemma

Example 1.2.4. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. The set of all regular open sets in X is the set $\{\phi, \{a\}, \{b\}, X\}$. Then $\tau_s = \tau_r = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$.

Remark. In a finite topological space, I_r -open sets coincide with regular open sets and hence τ_r coincide with τ_s .

Lemma 1.2.4. *If C is an I_r -closed set and B is an I_r -open set in a topological space X , then $B - C$ is an I_r -open set.*

Proof. If C is I_r -closed, then $X - C$ is I_r -open. Since $B - C = B \cap (X - C)$, it is I_r -open.

1.3 I_r -Closure of a Set in Topological Space

Definition 1.3.1. Let (X, τ) be a topological space and A be a subset of X . A point $x \in X$ is called an I_r -cluster point of A if $A \cap U \neq \phi$ for every I_r -open subset U of X containing x . The set of all I_r -cluster points of A is called the I_r -closure of A , denoted by $I_r\text{-cl}(A)$.

Example 1.3.1. Let $X = \mathbb{R}$ be equipped with the Sorgenfrey topology (lower limit topology) τ , $\tau = \{[a, b) : a < b, a, b \in \mathbb{R}\}$. Consider the subset $A = [0, 1) \subset X$. In this topology, the set $[1, 2)$ is regular open and hence I_r -open. Since $[1, 2) \cap A = \phi$ and $1 \in [1, 2)$, it follows that $1 \notin I_r\text{-cl}(A)$. Hence $I_r\text{-cl}(A) = [0, 1)$.

Proposition 1.3.1. *Let (X, τ) be a topological space and $A \subset X$. Then $I_r\text{-cl}(A) = \bigcap \{F \subset X : A \subset F \text{ and } F \text{ is } I_r\text{-closed}\}$.*

Proof. Let $C = \bigcap \{F \subset X : A \subset F \text{ and } F \text{ is } I_r\text{-closed}\}$. We have to show that $I_r\text{-cl}(A) = C$. Let $x \in I_r\text{-cl}(A)$. Then, by definition, for every I_r -open set U containing x , we have $A \cap U \neq \phi$. If possible suppose that $x \notin C$. Then $x \notin F$ for some I_r -closed set F containing A . Then $x \in X - F$, which is an I_r -open set containing x , and $A \cap (X - F) = \phi$, since $A \subset F$. This contradicts the assumption that every I_r -open containing x intersects A . Hence $x \in C$.

Let $x \in C$. If possible suppose that $x \notin I_r\text{-cl}(A)$. Then there exists an I_r -open set U such that $x \in U$ and $A \cap U = \phi$. Then $F = X - U$ is I_r -closed set containing A . Then $x \notin F$, this is a contradiction to the assumption that $x \in C$. Therefore, $I_r\text{-cl}(A) = \bigcap \{F \subset X : A \subset F \text{ and } F \text{ is } I_r\text{-closed}\}$.

Theorem 1.3.2. *Let (X, τ) be a topological space and $A \subset X$. Then $I_r\text{-cl}(A) \subset F$ for every I_r -closed set F containing A .*

Proof. Proof follows from the proposition 1.3.1.

Theorem 1.3.3. *Let (X, τ) be a topological space. If $A \subset B \subset X$, then $I_r\text{-cl}(A) \subset I_r\text{-cl}(B)$.*

Proof. Let $A \subset B$ and $x \in I_r\text{-cl}(A)$. Then, by definition, for every I_r -open set U containing x , $U \cap A \neq \phi$. Since $A \subset B$, it follows that $U \cap A \subset U \cap B$, and

hence $U \cap B \neq \phi$. This implies that x is also an I_r -cluster point of B , and so $x \in I_r\text{-cl}(B)$. Therefore $I_r\text{-cl}(A) \subset I_r\text{-cl}(B)$.

Theorem 1.3.4. *Let (X, τ) be a topological space and let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of subsets of X . Then $I_r\text{-cl}(\bigcup_{\alpha \in \Lambda} A_\alpha) = \bigcup_{\alpha \in \Lambda} I_r\text{-cl}(A_\alpha)$.*

Proof. Let $x \in I_r\text{-cl}(\bigcup_{\alpha \in \Lambda} A_\alpha)$. Then for every I_r -open set U containing x , $U \cap (\bigcup_{\alpha \in \Lambda} A_\alpha) \neq \phi$. So there exists some $\alpha_0 \in \Lambda$ such that $U \cap A_{\alpha_0} \neq \phi$, which implies $x \in I_r\text{-cl}(A_{\alpha_0})$. Therefore $x \in \bigcup_{\alpha \in \Lambda} I_r\text{-cl}(A_\alpha)$ and hence $I_r\text{-cl}(\bigcup_{\alpha \in \Lambda} A_\alpha) \subset \bigcup_{\alpha \in \Lambda} I_r\text{-cl}(A_\alpha)$.

Conversely, let $x \in \bigcup_{\alpha \in \Lambda} I_r\text{-cl}(A_\alpha)$. Then $x \in I_r\text{-cl}(A_{\alpha_0})$ for some $\alpha_0 \in \Lambda$. Hence every I_r -open set U containing x satisfies $U \cap A_{\alpha_0} \neq \phi$. Since $A_{\alpha_0} \subset \bigcup_{\alpha \in \Lambda} A_\alpha$, $U \cap (\bigcup_{\alpha \in \Lambda} A_\alpha) \neq \phi$, this implies that $x \in I_r\text{-cl}(\bigcup_{\alpha \in \Lambda} A_\alpha)$. Therefore $\bigcup_{\alpha \in \Lambda} I_r\text{-cl}(A_\alpha) \subset I_r\text{-cl}(\bigcup_{\alpha \in \Lambda} A_\alpha)$. Hence $I_r\text{-cl}(\bigcup_{\alpha \in \Lambda} A_\alpha) = \bigcup_{\alpha \in \Lambda} I_r\text{-cl}(A_\alpha)$.

Corollary 1.3.4.1. *Let (X, τ) be a topological space. For any subsets $A, B \subset X$, $I_r\text{-cl}(A \cup B) = I_r\text{-cl}(A) \cup I_r\text{-cl}(B)$.*

Theorem 1.3.5. *Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of subsets of a topological space (X, τ) . Then $I_r\text{-cl}(\bigcap_{\alpha \in \Lambda} A_\alpha) \subset \bigcap_{\alpha \in \Lambda} I_r\text{-cl}(A_\alpha)$.*

Proof. Let $x \in I_r\text{-cl}(\bigcap_{\alpha \in \Lambda} A_\alpha)$. Then for every I_r -open set U containing x , $U \cap (\bigcap_{\alpha \in \Lambda} A_\alpha) \neq \phi$. This implies that for each $\alpha \in \Lambda$, $U \cap A_\alpha \neq \phi$. Hence, $x \in I_r\text{-cl}(A_\alpha)$ for all $\alpha \in \Lambda$, and so $x \in \bigcap_{\alpha \in \Lambda} I_r\text{-cl}(A_\alpha)$. Therefore, $I_r\text{-cl}(\bigcap_{\alpha \in \Lambda} A_\alpha) \subset \bigcap_{\alpha \in \Lambda} I_r\text{-cl}(A_\alpha)$.

Remark. Reverse inclusion of the theorem 1.3.5 need not be true.

Example 1.3.2. Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{c\}, \{a, b\}, X\}$. The regular open sets in this topology are $\phi, \{c\}, \{a, b\}$ and X , so the set of I_r -open sets is $\{\phi, \{c\}, \{a, b\}, X\}$. Consider the sets $A = \{a, c\}$ and $B = \{b, c\}$, then $A \cap B = \{c\}$. We have $I_r\text{-cl}(A) = I_r\text{-cl}(B) = X$. But $I_r\text{-cl}(A \cap B) = I_r\text{-cl}(\{c\}) = \{c\}$. Thus, $I_r\text{-cl}(A \cap B) = \{c\} \subsetneq X = I_r\text{-cl}(A) \cap I_r\text{-cl}(B)$. Therefore reverse inclusion fails in general.

Theorem 1.3.6. *Let (X, τ) be a topological space and $A \subset X$. If A is I_r -closed, then $I_r\text{-cl}(A) = A$.*

Proof. Let A be an I_r -closed set in X and let $x \in I_r\text{-cl}(A)$. If possible assume that $x \notin A$. Then $x \in X - A$ and since $X - A$ is an I_r -open set containing x and x is an I_r -cluster point of A , $(X - A) \cap A \neq \phi$. This is a contradiction. Therefore $x \in A$ and hence $I_r\text{-cl}(A) \subset A$. Further, since $A \subset I_r\text{-cl}(A)$, $I_r\text{-cl}(A) = A$.

Theorem 1.3.7. *Let (X, τ) be an I_r -Alexandroff space and $A \subset X$. If $I_r\text{-cl}(A) = A$, then A is I_r -closed.*

Proof. Let (X, τ) be an I_r -Alexandroff space, $A \subset X$ and $I_r\text{-cl}(A) = A$. Let $x \in X - A$. Then $x \notin I_r\text{-cl}(A)$. Since x is not an I_r -cluster point of A , there exists an I_r -open set V_x containing x such that $V_x \cap A = \phi$. This implies that $V_x \subset X - A$ for every $x \in X - A$. Hence $\bigcup_{x \in (X - A)} V_x \subset X - A$. Also since $x \in V_x$ for every $x \in X - A$, $X - A \subset \bigcup_x V_x$. Therefore $X - A = \bigcup_x V_x$. Since X is an I_r -Alexandroff space, $X - A$ is I_r -open and hence A is I_r -closed.

Lemma 1.3.8. *Let (X, τ) be a topological space. Then $I_r\text{-cl}(I_r\text{-cl}(A)) = I_r\text{-cl}(A)$, for every subset A of X .*

Proof. *Proof.* We have $I_r(A) \subset I_r\text{-cl}(I_r\text{-cl}(A))$.

Now $x \notin I_r(A)$, this implies that there is an I_r -open set U containing x such that $U \cap A = \emptyset$. We have to prove that $U \cap I_r\text{-cl}(A) = \emptyset$. If possible suppose that $U \cap I_r\text{-cl}(A) \neq \emptyset$. Then there is an element $y \in U \cap I_r\text{-cl}(A)$, implies that $y \in I_r\text{-cl}(A)$ and $y \in U$, U is an I_r -open set, then $U \cap A \neq \emptyset$, this is a contradiction. Therefore $U \cap I_r\text{-cl}(A) = \emptyset$. This implies that $x \notin I_r\text{-cl}(I_r\text{-cl}(A))$. Therefore $I_r\text{-cl}(I_r\text{-cl}(A)) \subset I_r\text{-cl}(A)$. Hence $I_r\text{-cl}(I_r\text{-cl}(A)) = I_r\text{-cl}(A)$.

Theorem 1.3.9. *Let (X, τ) be an I_r -Alexandroff topological space. Then $I_r\text{-cl}(A)$ is an I_r -closed set for every subset A of X .*

Proof. Proof follows from the lemma 1.3.8 and the theorem 1.3.7, $I_r\text{-cl}(A)$ is I_r -closed. □

Theorem 1.3.10. *Let (X, τ) be an I_r -Alexandroff space and $A \subset X$. Then $I_r\text{-cl}(A)$ is the smallest I_r -closed set containing A .*

Proof. Proof follows from the theorem 1.3.2 $I_r\text{-cl}(A) \subset F$, for every I_r -closed set containing A and the theorem 1.3.9, $I_r\text{-cl}(A)$ is I_r -closed. Therefore $I_r\text{-cl}(A)$ is the smallest I_r -closed set containing A . □

Theorem 1.3.11. *Let (X, τ) be a topological space and A be a subset of X . Then the following statements are equivalent:*

(i) A contains all of its I_r -cluster points.

(ii) The intersection of all I_r -closed sets containing A is A itself.

(iii) The union of all I_r -open sets contained in A^C is A^C .

Proof.

(i) $\implies$ (ii)

Suppose that A contains all of its I_r -cluster points and let $\{R_\alpha\}_{\alpha \in J}$ be the collection of all I_r -closed sets containing A . Obviously $A \subset \bigcap_{\alpha \in J} R_\alpha$. On the other hand, let $x \in \bigcap_{\alpha \in J} R_\alpha$. Then we claim that $U \cap A \neq \emptyset$ for any I_r -open set U such that $x \in U$. To prove this, if possible suppose that $U \cap A = \emptyset$. Then $A \subset U^C$ where U^C is I_r -closed. Hence $U^C = R_{\alpha_0}$ for some $\alpha_0 \in J$, so that $x \in U^C$. This contradicts the fact that $x \in U$. Hence $U \cap A \neq \emptyset$. This implies that x is an I_r -cluster point of A . Since A contains all of its I_r -cluster points, $x \in A$ and therefore $\bigcap_{\alpha \in J} R_\alpha \subset A$. Hence $A = \bigcap_{\alpha \in J} R_\alpha$.

(ii) $\implies$ (i)

Suppose that $A = \bigcap_{\alpha \in J} R_\alpha$ where $\{R_\alpha\}_{\alpha \in J}$ is the collection of all I_r -closed sets containing A . Let x be an I_r -cluster point of A . If possible suppose that $x \notin A$. Then $x \notin \bigcap_{\alpha \in J} R_\alpha$. This implies that $x \notin R_{\alpha_0}$ for some $\alpha_0 \in J$ and $x \in R_{\alpha_0}^C$. Since R_{α_0} is an I_r -closed set containing A , $R_{\alpha_0}^C$ is an I_r -open set containing x . Since x is an I_r -cluster point of A , we have $R_{\alpha_0}^C \cap A \neq \emptyset$. This contradicts the fact that $A \subset R_{\alpha_0}$. Hence $x \in A$. Since x is arbitrary, A contains all of its I_r -cluster points.

$$(ii) \iff (iii)$$

The proof follows from the De Morgan's laws. $\square$

1.4 I_r -Interior of a Set in Topological Space

Definition 1.4.1. Let (X, τ) be a topological space and A be a subset of X .

A point $x \in X$ is called an I_r -interior point of A if there exists an I_r -open set U such that $x \in U \subset A$. The set of all I_r -interior points of A is called the I_r -interior of A , denoted by $I_r\text{-int}(A)$.

Example 1.4.1. Let $X = \mathbb{Q}$ with the co-countable topology, $\tau = \{U : U^c \text{ is countable}\}$. Consider the subset $\mathbb{N} \subset \mathbb{Q}$, $\mathbb{N}$ is regular open in this space. Then $I_r\text{-int}(\mathbb{N}) = \mathbb{N}$.

Proposition 1.4.1. Let (X, τ) be a topological space and $A \subset X$. Then $I_r\text{-int}(A) = \bigcup\{U \subset A : U \text{ is } I_r\text{-open in } X\}$.

Proof. Let $B = \bigcup\{U \subset A : U \text{ is } I_r\text{-open in } X\}$. We have to show that $B = I_r\text{-int}(A)$. Let $x \in B$. Then there exists an I_r -open set U such that $x \in U \subset A$. By the definition of I_r -interior point, $x \in I_r\text{-int}(A)$. Thus $B \subset I_r\text{-int}(A)$. Also let $x \in I_r\text{-int}(A)$. Then, by definition of I_r -interior, there exists an I_r -open set U such that $x \in U \subset A$. Then U is one of the sets in the union defining B , so $x \in B$. Hence, $I_r\text{-int}(A) \subset B$. Therefore, $I_r\text{-int}(A) = \bigcup\{U \subset A : U \text{ is } I_r\text{-open in } X\}$. $\square$

Theorem 1.4.2. *Let (X, τ) be a topological space and $A \subset X$. Then $U \subset I_r\text{-int}(A)$ for every I_r -open set U contained in A .*

Proof. Proof follows from the proposition 1.4.1. □

Theorem 1.4.3. *Let (X, τ) be a topological space and $A \subset X$. If A is I_r -open, then $I_r\text{-int}(A) = A$.*

Proof. Proof follows from the proposition 1.4.1. □

Theorem 1.4.4. *Let (X, τ) be a topological space and let $A, B \subset X$ such that $I_r\text{-int}(A) \subset B \subset A$. If A is I_r -open, then B is also I_r -open.*

Proof. Assume A is I_r -open. Then by theorem 1.4.3, $I_r\text{-int}(A) = A$. Since $I_r\text{-int}(A) \subset B \subset A$, we have $A \subset B \subset A$, which implies $B = A$. Hence B is I_r -open. □

Theorem 1.4.5. *For any subset A of a topological space X , $I_r\text{-int}(A) \subset A \subset I_r\text{-cl}(A)$.*

Proof. Proof follows from the proposition 1.4.1 and the proposition 1.3.1. □

Theorem 1.4.6. *Let (X, τ) be a topological space and $A, B \subset X$. Then, $I_r\text{-int}(A) \subset I_r\text{-int}(B)$.*

Proof. Let $x \in I_r\text{-int}(A)$. Then, by definition, there exists an I_r -open set U such that $x \in U \subset A$. Since $A \subset B$, it follows that $U \subset B$. Therefore, $x \in U \subset B$, which implies that x is an I_r -interior point of B . Thus, $x \in I_r\text{-int}(B)$. Hence,

every point of $I_r\text{-int}(A)$ is also a point of $I_r\text{-int}(B)$. Therefore $I_r\text{-int}(A) \subset I_r\text{-int}(B)$. $\square$

Corollary 1.4.6.1. *For any subset A of a topological space X , $I_r\text{-int}(A) \subset I_r\text{-int}(I_r\text{-cl}(A))$.*

Proof. Proof follows from the fact that $A \subset I_r\text{-cl}(A)$ and by the theorem 1.4.6. $\square$

Theorem 1.4.7. *Let (X, τ) be a topological space and let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of subsets of X . Then $I_r\text{-int}(\bigcap_{\alpha \in \Lambda} A_\alpha) = \bigcap_{\alpha \in \Lambda} I_r\text{-int}(A_\alpha)$.*

Proof. Let $x \in I_r\text{-int}(\bigcap_{\alpha \in \Lambda} A_\alpha)$. Then there exists an I_r -open set U such that $x \in U \subset (\bigcap_{\alpha \in \Lambda} A_\alpha)$. Since $U \subset (\bigcap_{\alpha \in \Lambda} A_\alpha)$, it follows that $U \subset A_\alpha$ for each $\alpha \in \Lambda$. Hence, $x \in I_r\text{-int}(A_\alpha)$ for each $\alpha \in \Lambda$ and therefore $\bigcap_{\alpha \in \Lambda} I_r\text{-int}(A_\alpha)$. Thus $I_r\text{-int}(\bigcap_{\alpha \in \Lambda} A_\alpha) \subset \bigcap_{\alpha \in \Lambda} I_r\text{-int}(A_\alpha)$.

Conversely, let $x \in \bigcap_{\alpha \in \Lambda} I_r\text{-int}(A_\alpha)$. Then $x \in I_r\text{-int}(A_\alpha)$ for each $\alpha \in \Lambda$, there exist I_r -open sets U_α such that $x \in U_\alpha \subset A_\alpha$ for each $\alpha \in \Lambda$. Then $x \in \bigcap_{\alpha \in \Lambda} U_\alpha \subset \bigcap_{\alpha \in \Lambda} A_\alpha$ and $\bigcap_{\alpha \in \Lambda} U_\alpha$ is I_r -open since the intersection of I_r -open sets is I_r -open. This implies $x \in I_r\text{-int}(\bigcap_{\alpha \in \Lambda} A_\alpha)$. Hence, $\bigcap_{\alpha \in \Lambda} I_r\text{-int}(A_\alpha) \subset I_r\text{-int}(\bigcap_{\alpha \in \Lambda} A_\alpha)$. Therefore $I_r\text{-int}(\bigcap_{\alpha \in \Lambda} A_\alpha) = \bigcap_{\alpha \in \Lambda} I_r\text{-int}(A_\alpha)$. $\square$

Corollary 1.4.7.1. *Let (X, τ) be a topological space. For any subsets $A, B \subset X$, $I_r\text{-int}(A \cap B) = I_r\text{-int}(A) \cap I_r\text{-int}(B)$.*

Theorem 1.4.8. *Let (X, τ) be a topological space. For any subsets $A, B \subset X$, $I_r\text{-int}(A) \cup I_r\text{-int}(B) \subset I_r\text{-int}(A \cup B)$.*

Proof. Let $x \in I_r\text{-int}(A) \cup I_r\text{-int}(B)$. Then either $x \in I_r\text{-int}(A)$ or $x \in I_r\text{-int}(B)$. Without loss of generality, assume $x \in I_r\text{-int}(A)$. Then there exists an I_r -open set U such that $x \in U \subset A$. Since $A \subset A \cup B$, $U \subset A \cup B$, this implies that $x \in I_r\text{-int}(A \cup B)$. The same argument applies if $x \in I_r\text{-int}(B)$, then $x \in I_r\text{-int}(A \cup B)$. Hence $I_r\text{-int}(A) \cup I_r\text{-int}(B) \subset I_r\text{-int}(A \cup B)$. $\square$

Remark. Converse of the theorem 1.4.8 need not be true.

Example 1.4.2. Let $X = \mathbb{R}$ with the finite complement topology,

$\tau = \phi \cup \{U \subset \mathbb{R} : X - U \text{ is finite}\}$. The only regular open sets are ϕ and X , So the only I_r -open sets are ϕ and X . Hence, for any subset $A \subset X$,

$$I_r\text{-int}(A) = \begin{cases} X & \text{if } A = X \\ \phi & \text{otherwise} \end{cases}.$$

Let $A = \mathbb{Q}$ and $B = \mathbb{R} - \mathbb{Q}$. So $A \cup B = \mathbb{R} = X$. Then $I_r\text{-int}(A) = \phi$, $I_r\text{-int}(B) = \phi$ and $I_r\text{-int}(A \cup B) = I_r\text{-int}(X) = X$. That is $I_r\text{-int}(A) \cup I_r\text{-int}(B) = \phi$. Thus, $I_r\text{-int}(A \cup B) \not\subset I_r\text{-int}(A) \cup I_r\text{-int}(B)$.

Chapter 2

Separation Axioms with respect to I_r -Open Sets in Topological Space

2.1 Introduction

The concept of I_r -open sets is utilized to establish a generalized framework for classical topological notions. In this chapter, we introduce and study the generalized separation axioms, namely the I_r - T_i spaces for $i = 0, 1, 2, 3, 4$ along with I_r -regular, I_r -normal, almost I_r -normal and completely I_r -normal spaces. Also I_r^* -regular, I_r^* -normal, almost I_r^* -normal and completely I_r^* -normal spaces. Various structural properties and characterizations of these separation axioms are explored, including their hereditary nature. In addition, we initiate the study of I_r -compactness with respect to I_r -open covers, leading to a broader generalization of classical compactness in topological spaces. Moreover, we develop

the notion of I_r -connectedness, which extends the classical idea of connectedness using the behavior of I_r -open sets. Also we introduce and explore the concept of the I_r -cover of a set.

Before delving into the main topic, we present a few key definitions for clarity:

Definition 2.1.1. [13] A topological space (X, τ) is said to be *locally indiscrete* if every open set in it is also closed.

Definition 2.1.2. [32] A topological space X is said to be *compact* if every open cover of X has a finite subcover. That is, if $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is a collection of open sets such that $X \subset \bigcup_{\alpha \in A} U_\alpha$, then there exists a finite subcollection $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\} \subset \mathcal{U}$ such that $X \subset \bigcup_{i=1}^n U_{\alpha_i}$. A subset $A \subset X$ is said to be **compact** if every open cover of A by open sets in X has a finite subcover.

Definition 2.1.3. [44] A topological space (X, τ) is *nearly compact* if and only if each regular open cover admits a finite subcover.

Definition 2.1.4. A topological space X is said to be

- (i) δT_0 [26] , if for each pair of distinct points x and y in X , there exists a regular open set which contains one of the points x and y , but not the other.
- (ii) δT_1 or rT_1 (respectively clopen T_1) [14], if for each pair of distinct points x and y in X , there exist regular open sets (respectively clopen sets) U and V containing x and y respectively such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Definition 2.1.5. [4] A topological space X is said to be

- (i) rT_2 , if every two distinct points of X can be separated by disjoint regular open sets.
- (ii) r -regular, if for each closed set F of X and a point $x \notin F$, there exist disjoint regular open sets U and V such that $F \subset U$ and $x \in V$.
- (iii) r -normal, if each pair of non-empty disjoint closed sets can be separated by disjoint regular open sets.

Definition 2.1.6. [40] A topological space (X, τ) is said to be *Nearly Hausdorff* if every pair of distinct points of X there exists a pair of regular closed sets separating them.

Definition 2.1.7. [22] A topology τ on a set X is called the *discrete topology* if every subset of X is open. That is, $\tau = \mathcal{P}(X)$.

Definition 2.1.8. [22] A topology τ on a set X is called the *indiscrete topology* if $\tau = \{\phi, X\}$. That is, the only open sets are ϕ and X .

Definition 2.1.9. [32] Let (X, τ) be a topological space. Then two subsets $A, B \subset X$ are called *separated* if $cl(A) \cap B = \phi$ and $A \cap cl(B) = \phi$.

Definition 2.1.10. [1] Let X be a non-empty set and let $\Sigma \subset \mathcal{P}(X)$. The collection Σ is called a *supra topology* on X if:

- (i) $X \in \Sigma$,
- (ii) Σ is closed under arbitrary unions. That is, for any family $\{U_i\}_{i \in I} \subset \Sigma$, we have $\bigcup_{i \in I} U_i \in \Sigma$.

Then (X, Σ) is called a *supra topological space*. Members of Σ are called *supra open sets*, and their complements in X are called *supra closed sets*.

Definition 2.1.11. [1] Let (X, Σ) be a supra-topological space. Define $\sigma^c = \{F \subset X : X \setminus F \in \Sigma\}$, the collection of all supra-closed subsets. Then for any $A \subset X$, the *supra-closure* of A is defined by $scl(A) = \bigcap \{F : A \subset F, F \in \sigma^c\}$. That is, $scl(A)$ is the smallest supra-closed set containing A .

Definition 2.1.12. [1] Let $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ be a mapping. Then c is said to be a *supra closure operator* if it satisfies the following axioms:

1. $c(\phi) = \phi$,
2. $A \subset c(A)$ for all $A \in \mathcal{P}(X)$,
3. $c(A) \cup c(B) \subset c(A \cup B)$ for all $A, B \in \mathcal{P}(X)$,
4. $c(c(A)) = c(A)$ for all $A \in \mathcal{P}(X)$.

Theorem 2.1.1. [1] Let X be a nonempty set and let the mapping $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ be a supra closure operator. Then the collection $\tau = \{G \subset X : c(G^c) = G^c\}$ is a supra topology on X induced by the supra closure operator c .

2.2 I_r -Separation axioms

Definition 2.2.1. A topological space (X, τ) is said to be

2.2. I_r -Separation axioms

- (i) I_rT_0 , if for each pair of distinct points x and y in X , there is an I_r -open set which contains one of them but not the other.
- (ii) I_rT_1 , if for each pair of distinct points x and y in X , there exists an I_r -open set U containing x but not y and an I_r -open set V containing y but not x .
- (iii) I_rT_2 (I_r -Hausdorff) , if for each pair of distinct points x and y in X , there exist an I_r -open set U containing x and an I_r -open set V containing y such that $U \cap V = \phi$.

Example 2.2.1. Let $X = \mathbb{N} \cup \{0\}$, and define a topology τ on X by $\tau = \{X\} \cup \{U \subset X : X - U \text{ is infinite}\}$.

For any $x, y \in X$ with $x \neq y$. Without loss of generality, $x < y$. Take $U = \{t \in X : t \equiv 0 \pmod{y}\}$ and $V = \{t \in X : t \equiv x \pmod{y}\}$. Then U and V are I_r -open sets such that $y \in U$ and $x \in V$, $U \cap V = \phi$. Then (X, τ) is an infinite I_rT_2 topological space. It is also I_rT_1 and I_rT_0 topological space.

Theorem 2.2.1. *Let (X, τ) be a topological space satisfying any of the separation axioms I_rT_0 , I_rT_1 , or I_rT_2 . Then every subspace $Y \subset X$, with the subspace topology, also satisfies the corresponding I_rT_i separation axiom for each $i = 0, 1, 2$.*

Proof. Let $Y \subset X$ and suppose X satisfies one of the I_rT_i conditions for $i = 0, 1, 2$. Let $x, y \in Y$ with $x \neq y$. Since X is I_rT_i , there exist I_r -open sets in X satisfying the corresponding separation condition. If U is an I_r -open set in X , then $U \cap Y$ is I_r -open in Y under the subspace topology.

(i) If X is I_rT_0 , then there exists an I_r -open set U contain one of them but not the other, without loss of generality $x \in U$, $y \notin U$. Then $U \cap Y$ is an I_r -open set in Y containing x but not y , so Y is I_rT_0 .

(ii) If X is I_rT_1 , there exist I_r -open sets U and V in X such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Then $U \cap Y$ and $V \cap Y$ are I_r -open sets in Y such that $x \in U \cap Y$, $y \notin U \cap Y$ and $y \in V \cap Y$, $x \notin V \cap Y$. Therefore Y is I_rT_1 .

(iii) If X is I_rT_2 , there exist disjoint I_r -open sets U and V in X such that $x \in U$, $y \in V$, and $U \cap V = \phi$. Then $U \cap Y$ and $V \cap Y$ are disjoint I_r -open sets in Y containing x and y , respectively. Hence, Y is I_rT_2 . $\square$

Theorem 2.2.2. *In a topological space (X, τ) , the I_r -separation axioms satisfy the implications*

$$I_rT_2 \Rightarrow I_rT_1 \Rightarrow I_rT_0.$$

Proof. (i) $I_rT_2 \Rightarrow I_rT_1$: Suppose that X is I_rT_2 . Then for any pair of distinct points $x, y \in X$, there exist I_r -open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \phi$. In particular, $x \in U$ and $y \notin U$, while $y \in V$ and $x \notin V$. Thus X is I_rT_1 .

(ii) $I_rT_1 \Rightarrow I_rT_0$: Suppose that X is I_rT_1 . Then for every pair of distinct points $x, y \in X$, there exist I_r -open sets U and V such that $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$. In particular, U separates x from y or V separates y from x . That is there is an I_r -open set contain one of them but not the other. Therefore X is I_rT_0 .

Theorem 2.2.3. *Let (X, τ) be an I_rT_0 space. Then $I_r\text{-cl}(\{x\}) \neq I_r\text{-cl}(\{y\})$ for every $x, y \in X$ with $x \neq y$.*

Proof. Let $x, y \in X$ with $x \neq y$. Since X is an I_rT_0 space, there is an I_r -open set containing one of them but not the other. Without loss of generality, U is an I_r -open set containing x but not y . Then U^C is an I_r -closed set containing y . Since by theorem 1.3.2, $I_r\text{-cl}(\{y\}) \subset U^C$. Therefore $I_r\text{-cl}(\{y\}) \cap U = \emptyset$. Since $I_r\text{-cl}(\{x\}) \cap U \neq \emptyset$, $I_r\text{-cl}(\{x\}) \neq I_r\text{-cl}(\{y\})$.

Theorem 2.2.4. *Let (X, τ) be a topological space. If X is an I_rT_1 space, then for any distinct points $x, y \in X$, $x \notin I_r\text{-cl}(\{y\})$ and $y \notin I_r\text{-cl}(\{x\})$.*

Proof. Suppose X is an I_rT_1 space. By definition, for each pair of distinct points $x, y \in X$, there exist I_r -open sets U and V such that $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$. If possible suppose that $x \in I_r\text{-cl}(\{y\})$, then every I_r -open set containing x must intersect $\{y\}$. That is every I_r -open set containing x must contain y . Then $y \in U$, this is a contradiction. Therefore, $x \notin I_r\text{-cl}(\{y\})$. A similar argument shows that $y \notin I_r\text{-cl}(\{x\})$.

Theorem 2.2.5. *Let (X, τ) be an I_r -Alexandroff space. If X is I_rT_1 , then every $\{x\}$ is I_r -closed.*

Proof. Let (X, τ) be an I_r -Alexandroff and an I_rT_1 space. Let $x \in X$. For each $y \in X - \{x\}$, there exist I_r -open sets U_y and V_y in X such that $x \in U_y$ with $y \notin U_y$ and $y \in V_y$ with $x \notin V_y$. Then $\bigcup_{y \in X - \{x\}} V_y \subset X - \{x\}$. Also for any

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$y \in X - \{x\}$, there exists an I_r -open set V_y containing y not containing x . This implies that $X - \{x\} = \bigcup_{y \in X - \{x\}} V_y$. Since X is an I_r -Alexandroff space, $X - \{x\}$ is an I_r -open set. Therefore $\{x\}$ is an I_r -closed set.

Theorem 2.2.6. *In an I_r -Hausdorff space, every subset is the intersection of a family of I_r -closed sets.*

Proof. Let X be an I_r -Hausdorff space and $A \subset X$. Let $x \in X - A$. For each point $a \in A$, since $x \neq a$ and the space is I_r -Hausdorff, there exist disjoint I_r -open sets U_a and V_a such that $a \in U_a$ and $x \in V_a$, with $U_a \cap V_a = \phi$. The collection $\{U_a : a \in A\}$ forms an I_r -open cover of A . Corresponding to each U_a , we have disjoint I_r -open sets $x \in V_a$. Let $V_x = \bigcap_a V_a$, which is an I_r -open set containing x and disjoint from each U_a . Hence, $V_x \cap A = \phi$, which implies that $V_x \subset X - A$. Then $X - A = \bigcup_{x \in X - A} V_x$, $A = \bigcap_{x \in X - A} V_x^C$ and V_x^C is I_r -closed. Hence A is the intersection I_r -closed sets.

Corollary 2.2.6.1. *In an I_r -Alexandroff I_r -Hausdorff space, every subset is I_r -closed.*

Proof. Proof follows from the fact that in I_r -Alexandroff space arbitrary intersection of I_r -closed sets is I_r -closed and by theorem 2.2.6.

Theorem 2.2.7. *Let (X, τ) be a topological space. If X is δT_0 , then it is $I_r T_0$.*

Proof. Proof follows from the fact that every regular open set is I_r -open.

Theorem 2.2.8. *Let (X, τ) be a topological space. If X is δT_1 (rT_1) or clopen T_1 , then it is $I_r T_1$.*

Proof. Proof follows from the fact that every clopen set is regular open and every regular open set is I_r -open.

Theorem 2.2.9. *Let (X, τ) be a topological space. If X is rT_2 , then it is $I_r T_2$.*

Proof. Proof follows from the fact that every regular open set is I_r -open.

Definition 2.2.2. A topological space (X, τ) is said to be I_r -regular at $x \in X$ if for every closed subset C of X not containing x , there exist disjoint I_r -open set containing x and an open set containing C . A space X is said to be I_r -regular if it is I_r -regular at each of its points.

Example 2.2.2. $\mathbb{R}$ with usual topology is an I_r -regular space.

Definition 2.2.3. A topological space (X, τ) is said to be I_r -normal if every pair of disjoint closed sets are contained in disjoint I_r -open sets.

Definition 2.2.4. A topological space (X, τ) is said to be *almost I_r -normal* if for every pair of disjoint closed set A and regular closed set B , there exist disjoint I_r -open sets U and V of X such that $A \subset U$ and $B \subset V$.

Definition 2.2.5. A topological space (X, τ) is said to be *completely I_r -normal* if for every pair of separated sets A and B , there exist disjoint I_r -open sets U and V of X such that $A \subset U$ and $B \subset V$.

Example 2.2.3. Let $X = \mathbb{Z}$ and $\tau = \{\emptyset, \mathbb{Z}, \{1\}, \mathbb{Z}^-, \mathbb{Z}^- \cup \{1\}, \mathbb{Z}^+, \mathbb{Z} - \{0\}\}$. Then (X, τ) is a completely I_r -normal.

Theorem 2.2.10. *Let (X, τ) be a topological space. Then X is I_r -regular if and only if given any point $x \in X$ and an open set U containing x , there exists an I_r -open set V containing x such that $V \subset cl(V) \subset U$.*

Proof. Suppose that X is I_r -regular. Let $x \in X$ and U be an open set containing x . Then $C = X - U$ is a closed set not containing x . Since X is I_r -regular, there exist disjoint I_r -open set V containing x and an open set W containing C . Then $cl(V)$ is disjoint from C and this implies that $cl(V) \subset X - C = U$. Thus $x \in V$ and $V \subset cl(V) \subset U$.

Conversely, suppose that $x \in X$ and C is a closed set not containing x . Then $U = X - C$ is an open set containing x . By the hypothesis, there is an I_r -open set V containing x such that $cl(V) \subset U$. Then the I_r -open set V containing x and the open set $X - cl(V)$ containing C are disjoint. Hence (X, τ) is I_r -regular.

Theorem 2.2.11. *Every nearly Hausdorff space is I_r -regular.*

Proof. Let (X, τ) be a nearly Hausdorff space. Let $x \in X$ and B be a closed set not containing x . Then for every $y \in B$, $x \neq y$ and hence there exist disjoint regular open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then $\{V_y : y \in B\}$ is an open cover of B and $U = \bigcap_{i=1}^{\infty} U_{y_i}$ is an I_r -open set containing x . Then there exists a pair of disjoint I_r -open set U containing x and an open set $V = \bigcup_{y \in B} V_y$ containing B . This implies that X is I_r -regular.

Theorem 2.2.12. *If (X_α, τ_α) is a topological space for each α , then any I_r -open set in the product topology $\prod_\alpha \tau_\alpha$ of $X = \prod_\alpha X_\alpha$ is a product of I_r -open sets in (X_α, τ_α) .*

Proof. Let (X_α, τ_α) be a family of topological spaces and A be an I_r -open set in the product topology of $X = \prod_\alpha X_\alpha$. Then there exist regular open sets U_i in X such that $A = \bigcap_i U_i$. Since each U_i is a regular open set in the product topology of X , there exist regular open sets $U_{i,\alpha} \in \tau_\alpha$ with $U_{i,\alpha} = X_\alpha$ for all but finitely many α 's such that $U_i = \prod_\alpha U_{i,\alpha}$. Hence $A = \bigcap_i (\prod_\alpha U_{i,\alpha})$. Using the fact that $\bigcap_\beta (\prod_\alpha A_{\alpha,\beta}) = \prod_\alpha (\bigcap_\beta A_{\alpha,\beta})$, we have $A = \prod_\alpha (\bigcap_i U_{i,\alpha}) = \prod_\alpha U_\alpha$, where $U_\alpha = \bigcap_i U_{i,\alpha}$, is an I_r -open set in (X_α, τ_α) . Hence the theorem.

Theorem 2.2.13. *Product of I_r -regular spaces is again I_r -regular.*

Proof. Let $\{X_\alpha\}$ be a family of I_r -regular spaces and $X = \prod_\alpha X_\alpha$. Let $x = (x_\alpha)$ be a point of X and $U = \prod_\alpha U_\alpha$ be an open set containing $x \in X$. If U_α is not open in X_α for some α , we choose a basis element of X containing x contained in U . Then $x_\alpha \in U_\alpha$ and U_α is open in X_α for each α . Since X_α is I_r -regular, by theorem 2.2.10, there exists an I_r -open set $V_\alpha \subset X_\alpha$ such that $x_\alpha \in V_\alpha$ and $V_\alpha \subset cl(V_\alpha) \subset U_\alpha$. Now V_α being I_r -open, we have $V_\alpha = \bigcap_j V_{\alpha,j}$ where $V_{\alpha,j}$ is regular open in X_α . If $U_\alpha = X_\alpha$, we simply choose $V_\alpha = X_\alpha$. Then $V = \prod_\alpha V_\alpha$ is an I_r -open set in $\prod_\alpha X_\alpha$. Since $cl(V) = cl(\prod_\alpha V_\alpha) = \prod_\alpha cl(V_\alpha)$, it follows that $x \in V$ and $V \subset cl(V) \subset \prod_\alpha U_\alpha \subset U$. Hence by theorem 2.2.10, X is I_r -regular.

Theorem 2.2.14. *Every r -regular space is I_r -regular.*

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Proof. Let (X, τ) be a r -regular topological space. Let V be an I_r -open subset of X containing x . Then we have $V = \bigcap_{j \in J} V_j$, where V_j 's are regular open sets in X . Since V_j is an open set containing x and X is r -regular, there exists a regular open set W_j such that $x \in W_j$ and $W_j \subset cl(W_j) \subset V_j$, for all $j \in J$. Take $W = \bigcap_{j \in J} W_j$. Then W is an I_r -open set in X such that $x \in W$ and $W \subset \bigcap_{j \in J} cl(W_j) \subset \bigcap_{j \in J} V_j = V$. Thus $x \in W$ and $W \subset cl(W) \subset V$ and hence X is I_r -regular by theorem 2.2.10.

Theorem 2.2.15. *Every I_r -Alexandroff I_r -regular space is I_r -normal.*

Proof. Let X be an I_r -Alexandroff I_r -regular space. Let A and B be disjoint closed subsets of X . Then for each $x \in A$, $X - B$ is an open set containing x . By theorem 2.2.10, there exists an I_r -open set V_x such that $x \in V_x$ and $V_x \subset cl(V_x) \subset X - B$. Therefore $\{V_x : x \in A\}$ forms an I_r -open covering of A . In the same way, we obtain $\{U_y : y \in B\}$ as an I_r -open covering of B . Define $U'_y = U_y - \bigcup_{x \in A} cl(V_x)$ and $V'_x = V_x - \bigcup_{y \in B} cl(U_y)$. Since arbitrary union of I_r -closed sets is I_r -closed, both $\bigcup_{x \in A} cl(V_x)$ and $\bigcup_{y \in B} cl(U_y)$ are I_r -closed. Therefore by lemma 1.2.4, $U'_y = U_y - \bigcup_{x \in A} cl(V_x)$ and $V'_x = V_x - \bigcup_{y \in B} cl(U_y)$ are I_r -open in X . Take $U' = \bigcup U'_y$ and $V' = \bigcup V'_x$. Then since X is an I_r -Alexandroff space, U' and V' are disjoint I_r -open sets containing A and B respectively. Hence X is I_r -normal.

Corollary 2.2.15.1. *Every I_r -Alexandroff r -regular space is I_r -normal.*

Proof. Proof follows from the fact that every r -regular space is I_r -regular.

Theorem 2.2.16. *In a T_1 and locally indiscrete space every subset is I_r -open.*

Proof. Let (X, τ) be a T_1 and locally indiscrete space and A be a subset of X . If $A = \phi$, the result is trivial. Otherwise every singleton set $\{x\} \subset X$ is closed and $A = \bigcap_{x \in X-A} (X - \{x\})$. Since X is locally indiscrete and $X - \{x\}$ is open, $X - \{x\}$ is regular open. Hence A is an arbitrary intersection of regular open sets and therefore it is I_r -open.

Corollary 2.2.16.1. *In a T_1 and locally indiscrete space every subset is I_r -clopen.*

Proof. Let (X, τ) be a T_1 and locally indiscrete space and A be a subset of X . If $A = \phi$, the result is trivial. Otherwise every singleton set $\{x\} \subset X$ is closed and $X - A = \bigcap_{x \in A} (X - \{x\})$. Since X is locally indiscrete and $X - \{x\}$ is open, $X - \{x\}$ is regular open. Hence $X - A$ is an arbitrary intersection of regular open sets and therefore it is I_r -open. Then A is I_r -closed. Also by theorem 2.2.16, we can conclude that A is I_r -clopen.

Theorem 2.2.17. *Every I_r -regular T_0 space is T_1 .*

Proof. Let (X, τ) be an I_r -regular T_0 space and $x, y \in X$ be a pair of distinct points in X . Then there exists an open set contain one of them but not the other. Without loss of generality there exists an open set V in X such that $x \in V$ and $y \notin V$. Since $x \in V$ and X is I_r -regular, by theorem 2.2.10, there exists an I_r -open set U in X such that $x \in U$ and $U \subset cl(U) \subset V$. Take $W = X - cl(U)$.

Then W is open in X such that $x \notin W$ and $y \in W$. Hence X is T_1 .

Theorem 2.2.18. *Let (X, τ) be an almost I_r -normal space. Then for a given closed set C and a regular open set D such that $C \subset D$, there is an I_r -open set G such that $C \subset G \subset I_r\text{-cl}(G) \subset D$.*

Proof. Suppose that X is almost I_r -normal. Let C be a closed set and D be a regular open set such that $C \subset D$. Then $X - D$ is regular closed and $C \cap (X - D) = \emptyset$. Since X is almost I_r -normal, there exist disjoint I_r -open sets G and V such that $C \subset G, X - D \subset V$. Then $X - V \subset D$ and so that $C \subset G \subset X - V \subset D$. By theorem 1.3.6, this implies that $C \subset G \subset I_r\text{-cl}(G) \subset I_r\text{-cl}(X - V) = X - V \subset D$.

Theorem 2.2.19. *Let (X, τ) be an I_r -Alexandroff space. Then X is almost I_r -normal if and only if for a given closed set C and a regular open set D such that $C \subset D$, there is an I_r -open set G such that $C \subset G \subset I_r\text{-cl}(G) \subset D$.*

Proof. The necessary part follows from the theorem 2.2.18.

To prove the sufficient part, consider a closed set C and a regular closed set D such that $C \cap D = \emptyset$. Then $X - D$ is a regular open set containing C . Then by the hypothesis, there is an I_r -open set G such that $C \subset G \subset I_r\text{-cl}(G) \subset X - D$. Take $V = X - I_r\text{-cl}(G)$. Then by the theorem 1.3.9 and $G \subset I_r\text{-cl}(G) = V^C$, V is an I_r -open set such that $D \subset V$ and $G \cap V = \emptyset$. Hence X is almost I_r -normal.

Theorem 2.2.20. *Every r -normal space is I_r -normal.*

Proof. Let (X, τ) be a r -normal space and let A and V be a regular closed subset of X with $A \cap V = \phi$. Since V is a closed set and X is r -normal then there exist two disjoint regular open sets G and H such that $A \subset G$ and $V \subset H$. Every regular open sets are I_r -open. Therefore X is almost I_r -normal.

Theorem 2.2.21. *Every I_r -normal space is almost I_r -normal.*

Corollary 2.2.21.1. *Every r -normal space is almost I_r -normal.*

Proof. Proof follows from theorem 2.2.21 and the theorem 2.2.20.

Theorem 2.2.22. *Arbitrary product of I_r -normal space is I_r -normal.*

Proof. Let $\{(X_\alpha, \tau_\alpha)\}$ be a family of I_r -normal spaces and $X = \prod_\alpha X_\alpha$. Let A and B be two closed subsets of X with $A \cap B = \phi$. Then $A = \prod_\alpha U_\alpha$ and $B = \prod_\alpha V_\alpha$, where U_α and V_α are closed subsets in X_α . This implies that $U_\alpha \cap V_\alpha = \phi$ for every α . Since each X_α is I_r -normal, there exist disjoint I_r -open sets G_α and H_α such that $U_\alpha \subset G_\alpha$ and $V_\alpha \subset H_\alpha$. Take $G = \prod_\alpha G_\alpha$ and $H = \prod_\alpha H_\alpha$. Then $G \cap H = \phi$. Otherwise, if there is an element $x = (x_\alpha) \in G \cap H$, then $x_\alpha \in G_\alpha \cap H_\alpha$ for every α and hence $G_\alpha \cap H_\alpha \neq \phi$ for some α , a contradiction. Therefore G and H are two disjoint I_r -open sets in X such that $U \subset G$ and $V \subset H$. Hence X is I_r -normal.

Definition 2.2.6. A topological space (X, τ) is said to be I_r^* -regular at $x \in X$ if for every I_r -closed subset C of X not containing x , there exists a pair of disjoint I_r -open sets U and V such that $x \in U$ and $C \subset V$. A space X is said to be I_r^* -

regular if it is I_r^* -regular at each of its points.

Example 2.2.4. 1) The space $\mathbb{R}$ with the usual topology is I_r^* -regular.

2) Consider the finite set $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{a, b\}, X\}$.

The space (X, τ) is I_r^* -regular

Theorem 2.2.23. *Let (X, τ) be a topological space. Then X is I_r^* -regular if and only if for any $x \in X$ and any I_r -open set U containing x , there is an I_r -open set V containing x such that $V \subset I_r\text{-cl}(V) \subset U$.*

Proof. Suppose that X is an I_r -Alexandroff I_r^* -regular space. Let $x \in X$ and U be an I_r -open set containing x . Then $C = X - U$ is an I_r -closed set not containing x . Since X is I_r^* -regular, there exist disjoint I_r -open sets V and W containing x and C respectively. Then $I_r\text{-cl}(V)$ is disjoint from C and this implies that $I_r\text{-cl}(V) \subset X - C = U$. Then there is an I_r -open set V containing x such that $V \subset I_r\text{-cl}(V) \subset U$.

Conversely, suppose that $x \in X$ and C is a I_r -closed set not containing x . Then $U = X - C$ is an I_r -open set containing x . By the hypothesis, there is an I_r -open set V containing x and an I_r -closed set F such that $V \subset F \subset U$. Then the I_r -open set V containing x and the I_r -open set $X - F$ containing C . Since $V \subset F$, $X - F \subset V^C$ and so $(X - F) \cap V \subset V^C \cap V = \phi$. Therefore V and $(X - F)$ are disjoint. Hence (X, τ) is I_r^* -regular.

Corollary 2.2.23.1. *Let (X, τ) be an I_r -Alexandroff topological space. Then X is I_r^* -regular if and only if for any $x \in X$ and any I_r -open set U containing x ,*

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there is an I_r -open set V containing x and an I_r -closed set F such that $V \subset F \subset U$.

Proof. Proof follows from theorem 2.2.23 and take $F = I_r\text{-cl}(V)$. Since X is an I_r -Alexandroff space, F is I_r -closed and $V \subset F \subset U$.

Theorem 2.2.24. *Let (X, τ) be an I_r -Alexandroff space. If X is I_r^* -regular and I_rT_1 space, then X is I_rT_2 .*

Proof. Let (X, τ) be a topological space and let $x, y \in X$ with $x \neq y$. Since X is I_r -Alexandroff, $\{x\}$ is an I_r -closed set. Since $\{x\}$ is an I_r -closed set not containing y . Also since X is an I_r^* -regular space, there exist disjoint I_r -open sets U and V such that $x \in U$ and $y \in V$. This implies that X is I_rT_2 .

Definition 2.2.7. A topological space (X, τ) is said to be I_r^* -normal if every pair of disjoint I_r -closed sets are contained in disjoint I_r -open sets.

Example 2.2.5. Let $X = \mathbb{R}$ with the lower limit topology (Sorgenfrey line). Here for any $a, b \in \mathbb{R}$, $[a, b)$ is regular open set. Therefore the Sorgenfrey line is I_r^* -normal.

Definition 2.2.8. A topological space (X, τ) is said to be

- (i) I_rT_3 if X is I_r^* -regular and I_rT_1 space.
- (ii) I_rT_4 if X is I_r^* -normal and I_rT_1 space.

Example 2.2.6. 1) The space $\mathbb{R}$ with the usual topology is I_rT_3 space.

2)The Sorgenfrey Line I_rT_1 . Let $x, y \in \mathbb{R}$ with $x \neq y$. Without loss of generality, assume $x < y$. In the Sorgenfrey line, the set $[x, x + \varepsilon)$ is a regular open set containing x and not y (take $\varepsilon < \frac{y-x}{2}$). The set $[y, y + \delta)$ is a regular open set containing y and not x , for $\delta \in \mathbb{R}^+$. Since both are regular open, they are I_r -open. Therefore, the Sorgenfrey line is an I_rT_1 space and then it is I_rT_4 space.

Definition 2.2.9. A topological space (X, τ) is said to be *almost I_r^* -normal* if for every pair of disjoint I_r -closed set A and regular closed set B , there exist disjoint I_r -open sets U and V of X such that $A \subset U$ and $B \subset V$.

Example 2.2.7. Let $X = \mathbb{Z}$ and $\tau = \{\emptyset, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^- \cup \{0\}, \mathbb{Z}^+, \mathbb{Z}^+ \cup \{0\}, \{0\}, \mathbb{Z} - \{0\}\}$. Then the (X, τ) is almost I_r^* -normal.

Theorem 2.2.25. Let (X, τ) be topological space. If X is I_r^* -normal, then almost I_r^* -normal.

Proof. Let (X, τ) be an I_r^* -normal topological space and let A be an I_r -closed set and B be a regular closed set such that $A \cap B = \emptyset$. Since B is also an I_r -closed set and X is I_r^* -normal space, there exist two disjoint I_r -open sets U and V such that $A \subset U$ and $B \subset V$. Hence X is almost I_r^* -normal space.

Theorem 2.2.26. Let (X, τ) be an I_r -Alexandroff space. If X is almost I_r^* -normal and I_rT_1 , then X is I_r^* -regular.

Proof. Let C be an I_r -closed set not containing x . Since X is I_rT_1 , $\{x\}$ is I_r -

closed set. Since X is almost I_r^* -normal, C is a I_r -closed set and $\{x\}$ is a regular closed set and are disjoint, there exist disjoint I_r -open sets U and V such that $x \in U$ and $C \subset V$. This implies that X is I_r^* -regular space.

Corollary 2.2.26.1. *Let (X, τ) be topological space. If X is I_r^* -normal and I_rT_1 , then I_r^* -regular.*

Proof. Proof follows from theorem 2.2.25, every I_r^* -normal is almost I_r^* -normal and the theorem 2.2.26, every almost I_r^* -normal and I_rT_1 space is I_r^* -regular

Corollary 2.2.26.2. *Let (X, τ) be topological space. If X is I_rT_4 space, then I_rT_3 space.*

Proof. Proof follows from the corollary 2.2.26.1 and the definition 2.2.8.

Definition 2.2.10. Let (X, τ) be a topological space. Then two subsets $A, B \subset X$ are called I_r -separated if $I_r\text{-cl}(A) \cap B = \phi$ and $A \cap I_r\text{-cl}(B) = \phi$.

Example 2.2.8. Let $X = \mathbb{Z}$ and $\tau = \{\phi, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, \mathbb{Z} - \{0\}\}$. Then the sets $\mathbb{Z}^-$ and $\mathbb{Z}^+$ are I_r -separated. Since $\mathbb{Z}^- \cap I_r\text{-cl}(\mathbb{Z}^+) = \mathbb{Z}^- \cap (\mathbb{Z}^+ \cup \{0\}) = \phi$ and $\mathbb{Z}^+ \cap I_r\text{-cl}(\mathbb{Z}^-) = \mathbb{Z}^+ \cap (\mathbb{Z}^- \cup \{0\}) = \phi$.

Theorem 2.2.27. *Let X be a topological space, $Y \subset X$ and $A, B \subset X$. If A and B are I_r -separated sets in X , then $A \cap Y$ and $B \cap Y$ are I_r -separated in the relative topology of the subspace Y .*

Proof. Let $A_Y = A \cap Y$ and $B_Y = B \cap Y$. Then $I_r\text{-cl}_Y(A_Y) \subseteq I_r\text{-cl}_X(A) \cap Y$ and $I_r\text{-cl}_Y(B_Y) \subseteq I_r\text{-cl}_X(B) \cap Y$. Since A and B are I_r -separated sets in X ,

$I_r\text{-cl}_X(A) \cap B = \phi$ and $A \cap I_r\text{-cl}_X(B) = \phi$. Then $I_r\text{-cl}_Y(A_Y) \cap B_Y \subseteq (I_r\text{-cl}_X(A) \cap Y) \cap (B \cap Y) = (I_r\text{-cl}_X(A) \cap B) \cap Y = \phi$. Similarly, $A_Y \cap I_r\text{-cl}_Y(B_Y) \subseteq (A \cap Y) \cap (I_r\text{-cl}_X(B) \cap Y) = (A \cap I_r\text{-cl}_X(B)) \cap Y = \phi$. Hence, A_Y and B_Y are I_r -separated in the subspace Y

Definition 2.2.11. A topological space (X, τ) is said to be *completely I_r^* -normal* if for every pair of I_r -separated sets A and B , there exist disjoint I_r -open sets U and V of X such that $A \subset U$ and $B \subset V$.

Example 2.2.9. Let $X = \mathbb{Z}$ and $\tau = \{\phi, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, \mathbb{Z} - \{0\}\}$. Then $\mathbb{Z}^-$ and $\mathbb{Z}^+$ are the only nonempty I_r -separated sets in $\mathbb{Z}$ and are also I_r -open sets. Therefore (X, τ) is a completely I_r^* -normal.

Theorem 2.2.28. Let (X, τ) be topological space. If X is completely I_r^* -normal, then I_r^* -normal.

Proof. Let (X, τ) be a completely I_r^* -normal topological space and let A be an I_r -closed set and B be a regular closed set such that $A \cap B = \phi$. Since $A \cap I_r\text{-cl}(B) = A \cap B = \phi$ and $I_r\text{-cl}(A) \cap B = A \cap B = \phi$, A and B are two I_r -separated sets. Since X is completely I_r^* -normal space, there exist two disjoint I_r -open sets U and V such that $A \subset U$ and $B \subset V$. Hence X is I_r^* -normal space .

Corollary 2.2.28.1. Let (X, τ) be topological space. If X is completely I_r^* -normal, then almost I_r^* -normal.

Proof. Proof is follows from the theorem 2.2.28 and the theorem 2.2.25.

Corollary 2.2.28.2. *Let (X, τ) be an I_r -Alexandroff space. If X is completely I_r^* -normal and $I_r T_1$, then almost I_r^* -regular.*

Proof. Proof is follows from the corollary 2.2.28.1 and the theorem 2.2.26. $\square$

2.3 I_r -Compactness

Definition 2.3.1. Let (X, τ) be a topological space and $A \subset X$. Then a collection $\{U_\alpha\}_{\alpha \in \Lambda}$ of I_r -open sets in X is said to be an I_r -open cover of A if $A \subset \bigcup_{\alpha \in \Lambda} U_\alpha$.

Definition 2.3.2. A topological space (X, τ) is said to be I_r -compact if every I_r -open cover of X has a finite subcover.

Definition 2.3.3. Let (X, τ) be a topological space. Then a subset $K \subset X$ is said to be I_r -compact if every I_r -open cover of K has a finite subcover.

Example 2.3.1. Let $X = \mathbb{Z}$ and $\tau = \{\phi, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, \mathbb{Z} - \{0\}\}$. Then (X, τ) is an I_r -compact topological space.

Theorem 2.3.1. *A topological space (X, τ) is I_r -compact if and only if every family of I_r -closed sets of X having finite intersection property has a nonempty intersection.*

Proof. Suppose (X, τ) is an I_r -compact space. Let $\{A_i : i \in I\}$ be a family of I_r -closed sets with finite intersection property. If possible suppose that $\bigcap_{i \in I} A_i = \phi$,

2.3. I_r -Compactness

then $(\bigcap_{i \in I} A_i)^C = X$. This implies that $\bigcup_{i \in I} A_i^C = X$. Thus $\{A_i^C : i \in I\}$ is an I_r -open cover of X . Since X is I_r -compact, there exists a finite subset I_0 of I such that $X \subset \bigcup_{i \in I_0} A_i^C$, $X \subset (\bigcap_{i \in I_0} A_i)^C$, $\bigcap_{i \in I_0} A_i \subset \phi$, then $\bigcap_{i \in I_0} A_i = \phi$, which is a contradiction. Therefore $\bigcap_{i \in I_0} A_i \neq \phi$.

Conversely, suppose that every family of I_r -closed sets of X having finite intersection property has a nonempty intersection. If possible, X is not I_r -compact. Then there exists an I_r -open cover $\{B_j : j \in J\}$ of X having no finite subcover. This implies for any finite subfamily $\{B_j : j = 1, 2, \dots, n\}$ of $\{B_j : j \in J\}$. Then we get $\bigcup_{j=1}^n B_j \neq X$, this implies that $\bigcap_{j=1}^n B_j^C \neq \phi$. Therefore the family $\{B_j^C : j \in J\}$ of I_r -closed sets have finite intersection property. By the assumption $\bigcap_{j \in J} B_j^C \neq \phi$. Hence $\bigcup_{j \in J} B_j \neq X$, this is a contradiction. Hence X is I_r -compact

Theorem 2.3.2. *If (X, τ) is a compact topological space and every I_r -closed set in X is closed, then X is I_r -compact space.*

Proof. Let $\{A_i : i \in I\}$ be an I_r -open cover of X . Since A_i^C is an I_r -closed set, for each $i \in I$. By the hypothesis A_i^C is closed set in X . Then $\{A_i : i \in I\}$ is an open cover of X . Since X is compact, there is a finite set I_0 of I such that $X \subset \bigcup_{i \in I_0} A_i$. Therefore X is I_r -compact.

Theorem 2.3.3. *Every I_r -compact space is nearly compact.*

Proof. Proof follows from the fact that every regular open set is I_r -open.

Theorem 2.3.4. *Every I_r -closed subset of a I_r -compact space is I_r -compact.*

Proof. Let A be an I_r -closed subset of an I_r -compact space X . Then A^C is an I_r -open subset of X . Let $\{A_i : i \in I\}$ be an I_r -open cover of A . Since $\{A_i : i \in I\} \cup A^C$ is an I_r -open cover of X and X is I_r -compact, there is a finite subcover $\{A_1, A_2, A_3, \dots, A_n\} \cup A^C$ for X . Then $\{A_1, A_2, A_3, \dots, A_n\}$ is a finite subcover of A . Hence A is I_r -compact.

2.4 I_r -Connected Space

Definition 2.4.1. Let (X, τ) be a topological space. The space X is said to be I_r -connected if it is not the union of two nonempty I_r -separated subsets. A subset C of X is said to be I_r -connected if C with relative topology is a I_r -connected space. If a space is not I_r -connected is called I_r -disconnected.

Example 2.4.1. 1) Let $X = \mathbb{Z}$ and $\tau = \{\phi, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, \mathbb{Z} - \{0\}\}$. Then (X, τ) is I_r -connected, Since $\mathbb{Z}^-$ and $\mathbb{Z}^+$ are the only nonempty I_r -separated sets in $\mathbb{Z}$ and $\mathbb{Z} \neq \mathbb{Z}^- \cup \mathbb{Z}^+$.

2) Let $X = \mathbb{R}$ with the usual topology. Since $\mathbb{R} = (-\infty, 1) \cup [1, \infty)$ and $(-\infty, 1)$ and $[1, \infty)$ are I_r -separated subsets of $\mathbb{R}$. Then the space $\mathbb{R}$ is I_r -disconnected.

Definition 2.4.2. An I_r -component of a point $x \in X$ is the union of all I_r -connected subsets of X containing x . It is the largest I_r -connected subset of X

that contains x .

Example 2.4.2. Let $X = \mathbb{R}$ with the usual topology. Every singleton set in $\mathbb{R}$ is I_r -component of $\mathbb{R}$.

Theorem 2.4.1. *Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of I_r -connected subsets of a topological space (X, τ) such that for every $\alpha, \beta \in \Lambda$ with $A_\alpha \cap A_\beta \neq \phi$. Then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is I_r -connected.*

Proof. Let $A = \bigcup_{\alpha \in \Lambda} A_\alpha$. Suppose that A is not I_r -connected. Then there exist nonempty subsets $U, V \subset A$ such that $A = U \cup V$, $I_r cl(U) \cap V = \phi$, and $U \cap I_r cl(V) = \phi$.

For each $\alpha \in \Lambda$, since $A_\alpha \subset A = U \cup V$, define $U_\alpha = A_\alpha \cap U$, $V_\alpha = A_\alpha \cap V$. Then $A_\alpha = U_\alpha \cup V_\alpha$, $U_\alpha \cap I_r cl(V_\alpha) \subset U \cap I_r cl(V) = \phi$ and $I_r cl(U_\alpha) \cap V_\alpha \subset I_r cl(U) \cap V = \phi$. Then U_α and V_α are I_r -separated in A_α . Since A_α is I_r -connected, one of them must be empty. Then, for each $\alpha \in \Lambda$, either $A_\alpha \subset U$ or $A_\alpha \subset V$.

Define $\Lambda_U = \{\alpha \in \Lambda : A_\alpha \subset U\}$, $\Lambda_V = \{\beta \in \Lambda : A_\beta \subset V\}$. If both Λ_U and Λ_V are nonempty, then there exist $\alpha \in \Lambda_U$, $\beta \in \Lambda_V$, with $A_\alpha \cap A_\beta \neq \phi$. Since $A_\alpha \subset U$ and $A_\beta \subset V$ for some α, β , $U \cap V \neq \phi$. This is a contradiction. Then, either all $A_\alpha \subset U$ or all $A_\alpha \subset V$. Since $A = \bigcup_{\alpha \in \Lambda} A_\alpha$ and $A = U \cup V$, either $U = \phi$ or $V = \phi$. This is a contradiction to the assumption that both U and V are nonempty. Hence, A is I_r -connected. $\square$

Corollary 2.4.1.1. *If A and B are I_r -connected subsets of X , and $A \cap B \neq \phi$,*

then $A \cup B$ is I_r -connected.

Theorem 2.4.2. *Let $A \subset X$ be an I_r -connected subset of a topological space (X, τ) . Then A is contained in a I_r - component of A .*

Proof. Let $\mathcal{F}$ be the family of all I_r -connected subsets of X that contain A , $\mathcal{F} = \{C \subset X \mid C \text{ is } I_r\text{-connected and } A \subset C\}$. This family is nonempty since $A \in \mathcal{F}$. Consider the union $C^* = \bigcup_{C \in \mathcal{F}} C$. We claim that C^* is I_r -connected. Since every $C \in \mathcal{F}$ contain A . So, $\mathcal{F}$ is a family of I_r -connected sets with a nonempty intersection. By theorem 2.4.1 C^* is I_r -connected. Suppose D is any I_r -connected set containing A . Then $D \in \mathcal{F}$. Therefore, $D \subset C^*$. Hence C^* is a maximal I_r -connected set containing A . $\square$

Theorem 2.4.3. *The I_r -components of a topological space X form a partition of X . Then,*

1. *every point $x \in X$ lies in a unique maximal I_r -connected subset of X ,*
2. *the I_r - components are pairwise disjoint,*
3. *and their union is the entire space X .*

Proof. Let $x \in X$. By the theorem 2.4.2, there exists a maximal I_r -connected subset $C_x \subset X$ containing x , I_r -component of x . Let C_x and C_y be two I_r -components with $C_x \cap C_y \neq \phi$. We claim that $C_x = C_y$. Since both are I_r -connected and intersect nontrivially, their union $C_x \cup C_y$ is I_r -connected by the theorem 2.4.1. But $C_x \subset C_x \cup C_y$, and C_x is maximal. So we must have $C_x =$

2.4. I_r -Connected Space

$C_x \cup C_y$, which implies $C_y \subset C_x$. Similarly, $C_x \subset C_y$, hence $C_x = C_y$. Therefore, any two I_r -components are either equal or disjoint. Since every point of X belongs to at least one such component, the family of I_r -components forms a partition of X . $\square$

Theorem 2.4.4. *Let (X, τ) be a topological space and $Y \subset X$ be a subspace with the subspace topology. If Y is I_r -connected in X , then Y is I_r -connected in the subspace topology.*

Proof. Suppose Y is I_r -connected in X , but not in the subspace topology. Then there exist nonempty, disjoint subsets $A, B \subset Y$ such that $Y = A \cup B$, $I_r cl_Y(A) \cap B = \phi$, $A \cap I_r cl_Y(B) = \phi$. Since $I_r cl_Y(A) = I_r cl_X(A) \cap Y$ and $I_r cl_Y(B) = I_r cl_X(B) \cap Y$, $I_r cl_X(A) \cap B = I_r cl_X(A) \cap Y \cap B = I_r cl_Y(A) \cap B = \phi$ and $A \cap I_r cl_X(B) = \phi$. This is a contradiction to the assumption that Y is I_r -connected in X . Hence Y is I_r -connected in subspace topology. $\square$

Theorem 2.4.5. *Let (X, τ) be a topological space and $C \subset X$. If C is a I_r -connected component of a topological space X , then $I_r-cl(C) = C$.*

Proof. Let C be an I_r -connected component of X , C is maximal I_r -connected. Let $x \in I_r cl(C)$. Define the set $C' = C \cup \{x\}$. We claim that C' is I_r -connected. If possible suppose that C' is not I_r -connected. Then C' can be written as a disjoint union of nonempty subsets A and B such that $C' = A \cup B$, $A \cap B = \phi$, $I_r cl(A) \cap B = \phi$, and $A \cap I_r cl(B) = \phi$. Since $x \notin C$, and $C \subset C'$, it follows that all points of C must lie entirely within either A or B . Without loss

of generality, suppose $C \subset A$, and hence $B = \{x\}$. Since $I_rcl(A) \cap B = \phi$, $x \notin I_rcl(A)$. Since $C \subset A$ implies $I_rcl(C) \subset I_rcl(A)$, $x \notin I_rcl(C)$. This is a contradiction to the assumption. Thus, C' must be I_r -connected. Since C is maximal, $C = C'$. Therefore $x \in C$. Hence $I_rcl(C) \subset C$. Since $C \subset I_rcl(C)$, $I_rcl(C) = C$. $\square$

Theorem 2.4.6. *A topological space X is I_r -connected, then the only I_r -clopen subsets of X are ϕ and X .*

Proof. Suppose X is I_r -connected and $A \subset X$ be I_r -clopen. Then $X - A$ is also I_r -clopen. If $A \neq \phi$ and $A \neq X$, then A and $X - A$ are nonempty. By theorem 1.3.6, $I_rcl(A) = A$ and $I_rcl(X - A) = X - A$, then A and $X - A$ are disjoint I_r -separated subsets covering X . This is a contradiction to the I_r -connectedness of X . Therefore $A = \phi$ or $A = X$. $\square$

2.5 I_r -Cover of a Set in Topological Space

Definition 2.5.1. Let (X, τ) be a topological space and $A \subset X$. Then I_r -cover of A , denoted by $I_r(A)$ is defined as $I_r(A) = \cap\{V : V \text{ is a regular open set containing } A\}$.

Remark. From the definition, it follows that $I_r(A)$ is the smallest I_r -open set containing A . If A is a I_r -open subset of X , then $I_r(A) = A$.

Theorem 2.5.1. *If X is an I_rT_2 -space, then $I_r(\{x\}) \cap I_r(\{y\}) = \phi$ for every $x, y \in X$ with $x \neq y$.*

Proof. Let X be an I_rT_2 -space and let $x, y \in X$ with $x \neq y$. Then there exist disjoint I_r -open sets U and V such that $x \in U$ and $y \in V$. Since $I_r(\{x\}) = \cap\{U' : U' \text{ is a regular open set and } x \in U'\}$, $I_r(\{y\}) = \cap\{V' : V' \text{ is a regular open set and } y \in V'\}$. Since I_r -open sets are the intersection of regular open sets, $I_r(\{x\}) \subset U$ and $I_r(\{y\}) \subset V$. Then $I_r(\{x\}) \cap I_r(\{y\}) = \phi$. $\square$

Corollary 2.5.1.1. *If X is an rT_2 -space, then $I_r(\{x\}) \cap I_r(\{y\}) = \phi$ for every $x, y \in X$ with $x \neq y$.*

Proof. Since every rT_2 space is I_rT_2 , the proof follows from the theorem 2.2.9 and theorem 2.5.1. $\square$

Theorem 2.5.2. *Let (X, τ) be a topological space and A, B be the subsets of X . Then the following holds:*

- (i) $A \subset I_r(A)$.
- (ii) $I_r(I_r(A)) = I_r(A)$.
- (iii) If $A \subset B$, then $I_r(A) \subset I_r(B)$.
- (iv) $I_r(\phi) = \phi$

Proof. The results (i),(iii) and (iv) are obvious from the definition of I_r - cover of A .

(ii). Let V be a regular open set containing A . Since $I_r(A) = \bigcap \{V' : V' \text{ is a regular open set containing } A\}$, $I_r(A) \subset V$. Then V is a regular open set also contains $I_r(A)$. This implies $I_r(I_r(A)) \subset I_r(A)$.

Let U be a regular open set containing $I_r(A)$. Since $A \subset I_r(A)$, $A \subset U$. Then U is a regular open set also contains A . This implies $I_r(I_r(A)) \supset I_r(A)$. Therefore $I_r(I_r(A)) = I_r(A)$. $\square$

Remark. Let (X, τ) be a topological space. Since the map I_r from $\mathcal{P}(X)$ into $\mathcal{P}(X)$ satisfies the following axioms:

For any $A \subset X$

- (i) $A \subset I_r(A)$
- (ii) $I_r(I_r(A)) = I_r(A)$
- (iii) $I_r(\phi) = \phi$,

then there exists a unique supra topology τ^* on X associated with the topology τ such that $I_r(A)$ is the supra-closure of the subset A of X .

Remark. I_r -open cover of a subset A of (X, τ) is same as the supra-closure of A with respect to the supra topology τ^* associated with τ .

Theorem 2.5.3. *Let (X, τ) be a topological space. We define $\tau_{I_r} = \{A^c : I_r(A) = A, A \subset X\}$. Then (X, τ_{I_r}) is a supra topological space and the I_r -cover, $I_r(A)$ of a subset A of X is the supra-closure of the subset A with respect τ_{I_r} .*

Proof. Since $I_r(\phi) = \phi$, $X \in \tau_{I_r}$. Further, since $X \subset I_r(X) \subset X$, $I_r(X) = X$. Therefore $\phi \in \tau_{I_r}$.

Suppose that $U_\alpha \in \tau_{I_r}$ for each $\alpha \in \Lambda$. Then $I_r(U_\alpha^C) = U_\alpha^C$ for each $\alpha \in \Lambda$. Since $\bigcap_{\alpha \in \Lambda} U_\alpha^C \subset U_\alpha^C$ for each $\alpha \in \Lambda$, $I_r(\bigcap_{\alpha \in \Lambda} U_\alpha^C) \subset I_r(U_\alpha^C) = U_\alpha^C$ for each $\alpha \in \Lambda$. Therefore $I_r(\bigcap_{\alpha \in \Lambda} U_\alpha^C) \subset \bigcap_{\alpha \in \Lambda} U_\alpha^C = (\bigcup_{\alpha \in \Lambda} U_\alpha)^C$. Hence $I_r((\bigcup_{\alpha \in \Lambda} U_\alpha)^C) \subset (\bigcup_{\alpha \in \Lambda} U_\alpha)^C$. On the other hand, from the theorem 2.5.2, we get $(\bigcup_{\alpha \in \Lambda} U_\alpha)^C \subset I_r((\bigcup_{\alpha \in \Lambda} U_\alpha)^C)$. Thus $(\bigcup_{\alpha \in \Lambda} U_\alpha)^C = I_r((\bigcup_{\alpha \in \Lambda} U_\alpha)^C)$. Hence $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau_{I_r}$ and therefore τ_{I_r} is a supra topology on X .

We prove that I_r is the supra closure operator with respect to τ_{I_r} . Since $I_r(I_r(A)) = I_r(A)$ from 2.5.2, it follows that $I_r(A)$ is a supra closed set containing A . Since $scl(A)$ is the smallest supra closed set containing A , $scl(A) \subset I_r(A)$. On the other hand, since $A \subset scl(A)$, $I_r(A) \subset I_r(scl(A)) = scl(A)$. Further, since $scl(A)$ is a supra closed set in (X, τ_{I_r}) . Therefore $I_r(A) \subset scl(A)$. Thus $I_r(A) = scl(A)$. Hence I_r is a supra closure operator with respect to (X, τ_{I_r}) . $\square$

Chapter 3

I_r^* -Open Sets in Topological Space

3.1 Introduction

The study of I_r^* -closed sets and I_r^* -open sets serves as a natural extension of I_r -closed sets and I_r -open sets, enabling the development of finer generalizations of classical topological properties. In this chapter, we introduce and investigate the concept of I_r^* -closed sets, defined via containment of I_r -closed set within I_r -open sets. We develop several characterizations, properties and theorems on I_r^* -closed sets and I_r^* -open sets, including their behavior under unions, hereditary properties, and their relationship with I_r -closure. In addition, we explore the interaction between I_r^* -closedness and I_r -compactness. Moreover, we study how I_r^* -closedness behaves under relative topology, singletons, and set-theoretic inclusion. Special attention is given to I_r -Alexandroff spaces, where alternate characterizations of I_r^* -closed sets are derived.

3.2 I_r^* -Open Sets and its Properties

Definition 3.2.1. Let (X, τ) be a topological space. A subset A of X is said to be I_r^* -closed if and only if there is an I_r -closed set F containing A such that $F \subset U$ whenever U is an I_r -open set containing A . Complement of I_r^* -closed set is I_r^* -open .

Theorem 3.2.1. Every I_r -closed set in a topological space (X, τ) is I_r^* -closed.

Remark. Converse of the theorem 3.2.1 need not be true.

Example 3.2.1. Let (X, τ) be a topological space where $X = \mathbb{Z}$ and $\tau = \{\phi, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, \mathbb{Z} - \{0\}\}$. Here $\mathbb{Z} - \{0\}$ is an I_r^* -closed, but not I_r -closed.

Theorem 3.2.2. Let (X, τ) be topological space. If $A \subset X$ is I_r^* -closed, then $I_r\text{-cl}(A)$ is I_r^* -closed.

Proof. Let (X, τ) be topological space and $A \subset X$ is I_r^* -closed. Let U be an I_r -open set containing $I_r\text{-cl}(A)$. Then U is an I_r -open set containing A , since $A \subset I_r\text{-cl}(A)$. Since A is I_r^* -closed, there exists an I_r -closed set F containing A such that $F \subset U$. Also by theorem 1.3.2, $I_r\text{-cl}(A) \subset F$. Hence $I_r\text{-cl}(A)$ is I_r^* -closed.

Theorem 3.2.3. Let (X, τ) be an I_r -Alexandroff space. A set A of X is I_r^* -closed if and only if there is an I_r -closed set F containing A such that $F - A$ does not contain any nonempty I_r -closed set.

3.2. I_r^* -Open Sets and its Properties

Proof. Let A be an I_r^* -closed subset of a topological space (X, τ) . Then there is an I_r -closed set F containing A such that $F \subset U$ whenever $A \subset U$ and U is I_r -open. If possible let there is an I_r -closed set F_1 in X such that $F_1 \subset F - A$. Since A is I_r^* -closed and F_1^C is an I_r -open set containing A , it follows that $F \subset F_1^C$. This implies that $F_1 \subset F^C$, and so $F_1 \subset F \cap F^C = \phi$. Hence $F - A$ does not contain any nonempty I_r -closed set.

Conversely, suppose that there is an I_r -closed set F containing A such that $F - A$ does not contain any nonempty I_r -closed set. Let U be an I_r -open set containing A . If possible suppose that $F \not\subset U$. Then $F \cap U^C \neq \phi$. Since (X, τ) is an I_r -Alexandroff space, $F \cap U^C = F - U$ is a nonempty I_r -closed set in X . Further since $A \subset U$, $F \cap U^C$ is contained in $F - A$, a contradiction. So A is I_r^* -closed.

Corollary 3.2.3.1. *Let (X, τ) be an I_r -Alexandroff space and $A \subset X$ be an I_r^* -closed set. Then A is I_r -closed if and only if both $I_r\text{-cl}(A)$ and $I_r\text{-cl}(A) - A$ are I_r -closed.*

Proof. If A is both I_r -closed and I_r^* -closed, then $I_r\text{-cl}(A) = A$ and $I_r\text{-cl}(A) - A = \phi$ are I_r -closed.

Conversely, suppose that A is an I_r^* -closed set such that both $I_r\text{-cl}(A)$ and $I_r\text{-cl}(A) - A$ are I_r -closed. Since A is I_r^* -closed, by Theorem 3.2.3, there is a I_r -closed set F containing A such that $F - A$ does not contain any nonempty I_r -closed set. Since by theorem 1.3.2, $I_r\text{-cl}(A) \subset F$ so that $I_r\text{-cl}(A) - A \subset F - A$.

Further since $I_r\text{-cl}(A) - A$ is I_r -closed, we get $I_r\text{-cl}(A) - A = \phi$. Therefore $A = I_r\text{-cl}(A)$ and hence by theorem 1.3.7, A is I_r -closed.

Theorem 3.2.4. *Let (X, τ) be a topological space. Then for each $x \in X$, $\{x\}$ is I_r -closed or its complement is I_r^* -closed.*

Proof. For each $x \in X$,

- Case-I: If $\{x\}$ is I_r -closed.

Then there is nothing to prove.

- Case-II: If $\{x\}$ is not I_r -closed.

Let $A = \{x\}^C$ and U be an I_r -open set containing A . Then $U^C \subset A^C = \{x\}$ and U^C is an I_r -closed set. Then either $U^C = \phi$ or $U^C = \{x\}$. If $U^C = \{x\}$, $\{x\}$ is an I_r -closed set, which is a contradiction. Therefore $U^C = \phi$ which implies $U = X$. Take $F = X$, an I_r -closed set containing A , $F \subset U$ and hence $A = \{x\}^C$ is I_r^* -closed.

□

Theorem 3.2.5. *Let (X, τ) be a topological space. Then the arbitrary union of I_r^* -closed sets is I_r^* -closed.*

Proof. Let $\{F_\alpha\}_{\alpha \in \Lambda}$ be a family of I_r^* -closed sets in X and $W = \bigcup_{\alpha \in \Lambda} F_\alpha$. Let U be an I_r -open set such that $W \subset U$. Since each F_α is I_r^* -closed and $F_\alpha \subset U$, there exists an I_r -closed set G_α such that $F_\alpha \subset G_\alpha \subset U$. Consider $G = \bigcup_{\alpha \in \Lambda} G_\alpha$,

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G is an I_r -closed set since arbitrary union of I_r -closed sets is I_r -closed and $W \subset G \subset U$. Hence, W is I_r^* -closed. $\square$

Corollary 3.2.5.1. *Let (X, τ) be a topological space. Then the union of two I_r^* -closed sets is I_r^* -closed.*

Corollary 3.2.5.2. *Let (X, τ) be a topological space. Then the arbitrary intersection of I_r^* -open sets is I_r^* -open.*

Proof. Proof follows from theorem 3.2.5 and De Morgan's Law. $\square$

Theorem 3.2.6. *Let (X, τ) be a topological space and $A, B \subset X$ such that $A \subset B \subset I_r\text{-cl}(A)$. If A is I_r^* -closed, then B is also I_r^* -closed.*

Proof. Let U be an I_r -open set in X such that $B \subset U$. Since $A \subset U$ and A is I_r^* -closed, there is an I_r -closed set F containing A such that $F \subset U$. Since by theorem 1.3.2, $I_r\text{-cl}(A) \subset F$ so that $B \subset F$. Hence B is I_r^* -closed. $\square$

Corollary 3.2.6.1. *Let (X, τ) be a topological space and $A \subset X$. If A is I_r^* -closed, then $I_r\text{-cl}(A)$ is I_r^* -closed.*

Proof. Since $A \subset I_r\text{-cl}(A) \subset I_r\text{-cl}(I_r\text{-cl}(A))$ and $I_r\text{-cl}(A) = I_r\text{-cl}(I_r\text{-cl}(A))$. Since A is I_r^* -closed and by theorem 3.2.6, $I_r\text{-cl}(A)$ is I_r^* -closed. $\square$

Theorem 3.2.7. *In a topological space an I_r -open I_r^* -closed set is I_r -closed.*

Proof. Let (X, τ) be a topological space and A be an I_r -open I_r^* -closed set in X . Then since A is I_r^* -closed set, there is an I_r -closed set F containing A such that

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$F \subset U$ whenever U is an I_r -open set containing A . Since A is an I_r -open set containing A , $F \subset A$. Then $A = F$ and hence A is I_r -closed.

□

Theorem 3.2.8. *Let (X, τ) be a topological space. If every subset of X is I_r^* -closed, then the collection of all I_r -closed sets = collection of all I_r -open sets.*

Proof. Proof directly follows by theorem 3.2.7.

□

Theorem 3.2.9. *Let (X, τ) be a topological space. Then a subset A of X is I_r^* -open if and only if there is an I_r -open set U of X contained in A such that $F \subset U$ whenever F is an I_r -closed set in X contained in A .*

Proof. First let A be an I_r^* -open set in X . Let F be an I_r -closed set contained in A . Then $A^C \subset F^C$ where F^C is an I_r -open set. Since A^C is I_r^* -closed, there is an I_r -closed set G containing A^C such that $G \subset F^C$. This implies $F \subset G^C$ where G^C is an I_r -open set contained in A . Take $U = G^C$.

Conversely, suppose that there is an I_r -open set U in X contained in A such that $F \subset U$ whenever F is an I_r -closed set in X contained in A . We claim that A^C is I_r^* -closed. Let V be an I_r -open set containing A^C . Then V^C is an I_r -closed set contained in A . Then by the hypothesis, there exists an I_r -open set U contained in A such that $V^C \subset U$. This implies that U^C is an I_r -closed set containing A^C such that $U^C \subset V$. Therefore A^C is I_r^* -closed and hence A is I_r^* -open.

□

Theorem 3.2.10. *Let (X, τ) be a topological space and $A \subset X$. If A is I_r^* -open, then there is an I_r -open set $V \subset A$ such that $V \cup A^C \subset U$ and U is an I_r -open set in X , implies $U = X$.*

Proof. Suppose that A is I_r^* -open. The by theorem 3.2.9 there is an I_r -open set V in X contained in A such that $F \subset V$ whenever F is an I_r -closed set in X contained in A . Suppose that $V \subset A$ such that $V \cup A^C \subset U$ for some I_r -open U in X . Then $U^C \subset V^C \cap A$. Since $U^C \subset A$, U^C is an I_r -closed contained in A , $U^C \subset V$. Hence $U^C \subset V \cap V^C = \phi$. Hence $U = X$. $\square$

Corollary 3.2.10.1. *Let (X, τ) be an I_r -Alexandroff topological space. Then a set A is I_r^* -open if and only if there is an I_r -open set $V \subset A$ such that $V \cup A^C \subset U$ and U is an I_r -open set in X , then $U = X$.*

Proof. Necessary part is follows from theorem 3.2.10. Conversely, suppose that there is an I_r -open set $V \subset A$ such that $V \cup A^C \subset U$ and U is I_r -open, then $U = X$. Let F be an I_r -closed set contained in A . Then $V \cup A^C \subset V \cup F^C$. Since X is an I_r -Alexandroff space, $V \cup F^C$ is I_r -open. So by the given condition applied to $V \cup F^C$ in place of U , $V \cup F^C = X$ it follows that $F \cap V^C = \phi$. This implies $F \subset V$. Therefore A is I_r^* -open by theorem 3.2.9. $\square$

Theorem 3.2.11. *Let (X, τ) be an I_r -Alexandroff topological space. Then A is I_r^* -closed if and only if $I_r\text{-cl}(A) - A$ is I_r^* -open.*

Proof. Let (X, τ) be an I_r -Alexandroff topological space and A is I_r^* -closed. Let

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F be an I_r -closed set such that $F \subset I_r\text{-cl}(A) - A$. Then $A \subset I_r\text{-cl}(A) - F = I_r\text{-cl}(A) \cap F^C$. Also F^C is I_r -open and $A \subset F^C$. Since A is I_r^* -closed, there is an I_r -closed set G containing A such that $G \subset F^C$. This implies that $F \subset G^C$ and G^C is an I_r -open set. Therefore by theorem 3.2.9, $I_r\text{-cl}(A) - A$ is I_r^* -open.

Conversely suppose that $I_r\text{-cl}(A) - A$ is I_r^* -open. In order to show that A is I_r^* -closed, let U be an I_r -open set such that $A \subset U$. Since $A \subset U$, $I_r\text{-cl}(A) - U \subset I_r\text{-cl}(A) - A$ and $I_r\text{-cl}(A) - U = I_r\text{-cl}(A) \cap U^C$ is I_r -closed. Since $I_r\text{-cl}(A) - A$ is I_r^* -open and $I_r\text{-cl}(A) - U$ is an I_r -closed set contained in $I_r\text{-cl}(A) - A$, by theorem 3.2.9, there is an I_r -open set V contained in $I_r\text{-cl}(A) - A$ such that $I_r\text{-cl}(A) - U \subset V$. This implies $I_r\text{-cl}(A) - V \subset U$. Take $F = I_r\text{-cl}(A) \cap V^C$. Since X is an I_r -Alexandroff, F is an I_r -closed set and also $F \subset U$. It remains to show that $A \subset F$. Since $V \subset I_r\text{-cl}(A) \cap A^C$, $A \subset (I_r\text{-cl}(A))^C \cup A \subset V^C$. Further since $A \subset I_r\text{-cl}(A)$ it follows that $A \subset I_r\text{-cl}(A) \cap V^C = F$. Hence A is an I_r^* -closed set. $\square$

Theorem 3.2.12. *Let (X, τ) be an I_r -Alexandroff I_r^* -regular space. Then I_r -compact subsets of X are I_r^* -closed if and only if the arbitrary union of I_r -open sets in X whose complement is the I_r -closure of an I_r -compact set is I_r^* -open.*

Proof. Suppose that every I_r -compact subset of X be I_r^* -closed. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an arbitrary collection of I_r -open sets and take $U = \bigcup_{\alpha \in \Lambda} U_\alpha$ such that $X - U = I_r\text{-cl}(A)$ (say) where A is an I_r -compact set. Now by hypothesis A is I_r^* -closed in X and so by theorem 3.2.6.1, $I_r\text{-cl}(A)$ is I_r^* -closed. Hence U is

I_r^* -open.

Conversely, suppose that the arbitrary union of I_r -open sets in X whose complement is the I_r -closure of an I_r -compact set is I_r^* -open. Let A be an I_r -compact set and U be an I_r -open set containing A . Let $x \in A$. Then $x \in U$. Since X is I_r^* -regular, by Theorem 2.2.23.1, there is an I_r -open set V_x containing x and an I_r -closed set F_x such that $V_x \subset F_x \subset U$. Then the collection $\{V_x : x \in A\}$ is an I_r -open cover of A . Since A is I_r -compact, there are finite number of points $x_1, x_2, \dots, x_n \in A$ such that $A \subset \bigcup_{i=1}^n V_{x_i} \subset \bigcup_{i=1}^n F_{x_i} \subset U$. Then since $I_r\text{-cl}(A)$ is a smallest I_r -closed set containing A , $A \subset I_r\text{-cl}(A) \subset \bigcup_{i=1}^n F_{x_i} \subset U$. Take $F = \bigcup_{i=1}^n F_{x_i}$, F is an I_r -closed set containing A such that $F \subset U$. Therefore by the definition, A is I_r^* -closed. $\square$

Definition 3.2.2. Let (X, τ) be a topological space and A be a subset of X . Then the I_r -kernel of A is the intersection of all I_r -open sets in X containing A and is denoted by $I_r\text{-ker}(A)$.

Example 3.2.2. In $X = \mathbb{R}$ with the usual topology, $I_r\text{-ker}([0, 1)) = [0, 1)$.

Theorem 3.2.13. Let (X, τ) be a topological space and $A \subset X$. If A is I_r -open, then $I_r\text{-ker}(A) = A$.

Proof. Proof follows from the definition. $\square$

Corollary 3.2.13.1. Let (X, τ) be a topological space and $A \subset X$. If A is regular open, then $I_r\text{-ker}(A) = A$.

Proof. Proof follows from the theorem 3.2.13 and remark 1.2. $\square$

Theorem 3.2.14. *Let (X, τ) be a topological space and $A \subset X$. Then I_r - $\ker(I_r(A)) = I_r(A)$.*

Proof. Proof follows from the theorem 3.2.13 and remark 2.5. $\square$

Theorem 3.2.15. *Let (X, τ) be a topological space and $A \subset X$. Then A is I_r^* -closed if and only if there is an I_r -closed set F such that $A \subset F \subset I_r\text{-}\ker(A)$.*

Proof. Let (X, τ) be a topological space and A be an I_r^* -closed subset of X . Then there is an I_r -closed set F in X containing A such that $F \subset U$ for every I_r -open subset U in X containing A . Therefore $A \subset F \subset \bigcap \{U : U \text{ is an } I_r\text{-open set containing } A\} = I_r\text{-}\ker(A)$.

Conversely suppose that there is an I_r -closed set F such that $A \subset F \subset I_r\text{-}\ker(A)$. Let U be an I_r -open set containing A . Then $A \subset F \subset I_r\text{-}\ker(A) \subset U$. This implies that $F \subset U$ and hence A is I_r^* -closed. $\square$

Chapter 4

I_r -Continuity and Related Function Classes

4.1 Introduction

The notion of I_r -open sets provides a broader framework for analyzing new continuity and related function properties in topological spaces. Traditionally, in classical topology, there exist well-established interrelations among continuity, openness, closedness, and invertibility of functions. These relationships play a central role in understanding the behavior of functions between topological spaces. The primary objective of this chapter is to investigate whether analogous relationships hold among various generalized function classes defined using I_r -open sets, I_r -continuous function. Beyond I_r -continuity, we introduce and analyze several associated function types within this framework, I_r -irresolute and

I_r-open functions. Throughout the chapter, we establish several fundamental theorems among these function types and their interaction with key topological properties such as *I_r-connectedness* and *I_r-compactness*. Particular focus is placed on the behavior of these functions under various structural constraints on the underlying spaces.

Before entering into the main topic, we present a few key definitions for clarity:

Definition 4.1.1. [22] A topological space X is said to be *compact* if every open cover of X has a finite subcover. That is, if $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets in X such that $X = \bigcup_{\alpha \in A} U_\alpha$, then there exists a finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset A$ such that $X = U_{\alpha_1} \cup U_{\alpha_2} \cup \dots \cup U_{\alpha_n}$.

Definition 4.1.2. [22] Let (X, τ) be a topological space and $A \subset X$. The *closure* of A , denoted by $cl(A)$ or $\overline{A}$, is defined as the smallest closed set containing A . Then, $cl(A) = \bigcap \{C \subset X : C \text{ is closed and } A \subset C\}$. A point $x \in X$ is called a *cluster point* of A if every open set U containing x satisfies $U \cap A \neq \emptyset$. Then $cl(A)$ is the set of all cluster points of A , $cl(A) = \{x \in X : \text{for every open set } U \text{ containing } x, U \cap A \neq \emptyset\}$.

Definition 4.1.3. [42] Let $f : X \rightarrow Y$ be a function between two topological spaces X and Y . Then f is said to be *almost continuous* if for every regular open set $V \subset Y$, the preimage $f^{-1}(V)$ is open in X .

4.2 I_r -Continuous Functions

Definition 4.2.1. Let (X, τ) and (Y, σ) be two topological spaces. Then a function $f : X \rightarrow Y$ is said to be I_r -continuous at a point $x \in X$, if for each I_r -open set V containing $f(x)$, there is an open set U containing x such that $f(U) \subset V$. A function $f : X \rightarrow Y$ is said to be I_r -continuous if it is I_r -continuous at each point x of X .

Example 4.2.1. Let (X, τ) and (Y, σ) be two topological spaces, where $X = \mathbb{R}$ with the usual topology τ , and $Y = \mathbb{Z}$ with topology $\sigma = \{\phi, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, \mathbb{Z} \setminus \{0\}\}$, where $\mathbb{Z}^- = \{\dots, -3, -2, -1\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$. Define a function $f : X \rightarrow Y$ by

$$f(x) = \begin{cases} \lceil x \rceil & \text{if } x \geq 0, \\ \lfloor x \rfloor & \text{if } x \leq 0, \end{cases}$$

where $\lfloor x \rfloor$ denotes the floor of x (the greatest integer less than or equal to x), and $\lceil x \rceil$ denotes the ceiling of x (the smallest integer greater than or equal to x).

Then the I_r -open sets in Y are $\phi, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+$. We find the inverse images:
 $f^{-1}(\phi) = \phi, \quad f^{-1}(\mathbb{Z}) = \mathbb{R}, \quad f^{-1}(\mathbb{Z}^-) = (-\infty, 0), \quad f^{-1}(\mathbb{Z}^+) = (0, \infty).$

Since $f^{-1}(\mathbb{Z}^-) = \{x \in \mathbb{R} : f(x) \in \mathbb{Z}^-\} = \{x \geq 0 : \lceil x \rceil \leq -1\} \cup \{x \leq 0 : \lfloor x \rfloor \leq -1\} = \phi \cup \{x \leq 0 : \lfloor x \rfloor \leq -1\} = \{x \in \mathbb{R} : x \leq 0 \text{ and } \lfloor x \rfloor \leq -1\} = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0)$ and $f^{-1}(\mathbb{Z}^+) = \{x \in \mathbb{R} : f(x) \in \mathbb{Z}^+\} = \{x \in \mathbb{R} : \lceil x \rceil \in \mathbb{Z}^+ \text{ if } x \geq 0, \text{ or } \lfloor x \rfloor \in \mathbb{Z}^+ \text{ if } x \leq 0\} = \{x \geq 0 : \lceil x \rceil \geq 1\} \cup \{x \leq 0 : \lfloor x \rfloor \geq 1\} = (0, \infty) \cup \phi = (0, \infty).$

Inverse of every I_r -open set in Y is open set in X . Therefore f is I_r -continuous.

Theorem 4.2.1. *Let $f : X \rightarrow Y$ be a function between topological spaces (X, τ) and (Y, σ) . If f is I_r -continuous, then for every subset $A \subset X$, $f(cl(A)) \subset I_r-cl(f(A))$.*

Proof. Let $f : X \rightarrow Y$ be I_r -continuous, $A \subset X$ and $x \in cl(A)$. Then $f(x) \in f(cl(A))$. Let V be any I_r -open set in Y containing $f(x)$. Since f is I_r -continuous at x , there exists an open set $U \subset X$ of x such that $f(U) \subset V$. Since $x \in cl(A)$, the set U must intersect A , so there exists $a \in A \cap U$. Then $f(a) \in f(A \cap U) \subset f(U) \subset V$, implies that $V \cap f(A) \neq \phi$. Hence $f(x) \in I_r-cl(f(A))$. Therefore, $f(cl(A)) \subset I_r-cl(f(A))$. $\square$

Theorem 4.2.2. *Every I_r -continuous function is almost continuous.*

Proof. Proof from the fact that every regular open set is I_r -open.

Remark. The converse of the above theorem need not be true.

Example 4.2.2. Let $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau)$ with τ as the usual topology, defined by $f(x) = x$. Then f is almost continuous. But not I_r -continuous, since $[1, 2]$ is I_r -open set and $f^{-1}([1, 2])$ is not open in $\mathbb{R}$.

Theorem 4.2.3. *Let X and Y be two topological spaces and $f : X \rightarrow Y$ be a function. Then the following are equivalent :*

i) f is I_r -continuous.

ii) Inverse image of every I_r -open subset of Y is open in X .

iii) Inverse image of every I_r -closed subset of Y is closed in X .

Proof. (i) $\iff$ (ii)

Suppose that f is I_r -continuous. Let U be an I_r -open subset of Y and $x \in f^{-1}(U)$.

Then $f(x) \in U$, there exists an open subset V in X such that $x \in V$ and $f(V) \subset U$. Thus $x \in V \subset f^{-1}(U)$ and therefore $f^{-1}(U)$ is open in X .

Conversely suppose that (ii) holds and let $x_0 \in X$. Given any I_r -open set V containing $f(x_0)$, $f^{-1}(V)$ is an open set containing x_0 and moreover $f(f^{-1}(V)) \subset V$. This shows that f is I_r -continuous at x_0 . Since x_0 is arbitrary, f is I_r -continuous.

(ii) $\iff$ (iii)

Let A be any I_r -closed subset of Y . Then $Y - A$ is I_r -open and $f^{-1}(Y - A) = X - f^{-1}(A)$ is open. Hence $f^{-1}(A)$ is closed.

Conversely suppose that (iii) holds and let V be an I_r -open subset of Y . Then $Y - V$ is I_r -closed and $f^{-1}(Y - V) = X - f^{-1}(V)$ is closed. Hence $f^{-1}(V)$ is open. □

Theorem 4.2.4. I_r -continuous image of a compact space is I_r -compact.

Proof. Let X and Y be two topological spaces, $f : X \rightarrow Y$ is I_r -continuous and X is compact. Let $\{A_i : i \in I\}$ be an I_r -open cover of $f(X)$. Then the collection $\{f^{-1}(A_i) : i \in I\}$ is an open cover of X . Since X is compact space, there exists a finite subset I_0 of I such that $X \subset \bigcup_{i \in I_0} f^{-1}(A_i)$. Therefore $f(X) \subset \bigcup_{i \in I_0} A_i$.

Hence $f(X)$ is I_r -compact. □

Theorem 4.2.5. *Let $f : X \rightarrow Y$ be an I_r -continuous function and $A \subset Y$. Then $cl(f^{-1}(A)) \subset f^{-1}(I_r-cl(A))$.*

Proof. Let $x \in cl(f^{-1}(A))$. If possible suppose that $f(x) \notin I_r-cl(A)$. Then there exists an I_r -open set $W \subset Y$ such that $f(x) \in W$ and $W \cap A = \phi$. Since f is I_r -continuous, $f^{-1}(W)$ is open in X and $x \in f^{-1}(W)$ and $f^{-1}(W) \cap f^{-1}(A) = f^{-1}(W \cap A) = \phi$. Since $f^{-1}(W)$ is an open set containing x disjoint from $f^{-1}(A)$, $x \notin cl(f^{-1}(A))$. This is a contradiction to the assumption. Therefore $f(x) \in I_r-cl(A)$, $x \in f^{-1}(I_r-cl(A))$. Hence $cl(f^{-1}(A)) \subset f^{-1}(I_r-cl(A))$. □

Theorem 4.2.6. *Let X and Y be two topological spaces and $f : X \rightarrow Y$ be I_r -continuous. If X is connected, then $f(X) \subset Y$ is I_r -connected.*

Proof. Let $f : X \rightarrow Y$ be an I_r -continuous and X is connected. If possible suppose that $f(X)$ is not I_r -connected. Then there exist nonempty subsets $A, B \subset f(X)$ such that $f(X) = A \cup B$, $I_r-cl(A) \cap B = \phi$ and $A \cap I_r-cl(B) = \phi$. Since $f^{-1}(A)$ and $f^{-1}(B)$ are the subsets of X and satisfy $X = f^{-1}(A) \cup f^{-1}(B)$. Since f is I_r -continuous, by theorem 4.2.5 $cl(f^{-1}(A)) \subset f^{-1}(I_r-cl(A))$, $cl(f^{-1}(B)) \subset f^{-1}(I_r-cl(B))$, $cl(f^{-1}(A)) \cap f^{-1}(B) \subset f^{-1}(I_r-cl(A) \cap B) = \phi$, $f^{-1}(A) \cap cl(f^{-1}(B)) \subset f^{-1}(A \cap I_r-cl(B)) = \phi$. Thus $f^{-1}(A)$ and $f^{-1}(B)$ are the separated subsets of X , contradicting the connectedness of X . Hence $f(X)$ is I_r -connected. □

4.3 I_r -Irresolute and I_r -Open Functions

Definition 4.3.1. A function $f : X \rightarrow Y$ is said to be I_r -irresolute if for every I_r -open set $V \subset Y$, the set $f^{-1}(V)$ is I_r -open in X .

Example 4.3.1. Let (x, τ) , (Y, σ) be two topological spaces, where $X = \mathbb{R}$, τ be the usual topology and $Y = \mathbb{Z}$, $\sigma = \{\phi, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, \mathbb{Z} - \{0\}\}$. Define function $f : X \rightarrow Y$ by $f(x) = [x]$, where $[x]$ denotes the floor of x , the greatest integer less than or equal to x .

Then I_r -open sets in Y are $\phi, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+$ and $f^{-1}(\phi) = \phi$, $f^{-1}(\mathbb{Z}) = \mathbb{R}$, $f^{-1}(\mathbb{Z}^-) = (-\infty, 0)$, $f^{-1}(\mathbb{Z}^+) = [1, \infty)$. Since $[1, \infty) = \bigcap_{n \in \mathbb{N}} (1 - \frac{1}{n}, \infty)$, is I_r -open set in $\mathbb{R}$. Then inverse of every I_r -open set in Y is I_r -open set in X . Therefore f is I_r -irresolute.

Theorem 4.3.1. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a I_r -irresolute. Then for every subset $A \subset X$, $f(I_r\text{-cl}(A)) \subset I_r\text{-cl}(f(A))$.

Proof. Let $f : X \rightarrow Y$ be I_r -irresolute and $A \subset X$. Let $x \in I_r\text{-cl}(A)$. Then $f(x) \in f(I_r\text{-cl}(A))$. Let V be any I_r -open set in Y containing $f(x)$. Since f is I_r -irresolute, $f^{-1}(V)$ is I_r -open set in X containing x . Since $x \in I_r\text{-cl}(A)$, the set $f^{-1}(V)$ must intersect A , so there exists some $a \in A \cap f^{-1}(V)$. Then $f(a) \in f(A \cap f^{-1}(V)) \subset f(f^{-1}(V)) \subset V$, implies that $V \cap f(A) \neq \phi$. Hence $f(x) \in I_r\text{-cl}(f(A))$. Therefore $f(I_r\text{-cl}(A)) \subset I_r\text{-cl}(f(A))$. $\square$

Theorem 4.3.2. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a I_r -irresolute. Then for every

subset $A \subset Y$, $f^{-1}(I_r\text{-int}(A)) \subset I_r\text{-int}(f^{-1}(A))$,

Proof. Let $A \subset Y$. Then by Proposition 1.4.1, $I_r\text{-int}(A) = \bigcup \{U \subset A \mid U \text{ is } I_r\text{-open in } Y\}$. Let $x \in f^{-1}(I_r\text{-int}(A))$. Then $f(x) \in I_r\text{-int}(A)$, so there exists an I_r -open set $V \subset A$ such that $f(x) \in V$. Since f is I_r -irresolute, $f^{-1}(V)$ is I_r -open in X , and $x \in f^{-1}(V) \subset f^{-1}(A)$. Hence $x \in I_r\text{-int}(f^{-1}(A))$. Therefore $f^{-1}(I_r\text{-int}(A)) \subset I_r\text{-int}(f^{-1}(A))$. $\square$

Theorem 4.3.3. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an I_r -irresolute. Then a subset F of X is I_r -compact relative to (X, τ) if and only if the image $f(F)$ is I_r -compact relative to (Y, σ) .*

Proof. Suppose that a subset F of X is I_r -compact relative to (X, τ) . Let $\{A_i : i \in I\}$ be an I_r -open cover of $f(F)$ by I_r -open subsets of (Y, σ) . Then $\{f^{-1}(A_i) : i \in I\}$ be an I_r -open cover of F by I_r -open subsets of (X, τ) . Since F is I_r -compact relative to (X, τ) , there exists a finite subset I_0 of I such that $F \subset \bigcup_{i \in I_0} f^{-1}(A_i)$. This implies that $f(F) \subset \bigcup_{i \in I_0} A_i$. Hence $f(F)$ is I_r -compact relative to (Y, σ) .

Conversely, suppose that $f(F)$ is I_r -compact relative to (Y, σ) . Let $\{B_j : j \in J\}$ be an I_r -open cover of F by I_r -open subsets of (X, τ) . Then $\{f(B_j) : j \in J\}$ be an I_r -open cover of $f(F)$ by I_r -open subsets of (Y, σ) . Since $f(F)$ is I_r -compact relative to (Y, σ) , there exists a finite subset J_0 of J such that $f(F) \subset \bigcup_{j \in J_0} f(B_j)$. This implies that $F \subset \bigcup_{j \in J_0} B_j$. Hence F is I_r -compact relative to (X, τ) . $\square$

Theorem 4.3.4. *Let $f : X \rightarrow Y$ be an I_r -irresolute and $A \subset Y$. Then $I_r\text{-}cl(f^{-1}(A)) \subset f^{-1}(I_r\text{-}cl(A))$.*

Proof. Let $x \in I_r\text{-}cl(f^{-1}(A))$. If possible suppose that $f(x) \notin I_r\text{-}cl(A)$. Then there exists an I_r -open set $W \subset Y$ such that $f(x) \in W$ and $W \cap A = \phi$. Since f is I_r -irresolute, $f^{-1}(W)$ is I_r -open in X and $x \in f^{-1}(W)$, and $f^{-1}(W) \cap f^{-1}(A) = f^{-1}(W \cap A) = \phi$. Since $f^{-1}(W)$ is an I_r -open set containing x disjoint from $f^{-1}(A)$, $x \notin I_r\text{-}cl(f^{-1}(A))$. This is a contradiction to the assumption. Therefore $f(x) \in I_r\text{-}cl(A)$, $x \in f^{-1}(I_r\text{-}cl(A))$. Hence $I_r\text{-}cl(f^{-1}(A)) \subset f^{-1}(I_r\text{-}cl(A))$. $\square$

Theorem 4.3.5. *Let X and Y be two topological spaces and $f : X \rightarrow Y$ be an I_r -irresolute. If X is I_r -connected, then $f(X) \subset Y$ is also I_r -connected.*

Proof. Let $f : X \rightarrow Y$ be an I_r -continuous and X is I_r -connected. If possible assume that $f(X)$ is not I_r -connected. Then there exist nonempty subsets $A, B \subset f(X)$ such that $f(X) = A \cup B$, $I_r\text{-}cl(A) \cap B = \phi$ and $A \cap I_r\text{-}cl(B) = \phi$. Since $f^{-1}(A)$ and $f^{-1}(B)$ are the subsets of X and satisfy $X = f^{-1}(A) \cup f^{-1}(B)$. Since f is I_r -irresolute, by theorem 4.3.4 $I_r\text{-}cl(f^{-1}(A)) \subset f^{-1}(I_r\text{-}cl(A))$, $I_r\text{-}cl(f^{-1}(B)) \subset f^{-1}(I_r\text{-}cl(B))$, $I_r\text{-}cl(f^{-1}(A)) \cap f^{-1}(B) \subset f^{-1}(I_r\text{-}cl(A) \cap B) = \phi$, $f^{-1}(A) \cap I_r\text{-}cl(f^{-1}(B)) \subset f^{-1}(A \cap I_r\text{-}cl(B)) = \phi$. Thus $f^{-1}(A)$ and $f^{-1}(B)$ are the I_r -separated subsets of X , contradicting the I_r -connectedness of X . Hence $f(X)$ is I_r -connected. $\square$

Definition 4.3.2. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be I_r -open if the

image of every open set in X is I_r -open in Y

Theorem 4.3.6. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a I_r -open function. Then for every subset $A \subset X$, $f(int(A)) \subset I_r-int(f(A))$.*

Proof. Let $A \subset X$ and $x \in int(A)$. Then there exists an open set V in X such that $x \in V \subset A$. Also $f(x) \in f(V) \subset f(A)$. Since f is I_r -open, $f(V)$ is I_r -open in Y . Hence $f(x) \in I_r-int(f(A))$. Therefore $f(int(A)) \subset I_r-int(f(A))$. $\square$

Theorem 4.3.7. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a I_r -open and onto function. Then for every subset $F \subset X$, $f^{-1}(I_rcl(F)) \subset cl(f^{-1}(F))$.*

Proof. Let $x \in f^{-1}(I_rcl(F))$, then $f(x) \in I_rcl(F)$. If possible suppose that $x \notin cl(f^{-1}(F))$, then there exists an open set V in X such that $f^{-1}(F) \cap V = \phi$. Then $V \subset (f^{-1}(F))^C$, $f(V) \subset F^C$. Since f is I_r -open, $f(V)$ is I_r -open in Y . Also since $f(x) \in f(V)$ and $f(V) \cap F = \phi$, $f(x) \notin I_rcl(F)$. This is a contradiction. Therefore $x \in cl(f^{-1}(F))$. Hence $f^{-1}(I_rcl(F)) \subset cl(f^{-1}(F))$. $\square$

Minimal Regular Open Sets

5.1 Introduction

In 2015 N. Anuradha and Baby Chacko [5] defined the concepts of minimal regular open sets and maximal regular closed sets, which extended the study of open and closed sets within the topological space. This chapter aims to investigate and explores the significant properties and results concerning the minimal regular open sets and maximal regular open sets.

The following definitions serve as foundational concepts required for the subsequent discussion:

Definition 5.1.1. [35] A proper nonempty open subset U of a topological space X is said to be *minimal open* if any open set which is contained in U is ϕ or U .

Definition 5.1.2. [5] A proper nonempty regular open subset U of a topological

space X is said to be *minimal regular open* if any regular open set which is contained in U is ϕ or U .

Definition 5.1.3. [5] A proper nonempty regular open subset U of a topological space X is said to be *maximal regular open* if any regular open set which contains U is X or U .

Definition 5.1.4. [5] A proper nonempty regular closed subset F of a topological space X is said to be *minimal regular closed* if any regular closed set which contained in F is ϕ or F .

Definition 5.1.5. [5] A proper nonempty regular closed subset F of a topological space X is said to be *maximal regular closed* if any regular closed set which contains F is X or F .

Lemma 5.1.1. [5] 1) If U is a minimal regular open set and W is a regular open set, then $W \cap U = \phi$ or $U \subset W$.

2) If U and V are minimal regular open sets, then $U \cap V = \phi$ or $U = V$.

5.2 Properties of Minimal Regular Open Sets

Definition 5.2.1. Let X be a topological space and $U \subset X$. Then U is said to be a *r-neighbourhood* of $x \in X$ if U is a regular open set containing x .

Proposition 5.2.1. If U is a minimal regular open set and x is an element of U , then $U \subset W$ for any *r-neighbourhood* W of x .

Proof. Let U be a minimal regular open set and $x \in U$. If W is any regular open set and $x \in W$, then $U \cap W$ is also regular open and $U \cap W \subset U$. Since $W \cap U \neq \phi$, $U \subset W$ for any r -neighbourhood W of x . $\square$

Theorem 5.2.2. *If U is a maximal regular closed set and x is an element of $X - U$, then $X - U \subset W$ for any r -neighbourhood W of x .*

Proof. Let W be any r -neighbourhood of x and $X - U$ is a minimal regular open set containing x . Since $x \in W \cap (X - U)$, $W \cap (X - U) \neq \phi$. By proposition 5.2.1, $X - U \subset W$ for any r -neighbourhood W of x . $\square$

Corollary 5.2.2.1. *If U is a maximal regular closed set, then either of the following (1) and (2) holds:*

- (1) *for each $x \in X - U$ and each r -neighbourhood W of x , $W = X$.*
- (2) *there exists a proper regular open set W such that $X - U \subset W$ and $W \subset X$.*

Proof. If (1) doesnot hold, then there exist an element x of $X - U$ and a r -neighbourhood W of x such that $W \subsetneq X$. Then by theorem 5.2.2, it follows that $X - U \subset W$. $\square$

Corollary 5.2.2.2. *If U is a maximal regular closed set, then either of the following (1) and (2) holds:*

- (1) *for each $x \in X - U$ and each r -neighbourhood W of x , $X - U \subsetneq W$.*
- (2) *there exists a proper regular open subset W of X such that $X - U = W$.*

Proof. Assume that (2) doesnot holds. Then by theorem 5.2.2, we have $X - U \subset$

W for each $x \in X - U$ and each r -neighbourhood W of x . Hence $X - U \subseteq W$. $\square$

Theorem 5.2.3. *Let U be a minimal regular open set and x be an element of $X - U$. Define $U_x = \cap\{W : W \text{ is a } r\text{-neighbourhood of } x\}$. Then either $U_x \cap U = \phi$ or $U \subset U_x$.*

Proof. By lemma 5.1.1, $W \cap U = \phi$ or $U \subset W$ for any r -neighbourhood W of x . That is, $W \cap U = \phi$ or $U \subset \cap\{W : W \text{ is a } r\text{-neighbourhood of } x\}$. Now $U_x \cap U = \cap\{W : W \text{ is } r\text{-neighbourhood of } x\} \cap U \subset W \cap U$. Hence either $U_x \cap U = \phi$ or $U \subset U_x$. $\square$

Theorem 5.2.4. *Let $U_\alpha, U_\beta, U_\gamma$ be minimal regular open sets such that $U_\alpha \neq U_\beta$. If $U_\gamma \subset (U_\alpha \cup U_\beta)$, then either $U_\gamma = U_\alpha$ or $U_\gamma = U_\beta$.*

Proof. We have

$$\begin{aligned}
 U_\alpha \cup U_\gamma &= U_\alpha \cup (U_\gamma \cup \phi) \\
 &= U_\alpha \cup (U_\gamma \cup (U_\alpha \cap U_\beta)) \\
 &= U_\alpha \cup ((U_\gamma \cup U_\alpha) \cap (U_\gamma \cup U_\beta)) \\
 &= (U_\alpha \cup U_\gamma) \cap (U_\alpha \cup U_\beta \cup U_\gamma) \\
 &= (U_\alpha \cup U_\gamma) \cap (U_\alpha \cup U_\beta) \\
 &= U_\alpha \cup (U_\gamma \cap U_\beta)
 \end{aligned}$$

If $U_\gamma \neq U_\beta$, then $U_\gamma \cap U_\beta = \phi$. Hence $U_\alpha \cup U_\gamma = U_\alpha$ which implies $U_\gamma \subset U_\alpha$.

Since U_γ and U_α are minimal regular open sets, it follows that $U_\gamma = U_\alpha$.

By a similar argument, if $U_\gamma \neq U_\alpha$ we get $U_\gamma = U_\beta$. $\square$

Theorem 5.2.5. *If U_α, U_β and U_γ are minimal regular open sets, which are different from each other, then $U_\alpha \cup U_\beta \not\subset U_\alpha \cup U_\gamma$.*

Proof. If possible suppose that $U_\alpha \cup U_\beta \subset U_\alpha \cup U_\gamma$. Then $(U_\alpha \cup U_\beta) \cap (U_\beta \cup U_\gamma) \subset (U_\alpha \cup U_\gamma) \cap (U_\beta \cup U_\gamma)$. This implies that $U_\beta \cup (U_\alpha \cap U_\gamma) \subset U_\gamma \cup (U_\alpha \cap U_\beta)$. Since $U_\alpha \cap U_\gamma = \phi = U_\alpha \cap U_\beta$, it follows that $U_\beta \subset U_\gamma$. Then $U_\beta = U_\gamma$, this is a contradiction and hence $U_\alpha \cup U_\beta \not\subset U_\alpha \cup U_\gamma$. $\square$

Theorem 5.2.6. *Let U_λ and U_μ be two distinct minimal regular open sets where $\lambda, \mu \in \Lambda$ with $|\Lambda| \geq 2$. Then,*

$$(i) \ U_\mu \subset X - \bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda.$$

$$(ii) \ \bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda \neq X.$$

Proof. (i) Let μ be any element of Λ . Then by lemma 5.2.1(2), we have $U_\mu \cap U_\lambda = \phi$, which implies that $U_\lambda \subset X - U_\mu$, for any element λ of Λ with $\lambda \neq \mu$. Then $\bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda \subset X - U_\mu$. Therefore $U_\mu \subset X - \bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda$.

(ii) Let μ be any element of Λ . If possible suppose that $\bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda = X$, we have $U_\mu = \phi$ by (i). This contradicts our assumption that U_μ is a minimal regular open set. Therefore $\bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda \neq X$. $\square$

Corollary 5.2.6.1. *If $|\Lambda| \geq 3$ and $U_\lambda \neq U_\mu$ for any two distinct minimal regular open sets U_λ and U_μ for $\lambda, \mu \in \Lambda$, then $U_\lambda \cup U_\mu \neq X$.*

Proof. The result follows by the theorem 5.2.6(2). $\square$

Theorem 5.2.7. *Let $|\Lambda| \geq 2$ and U_λ be a minimal regular open set for any $\lambda \in \Lambda$.*

If $U_\lambda \neq U_\mu$ for $\mu \in \Lambda$ with $\lambda \neq \mu$, then $\bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda \not\subset U_\mu \not\subset \bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda$.

Proof. Let μ be any element of Λ . If $U_\mu \subset \bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda$, then by theorem 5.2.6(1) $U_\mu \subset (X - \bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda) \cap (\bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda)$. This implies that $U_\mu = \phi$, which is a contradiction.

Also $\bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda \subset U_\mu \Rightarrow U_\lambda \subset U_\mu \Rightarrow U_\lambda = U_\mu$, for any $\lambda \in \Lambda - \{\mu\}$. This contradicts our assumption that $U_\lambda \neq U_\mu$. $\square$

Theorem 5.2.8. *If V is a nonempty finite regular open set, then there exists atleast one minimal regular open set U such that $U \subset V$.*

Proof. If V is a nonempty finite minimal regular open set, we may set $U = V$. If V is not a minimal regular open set, then there exists a nonempty regular open set V_1 such that $V_1 \subsetneq V$. If V_1 is a minimal regular open set, we may set $U = V_1$. If V_1 is not a minimal regular open set, then there exists a nonempty regular open set V_2 such that $V_2 \subsetneq V_1 \subsetneq V$. Continuing this process, we have a sequence of nonempty regular open sets $V_1, V_2, V_3, \dots$ such that $V \supsetneq V_1 \supsetneq V_2 \supsetneq \dots \supsetneq V_k \supsetneq \dots$. Since V is a finite set, this process repeats only finitely many times. Then we get a minimal regular open set $U = V_n$ for some positive integer n . $\square$

Remark. If U is an infinite regular open set, then there may not exist a minimal regular open set contained in U .

Example 5.2.1. Let $X = \mathbf{R}$, and $U = (a, b)$ where $a, b \in \mathbf{R}$.

Then U is regular open and every regular open set contained in U contains more than one regular open set.

Definition 5.2.2. A topological space X is said to be *r-locally finite* if each of its elements is contained in a finite regular open set. In other words, if every element of X has a finite r -neighbourhood.

Corollary 5.2.8.1. *If X is a r -locally finite space and V is a nonempty regular open set in X , then there exists at least one minimal regular open set U such that $U \subset V$.*

Proof. Since V is a nonempty regular open set, there exists an element $x \in V$. Since X is a r -locally finite space, there exists a finite regular open set V_x such that $x \in V_x$. Since $V \cap V_x$ is a finite regular open set, by theorem 5.2.8 there exists a minimal regular open set U such that $U \subset V \cap V_x \subset V$. $\square$

Theorem 5.2.9. *If V_λ is a regular open set for $\lambda \in \Lambda$ and W is a nonempty finite regular open set, then $W \cap (\cap_{\lambda \in \Lambda} V_\lambda)$ is a finite regular open set.*

Proof. Since W is a nonempty finite regular open set, there exists an integer n such that $W \cap (\cap_{\lambda \in \Lambda} V_\lambda) = W \cap (\cap_{i=1}^n V_{\lambda_i})$. Also Since finite intersection of regular open sets is regular open. Therefore $W \cap (\cap_{\lambda \in \Lambda} V_\lambda)$ is a finite regular open set. $\square$

Theorem 5.2.10. *Let U be a nonempty finite regular open set and $\{U_1, U_2, \dots, U_n\}$ be the set of all minimal regular open sets in U . Then the following two condi-*

tions hold:

- (1) $U_i \cap U_j = \phi$ for any i and j with $1 \leq i, j \leq n$ and $i \neq j$.
- (2) If U' is a minimal regular open set in U , then there exists i with $1 \leq i \leq n$ such that $U' = U_i$.

Proof. (1) Let U be a nonempty finite regular open set, and $\{U_1, U_2, \dots, U_n\}$ be the set of all minimal regular open sets in U . Suppose that there exist indices i and j with $1 \leq i, j \leq n$ and $i \neq j$ such that $U_i \cap U_j \neq \phi$.

Since U_i and U_j are both regular open sets, $U_i \cap U_j$ is also a regular open set. However, U_i and U_j are minimal regular open sets and $U_i \cap U_j$ is a nonempty regular open set contained in both U_i and U_j . This is a contradiction. Thus, our assumption must be false, and we conclude that $U_i \cap U_j = \phi$ for all $i \neq j$.

(2) Let U' be any minimal regular open set contained in U . By definition, U' is a regular open set, and since $\{U_1, U_2, \dots, U_n\}$ is the collection of all such minimal regular open sets in U , there must exist some index i such that $U' = U_i$. This completes the proof. $\square$

Theorem 5.2.11. *Let U be a nonempty finite regular open set which is not minimal regular open. Let $\{U_1, U_2, \dots, U_n\}$ be the set of all minimal regular open sets in U and x is an element of $U - (U_1 \cup U_2 \cup \dots \cup U_n)$. Define $U_x = \cap \{W : W \text{ is a } r\text{-neighbourhood of } x\}$. Then there exists a positive integer i , $1 \leq i \leq n$ such that $U_i \subset U_x$.*

Proof. Assume that $U_i \not\subset U_x$ for any positive integer i , $1 \leq i \leq n$. Since

$x \in U - (U_1 \cup U_2 \cup \dots \cup U_n) \subset U - U_i$ and corollary 5.2.3, $U_i \cap U_x = \phi$ for minimal regular open set U_i in U . Since U_x is the intersection of finite regular open sets and by theorems 5.2.8, there exists a minimal regular open set U' such that $U' \subset U_x$. Then $U_i \cap U' \subset U_i \cap U_x = \phi$ for minimal regular open set U_i . Therefore $U' \neq U_i$ for each positive integer i , $1 \leq i \leq n$. This is a contradiction to the fact that U' is a minimal regular open set contained in U . Therefore $U_i \subset U_x$ for $1 \leq i \leq n$. $\square$

Corollary 5.2.11.1. *Let U be a nonempty finite regular open set which is not a minimal regular open. Let $\{U_1, U_2, \dots, U_n\}$ be the set of all minimal regular open sets in U and x is an element of $U - (U_1 \cup U_2 \cup \dots \cup U_n)$. Then there exists a positive integer i , $1 \leq i \leq n$ such that $U_i \subset W_x$, for any r -neighbourhood W_x of x .*

Proof. Since $\cap\{W : W \text{ is an } r\text{-neighbourhood of } x\} \subset W_x$ for any r -neighbourhood W_x of x and by theorem 5.2.11, $U_i \subset W_x$ for any r -neighbourhood W_x of x . $\square$

Chapter 6

Somewhat mr -Continuous and Somewhat mr -Open Functions

6.1 Introduction

Gentre and Hoyle [27] explored the concepts of *somewhat continuous functions* and *somewhat open functions*. Anuradha and Baby Chacko [4] introduced and discussed about *minimal regular open sets* and *maximal regular open sets* in topological space. In this chapter, we introduce a new class of functions, namely *somewhat mr -continuous* and *somewhat mr -open functions*. we explores their properties and provide characterizations. The relationship between somewhat mr -continuous functions and other types of functions are studied, along with a discussion on *mr -weakly stronger topologies*. Also we focus on the definition and properties of somewhat mr -open functions.

Definition 6.1.1. [17] A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *somewhat continuous* if for each $U \in \sigma$ and $f^{-1}(U) \neq \phi$, there exists a nonempty open set V in X such that $V \subset f^{-1}(U)$.

Definition 6.1.2. [6] A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *somewhat r -continuous* if for each $U \in \sigma$ with $f^{-1}(U) \neq \phi$, there exists a nonempty regular open set V in X such that $V \subset f^{-1}(U)$.

Definition 6.1.3. [4] A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *somewhat open* if for each $U \in \tau$ with $U \neq \phi$, there exists a nonempty open set V in Y such that $V \subset f(U)$.

Definition 6.1.4. [4] A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *somewhat clopen* if for each $U \in \tau$ with $U \neq \phi$, there exists a clopen set V in Y with $V \neq \phi$ such that $V \subset f(U)$.

Definition 6.1.5. A function $f : X \rightarrow Y$ is said to be *cl-super continuous* [45] (or *clopen continuous* [39]) if for each $x \in X$ and for each open set V containing $f(x)$, there exists a clopen set U containing x such that $f(U) \subset V$.

Definition 6.1.6. A function $f : X \rightarrow Y$ is said to be *δ -continuous* [39] if for each $x \in X$ and for each regular open set V in Y containing $f(x)$, there exists a regular open set U in X containing x such that $f(U) \subset V$.

Definition 6.1.7. A function $f : X \rightarrow Y$ is said to be *completely continuous* [7] if $f^{-1}(V)$ is a regular open set in X for every open set $V \subset Y$.

Definition 6.1.8. A function $f : X \rightarrow Y$ is said to be *almost completely continuous* [27] if $f^{-1}(V)$ is a regular open set in X for every regular open set $V \subset Y$.

Theorem 6.1.1. [5] Let X be a topological space and $F \subset X$. Then F is minimal regular closed if and only if $X - F$ is maximal regular open.

Theorem 6.1.2. [5] Let X be a topological space and $U \subset X$. Then U is minimal regular open if and only if $X - U$ is maximal regular closed.

Definition 6.1.9. [22] Let (X, τ) be a topological space and let $A \subset X$. Then A is said to be *dense* in X if $\overline{A} = X$,. That is every point of X is either in A or is a limit point of A . Equivalently, A is dense in X if every non-empty open set in X intersects A , for every $U \in \tau$, $U \neq \phi \Rightarrow U \cap A \neq \phi$.

Definition 6.1.10. [21] Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same set X . Then $\mathcal{T}_1$ is said to be *stronger* (or *finer*) than $\mathcal{T}_2$ or $\mathcal{T}_2$ is said to be *weaker* (or *coarser*) than $\mathcal{T}_1$, if $\mathcal{T}_2 \subset \mathcal{T}_1$. That is, every open set in $\mathcal{T}_2$ is also an open set in $\mathcal{T}_1$. If $\mathcal{T}_2 \subsetneq \mathcal{T}_1$, we say that $\mathcal{T}_1$ is *strictly stronger* (or *strictly finer*) than $\mathcal{T}_2$.

6.2 Somewhat *mr*- continuous functions

Definition 6.2.1. Let X and Y be any two topological spaces. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *somewhat mr-continuous* if for each $U \in \sigma$ with $f^{-1}(U) \neq \phi$, there exists a minimal regular open set V in X and $V \subset f^{-1}(U)$.

Example 6.2.1. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{a, c\}\}$ and $\sigma = \{X, \phi, \{a, c\}\}$.

Define $f : X \rightarrow Y$ by $f(a) = c, f(b) = b$ and $f(c) = a$.

Here $f^{-1}(\phi) = \phi$, $f^{-1}(X) = X$, $f^{-1}(\{a, c\}) = \{a, c\}$ and $\{a, c\}$ is a minimal regular open set. Then f is somewhat mr -continuous.

Theorem 6.2.1. *Every somewhat mr -continuous functions is somewhat r -continuous.*

Proof. Since minimal regular open sets are regular open, the proof follows.

Remark. The converse of the above theorem need not be true.

Example 6.2.2. Let $X = Y = \mathbf{R}$ and $\tau = \sigma =$ The usual topology on $\mathbf{R}$.

Define $f(x) = x, x \in \mathbf{R}$. Then f is somewhat r -continuous but is not somewhat mr -continuous, since for $f^{-1}(a, b) = (a, b)$, there doesnot exists a minimal regular open set contained in (a, b) .

Remark. A minimal regular open set need not be a minimal open set and vice versa.

Example 6.2.3. Let (X, τ) be a topological space where $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}, \{c, d\}\}$.

Then $\{a, b\}$ is minimal regular open but is not minimal open.

Theorem 6.2.2. *Any function from a discrete space to any other space is somewhat mr -continuous.*

Proof. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ where τ be the discrete topology and σ be any topology on Y . For each $U \in \sigma$ with $f^{-1}(U) \neq \phi$, there exists some $x \in f^{-1}(U)$

such that $\{x\}$ is a minimal regular open set contained in $f^{-1}(U)$. Hence f is somewhat mr -continuous.

Definition 6.2.2. A space X is said to be a mr -space if for each $x \in X$ and for every r -neighbourhood U of x , there exists a minimal regular open set W containing x such that $W \subset U$.

Example 6.2.4. 1). Every discrete space is a mr -space.

2) Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{b\}, \{a, c\}, \{c\}\}$. Then (X, τ) is a mr -space.

Theorem 6.2.3. Every finite space is a mr -space.

Proof. By theorem 5.2.8, the proof follows.

Theorem 6.2.4. Every cl -super continuous function on a mr -space is somewhat mr -continuous.

Proof. Let $f : X \rightarrow Y$ be cl -super continuous and X be a mr -space. For each $x \in X$ and for each open set V containing $f(x)$, there exists a clopen set U containing x such that $f(U) \subset V$. Since X is a mr -space and U is regular open, there exists a minimal regular open set W containing x and $W \subset U$. Hence there exists a minimal regular open set W such that $x \in W$ and $f(W) \subset V$. Then $W \subset f^{-1}(V)$ and it follows that f is somewhat mr -continuous.

Remark. The converse of the theorem 6.2.4 need not be true.

Example 6.2.5. Let $X = \{p, q, r, s\}$, $Y = \{p, q, r\}$, $\tau = \{X, \phi, \{p, q\}, \{p\}, \{q\}, \{p, q, s\}\}$ and $\sigma = \{Y, \phi, \{p\}, \{q\}\}$.

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is somewhat *mr*-continuous, but is not *cl*-super continuous.

Theorem 6.2.5. *If $f : X \rightarrow Y$ is somewhat *mr*-continuous and X is locally indiscrete, then f is *cl*-super continuous.*

Proof. The proof follows from the fact that every minimal regular open set is open and every open set in a locally indiscrete space is clopen. $\square$

Theorem 6.2.6. *Every completely continuous function on a *mr*-space is somewhat *mr*-continuous.*

Proof. Let $U \in \sigma$ with $f^{-1}(U) \neq \phi$. Since $f : (X, \tau) \rightarrow (Y, \sigma)$ is a completely continuous function, $f^{-1}(U)$ is regular open in X . Further since X is a *mr*-space, there exists a minimal regular open set V such that $V \subset f^{-1}(U)$. Hence f is somewhat *mr*-continuous. $\square$

Remark. The converse of the above theorem need not be true.

Example 6.2.6. Let $X = Y = \{p, q, r\}$, $\tau = \{X, \phi, \{p\}, \{q\}, \{p, q\}\}$ and $\sigma = \{Y, \phi, \{p, q\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is somewhat *mr*-continuous. Since $\text{int cl}\{f^{-1}(\{p, q\})\} = X$, $f^{-1}(\{p, q\})$ is not regular. So f is not completely continuous.

Theorem 6.2.7. *Every onto somewhat *mr*-continuous function is δ -continuous.*

Proof. Let $f : X \rightarrow Y$ be somewhat *mr*-continuous and V be a nonempty regular open set in Y . Since f is somewhat *mr*-continuous and V is open with

$f^{-1}(V) \neq \phi$, there exists a minimal regular open set U such that $U \subset f^{-1}(V)$.

Hence f is δ -continuous. $\square$

Remark. The converse of the above theorem need not be true.

Example 6.2.7. Let $f : (X, \tau) \rightarrow (Y, \sigma)$, $X = \mathbb{R}$, $Y = \{p, q, r\}$, τ = Usual topology on $\mathbb{R}$ and $\sigma = \{Y, \phi, \{p\}, \{r\}, \{p, r\}\}$. Define $f(x) = \begin{cases} p, & \text{if } x > 0 \\ q, & \text{if } x = 0 \\ r, & \text{if } x < 0 \end{cases}$.

Then f is δ -continuous, but not somewhat mr -continuous.

Theorem 6.2.8. *Let X be finite. Then $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat r -continuous if and only if f is somewhat mr -continuous.*

Proof. Suppose that X is finite and $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat r -continuous. Let $U \in \sigma$ with $f^{-1}(U) \neq \phi \in \tau$. Then there exists a regular open set V in X such that $V \subset f^{-1}(U)$. Since X is finite, V is a finite regular open set. Then by Theorem 5.2.8, there exists a minimal regular open set W in X such that $W \subset V \subset f^{-1}(U)$. This implies f is somewhat mr -continuous. The converse part follows from the definition. $\square$

Theorem 6.2.9. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is almost completely continuous, Y is locally indiscrete and X is mr -space, then f is somewhat mr -continuous.*

Proof. Let V be open in Y with $f^{-1}(V) \neq \phi$. Since Y is locally indiscrete, V is regular open and since f is almost completely continuous, $f^{-1}(V)$ is regular

open. As X is a *mr*-space, there exists a minimal regular open set W in X such that $W \subset f^{-1}(V)$. Hence f is somewhat *mr*-continuous. $\square$

Theorem 6.2.10. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat continuous and X is finite and locally in-discrete, then f is somewhat *mr*-continuous.*

Proof. Since f is somewhat continuous, for each $U \in \sigma$ with $f^{-1}(U) \neq \phi$, there exists an open set $V \neq \phi$ such that $V \subset f^{-1}(U)$. Since X is locally in-discrete, V is closed and hence V is regular open. Since X is finite, by theorem 5.2.8 there exists a minimal regular open set W such that $W \subset V$. Thus $W \subset V \subset f^{-1}(U)$. This implies $W \subset f^{-1}(U)$ for each $U \in \sigma$ and therefore f is somewhat *mr*-continuous. $\square$

Definition 6.2.3. A topological space is said to be a *minimal space* (*m-space*) if every open set in X is minimal open.

Theorem 6.2.11. *Every *m-space* X contains at most two nonempty disjoint proper open sets.*

Proof. On the contrary suppose that X contains more than two open sets. Without loss of generality let X contain three distinct nonempty proper disjoint open sets A, B and C . Let $D = A \cup B$.

Case I: If $D \neq X$

Since X is a *m-space* and D is open, D is minimal open. This is a contradiction to the fact that $A \subset D$ and $B \subset D$.

Case II: If $D = X$

Then $C \subset A \cup B$. Since A and B are disjoint, either $C \subset A$ and $C \subset B$. This is a contradiction to the fact that A and B are minimal open. $\square$

Theorem 6.2.12. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is open and δ -continuous and Y is locally indiscrete and m -space, then f is somewhat mr -continuous.*

Proof. Let $U \in \sigma$ with $f^{-1}(U) \neq \phi$. Since Y is locally indiscrete, U is closed. Since X is m -space, U is minimal regular open. Since f is δ -continuous and U is regular open, for each $x \in f^{-1}(U)$ there exists a regular open set V containing x in X such that $f(V) \subset U$. Thus $V \subset f^{-1}(U)$ and $V \neq \phi$. If V is minimal regular open, then f is somewhat mr -continuous. If V is not minimal regular open, then there exists a nonempty regular open set W such that $W \subset V$. Thus $W \subset V \subset f^{-1}(U)$. This implies that $f(W) \subset U$ and also $f(W)$ is open, since f is open. Hence $f(W) \neq \phi$ and $f(W) \subset U$, which is a contradiction to the fact that U is minimal open. $\square$

Theorem 6.2.13. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be two functions. If f is somewhat mr -continuous and g is continuous, then $g \circ f$ is somewhat mr -continuous.*

Proof. Let $U \in \eta$ and $(g \circ f)^{-1}(U) \neq \phi$. Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$, $g^{-1}(U) \neq \phi$. Since f is somewhat mr -continuous and g is continuous, there exists a minimal regular open set V in X such that $V \subset f^{-1}(g^{-1}(U))$. Thus $V \subset (g \circ f)^{-1}(U)$. Therefore $g \circ f$ is somewhat mr -continuous. $\square$

Remark. Somewhat mr -continuous function need not be continuous.

Let $X = Y = \{a, b, c, d\}$, $\tau = \{X, \phi, \{b\}, \{a, c\}, \{a, b, c\}\}$ and $\sigma = \{X, \phi, \{a, b\}, \{a, b, c\}\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = b, f(c) = d, f(d) = c$. Here $f^{-1}(\{a, b, c\}) = \{a, b, d\}$, $f^{-1}(\{a, b\}) = \{a, b\}$ are not open in (X, τ) . But $\{b\}$ is a minimal regular open set in (X, τ) contained in both $\{a, b, d\}$ and $\{a, b\}$. Therefore f is somewhat mr -continuous, but not continuous.

Theorem 6.2.14. *If $f : X \rightarrow Y$ is continuous and $g : Y \rightarrow Z$ is somewhat mr -continuous, then $g \circ f : X \rightarrow Z$ is somewhat continuous.*

Proof. Let $U \subset Z$ be open with $g^{-1}(U) \neq \phi$. Since g is somewhat mr -continuous function, there exist minimal regular open set V such that $V \subset g^{-1}(U)$. Since $V \neq \phi$ is open and f is continuous, $f^{-1}(V) \neq \phi$ is open. Also $f^{-1}(V) \subset f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. Therefore $g \circ f : X \rightarrow Z$ is somewhat continuous. □

Theorem 6.2.15. *Let (X, τ) be a topological space, A be a regular open subset of X and V be a minimal regular open subset of $(A, \tau/A)$. Then V is a minimal regular open subset of X .*

Proof. Let A be a regular open subset of X and V be a minimal regular open subset of $(A, \tau/A)$. Since V is a regular open subset of A , $V = W \cap A$ for some regular open subset W of X and so V is a regular open subset of X . If possible assume that V is not a minimal regular open subset of X . Then there exists a nonempty regular open subset W_1 of X such that $W_1 \subset V$ and then

$(W_1 \cap A) \subset (V \cap A) = W \cap A = V$. Since $W_1 \cap A$ is a nonempty regular open subset of A , V is not a minimal regular open subset of A . This is a contradiction and hence V is a minimal regular open subset of X . $\square$

Theorem 6.2.16. *Let (X, τ) and (Y, σ) be two topological spaces. Let A be a regular open set of (X, τ) and $f : (A, \tau/A) \rightarrow (Y, \sigma)$ be somewhat mr -continuous such that $f(A)$ is dense in Y . Then any extension $F : (X, \tau) \rightarrow (Y, \sigma)$ of f is somewhat mr -continuous.*

Proof. Let U be an open set in (Y, σ) such that $F^{-1}(U) \neq \phi$. Since $f(A) \subset Y$ is dense in Y , $U \cap f(A) \neq \phi$. So $f^{-1}(U) \cap A \neq \phi$. Since f is somewhat mr -continuous, there exist a minimal regular open set V in A such that $V \subset f^{-1}(U)$. Since by theorem 6.2.15, V is a minimal regular open set in A and A is a regular open subset of X , then V is a minimal regular open subset of X and also $V \subset F^{-1}(U)$. Hence F is somewhat mr -continuous. $\square$

Definition 6.2.4. Let M be a subset of a topological space (X, τ) . Then M is said to be mr -dense in X , if there does not exist a maximal regular closed set C in X such that $M \subset C \subset X$.

Theorem 6.2.17. [5] *Let X be a topological space and $F \subset X$. Then F is minimal regular closed (open resp.) if and only if $X - F$ is maximal regular open (closed resp.).*

Theorem 6.2.18. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an onto function, then the following are equivalent :*

(1) *f* is somewhat *mr*-continuous.

(2) If *C* is a closed subset of *Y* such that $f^{-1}(C) \neq X$, then there is a maximal regular closed subset *D* of *X* such that $D \supset f^{-1}(C)$.

(3) If *M* is *mr*-dense in *X*, then *f*(*M*) is dense in *Y*.

Proof. If σ is an indiscrete topology, the proof is trivial.

Otherwise :

(1) $\implies$ (2)

Let *C* be a closed subset of *Y* such that $f^{-1}(C) \neq X$. Then $Y - C$ is open in *Y* such that $f^{-1}(Y - C) = X - f^{-1}(C) \neq \phi$. Since *f* is somewhat *mr*-continuous, there exists a minimal regular open set *V* such that $V \subset X - f^{-1}(C)$. Thus $f^{-1}(C) \subset X - V$. Take $D = X - V$. Since *V* is minimal regular open, *D* is maximal regular closed. Hence there exists a maximal regular closed set *D* in *X* such that $D \supset f^{-1}(C)$.

(2) $\implies$ (1)

Let $U \in \sigma$ with $f^{-1}(U) \neq \phi$. Then $Y - U$ is closed in *Y* and $f^{-1}(Y - U) = X - f^{-1}(U) \neq X$. So by (2) there exists a maximal regular closed subset *D* of *X* such that $D \supset f^{-1}(Y - U) = X - f^{-1}(U)$. Then $X - D$ is minimal regular open in *Y* and $X - D \subset f^{-1}(U)$. Hence *f* is somewhat *mr*-continuous.

(2) $\implies$ (3)

Let *M* be *mr*-dense in *X*. Suppose that *f*(*M*) is not dense in *Y*. Then there exists a proper closed set *C* in *Y* such that $f(M) \subset C \subset Y$. Clearly $f^{-1}(C) \neq X$. Hence by (2), there exists a maximal regular closed set *D* in *X* such that

$D \supset f^{-1}(C)$. Then $M \subset f^{-1}(C) \subset D \subset X$. This contradicts the fact that M is mr -dense in X . So $f(M)$ is dense in Y .

(3) $\implies$ (2)

Suppose (2) is not true and let C be a closed set C in Y such that $f^{-1}(C) \neq X$. Then there is no maximal regular closed set D in X such that $f^{-1}(C) \subset D$. This implies that $f^{-1}(C)$ is mr -dense in X . But by (3), $f(f^{-1}(C)) = C$ must be dense in Y . This is a contradiction to the choice of C . $\square$

Theorem 6.2.19. *Every dense subset in a topological space (X, τ) is mr -dense in X .*

Proof. Let (X, τ) be any topological space and C be any dense subset of X . Then by the definition of dense subset there does not exist a proper closed set V in X such that $C \subset V$. This implies that there does not exist a maximal regular closed set F in X satisfying the property $C \subset F \subset X$ since every maximal regular closed set is a closed set. Hence C is mr -dense in X . $\square$

Remark. A mr -dense subset a topological space X need not be a dense subset.

Example 6.2.8. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then $\overline{\{b, c\}} = \{b, c, d\}$, $\{b, c\}$ is not dense in X . But $\{b, c\}$ is mr -dense in X , since there is no maximal regular closed set in X containing $\{b, c\}$.

Theorem 6.2.20. *Let (X, τ) and (Y, σ) be any two topological spaces, $X = A \cup B$ where A and B are regular open subsets of X and let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function such that f/A and f/B are somewhat mr -continuous. Then f is also*

somewhat mr -continuous.

Proof. Let $U \in \sigma$ such that $f^{-1}(U) \neq \phi$. Then either $(f/A)^{-1}(U) \neq \phi$ or $(f/B)^{-1}(U) \neq \phi$

Case-1: $(f/A)^{-1}(U) \neq \phi$

Since f/A is somewhat mr -continuous, there exists a minimal regular open set V in A such that $V \subset (f/A)^{-1}(U) \subset (f)^{-1}(U)$. By theorem 6.2.15, since V is minimal regular open in A and A is regular open in X , V is minimal regular open in X . Hence f is somewhat mr -continuous.

Case-2: $(f/B)^{-1}(U) \neq \phi$

This can be proved by using the same argument as in case-1.

Case-3: $(f/A)^{-1}(U) \neq \phi$ and $(f/B)^{-1}(U) \neq \phi$

The proof follows from the proofs of case-1 and case-2. □

Definition 6.2.5. A topological space X is said to be mr -separable if there exists a countable mr -dense subset B of X .

Example 6.2.9. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. Since $\{a, b\}$ is mr -dense in X and countable, (X, τ) is mr -separable.

Theorem 6.2.21. *If f is a somewhat mr -continuous function from X onto Y and X is mr -separable, then Y is separable.*

Proof. Let $f : X \longrightarrow Y$ be somewhat mr -continuous and X be mr -separable. Then there exists a countable subset B of X which is mr -dense in X and by

theorem 6.2.18, $f(B)$ is dense in Y . Since B is countable, $f(B)$ is countable and hence Y is separable. $\square$

Definition 6.2.6. Let τ and σ be two topologies on a set X . Then σ is said to be *mr-weakly stronger than τ* (or τ is *mr-weakly weaker than σ*) if for each $U \in \tau$ with $U \neq \phi$, there is a minimal regular open set V in σ such that $V \subset U$.

Example 6.2.10. Let $X = \{a, b, c\}$ and $\sigma = \{X, \phi, \{b, c\}, \{a\}\}$, $\tau = \{X, \phi, \{a, b\}\}$ be two topologies on X . Then $\{a, b\}$ is the only proper nonempty open set in (X, τ) . Since $\text{intcl}\{a\} = \text{int}\{a\} = \{a\}$, the set $\{a\}$ is minimal regular open in (X, σ) such that $\{a\} \subset \{a, b\}$. Hence σ is *mr-weakly stronger than τ* .

Theorem 6.2.22. Let τ and τ^* be two topologies on X such that τ^* is *mr-weakly stronger than τ* . If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is somewhat continuous, then $f : (X, \tau^*) \longrightarrow (Y, \sigma)$ is somewhat *mr*-continuous.

Proof. Let U be an open set in Y such that $f^{-1}(U) \neq \phi$. Since $f : (X, \tau) \longrightarrow (Y, \sigma)$ is somewhat continuous, there exists an open set V in τ such that $V \neq \phi$ and $V \subset f^{-1}(U)$. Since τ^* is *mr-weakly stronger than τ* , there exists a minimal regular open set W in τ^* such that $W \subset V$. Then $W \subset V \subset f^{-1}(U)$ and hence $f : (X, \tau^*) \longrightarrow (Y, \sigma)$ is somewhat *mr*-continuous. $\square$

Theorem 6.2.23. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be somewhat *mr*-continuous and onto. If σ^* is a topology on Y such that σ is stronger than σ^* , then $f : (X, \tau) \longrightarrow (Y, \sigma^*)$ is somewhat *mr*-continuous.

Proof. Let $U \in \sigma^*$ such that $f^{-1}(U) \neq \phi$. Since $U \neq \phi$ and σ is stronger than σ^* , there exists a nonempty open set V in σ such that $V \subset U$. Since f is onto, $f^{-1}(V) \neq \phi$. Further since $f : (X, \tau) \longrightarrow (Y, \sigma)$ is somewhat *mr*-continuous, there exists a minimal regular open set W in τ such that $W \subset f^{-1}(V)$. Then $W \subset f^{-1}(U)$ and hence $f : (X, \tau) \longrightarrow (Y, \sigma^*)$ is somewhat *mr*-continuous. $\square$

6.3 Somewhat *mr*-Open Functions

Definition 6.3.1. A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be *somewhat mr-open* if for each $U \in \tau$ with $U \neq \phi$, there exists a minimal regular open set $V \in \sigma$ such that $V \subset f(U)$.

Example 6.3.1. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}\}$. Define a function $f : (X, \tau) \longrightarrow (Y, \sigma)$ by $f(a) = b, f(b) = c$ and $f(c) = a$. Then f is somewhat *mr*-open.

Theorem 6.3.1. *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a somewhat clopen function and Y is finite, then f is somewhat *mr*-open.*

Proof. Let $U \in \tau$ with $U \neq \phi$. Since f is somewhat clopen, there exists a clopen set V in Y such that $V \subset f(U)$. Then V is regular open. Since Y is finite, V is a finite regular open set. Then by theorem 5.2.8, there exists a minimal regular open set W such that $W \subset V \subset f(U)$. This implies that f is somewhat *mr*-open. $\square$

Theorem 6.3.2. *If $f : X \longrightarrow Y$ is somewhat *mr*-open and X is locally indiscrete, then f is somewhat clopen.*

Proof. Let $U \neq \phi$ be open in X . Since f is somewhat *mr*-open, there exists a minimal regular open set V in Y such that $V \subset f(U)$. Since X is locally indiscrete, V is clopen. This implies that f is somewhat clopen. $\square$

Theorem 6.3.3. *If $f : X \longrightarrow Y$ is somewhat *mr*-open, then f is somewhat open.*

Proof. The proof follows from the fact that every regular open set is open. $\square$

Remark. The converse of the theorem 6.3.3 need not be true.

Example 6.3.2. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{X, \phi, \{b\}\}$. Define $f : (X, \tau) \longrightarrow (Y, \sigma)$ by $f(a) = b, f(b) = c, f(c) = a, f(\{a\}) = \{b\}, f(\{a, b\}) = \{b, c\}$. Here f is somewhat open, but not somewhat *mr*-open, since $\{b\}$ is not regular open.

Theorem 6.3.4. *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is an open function and $g : (Y, \sigma) \longrightarrow (Z, \eta)$ is a somewhat *mr*-open function, then $g \circ f : (X, \tau) \longrightarrow (Z, \eta)$ is somewhat *mr*-open.*

Proof. Let $U \in \tau$ and $U \neq \phi$. Since f is an open function, $f(U)$ is open with $f(U) \neq \phi$. Since g is somewhat *mr*-open and $f(U) \in \sigma$ with $f(U) \neq \phi$, there exists a minimal regular open set $V \in \eta$ such that $V \subset g(f(U)) = (g \circ f)(U)$. Hence $g \circ f$ is somewhat *mr*-open. $\square$

Theorem 6.3.5. *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a one-one and onto function, then the following are equivalent:*

- (1) *f is somewhat *mr*-open.*
- (2) *If C is a closed subset of X such that $f(C) \neq Y$, then there is a maximal regular closed subset D of Y such that $D \supset f(C)$.*

Proof. (1) $\implies$ (2)

Let C be a closed subset of X such that $f(C) \neq Y$. Then $X - C$ is open in X and $X - C \neq \phi$. Since f is somewhat *mr*-open, there exists a minimal regular open set V in Y such that $V \subset f(X - C)$. Take $D = Y - V$ so that D is a maximal regular closed set in Y . Also $D = Y - V \supset Y - f(X - C) = f(C)$.

(2) $\implies$ (1).

Let $U \in \tau$ and $U \neq \phi$. Take $C = X - U$ so that C is a closed subset of X . Since $f(U) \neq \phi$, $f(C) = f(X - U) = Y - f(U) \neq Y$. So by (2), there exists a maximal regular closed subset D of Y such that $D \supset f(C)$. Choose $V = Y - D$ so that V is minimal regular open. Also $V = (Y - D) \subset (Y - f(C)) = Y - (Y - f(U)) = f(U)$. Hence f is somewhat *mr*-open. $\square$

Theorem 6.3.6. *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a somewhat *mr*-open function and A is an open subset of X , then $f|_A : (A, \tau|_A) \longrightarrow (Y, \sigma)$ is also somewhat *mr*-open.*

Proof. Let $U \in \tau|_A$ and $U \neq \phi$. Since U is open in A and A is open in X , U is open in X . Since $f : (X, \tau) \longrightarrow (Y, \sigma)$ is somewhat *mr*-open, there exists a minimal regular open set V in Y such that $V \subset f(U) = (f|_A(U))$. Hence $f|_A$ is

somewhat *mr*-open. □

Theorem 6.3.7. *Let (X, τ) and (Y, σ) be any two topological spaces, $X = A \cup B$ where A and B are open subsets of X . If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a function such that $f|_A$ and $f|_B$ are somewhat *mr*-open, then f is also somewhat *mr*-open.*

Proof. Let $U \in \tau$ with $U \neq \phi$. Then $f(U) \neq \phi$.

Case-1: $(f|_A)(U) \neq \phi$

Since $f|_A$ is a somewhat *mr*-open, there exists a minimal regular open set V in Y such that $V \subset (f|_A)(U) \subset f(U)$. So f is somewhat *mr*-open function.

Case-2: $(f|_B)(U) \neq \phi$

This can be proved by the same argument as in case-1.

Case-3: $(f|_A)(U) \neq \phi$ and $(f|_B)(U) \neq \phi$

The proof follows from the proofs of case-1 and case-2. □

Minimal Regular Open Sets in Bitopological Space

7.1 Introduction

In 1963, J. C. Kelly [23] defined a *bitopological space* (X, τ_1, τ_2) as a set X equipped with two topologies τ_1 and τ_2 . The study of bitopological spaces has since attracted significant interest, particularly in exploring how various generalizations of open and closed sets behave under two topological structures. In 2012, S. S. Benchalli, Basavaraj M. Ittanagi, and R. S. Wali [10] introduced the concepts of *pairwise minimal open sets* (respectively (i, j) -minimal open sets) and *pairwise maximal open sets* (respectively (i, j) -maximal open sets) in the context of bitopological spaces. These notions further generalized the idea of minimality and maximality of sets relative to each topology in the pair. In 2015,

N. Anuradha and Baby Chacko studied *minimal regular open* (respectively, *minimal regular closed*) sets and *maximal regular open* (respectively, *maximal regular closed*) sets. These sets combine the properties of regularity with minimal and maximal structures, providing a finer classification within the topology.

In this chapter, we introduce and investigate *pairwise minimal regular open sets* (respectively (i, j) -minimal regular open sets) and *pairwise maximal regular open sets* (respectively (i, j) -maximal regular open sets) in bitopological spaces. We explore their properties and analyze how they relate to other types of open sets within the bitopological framework. We also introduce and examine new separation axioms, namely $Pairwise-rT_{\min}$ and $Pairwise-rT_{\max}$ spaces, as well as the concepts of (i, j) - r -door spaces and (i, j) - $rT_{\frac{1}{2}}$ -spaces. These spaces provide further insights into the structure and behavior of bitopological spaces under regular open and closed conditions. In addition, we define and study several types of continuous functions such as (i, j) -almost continuous, (i, j) -almost completely continuous, and (i, j) -almost perfectly continuous functions. We also consider functions defined with respect to a single topology, such as i -completely continuous, i -strongly continuous, i -almost continuous, i -almost completely continuous, i -almost perfectly continuous, and i -totally continuous functions. The relationships among these classes of functions are thoroughly explored. Finally, we introduce two new classes of functions p -Min r -Continuous Maps and p -Max r -Continuous Maps and study their properties and interconnections with the existing classes of functions in bitopological spaces. These investigations aim to

enhance the understanding of continuity in the setting of bitopological structures. Throughout this chapter, a subset A of a bitopological space (X, τ_1, τ_2) is said to be τ_i -open (regular open, minimal open, minimal regular open respectively) if it is open (regular open, minimal open, minimal regular open respectively) with respect to the topology τ_i for $i \in \{1, 2\}$. Also interior (respectively closure) of A with respect to the topology $\tau_i, i \in \{1, 2\}$ denoted by $i\text{-int}(A)$ (respectively $i\text{-cl}(A)$).

Before proceeding to the main results, we present essential preliminary definitions that will be used throughout this chapter.

Definition 7.1.1. [23] A set X equipped with two (arbitrary) topologies τ_1 and τ_2 is called a *bitopological space* and is denoted by (X, τ_1, τ_2) .

Definition 7.1.2. ([41], [24]) A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j) -regular open (resp. (i, j) -regular closed) if $i\text{-int}(j\text{-cl}(A)) = A$ (resp. $i\text{-cl}(j\text{-int}(A)) = A$) for each $i, j \in \{1, 2\}$ with $i \neq j$. A is called pairwise regular open (resp. pairwise regular closed) if it is both $(1, 2)$ -regular open and $(2, 1)$ -regular open (resp. $(1, 2)$ -regular closed and $(2, 1)$ -regular closed).

Definition 7.1.3. [30] A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j) -pre open (resp. (i, j) -pre closed) if $A \subset i\text{-int}(j\text{-cl}(A))$ (resp. $i\text{-cl}(j\text{-int}(A)) \subset A$) for each $i, j \in \{1, 2\}$ with $i \neq j$.

Definition 7.1.4. [10] Let $i, j \in \{1, 2\}, i \neq j$ and (X, τ_1, τ_2) be a bitopological space.

- (i) A proper nonempty τ_i -open subset M in X is said to be a (i, j) -*minimal open set* if any τ_j -open set which is contained in M is either ϕ or M itself.
- (ii) A proper nonempty τ_i -open subset M in X is said to be a (i, j) -*maximal open set* if any τ_j -open set which contains M is either X or M itself.
- (iii) A proper nonempty τ_i -closed subset F in X is said to be a (i, j) -*minimal closed set* if any τ_j -closed set which is contained in F is either ϕ or F itself.
- (iv) A proper nonempty τ_i -closed subset F in X is said to be a (i, j) -*maximal closed set* if any τ_j -closed set which contains F is either X or F itself.

7.2 p -Minimal Regular Open Sets and p -Maximal Regular Open Sets

Definition 7.2.1. Let $i, j \in \{1, 2\}, i \neq j$ and (X, τ_1, τ_2) be a bitopological space.

- (i) A subset M in X is said to be a (i, j) -*minimal regular open* if M is a proper nonempty τ_i -regular open set and any τ_j -regular open set which is contained in M is either ϕ or M itself. A subset M is said to be *pairwise minimal regular open* (*p -minimal regular open*), if it is both $(1, 2)$ -minimal regular open and $(2, 1)$ -minimal regular open.
- (ii) A subset M in X is said to be a (i, j) -*maximal regular open* if M is a proper nonempty τ_i -regular open set and any τ_j -regular open set which contains

M is either X or M itself. A subset M is said to be *pairwise maximal regular open* (*p -maximal regular open*), if it is both $(1, 2)$ -maximal regular open and $(2, 1)$ -maximal regular open.

(iii) A subset F in X is said to be a (i, j) -*minimal regular closed* if F is a proper nonempty τ_i -regular closed set and any τ_j -regular closed set which is contained in F is either ϕ or F itself. A subset F is said to be *pairwise minimal regular closed* (*p -minimal regular closed*), if it is both $(1, 2)$ -minimal regular closed and $(2, 1)$ -minimal regular closed.

(iv) A subset F in X is said to be a (i, j) -*maximal regular closed* set if F is a proper nonempty τ_i -regular closed set and any τ_j -regular closed set which contains F is either X or F itself. A subset F is said to be *pairwise maximal regular closed* (*p -maximal regular closed*), if it is both $(1, 2)$ -maximal regular closed and $(2, 1)$ -maximal regular closed.

The family of all (i, j) -minimal regular (respectively maximal regular) open sets is denoted by (i, j) - $M_iRO(X)$ (respectively (i, j) - $M_aRO(X)$) and the family of all (i, j) -minimal regular (respectively maximal regular) closed sets is denoted by (i, j) - $M_iRC(X)$ (respectively (i, j) - $M_aRC(X)$). The family of all p -minimal regular (respectively maximal regular) open sets is denoted by $PM_iRO(X)$ (respectively $PM_aRO(X)$) and the family of all p -minimal regular (respectively maximal regular) closed sets is denoted by $PM_iRC(X)$ (respectively $PM_aRC(X)$).

Result 7.2.1. *Let $i, j \in \{1, 2\}, i \neq j$ and (X, τ_1, τ_2) be a bitopological space.*

1. Every (i, j) -minimal regular open set is τ_i -regular open, but not conversely.
2. Every (i, j) -maximal regular open set is τ_i -regular open, but not conversely.

Example 7.2.1. Let (X, τ_1, τ_2) be a bitopological space, where $X = \{a, b, c, d\}$ and $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

Collection of τ_1 -regular open sets = Collection of τ_1 -regular closed sets = $\{\{a, b\}, \{c, d\}, \phi, X\}$,

Collection of τ_2 -regular open sets = Collection of τ_2 -regular closed sets = $\{\{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \phi, X\}$.

$$(1, 2)-M_iRO(X) = (1, 2)-M_iRC(X) = \{\{c, d\}\},$$

$$(2, 1)-M_iRO(X) = (2, 1)-M_iRC(X) = \{\{a\}, \{b\}, \{a, b\}, \{c, d\}\},$$

$$(1, 2)-M_aRO(X) = (1, 2)-M_aRC(X) = \{\{a, b\}\},$$

$$(2, 1)-M_aRO(X) = (2, 1)-M_aRC(X) = \{\{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$$

Counterexamples for the results in Result 7.2.1 follow in the respective order.

1. The set $\{a, b\}$ is τ_1 -regular open, but is not $(1, 2)$ -minimal regular open.
2. The set $\{c, d\}$ is τ_1 -regular open, but is not $(1, 2)$ -maximal regular open.

Result 7.2.2. Let $i, j \in \{1, 2\}, i \neq j$ and (X, τ_1, τ_2) be a bitopological space.

1. (i, j) -minimal regular open sets and τ_j -regular open sets are independent of each other.

2. (i, j) -maximal regular open sets and τ_j -regular open sets are independent of each other.

Example 7.2.2. Let (X, τ_1, τ_2) be a bitopological space where $X = \{a, b, c, d, e\}$, $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, e\}, X\}$ and $\tau_2 = \{\phi, \{c\}, \{d\}, \{c, d\}, \{c, d, e\}, X\}$.

Then

Collection of τ_1 -regular open sets = $\{\{a\}, \{b\}, \phi, X\}$, , Collection of τ_2 -regular open sets = $\{\{c\}, \{d\}, \phi, X\}$,

Collection of τ_1 -minimal regular open sets = Collection of τ_1 -maximal regular open sets = $\{\{a\}, \{b\}\}$,

Collection of τ_2 -minimal regular open sets = Collection of τ_2 -maximal regular open sets = $\{\{c\}, \{d\}\}$.

$$(1, 2) - M_i RO(X) = (1, 2) - M_a RO(X) = \{\{a\}, \{b\}\},$$

$$(2, 1) - M_i RO(X) = (2, 1) - M_a RO(X) = \{\{c\}, \{d\}\}.$$

Counter examples for the results in Result 7.2.2 follow in the respective order.

1. The set $\{a\}$ is $(1, 2)$ -minimal regular open, but is not τ_2 -regular open and the set $\{c\}$ is τ_2 -regular open, but is not $(1, 2)$ -minimal regular open.
2. The set $\{c\}$ is $(2, 1)$ -minimal regular open, but is not τ_1 -regular open and the set $\{b\}$ is τ_1 -regular open, but is not $(2, 1)$ -minimal regular open.

Result 7.2.3. Let $i, j \in \{1, 2\}, i \neq j$ and (X, τ_1, τ_2) be a bitopological space.

1. (i, j) -minimal regular open sets and τ_i -minimal regular open sets are independent of each other.

2. (i, j) -maximal regular open sets and τ_i -maximal regular open sets are independent of each other.
3. (i, j) -minimal regular open sets and (i, j) -maximal regular open sets are independent of each other.

Example 7.2.3. Let (X, τ_1, τ_2) be a bitopological space, where $X = \{a, b, c, d\}$ and $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Collection of τ_1 -minimal regular open sets = Collection of τ_1 -maximal regular open sets = $\{\{a, b\}, \{c, d\}\}$,

Collection of τ_2 -minimal regular open sets = $\{\{a\}, \{b\}, \{c, d\}\}$,

Collection of τ_2 -maximal regular open sets = $\{\{a, b\}, \{a, c, d\}, \{b, c, d\}\}$.

Counterexamples for the results in Result 7.2.3 follow in the respective order.

1. The set $\{a, b\}$ is $(2, 1)$ -minimal regular open, but is not τ_2 -minimal regular open and the set $\{a, b\}$ is τ_1 -minimal regular open, but is not $(1, 2)$ -minimal regular open.
2. The set $\{c, d\}$ is $(2, 1)$ -maximal regular open, but is not τ_2 -maximal regular open and the set $\{c, d\}$ is τ_1 -maximal regular open, but is not $(1, 2)$ -maximal regular open.
3. The set $\{c, d\}$ is $(1, 2)$ -minimal regular open but is not $(1, 2)$ -maximal regular open and the set $\{a, c, d\}$ is $(2, 1)$ -maximal regular open, but is not $(2, 1)$ -minimal regular open.

Result 7.2.4. *Let $i, j \in \{1, 2\}, i \neq j$ and (X, τ_1, τ_2) be a bitopological space.*

1. *Every (i, j) -minimal regular closed set is τ_i -regular closed but not conversely.*
2. *Every (i, j) -maximal regular closed set is τ_i -regular closed but not conversely.*

Example 7.2.4. Let (X, τ_1, τ_2) be a bitopological space, where $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

Collection of τ_1 -regular open sets = Collection of τ_1 -regular closed sets = $\{\{a, b\}, \{c, d\}, \phi, X\}$,

$(1, 2)$ - $M_i RC(X) = \{\{c, d\}\}$, $(1, 2)$ - $M_a RC(X) = \{\{a, b\}\}$,

Counterexamples for the results in the Result 7.2.4 follow in the respective order.

1. The set $\{a, b\}$ is τ_1 -regular closed, but is not $(1, 2)$ -minimal closed.
2. The set $\{c, d\}$ is τ_1 -regular closed, but is not $(1, 2)$ -maximal closed.

Result 7.2.5. *Let $i, j \in \{1, 2\}, i \neq j$ and (X, τ_1, τ_2) be a bitopological space.*

1. *(i, j) -minimal regular closed sets and τ_j -regular closed sets are independent of each other.*
2. *(i, j) -maximal regular closed sets and τ_j -regular closed sets are independent of each other.*

Example 7.2.5. Let (X, τ_1, τ_2) be a bitopological space where $X = \{a, b, c, d, e\}$, $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, e\}, X\}$ and $\tau_2 = \{\phi, \{c\}, \{d\}, \{c, d\}, \{c, d, e\}, X\}$.

Then

Collection of τ_1 -regular closed sets = $\{\phi, \{b, c, d, e\}, \{a, c, d, e\}, X\}$,

Collection of τ_2 -regular closed sets = $\{\phi, \{a, b, d, e\}, \{a, b, c, e\}, X\}$,

Collection of τ_1 -minimal regular closed sets = Collection of τ_1 -maximal regular closed sets = $\{\{b, c, d, e\}, \{a, c, d, e\}\}$,

Collection of τ_2 -minimal regular closed sets = Collection of τ_2 -maximal regular closed sets = $\{\{a, b, d, e\}, \{a, b, c, e\}\}$.

$(1, 2)$ - $M_iRC(X) = \{\{b, c, d, e\}, \{a, c, d, e\}\} = (1, 2)$ - $M_aRC(X)$,

$(2, 1)$ - $M_iRC(X) = \{\{a, b, d, e\}, \{a, b, c, e\}\} = (2, 1)$ - $M_aRC(X)$.

Counterexamples for the results in Result 7.2.5 follow in the respective order.

1. The set $\{a, b, d, e\}$ is $(2, 1)$ -minimal regular closed, but is not τ_1 -regular closed and the set $\{b, c, d, e\}$ is τ_1 -regular closed, but is not $(2, 1)$ -minimal regular closed.
2. The set $\{a, c, d, e\}$ is $(1, 2)$ -maximal regular closed, but is not τ_2 -regular closed and the set $\{a, b, d, e\}$ is τ_2 -regular closed, but is not $(1, 2)$ -maximal regular closed.

Result 7.2.6. Let $i, j \in \{1, 2\}, i \neq j$ and (X, τ_1, τ_2) be a bitopological space.

1. (i, j) -minimal regular closed sets and τ_i -minimal regular closed sets are independent of each other.

2. (i, j) -maximal regular closed sets and τ_i -maximal regular closed sets are independent of each other.
3. (i, j) -minimal regular closed sets and (i, j) -maximal regular closed sets are independent of each other.

Example 7.2.6. Let (X, τ_1, τ_2) be a bitopological space, where $X = \{a, b, c, d\}$ and $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

Collection of τ_1 -minimal regular closed sets = $\{\{a, b\}, \{c, d\}\}$,

Collection of τ_1 -maximal regular closed sets = $\{\{a, b\}, \{c, d\}\}$,

Collection of τ_2 -minimal regular closed sets = $\{\{c, d\}, \{a\}, \{b\}\}$,

Collection of τ_2 -maximal regular closed sets = $\{\{b, c, d\}, \{a, c, d\}, \{a, b\}\}$.

Counterexamples for the results in Result 7.2.6 follow in the respective order.

1. The set $\{a, b\}$ is $(2, 1)$ -minimal regular closed, but is not τ_2 -minimal regular closed and the set $\{a, b\}$ is τ_1 -minimal regular closed, but is not $(1, 2)$ -minimal regular closed.
2. The set $\{c, d\}$ is $(2, 1)$ -maximal regular closed, but is not τ_2 -maximal regular closed and the set $\{c, d\}$ is τ_1 -maximal regular closed, but is not $(1, 2)$ -maximal regular closed.
3. The set $\{c, d\}$ is $(1, 2)$ -minimal regular closed, but is not $(1, 2)$ -maximal regular closed and the set $\{a, c, d\}$ is $(2, 1)$ -maximal regular closed, but is not $(2, 1)$ -minimal regular closed.

Theorem 7.2.7. *A proper nonempty subset U of a bitopological space (X, τ_1, τ_2) is (i, j) -minimal regular open if and only if $X - U$ is (i, j) -maximal regular closed for $i, j \in \{1, 2\}, i \neq j$.*

Proof. Let U be a (i, j) -minimal regular open set in X . Suppose that $X - U$ is not (i, j) -maximal regular closed. Then there exists a proper τ_j -regular closed set F in X such that $(X - U) \subset F$. Then $(X - F) \subset U$ and $X - F$ is a nonempty τ_j -regular open set in X . This is a contradiction to the fact that U is (i, j) -minimal regular open. Therefore $X - U$ is a (i, j) -maximal regular closed set in X .

Conversely suppose that $X - U$ is a (i, j) -maximal regular closed set in X . If possible, assume that U is not (i, j) -minimal regular open. Then there exists a nonempty τ_j -regular open set V in X such that $V \subset U$. Then $(X - U) \subset (X - V)$ and $X - V$ is a proper τ_j -regular closed set in X . This is a contradiction to the fact that $X - U$ is (i, j) -maximal regular closed. Therefore U is a (i, j) -minimal regular open set in X . $\square$

Corollary 7.2.7.1. *A proper nonempty subset U of a bitopological space (X, τ_1, τ_2) is p -minimal regular open if and only if $X - U$ is p -maximal regular closed.*

Theorem 7.2.8. *A proper nonempty subset U of a bitopological space (X, τ_1, τ_2) is (i, j) -maximal regular open if and only if $X - U$ is a (i, j) -minimal regular closed for $i, j \in \{1, 2\}, i \neq j$.*

Proof. Let U be a (i, j) -maximal regular open set in X . Suppose $X - U$ is not

(i, j) -minimal regular closed. Then there exists a nonempty τ_j -regular closed set F in X such that $F \subset (X - U)$. Then $U \subset (X - F)$ and $X - F$ is a proper τ_j -regular open. This is a contradiction to the fact that U is a (i, j) -maximal regular open set in X . Therefore $X - U$ is a (i, j) -minimal regular closed set in X .

Conversely, suppose that $X - U$ is a (i, j) -minimal regular closed set in X . If possible, assume that U is not (i, j) -maximal regular open. Then there exists a proper τ_j -regular open set V in X such that $U \subset V$. Then $(X - V) \subset (X - U)$ and $X - V$ is a nonempty τ_j -regular closed set in X . This is a contradiction to the fact that $X - U$ is (i, j) -minimal regular closed. Therefore U is a (i, j) -maximal regular open set in X . $\square$

Corollary 7.2.8.1. *A proper nonempty subset U of a bitopological space (X, τ_1, τ_2) is p -maximal regular open if and only if $X - U$ is p -minimal regular closed.*

Result 7.2.9. *Let $i, j \in \{1, 2\}, i \neq j$ and (X, τ_1, τ_2) be a bitopological space. Then,*

1. *Union of (i, j) -minimal regular open (respectively (i, j) -minimal regular closed) sets need not be a (i, j) -minimal regular open (respectively (i, j) -minimal regular closed) set.*
2. *Union of (i, j) -maximal regular open (respectively (i, j) -maximal regular closed) sets need not be a (i, j) -maximal regular open (respectively (i, j) -maximal regular closed) set.*

3. *Intersection of (i, j) -minimal regular open (respectively (i, j) -minimal regular closed) sets need not be a (i, j) -minimal regular open (respectively (i, j) -minimal regular closed) set.*
4. *Intersection of (i, j) -maximal regular open (respectively (i, j) -maximal regular closed) sets need not be a (i, j) -maximal regular open (respectively (i, j) -maximal regular closed) set.*

Example 7.2.7. Let (X, τ_1, τ_2) be a bitopological space, where $X = \{a, b, c, d\}$ and $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

$$(1, 2)\text{-}M_iRO(X) = (1, 2)\text{-}M_iRC(X) = \{\{c, d\}\},$$

$$(2, 1)\text{-}M_iRO(X) = (2, 1)\text{-}M_iRC(X) = \{\{a\}, \{b\}, \{a, b\}, \{c, d\}\},$$

$$(1, 2)\text{-}M_aRO(X) = (1, 2)\text{-}M_aRC(X) = \{\{a, b\}\},$$

$$(2, 1)\text{-}M_aRO(X) = (2, 1)\text{-}M_aRC(X) = \{\{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$$

Counterexamples for the results in Result 7.2.9 follow in the respective order.

1. The sets $\{a, b\}, \{c, d\} \in (2, 1)\text{-}M_iRO(X)$, but then $\{a, b\} \cup \{c, d\} = X \notin (2, 1)\text{-}M_iRO(X)$ and the sets $\{a, b\}, \{c, d\} \in (2, 1)\text{-}M_iRC(X)$, but then $\{a, b\} \cup \{c, d\} = X \notin (2, 1)\text{-}M_iRC(X)$.
2. The sets $\{a, b\}, \{a, c, d\} \in (2, 1)\text{-}M_aRO(X)$, but then $\{a, b\} \cup \{a, c, d\} = X \notin (2, 1)\text{-}M_aRO(X)$ and the sets $\{a, b\}, \{c, d\} \in (2, 1)\text{-}M_aRC(X)$, but then $\{a, b\} \cup \{c, d\} = X \notin (2, 1)\text{-}M_aRC(X)$.

3. The sets $\{a, b\}, \{c, d\} \in (2, 1)\text{-}M_iRO(X)$, but then $\{a, b\} \cap \{c, d\} = \phi \notin (2, 1)\text{-}M_iRO(X)$ and the sets $\{a, b\}, \{c, d\} \in (2, 1)\text{-}M_iRC(X)$, but then $\{a, b\} \cap \{c, d\} = \phi \notin (2, 1)\text{-}M_iRC(X)$.
4. The sets $\{a, b\}, \{a, c, d\} \in (2, 1)\text{-}M_aRO(X)$, but then $\{a, b\} \cap \{a, c, d\} = \{a\} \notin (2, 1)\text{-}M_aRO(X)$ and the sets $\{a, b\}, \{c, d\} \in (2, 1)\text{-}M_aRC(X)$, but then $\{a, b\} \cap \{c, d\} = \phi \notin (2, 1)\text{-}M_aRC(X)$.

Result 7.2.10. *Let $i, j \in \{1, 2\}, i \neq j$ and (X, τ_1, τ_2) be a bitopological space.*

Then,

1. *The family $(i, j)\text{-}M_iRO(X)$ is generally not equal to the family $(j, i)\text{-}M_iRO(X)$.*
2. *The family $(i, j)\text{-}M_aRO(X)$ is generally not equal to the family $(j, i)\text{-}M_aRO(X)$.*
3. *The family $(i, j)\text{-}M_iRC(X)$ is generally not equal to the family $(j, i)\text{-}M_iRC(X)$.*
4. *The family $(i, j)\text{-}M_aRC(X)$ is generally not equal to the family $(j, i)\text{-}M_aRC(X)$.*

Example 7.2.8. Let (X, τ_1, τ_2) be a bitopological space, where $X = \{a, b, c, d\}$ and $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Counterexamples for the results in Result 7.2.10 follow in the respective order.

1. $(1, 2)\text{-}M_iRO(X) = \{\{c, d\}\} \neq \{\{a\}, \{b\}, \{a, b\}, \{c, d\}\} = (2, 1)\text{-}M_iRO(X)$
2. $(1, 2)\text{-}M_aRO(X) = \{\{a, b\}\} \neq \{\{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\} = (2, 1)\text{-}M_aRO(X)$

3. $(1, 2)\text{-}M_iRC(X) = \{\{c, d\}\} \neq \{\{a\}, \{b\}, \{a, b\}, \{c, d\}\} = (2, 1)\text{-}M_iRC(X)$
4. $(1, 2)\text{-}M_aRC(X) = \{\{a, b\}\} \neq \{\{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\} = (2, 1)\text{-}M_aRC(X).$

Theorem 7.2.11. *Let (X, τ_1, τ_2) be a bitopological space and $U \subset X$. If U is a (i, j) -minimal regular open set in X , then U is a (i, j) -pre open set for each $i, j \in \{1, 2\}, i \neq j$.*

Proof. We have, $U \subset j\text{-cl}(U)$. Then $i\text{-int}(U) \subset i\text{-int}(j\text{-cl}(U))$, $U \subset i\text{-int}(j\text{-cl}(U))$. Therefore U is (i, j) -pre open. $\square$

Theorem 7.2.12. *Let (X, τ_1, τ_2) be a bitopological space and $U \subset X$.*

- (i) *If U is a (i, j) -minimal regular open set, then U is τ_j -minimal regular open set.*
- (ii) *If U is a (i, j) -minimal regular open set and U is τ_i -closed, then U is (i, j) -regular open set.*

Proof. (i) If possible U is not τ_j -minimal regular open set, then there exists a nonempty τ_j -regular open set such that $V \subset U$. This is a contradiction to the assumption U is (i, j) -minimal regular open set. Therefore U is τ_j -minimal regular open.

(ii) We have $U \subset j\text{-cl}(U)$. Then $i\text{-int}(U) \subset i\text{-int}(j\text{-cl}(U))$ and hence $U \subset i\text{-int}(j\text{-cl}(U))$.

Also we have $j-cl(U) = U$. Then $i-int(j-cl(U)) = i-int(U) \subset U$. Therefore $i-int(j-cl(U)) = U$. $\square$

Theorem 7.2.13. *Let (X, τ_1, τ_2) be a bitopological space and $U \subset X$. If U is a τ_i -regular open set and τ_j -minimal regular open set in X , then U is (i, j) -minimal regular open set in X .*

Proof. Let U be a τ_i -regular open set and τ_j -minimal regular open set in X and let V be any nonempty τ_j -regular open set in X contained in U . Since U is τ_j -minimal regular open and $V \neq \phi$, $V = U$. Therefore by the definition of (i, j) -minimal regular open set, U is (i, j) -minimal regular open set in X . $\square$

7.3 Pairwise- rT_{min} and Pairwise- rT_{max} Spaces

Definition 7.3.1. Let $i, j \in \{1, 2\}, i \neq j$. A bitopological space (X, τ_1, τ_2) is said to be

- (i) (i, j) - rT_{min} space if every nonempty proper τ_i -regular open set is (i, j) -minimal regular open set and X is said to be *pairwise- rT_{min} space* if it is both $(1, 2)$ - rT_{min} and $(2, 1)$ - rT_{min} space .
- (ii) (i, j) - rT_{max} space if every nonempty proper τ_i -regular open set is (i, j) -maximal regular open set and X is said to be *pairwise- rT_{max} space* if it is both $(1, 2)$ - rT_{max} and $(2, 1)$ - rT_{max} .

Example 7.3.1. Let (X, τ_1, τ_2) be a bitopological space where $X = \{a, b, c\}$,

$\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{c\}, \{a, b\}, X\}$. Then,

Collection of τ_1 -regular open sets = $\{\phi, \{a\}, \{b\}, X\}$,

Collection of τ_2 -regular open sets = $\{\phi, \{c\}, \{a, b\}, X\}$.

$(1, 2)$ - $M_iRO(X) = \{\{a\}, \{b\}\}$,

$(2, 1)$ - $M_iRO(X) = \{\{c\}\}$,

$(1, 2)$ - $M_aRO(X) = \phi$,

$(2, 1)$ - $M_aRO(X) = \{\{c\}, \{a, b\}\}$.

Here X is $(1, 2)$ - rT_{min} space, but is not $(1, 2)$ - rT_{max} space. Also X is $(2, 1)$ - rT_{max} space, but is not $(2, 1)$ - rT_{min} space.

Definition 7.3.2. Let $i, j \in \{1, 2\}, i \neq j$. A bitopological space (X, τ_1, τ_2) is said to be

- (i) (i, j) - r -door space if every subset of X is either (i, j) -regular open or (i, j) -regular closed and X is said to be *pairwise- r -door space*, if it is both $(1, 2)$ - r -door space and $(2, 1)$ - r -door space.
- (ii) (i, j) - $rT_{\frac{1}{2}}$ -space if every τ_i -regular closed subset of X is (i, j) -regular closed and X is said to be *pairwise- $rT_{\frac{1}{2}}$ space*, if it is both $(1, 2)$ - $rT_{\frac{1}{2}}$ space and $(2, 1)$ - $rT_{\frac{1}{2}}$ space.

Definition 7.3.3. Let $i, j \in \{1, 2\}, i \neq j$ and let $(X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)$ be two bitopological spaces. A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be

- (i) (i, j) -almost continuous if $f^{-1}(V)$ is a τ_i -open set of X for every (i, j) -

regular open set V of Y and f is said to be *pairwise-almost continuous* (*p-almost continuous*) if it is both $(1, 2)$ -almost continuous and $(2, 1)$ -almost continuous.

- (ii) (i, j) -almost completely continuous if $f^{-1}(V)$ is a τ_i -regular open set of X for every (i, j) -regular open set V of Y and f is said to be *pairwise-almost completely continuous* (*p-almost completely continuous*) if it is both $(1, 2)$ -almost completely continuous and $(2, 1)$ -almost completely continuous.
- (iii) (i, j) -almost perfectly continuous if $f^{-1}(V)$ is a τ_i -clopen set of X for every (i, j) -regular open set V of Y and f is said to be *pairwise-almost perfectly continuous* (*p-almost perfectly continuous*) if it is both $(1, 2)$ -almost perfectly continuous and $(2, 1)$ -almost perfectly continuous.
- (iv) i -completely continuous if $f^{-1}(V)$ is a τ_i -regular open set of X for every σ_i -regular open set V of Y and f is said to be completely continuous if it is both 1-completely continuous and 2-completely continuous.
- (v) i -strongly continuous if $f^{-1}(V)$ is a τ_i -clopen set of X for every subset V of Y and f is said to be strongly continuous if it is both 1-strongly continuous and 2-strongly continuous.
- (vi) i -almost continuous if $f^{-1}(V)$ is a τ_i -open set of X for every σ_i -regular open set V of Y and f is said to be almost continuous if it is i -almost continuous for each i .

- (vii) *i-almost completely continuous* if $f^{-1}(V)$ is a τ_i -regular open set of X for every σ_i -regular open set V of Y and f is said to be almost completely continuous if it is *i-almost completely continuous* for each i .
- (viii) *i-almost perfectly continuous* if $f^{-1}(V)$ is a τ_i -clopen set of X for every σ_i -regular open set V of Y and f is said to be almost perfectly-continuous if it is *i-almost perfectly continuous* for each i .
- (ix) *i-totally continuous* if $f^{-1}(V)$ is a τ_i -clopen set of X for every σ_i -open set V of Y and f is said to be totally continuous, if it is *i-totally continuous* for each i .

Definition 7.3.4. A topological space (X, τ_1, τ_2) is said to be

- (i) δT_0 if for each pair of distinct points x and y in X , there is a set which is either τ_1 -regular open or τ_2 -regular open containing one but not the other.
- (ii) δT_1 if for each pair of distinct points x and y in X , there exists a τ_1 -regular open set U containing x but not y or a τ_2 -regular open set V containing y but not x .
- (iii) δT_2 if for each pair of distinct points x and y in X , there exist a τ_1 -regular open set U containing x and a τ_2 -regular open set V containing y such that $U \cap V = \phi$.

Theorem 7.3.1. Let $i, j \in \{1, 2\}, i \neq j$. Then a bitopological space (X, τ_1, τ_2) is (i, j) - rT_{min} space (respectively (i, j) - rT_{max} space) if and only if every nonempty

proper τ_i -regular closed subset of X is (i, j) -maximal regular closed (respectively (i, j) -minimal regular closed).

Proof. The proof follows from the definition of (i, j) - rT_{min} space (respectively (i, j) - rT_{max} space) and the fact that the complement of every (i, j) -minimal regular open (respectively (i, j) -maximal regular open) set is a (i, j) -maximal regular closed (respectively (i, j) -minimal regular closed) set. $\square$

Corollary 7.3.1.1. *Let (X, τ_1, τ_2) be a Pairwise- rT_{min} (respectively Pairwise- rT_{max} space) bitopological space if and only if every nonempty proper subset of X which is τ_i -regular closed for each $i \in \{1, 2\}$ is p -maximal regular closed (respectively p -minimal regular closed).*

Theorem 7.3.2. *Every pair of distinct (i, j) -minimal regular open sets in a (i, j) - rT_{min} space X is either disjoint or their intersection is a (i, j) -minimal regular open set.*

Proof. Let U and V be two distinct (i, j) -minimal regular open sets in a (i, j) - rT_{min} space X . Suppose that $U \cap V \neq \emptyset$. Then $U \cap V$ is a nonempty τ_i -regular open set. If $U \cap V$ is not a (i, j) -minimal regular open set, then there exists a nonempty τ_j -regular open set W such that $W \subset U \cap V$. Then $W \subset U$, which is a contradiction to the fact that U is (i, j) -minimal regular open. Hence $U \cap V$ is (i, j) -minimal regular open. $\square$

Theorem 7.3.3. *Union of every pair of distinct (i, j) -maximal regular closed sets in a (i, j) - rT_{min} space X is either X or their union is a (i, j) -maximal regular*

closed set.

Proof. Let U and V be two distinct (i, j) -maximal regular closed sets in a (i, j) - rT_{min} space X . Suppose $U \cup V \neq X$, then $U \cup V$ is a τ_i -regular closed set. If $U \cup V$ is not a (i, j) -maximal regular closed set, then there exists a proper τ_j -regular closed set W such that $U \cup V \subset W$. This implies that $U \subset W$, which is a contradiction to the fact that U is (i, j) -maximal regular closed. Hence $U \cup V$ is (i, j) -maximal regular closed. $\square$

Theorem 7.3.4. *Every (i, j) -almost completely continuous function is (i, j) -almost continuous.*

Proof. Proof follows from the fact that every regular open set is open. $\square$

Corollary 7.3.4.1. *Every p -almost completely continuous function is p -almost continuous.*

Theorem 7.3.5. *Let (X, τ_1, τ_2) be an (i, j) - rT_{min} space and Y be a τ_j -regular open subset of the X . Then Y is an (i, j) - rT_{min} space.*

Proof. Let Y be a τ_j -regular open subset of the (i, j) - rT_{min} space X . Let U be a nonempty proper τ_i -regular open subset of Y . If possible suppose that U is not an (i, j) -minimal regular open set in Y . Then there exist a nonempty proper τ_j -regular open subset V in Y such that $V \subset U \subset Y$. Therefore V is a τ_j -regular open set in Y . This implies that V is a τ_j -regular open set in X , which is a contradiction to the fact that U is an (i, j) -minimal regular open set

in X . Therefore V is an (i, j) -minimal regular open set in Y and hence Y is an (i, j) - rT_{min} space. $\square$

7.4 p -Min r -Continuous Maps and p -Max r -Continuous Maps

Definition 7.4.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A map $f : X \longrightarrow Y$ is said to be

- (i) (i, j) -minimal r -continuous, if $f^{-1}(U)$ is a τ_i -regular open set in X for every (i, j) -minimal regular open set U in Y and f is called *pairwise-minimal r -continuous* (*p -minimal r -continuous* or *p -min r -continuous*) if it is both $(1, 2)$ -minimal r -continuous and $(2, 1)$ -minimal r -continuous.
- (ii) (i, j) -maximal r -continuous, if $f^{-1}(U)$ is a τ_i -regular open set in X for every (i, j) -maximal regular open set U in Y and f is called *pairwise-maximal r -continuous* (*p -maximal r -continuous* or *p -max r -continuous*) if it is both $(1, 2)$ -maximal r -continuous and $(2, 1)$ -maximal r -continuous.
- (iii) (i, j) -minimal r -irresolute, if $f^{-1}(U)$ is an (i, j) -minimal regular open set in X for every (i, j) -minimal regular open set U in Y and f is called *pairwise-minimal r -irresolute* (*p -minimal r -irresolute*) if it is both $(1, 2)$ -minimal r -irresolute and $(2, 1)$ -minimal r -irresolute.

(iv) (i, j) -maximal r -irresolute, if $f^{-1}(U)$ is an (i, j) -maximal regular open set in X for every (i, j) -maximal regular open set U in Y and f is called *pairwise-maximal r -irresolute* (*p -maximal r -irresolute*) if it is both $(1, 2)$ -maximal r -irresolute and $(2, 1)$ -maximal r -irresolute.

Theorem 7.4.1. *Every i -almost completely continuous function on a bitopological space is (i, j) -minimal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. If $f : X \longrightarrow Y$ is i -almost completely continuous for $i, j \in \{1, 2\}$. Suppose that U be a (i, j) -minimal regular open set in Y . Since U is σ_i -regular open in Y and f is i -almost completely continuous, $f^{-1}(U)$ is τ_i -regular open in X . Therefore f is (i, j) -minimal r -continuous. $\square$

Remark. Converse of the theorem 7.4.1 need not be true.

Example 7.4.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces where $X = Y = \{a, b, c\}$, $\tau_1 = \sigma_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\tau_2 = \sigma_2 = \{\phi, \{a\}, X\}$. Let $f : X \longrightarrow Y$ defined by $f(a) = b, f(b) = a, f(c) = c, f(d) = d$. Then f is $(1, 2)$ -minimal r -continuous, but not 1-almost completely continuous.

Corollary 7.4.1.1. *Every almost completely continuous function on a bitopological space is p -minimal r -continuous.*

Theorem 7.4.2. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. If $f : X \longrightarrow Y$ is (i, j) -minimal r -continuous and Y is (i, j) - rT_{min} space for $i, j \in \{1, 2\}$ with $i \neq j$, then f is i -almost continuous.*

Proof. Let U be a σ_i -regular open set in Y . Since Y is (i, j) - rT_{min} space, U is an (i, j) -minimal regular open set in Y . Since f is an (i, j) -minimal r -continuous map, $f^{-1}(U)$ is a τ_i -regular open set in X . Hence f is i -almost continuous function. $\square$

Theorem 7.4.3. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. If $f : X \longrightarrow Y$ is a (i, j) -minimal r -continuous function and Y is an (i, j) - rT_{min} space for $i, j \in \{1, 2\}$ with $i \neq j$, then f is i -almost completely continuous.*

Proof. Let V be a σ_i -regular open set in Y . Since Y is an (i, j) - rT_{min} space, V is (i, j) -minimal regular open. Also since f is an (i, j) -minimal r -continuous function, $f^{-1}(V)$ is τ_i -regular open set in X . Hence f is i -almost completely continuous. $\square$

Theorem 7.4.4. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. Let $f : X \longrightarrow Y$ be a (i, j) -maximal r -continuous map and Y be an (i, j) - rT_{max} space for $i, j \in \{1, 2\}$ with $i \neq j$, then f is i -almost completely continuous.*

Proof. Proof is similar as the proof of theorem 7.4.3. $\square$

Theorem 7.4.5. *Every i -strongly continuous function is (i, j) -minimal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and $f : X \longrightarrow Y$ be an i -strongly continuous function. Let U be an (i, j) -minimal regular open set in Y . Since $f : X \longrightarrow Y$ is an i -strongly continuous function, $f^{-1}(U)$ is

τ_i -clopen in X . Then $f^{-1}(U)$ is τ_i -regular open in X . Hence f is (i, j) -minimal r -continuous. $\square$

Theorem 7.4.6. *Every i -completely continuous function is (i, j) -minimal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and $f : X \rightarrow Y$ be i -completely continuous function. Let U be an (i, j) -minimal regular open set in Y . Since $f : X \rightarrow Y$ is an i -completely continuous function and U is σ_i -regular open in Y , $f^{-1}(U)$ is τ_i -regular open in X . Hence f is (i, j) -minimal r -continuous. $\square$

Corollary 7.4.6.1. *Every completely continuous function on a bitopological space is p -minimal r -continuous.*

Theorem 7.4.7. *Every i -almost perfectly continuous function is (i, j) -minimal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and $f : X \rightarrow Y$ be i -almost perfectly continuous function. Let U be an (i, j) -minimal regular open set in Y . Since $f : X \rightarrow Y$ is an i -almost perfectly continuous function and U is σ_i -regular open in Y , $f^{-1}(U)$ is τ_i -clopen in X . Then $f^{-1}(U)$ is τ_i -regular open in X . Hence f is (i, j) -minimal r -continuous. $\square$

Definition 7.4.2. A space (X, τ_1, τ_2) is said to be *locally indiscrete* if it is locally indiscrete with respect to each $\tau_i, i = 1, 2$.

Example 7.4.2. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a, b\}, \{c, d\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b, c, d\}\}$. Then the space (X, τ_1, τ_2) is locally indiscrete bitopological space.

Theorem 7.4.8. *Every locally indiscrete bitopological space (X, τ_1, τ_2) is an (i, j) - r -door space for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Since X is locally indiscrete, every open set with respect to τ_i is clopen for each $i \in \{1, 2\}$. Then X is (i, j) - r -door space for $i, j \in \{1, 2\}$ with $i \neq j$. $\square$

Theorem 7.4.9. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. If $f : X \longrightarrow Y$ is i -almost continuous and X is locally indiscrete with respect to τ_i , then f is (i, j) -minimal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let U be an (i, j) -minimal regular open set in Y . Since $f : X \longrightarrow Y$ is an i -almost continuous function and U is σ_i -regular open in Y , $f^{-1}(U)$ is τ_i -open in X . Since X is locally indiscrete with respect to τ_i , $f^{-1}(U)$ is τ_i -closed. Then $f^{-1}(U)$ is τ_i -regular open in X . Hence f is (i, j) -minimal r -continuous. $\square$

Corollary 7.4.9.1. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. If $f : X \longrightarrow Y$ is almost continuous and X is locally indiscrete, then f is p -minimal r -continuous.*

Theorem 7.4.10. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. If $f : X \longrightarrow Y$ is p -minimal r -continuous function and Y is a pairwise- rT_{min} space, then f is completely continuous.*

Proof. For $i \in \{1, 2\}$, let U be σ_i -regular open set of Y . Since Y is a pairwise- rT_{min} space, U is an (i, j) -minimal regular open set in Y . Since f is (i, j) -minimal r -continuous, $f^{-1}(U)$ is τ_i -regular open set in X . Therefore f is completely continuous. $\square$

Theorem 7.4.11. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. If $f : X \longrightarrow Y$ is (i, j) -maximal r -continuous, X is locally indiscrete with respect to τ_i and Y is (i, j) - rT_{max} space for $i, j \in \{1, 2\}$ with $i \neq j$, then f is i -almost perfectly continuous.*

Theorem 7.4.12. *Let (X, τ_1, τ_2) be a locally indiscrete space with respect to τ_i and (Y, σ_1, σ_2) be an (i, j) - rT_{min} space which is locally indiscrete with respect to σ_i . If $f : X \longrightarrow Y$ is (i, j) -minimal r -continuous, then f is i -totally continuous.*

Proof. Let U be a σ_i -open set in Y . Since Y is an (i, j) - rT_{min} space and locally indiscrete with respect to σ_i , U is (i, j) -minimal regular open in Y . Since f is (i, j) -minimal r -continuous, $f^{-1}(U)$ is a τ_i -regular open set in X . Since X is a locally indiscrete space with respect to τ_i , $f^{-1}(U)$ is τ_i -clopen in X . Hence f is i -totally continuous. $\square$

Theorem 7.4.13. *Every i -totally continuous function is (i, j) -maximal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and $f : X \longrightarrow Y$ be an i -totally continuous function. Let U be an (i, j) -maximal regular open

set in Y . Since f is i -totally continuous, $f^{-1}(U)$ is τ_i -clopen in X . Then $f^{-1}(U)$ is τ_i -regular open in X . Hence f is (i, j) -maximal r -continuous. $\square$

Theorem 7.4.14. *Let (X, τ_1, τ_2) be a locally indiscrete space with respect to τ_i and (Y, σ_1, σ_2) be an (i, j) - rT_{max} space which is locally indiscrete with respect to σ_i for $i, j \in \{1, 2\}$ with $i \neq j$. If $f : X \rightarrow Y$ is (i, j) -maximal r -continuous, then f is i -totally continuous.*

Proof. Let U be a σ_i -open set in Y . Since Y is an (i, j) - rT_{max} space and locally indiscrete with respect to σ_i , U is (i, j) -maximal regular open in Y . Since f is (i, j) -maximal r -continuous, $f^{-1}(U)$ is a τ_i -regular open set in X . Since X is a locally indiscrete space with respect to τ_i , $f^{-1}(U)$ is τ_i -clopen in X . Hence f is i -totally continuous. $\square$

Remark. (i, j) -minimal r -continuous and (i, j) -maximal r -continuous maps are independent to each other.

Example 7.4.3. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces where $X = Y = \{a, b, c, d\}$, $\tau_1 = \sigma_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \sigma_2 = \{\phi, \{b\}, \{b, c\}, \{d\}, \{b, d\}, \{b, c, d\}, X\}$. Then

τ_1 -regular open sets in $X = \{\phi, \{a\}, \{b\}, X\}$, τ_2 -regular open sets in $X = \{\phi, \{b, c\}, \{d\}, X\}$

$$(1, 2)\text{-}M_iRO(X) = \{\{a\}, \{b\}\}$$

$$(2, 1)\text{-}M_iRO(X) = \{\{d\}\}$$

$$(1, 2)\text{-}M_aRO(X) = \{\{a\}\}$$

$$(2, 1)\text{-}M_aRO(X) = \{\{b, c\}, \{d\}\}.$$

Let $f : X \longrightarrow Y$ be defined by $f(a) = b, f(b) = a, f(c) = c, f(d) = d$ and $g : X \longrightarrow Y$ be defined by $g(a) = c, g(b) = a, g(c) = b, g(d) = d$. Then f is $(2, 1)$ -minimal r -continuous, but is not $(2, 1)$ -maximal r -continuous and g is $(1, 2)$ -maximal r -continuous, but is not $(1, 2)$ -minimal r -continuous.

Theorem 7.4.15. *Every (i, j) -minimal r -irresolute is (i, j) -minimal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and $f : X \longrightarrow Y$ be an (i, j) -minimal r -irresolute. Let U be an (i, j) -minimal regular open set in Y . Since f is an (i, j) -minimal r -irresolute, $f^{-1}(U)$ is (i, j) -minimal regular open in X . This implies $f^{-1}(U)$ is τ_i -regular open in X . Hence f is (i, j) -minimal r -continuous. $\square$

Theorem 7.4.16. *Let (X, τ_1, τ_2) be an (i, j) - rT_{min} space and (Y, σ_1, σ_2) be any bitopological space. If $f : X \longrightarrow Y$ is (i, j) -minimal r -continuous, then f is (i, j) -minimal r -irresolute for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let U be an (i, j) -minimal regular open set in Y . Since f is (i, j) -minimal r -continuous, $f^{-1}(U)$ is τ_i -regular open in X . Since X is an (i, j) - rT_{min} space, $f^{-1}(U)$ is (i, j) -minimal regular open in X . Hence f is an (i, j) -minimal r -irresolute. $\square$

Theorem 7.4.17. *Let (X, τ_1, τ_2) be an (i, j) - rT_{max} space and (Y, σ_1, σ_2) be any*

bitopological space. If $f : X \longrightarrow Y$ be (i, j) -maximal r -continuous, then f is (i, j) -maximal r -irresolute for $i, j \in \{1, 2\}$ with $i \neq j$.

Proof. Let U be an (i, j) -maximal regular open set in Y . Since f is (i, j) -maximal r -continuous, $f^{-1}(U)$ is τ_i -regular open in X . Since X is an (i, j) - rT_{max} space, $f^{-1}(U)$ is (i, j) -maximal regular open in X . Hence f is an (i, j) -maximal r -irresolute. $\square$

Theorem 7.4.18. *Every (i, j) -minimal r -irresolute map into an (i, j) - rT_{min} space is i -almost completely continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and $f : X \longrightarrow Y$ be (i, j) -minimal r -irresolute. Let U be σ_i -regular open set in Y . Since Y is an (i, j) - rT_{min} space, U is (i, j) -minimal regular open set in Y . Since f is an (i, j) -minimal r -irresolute, $f^{-1}(U)$ is (i, j) -minimal regular open set in X . Then f^{-1} is a τ_i -regular open in X . Hence f is i -almost completely continuous. $\square$

Theorem 7.4.19. *Every (i, j) -maximal r -irresolute map into an (i, j) - rT_{max} space is i -almost completely continuous $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and $f : X \longrightarrow Y$ be (i, j) -maximal r -irresolute. Let U be σ_i -regular open set in Y . Since Y is an (i, j) - rT_{max} space, U is (i, j) -maximal regular open set in Y . Since f is an (i, j) -maximal r -irresolute, $f^{-1}(U)$ is (i, j) -maximal regular open set in X . Then f^{-1} is a τ_i -regular open in X . Hence f is i -almost completely continuous. $\square$

Theorem 7.4.20. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. If $f : X \longrightarrow Y$ is i -almost completely continuous and X is (i, j) - rT_{min} , then f is an (i, j) -minimal r -irresolute for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let U be an (i, j) -minimal regular open set in Y . Since f is i -almost completely continuous and U is σ_i -regular open in Y , $f^{-1}(U)$ is τ_i -regular open in X . Since X is an (i, j) - rT_{min} space, $f^{-1}(U)$ is (i, j) -minimal regular open in X . Therefore f is an (i, j) -minimal r -irresolute. $\square$

Theorem 7.4.21. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. If $f : X \longrightarrow Y$ is i -almost completely continuous and X is (i, j) - rT_{max} , then f is an (i, j) -maximal r -irresolute for $i, j \in \{1, 2\}$ with $i \neq j$.*

Theorem 7.4.22. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A map $f : X \longrightarrow Y$ is (i, j) -minimal r -continuous if and only if the inverse image of each (i, j) -maximal regular closed set in Y is τ_i -regular closed in X for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let U be (i, j) -maximal regular closed set in Y . Then U^C is an (i, j) -minimal regular open set in Y . Since f is (i, j) -minimal r -continuous, $f^{-1}(U^C)$ is τ_i -regular open. Then $f^{-1}(U)$ is τ_i -regular closed in X .

Conversely, let U be an (i, j) -minimal regular open set in Y . Then U^C is an (i, j) -maximal regular closed set in Y . By the assumption $f^{-1}(U^C)$ is τ_i -regular closed in Y . Then $f^{-1}(U)$ is τ_i -regular open in X . Hence f is (i, j) -minimal r -continuous. $\square$

Theorem 7.4.23. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A map $f : X \longrightarrow Y$ is (i, j) -minimal r -irresolute if and only if the inverse image of each (i, j) -maximal regular closed set in Y is (i, j) -maximal regular closed in X for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let U be an (i, j) -maximal regular closed set in Y . Then U^C is an (i, j) -minimal regular open set in Y . Since f is an (i, j) -minimal r -irresolute, $f^{-1}(U^C)$ is an (i, j) -minimal regular open set in X . Then $f^{-1}(U)$ is an (i, j) -maximal regular closed set in X .

Conversely, let U be an (i, j) -minimal regular open set in Y . Then U^C is an (i, j) -maximal regular closed set in Y . By the assumption $f^{-1}(U^C)$ is an (i, j) -maximal regular closed set in Y . Then $f^{-1}(U)$ is an (i, j) -minimal regular open in X . Hence f is an (i, j) -minimal r -irresolute. $\square$

Theorem 7.4.24. *Let (X, τ_1, τ_2) be an (i, j) - rT_{min} space and (Y, σ_1, σ_2) be any bitopological space. A map $f : X \longrightarrow Y$ is (i, j) -minimal r -continuous if and only if the inverse image of each (i, j) -minimal regular closed set in Y is (i, j) -maximal regular closed in X for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let U be an (i, j) -minimal regular closed set in Y . Then U^C is an (i, j) -maximal regular open set in Y . Since f is (i, j) -minimal r -continuous, $f^{-1}(U^C)$ is τ_i -regular open in X . Since X is an (i, j) - rT_{min} space, $f^{-1}(U^C)$ is an (i, j) -minimal regular open set in X . Then $f^{-1}(U)$ is an (i, j) -maximal regular closed set in X .

Conversely, let U be an (i, j) -minimal regular open set in Y . Then U^C is an (i, j) -maximal regular closed set in Y . By the assumption $f^{-1}(U^C)$ is (i, j) -minimal regular closed set in Y . Then $f^{-1}(U)$ is an (i, j) -maximal regular open in X . This implies that $f^{-1}(U)$ is τ_i -regular open in X and hence f is (i, j) -minimal r -continuous. $\square$

Theorem 7.4.25. *Let (X, τ_1, τ_2) , (Y, σ_1, σ_2) and (Z, γ_1, γ_2) be bitopological spaces. If $f : X \longrightarrow Y$ is i -almost completely continuous and $g : Y \longrightarrow Z$ is (i, j) -minimal r -continuous, then $g \circ f : X \longrightarrow Z$ is (i, j) -minimal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let U be an (i, j) -minimal regular open set in Z . Since g is (i, j) -minimal r -continuous, $g^{-1}(U)$ is σ_i -regular open in Y . Further since f is i -almost completely continuous, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is τ_i -regular open in X . Hence $g \circ f$ is (i, j) -minimal r -continuous. $\square$

Theorem 7.4.26. *Let (X, τ_1, τ_2) , (Y, σ_1, σ_2) and (Z, γ_1, γ_2) be bitopological spaces. If $f : X \longrightarrow Y$ is i -almost completely continuous, $g : Y \longrightarrow Z$ is (i, j) -maximal r -continuous, then $g \circ f : X \longrightarrow Z$ is (i, j) -maximal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Theorem 7.4.27. *Let (X, τ_1, τ_2) , (Y, σ_1, σ_2) and (Z, γ_1, γ_2) be bitopological spaces. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are (i, j) -minimal r -continuous functions and Y is an (i, j) - rT_{min} space, then $g \circ f : X \longrightarrow Z$ is (i, j) -minimal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. Let U be an (i, j) -minimal regular open set in Z . Since g is (i, j) -minimal r -continuous, $g^{-1}(U)$ is a σ_i -regular open set in Y . Since Y is an (i, j) - rT_{min} space, $g^{-1}(U)$ is an (i, j) -minimal regular open set in Y . Again since f is (i, j) -minimal r -continuous, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is τ_i -regular open in X . Hence $g \circ f$ is (i, j) -minimal r -continuous. $\square$

Theorem 7.4.28. *Let (X, τ_1, τ_2) , (Y, σ_1, σ_2) and (Z, γ_1, γ_2) be bitopological spaces. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are (i, j) -maximal r -continuous functions and Y is an (i, j) - rT_{max} space, then $g \circ f : X \longrightarrow Z$ is (i, j) -maximal r -continuous for $i, j \in \{1, 2\}$ with $i \neq j$.*

Remark. Composition of (i, j) -minimal r -continuous (respectively (i, j) -maximal r -continuous) functions need not be (i, j) -minimal r -continuous (respectively (i, j) -maximal r -continuous) for $i, j \in \{1, 2\}$ with $i \neq j$.

Example 7.4.4. Let (X, τ_1, τ_2) , (Y, σ_1, σ_2) and (Z, ρ_1, ρ_2) be bitopological spaces where $X = Y = \{a, b, c, d\}$, $\tau_1 = \sigma_1 = \rho_1 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\}$ and $\tau_2 = \sigma_2 = \rho_2 = \{\phi, \{b\}, \{b, c\}, \{d\}, \{b, d\}, \{b, c, d\}, X\}$.

Then the collection of τ_1 -regular open sets in $X = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$ and the collection of τ_2 -regular open sets in $X = \{\phi, \{b, c\}, \{d\}, X\}$.

$$(1, 2)\text{-}M_iRO(X) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\},$$

$$(2, 1)\text{-}M_iRO(X) = \{\{d\}\}$$

$$(1, 2)\text{-}M_aRO(X) = \{\{a\}, \{a, b\}, \{b, c\}, \{a, c\}\},$$

$$(2, 1)\text{-}M_aRO(X) = \{\{b, c\}, \{d\}\}.$$

Let $f : X \longrightarrow Y$ be defined by $f(a) = b, f(b) = a, f(c) = c, f(d) = d$ and $g : Y \longrightarrow Z$ be defined by $g(a) = a, g(b) = d, g(c) = d, g(d) = b$. Then f and g are $(2, 1)$ -minimal r -continuous, but $g \circ f$ is not $(2, 1)$ -minimal r -continuous, since $(g \circ f)^{-1}(\{d\}) = f^{-1}(g^{-1}(\{d\})) = \{a, c\}$, is not τ_2 -regular open.

Theorem 7.4.29. *Let (X, τ_1, τ_2) , (Y, σ_1, σ_2) and (Z, γ_1, γ_2) be bitopological spaces. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are minimal r -irresolutes, then $g \circ f : X \longrightarrow Z$ is a minimal r -irresolute.*

Proof. Let U be an (i, j) -minimal regular open set in Z . Since g is (i, j) -minimal r -irresolute, $g^{-1}(U)$ is (i, j) -minimal regular open in Y . Further since f is an (i, j) -minimal r -irresolute, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is (i, j) -minimal regular open in X . Hence $g \circ f$ is an (i, j) -minimal r -irresolute. $\square$

Theorem 7.4.30. *Let (X, τ_1, τ_2) , (Y, σ_1, σ_2) and (Z, γ_1, γ_2) be bitopological spaces. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are (i, j) -maximal r -irresolutes, then $g \circ f : X \longrightarrow Z$ is an (i, j) -maximal r -irresolute.*

Chapter 8

Conclusion

The study of regular open sets, initiated by M.H. Stone in 1937, has served as a foundational milestone in the development of general topology. Stone's observation that the collection of all regular open sets forms a complete lattice and his pioneering work connecting Boolean algebras to topological structures opened up a wealth of new directions for topological investigation. This thesis is extending the theory of regular open sets through the formulation and application of new classes of sets and functions, with particular emphasis on I_r -open sets.

The initial chapters laid the groundwork by introducing the concept of I_r -open sets: sets that can be expressed as arbitrary intersections of regular open sets. This generalization, presented in Chapter 1, derived from the well-known limitation that arbitrary intersections of regular open sets are not necessarily regular open. The introduction of I_r -open sets leads naturally to a broader framework in which classical topological notions such as interior, closure, continuity, and

separation axioms are generalized and examined using I_r -open sets. Also their interaction with standard set operations was rigorously analyzed, accompanied by examples and counterexamples.

In Chapter 2, attention was directed to the redefinition of classical separation axioms in terms of I_r -open sets. The generalized I_r - T_i ($i = 0, 1, 2, 3, 4$) axioms were introduced, and several new classes of spaces, including I_r -regular, I_r -normal, almost I_r -normal, and completely I_r -normal spaces, were defined and studied. Also I_r^* -regular and I_r^* -normal spaces, were introduced and studied. Through the propositions, the interrelations, characterizations, and hereditary nature of these spaces were investigated. The notions of I_r -connectedness and I_r -compactness were also introduced and studied in this chapter.

Chapter 3 introduced the notion of I_r^* -closed sets, a generalization to I_r -closed sets. We established various structural properties and characterizations of I_r^* -closed sets, such as their behavior under union and intersection, hereditary properties, and their relation to the I_r -closure operator. Furthermore, the concept of the I_r -cover was developed, and its foundational role in defining a new supra topological structure, τ_{I_r} , was established.

In Chapter 4, we introduced I_r -continuous functions and examined various related function classes such as I_r -irresolute and I_r -open. These function types were studied in terms of their ability to preserve the properties like I_r -connectedness and I_r -compactness. Several equivalence theorems and implications were established.

In Chapter 5, we presented the theory of minimal and maximal regular open sets. By extending earlier notions of minimal and maximal open sets, this chapter focused on the characterizations and properties of minimal and maximal regular open sets and their interactions with other topological constructs.

In Chapter 6, we introduced a new class of functions termed *somewhat mr -continuous* and *somewhat mr -open* functions. Their properties and relationships to existing function types were explored, and new results were obtained. Also, we defined and studied mr -weakly stronger topologies and related theorems.

Finally, in Chapter 7, we explored on *bitopological spaces*, sets consisting of two topologies. This framework enabled the study of pairwise generalizations of regularity, continuity, and separation. Concepts such as pairwise minimal and maximal regular open sets were introduced, and their role in defining pairwise- $rT_{\min}$ and pairwise- $rT_{\max}$ spaces was thoroughly investigated. New function classes such as (i, j) -almost continuous, (i, j) -almost perfectly continuous, and p -Min/ p -Max r -continuous functions, etc were introduced and analyzed.

Through this systematic investigation, the thesis affirms that the generalization of regular open sets via I_r -open sets is not only a natural and elegant extension of classical topology but also a fertile ground for new theoretical and applied developments.

Further work

Future research directions stemming from this work may include:

- **Study of I_r -quotient spaces and quotient mappings:** In classical topology, quotient spaces play a vital role in constructing new spaces and simplifying complex structures. Extending the concept to the I_r -setting involves analyzing how the quotient topology behaves when induced by I_r -open sets. This includes identifying whether the image of an I_r -open set under a quotient map remains I_r -open and characterizing conditions under which I_r -continuity is preserved under quotient maps. This could lead to new insights into identification spaces and fiber bundles under the I_r framework.
- **Further studies on generalized separation axioms, regularity, and continuity via I_r^* -closures:** The concept of I_r^* -closed sets provides a refined tool to extend classical closedness using containment relations in-

volution both I_r -closed and I_r -open sets. A promising direction is to utilize the I_r^* -closure operator to formulate new types of separation axioms, such as $I_r^*-T_i$ ($i = 0, 1, 2$), and examine their implications in terms of space regularity and normality. These generalized axioms may offer more flexibility in dealing with topological structures that do not satisfy classical separation conditions. Furthermore, the notion of I_r^* -continuity—where preimages of I_r^* -closed sets are I_r^* -closed—can be developed to investigate whether classical function-theoretic properties like open mapping, closed mapping, and compact mapping theorems hold in this new setting. These investigations would deepen the understanding of the structural behavior of spaces under the influence of generalized closure operations.

- **Construction and analysis of I_r -product spaces:** Another promising direction is the definition and examination of product topologies using I_r -open bases. Since classical product topologies preserve properties like compactness and continuity under certain conditions, it is natural to ask whether I_r -compactness, I_r -connectedness, or other I_r -properties are preserved under products. This would involve defining appropriate base structures and studying their behavior in finite and infinite products.
- **Characterization of I_r -convergence, I_r -filters, and I_r -nets:** The study of convergence using nets and filters is a powerful tool in topology. A natural extension is to define I_r -nets and I_r -convergent filters using I_r -open sets. This could yield alternative characterizations of I_r -

compactness, I_r -continuity, and separation axioms. These generalized convergence structures may also reveal hidden properties in non-Hausdorff or infra-topological spaces.

- **Development of a categorical framework for I_r -topologies:** From a categorical point of view, one could define a category $\mathbf{Top}_{I_r}$, whose objects are I_r -topological spaces and morphisms are I_r -continuous maps. Investigating limits, colimits, initial and final structures in this category could connect the study of I_r -topologies with other branches of abstract mathematics. It would also help in understanding the algebraic behavior of I_r -structures under composition and functorial transformations.
- **Applications in digital topology and image analysis:** In digital topology, where the underlying space consists of pixels or voxels, standard open sets often fail to capture discrete continuity or adjacency. Since I_r -open sets are built from intersections of regular open sets, their application in modeling digital images or representing object boundaries in digital space could improve algorithms in computer graphics, image processing, and pattern recognition.
- **Extension to fuzzy, soft, and intuitionistic fuzzy topologies:** In soft set theory and fuzzy topology, uncertainty and vagueness are captured more precisely than in classical set theory. Defining fuzzy I_r -open sets or intuitionistic fuzzy I_r -open sets could generate new classes of fuzzy topolog-

ical spaces with generalized separation and compactness properties. This would have potential applications in decision-making, artificial intelligence, and uncertain systems modeling.

- **Classification and refinement of function types using I_r -structures:**

In addition to I_r -continuous and I_r -irresolute functions, further research could involve defining and analyzing I_r -semi-continuous, I_r -perfect, I_r -almost open, and I_r -weakly open mappings. Establishing equivalence theorems, hereditary properties, and preservation under compositions for these function types would deepen the functional aspect of I_r -topology.

- **Dualization via I_r -closure and I_r^* -closed sets:**

Many current generalizations focus on interior-based constructions. Investigating closure-based duals such as I_r -closure systems, I_r^* -closure operators, and their fixed-point structures could offer a parallel framework that generalizes the notion of closedness. This may lead to new dual theorems for compactness, separation, and connectedness.

- **Incorporation into algebraic topology:**

Concepts such as I_r -path-connectedness, I_r -homotopy, and I_r -fundamental groups could be developed by redefining paths and continuous deformations using I_r -open sets. Studying covering spaces and homology theories in the context of I_r -topology could create new algebraic invariants that are more sensitive to subtle topological features.

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- **Topological algebra under I_r -structures:** Topological groups, rings, and vector spaces form a central part of topological algebra. Introducing I_r -topologies into these structures would involve studying group operations, ring homomorphisms, and scalar multiplications that are I_r -continuous. This could potentially lead to the definition of I_r -topological groups and modules with new completeness or compactness properties.
 - **I_r -compactifications and generalizations of classical extensions:** Just as Stone–Čech and Alexandroff compactifications extend topological spaces to compact spaces, similar constructions using I_r -closure and I_r -covers could be developed. An I_r -compactification would aim to embed a space densely in an I_r -compact space, possibly preserving certain I_r -properties like connectedness or regularity.
 - **Applications in topological data analysis and machine learning:** Topological data analysis (TDA) relies on persistent homology and neighborhood systems. I_r -open sets, being based on regular open intersections, could offer new methods for defining neighborhoods and clusters in high-dimensional data. This may provide robust techniques for feature detection, classification, and manifold learning.

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