Ph.D. THESIS **MATHEMATICS**

A STUDY OF LATTICE OF c-STRUCTURES AND HOMOGENEOUS c-SPACES

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by

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CERTIFICATE

I hereby certify that the thesis entitled “A STUDY OF LATTICE OF c-STRUCTURES AND HOMOGENEOUS c-SPACES” is a bonafide work carried out by **Ms. DARSANA C.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Dr. Sini P.

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DECLARATION

I hereby declare that the work presented in the thesis entitled "**A STUDY OF LATTICE OF c-STRUCTURES AND HOMOGENEOUS c-SPACES**" is based on the original work done by me under the guidance of **Dr. Sini P.**, Assistant Professor, Department of Mathematics, University of Calicut and has not been included in any other thesis submitted previously for the award of any degree. The contents of the thesis are undergone plagiarism check using iThenticate software at C.H.M.K Library, University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.

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ABSTRACT

The concept of connected sets is defined in several mathematical fields, including topology, graph theory, and their fuzzy analogues. Connectivity plays a crucial role in digital image processing. However, the approaches to connectedness in topology and graph theory do not necessarily coincide, but the compatibility of these two connectedness is essential as discrete image can be obtained from continuous scenes. To overcome these limitations, an integrated approach is required. All meaningful notions of connectivity share the following properties: (i) The empty set and the points are connected. (ii) The union of overlapping connected objects is connected. In 1983, R. Börger introduced an axiomatic approach to connectivity: the theory of connectivity classes, or c-structures which adopts properties (i) and (ii) as axioms. This approach broadens the notion of connectedness in both graph theory and topology. Pioneering contributions in this area have been made by J. Serra, S. Dugowson, J. Muscat and D. Buhagiar, H. J. A. M. Heijmans and C. Ronse, among others.

The theoretical development of the theory of connectivity classes is essential for its advancement in application-based studies. The primary goal of this thesis is to present novel contributions to the theory of c-spaces. The thesis mainly examines the properties of the lattice of c-structures and homogeneous c-spaces. The concept of simple expansion is defined and also study the relationship between simple expansion and upper neighbors of a c-structure. The automorphism group of the lattice of c-structures and the fixed points of the automorphism group of the lattice of c-structures are determined. We introduce and examine three distinct types of homogeneity in c-spaces namely; n-homogeneity, strongly n-homogeneity, and local homogeneity. Several characteristics of hereditarily homogeneous c-spaces are examined. Additionally, hereditarily homogeneous c-spaces are characterized in terms of c-automorphisms. Extending the concept of c-spaces, we define fuzzy c-spaces. Subsequently, it is shown that the collection of all fuzzy c-structures on a given set forms a complete lattice, and its lattice properties are explored.

Key Words: c-spaces, lattice of c-structures, homogeneous c-spaces, hereditarily homogeneous c-spaces, fuzzy c-spaces.

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സംഗ്രഹം

ടോപ്പോളജി, ഗ്രാഫ് തിയറി അവയുടെ ഫസി അനലോഗുകൾ എന്നിവയുൾപ്പെടെ നിരവധി ഗണിതശാസ്ത്ര മേഖലകളിൽ കണക്റ്റഡ് സെറ്റുകളുടെ ആശയം നിർവചിക്കപ്പെട്ടിട്ടുണ്ട്. കണക്റ്റിവിറ്റിക്ക് ഡിജിറ്റൽ ഇമേജ് പ്രോസസിംഗിൽ നിർണായക പങ്കുണ്ട്. ടോപ്പോളജിയിലും ഗ്രാഫ് തിയറിയിലുമുള്ള കണക്റ്റിവിറ്റിയിലേക്കുള്ള സമീപനങ്ങൾ ഒരു പോലെ യോജിക്കുന്നതല്ല. പക്ഷെ കണ്ടിന്യൂവസ് സീനുകളിൽ നിന്നും ഡിസ്ക്രീറ്റ് ചിത്രം ലഭിക്കുമെന്നതിനാൽ ഈ പരിമിതികളെ മറി കടക്കാൻ രണ്ടു കണക്റ്റിവിറ്റികളുടെയും ഒരു സംയോജിത സമീപനം ആവശ്യമാണ്. കണക്റ്റിവിറ്റിയുടെ എല്ലാ അർത്ഥവത്തായ ആശയങ്ങളും താഴെ പറയുന്ന സവിശേഷതകൾ പങ്കിടുന്നു. (i) ശൂന്യഗണവും ഏകാംഗ ഗണങ്ങളും കണക്റ്റഡ് ആണ്. (ii) സംഗമം ശൂന്യഗണമല്ലാത്ത ഗണങ്ങളുടെ യോഗം കണക്റ്റഡ് ആണ്. (i) ഉം (ii) ഉം സവിശേഷതകളെ പ്രമാണങ്ങളായി സ്വീകരിച്ച് 1983-ൽ ആർ. ബോർഗർ കണക്റ്റിവിറ്റിക്ക് ഒരു ആക്സിയോമാറ്റിക് സമീപനം അവതരിപ്പിച്ചു. ഇത് കണക്റ്റിവിറ്റി ക്ലാസുകൾ അഥവാ സി-സ്ട്രക്ചറുകൾ എന്ന് അറിയപ്പെടുന്നു. ഈ സമീപനം ഗ്രാഫ് തിയറിയിലെയും ടോപ്പോളജിയിലെയും കണക്റ്റഡ്നെസ് എന്ന ആശയത്തെ വികസിപ്പിക്കുന്നു. തുടർന്ന് ഈ മേഖലയിൽ ജെ. സേറ, എസ്. ഡുഗൗസൺ, ജെ. മസ്ക്കറ്റ്, ഡി. ബൃഹാഗിയർ, എച്ച്. ജെ. എ. എം. ഹെയ്ജ്മാൻ, സി. റോൺസ് തുടങ്ങി നിരവധി പേർ ഗവേഷണം നടത്തിയിട്ടുണ്ട്.

കണക്റ്റിവിറ്റി ക്ലാസുകളുടെ സിദ്ധാന്തപരമായ വികസനം അതിൻ്റെ പ്രായോഗിക പഠനങ്ങളുടെ തുടർച്ചയ്ക്ക് അനിവാര്യമാണ്. അതുകൊണ്ടുതന്നെ ഈ പഠനത്തിലെ നമ്മുടെ പ്രാഥമിക ലക്ഷ്യം സി-സ്പെയ്സുകളുടെ സിദ്ധാന്തത്തിന് പുതിയ സംഭാവനകൾ നൽകുക എന്നതാണ്. സി-സ്ട്രക്ചറുകളുടെ ലാറ്റിസിൻ്റെയും ഹോമോജിനസ് സി-സ്പെയ്സുകളുടെയും സവിശേഷതകളാണ് ഈ തീസിസ് പ്രധാനമായും പരിശോധിക്കുന്നത്. സിമ്പിൾ എക്സ്പാൻഷൻ എന്ന ആശയത്തെ നിർവചിക്കുകയും കൂടാതെ ഒരു സി-സ്ട്രക്ചറിൻ്റെ സിമ്പിൾ എക്സ്പാൻഷനും അപ്പർ നൈബറും തമ്മിലുള്ള ബന്ധത്തെകുറിച്ച് പഠിക്കുകയും ചെയ്യുന്നു. സി-സ്ട്രക്ചറുകളുടെ ലാറ്റിസിൻ്റെ ഓട്ടോമോർഫിസം ഗ്രൂപ്പും ഈ ഗ്രൂപ്പിൻ്റെ ഫിക്സഡ് പോയൻ്റുകളും നിർണയിക്കുന്നു. എൻ-ഹോമോജിനിറ്റി, സ്ട്രോംഗ്ലി എൻ-ഹോമോജിനിറ്റി, ലോകൽ ഹോമോജിനിറ്റി എന്നീ മൂന്നുതരം ഹോമോജിനിറ്റികളെ ഞങ്ങൾ പരിചയപ്പെടുത്തുകയും പരിശോധിക്കുകയും ചെയ്യുന്നു. ഹെറിഡിറ്ററിലി ഹോമോജിനസ് സി-സെപ്യ്സുകളുടെ പലതരം സവിശേഷതകൾ വിശകലനം ചെയ്യുന്നു. കൂടാതെ സി-ഓട്ടോമോർഫിസവുമായി ബന്ധപ്പെടുത്തി ഹെറിഡിറ്ററിലി ഹോമോജിനസ് സി-സ്പെയ്സുകളെ വിശേഷിപ്പിക്കുന്നു. സി-സ്പെയ്സുകളുടെ ആശയം വികസിപ്പിച്ചു കൊണ്ട് ഫസി സി-സ്പെയ്സുകളെ നിർവചിക്കുന്നു. തുടർന്ന്, ഒരു നിർദ്ദിഷ്ട സെറ്റിലെ എല്ലാ ഫസി സി- സ്ട്രക്ചറുകളുടെയും ശേഖരം ഒരു സമ്പൂർണ ലാറ്റിസ് രൂപപ്പെടുമെന്ന് തെളിയിക്കുകയും അതിൻ്റെ ലാറ്റിസ് സംബന്ധമായ സവിശേഷതകൾ പര്യവേഷണം ചെയ്യുകയും ചെയ്യുന്നു.

സൂചകപദങ്ങൾ: സി - സ്പെയ്സസ്, ലാറ്റിസ് ഓഫ് സി-സ്ട്രക്ചേർസ്, ഹോമോജിനസ് സി- സ്പെയ്സസ്, ഹെറിഡിറ്ററിലി ഹോമോജിനസ് സി-സ്പെയ്സസ്, ഫസി സി-സ്പെയ്സസ്.

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Contents

Chapter 0

Introduction

Numerous mathematical fields, including topology, graph theory, and its fuzzy analogues, discuss the concept of connectedness. Several areas have found significant applications for connectedness, such as mathematical morphology, pattern recognition, image filtering, ultrasound image segmentation, tumor detection, biomedical image analysis, digital topology, link theory, and so forth. To further advance these application-based studies, this thesis aims to provide novel contributions to the theoretical development of the concept of connectedness. In this work, set theoretical, fuzzy set theoretical, topological, and lattice theoretical methods are adopted.

0.1 Motivation and Survey of Literature

The concept of a connected set that originates from point set topology appears in a different form in graph theory. In general, topological connectivity is useful for images defined over a continuous space, while graph-theoretic connectivity applies to images defined over a discrete space. Compatibility is essential, as discrete images are often obtained from the discretization of continuous scenes. Thus, the topological or graph-theoretical approaches to connectivity limit its practical application. The primary motivation behind formalizing the concept of connectedness is that it has been given multiple definitions.

Here, we demonstrate this using two classical methods from graph theory and topology.

In topology, connectedness is defined in terms of separation, while in graph theory, it is defined in terms of paths. A separation [22] on a topological space (X, τ) is a pair of disjoint nonempty open subsets of X whose union is X. The space (X, τ) is said to be connected [22] if there does not exist a separation of X. A graph G is said to be connected [3] if for every pair of vertices u, $v \in$ G, there is a path with end vertices u and v. However, the scenarios covered by these two connectivity approaches are not necessarily the same. For example, consider the usual topology on the set of real numbers $\mathbb{R}$. Here, every two element subset of $\mathbb{R}$ is disconnected. Consequently, the collection of all connected subsets of any graph is incompatible with the collection of all connected sets with respect to this topology. It has been proved that there does not exist a topology for which the collection of all connected sets is equivalent to the connected subsets of the cyclic graph with n vertices for any $n \geq 5$. To clarify this incompatibility, consider the following example:

Consider the classical notions of 4-connectivity and 8-connectivity subsets of $\mathbb{Z}^2$. Here, two graphs are formed with $\mathbb{Z}^2$ as the set of vertices, where the edges are defined by the following adjacency relations:

$$(x_1, y_1) \text{ is } 4-\text{adjacent to } (x_2, y_2) \ if \ |x_1 - x_2| + |y_1 - y_2| = 1, \text{ and}$$
$$(x_1, y_1) \text{ is } 8-\text{adjacent to } (x_2, y_2) \ if \ max\{|x_1 - x_2|, |y_1 - y_2|\} = 1.$$

It has been proved that there is a topology on $\mathbb{Z}^2$ for which connected sets are precisely the 4-connected sets, whereas no topology on $\mathbb{Z}^2$ exists for which connectedness is equivalent to 8-connectedness.

There are difficulties in using these two approaches independently, and this limitation need to be resolved through an integrated method. By unifying key notions of connectedness from both topology and graph theory, R. Börger [7] proposed a framework for connectivity known as the theory of connectivity classes, or c-structures. The axioms refer to the minimum properties of connected sets. That is, singletons are connected, and the union of connected sets with a nonempty intersection is connected. Subsequently, a standard notion of connectedness was established. We adopt the definition of c-space from [23], which includes an additional condition: the empty set is connected. Thus, a c-space can be defined as follows:

A c-space [23] is an ordered pair (X, C_X), where C_X is a collection of subsets of X satisfying the following properties:

1. $\emptyset \in C_X$ and $\{x\} \in C_X$ for all $x \in X$.
2. If $\{C_j : j \in J\}$ is a collection of subsets in C_X with $\bigcap_{j \in J} C_j \neq \emptyset$, then $\bigcup_{j \in J} C_j \in C_X$.

Here, C_X is called a c-structure on X, and elements of C_X are called connected sets. J. Serra [31, 33] undertakes a thorough analysis of connectivity classes. Since then, many mathematicians have studied it extensively, including C. Ronse [26], S. Dugowson [12, 13], J. Muscat and D. Buhagiar [23], P. K. Santhosh [27–30], P. Sini and A. K. Sruthi [35, 37–39], among others. For more details, see [14, 25, 36, 40, 41].

C. Ronse [26] characterized connectivity in terms of separating pairs of sets. He studied the system of connectivity openings and c-spaces. The concept of touching points in a c-space was first introduced by J. Muscat and D. Buhagiar [23], who further investigated its properties. They also proved that every finite connective space is a simple graph.

Finite topological c-spaces and connective spaces were characterized by K. P. Ratheesh and N. M. Madhavan Namboothiri [25], who also introduced the notion of α-generated c-structure. In addition, they studied the properties of the lattice of c-structures. They denoted this lattice by $LCS(X)$ and examined its atomic and dually atomic properties.

P. K. Santhosh [28] investigated the properties of homogeneity and bihomogeneity in c-spaces. Additionally, he studied the product, the sum, and the quotient of c-spaces.

P. Sini [35] determined completely homogeneous c-spaces. She investigated the relationship between hereditarily homogeneous c-spaces and completely homogeneous c-spaces.

A. K. Sruthi [39] focused on order-induced c-spaces. She further studied the reversible properties of c-spaces and the characteristics of cut-point c-spaces. Additionally, she investigated the relation between the group of all c-automorphisms of a c-space and its associated hypergraph.

0.2 Organization of the Thesis

Apart from the introduction, this dissertation comprises six chapters. **Chapter 1** provides the basic definitions and results for the development of the thesis. It covers foundational concepts in lattice theory, topology, graph theory, fuzzy set theory, c-spaces, and the system of connectivity openings.

Chapter 2 focuses on the properties of the lattice of c-structures. In this chapter, we characterize dually atomic c-structures in $LCS(X)$. Furthermore, we characterize the c-spaces for which a complement exists. The concept of simple expansion is defined as the method for generating a finer c-structure from a given c-structure. We further investigate a more general question: under what conditions does a simple expansion of a given c-structure C_X preserve a given property? The relationship between the upper neighbor and the simple expansion of a c-structure is investigated, and it is proved that each c-structure on a finite set has an upper neighbor. Furthermore, we determine the automorphism group of $LCS(X)$ and also the fixed points of the automorphism group of $LCS(X)$.

Chapter 3 explores homogeneity in the context of c-spaces. Several features associated with homogeneous c-spaces are discussed. We define a special class of homogeneous c-structures on finite sets and also introduce and investigate some characteristics of three different forms of homogeneity in c-spaces: n-homogeneity, strongly n-homogeneity, and local homogeneity. Also, we analyze how different types of homogeneity in c-spaces relate to each other. It is shown that, every strongly n-homogeneous c-space is n-homogeneous. However, the converse does not hold. In addition, we prove that every invertible locally homogeneous c-space is homogeneous.

Chapter 4 discusses several characteristics of hereditarily homogeneous c-spaces. We examine the properties of touching points of a hereditarily homogeneous c-space and also characterize hereditarily homogeneous c-spaces in terms of c-automorphisms. Specifically, a c-space is hereditarily homogeneous if and only if every transposition is a c-automorphism. Furthermore, we prove that hereditary homogeneity and complete homogeneity are equivalent notions for connective spaces.

In our daily lives, we encounter numerous challenges, many of them are uncertain or imprecise in nature. If we represent such problems using the idea of classical set theory, the solution obtained may be far away from the reality. The response that results from expressing such situations using the idea of classical set theory may differ significantly from the original situation. This underscores the relevance of a set theory that can assist in resolving this. L. A. Zadeh [43] introduced fuzzy set theory in 1965, which provided a proper framework for dealing with imprecision and uncertainty in real applications. The theory of c-spaces is limited to binary pictures and is set oriented. Fuzzy set theory allowed classical notions of connectedness to be extended to fuzzy circumstances. By generalizing the idea of c-spaces, we introduce fuzzy context to the idea of c-spaces in **Chapter 5**. This eventually results in the establishment of a concept of connectivity that works with binary and gray scale images. Fuzzy t-closed sets and fuzzy touching points are then introduced. In addition, we define the fuzzy c-continuous mapping and explore some of its characteristics. Furthermore, we prove that the collection of fuzzy c-structures on a set forms a lattice under the usual set inclusion.

Chapter 6 presents the conclusion of the thesis along with a discussion of several open problems. The bibliography is provided at the end.

Chapter 1

Preliminaries

In this chapter, we list the fundamental definitions and results that are used in the subsequent chapters. This chapter is divided into six sections. Section 1 contains definitions of basic terminologies associated with lattice theory. Section 2 deals with topology. Section 3 discusses the fundamental concepts of graph theory. Section 4 focuses on fuzzy set theory. We go through the fundamentals of c-spaces in Section 5. In the final section, the basics of the system of connectivity openings and its relation with c-spaces are discussed.

1.1 Lattice Theory

In this section, we go through a few basic definitions of lattice theory.

A lattice [6] is a partially ordered set in which any two elements have a meet and a join in it.

Let $(\mathcal{L}, \leq)$ be a lattice with the smallest element 0 and the largest element 1. The elements 0 and 1 are called universal bounds of the lattice $\mathcal{L}$ [6]. A lattice $\mathcal{L}$ is said to be bounded if it has universal bounds. For $a_1, a_2 \in \mathcal{L}$, we say that a_1 is an upper neighbor of a_2 or a_1 covers a_2 [6] if $a_2 < a_1$ and there is no $a \in \mathcal{L}$ such that $a_2 < a < a_1$.

An atom of the lattice $\mathcal{L}$ [6] is an element that covers the smallest element 0. A lattice is atomic [6] if every element other than the smallest element can be written as the join of atoms. A dual atom or an anti-atom [6] is an element which is covered by the largest element 1 in the lattice. A lattice is dually atomic if every element other than the largest element can be written as the meet of dual atoms [6].

Let a_1, $a_2 \in \mathcal{L}$. Then a_1 is said to be a complement of a_2 [6] if $a_1 \vee a_2 = 1$ and $a_1 \wedge a_2 = 0$. Also, $\mathcal{L}$ is called complemented [6] if all its elements have complements.

Let $\mathcal{L}$ be a lattice. Then

1. $\mathcal{L}$ is called distributive [6] if for all a_1, a_2, $a_3 \in \mathcal{L}$, $a_1 \wedge (a_2 \vee a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3)$. This is equivalent to $a_1 \vee (a_2 \wedge a_3) = (a_1 \vee a_2) \wedge (a_1 \vee a_3)$
2. $\mathcal{L}$ is called modular [6] if for all a_1, a_2, $a_3 \in \mathcal{L}$, $a_1 \leq a_3$ implies $a_1 \vee (a_2 \wedge a_3) = (a_1 \vee a_2) \wedge a_3$.

A lattice $\mathcal{L}$ is called lower semi-modular [6] if and only if for any two distinct elements a_1 and a_2 in $\mathcal{L}$ which are both covered by a third element a_3, it follows that a_1 and a_2 both cover $a_1 \wedge a_2$. The dual notion is called upper semi-modular.

A sublattice [6] of a lattice $\mathcal{L}$ is a subset M of $\mathcal{L}$ such that $a_1 \in M$, $a_2 \in M$ imply $a_1 \wedge a_2 \in M$ and $a_1 \vee a_2 \in M$. A lattice $\mathcal{L}$ is said to be complete [6] if every subset of $\mathcal{L}$ has a meet and join in $\mathcal{L}$.

The lattices $(\mathcal{L}_0, \leq)$ and $(\mathcal{L}_1, \leq^{'})$ are isomorphic, and the map $\psi : \mathcal{L}_0 \to \mathcal{L}_1$ is an isomorphism [6] if and only if ψ is one to one and onto and

$$a_1 \leq a_2 \ in \ \mathcal{L}_0 \ if \ and \ only \ if \ \psi(a_1) \leq^{'} \psi(a_2) \ in \ \mathcal{L}_1$$

An isomorphism from a lattice on to itself is called an automorphism [6] and the set of all automorphisms of a lattice onto itself is a group under function composition.

1.2 Topology

We review some topological definitions here, taken from [15] and [22].

Let (X, τ) be a topological space. A subfamily $\mathbf{B}$ is said to be a base for τ [15] if every member of τ can be expressed as the union of some members of $\mathbf{B}$.

Theorem 1.2.1. *Let (X, τ) be a topological space, and $\mathbf{B} \subseteq \tau$. Then $\mathbf{B}$ is a base for τ if and only if for any $x \in X$ and any open set U containing x, there exists $B \in \mathbf{B}$ such that $x \in B$ and $B \subseteq U$ [15].*

Let X be a topological space. A separation of X [22] is a pair A, B of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X.

Theorem 1.2.2. *[15] Every continuous image of a connected set is connected.*

However, there are functions that map connected sets to connected sets, but are not continuous. One example is shown below.
Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) = \begin{cases} sin(\frac{1}{t}) & \text{if } t \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently, h maps connected sets to connected sets but is not continuous.

A topological space (X, τ) is said to be homogeneous [17] if for any x and y in X there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.

1.3 Graph Theory

Let us go through some basic definitions in graph theory.

A graph G [3] is a triple consisting of a vertex set V(G), an edge set E(G), and a relation that associates with each edge to vertices called its end points.

A walk in G [3] is an alternating sequence W : $v_0e_1v_1e_2v_2...e_nv_n$ of vertices and edges beginning and ending with vertices in which v_{j-1} and v_j are the ends of e_j. v_0 is the origin and v_n is the terminus of W.

A walk is called a path if all vertices are distinct [3].

The graph G is said to be connected [3] if any two vertices u and v in it are connected by a path. Otherwise, it is disconnected.

1.4 Fuzzy Set Theory

The theory of fuzzy sets, first introduced by L. A. Zadeh in 1965 [43], is a mathematical description of fuzziness. Throughout this section, X is a nonempty set, I will denote the unit interval [0,1], and I^X is the collection of all functions from X to I. See [21, 42] for further details.

A fuzzy set in X [42] is a function with domain X and values in I.

Let $f \in I^X$. Then the support of f [42] denoted by $supp(f)$ is the set $\{x \in X : f(x) > 0\}$. Here, $f(x)$ is called the grade of membership of x in f and X is the carrier of the fuzzy set f.

If f takes only the values 0 and 1, then f is called a crisp set in X [42]. If $f(x) = a$ for all $x \in X$, then we denote the fuzzy set f by $\underline{a}$.

Let f_1, $f_2 \in I^X$. Then f_1 is said to be contained in f_2 denoted by $f_1 \leq f_2$ [42] if $f_1(x) \leq f_2(x)$ for all $x \in X$. Then the join of f_1 and f_2, $f_1 \vee f_2$ [42] is defined as $(f_1 \vee f_2)(x) = max\{f_1(x), f_2(x)\}$. The meet of f_1 and f_2 [42] is given by $(f_1 \wedge f_2)(x) = min\{f_1(x), f_2(x)\}$. The complement of f [42] is the fuzzy set f' defined as $f'(x) = 1 - f(x)$ for all $x \in X$.

Let $h : X \to Y$, $f \in I^X$, and $g \in I^Y$. Then $h(f)$ is a fuzzy set in Y defined as

$$h(f)(y) = \begin{cases} \bigvee\{f(x) : x \in h^{-1}(y)\} & \text{if } h^{-1}(y) \neq \emptyset, \\ 0 & \text{if } \ h^{-1}(y) = \emptyset. \end{cases}$$

and $h^{-1}(g)$ is a fuzzy set in X, $h^{-1}(g)(x)$ defined by $g(h(x))$ for $x \in X$ [20].

A fuzzy set p in X is called a fuzzy point [42] if

$$p(x) = \begin{cases} a & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

where $0 < a < 1$. We denote this fuzzy point by y_a. A fuzzy point $p \in I^X$ is said to be a fuzzy point in a fuzzy set f if $p \leq f$.

A fuzzy set p is called a crisp point if

$$p(x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

We denote this crisp point by y_1.

1.5 c-spaces

We provide here a few definitions and notations pertaining to c-spaces. For additional information, see [23, 35].

A c-space [23] is an ordered pair (X, C_X), where C_X is a collection of subsets of X such that the following properties hold.

(1) $\emptyset \in C_X$ and $\{x\} \in C_X$ for all $x \in X$.

(2) If $\{C_j : j \in J\}$ is a collection of subsets in C_X with $\bigcap_{j\in J} C_j \neq \emptyset$, then $\bigcup_{j\in J} C_j \in C_X$.

Here, C_X is called a c-structure on X and elements of C_X are called connected sets.
A connected set containing more than one element is called a non-degenerate connected set.

Examples 1.5.1. Let X be any set. Then

1. $C_X = \{\emptyset\} \cup \{\{x\} : x \in X\}$ is a c-structure on X, called the discrete c-structure on X and is denoted by D_X. The space (X, D_X) is called the discrete c-space.

2. $C_X = P(X)$, the power set of X, is also a c-structure on X, called the indiscrete c-structure on X and the space (X, C_X) is called the indiscrete c-space.

3. $C_X = D_X \cup \{B \subseteq X : B \text{ is infinite } \}$ is a c-structure on X, called the co-finite c-structure on X.

4. For any $x \in X$, $C_X = D_X \cup \{B \subseteq X : x \in B\}$ is called the rooted c-structure on X.

5. Consider the set of real numbers $\mathbb{R}$. Then the collection of all the intervals in $\mathbb{R}$ is a c-structure, called the standard c-structure on $\mathbb{R}$.

6. The set of all intervals in $\mathbb{Z}$ and the set of all intervals in $\mathbb{N}$ each form a c-structure, called the standard c-structure on the respective sets.

Definition 1.5.1. Let C_X be a c-structure on X. Then C_X is called a connective structure or a connectology [23] if it satisfies two more conditions.

(3) Given any nonempty sets $G, H \in C_X$ with $G \cup H \in C_X$, there exists $x \in G \cup H$ such that $\{x\} \cup G \in C_X$ and $\{x\} \cup H \in C_X$.

(4) If $G, H, C_i \in C_X$ are disjoint and $G \cup H \cup (\bigcup_{i \in I} C_i) \in C_X$, then there exists $J \subseteq I$ such that $G \cup (\bigcup_{j \in J} C_j) \in C_X$ and $H \cup (\bigcup_{i \in I \setminus J} C_i) \in C_X$.

The ordered pair (X, C_X) is called a connective space.

The following is an example of a c-space that is not a connective space.

Example 1.5.1. Let $X = \{a, b, c, d\}$ and $C_X = D_X \cup \{\{a, c\}, \{b, d\}\}$. Here, (X, C_X) is not a connective space. Condition (3) does not hold here.

Theorem 1.5.1. *Every finite connective space is a simple graph [23].*

Let X be any set and $\mathbf{B} \subseteq P(X)$. Then the intersection of all c-structures on X containing $\mathbf{B}$ is a c-structure, called the c-structure generated by $\mathbf{B}$ [23], and is denoted by $\langle \mathbf{B} \rangle$. Let (X, C_X) be a c-space and α be any cardinal with $\alpha \leq |X|$. Then C_X is said to be α-generated [25] if there is a sub collection $\mathbf{B} \subseteq \{C \in C_X : |C| \leq \alpha\}$ such that $C_X = \langle \mathbf{B} \rangle$.

Theorem 1.5.2. *Finite connective spaces are precisely the finite* 2*-generated* *c-spaces [25].*

A c-space (Y, C_Y) is said to be a sub c-space [23], of the c-space (X, C_X) if $Y \subseteq X$ and $C_Y = \{B \in C_X : B \subseteq Y\}$.

Example 1.5.2. Let $X = \{a, b, c, d, e\}$ and $C_X = D_X \cup \{\{a, d\}, \{a, b, c\}, \{a, b, c, d\}\}$.

1. If $Y = \{b, c\}$, then $C_Y = D_Y$ is the sub c-structure on Y.

2. If $Y = \{a, c, d\}$, then $C_Y = D_Y \cup \{\{a, d\}\}$ is the sub c-structure on Y.

Definition 1.5.2. Let (X, C_X) be a c-space. Then the c-space $(X, C_X \cup \{X\})$ is called the Brunnian closure [13] of the given c-space.

Let (X, C_X) and (Y, C_Y) be two c-spaces and $h : X \to Y$ be a function. Then h is said to be c-continuous [23] if it maps connected sets in X to connected sets in Y and h is said to be a c-isomorphism or a catenomorphism [23] if it is bijective and both h and h^{-1} are c-continuous. A c-isomorphism from a c-space onto itself is called a c-automorphism. Note that the set of all c-automorphisms of a c-space is a group under composition. We denote this group by $C(X, C_X)$

Example 1.5.3. 1. Any function from a discrete c-space is c-continuous.

2. Any function into an indiscrete c-space is c-continuous.

3. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Consider the c-structures $C_X = D_X \cup \{\{1, 3\}, \{3, 4\}, \{1, 3, 4\}\}$ and $C_Y = D_Y \cup \{\{a, b\}, \{b, c\}, \{a, b, c\}\}$ on X and Y, respectively. Define $h : X \to Y$ by $h(1) = c$, $h(2) = c$, $h(3) = b$, and $h(4) = a$. Then h is c-continuous.

4. In the above example, if $h(3) = a$, then h is not c-continuous.

5. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$. Consider the c-structures $C_X = D_X \cup \{\{1, 3\}, \{2, 4\}\}$ and $C_Y = D_Y \cup \{\{a, b\}, \{c, d\}\}$ on X and Y, respectively. Define $h : X \to Y$ by $h(1) = a$, $h(2) = c$, $h(3) = b$, and $h(4) = d$. Then h is a c-isomorphism.

Let (X, C_X) be a c-space and $B \subseteq X$. Then a point $x \in X$ is said to touch the set B if there is a nonempty subset $C \subseteq B$ such that $\{x\} \cup C$ is connected [23]. The set of all points touching the set B is denoted by $t(B)$. The set B is said to be t-closed [23] if $t(B) = B$. The t-closure of a set B is defined to be the smallest t-closed set containing B and is denoted by $\overline{B}$. Now, B is said to be c-dense in X [35] if $t(B) = X$.

The subsets B_1 and B_2 are said to touch [23] if there is a point $x \in B_1 \cup B_2$ that touches both B_1 and B_2.

A c-space (X, C_X) is said to be C_1 [23] if distinct points of X are disconnected. That is $\overline{\{x\}} = \{x\}$ for all $x \in X$.

The collection of all connected subsets of a topological space (X, τ) will form a c-structure on X, called the associated c-structure [28], and X with this associated c-structure is called the associated c-space of (X, τ).
A c-space (X, C_X) is said to be topological [25] if there exists a topology τ on X such that the associated c-space of (X, τ) is (X, C_X). It is said to be graphical [28] if there exists a graph G such that the collection of all connected sets of G coincides with C_X.

Theorem 1.5.3. *A finite topological c-space is* 2*-generated [25].*

Let X be a given set. Then the set of all c-structures on X is a lattice under the usual set inclusion. We denote this lattice by $LCS(X)$. Here, the join of two c-structures C_1 and C_2, is the c-structure generated by $C_1 \cup C_2$ and the meet of C_1 and C_2 is $C_1 \cap C_2$. The discrete c-structure D_X is the least element and the indiscrete c-structure $P(X)$ is the greatest element in $LCS(X)$.

Let (X_1, C_{X_1}) and (X_2, C_{X_2}) be two disjoint c-spaces. Then its sum space [28] is defined to be the c-space (X, C_X), where $X = X_1 \cup X_2$ and $C_X = C_{X_1} \cup C_{X_2}$.

Let $\{X_j : j \in J\}$ be an indexed family of sets. Then its direct sum or coproduct is defined as the set $\{(x, j) \in (\bigcup\limits_{j \in J} X_j) \times J : x \in X_j\}$ [28] and is denoted by $\sum\limits_{j \in J} X_j$.
Obviously, $\sum\limits_{j \in J} X_j = \bigcup\limits_{j \in J} (X_j \times \{j\})$. If X_j's are mutually disjoint, then $\sum\limits_{j \in J} X_j = \bigcup\limits_{j \in J} X_j$.

The concept of sum space can be expanded to an arbitrary family of c-spaces by the following definition.

Let $\{(X_j, C_{X_j}) : j \in J\}$ be a family of c-spaces and $X = \sum_{j\in J} X_j$ be the set theoretical sum of the sets $\{X_j : j \in J\}$. Define a c-structure C_X on X as the weak c-structure on X generated by the family of injective functions $\{\lambda_j : X_j \to X : \lambda_j(x) = (x, j) \textit{ for each } j \in J\}$. Then the c-space (X, C_X) is defined to be the sum or coproduct [28] of the given family of c-spaces and is denoted by $\sum_{j\in J} X_j$.

Theorem 1.5.4. *[28] Let (X, C_X) be the sum of the family of c-spaces $\{(X_j, C_{X_j}) : j \in J\}$. Then $C_X = \bigcup_{j\in J} \{C \times \{j\} : C \in C_{X_j}\}$.*

The following definition gives the structure of connected sets in the product c-structure.

Let $\{(X_j, C_{X_j}) : j \in J\}$ be an indexed family of c-spaces and $X = \prod_{j\in J} X_j$. Let C_X be the collection of all subsets of X such that $B \in C_X$ if and only if $\pi_j(B) \in C_{X_j}$ for every $j \in J$. Then C_X is a c-structure on X, called the product c-structure and (X, C_X) is called the product c-space [23].

Let (X, C_X) and (Y, C_Y) be any two c-spaces. Let $h : X \to Y$ be an onto function. Then h is said to be a quotient map or Y is said to be a quotient space of X with respect to h [7] if C_Y is the weak c-structure on Y generated by $\{h\}$.

1.6 System of Connectivity Openings

In this section, we define some basic concepts related to openings. The definitions and results are taken from [26] and [32].

Given two sets X and Y, let η be a map $X \to Y$, we call it a set operator, or simply an operator.

- η is said to be increasing if $A \subseteq B$ implies $\eta(A) \subseteq \eta(B)$.
- η is said to be idempotent if $\eta(\eta(A)) = \eta(A)$.
- η is said to be is extensive if $A \subseteq \eta(A)$.
- η is said to be is anti-extensive if $\eta(A) \subseteq A$.

An opening on $P(X)$ [26] is an increasing, idempotent, and anti-extensive operator.

Definition 1.6.1. Let X be any set. A family of operators γ_p, $p \in X$, satisfying the following four conditions is called a system of connectivity openings [32] on $P(X)$.

(C_1) γ_p is an opening for all $p \in X$.

(C_2) $\gamma_p(\{p\}) = \{p\}$ for all $p \in X$.

(C_3) For every p, $q \in X$ and $A \subseteq X$; either $\gamma_p(A) \cap \gamma_q(A) = \emptyset$ or $\gamma_p(A) = \gamma_q(A)$.

(C_4) For $A \subseteq X$ and $p \in X$ such that $p \notin A$ we have $\gamma_p(A) = \emptyset$.

Theorem 1.6.1. *There is a one-to-one correspondence between connectivity classes on $P(X)$ and systems of connectivity openings on $P(X)$. A connectivity class C_X and the corresponding family of connectivity openings γ_p define each other by the following two equivalent relations [32]:*

(i) For $A \subseteq X$, $\gamma_p(A)$ is the union of all $C \in C_X$ such that $p \in C \subseteq A$; in other words, it is $\emptyset$ for $p \notin A$, while for $p \in A$ it is the greatest $C \in C_X$ such that $p \in C \subseteq A$.

(ii) C_X is the set of all $\gamma_p(A)$ for $p \in X$ and $A \subseteq X$.

For example, if $C_X = D_X$, then for $A \subseteq X$ and $x \in A$, we have

$$\gamma_p(A) = \begin{cases} \{p\} & \text{if } p \in A, \\ \emptyset & \text{if } p \notin A. \end{cases}$$

If $C_X = P(X)$, then

$$\gamma_p(A) = \begin{cases} A & \text{if } p \in A, \\ \emptyset & \text{if } p \notin A. \end{cases}$$

If C_X is a co-finite c-structure on X, then

$$\gamma_p(A) = \begin{cases} A & \text{if } p \in A \text{ and } A \text{ is infinite}, \\ \{p\} & \text{if } p \in A \text{ and } A \text{ is finite}, \\ \emptyset & \text{if } p \notin A. \end{cases}$$

Chapter 2

The Lattice of c-structures

2.1 Introduction

Lattice theory has been shown to be useful in many cases when analyzing the totality of mathematical systems of a given type and it gives a framework for organizing the study of a particular class in mathematics. Many authors studied the lattice properties of different set structures, including topology, generalized topology, and fuzzy generalized topology [4, 11, 19].

In [25], K. P. Ratheesh and N. M. Madhavan Namboothiri proved that the lattice of c-structures, $LCS(X)$ is an atomic lattice whose atoms are of the form $D_X \cup \{C\}$, where $C \subseteq X$, $|C| \geq 2$ and the dual atoms in $LCS(X)$ are of the form $P(X) \setminus \{\{a, b\}\}$, where a, $b \in X$ and $a \neq b$. Moreover, they proved that $LCS(X)$ forms a complete lattice.

We explore the characteristics of the lattice of c-structures in this chapter. We define the notion of simple expansion and study the relationship between simple expansion and upper neighbors of c-spaces. In addition, we characterize the c-spaces for which complement exists. The automorphism group of the lattice of c-structures and the fixed points of the group of automorphisms are determined. A part of this chapter was published in Gulf Journal of Mathematics [10].

2.2 Lattice of c-structures

Here, we continue the study of the lattice of c-structures.
The complement of a c-structure in $LCS(X)$ is defined as follows.

Definition 2.2.1. Let X be a nonempty set and $C_X \in LCS(X)$. Then $C'_X \in LCS(X)$ is said to be a complement of C_X if $C_X \vee C'_X = P(X)$ and $C_X \wedge C'_X = D_X$.

Example 2.2.1.

- Let $X = \{a, b, c\}$ and $C_X = D_X \cup \{\{a, b\}, \{b, c\}, \{a, b, c\}\}$. Then $C'_X = D_X \cup \{\{a, c\}\}$ is the complement of C_X.
- Let X be a nonempty set and $x \in X$. Consider $C_X = D_X \cup \{B \subseteq X : x \in B\}$ and $C'_X = D_X \cup \{B \subseteq X : x \notin B\}$. Clearly, C_X and C'_X are c-structures on X and C'_X is the complement of C_X.

Let $C_X \neq D_X$ be a c-structure on X and there exists no $C \in C_X$ such that $|C| = 2$. If C_X has a complement, say C'_X, then C'_X contains all two element subsets of X and hence $C'_X = P(X)$. So C_X has no complement. That is, if there exists no two element subset of X in C_X, then C_X has no complement.

In [25], it is proved that $LCS(X)$ is not complemented. Now, we characterize the c-spaces for which complement exists.

Theorem 2.2.1. *Let $C_X \in LCS(X)$ and $\mathcal{C}$ be the collection of all two element subsets of X. Then C'_X exists if and only if there exists some $B \subseteq \mathcal{C}$ which generates C_X and $\mathcal{C} \setminus B$ generates $P(X) \setminus C_X$. Furthermore, $C'_X = D_X \cup (P(X) \setminus C_X)$.*

Proof. Let $C_X \in LCS(X)$ be such that C'_X exists. Since $C_X \vee C'_X = P(X)$, $C_X \cup C'_X$ contains all two element subsets of X. Also, $C_X \wedge C'_X = D_X$, there exists no two element subset in common. Now, let B be the collection of all two element subsets of X in C_X and $B^{'}$ be the collection of all two element subsets of X in C'_X. Then $B \cap B^{'} = \emptyset$ and $B \cup B^{'} = \mathcal{C}$. So $B^{'} = \mathcal{C} \setminus B$. Suppose that there exists some $C \in C_X$ with $|C| \geq 3$ such that $C \notin \langle B \rangle$. Then B generates some proper subsets of C. Otherwise, $C \in C'_X$, which is not possible.

Let A be a proper subset of C generated by B. Then there exists some $y \in C \setminus A$ such that $\{x, y\} \notin C_X$ for all $x \in A$. Let $\mathcal{A} = \{A \subsetneq C\colon\ A \in \langle B \rangle\}$ and for each $A \in \mathcal{A}$, define $C(A) = \bigcup \{\{x, y\} : x \in A,\ y \in C \setminus A$ and $\{x, y\} \notin C_X\}$. Clearly, $C(A) \in C'_X$ for each $A \in \mathcal{A}$. We prove $C = C(A)$ for some $A \in \mathcal{A}$. Let $z \in C$. Then either $z \in A$ or $z \in C \setminus A$. If $z \in A$, then there exists some $y \in C \setminus A$ such that $\{z, y\} \notin C_X$. So $z \in C(A)$. If $z \in C \setminus A$, then either $\{x, z\} \notin C_X$ for all $x \in A$ or there exists a $x \in A$ such that $\{x, z\} \in C_X$.

Now, $\{x, z\} \notin C_X$ for all $x \in A$ implies $z \in C(A)$. If $\{x, z\} \in C_X$, then there exists some $A_1 \in \mathcal{A}$ where $A \subsetneq A_1$ such that $z \in A_1$ and $\{z, y\} \notin C_X$ for some $y \in C \setminus A_1$. So $z \in C(A_1)$. Hence $C = C(A_1)$. Thus, $C \in C'_X$, which is a contradiction. Thus, any $C \in C_X$ with $|C| \geq 3$ is generated by elements of B. Similarly, we can prove that any $C' \in P(X) \setminus C_X$ is generated by elements of $\mathcal{C} \setminus B$. Since $C_X \wedge C'_X = D_X$, C'_X contains none of the elements of C_X except D_X. This implies that $C'_X \subseteq D_X \cup (P(X) \setminus C_X)$. Since $C_X \vee C'_X = P(X)$, $C_X \cup C'_X$ contains all two element subsets of X. We have $C_X = \langle B \rangle$, it follows that C'_X contains $\mathcal{C} \setminus B$. So $\langle \mathcal{C} \setminus B \rangle \in C'_X$. But $\langle \mathcal{C} \setminus B \rangle = P(X) \setminus C_X$ implies $C'_X = D_X \cup (P(X) \setminus C_X)$.
The converse is trivial. Hence the theorem. □

By Theorem 2.2.1, for any c-structure C_X, if the complement exists, then it is unique.

Corollary 2.2.1. *Let $C_X \neq D_X$ be a C_1 c-structure. Then C'_X does not exist.*

Proof. Since the c-space (X, C_X) is C_1, there do not exist two element connected sets in C_X. It follows that C'_X does not exist. □

The converse of Corollary 2.2.1 does not hold in general. Observe that the standard c-structure on $\mathbb{Z}$ does not have a complement and is not C_1.

Proposition 2.2.1. *Let C_X be a c-structure on X such that C'_X exists. Then the union of touching points of any subset of X with respect to C_X and C'_X is X.*

Proof. Let $A \subseteq X$ and $x \in X$. Suppose that x is not a touching point of A in C_X. Then $\{x, y\} \notin C_X$ for any $y \in A$. Thus, $\{x, y\} \in C'_X$ and hence x is a touching point of A in C'_X. Since x is arbitrary, the proof follows. $\square$

In [25], it is proved that $LCS(X)$ is not dually atomic whenever $|X| \geq 3$. Now, let us go through the definition of a dually atomic c-structure.

Definition 2.2.2. A c-structure $C_X \in LCS(X)$, where $C_X \neq P(X)$ is said to be dually atomic if it can be written as the meet of dual atoms in $LCS(X)$.

Let $X = \mathbb{R}$ and C_X be the standard c-structure on $\mathbb{R}$. Then we cannot write C_X as the meet of dual atoms in $LCS(X)$. So C_X is not dually atomic.

We can easily characterize the dually atomic c-structures in $LCS(X)$.

Theorem 2.2.2. *Let $C_X \neq P(X)$ be a c-structure on X, where $|X| \geq 3$ and $\mathcal{B} = \{B \subseteq X : |B| \geq 3\}$. Then C_X is dually atomic if and only if $\mathcal{B} \subseteq C_X$.*

Proof. Suppose that C_X is dually atomic. Then C_X can be written as the meet of dual atoms. Since the dual atoms in X are of the form $P(X) \setminus \{\{a, b\}\}$, where $a, b \in X$ and $a \neq b$, the dual atom contains all subsets of X having cardinality greater than 2 and so is their meet. Hence $\mathcal{B} \subseteq C_X$.

Conversely, assume that C_X contains all subsets of X that have cardinality greater than 2. Let $\mathcal{A} = \{\{a, b\} \subseteq X : \{a, b\} \notin C_X\}$. Then we have $C_X = \bigwedge_{\{a,b\} \in \mathcal{A}} (P(X) \setminus \{\{a, b\}\})$. Hence C_X is dually atomic. $\square$

2.3 Simple Expansion of c-structures

The simple expansion is the method used to generate finer c-structures from a given c-structure. More precisely, for a given c-structure C_X and a subset G of X, where $G \notin C_X$, by using the idea of simple expansion, we determine the smallest c-structure on X containing both C_X and G.

Theorem 2.3.1. *Let C_X be a c-structure on a nonempty set X and $G \subseteq X$ with $G \notin C_X$. Let $\{C_i : i \in I\}$ be the collection of sets in C_X with $C_i \cap G \neq \emptyset$ for all $i \in I$. Then $C_X(G) = C_X \cup \{(\bigcup_{j\in J} C_j) \cup G$ for all $J \subseteq I\}$ is the smallest c-structure on X containing both C_X and G.*

Proof. Clearly, $C_X \subseteq C_X(G)$. Since every singleton in G intersects with G, $G \in C_X(G)$. Let $\{K_l : l \in L\}$ be a collection of sets in $C_X(G)$ with $\bigcap_{l\in L} K_l \neq \emptyset$. If $K_l \in C_X$ for all l, then $\bigcup_{l\in L} K_l \in C_X$. Otherwise, we have

$$\bigcup_{l\in L} K_l = K \cup ((\bigcup_{j\in J} C_j) \cup G),$$

where $K \in C_X$. Since $\bigcap_{l\in L} K_l \neq \emptyset$, $K \cap ((\bigcup_{j\in J} C_j) \cup G) \neq \emptyset$ and hence either $K \cap (\bigcup_{j\in J} C_j) \neq \emptyset$ or $K \cap G \neq \emptyset$.

If $K \cap G \neq \emptyset$, then $K \cup ((\bigcup_{j\in J} C_j) \cup G) \in C_X(G)$. If $K \cap (\bigcup_{j\in J} C_j) \neq \emptyset$, then there exists some $C_{j'}$, where $j' \in J$ such that $K \cap C_{j'} \neq \emptyset$. It follows that $K \cup C_{j'} \in C_X$. Since $C_{j'} \cap G \neq \emptyset$, $(K \cup C_{j'}) \cap G \neq \emptyset$. So $\bigcup_{l\in L} K_l = (K \cup C_{j'}) \cup ((\bigcup_{\substack{j\in J \\ j\neq j'}} C_j) \cup G) \in C_X(G)$. If $K_l \in C_X(G) \setminus C_X$ for all $l \in L$, then it is clear that $\bigcup_{l\in L} K_l \in C_X(G)$. Thus,

$C_X(G)$ is a c-structure on X. Obviously, $C_X(G)$ is the smallest c-structure containing both C_X and G. This completes the proof. $\square$

Definition 2.3.1. Let C_X be a c-structure on a nonempty set X and $G \subseteq X$ with $G \notin C_X$. Then $C_X(G)$ is called the simple expansion of C_X by G.

Corollary 2.3.1. *$C_X(G) = C_X \vee (D_X \cup \{G\})$.*

Example 2.3.1. Let $C_{\mathbb{Z}}$ be the standard c-structure on $\mathbb{Z}$ and $G = 2\mathbb{Z}$. Then $G \notin C_{\mathbb{Z}}$.

Now, the simple expansion of $C_{\mathbb{Z}}$ by G is given by $C_{\mathbb{Z}}(G) = C_{\mathbb{Z}} \cup \{C : C \supseteq 2\mathbb{Z}\}$.

2.3.1 Upper Neighbor of a c-space

The concept of upper neighbor of an element in a lattice is already discussed in section 1.1. For $C_X, K_X \in LCS(X)$, we say that C_X is an upper neighbor of K_X if $K_X < C_X$ and there is no $K_X^{'} \in LCS(X)$ such that $K_X < K_X^{'} < C_X$.

Theorem 2.3.2. *Let (X, C_X) be a c-space. Then every upper neighbor of C_X is a simple expansion of C_X.*

Proof. Suppose that $\tilde{C_X}$ is an upper neighbor of C_X. Let $G \subseteq X$ be such that $G \in \tilde{C_X} \setminus C_X$. Then $C_X(G)$ is the smallest c-structure containing both C_X and G. Thus, $C_X \subsetneq C_X(G) \subseteq \tilde{C_X}$. Since $\tilde{C_X}$ is an upper neighbor, $C_X(G) = \tilde{C_X}$. It follows that $\tilde{C_X}$ is a simple expansion of C_X by G. Hence the result. $\square$

A simple expansion of a c-space need not be an upper neighbor. See the following example.

Example 2.3.2. Let $C_{\mathbb{Z}} = D_{\mathbb{Z}} \cup \{\mathbb{N}, \mathbb{Z} \setminus \mathbb{N}\}$ and $G = \{0, 1\}$. Then

$$C_{\mathbb{Z}}(G) = D_{\mathbb{Z}} \cup \{\{0,1\}, \mathbb{N}, \mathbb{Z} \setminus \mathbb{N}, \mathbb{N} \cup \{0\}, (\mathbb{Z} \setminus \mathbb{N}) \cup \{1\}, \mathbb{Z}\}.$$

Let $K_{\mathbb{Z}} = D_{\mathbb{Z}} \cup \{\mathbb{N}, \mathbb{Z} \setminus \mathbb{N}, \mathbb{N} \cup \{0\}, \mathbb{Z}\} \in LCS(\mathbb{Z})$. Then $C_{\mathbb{Z}} \subsetneq K_{\mathbb{Z}} \subsetneq C_{\mathbb{Z}}(G)$. So $C_{\mathbb{Z}}(G)$ is not an upper neighbor of $C_{\mathbb{Z}}$.

Remark 2.3.1. *Let (X, C_X) be a c-space and $G \subseteq X$ be such that $G \cap C = \emptyset$ for all non-degenerate connected sets $C \in C_X$. Then the simple expansion of C_X by G given by $C_X(G) = C_X \cup \{G\}$ is an upper neighbor of C_X. If (X, C_X) is a non connected c-space, then the simple expansion of C_X by X is an upper neighbor of C_X.*

Let C_X be a c-structure on X such that C'_X exists. Then for any $G \notin C_X$, the complement of $C_X(G)$, denoted by $C'_X(G)$ need not exist. See the example below.

Example 2.3.3. Let $X = \{a, b, c, d\}$ and $C_X = D_X \cup \{\{a, b\}, \{a, c\}, \{a, b, c\}\}$. Let $G = \{a, c, d\} \notin C_X$. Then

$$C_X(G) = D_X \cup \{\{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}.$$

Note that $C_X(G)$ has no complement, since $\{a, c, d\}$ cannot be generated by two element subsets of X in C_X.

Remark 2.3.2. *Let C_X be a c-structure on X such that C'_X exists. If $C'_X(G)$ exists, then $|G| = 2$. But $|G| = 2$ does not necessarily mean that $C'_X(G)$ exists.*

Example 2.3.4. Consider the c-space $(\mathbb{Z}, C_{\mathbb{Z}})$, where $C_{\mathbb{Z}} = D_{\mathbb{Z}} \cup \{(n, \infty) : n \in \mathbb{Z}\}$. Let $G = \{1, 2, 3\}$. Then $C_{\mathbb{Z}}(G) = C_{\mathbb{Z}} \cup \{G\}$. Let $A = \{2, 3, 4\}$. Then $t(A) = A$ in $C_{\mathbb{Z}}$ and $t(A) = \{1, 2, 3, 4\}$ in $C_{\mathbb{Z}}(G)$. If $A = \{-2, -1, 0\}$, then $t(A) = A$ in both $C_{\mathbb{Z}}$ and $C_{\mathbb{Z}}(G)$.

The following theorem provides a sufficient condition for each subset of X to have the same touching points in both C_X and $C_X(G)$.

Theorem 2.3.3. *Let (X, C_X) be a c-space and $G \subseteq X$ with $G \notin C_X$. If $A \subseteq X$ such that $G \cap A = \emptyset$, then $t(A)$ is the same in both C_X and $C_X(G)$.*

Proof. Let $t(A) = Y$ in C_X, where $Y \subseteq X$. Suppose that $x \in X \setminus Y$ is a touching point of A in $C_X(G)$. Then there exists a nonempty subset $C \subseteq A$ such that $C \cup \{x\} \in C_X(G)$. Thus, $C \cup \{x\}$ can be written as $C \cup \{x\} = (\bigcup_{j \in J} C_j) \cup G$, where $\{C_j : j \in J\} \subseteq C_X$ with $C_j \cap G \neq \emptyset$ for each $j \in J$, which implies $G \subseteq C \cup \{x\}$. It follows that $G \cap C \neq \emptyset$ and hence $G \cap A \neq \emptyset$. This is a contradiction. □

Definition 2.3.2. A subfamily $\mathbf{B}$ of $P(X)$ is said to form an upper family [35] if $B \in \mathbf{B}$ and $B \subseteq C \subseteq X$ implies $C \in \mathbf{B}$. Obviously, $C_X = D_X \cup \mathbf{B}$ is a c-structure on X and is called an upper c-structure [35].

If C_X is an upper c-structure, then $C_X(G)$ need not be an upper c-structure. Example: Consider the c-structure $C_{\mathbb{Z}} = D_{\mathbb{Z}} \cup \{\mathbb{Z} \setminus \{1\}, \mathbb{Z}\}$ on $\mathbb{Z}$. Let $G = \mathbb{Z} \setminus \{1, 2\}$. Then $C_{\mathbb{Z}}(G) = D_{\mathbb{Z}} \cup \{\mathbb{Z} \setminus \{1, 2\}, \mathbb{Z} \setminus \{1\}, \mathbb{Z}\}$ is not an upper c-structure, since $\mathbb{Z} \setminus \{2\} \notin C_{\mathbb{Z}}$.

A necessary and sufficient condition for the simple expansion of an upper c-space to be an upper c-space is established by the following theorem.

Theorem 2.3.4. *Let (X, C_X) be an upper c-space and $G \subseteq X$. Then $(X, C_X(G))$ is an upper c-space if and only if $G \cup \{x\} \in C_X$ for all $x \in X \setminus G$.*

Proof. Suppose that $G \cup \{x\} \in C_X$ for all $x \in X \setminus G$. This implies $G \cup \{x\} \in C_X(G)$. Since C_X is an upper c-structure, $G' \in C_X$ for any G' such that $G \cup \{x\} \subseteq G' \subseteq X$. Thus, $G' \in C_X(G)$. So any superset of $G \in C_X(G)$. It follows that $C_X(G)$ is an upper c-structure.

Conversely, let $C_X(G)$ be an upper c-structure. Since $G \in C_X(G)$, $G \cup \{x\} \in C_X(G)$ for all $x \in X \setminus G$. If $G \cup \{x\} \in C_X$ for all $x \in X \setminus G$, then the proof follows. If not, $G \cup \{x\} = (\bigcup_{j \in J} C_j) \cup G$, where $C_j \in C_X$ and $C_j \cap G \neq \emptyset$ for each $j \in J$. Thus, $x \in \bigcup_{j \in J} C_j$, which implies $x \in C_{j'}$ for some $j' \in J$. Also, $\bigcup_{j \in J} C_j$ contains exactly one point which does not belong to G. So there exists some $G_1 \subseteq G$ such that $C_{j'}$ can be written as $C_{j'} = G_1 \cup \{x\}$. So $G_1 \cup \{x\} \in C_X$, since C_X is an upper c-structure, it follows that $G \cup \{x\} \in C_X$. This completes the proof. □

Definition 2.3.3. [37] Let $C_{\leq}$ be the collection of all intervals of a linearly ordered set $(X, \leq)$. Then $C_{\leq}$ is c-structure on X, called the order-induced c-structure. The ordered pair $(X, C_{\leq})$ is called the order-induced c-space corresponding to the linearly ordered set $(X, \leq)$.

Theorem 2.3.5. *The simple expansion of an order-induced c-space is not an order-induced c-space.*

Proof. Let $(X, C_{\leq})$ be an order-induced c-space and $G \subseteq X$ be such that $G \notin C_{\leq}$. Then the simple expansion $C_{\leq}(G)$ of $C_{\leq}$, is not an order-induced c-structure, since G is not an interval. $\square$

The next theorem asserts the existence of an upper neighbor of a c-space if X is finite.

Theorem 2.3.6. *Let X be a finite set and $C_X \neq P(X)$ be a c-structure on X. Then there exists some $G \subseteq X$ with $G \notin C_X$ such that $C_X(G) = C_X \cup \{G\}$.*

Proof. Let $|X| = n$. Here, we prove $C_X(G) = C_X \cup \{G\}$ is a c-structure on X for some $G \subseteq X$. If $X \notin C_X$, then $C_X \cup \{X\}$ is a c-structure on X. If $X \in C_X$, then consider the collection, $\mathcal{G}_{n-1} = \{G \subseteq X : |G| = n-1\}$. If $\mathcal{G}_{n-1} \nsubseteq C_X$, then take some $G \in \mathcal{G}_{n-1}$ such that $G \notin C_X$. Consider $C_X \cup \{G\}$. Let C_1, $C_2 \in C_X \cup \{G\}$ with $C_1 \cap C_2 \neq \emptyset$. If C_1, $C_2 \in C_X$, then $C_1 \cup C_2 \in C_X$. If $C_1 \in C_X$ and $C_2 = G$, then $C_1 \cup C_2 = X$ or $C_1 \cup C_2 = G$, which implies $C_1 \cup C_2 \in C_X(G)$. Similarly, if $\mathcal{G}_{n-1} \subseteq C_X$ and $X \in C_X$, then consider $\mathcal{G}_{n-2} = \{G \subseteq X : |G| = n-2\}$. If $\mathcal{G}_{n-2} \nsubseteq C_X$, then take some $G \in \mathcal{G}_{n-2}$ such that $G \notin C_X$. Then $C_X \cup \{G\}$ is a c-structure on X. Proceeding like this, if C_X contains all sets with cardinality greater than 2 and its supersets and $C_X \neq P(X)$, there exists some $\{x, y\} \subseteq X$ with $\{x, y\} \notin C_X$, then we can take G as $\{x, y\}$ and $C_X \cup \{G\}$ is a c-structure on X. Hence the proof. $\square$

A necessary and sufficient condition for a c-structure $\tilde{C_X}$ to be an upper neighbor of C_X is provided by the following theorem.

Theorem 2.3.7. *Let C_X and $\tilde{C_X}$ be two c-structures on X. Then $\tilde{C_X}$ is an upper neighbor of C_X if and only if $\tilde{C_X} = C_X(G)$ for all $G \in \tilde{C_X} \setminus C_X$.*

Proof. Suppose that $\tilde{C_X}$ is an upper neighbor of C_X and $G \in \tilde{C_X} \setminus C_X$. Since $C_X(G)$ is the smallest c-structure containing C_X and G, we can write $C_X \subseteq C_X(G) \subseteq \tilde{C_X}$. By our assumption, we get $\tilde{C_X} = C_X(G)$. Thus, for all $G \in \tilde{C_X} \setminus C_X$ we get $\tilde{C_X} = C_X(G)$.

Conversely, let $\tilde{C_X} = C_X(G)$ for all $G \in \tilde{C_X} \setminus C_X$. Suppose that $\tilde{C_X}$ is not an upper neighbor of C_X. Then there exists some $K_X \in LCS(X)$ such that $C_X \subsetneq K_X \subsetneq \tilde{C_X}$. Since $K_X \neq C_X$, there exists some $H \in K_X$ such that $H \notin C_X$. Now, $H \in K_X$ implies $H \in \tilde{C_X}$. So $H \in \tilde{C_X} \setminus C_X$ implies $\tilde{C_X} = C_X(H)$. That is, $\tilde{C_X}$ is the smallest c-structure containing both C_X and H. It follows that $K_X = \tilde{C_X}$, a contradiction. Thus, $\tilde{C_X}$ is an upper neighbor of C_X. $\square$

Theorem 2.3.8. *Let X be a set and C_X be a c-structure on X. Then every upper neighbor of C_X is of the form $C_X \cup \{G\}$, where $G \subseteq X$ and $G \notin C_X$.*

Proof. Suppose $\tilde{C_X} = C_X \cup \mathcal{B}$ is an upper neighbor of C_X, where $\mathcal{B}$ contains at least two subsets of X which do not belong to C_X. Now, for a fixed set $G_k \in \mathcal{B}$, let $\mathcal{B}_k = \{G_h \in \mathcal{B} : G_h \subsetneq G_k\}$. First we consider the case $\mathcal{B}_k \neq \emptyset$. Let $\mathcal{B}'_k = \mathcal{B} \setminus \mathcal{B}_k$. We prove $C_X \cup \mathcal{B}'_k$ is a c-structure on X. Let $\{C_i : i \in I\}$ be a collection of sets in $C_X \cup \mathcal{B}'_k$ with $\bigcap\limits_{i\in I} C_i \neq \emptyset$. If $C_i \in C_X$ for all i, then $\bigcup\limits_{i\in I} C_i \in C_X$. Otherwise, let $\bigcup\limits_{i\in I} C_i = C \cup (\bigcup\limits_{j\in J} C_j)$ where $C \in C_X$, $J \subseteq I$ and $C_j \in \mathcal{B}'_k$ for each $j \in J$. If $\bigcup\limits_{i\in I} C_i \notin C_X \cup \mathcal{B}'_k$, then $\bigcup\limits_{i\in I} C_i \in \mathcal{B}_k$ which implies $C_j \in \mathcal{B}_k$ for each $j \in J$, which is a contradiction. So $\bigcup\limits_{i\in I} C_i \in C_X \cup \mathcal{B}'_k$. If $C_i \in \mathcal{B}'_k$ for all i, then also we can prove that $\bigcup_{i\in I} C_i \in C_X \cup \mathcal{B}'_k$. Thus, $C_X \cup \mathcal{B}'_k$ is a c-structure on X. If $\mathcal{B}_k = \emptyset$, then consider $\tilde{C_X} \setminus \{G_k\}$. Similarly, we can prove that $\tilde{C_X} \setminus \{G_k\}$ is a c-structure on X. In each of the cases, $\tilde{C_X}$ is not an upper neighbor of C_X. Thus, an upper neighbor of C_X is of the form $C_X \cup \{G\}$, $G \notin C_X$. □

The preceding theorem gives the form of an upper neighbor of a c-space. If X is finite, then by Theorem 2.3.6, every c-structure $C_X \neq P(X)$ on X has an upper neighbor. But in general, this is not true. The following example illustrates this.

Example 2.3.5. Let $C_{\mathbb{Z}}$ be a c-structure on $\mathbb{Z}$, the set of integers given by

$$C_{\mathbb{Z}} = D_{\mathbb{Z}} \cup \{P(2\mathbb{Z}) \cup P(2\mathbb{Z}+1)\} \cup \{C \subseteq \mathbb{Z} : C \cap \mathbb{Z} \text{ is a co-finite subset of } \mathbb{Z}\},$$

where $P(2\mathbb{Z})$ is the power set of $2\mathbb{Z}$ and $P(2\mathbb{Z}+1)$ is the power set of $2\mathbb{Z}+1$.

Let $G \subseteq \mathbb{Z}$ and $G \notin C_{\mathbb{Z}}$. Then there exist some $x \in 2\mathbb{Z}$ and $y \in 2\mathbb{Z}+1$ such that $\{x, y\} \subseteq G$. Let $z \in \mathbb{Z} \setminus G$. Then either $\{x, z\}$ or $\{y, z\} \in C_{\mathbb{Z}}$. So $G \cup \{z\} \in C_{\mathbb{Z}}(G)$. But $G \cup \{z\} \notin C_{\mathbb{Z}} \cup \{G\}$. Thus, $C_{\mathbb{Z}} \cup \{G\}$ is not an upper neighbor. So $C_{\mathbb{Z}}$ has no upper neighbor.

Note that $LCS(X)$ is non-modular and hence non distributive when $|X| > 2$ [25]. But we can show that it is lower semi-modular.

Theorem 2.3.9. *For any set X, $LCS(X)$ is lower semi-modular.*

Proof. Suppose C_X, $K_X \in LCS(X)$ and $C_X \neq K_X$. Let $\mathcal{K}_X \in LCS(X)$ be an upper neighbor of both C_X and K_X. Then $\mathcal{K}_X$ is of the form $C_X \cup \{G\}$,

$G \notin C_X$ or $K_X \cup \{H\}$, $H \notin K_X$ which implies $G \in K_X$ and $H \in C_X$. Thus, $C_X \wedge K_X = C_X \setminus \{H\} = K_X \setminus \{G\}$, implies both C_X and K_X are upper neighbor of $C_X \wedge K_X$. □

Remark 2.3.3. *$LCS(X)$ is not upper semi-modular.*

Example 2.3.6. Let $X = \{a, b, c, d\}$. Consider the c-structures

$$C_X = D_X \cup \{\{a, b, c\}\} \text{ and } K_X = D_X \cup \{\{a, c, d\}\}.$$

Here, both C_X and K_X are upper neighbors of D_X and $C_X \vee K_X = D_X \cup \{\{a, b, c\}, \{a, c, d\}, X\}$. But $C_X \vee K_X$ is neither an upper neighbor of C_X nor K_X.

2.4 A Sublattice of $LCS(X)$

Let $\mathcal{F}(X)$ denotes the collection of all C_1 c-structures on a given set X. Then $\mathcal{F}(X)$ is a sublattice of $LCS(X)$. Clearly, D_X is C_1 and is the least element of $\mathcal{F}(X)$. Let $\mathcal{B} = \{B \subseteq X : |B| = 2\}$. Then $(P(X) \setminus \mathcal{B}) \in \mathcal{F}(X)$ and is the largest element in $\mathcal{F}(X)$. Obviously, $\mathcal{F}(X)$ is an atomic lattice. The atoms are of the form $D_X \cup \{C\}$ where $C \subseteq X$ and $|C| \geq 3$.

Remark 2.4.1. *The dual atoms in $\mathcal{F}(X)$ are of the form $P(X) \setminus (\mathcal{B} \cup \{\{a, b, c\}\})$, where a, b and c are distinct points of X. But $\mathcal{F}(X)$ is not dually atomic.*

Example 2.4.1. Let $X = \{a, b, c, d\}$. Then the dual atoms in $\mathcal{F}(X)$ are $D_X \cup \{\{b, c, d\}, \{a, c, d\}, \{a, b, d\}, X\}$, $D_X \cup \{\{a, b, c\}, \{a, c, d\}, \{a, b, d\}, X\}$, $D_X \cup$

$\{\{a,b,c\},\{b,c,d\},\{a,c,d\},X\}$, $D_X \cup \{\{a,b,c\},\{b,c,d\},\{a,b,d\},X\}$. Note that $D_X \cup \{\{a,b,c\}\}$ cannot be written as a meet of dual atoms.

Obviously, $\mathcal{F}(X)$ is not complemented.

The collection of topological c-structure (or graphical c-structures) on a set X is not a sublattice of $LCS(X)$. Since the meet of topological c-structures (or graphical c-structures) need not be topological (or graphical).

Definition 2.4.1. [6] An ideal is a nonempty subset J of a lattice $\mathcal{L}$ with the properties

1. $a \in J$, $x \in \mathcal{L}$, $x \leq a$ implies $x \in J$.

2. $a \in J$, $b \in J$ implies $a \vee b \in J$.

For any set X, consider $LCS(X)$. Let $C_X \in LCS(X)$. Then $J(X) = \{K_X \in LCS(X) : K_X \subseteq C_X\}$ is an ideal in $LCS(X)$.

Theorem 2.4.1. *The collection of all C_1 c-structures on a set X is an ideal.*

Proof. Let $\mathcal{F}(X)$ denote the collection of all C_1 c-structures on a set X. Then $C_X \vee K_X \in \mathcal{F}(X)$ for any C_X, $K_X \in \mathcal{F}(X)$. Let $C_X \in \mathcal{F}(X)$, $\mathcal{K}_X \in LCS(X)$ such that $\mathcal{K}_X \subseteq C_X$. So $\mathcal{K}_X$ is also C_1, which implies $\mathcal{K}_X \in \mathcal{F}(X)$. Hence the proof. □

A simple expansion of a C_1 c-structure is not always C_1. However, if we restrict the cardinality of G, we have the following theorem.

Theorem 2.4.2. *Let (X, C_X) be a C_1 c-space. Then $C_X(G)$ is C_1 if and only if $|G| \geq 3$.*

Proof. Suppose that $C_X(G)$ is C_1. Let $G = \{a, b\} \subseteq X$, where $a, b \in X$ such that $a \neq b$. Then $C_X(G)$ is not C_1. Conversely, suppose that $|G| \geq 3$. It is clear that $C_X(G)$ does not contains any two element subsets of X. Hence $C_X(G)$ is C_1. □

2.5 Automorphism Group of Lattice of c-structures

Here, we determine the automorphism group of $LCS(X)$. Throughout this section, $S(X)$ denotes the symmetric group on X.

Lemma 2.5.1. *Let $f \in S(X)$ and $C_X \in LCS(X)$. Then $f(C_X) = \{f(C) : C \in C_X\}$ is a c-structure on X, where $f(C) = \{f(x) : x \in C\}$.*

Proof. Since f is a bijection, $f(D_X) = D_X$ and hence $D_X \subseteq f(C_X)$. Let $\{C_i : i \in I\}$ be an arbitrary family of sets in $f(C_X)$ with $\bigcap\limits_{i\in I} C_i \neq \emptyset$. Then $C_i = f(K_i)$, where $K_i \in C_X$. Since $\bigcap\limits_{i\in I} C_i \neq \emptyset$, there exists some $c \in X$ such that $c \in C_i$ for all $i \in I$. Then $f^{-1}(c) \in K_i$ for all $i \in I$. It follows that $\bigcap\limits_{i\in I} K_i \neq \emptyset$ and hence $\bigcup\limits_{i\in I} K_i \in C_X$. Now, $\bigcup\limits_{i\in I} C_i = \bigcup\limits_{i\in I} f(K_i) = f(\bigcup\limits_{i\in I} K_i) \in f(C_X)$. Hence the proof. □

Let $f \in S(X)$. Define f_c, a map on $LCS(X)$ by $f_c(C_X) = f(C_X)$, $C_X \in LCS(X)$.

Lemma 2.5.2. *f_c is an automorphism on $LCS(X)$.*

Proof. Let $C_X, K_X \in LCS(X)$ be such that $f_c(C_X) = f_c(K_X)$. That is, $f(C_X) = f(K_X)$. Now, $C \in C_X$ if and only if $f(C) \in f(C_X)$ if and only if $f(C) \in f(K_X)$ if and only if $C \in K_X$. So $C_X = K_X$, which implies that f_c is injective.
Now, for any $K_X \in LCS(X)$, let $C_X = \{f^{-1}(C) : C \in K_X\}$ where $f^{-1}(C) = \{f^{-1}(x) : x \in C\}$. Then $C_X = f^{-1}(K_X)$. It follows that $C_X = f_c^{-1}(K_X)$. So f_c is surjective and hence it is bijective.

Suppose that $C_X \subseteq K_X$. Let $K \in f_c(C_X)$. Then there exists some $C \in C_X$ such that $K = f_c(C)$ implies $K = f(C)$. Since $C \in C_X$, we have $C \in K_X$ which implies $K = f(C) \in f(K_X)$, it follows that $K \in f_c(K_X)$. Thus, $f_c(C_X) \subseteq f_c(K_X)$. So f_c is an automorphism. □

It is easy to prove that an automorphism on $LCS(X)$ maps atoms to atoms and dual atoms to dual atoms.

Lemma 2.5.3. *Let F be an automorphism on $LCS(X)$ and $C_X, K_X \in LCS(X)$. Then C_X is the complement of K_X if and only if $F(C_X)$ is the complement $F(K_X)$.*

Proof. The c-structures C_X and K_X are complements to each other

$$\begin{aligned}
&\iff C_X \vee K_X = P(X) \text{ and } C_X \wedge K_X = D_X \\
&\iff F(C_X \vee K_X) = F(P(X)) \text{ and } F(C_X \wedge K_X) = F(D_X) \\
&\iff F(C_X) \vee F(K_X) = P(X) \text{ and } F(C_X) \wedge F(K_X) = D_X \\
&\iff F(C_X) \text{ and } F(K_X) \text{ are complements to each other.}
\end{aligned}$$

□

Theorem 2.5.1. *The set of all automorphisms of $LCS(X)$ is precisely $\{f_c : f \in S(X)\}$.*

Proof. By Lemma 2.5.2, f_c is an automorphism on $LCS(X)$.

Conversely, let F be an automorphism on $LCS(X)$. If $|X| \leq 2$, then there is nothing to prove. Suppose that $|X| \geq 3$ and let $\mathcal{A}$ denote the set of all atoms of the form $\alpha_A = D_X \cup \{A\}$, $A \subseteq X$, and $|A| = 2$. Consider the dual atom of α_A, which is denoted by α'_A and $\alpha'_A = P(X)\setminus\{A\}$. Since α_A and α'_A are complements to each other by Lemma 2.5.3, $F(\alpha_A)$ and $F(\alpha'_A)$ are complements to each other. As $F(\alpha'_A)$ is a dual atom, $F(\alpha'_A)$ is of the form $F(\alpha'_A) = P(X) \setminus \{B\}$, where $|B| = 2$. This implies $B \in F(\alpha_A)$. Now, $F(\alpha'_A) \vee F(\alpha_A) = P(X)$ implies $F(\alpha_A) = D_X \cup \{B\}$. So F maps $\mathcal{A}$ into $\mathcal{A}$.

Now, for any $\alpha_G = D_X \cup \{G\} \in \mathcal{A}$, $|G| = 2$, consider the corresponding dual atom $\alpha'_G = P(X)\setminus\{G\}$. Since F is onto, there exists a dual atom $\alpha'_H \in LCS(X)$ where $H \subseteq X$, $|H| = 2$ such that $F(\alpha'_H) = \alpha'_G$, where $\alpha'_H = P(X) \setminus \{H\}$. Let $\alpha_H = D_X \cup \{H\}$.

Now,

$$\begin{aligned}
F(\alpha_H) \vee \alpha'_G &= F(\alpha_H) \vee F(\alpha'_H) \\
&= F(\alpha_H \vee \alpha'_H) \\
&= F(P(X)) \\
&= P(X)
\end{aligned}$$

which implies $G \in F(\alpha_H)$. It follows that $F(\alpha_H) = \alpha_G$. Thus, F maps $\mathcal{A}$ onto $\mathcal{A}$.

Claim: $F = f_c$.

Let $C_1 = D_X \cup \{A\}$, $C_2 = D_X \cup \{B\} \in \mathcal{A}$, $A \cap B \neq \emptyset$, $A \neq B$. Let $F(C_1) = D_X \cup \{C\}$, $F(C_2) = D_X \cup \{D\}$. Suppose that $C \cap D = \emptyset$. Then $F(C_1 \vee C_2) = F(C_1) \vee F(C_2) = D_X \cup \{C, D\}$. Now,

$$\begin{aligned} F(C_1 \vee C_2) &= F(D_X \cup \{A, B, A \cup B)\} \\ &= F((D_X \cup \{A\}) \vee (D_X \cup \{B\}) \vee (D_X \cup \{A \cup B\})) \\ &= F(D_X \cup \{A\}) \vee F(D_X \cup \{B\}) \vee F(D_X \cup \{A \cup B\}) \\ &= (D_X \cup \{C\}) \vee (D_X \cup \{D\}) \vee F(D_X \cup \{A \cup B\}) \\ &= D_X \cup \{C, D\} \vee F(D_X \cup \{A \cup B\}) \end{aligned}$$

But $F(C_1 \vee C_2) = D_X \cup \{C, D\}$ implies that $F(D_X \cup \{A \cup B\})$ can be either of D_X, $D_X \cup \{C\}$, $D_X \cup \{D\}$ or $D_X \cup \{C, D\}$. In each case, we will get a contradiction. So $C \cap D \neq \emptyset$ and it follows that $F(D_X \cup \{A \cup B\}) = D_X \cup \{C \cup D\}$.

Now, for any $x \in X$, there exist at least two distinct elements α_{A_i} and $\alpha_{A_j} \in \mathcal{A}$ such that $x \in A_i \cap A_j$. Let $F(\alpha_{A_i}) = \alpha_{B_i}$ and $F(\alpha_{A_j}) = \alpha_{B_j}$. Then obviously α_{B_i} and $\alpha_{B_j} \in \mathcal{A}$. Corresponding to each $x \in X$, we can find a $y \in X$ such that $y \in B_i \cap B_j$. Now, for each $x \in X$, define $f : X \to X$ as $f(x) = y$. Let $A_i = \{x, a\}$, $B_i = \{y, c\}$, $A_j = \{x, b\}$, $B_j = \{y, d\}$. Suppose that $A_k = \{x, e\}$. Let $F(\alpha_{A_k}) = \alpha_{B_k}$, $|B_k| = 2$. If $y \notin B_k$, then $B_k = \{c, d\}$. Since $|B_j \cap B_k| = 1$ and $|B_i \cap B_k| = 1$, we have

$$F(D_X \cup \{\{x, a, b\}\}) = D_X \cup \{\{y, c, d\}\}$$

$$F(D_X \cup \{\{x, a, e\}\}) = D_X \cup \{\{y, c, d\}\}$$

a contradiction. Thus, $y \in B_k$. So f is well defined.

To prove f is a bijection on X, note that F is a bijection on $\mathcal{A}$ also. Suppose that x_1, $x_2 \in X$ with $x_1 \neq x_2$. If $f(x_1) = f(x_2) = y$, then $F(D_X \cup \{\{x_1, a\}\}) = D_X \cup \{\{y, f(a)\}\}$ and $F(D_X \cup \{\{x_2, a\}\}) = D_X \cup \{\{y, f(a)\}\}$, a contradiction. So $f(x_1) \neq f(x_2)$ implies f is injective.
Now, consider $D_X \cup \{\{y, c\}\}$, $D_X \cup \{\{y, d\}\}$. Since F is bijective there exists $D_X \cup \{\{x, a\}\}$, $D_X \cup \{\{x, b\}\} \in \mathcal{A}$ such that $F(D_X \cup \{\{x, a\}\}) = D_X \cup \{\{y, c\}\}$ and $F(D_X \cup \{\{x, b\}\}) = D_X \cup \{\{y, d\}\}$ implies $f(x) = y$. So f is surjective. Thus, the proof follows. □

2.5.1 Fixed Points of the Group of Automorphisms of $LCS(X)$

Here, we determine the fixed points of the group of all automorphisms of $LCS(X)$. That is, the set of all c-structures that are fixed by all automorphisms on $LCS(X)$.

Definition 2.5.1. Let X be a nonempty set and $h : X \to X$ be a function. A point $x \in X$ is said to be a fixed point of h, if $h(x) = x$ [15].

It is clear that for any set X, every automorphism on $LCS(X)$ maps D_X

onto itself and $P(X)$ onto itself. So D_X and $P(X)$ are fixed points of any automorphism on $LCS(X)$.

Definition 2.5.2. A c-structure C_X on X is called completely homogeneous [35] if every bijection of X is a c-automorphism.

Theorem 2.5.2. *Let $\mathcal{F}$ denote the group of all automorphisms on $LCS(X)$. Then the fixed points of $\mathcal{F}$ are precisely the completely homogeneous c-structures.*

Proof. Suppose that C_X is a fixed point of $\mathcal{F}$. Then $F(C_X) = C_X$ for all $F \in \mathcal{F}$. Since $F = f_c$, it follows that $f_c(C_X) = C_X$, which implies $f(C_X) = C_X$. That is, f maps C_X onto itself, which implies that f is a c-automorphism on X. Since f is arbitrary, each bijection on X is a c-automorphism, which implies that C_X is completely homogeneous. Thus, the fixed points of $\mathcal{F}$ are the completely homogeneous c-structures. The converse part is obvious. □

In [35] (Theorem 2.21), completely homogeneous c-structures are listed. So the fixed points of the group of all automorphisms of $LCS(X)$ are the following when $|X| = n$.

1. D_X.

2. $D_X \cup \{C \subseteq X : |C| \geq k\}$, $1 < k \leq n$.

3. $D_X \cup \{C \subseteq X : |X \setminus C| \leq k\}$, where $k < n$.

4. $D_X \cup \{C \subseteq X : |X \setminus C| < k\}$, where $k \leq n$ and k is a limit cardinal.

2.6 c-spaces and the System of Connectivity Openings

The characteristics of the system of connectivity openings that correspond to numerous concepts pertaining to c-spaces such as simple expansion, complement etc. are examined in this section.

Let X be a set. For $x \in X$, let $\tilde{\gamma}_x$ be the system of connectivity openings corresponding to the simple expansion $C_X(G)$ of the c-structure C_X.
For $A \subseteq X$ and $x \in A$, the following theorem discusses the relation between $\gamma_x(A)$ and $\tilde{\gamma}_x(A)$ and also provides a necessary and sufficient condition for $\gamma_x(A) = \tilde{\gamma}_x(A)$.

Theorem 2.6.1. *Let X be a set, $A \subseteq X$ and $x \in A$. Then*

(i) $\gamma_x(A) \subseteq \tilde{\gamma}_x(A)$.

(ii) $\gamma_x(A) = \tilde{\gamma}_x(A)$ *if and only if one of the following conditions holds.*

- *(a)* $G \subseteq \gamma_x(A)$
- *(b)* $G \nsubseteq A$
- *(c)* $G \subseteq (A \setminus \gamma_x(A))$

Proof. (i) If $A \in C_X$, then $\gamma_x(A) = A = \tilde{\gamma}_x(A)$. If $A \notin C_X$, then $\gamma_x(A)$ is the largest $C \in C_X$ such that $x \in C \subseteq A$. Since $C \in C_X(G)$, $\gamma_x(A) \subseteq \tilde{\gamma}_x(A)$.

(ii) For $x \in A$, let $\gamma_x(A) = \tilde{\gamma}_x(A)$.

Case 1: $G \nsubseteq \gamma_x(A)$ and $G \subseteq A$.

Since $G \nsubseteq \gamma_x(A)$, there exists a $y \in G$ such that $y \notin \gamma_x(A)$. Obviously, $\gamma_x(A) \cap G = \emptyset$, otherwise $\gamma_x(A) \cup G \in C_X(G)$, a contradiction to $\gamma_x(A) = \tilde{\gamma}_x(A)$. Now, $\gamma_x(A) \cap G = \emptyset$ and $G \subseteq A$ implies $G \subseteq (A \setminus \gamma_x(A))$.

Case 2: $G \nsubseteq (A \setminus \gamma_x(A))$ and $G \subseteq A$.

Since $G \nsubseteq (A \setminus \gamma_x(A))$, there exists a $y \in G$ such that $y \notin (A \setminus \gamma_x(A))$. Consequently, $y \in \gamma_x(A)$ and so $\gamma_x(A) \cap G \neq \emptyset$. Thus, $\gamma_x(A) \cup G \in C_X(G)$. Suppose that $G \nsubseteq \gamma_x(A)$. Then there exists a $z \in G$ such that $z \notin \gamma_x(A)$. It follows that $\gamma_x(A) \neq \tilde{\gamma}_x(A)$. So $G \subseteq \gamma_x(A)$.

Case 3: $G \nsubseteq \gamma_x(A)$ and $G \nsubseteq (A \setminus \gamma_x(A))$.

Suppose that $G \subseteq A$. Then there exist a $y \in \gamma_x(A)$ such that $y \in G$ and $z \in A \setminus \gamma_x(A)$ such that $z \in G$. Thus, $\{y, z\} \subseteq G$. Note that $\gamma_x(A) \cap G \neq \emptyset$. So $\gamma_x(A) \cup G \in C_X(G)$. Since $G \nsubseteq \gamma_x(A)$ and $G \subseteq A$, $\gamma_x(A) \neq \tilde{\gamma}_x(A)$, which is a contradiction. Hence $G \nsubseteq A$.

Conversely, suppose G satisfies one of the given conditions and $\gamma_x(A) \neq \tilde{\gamma}_x(A)$. Then $\tilde{\gamma}_x(A)$ can be written as $\tilde{\gamma}_x(A) = (\bigcup_{j \in J} C_j) \cup G$, where $C_j \in C_X$ and $C_j \cap G \neq \emptyset$ for each $j \in J$.

Case 1: $G \subseteq \gamma_x(A)$.

Since $\gamma_x(A) \neq \tilde{\gamma}_x(A)$. There exists some $y \in \tilde{\gamma}_x(A)$ such that $y \notin \gamma_x(A)$. Since $G \subseteq \gamma_x(A)$, there exist some $C_{j'}$, $j' \in J$ such that $y \in C_{j'}$. Also note that $C_{j'} \subseteq \tilde{\gamma}_x(A) \subseteq A$. Now, $C_{j'} \cap G \neq \emptyset$ implies $C_{j'} \cap \gamma_x(A) \neq \emptyset$. Thus, $C_{j'} \cup \gamma_x(A) \in C_X$, a contradiction to the fact that $\gamma_x(A)$ is the largest connected subset of A containing x. So $\gamma_x(A) = \tilde{\gamma}_x(A)$.

Case 2: $G \nsubseteq A$.

Then there exists a $y \in G$ such that $y \notin A$. Since $y \in G$, $y \in \tilde{\gamma}_x(A)$. So $y \in A$, which is a contradiction. Thus, $\tilde{\gamma}_x(A)$ is an element of C_X. So $\gamma_x(A) = \tilde{\gamma}_x(A)$.

Case 3: $G \subseteq (A \setminus \gamma_x(A))$.

Thus, $x \notin G$. So there exists C_{j_0}, $j_0 \in J$ such that $x \in C_{j_0}$ and $C_{j_0} \in C_X$. It follows that $C_{j_0} \subseteq \gamma_x(A)$. Since $C_{j_0} \cap G \neq \emptyset$, $\gamma_x(A) \cap G \neq \emptyset$, which is a contradiction. Thus, $\gamma_x(A) = \tilde{\gamma}_x(A)$. Hence the theorem.

□

Corollary 2.6.1. *Let X be a set, $A \subseteq X$ and $x \in A$. If $A \notin C_X$ and $A \cap G = \emptyset$, then $\gamma_x(A) = \tilde{\gamma}_x(A)$.*

Proof. Since $A \notin C_X$ and $A \cap G = \emptyset$, $A \notin C_X(G)$. Let $\gamma_x(A) = C_1$ and $\tilde{\gamma}_x(A) = C_2$. Suppose that $C_1 \neq C_2$. Since $C_2 \in C_X(G)$, $C_2 = (\bigcup_{j \in J} C_j) \cup G$, where $\{C_j : j \in J\} \subseteq C_X$ with $C_j \cap G \neq \emptyset$ for each $j \in J$. Thus, $C_2 \cap G \neq \emptyset$. Since $C_2 \subseteq A$, it follows that $A \cap G \neq \emptyset$, a contradiction. Thus, $\gamma_x(A) = \tilde{\gamma}_x(A)$. □

Theorem 2.6.2. *Let X be a set and $G \subseteq X$. If $\tilde{\gamma}_x(G') = G$ for all $G' \supseteq G$ and $x \in X$, then $C_X(G)$ is an upper neighbor of C_X.*

Proof. For all $x \in G$ and $G' \supseteq G$, let $\tilde{\gamma}_x(G') = G$. Suppose that $C_X(G)$ is not an upper neighbor of C_X. By Theorem 2.3.8, we have if C_X has an upper neighbor, then it is of the form $C_X \cup \{G\}$. Thus, there exists an $H \in C_X(G)$ where $H \neq G$

and $H \notin C_X$. So H can be written as $H = (\bigcup_{j\in J} C_j) \cup G$, where $C_j \in C_X$ and $C_j \cap G \neq \emptyset$ for all $j \in J$. Thus, $H \supseteq G$ and $\tilde{\gamma}_x(H) = H$, which is a contradiction. So $C_X(G)$ is an upper neighbor of C_X. □

Let X be a set. For $x \in X$, let $\gamma_x^{'}$ be the system of connectivity openings corresponding to the complement $C_X^{'}$ of C_X. Then we get the following results.

Theorem 2.6.3. *Let X be a set and $\mathcal{A} = \{A \subseteq X : |A| = 2\}$. Then for each $A \in \mathcal{A}$ and $x \in A$ either $\gamma_x(A) = A$ or $\gamma_x^{'}(A) = A$.*

Proof. Let B be the collection of all two element subsets of X in C_X and $B^{'}$ be the collection of all two element subsets of X in $C_X^{'}$. Then $B \cup B^{'} = \mathcal{A}$. That is, for any $A \in \mathcal{A}$, either $A \in C_X$ or $A \in C_X^{'}$. If $A \in C_X$, then $\gamma_x(A) = A$. Similarly, if $A \in C_X^{'}$, then $\gamma_x^{'}(A) = A$. Hence the proof. □

Theorem 2.6.4. *Let $A \subseteq X$ and $x \in A$.*

1. *Then $\gamma_x(A) \cup \gamma_x^{'}(A) = A$.*
2. *If $\gamma_x(A) = \gamma_x^{'}(A)$, then $A = \{x\}$.*

Proof. 1. If $A \in C_X$ or $A \in C_X^{'}$, then obviously $\gamma_x(A) \cup \gamma_x^{'}(A) = A$. Suppose that $A \notin C_X$ and $A \notin C_X^{'}$. Let $\gamma_x(A) = C$. Then C is the largest connected set such that $x \in C \subseteq A$. Thus, $C \cup \{y\} \notin C_X$ for all $y \in A \setminus C$. Consequently, $C \cup \{y\} \in C_X^{'}$ for all $y \in A \setminus C$. So $\bigcup_{y\in A\setminus C} (C \cup \{y\}) \in C_X^{'}$. But $\bigcup_{y\in A\setminus C} (C \cup \{y\}) = A$, which implies $A \in C_X^{'}$. Thus, $\gamma_x^{'}(A) = A$. Hence $\gamma_x(A) \cup \gamma_x^{'}(A) = A$.

2. For $x \in A$, let $\gamma_x(A) = \gamma_x^{'}(A)$. Suppose that $A \neq \{x\}$. Then there exists some $y \in A$ such that $y \neq x$. It follows that either $\{x, y\} \subseteq \gamma_x(A)$ or $\{x, y\} \subseteq \gamma_x^{'}(A)$. Let $\gamma_x(A) = C$. Then $C \in C_X$ and $|C| \geq 2$. But by our assumption $\gamma_x(A) = \gamma_x^{'}(A)$, Thus, $\gamma_x^{'}(A) = C$. So $C \in C_X^{'}$, which is a contradiction. Hence $A = \{x\}$.

□

Obviously, for a connected set C and for all $x \in C$, $\gamma_x(C) = C$, and γ_x is increasing. Also, if $A = B$, then $\gamma_x(A) = \gamma_x(B)$ for all $x \in X$. But $\gamma_x(A) = \gamma_x(B)$ need not implies that $A = B$. If $\gamma_x(A) = \gamma_x(B)$ and A and B are connected, then $A = B$.

Theorem 2.6.5. *Let $A, B \subseteq X$.*

1. *If x is a touching point of A, then $\gamma_x(C \cup \{x\}) = C \cup \{x\}$ for some $C \subseteq A$.*

2. *If $\gamma_x(A)$ and $\gamma_x(B)$ are nonempty, then A and B touch.*

Proof. 1. Since x is a touching point of A, there exists a nonempty set $C \subseteq A$ such that $\{x\} \cup C$ is connected. So it follows that the greatest connected set containing x in $C \cup \{x\}$ is $C \cup \{x\}$. Hence $\gamma_x(C \cup \{x\}) = C \cup \{x\}$.

2. Let $\gamma_x(A) = C_1$ and $\gamma_x(B) = C_2$. Then $C_1 \subseteq A$, $C_2 \subseteq B$ and $x \in C_1 \cap C_2$. Now, $C_1 \cup \{x\} = C_1 \in C_X$ and $C_2 \cup \{x\} = C_2 \in C_X$. Thus, x is a touching point of A and B. Hence the proof.

□

It is clear that for all non-degenerate connected set C of a c-space (X, C_X) with $A \cap C = \emptyset$, $A \subseteq X$, we have $\gamma_x(A) = \{x\}$ for all $x \in A$.

Theorem 2.6.6. *Let (X, C_X) be an upper c-space, $A \subseteq X$ and $x \in A$. Then either $\gamma_x(A) = \{x\}$ or $\gamma_x(A) = A$.*

Proof. If $A = \{x\}$, then the proof is obvious. So consider the case $|A| \geq 2$. Suppose that $\gamma_x(A) \neq \{x\}$. If A is connected, then $\gamma_x(A) = A$ and the proof follows. If $A \notin C_X$, then $\gamma_x(A) = C$, where $|C| \geq 2$. Consequently $C \in C_X$. But C_X is an upper c-structure and $C \subseteq A$. So $A \in C_X$. Thus, $\gamma_x(A) = A$. □

Let C_X be a c-structure on X, $x \in X$ and $Y \subseteq X$. Then $\gamma_x|_Y$ denote the system of connectivity openings corresponding to the sub c-structure C_Y of C_X. If $A \subseteq Y$, then $\gamma_x|_Y(A)$ exists and $\gamma_x|_Y(A) = \gamma_x(A)$.

Let $(X, C_{\leq})$ be an order-induced c-space and $Y \subseteq X$. Then we get two sub c-structures of $C_{\leq}$. One is the sub c-structure $C_Y = \{A \in C_{\leq} : A \subseteq Y\}$ and the second one is the order-induced c-structure $C_{\leq_Y}$, where $C_{\leq_Y}$ is the order of Y inherited from the order $\leq$ on X [37]. Let $\gamma_x|_{Y_{\leq}}$ denote the system of connectivity openings corresponding to the order-induced c-space $(Y, C_{\leq_Y})$.

Remark 2.6.1. *Let X be a set and $Y \subseteq X$. In general, $A \subseteq Y$ need not imply $\gamma_x|_{Y_{\leq}}(A) = \gamma_x(A)$, where $\gamma_x(A)$ is the system of connectivity openings corresponding to the order-induced c-space $(X, C_{\leq})$. If $A \subseteq Y$ and Y is an interval in X, then $\gamma_x|_{Y_{\leq}}(A) = \gamma_x(A)$.*

Now, $\gamma_x|_{Y_{\leq}}(A)$ can be the same as $\gamma_x(A)$, even if Y is not an interval. For example, consider the set of all integers $\mathbb{Z}$ with the standard c-structure on $\mathbb{Z}$, that is the c-structure generated by intervals in $\mathbb{Z}$. Let $Y = \{1, 2, 4\}$. Then $C_Y = D_Y \cup \{\{1, 2\}\}$ and $C_{Y_\leq} = D_Y \cup \{\{1, 2\}, \{2, 4\}, \{1, 2, 4\}\}$. If $A = \{1, 2\}$, then for $x \in A$, $\gamma_x|_{Y_\leq}(A) = A$ and $\gamma_x(A) = A$.

That is, if $A \subseteq Y$ and A is an interval in X, then $A \in C_{Y_\leq}$. So $\gamma_x|_{Y_\leq}(A) = \gamma_x(A) = \gamma_x|_Y(A) = A$.

Theorem 2.6.7. *Let $(X, C_\leq)$ be a finite order-induced c-space and A be a proper non trivial subset of X. Then there exists some $x \notin A$ such that $\gamma_x(C \cup \{x\}) = C \cup \{x\}$, where $C \subseteq A$.*

Proof. Since $(X, C_\leq)$ is a finite order-induced c-space, it follows that $t(A) \neq A$, for any proper non trivial subset A of X. Thus, there exist some $x \in X$, $x \notin A$ and $C \subseteq A$ such that $C \cup \{x\} \in C_\leq$. So $\gamma_x(C \cup \{x\}) = C \cup \{x\}$. □

Chapter 3

Homogeneous c-spaces

3.1 Introduction

W. Sierpinsky introduced the notion of homogeneity in topological spaces. The properties of various forms of homogeneity in topological spaces and generalized topological spaces are discussed in [1, 2, 5]. In [28], P. K. Santhosh extended the notion of homogeneity to c-spaces. Here, we continue the study of homogeneous c-spaces. In addition, we present and investigate certain characteristics of n-homogeneity, strongly n-homogeneity, and local homogeneity in c-spaces. Furthermore, the relationship between various forms of homogeneity in c-spaces is investigated and examples are given.

3.2 Homogeneous c-spaces

Here, we study some properties of homogeneous c-spaces.

A c-space (X, C_X) is said to be homogeneous [28] if for any two points $x, y \in X$, there exists a c-automorphism $h : X \to X$ such that $h(x) = y$. The following are some examples of homogeneous c-spaces.

Example 3.2.1.

1. All discrete and indiscrete c-spaces.
2. The c-structure $C_{\mathbb{Z}} = D_{\mathbb{Z}} \cup \{C \subseteq \mathbb{Z} : C \cap 2\mathbb{Z} \text{ is a co-finite subset of } 2\mathbb{Z}\}$ on $\mathbb{Z}$.
3. $\mathbb{R}$ with the standard c-structure.

Note that a c-space is homogeneous if and only if its Brunnian closure is homogeneous.

Theorem 3.2.1. *Let (X, C_X) be a homogeneous c-space and $p \in X$. If $X \setminus \{p\}$ is connected, then $X \setminus \{x\}$ is connected for all $x \in X$.*

Proof. Choose $y \in X \setminus \{p\}$. Since (X, C_X) is a homogeneous c-space, we can find a c-automorphism $h : X \to X$ such that $h(p) = y$. As h maps connected sets to connected sets, $X \setminus \{y\}$ is connected. Since y is arbitrary, $X \setminus \{x\}$ is connected for all $x \in X$. □

Let (X, C_X) be a homogeneous c-space and $\gamma_x(X) = \{x\}$ for some $x \in X$. Then there does not exist a non-degenerate connected set containing x. Since X

is homogeneous, there does not exist a non-degenerate connected set containing any point of X. So for any $x \in X$, $\gamma_x(X) = \{x\}$. Consequently, $C_X = D_X$.

Definition 3.2.1. Let (X, C_X) be a c-space and $A \subseteq X$ be a non-degenerate connected set. Then A is said to be a minimal connected set if the only non-degenerate connected set contained in A is A itself. The collection of all minimal connected sets is denoted by $min(X, C_X)$.

Theorem 3.2.2. *Let $h : (X, C_X) \to (X, C_X)$ be a c-automorphism. If A is a minimal connected set, then $h(A)$ is also a minimal connected set.*

Proof. Let $A \in min(X, C_X)$. If $h(A)$ is not a minimal connected set, then we get a proper non-degenerate connected subset of $h(A)$, say B. Now, since h^{-1} is a c-continuous function, $h^{-1}(B)$ is also connected and is a proper non-degenerate subset of A. This is a contradiction. Thus, $h(A)$ is minimal. □

Theorem 3.2.3. *Let (X, C_X) be a homogeneous c-space and $p \in X$. If there exists a minimal connected set containing p, then there exists a minimal connected set containing x for all $x \in X$.*

Proof. Let $\{V_i : i \in I\}$ be the collection of all minimal connected sets containing p. Choose an arbitrary element $q \in X \setminus \{p\}$. Since (X, C_X) is homogeneous, there exists a c-automorphism h on (X, C_X) such that $h(p) = q$. Thus, $h(V_i)$ is a minimal connected set and $q \in h(V_i)$ for all $i \in I$. Suppose that H is a minimal connected set containing q and $H \neq h(V_i)$ for all $i \in I$. Now, consider $h^{-1}(H)$, which is a minimal connected set containing p. Thus, $h^{-1}(H) = V_i$ for some $i \in I$, which implies that $H = h(V_i)$ for some i. □

3.3 Finite Homogeneous c-spaces

Here, we discuss some properties of finite homogeneous c-spaces.

It can be observed from [28] that the sum of homogeneous c-spaces is not homogeneous. The following theorem provides a necessary and sufficient condition for the sum of disjoint collection of finite homogeneous c-spaces to be homogeneous.

Theorem 3.3.1. *Let $\{(X_i, C_{X_i}) : i \in I\}$ be a collection of disjoint finite homogeneous c-spaces, and (X, C_X) be their sum. Then (X, C_X) homogeneous if and only if there exists a c-isomorphism from each X_j to $X_{j'}$, where j, $j^{'} \in I$.*

Proof. Suppose that $\{(X_i, C_{X_i}) : i \in I\}$ is a collection of disjoint finite homogeneous c-spaces, and let (X, C_X) be their sum. Suppose that (X, C_X) is homogeneous. Then for each x, $y \in X$, there exists a c-automorphism $h : X \to X$ such that $h(x) = y$. Let $x \in X_j$ and $y \in X_{j'}$, where j, $j^{'} \in I$ and let C be a connected set containing x in X_j. So $C \in C_{X_j}$. Consequently, $h(C)$ is a connected set containing y in $X_{j'}$. That is, h maps connected sets in X_j to connected sets in $X_{j'}$. Similarly h^{-1} maps connected sets in $X_{j'}$ to connected sets in X_j. Thus, C_{X_j} and $C_{X_{j'}}$ are c-isomorphic.

Conversely, suppose that $\{(X_i, C_{X_i}) : i \in I\}$ is a collection of finite homogeneous c-spaces and X_j is c-isomorphic to $X_{j'}$, j, $j^{'} \in I$. To prove C_X is homogeneous. Let x, $y \in X$.

Case 1: If x, $y \in X_j$.

Since C_{X_j} is homogeneous, there exists a c-automorphism $f : X_j \to X_j$ such that $f(x) = y$. Define $h : X \to X$ as follows

$$h(a) = \begin{cases} f(a) & \text{if } a \in X_j, \\ a & \text{if } a \notin X_j. \end{cases}$$

Then h is a c-automorphism on X mapping x to y.

Case 2: If $x \in X_j$ and $y \in X_{j'}$.

Since C_{X_j} and $C_{X_{j'}}$ are c-isomorphic, there exists a c-isomorphism $h : X_j \to X_{j'}$.

Let C be a connected set containing x in X_j. Then $h(C) \in C_{X_{j'}}$.

If $h(x) = y$. Define $g : X \to X$ as

$$g(a) = \begin{cases} h(a) & \text{if } a \in X_j, \\ h^{-1}(a) & \text{if } a \in X_{j'}, \\ a & \text{otherwise.} \end{cases}$$

Then g is a c-automorphism on X mapping x to y.

If $h(x) \neq y$, then let $h(x) = y^{'}$, where $y^{'} \in X_{j'}$. Since $C_{X_{j'}}$ is homogeneous, there exists a c-automorphism $g : X_{j'} \to X_{j'}$ such that $g(y^{'}) = y$. Consider $g \circ h : X_j \to X_{j'}$. Then $g \circ h(x) = g(h(x)) = g(y^{'}) = y$. Define $k : X \to X$ as

$$k(a) = \begin{cases} g \circ h(a) & \text{if } a \in X_j, \\ (g \circ h)^{-1}(a) & \text{if } a \in X_{j'}, \\ a & \text{otherwise.} \end{cases}$$

Then k is a c-automorphism on X mapping x to y. Thus, (X, C_X) is homogeneous.

□

Definition 3.3.1. Let (X, C_X) be a c-space. A cover $\mathcal{U}$ of X is a collection of subsets $\{U_\alpha\}_{\alpha \in A}$ of X whose union is the whole space X and $\mathcal{U}$ is said to be a regular cover if each U_α, $\alpha \in A$ has the same cardinality.

The following theorem establishes that the collection of minimal connected sets forms a regular cover for a finite homogeneous c-space.

Theorem 3.3.2. *Let (X, C_X) be a finite homogeneous c-space. Then $min(X, C_X)$ is a regular cover of X.*

Proof. Let $A \in min(X, C_X)$ and $|A| = k$. Since the c-space (X, C_X) is homogeneous, for any two points x and y in X, there exists a c-automorphism $h : X \to X$ such that $h(x) = y$. Now, h maps minimal connected sets to minimal connected sets, $h(A)$ is also a minimal connected set, thus $h(A) \in min(X, C_X)$. So, for all $x \in X$ we can find $A \in min(X, C_X)$ such that $x \in A$. Thus, union of $min(X, C_X)$ is X. Since h is a bijection, $|A| = |h(A)| = k$. That is, $|A| = k$ for all $A \in min(X, C_X)$. Hence $min(X, C_X)$ forms a regular cover of X. □

Remark 3.3.1. *Let X be any set and $\mathcal{P}$ be a partition of X. Then $C_X = D_X \cup \mathcal{P}$ is a c-structure, which is called a partition c-structure. If $\mathcal{P}$ is a partition of X with $|A| = n$ for all $A \in \mathcal{P}$, where $1 \leq n \leq |X|$, then obviously the c-structure $C_X = D_X \cup \mathcal{P}$ is homogeneous. Note that the partition c-structure is not connected.*

In [28], P. K. Santhosh provided a characterization theorem for homogeneous c-spaces. That is, "A c-space (X, C_X) is homogeneous if and only if it can be written as a product of a discrete c-space and a connected homogeneous c-space." This theorem describes a homogeneous c-space in terms of connected homogeneous c-space, but provides no specific features of it. Thus, the characterization of homogeneous c-spaces is still an open problem.

Next, we construct a particular class of homogeneous c-structures on finite sets using the cyclic order in the underlying set.

Definition 3.3.2. Let $X = \{x_1, x_2, ..., x_n\}$. For an integer $1 \leq k \leq n$, define $C_i = \{x_i, x_{i\oplus 1}, .., x_{i\oplus k-1}\}$ for $i = 1, 2, .., n$ where $\oplus$ denotes the addition modulo n and let us denote the c-space generated by $C_1, C_2, .., C_n$ by (X, C_{X_k}).

The following theorem demonstrates that the above-defined c-space is homogeneous.

Theorem 3.3.3. *Let* $X = \{x_1, x_2, ..., x_n\}$. *Then* C_{X_k} *for* $1 \leq k \leq n$ *is a homogeneous c-structure on* X.

Proof. Let $X = \{x_1, x_2, ..., x_n\}$ and C_{X_k} be the c-structure generated by the sets $C_1, C_2, .., C_n$, where $|C_i| = k$ for each $i = 1, 2, ..., n$. Now, for each x_i, $x_j \in X$, we want to find a c-automorphism on (X, C_{X_k}) that maps x_i to x_j. Let $h = (x_1, x_2, ..., x_n)$ be a cycle on X. Obviously, h and all its powers are c-automorphisms on X.

If $i < j$, take $m = j - i$, then $h^m(x_i) = x_{i+m} = x_j$. If $i > j$, take $m = n - i$, then $h^{m+j}(x_i) = x_j$. Hence the proof. $\square$

Example 3.3.1. Let $X = \{a, b, c, d\}$ and $k = 2$. We have $C_1 = \{a, b\}$, $C_2 = \{b, c\}$, $C_3 = \{c, d\}$ and $C_4 = \{a, d\}$.
Here, $C_{X_2} = D_X \cup \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, X\}$ is a homogeneous c-structure on X.

Remark 3.3.2.

- *If X and Y are finite sets with same cardinality n, then for $1 \le k \le n$, (X, C_{X_k}) is c-isomorphic to (Y, C_{Y_k}).*
- *Let X be a finite set and consider the c-structure C_{X_k}. Then the number of minimal connected sets containing each $x \in X$ is k.*

Thus, by choosing alternative values of k, we may construct distinct homogeneous c-structures for any finite set X. In the following theorem, we attempt to construct a new homogeneous c-structure from the existing ones.

Theorem 3.3.4. *Let X and Y be two disjoint finite sets. For $1 \le k \le |X|$ and $1 \le l \le |Y|$, consider the c-structures C_{X_k} and C_{Y_l} on X and Y respectively. Then the union of C_{X_k} and C_{Y_l} is a homogeneous c-structure on $X \cup Y$ if and only if*

1. $|X| = |Y|$.
2. $k = l$.

Proof. Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$. Now, consider the collections $B = \{C_1, C_2, ..., C_m\}$, $B' = \{C'_1, C'_2, ..., C'_n\}$ where $|C_i| = k$ for $i = 1, 2, .., m$ and $|C'_j| = l$ for $j = 1, 2, .., n$. Let B and B' generate the c-structures

C_{X_k} and C_{Y_l} on X and Y respectively. Suppose that $C_{X_k} \cup C_{Y_l}$ is a homogeneous c-structure on $X \cup Y$. Then for any $x \in X$ and $y \in Y$, we can find a c-automorphism h on $X \cup Y$ such that $h(x) = y$. Now, $x \in X$ implies $x \in C_i$ for some $i \in \{1, 2, .., m\}$. Since c-automorphism maps a minimal connected set to a minimal connected set, $h(C_i)$ is a minimal connected set and $y \in h(C_i)$. Now, $h(C_i) \in B'$ implies $h(C_i) = C'_r$ for some $r \in \{1, 2, .., n\}$. Since h is bijection, $|C_i| = |C'_r|$. Thus, $k = l$. Also, $h(\bigcup_{i=1}^{m} C_i) = \bigcup_{j=1}^{n} C'_j$, which implies $h(X) = Y$. Since $X \cap Y = \emptyset$, we have $|X| = |Y|$.

Conversely, suppose that $n = m$ and $k = l$. Let $p, q \in X \cup Y$.

Case 1: $p, q \in X$. Let $p = x_i, q = x_j$ for $i, j \in \{1, 2, .., m\}$. Consider the cycle $f = (x_1, x_2, .., x_m) \in S(X \cup Y)$, where $S(X \cup Y)$ is the group of all permutations of $X \cup Y$. Obviously, f is a c-automorphism on $X \cup Y$. Then f^{j-i} and f^{m-i+j} are c-automorphism on $X \cup Y$ which maps x_i to x_j when $i < j$ and $i > j$ respectively.

Case 2: $p, q \in Y$. Similar to case 1.

Case 3: $p \in X$ and $q \in Y$. Let $p = x_i$, $q = y_j$, $i, j \in \{1, 2, .., m\}$. Define h as $h(x_{i \oplus u}) = y_{j \oplus u}$, where u is a natural number. Obviously, h is a c-automorphism on $X \cup Y$ which maps x_i to y_j. This completes the proof. □

A necessary condition for the union of two c-structures, C_{X_k} and C_{Y_l}, defined on two finite sets X and Y, respectively with $X \cap Y \neq \emptyset$, is homogeneous on $X \cup Y$ is proved in the following theorem.

Theorem 3.3.5. *Let X and Y be two finite sets with $X \cap Y \neq \emptyset$. For $1 \leq k \leq |X|$ and $1 \leq l \leq |Y|$, consider the c-structures C_{X_k} and C_{Y_l} on X and Y respectively.*

If the union of C_{X_k} *and* C_{Y_l} *is a homogeneous c-structure on* $X \cup Y$*, then* $X = Y$*.*

Proof. Let $B = \{C_1, C_2, ..., C_m\}$ and $B' = \{C'_1, C'_2, ..., C'_n\}$ where $|C_i| = k$, for $i = 1, 2, .., m$ and $|C'_j| = l$, for $j = 1, 2, .., n$. Suppose B and B' generate the c-structures C_{X_k} and C_{Y_l} respectively. Then the number of minimal connected sets containing each $x \in X$ is k and $y \in Y$ is l. Suppose that the union of C_{X_k} and C_{Y_l} is a homogeneous c-structure on $X \cup Y$ and $X \neq Y$. Since $X \neq Y$, without loss of generality, assume that there exists some $y \in Y$ such that $y \notin X$. Let $x \in X$ and C_i be a minimal connected set containing x. Since $C_{X_k} \cup C_{Y_l}$ is a homogeneous c-structure on $X \cup Y$, there exists a c-automorphism h on $X \cup Y$ such that $h(x) = y$. Let C'_j be a minimal connected set containing y and $h(C_i) = C'_j$. Consequently, $k = l$. Since $X \cap Y \neq \emptyset$, there exist some $z \in X \cap Y$. Since $y \in Y$ and $y \notin X$, the number of minimal connected sets containing y is strictly less than the number of minimal connected sets containing z. But $C_{X_k} \cup C_{Y_l}$ is a homogeneous c-structure, it follows that the minimal connected sets containing z in B and B' are the same. Thus, the minimal connected sets containing each element in $X \cap Y$ are the same in both B and B'. Now, $z \in X$ implies $z \in C_r$ for some $r \in \{1, 2, .., m\}$. But in B, there are k minimal connected sets containing z. Let $C_1, C_2, ..., C_k$ be the minimal connected sets containing z. Thus, $C_1, C_2, ..., C_k \in B'$ also. Thus, each of these minimal connected set is contained in $X \cap Y$. Now, C_{k+1} is also a minimal connected set for all elements in $C_k \cap C_{k+1} \subseteq C_k$. So $C_{k+1} \in B'$. Proceeding like this, we get $C_i \in B'$ for all $i \in \{1, 2, .., m\}$. Thus, $B \subseteq B'$. Similarly, we can prove that $B' \subseteq B$. So $B = B'$. Thus, $X = Y$. This completes the proof. □

Definition 3.3.3. Let (X, C_X) be a c-space. Define a relation $\sim$ on X as follows. For x, $y \in X$, $x \sim y$ if there exists a c-automorphism $h : X \to X$ such that $h(x) = y$. The set $\{y \in X : x \sim y\}$ is called the orbit of x and we denote this set by O_x.

Clearly, a c-space (X, C_X) is homogeneous if and only if it has exactly one orbit. Now, O_x need not be connected. See the following example.

Example 3.3.2. Let $X = \{a, b, c, d\}$ and $C_X = D_X \cup \{\{a, b\}, \{c, d\}\}$.
Here, the group of c-automorphism is given by

$$C(X, C_X) = \{I, (a\,b), (a\,b)(c\,d), (c\,d), (a\,c)(b\,d), (a\,c\,b\,d), (a\,d\,b\,c), (a\,d)(b\,c)\}.$$

Thus, for any $x \in X$, $O_x = X$ but $X \notin C_X$.

Theorem 3.3.6. *If h is a c-automorphism on X, then for any $x \in X$, $h(O_x) = O_x$.*

Proof. Let $y \in h(O_x)$. So, there exists a $x_1 \in O_x$ such that $h(x_1) = y$ which implies that $x_1 \sim y$. Since $x_1 \in O_x$, $x \sim x_1$. So $x \sim y$, implies $y \in O_x$. Thus, $h(O_x) \subseteq O_x$. Since h^{-1} is also a c-automorphism, $O_x \subseteq h(O_x)$. Hence the proof. $\square$

Note that a sub c-space of a homogeneous c-space need not be homogeneous. The following theorem explains how to find a homogeneous sub c-space from a given c-space.

Theorem 3.3.7. *Let (X, C_X) be a c-space. Then for any $x \in X$, $(O_x, C_X|_{O_x})$ is a homogeneous sub c-space of (X, C_X).*

Proof. Let $y,\ z \in O_x$. We want to find a c-automorphism $h : O_x \to O_x$ such that $h(y) = z$. Since $y,\ z \in O_x$, there exist c-automorphisms f and g on X such that $f(x) = y$ and $g(x) = z$. Since the composition of two c-automorphism is again a c-automorphism, $g \circ f^{-1}$ is a c-automorphism on X. By Theorem 3.3.6, $g \circ f^{-1}$ maps O_x on to itself. Now, $g \circ f^{-1}(y) = g(f^{-1}(y)) = g(x) = z$. Since the restriction of c-continuous function to a sub c-space remains c-continuous, $g \circ f^{-1}$ is a c-automorphism on O_x also. This completes the proof. □

Theorem 3.3.8. *Let h be a c-automorphism on X and $x_1,\ x_2 \in X$. If $x_2 \in O_{x_1}$, then $h(x_2) \in O_{h(x_1)}$.*

Proof. Since $x_2 \in O_{x_1}$, there exists a c-automorphism f on X such that $f(x_1) = x_2$. Let $g = h \circ f \circ h^{-1}$. Then g is a c-automorphism on X. Now,

$$\begin{aligned} g(h(x_1)) &= (h \circ f \circ h^{-1})(h(x_1)) \\ &= (hof)(h^{-1}(h(x_1)) \\ &= (hof)(x_1) \\ &= h(x_2)) \end{aligned}$$

Thus, $h(x_2) \in O_{h(x_1)}$. □

3.4 n-homogeneous c-spaces and Strongly n-homogeneous c-spaces

In this section, we introduce the notions of n-homogeneity and strongly n-homogeneity in c-spaces.
Let $A \subseteq X$ and $n \in \mathbb{N}$, the set of natural numbers. Then A is called an n-ton of X if $|A| = n$.

Definition 3.4.1. A c-space (X, C_X) is said to be

1. n-homogeneous if for any two n-tons A and B of X, there exists a c-automorphism h on X such that $h(A) = B$.
2. strongly n-homogeneous if for any two n-tuples of distinct points $A = (a_1, a_2, .., a_n)$ and $B = (b_1, b_2, .., b_n)$ of X, there exists a c-automorphism h on X such that $h(a_i) = b_i$, for all $i = 1, 2, .., n$.

Example 3.4.1. Let $X = \mathbb{N}$ and $X' = 2\mathbb{N}$. Define a c-structure C_X on X as follows.

$$C_X = D_X \cup \{C \subseteq X : C \cap X' \text{is a co-finite subset of } X'\}$$

Let $A = (a_1, a_2, .., a_n)$ and $B = (b_1, b_2, .., b_n)$ be two n-tuples of distinct points of X. Take $h = (a_1\ b_1)(a_2\ b_2)....(a_n\ b_n)$. Then h is a c-automorphism on X. So C_X is strongly n-homogeneous for all $n \in \mathbb{N}$.

Obviously, every strongly n-homogeneous c-space is n-homogeneous.

But the converse is not true. The following example illustrates this.

Example 3.4.2. Let $X = \mathbb{R}$ and $C_X = D_X \cup \{[x,\infty), (-\infty,x), (x,\infty) : x \in \mathbb{R}\}$. Obviously, C_X is a c-structure on $\mathbb{R}$.

First we claim that (X, C_X) is n-homogeneous for all $n \in \mathbb{N}$. Let $A = \{x_1, x_2, .., x_n\}$ and $B = \{y_1, y_2, .., y_n\}$ be two n-tons of distinct points of X. Without loss of generality, assume that $x_1 < x_2 < ... < x_n$ and $y_1 < y_2 < ... < y_n$. Let

$$A_0 = (-\infty, x_1),\ A_i = (x_i, x_{i+1}) \text{ for } i = 1, 2, ..., n-1 \text{ and } A_n = (x_n, \infty)$$

and

$$B_0 = (-\infty, y_1),\ B_i = (y_i, y_{i+1}) \text{ for } i = 1, 2, ..., n-1 \text{ and } B_n = (y_n, \infty).$$

Now, define $h : X \to X$ as

$$h(x) = \begin{cases} y_1 + (x - x_1) & \text{if } x \in A_0, \\ y_i + \dfrac{(x - x_i)(y_{i+1} - y_i)}{(x_{i+1} - x_i)} & \text{if } x \in A_i, i = 1, 2, .., n-1, \\ y_n + (x - x_n) & \text{if } x \in A_n, \\ y_i & \text{if } x = x_i, i = 1, 2, .., n. \end{cases}$$

Clearly, h is a c-automorphism on X and $h(x_i) = y_i$, for $i = 1, 2, .., n$. Hence (X, C_X) is n-homogeneous for all $n \in \mathbb{N}$.

Next we claim that (X, C_X) is not strongly 2-homogeneous. Suppose that (X, C_X) is strongly 2-homogeneous. Consider the ordered pairs $(1, 3)$ and $(3, 0)$ of X, we can find a c-automorphism h on X such that $h(1) = 3$ and $h(3) = 0$. Now, $[2, \infty) \in C_X$ implies $h^{-1}([2, \infty)) \in C_X$. Since h is a bijection, $h^{-1}([2, \infty)) \neq \mathbb{R}$. Thus, $h^{-1}([2, \infty))$ is either of the form $[x, \infty)$, $(-\infty, x)$ or (x, ∞) for some $x \in \mathbb{R}$.

Case(i): If $h^{-1}([2, \infty)) = [x, \infty)$.

Note that $1 \in h^{-1}([2, \infty))$, which implies $x \leq 1$. So $3 \in [x, \infty)$, it follows that $h(3) = 0 \in [2, \infty)$, which is a contradiction.

Case (ii): If $h^{-1}([2, \infty)) = (x, \infty)$.

Similar to Case i.

Case (iii): If $h^{-1}([2, \infty)) = (-\infty, x)$.

We have $(2, \infty) \in C_X$, which implies $h^{-1}((2, \infty)) \in C_X$ and $h^{-1}((2, \infty)) = (-\infty, h^{-1}(2)) \cup (h^{-1}(2), x) \notin C_X$. Hence (X, C_X) is not strongly 2-homogeneous. Thus, the c-space is n-homogeneous for all $n \in \mathbb{N}$. But it is not even strongly 2-homogeneous.

Theorem 3.4.1. *The associated c-space of an n-homogeneous topological space is n-homogeneous. Also, the associated c-space of a strongly n-homogeneous topological space is strongly n-homogeneous.*

Proof. As any homeomorphism maps connected sets to connected sets, it is a c-automorphism, the proof is straightforward. □

Theorem 3.4.2. *If a c-space (X, C_X) is a strongly n-homogeneous where $|X| \geq n$, then it is strongly m-homogeneous for $m < n$.*

Proof. Let $A = \{a_1, a_2, .., a_m\}$ and $B = \{b_1, b_2, .., b_m\}$ be two m-tons of X and $m < n$. Consider the subsets $\{a_{m+1}, a_{m+2}, .., a_n\}$ and $\{b_{m+1}, b_{m+2}, .., b_n\}$ of $X \setminus A$ and $X \setminus B$ respectively, where a_i's and b_i's are distinct. Now, let $G = (a_1, a_2, .., a_n)$ and $H = (b_1, b_2, .., b_n)$. Since (X, C_X) is strongly n-homogeneous, there exists $h \in C(X, C_X)$ such that $h(a_i) = b_i$, for all $i = 1, 2, ..., n$, which implies $h(a_i) = b_i$ for all $i = 1, 2, .., m$. □

Observe that, the converse of Theorem 3.4.2 is not true. For example, the standard c-structure on $\mathbb{R}$ is strongly 2-homogeneous but it is not strongly 3-homogeneous.

Theorem 3.4.3. *If (X, C_X) is a 2-homogeneous c-space, $|X| > 2$, then (X, C_X) is homogeneous.*

Proof. For any x, $y \in X$, we want to prove there exists a $h \in C(X, C_X)$ such that $h(x) = y$. Let $z \in X \setminus \{x, y\}$. Since (X, C_X) is 2-homogeneous, there exists a $h_1 \in C(X, C_X)$ such that $h_1(\{x, z\}) = \{y, z\}$. If $h_1(x) = y$, then the proof is clear. If $h_1(x) = z$, then $h_1(z) = y$ and hence $(h_1^2(x)) = h_1(z) = y$. Hence (X, C_X) is homogeneous. □

However, in general, a c-space is n-homogeneous need not imply that it is m-homogeneous for any m. See the following examples.

Example 3.4.3. Let $X = \{a, b, c, d\}$ and $C_X = D_X \cup \{\{a, b\}, \{c, d\}\}$. Here, (X, C_X) is 3-homogeneous, but is not 2-homogeneous.

Example 3.4.4. Consider $(\mathbb{Z}, C_{\mathbb{Z}})$, where $C_{\mathbb{Z}}$ is the standard c-structure on $\mathbb{Z}$. Let $a, b \in Z$, define $h : \mathbb{Z} \to \mathbb{Z}$ by $h(x) = x + (b - a)$. Then h is a c-automorphism on $\mathbb{Z}$ and $h(a) = b$. So $C_{\mathbb{Z}}$ is a homogeneous c-structure on $\mathbb{Z}$. Here, $\{1, 2\}$ is connected and $\{2, 4\}$ is not connected. So, there does not exist a c-automorphism mapping $\{1, 2\}$ to $\{2, 4\}$. This implies that $C_{\mathbb{Z}}$ is not a 2-homogeneous c-structure on $\mathbb{Z}$.

Theorem 3.4.4. *The arbitrary product of strongly n-homogeneous c-spaces is strongly n-homogeneous.*

Proof. Let $\{X_i : i \in I\}$ be an indexed family of strongly n-homogeneous c-spaces and $X = \prod_{i \in I} X_i$. Let $A = (((a_1)_i)_{i \in I}, ((a_2)_i)_{i \in I}, ..., ((a_n)_i)_{i \in I})$ and $B = (((b_1)_i)_{i \in I}, ((b_2)_i)_{i \in I}, ..., ((b_n)_i)_{i \in I})$ be two n-tuples of distinct points of X. This implies that $\{(a_1)_i, (a_2)_i, ..., (a_n)_i\}$ and $\{(b_1)_i, (b_2)_i, ..., (b_n)_i\}$ are two n-tuples of distinct points of X_i. Since X_i is strongly n-homogeneous, there exists some $h_i \in C(X_i, C_{X_i})$ such that $h_i((a_j)_i) = (b_j)_i$, $j = 1, 2, ..., , n$. Now, define $h : X \to X$ as follows,

$$h(((a_j)_i)_{i \in I}) = (h_i((a_j)_i))_{i \in I} = ((b_j)_i)_{i \in I}$$

Clearly, h is a c-automorphism on X. Hence X is strongly n-homogeneous. □

3.5 Locally Homogeneous c-spaces

Here, we study locally homogeneous c-spaces.

Definition 3.5.1. Let (X, C_X) be a c-space. Then (X, C_X) is called locally homogeneous at x provided, there exists a non-degenerate connected set C containing x such that for any $y \in C$ there exists some $h \in C(X, C_X)$ such that $h(x) = y$. A c-space (X, C_X) is called locally homogeneous if it is locally homogeneous at each of its points.

Example 3.5.1.

1. Every partition c-space is locally homogeneous.
2. Let $X = \mathbb{R}$ and A be a nonempty finite subset of $\mathbb{R}$. Consider the c-structure $C_X = D_X \cup \{A, \mathbb{R}\}$. Here, (X, C_X) is locally homogeneous at each point of A. But it is not locally homogeneous at each $y \in \mathbb{R} \setminus A$.

From the definition it follows that every homogeneous c-space is locally homogeneous.

But the converse is not true. See the following example.

Example 3.5.2. Let $X = \mathbb{N}$ and $C_X = D_X \cup \{\{1, 2, 3\}, \mathbb{N} \setminus \{1, 2, 3\}\}$. Here, C_X is locally homogeneous at all points of X, but it is not homogeneous.

Definition 3.5.2. A c-space (X, C_X) is said to be invertible if, for every non-degenerate connected set $C \subseteq X$, there exists a c-automorphism h on X such that $h(X \setminus C) \subseteq C$.

Example 3.5.3. The c-space given in Example 3.3.2 is invertible.

Theorem 3.5.1. *If $C_X \neq D_X$ is an invertible c-structure on X, then for all $x \in X$ there exists at least one non-degenerate connected set C containing x.*

Proof. Let $x \in X$. Since $C_X \neq D_X$, there exists at least one non-degenerate connected subset of X, say C. Let $x \in C$. Then there is nothing to prove. Otherwise, $x \in (X \setminus C)$. Since the c-space (X, C_X) is invertible, there exists some $h \in C(X, C_X)$ such that $h(X \setminus C) \subseteq C$. This implies that $h(x) \in C$ and hence $x \in h^{-1}(C)$. □

Theorem 3.5.2. *Every invertible locally homogeneous c-space is homogeneous.*

Proof. For any x, $y \in X$, we want to prove that there exists some $h \in C(X, C_X)$ such that $h(x) = y$. Since (X, C_X) is invertible, by Theorem 3.5.1, for each $x \in X$, there exists at least one non-degenerate connected set containing x. Since C_X is locally homogeneous, there exists a non-degenerate connected set C containing x such that, for any $y \in C$, there exists some $h \in C(X, C_X)$ such that $h(x) = y$. If $y \notin C$, since C_X is invertible, there exists $h_1 \in C(X, C_X)$ such that $h_1(X \setminus C) \subseteq C$, which implies $h_1(y) \in C$. Thus, we can find a c-automorphism $g \in C(X, C_X)$ such that $g(x) = h_1(y)$. Clearly, $h_1^{-1} \circ g$ is a c-automorphism on X, also $h_1^{-1} \circ g(x) = y$. Hence the proof. □

Remark 3.5.1.

- *An invertible c-space need not be connected. See Example 3.3.2.*
- *A homogeneous c-space need not be invertible. See the following example.*

Example 3.5.4. Let $X = \{a, b, c, d, e\}$ and C_X be the c-structure generated by the set $\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{a, e\}\}$. Thus, (X, C_X) is homogeneous. Let $C = \{a, b\}$. Then $X \setminus C = \{c, d, e\}$. So there does not exist an $h \in C(X, C_X)$ such that $h(X \setminus C) \subseteq C$.

Remark 3.5.2. *Since every homogeneous c-space is locally homogeneous, it is obvious that the sum of locally homogeneous c-spaces is not locally homogeneous. Similarly, the quotient space of a locally homogeneous c-space is not locally homogeneous.*

Next, we prove that the arbitrary product of locally homogeneous c-spaces is locally homogeneous.

Theorem 3.5.3. *Let $\{X_i : i \in I\}$ be an indexed family of locally homogeneous c-spaces. Then, $\prod\limits_{i \in I} X_i$ is also locally homogeneous.*

Proof. Let $X = \prod\limits_{i \in I} X_i$. Let $x = (x_i)_{i \in I}$, $y = (y_i)_{i \in I} \in X$, where x_i, $y_i \in X_i$ for each $i \in I$. Since each X_i is locally homogeneous, there exists a non-degenerate connected set $C_i \subseteq X_i$ containing x_i and for all $y_i \in C_i$, there exists a c-automorphism $h_i : X_i \to X_i$ such that $h_i(x_i) = y_i$. Consider $\prod\limits_{i \in I} C_i$, which is connected in X. Also note that $x \in \prod\limits_{i \in I} C_i$.
Now, define $h : X \to X$ as follows,

$$h(x) = h((x_i)_{i \in I}) = (h_i(x_i))_{i \in I} = (y_i)_{i \in I} = y$$

and $y \in \prod\limits_{i \in I} C_i$. So X is locally homogeneous. □

Definition 3.5.3. Let X be a set. A topology τ on X is called a partition topology if a partition of X forms a basis for τ.

Recall that a c-space (X, C_X) is said to be topologizable [25] if there exists a topology τ on X such that the associated c-space of (X, τ) is (X, C_X).

We know that every continuous function from a topological space to another topological space maps connected sets to connected sets. Consequently, every homeomorphism is a c-automorphism. If $H(X, \tau)$ denotes the collection of all homeomorphisms of (X, τ), then we have $H(X, \tau) \subseteq C(X, C_X)$. But the converse is not true. The following theorem gives a sufficient condition for $C(X, C_X) = H(X, \tau)$.

Theorem 3.5.4. *Let (X, τ) be a partition topological space and (X, C_X) be the associated c-space of (X, τ). Then $C(X, C_X) = H(X, \tau)$.*

Proof. Let $\{P_i : i \in I\}$ be a partition X, which forms the basis for (X, τ). Let $\mathbf{B} = \{P_i : i \in I\}$. Then each P_i is connected in its subspace topology. Otherwise, there exists U, $V \in \tau$ such that $U \cap P_i$ and $V \cap P_i$ form a separation for P_i. Consequently, $U \cap P_i$ and $V \cap P_i$ are nonempty, disjoint, open subsets of P_i in the subspace topology on P_i. Since $U \cap P_i \neq \emptyset$, let $x \in U \cap P_i$. So $x \in U$. Then there exists $P_i \in \mathbf{B}$ such that $x \in P_i$ and $P_i \subseteq U$. Note that $P_i \cap P_j = \emptyset$ for all $i \neq j$. So $x \notin P_j$ for all $i \neq j$. Hence $V \cap P_i = \emptyset$, which is a contradiction. So P_i is connected. Similarly, we can prove that every subsets of P_i is also connected.

Let h be a c-automorphism on (X, C_X). We claim that $h(P_i) = P_j$. Suppose not. Let $h(P_i) = A$, where $A \subsetneq P_j$. Now,

$$\begin{aligned} & h(P_i) = A \\ \implies & P_i = h^{-1}(A) \subseteq h^{-1}(P_j) \\ \implies & P_i \subseteq h^{-1}(P_j) \end{aligned}$$

which is not possible, since there does not exist a superset of P_i in C_X. So h maps each P_i to some P_j.

Let $U \in \tau$. Then U can be written as $U = \bigcup_{j \in J} P_j$, where $J \subseteq I$. Now,

$$h(U) = h(\bigcup_{j \in J} P_j) = \bigcup_{j \in J} h(P_j) \in \tau$$

So h^{-1} is continuous. Similarly, we can prove that h is continuous. Thus, h is a homeomorphism. So $C(X, C_X) \subseteq H(X, \tau)$. Hence the proof. □

Remark 3.5.3. *The converse of the above theorem does not hold in general. For example, let* $X = \{1, 2, 3\}$*. Consider the topology,* $\tau = \{\emptyset, X, \{2\}, \{3\}, \{2, 3\}\}$ *on* X*. The associated c-structure of* τ *is given by* $C_X = D_X \cup \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$*. Here,* $C((X, C_X)) = H((X, \tau))$*. But* τ *is not a partition topology.*

Chapter 4

Hereditarily Homogeneous c-spaces

4.1 Introduction

Numerous researchers investigated homogeneity, complete homogeneity, and hereditary homogeneity in different structures, including topology, generalized topology, and Čech closure operators [17, 18, 34]. P. Sini studied completely homogeneous c-spaces [35]. Furthermore, she investigated the relationship between hereditarily homogeneous c-spaces and completely homogeneous c-spaces.

In this chapter, various properties of hereditarily homogeneous c-spaces are discussed. We provide a characterization for hereditarily homogeneous c-spaces in terms of c-automorphisms. Further, we prove that hereditary homogeneity and complete homogeneity are equivalent concepts for connective spaces and finite

c-spaces. A part of this chapter was published in Research in Mathematics [9].

4.2 Hereditarily Homogeneous c-spaces

First, we define a hereditarily homogeneous c-space.

Definition 4.2.1. A c-space (X, C_X) is said to be hereditarily homogeneous [35] if every sub c-space of (X, C_X) is homogeneous.

Here, we list some examples and non- examples of hereditarily homogeneous c-spaces.

1. Let $X = \{a, b, c, d\}$. Then
   $C_X = D_X \cup \{\{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}, X\}$ is a hereditarily homogeneous c-structure on X.

2. Let $X = \mathbb{R}$. Then by using Theorem 4.3.4, we can prove that $C_X = D_X \cup \{A \subseteq \mathbb{R} : (X \setminus A) \cap \mathbb{Q}$ is finite$\}$ is a hereditarily homogeneous c-structure on X.

3. By Theorem 4.3.4, the c-structure $C_{\mathbb{Z}} = D_{\mathbb{Z}} \cup \{C \subseteq \mathbb{Z} : C \cap 2\mathbb{Z}$ is a co-finite subset of $2\mathbb{Z}\}$ on $\mathbb{Z}$ is hereditarily homogeneous c-structure on $\mathbb{Z}$.

4. $\mathbb{R}$ with the standard c-structure is not hereditarily homogeneous.

5. Consider the set of all integers $\mathbb{Z}$ with the standard c-structure on $\mathbb{Z}$. Then clearly $(\mathbb{Z}, C_{\leq})$ is a homogeneous c-space. Let $A = \{1, 2, 3\}$, consider the

sub c-space $(A, C_{\leq_A})$ of $\mathbb{Z}$, where $C_{\leq_A}$ is the restriction of $C_{\leq}$ into A. Thus, $C_{\leq_A} = D_A \cup \{\{1,2\},\{2,3\},\{1,2,3\}\}$. It is clear that $C_{\leq_A}$ is not homogeneous. That is, $(\mathbb{Z}, C_{\leq})$ is not hereditarily homogeneous.

4.3 Properties of Hereditarily Homogeneous c-spaces

First, we prove that a c-space that is hereditarily homogeneous is either C_1 or indiscrete.

Theorem 4.3.1. *Let (X, C_X) be a hereditarily homogeneous c-space. Then $C_X = P(X)$ or C_X is C_1.*

Proof. If $|X| < 4$, then the only homogeneous c-structures on X are $P(X)$ and D_X. So in this case, there is nothing to prove.

Now, consider the case $|X| \geq 4$. Assume that (X, C_X) is not C_1. Then there exists $x \in X$ such that $\overline{\{x\}} \neq \{x\}$. Therefore, there exists $y \in \overline{\{x\}}$ such that $y \neq x$ and $\{x, y\}$ is a connected set. Choose $z \in X \setminus \{x, y\}$. Now, consider the sub c-structure C_Y on $Y = \{x, y, z\}$, which is a homogeneous c-structure on Y. Since $\{x, y\} \in C_X$, $\{x, y\} \in C_Y$. Since C_Y is homogeneous, $\{x, z\}$ and $\{y, z\}$ are connected in Y. Let $w \in X \setminus \{x, y, z\}$. Again, consider the sub c-structure C_Z on $Z = \{x, y, w\}$, which is also homogeneous. Thus, we get $\{x, y\}$, $\{x, w\}$ and $\{y, w\}$ are connected in Z and hence in X. Now, let $W = \{x, z, w\}$. Then $\{x, z\}$ and $\{x, w\}$ are connected in W also. Since W is homogeneous, $\{z, w\}$ is connected in W. Thus, $\{z, w\}$ is connected in X. It follows that any two element set is connected in X. So (X, C_X) is an indiscrete c-space. This completes the proof. □

Corollary 4.3.1. *Let (X, C_X) be a finite hereditarily homogeneous c-space and $C_X \neq P(X)$. Then C_X is neither topological nor graphical.*

Proof. Suppose that (X, C_X) is a finite hereditarily homogeneous c-space and $C_X \neq P(X)$. Then by Theorem 4.3.1, C_X is C_1. So every two element subset of X is disconnected. Since every finite topological c-space and every finite graphical c-space is 2-generated, it follows that C_X is neither topological nor graphical. □

Theorem 4.3.2. *Let (X, C_X) be a hereditarily homogeneous c-space, $A \subseteq X$, $|A| \geq 2$ and $x \in A$. If $\gamma_x(A) = \{x\}$, then C_X is C_1.*

Proof. If $C_X = D_X$, then the proof follows. Suppose that $C_X \neq D_X$ and $\gamma_x(A) = \{x\}$. Consequently $A \notin C_X$. By Theorem 4.3.1, every hereditarily homogeneous c-space is either indiscrete or C_1. Since $A \notin C_X$, $C_X \neq P(X)$. Thus, C_X is C_1. Hence the proof. □

Theorem 4.3.3. *Let (X, C_X) be a hereditarily homogeneous c-space. Let x, y be two distinct elements of X and A be a nonempty connected subset of X which contains neither x nor y. Then $x \in t(A)$ if and only if $y \in t(A)$.*

Proof. By Theorem 4.3.1, (X, C_X) is either indiscrete or C_1. If (X, C_X) is indiscrete, then there is nothing to prove. So suppose that (X, C_X) is C_1 and x is a touching point of A.

Let $Y = A \cup \{x, y\}$. Then (Y, C_Y) is a homogeneous c-space. So, there exists a c-automorphism $h : Y \to Y$ such that $h(x) = y$. Now,

$$h(A) = \begin{cases} A & \text{if } h(y) = x, \\ (A \setminus \{h(y)\}) \cup \{x\} & \text{if } h(y) \neq x. \end{cases}$$

Case 1 : $h(A) = A$

Since x is a touching point of A, there exists a nonempty subset $C \subsetneq A$ such that $\{x\} \cup C$ is connected. Then $h(\{x\} \cup C)$ is connected. That is, $\{y\} \cup h(C)$ is connected. Since $h(A) = A$, $h(C) \subsetneq A$. Therefore, y is a touching point of A.

Case 2: $h(A) = (A \setminus \{h(y)\}) \cup \{x\}$

Now, x is a touching point of A implies there exists a nonempty subset $C \subsetneq A$ such that $\{x\} \cup C$ is connected, so it follows that $\{x\} \cup A$ is connected. Suppose $y \notin t(A)$. Then for any nonempty subset $C \subsetneq A$, $C \cup \{y\}$ is not connected, which implies $A \cup \{y\}$ is not connected. It follows that $h(A \cup \{y\})$ is not connected. Now,

$$\begin{aligned} h(A \cup \{y\}) &= h(A) \cup h(\{y\}) \\ &= ((A \setminus \{h(y)\}) \cup \{x\}) \cup \{h(y)\} \\ &= A \cup \{x\}. \end{aligned}$$

That is, $A \cup \{x\}$ is not connected, which is a contradiction. So $y \in t(A)$. Hence the proof. □

Remark 4.3.1. *In a hereditarily homogeneous c-space, a c-dense subset need not be connected. The following example illustrates this.*

Example 4.3.1. Consider the c-space (X, C_X) where $X = \{a, b, c, d, e\}$ and $C_X = D_X \cup \{A \subseteq X : |A| \geq 4\}$. It is easy to show that (X, C_X) is hereditarily homogeneous. Now, let $B = \{a, b, c\} \subseteq X$. Then $t(B) = X$ but $B \notin C_X$.

4.3.1 A Characterization Theorem for Hereditarily Homogeneous c-spaces

In the next theorem, we characterize hereditarily homogeneous c-spaces.

Theorem 4.3.4. *Let (X, C_X) be a c-space. Then (X, C_X) is hereditarily homogeneous if and only if every transposition of X is a c-automorphism on X.*

Proof. Let (X, C_X) be a hereditarily homogeneous c-space. Let x and y be two distinct elements of X and h be the transposition $(x\ y)$ mapping x to y, y to x, and keeping all other elements fixed. To prove h is a c-automorphism of (X, C_X) on to itself, it is enough to prove that C is connected in X if and only if $h(C)$ is connected for every $C \in C_X$.

Let $C \in C_X$.

Case 1: $x \notin C$, $y \notin C$

Then $h(C) = C \in C_X$

Case 2: $x \in C$, $y \in C$

Then also $h(C) = C \in C_X$

Case 3: $x \in C$, $y \notin C$

In this case

$$h(C) = (C \setminus \{x\}) \cup \{y\}.$$

Let $Y = C \cup \{y\}$. Now, consider the sub c-space (Y, C_Y). Since (X, C_X) is a hereditarily homogeneous c-space, (Y, C_Y) is a homogeneous c-space. Since $C = Y \setminus \{y\}$ is connected in X, C is connected in Y also. Then by Theorem 3.2.1, we have $h(C) = Y \setminus \{x\}$ is connected in Y and so in X. That is, $h(C) = (C \setminus \{x\}) \cup \{y\}$ is connected in X.

Case 4: $x \notin C$, $y \in C$.

Similar to case 3. Thus, we have if C is connected, then $h(C)$ is connected.

Now, let $h(C)$ be connected. Then $h(h(C))$ is connected. But $h(h(C)) = C$, which implies that C is a connected subset of X. So $h = (x\ y)$ is a c-automorphism on X. Thus, every transposition of X is a c-automorphism of X onto itself.

Conversely, assume that every transposition of X is a c-automorphism on X. It follows that (X, C_X) is a homogeneous c-space. Let (Y, C_Y) be a subspace of X and $x, y \in Y$. Then by our assumption $h = (x\ y)$ is a c-automorphism on X. Let $A \in C_Y$. Then $A \in C_X$ and also $A \subseteq Y$. Since h is a c-automorphism on X, $h(A)$ is connected in X. Clearly, $h(A) \subseteq Y$ and so $h(A) \in C_Y$. Thus, $h = (x\ y)$ is a c-automorphism on Y. So (Y, C_Y) is a homogeneous c-space. It follows that (X, C_X) is a hereditarily homogeneous c-space. This completes the proof. □

Corollary 4.3.2. *Let (X, C_X) be a hereditarily homogeneous c-space and A be* *a non-degenerate connected subset of* X*. Then the* t*-closure of* A*,* $\overline{A} = X$*.*

Proof. Let $x \in A$ and $y \in X \setminus A$. By Theorem 4.3.4, the transposition $(x\ y)$ is a c-automorphism on X. Let $h = (x\ y)$. Then $h(A) = (A \setminus \{x\} \cup \{y\}) \in C_X$. Also, $A \cap h(A) \neq \emptyset$. It follows that $A \cup h(A) = A \cup \{y\} \in C_X$. Since y is arbitrary, $A \cup \{y\} \in C_X$ for all $y \in X \setminus A$. Thus, $\overline{A} = X$. □

Remark 4.3.2. *In a hereditarily homogeneous c-space, any two non-degenerate connected sets touch. But any two non-degenerate subsets of a hereditarily homogeneous c-space touch need not imply that they are connected.*

Example 4.3.2. Let X be any set with $|X| \geq 3$. Consider the c-structure $C_X = D_X \cup \{A \subseteq X : |A| \geq 3\}$. For A, $B \in C_X \setminus D_X$, we have $t(A) = t(B) = X$. Let $C, D \subseteq X$ with $|C| = |D| = 2$. Then $t(C) = t(D) = X$. So C and D touch, but $C, D \notin C_X$.

Definition 4.3.1. [8] A permutation on an infinite set X is finitary if it moves only finitely many points.

It is very easy to show that the set of all finitary permutations of X forms a subgroup of the symmetric group of X.

We can characterize hereditarily homogeneous c-spaces in terms of finitary permutations in the following way.

Corollary 4.3.3. *A c-space* (X, C_X) *is hereditarily homogeneous if and only if the group of all finitary permutations on* X *is a subgroup of the group of c-automorphisms on* X*.*

Proof. Since (X, C_X) is hereditarily homogeneous, by Theorem 4.3.4, every transposition on X is a c-automorphism on X. Let h be a finitary permutation on X, so it moves only finitely many points. Thus, h can be written as the finite product of transpositions. It follows that h is a c-automorphism on X. Conversely, if the group of all finitary permutations is a subgroup of the group of c-automorphisms, then every transposition is a c-automorphism. Thus, (X, C_X) is a hereditarily homogeneous c-space. □

Corollary 4.3.4. *Let (X, C_X) be a hereditarily homogeneous c-space. Then every subset of X is either t-closed or c-dense in X.*

Proof. Let $A \subseteq X$. It is enough to prove that either $t(A) = A$ or $t(A) = X$. Suppose that $t(A) \neq A$. Then there exists $x \in X \setminus A$ such that x is a touching point of A. Let $y \in X \setminus A$ and $x \neq y$. Consider the transposition $(x\ y)$ on X. By Theorem 4.3.4, $(x\ y)$ is a c-automorphism on X. Let $Y = A \cup \{x, y\}$. Then (Y, C_Y) is a homogeneous sub c-space of X. Then the transposition $h = (x\ y)$ is a c-automorphism on Y. Since x is a touching point of A, $h(x) = y$ is a touching point of $h(A)$. But $h(A) = A$ and so y is a touching point of A. Since y is arbitrary, it follows that $t(A) = X$. This completes the proof. □

Theorem 4.3.5. *[17] Let (X, τ) be a hereditarily homogeneous topological space. Then every transposition of X is a homeomorphism of (X, τ) onto itself.*

Proposition 4.3.1. *The associated c-space of a hereditarily homogeneous topological space is hereditarily homogeneous.*

Proof. Let (X, τ) be a hereditarily homogeneous topological space and (X, C_X)

be the associated c-space of (X, τ). Then by Theorem 4.3.5, every transposition of X is a homeomorphism. Since a continuous function on a topological space maps connected sets to connected sets, every homeomorphism of (X, τ) is a c-automorphism on (X, C_X). Thus, by Theorem 4.3.4, (X, C_X) is a hereditarily homogeneous c-space. □

But the converse of Proposition 4.3.1 is not true. In [16], V. Kannan and M. Rajagopalan proved that there exists a totally disconnected topological space (X, τ) such that the only homeomorphism on (X, τ) is the identity permutation. So (X, τ) is not even homogeneous. Here, only the singleton sets are connected and it follows that the associated c-space of (X, τ) is discrete which is hereditarily homogeneous.

Theorem 4.3.6. *Every hereditarily homogeneous c-space is either connected or discrete.*

Proof. Let (X, C_X) be a hereditarily homogeneous c-space. Suppose that $C_X \neq D_X$, then there exists an $A \in C_X$ such that $|A| > 1$. If $A = X$, then there is nothing to prove. For $A \neq X$, choose $x \in X$ such that $x \notin A$. Then by Corollary 4.3.2, x is a touching point of A. So, there exists a nonempty subset C of A such that $C \cup \{x\}$ is connected. Since A is connected, $A \cup C \cup \{x\}$ is connected, which implies that $A \cup \{x\}$ is connected. So for each $x \in (X \setminus A)$, $A \cup \{x\}$ is connected. This implies that $X = \cup_{x \in (X \setminus A)}(A \cup \{x\})$ is connected. This completes the proof. □

The following is an immediate consequence of the preceding theorem.

Corollary 4.3.5. *Let $C_X \neq D_X$ be a hereditarily homogeneous c-structure on X. Then C_X is an upper c-structure.*

Proof. Let A be a non-degenerate connected subset of X and $A \subseteq B$. Then $A \cup \{x\}$ is connected for all $x \in B \setminus A$. Thus, B can be written as $B = \cup_{x \in (B \setminus A)}(A \cup \{x\})$. So B is connected. □

Theorem 4.3.7. *Let $C_X \neq D_X$ be a hereditarily homogeneous c-structure on X and $A \subseteq X$. If A is connected and $x \in A$, then $\gamma_x(A_1) = A_1$ for any $A_1 \supseteq A$.*

Proof. Since C_X is a hereditarily homogeneous c-structure, it is an upper c-structure. Now, $A \in C_X$ implies $A_1 \in C_X$ for all $A_1 \supseteq A$. Thus, $\gamma_x(A_1) = A_1$. □

Corollary 4.3.6. *If $C_X \neq D_X$ is a hereditarily homogeneous c-structure on X, then $\gamma_x(X) = X$, for all $x \in X$.*

Proof. Since every hereditarily homogeneous c-space is an upper c-space, the proof is obvious. □

If (X, C_X) is a homogeneous c-space, then it is not always true that $\gamma_x(X) = X$. For example consider the c-structure $C_X = D_X \cup \mathcal{P}$, where $\mathcal{P}$ is a partition of X with $|A| = n$ for all $A \in \mathcal{P}$ and $1 < n < |X|$. Then (X, C_X) is homogeneous. But here $\gamma_x(X) = \{A\}$ for $x \in A$.

Every completely homogeneous c-space is hereditarily homogeneous. But the converse is not true [35]. But in the finite case, using Theorem 4.3.4, we can easily prove these two notions are equivalent.

Theorem 4.3.8. *Let (X, C_X) be a finite c-space. Then (X, C_X) is completely homogeneous if and only if it is hereditarily homogeneous.*

Proof. We have every completely homogeneous c-space is hereditarily homogeneous. To prove the converse, let (X, C_X) be a hereditarily homogeneous c-space. Then by Theorem 4.3.4, every transposition of X is a c-automorphism on X. Since X is finite, every permutation of X can be written as a composition of transpositions. Thus, every permutation is a c-automorphism on X. This completes the proof. □

From [28] it can be observed that the quotient space of a homogeneous c-space need not be homogeneous and also the sum of two homogeneous c-spaces need not be homogeneous. Hence, the quotient space and the sum of hereditarily homogeneous c-spaces need not be hereditarily homogeneous. However, an arbitrary product of the homogeneous c-space is homogeneous.

In the next theorem, we prove that arbitrary product of hereditarily homogeneous c-space is hereditarily homogeneous.

Theorem 4.3.9. *The arbitrary product of hereditarily homogeneous c-spaces is hereditarily homogeneous.*

Proof. Let $\{X_i : i \in I\}$ be an indexed family of hereditarily homogeneous c-spaces and $X = \prod_{i \in I} X_i$. Let $x, y \in X$. Then $h = (x\ y)$ is a transposition of X. Let $C \in C_X$. Then $C = \prod_{i \in I} C_i$, where C_i is connected in X_i. Since $\pi_i(x)$, $\pi_i(y) \in X_i$ and X_i is hereditarily homogeneous, $(\pi_i(x)\ \pi_i(y))$ is a c-automorphism on X_i. Let $h_i = (\pi_i(x)\ \pi_i(y))$. Then h_i maps each $C_i \in X_i$ to some $B_i \in X_i$, where B_i is connected in X_i. Now,

$$\begin{aligned} h(C) &= h(\prod_{i\in I} C_i) \\ &= \prod_{i\in I}(h_i(C_i)) \\ &= \prod_{i\in I} B_i \in C_X \end{aligned}$$

That is, h is a c-automorphism on X. Since h is arbitrary, every transposition of X is a c-automorphism on X. Hence, the proof. □

Now, we discuss the relation between the hereditarily homogeneous c-space and the bihomogeneous c-space.

Definition 4.3.2. [28] A c-space (X, C_X) is said to be bihomogeneous if for any two points $x, y \in X$ there exists a c-automorphism $h : X \to X$ such that $h(x) = y$ and $h(y) = x$.

Obviously, every hereditarily homogeneous c-space is bihomogeneous. But a bihomogeneous c-space need not be hereditarily homogeneous. See the following example.

Example 4.3.3. Let $X = \{a, b, c, d\}$ and $C_X = D_X \cup \{\{a, b\}, \{c, d\}\}$. Then

$$C(X, C_X) = \{I, (a\,b), (a\,b)(c\,d), (c\,d), (a\,c)(b\,d), (a\,c\,b\,d), (a\,d\,b\,c), (a\,d)(b\,c)\}.$$

It is clear that (X, C_X) is bihomogeneous. Consider the sub c-space (Y, C_Y) where $Y = \{a, b, c\}$ and $C_Y = D_Y \cup \{\{a, b\}\}$. This space is not homogeneous.

4.4 Hereditarily Homogeneous Connective Spaces

Finally, we study hereditary homogeneity in connective spaces.

We need the following Theorem in [35].

Theorem 4.4.1. *[35] Let (X, C_X) be an upper c-space. Then C_X is connective if and only if for every connected set A and distinct points $x, y \in A$, $A \setminus \{x\}$ or $A \setminus \{y\}$ is connected.*

Using Theorem 4.4.1, we characterize hereditarily homogeneous connective structures on a non-empty set X.

Proposition 4.4.1. *Let X be a set and C_X be a hereditarily homogeneous c-structure on X. Then C_X is connective if and only if for any $A \in C_X$, $A \setminus \{z\} \in C_X$ for all $z \in X$.*

Proof. Let C_X be a hereditarily homogeneous c-structure on X. Then by Corollary 4.3.5, we have C_X is an upper c-structure on X. Let x, $y \in X$. By Theorem 4.4.1, C_X is connective if and only if for every connected set A and distinct points $x, y \in A$, $A \setminus \{x\}$ or $A \setminus \{y\}$ is connected. In addition, we see that the transposition $(x\ y)$ is a c-automorphism on X. Thus, if $A \setminus \{x\}$ is connected, then $A \setminus \{y\}$ is also connected and conversely. That is, for any x and y, $x \neq y$

in X, both $A \setminus \{x\}$ and $A \setminus \{y\}$ are connected. If $|A| \leq 1$, then $A \setminus \{z\} \in C_X$ for all $z \in X$. Thus, C_X is connective if and only if for any $A \in C_X$, $A \setminus \{z\} \in C_X$ for all $z \in X$. □

Proposition 4.4.2. *[35] Let X be a set and C_X be a completely homogeneous c-structure on X. Then C_X is connective if and only if for any $C \in C_X$, $C \setminus \{x\} \in C_X$ for all $x \in X$.*

Using Propositions 4.4.1 and 4.4.2, we have complete homogeneity and hereditary homogeneity are equivalent in connective spaces.

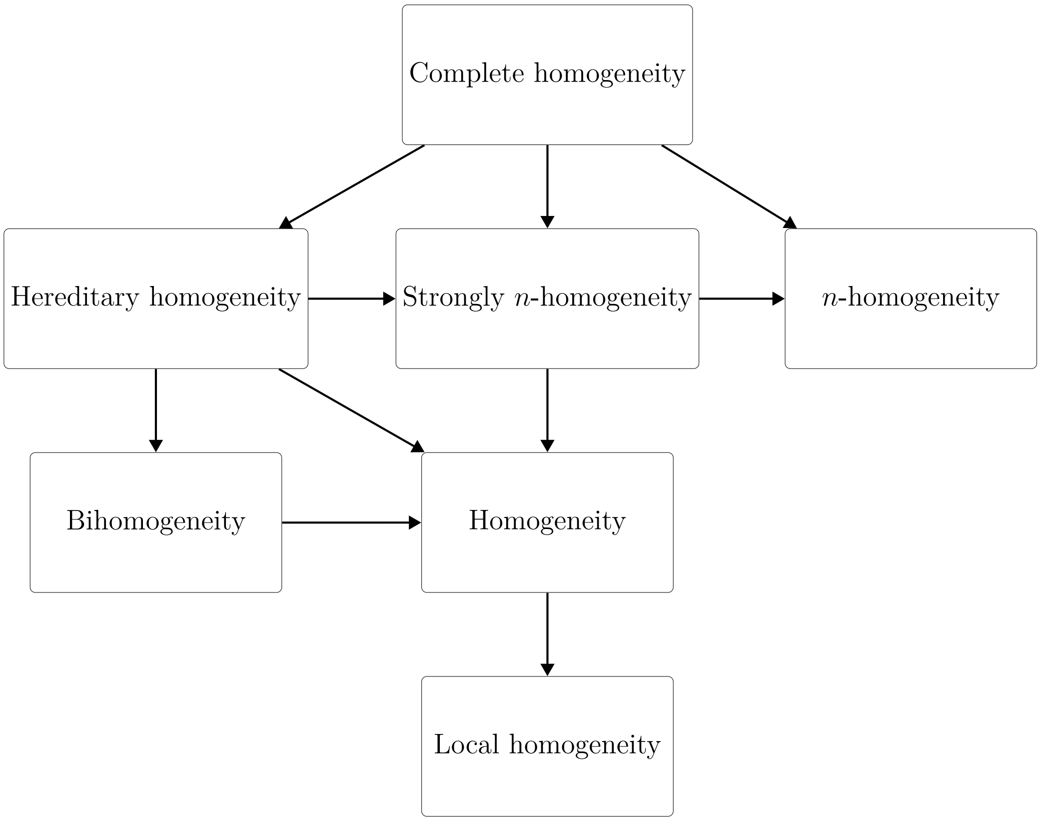
Corollary 4.4.1. *Let X be a set and C_X be a connective structure on X. Then (X, C_X) is hereditarily homogeneous if and only if (X, C_X) is completely homogeneous.*

Proof. The proof is obvious. □

In [35] (Theorem 2.26), the completely homogeneous connective structures are listed. So the only hereditarily homogeneous connective structures on a nonempty set X are the following.

1. D_X.
2. $D_X \cup \{A \subseteq X : |A| \geq n\}$ where $n = 2$ or is an infinite cardinal, $n \leq |X|$.
3. $D_X \cup \{A \subseteq X : |X \setminus A| \leq n\}$ where n is an infinite cardinal, $n < |X|$.
4. $D_X \cup \{A \subseteq X : |X \setminus A| < n\}$ where $n \leq |X|$ and n is a limit cardinal.

The following figure shows how different types of homogeneity in c-spaces relate to one another.



Chapter 5

Fuzzy c-spaces

5.1 Introduction

The idea of a c-space broadens the concept of connectedness in both topology and graph theory. However, this approach was limited to binary pictures and set-oriented. In 1965, L. A. Zadeh proposed fuzzy set theory [43]. With the introduction of fuzzy set theory, traditional conceptions of connectedness are being expanded to include fuzzy contexts. This eventually leads to the formulation of connectivity criteria for grayscale images, as well as the establishment of a suitable theoretical framework for studying grayscale images. Here, we expand the concept of c-spaces to establish fuzzy c-space theory. As a result, a definition of connectivity that is compatible with grayscale and binary images is developed.

In the first section of this chapter, we define fuzzy c-spaces and obtain some of their properties. The concept of fuzzy touching points and fuzzy t-closed sets are introduced in the next section. Furthermore, we define fuzzy c-continuous mapping and investigate some of its properties. In the last section, we study the properties of the lattice of fuzzy c-structures.

5.2 Fuzzy c-spaces

Here, we introduce the notion of a fuzzy c-space.

Definition 5.2.1. Let X be a set, and $\mathcal{F} \subseteq I^X$. Then $(X, \mathcal{F})$ is said to be a fuzzy c-space if it satisfies the following conditions

1. $\underline{0} \in \mathcal{F}$.
2. If $\mathcal{A}$ is a family of fuzzy subsets in $\mathcal{F}$ with $\wedge\mathcal{A} \neq \underline{0}$, then $\vee\mathcal{A} \in \mathcal{F}$.
3. $x_1 \in \mathcal{F}$ for all $x \in X$.

Here, $\mathcal{F}$ is called a fuzzy c-structure on X and elements of $\mathcal{F}$ are called fuzzy connected sets.

Here onwards, D_X denotes the set of all crisp points together with $\underline{0}$.

The following are some examples of fuzzy c-spaces.

Example 5.2.1. 1. Let X be any set. Then $\mathcal{F} = D_X$ is a fuzzy c-structure on X and is called the trivial fuzzy c-structure.

2. Let X be any set and $\mathcal{F} = I^X$. Then $(X, \mathcal{F})$ is a fuzzy c-space called the indiscrete fuzzy c-space.

3. Let (X, δ) be a fuzzy topological space. Then the collection of all fuzzy connected subsets of X forms a fuzzy c-structure on X.

4. Let $\mathcal{F}$ be a fuzzy c-structure on X. Then $\mathcal{F} \cup \{\underline{1}\}$ is also a fuzzy c-structure on X, called the Brunnian closure of $\mathcal{F}$.

5. Let $f \in I^X$. Then $\mathcal{F} = D_X \cup \{g \in I^X : f \leq g\}$ is a fuzzy c-structure on X, called the fuzzy c-structure rooted at f.

Theorem 5.2.1. *Let $(X, \mathcal{F})$ be a fuzzy c-space and $Y \subseteq X$. Then the collection $\mathcal{F}_Y = \{f \restriction_Y : f \in \mathcal{F}$ and $f(X \setminus Y) = \{0\}\}$ is a fuzzy c-structure on Y.*

Proof. It is clear that $\underline{0} \in \mathcal{F}_Y$. Let $f = y_1 \in \mathcal{F}$, where $y \in Y$ and $g = y_1 \restriction_Y$. Then $g \in \mathcal{F}_Y$. Let $\{g_i : i \in I\}$ be a collection of fuzzy sets in $\mathcal{F}_Y$ with $\bigwedge_{i \in I} g_i \neq \underline{0}$. Then for each i, there exists $f_i \in \mathcal{F}$ such that $g_i = f_i \restriction_Y$ and $f_i(X \setminus Y) = \{0\}$. Now, $\bigwedge_{i \in I} g_i \neq \underline{0}$ implies $\bigwedge_{i \in I} f_i \neq \underline{0}$, it follows that $\bigvee_{i \in I} f_i \in \mathcal{F}$. Also, note that for each $x \in X \setminus Y$, $f_i(x) = 0$ for all $i \in I$, which implies $(\bigvee_{i \in I} f_i)(x) = 0$ for each $x \in X \setminus Y$. Thus, $\bigvee_{i \in I} g_i$ can be written as $\bigvee_{i \in I} g_i = (\bigvee_{i \in I} f_i) \restriction_Y$. So $\bigvee_{i \in I} g_i \in \mathcal{F}_Y$. Hence the proof. $\square$

Now, we define the concept of a sub fuzzy c-space.

Definition 5.2.2. Let $(X, \mathcal{F})$ be a fuzzy c-space and $Y \subseteq X$. Define $\mathcal{F}_Y = \{f \restriction_Y : f \in \mathcal{F}$ and $f(X \setminus Y) = \{0\}\}$. Then $(Y, \mathcal{F}_Y)$ is a fuzzy c-space called the sub fuzzy c-space of $(X, \mathcal{F})$ and $\mathcal{F}_Y$ is called the sub fuzzy c-structure on Y.

Definition 5.2.3. Let $(X, \mathcal{F})$ be a fuzzy c-space and $\mathbf{B} \subseteq I^X$. Then the intersection of all fuzzy c-structures containing $\mathbf{B}$, denoted by $\langle \mathbf{B} \rangle$ is a fuzzy c-structure on X, and is called the fuzzy c-structure generated by $\mathbf{B}$. The elements of $\mathbf{B}$ are called basic fuzzy connected sets.

Definition 5.2.4. Let $(X, \mathcal{F})$ be a fuzzy c-space. Let K be a finite index set and $H : K \to \mathcal{F}$ be a function such that $H(k) \wedge H(k+1) \neq \underline{0}$ for each $k \in K$. Then $\{H(k) : k \in K\}$ is called a chain.

For our convenience, we denote $H(k)$ and $H(k+1)$ by H_k and H_{k+1}, respectively.

Let $\mathcal{G}$ be the collection of all $g \in I^X$ such that for any $g(x)$ and $g(y)$, there exists a chain $B : K \to \mathbf{B}$ such that each $B_k \leq g$, $k \in K$ and $B_{k'}(x) = g(x)$ and $B_{k''}(y) = g(y)$, together with D_X. Now, we claim that $\mathcal{G}$ is a fuzzy c-structure on X.

Let $\{g_j : j \in J\}$ be the collection of sets in $\mathcal{G}$ with $\bigwedge_{j \in J} g_j \neq \underline{0}$. Consider $\bigvee_{j \in J} g_j$. Let $\bigvee_{j \in J} g_j(x) = a$ and $\bigvee_{j \in J} g_j(y) = b$. Then there exists some g_1, g_2 with $g_1(x) = a$ and $g_2(y) = b$. Since $\bigwedge_{j \in J} g_j \neq \underline{0}$, there exists some $z \in X$ such that $g_j(z) \neq 0$ for all j. Consider $g_1(x)$ and $g_1(z)$. Then by our assumption, there exists a chain of basic fuzzy connected sets, say $B_1, B_2, .., B_m$ contained in g_1 such that $B_i(x) = g_1(x) = a$ and $B_l(z) = g_1(z)$ for some i, $l \in \{1, 2, .., m\}$. Similarly, consider $g_2(y)$ and $g_2(z)$. Then there exists a chain of basic fuzzy connected sets, say $B_1^{'}, B_2^{'}, .., B_n^{'}$ contained in g_2 such that $B_i^{'}(y) = g_2(y) = b$ and $B_l^{'}(z) = g_2(z)$ for some $i^{'}$, $l^{'} \in \{1, 2, .., n\}$. Observe that $B_l \wedge B_l^{'} \neq \underline{0}$, since $g_j(z) \neq 0$ for all $j \in J$. Consequently, there exists a chain of basic fuzzy connected sets $B_1, B_2, .., B_m, B_1^{'}, B_2^{'}, .., B_n^{'}$ such that $B_i(x) = \bigvee_{j \in J} g_j(x) = a$ and $B_i^{'}(y) = \bigvee_{j \in J} g_j(y) = b$. That is, $\bigvee_{j \in J} g_j \in \mathcal{G}$. Obviously, $\mathbf{B} \subseteq \mathcal{G}$. Also, $\mathcal{G} \subseteq \mathcal{F}$ for all

$\mathcal{F}$ containing $\mathbf{B}$. Thus, $\mathcal{G} = \langle \mathbf{B} \rangle$.

We summarize these observations in the following theorem.

Theorem 5.2.2. *Let $\mathcal{F}$ be a fuzzy c-structure on X generated by $\mathbf{B}$. Then any $f \in \mathcal{F} \setminus D_X$ is characterized by the condition that for any $f(x)$ and $f(y)$, there exists a chain $B : K \to \mathbf{B}$ such that each $B_k \leq f$, $k \in K$ and $B_{k'}(x) = f(x)$ and $B_{k''}(y) = f(y)$.*

5.3 Fuzzy Touching Points

In [23] J. Muscat and D. Buhagiar proposed the notion of touching points and examined its different features. Analogously, here, we introduce the concept of fuzzy touching points and investigate its characteristics.

Definition 5.3.1. Let $(X, \mathcal{F})$ be a fuzzy c-space and $f \in I^X$. Then a fuzzy point p is said to touch f, if there exists $\underline{0} \neq f' \in I^X$ with $f' \leq f$ such that $p \vee f' \in \mathcal{F}$.

The set of all fuzzy points touching f is denoted by $Ft(f)$.

Definition 5.3.2. Two fuzzy sets f and g are said to touch if there exists a fuzzy point $p \leq f \vee g$, which touches both f and g.

The following example demonstrates this.

Example 5.3.1. Let $X = \{a, b, c\}$ and $\mathcal{F}$ be a fuzzy c-structure on X having base $\mathbf{B} = D_X \cup \{f_1, f_2\}$, where f_1 and f_2 are defined by

$$f_1(a) = 0, f_1(b) = 0.5, f_1(c) = 0.25$$

$$f_2(a) = 0, f_2(b) = 0.5, f_2(c) = 0.5$$

Consider the fuzzy sets f and g given by

$$f(a) = 0.25, f(b) = 0.5, f(c) = 0.5$$

$$g(a) = 0.25, g(b) = 0.25, g(c) = 1$$

Then $f \vee g$ is given by

$(f \vee g)(a) = 0.25, (f \vee g)(b) = 0.5, (f \vee g)(c) = 1.$

Let $p = b_{0.5} \leq f \vee g$. Take $f^{'}$ and $g^{'}$, where

$$f^{'}(a) = 0, f^{'}(b) = 0.25, f^{'}(c) = 0.5$$

$$g^{'}(a) = 0, g^{'}(b) = 0.25, g^{'}(c) = 1$$

Clearly, $f^{'} \leq f$ and $g^{'} \leq g$. Now, $(f^{'} \vee p)(a) = 0, (f^{'} \vee p)(b) = 0.5, (f^{'} \vee p)(c) = 0.5$ and $(g^{'} \vee p)(a) = 0, (g^{'} \vee p)(b) = 0.5, (g^{'} \vee p)(c) = 1$. Note that $f^{'} \vee p$ and $g^{'} \vee p$ are fuzzy connected. Thus, f and g touch.

Remark 5.3.1. *Let $(X, \mathcal{F})$ be a fuzzy c-space and f, g, and $k \in I^X$. If f touches g and g touches k, then f need not touch k. See the following example.*

Example 5.3.2. Let $X = \{a, b, c, d\}$. Consider the fuzzy c-structure generated by $\mathbf{B} = D_X \cup \{f, g, k\}$, where

$$f(a) = 0.4, f(b) = 0.2, f(c) = 0, f(d) = 0$$

$$g(a) = 0, g(b) = 0.5, g(c) = 0.3, g(d) = 0$$

$$k(a) = 0, k(b) = 0, k(c) = 0.1, k(d) = 0.5$$

Here, f touches g and g touches k. But f does not touch k.

The following theorem establishes the condition under which f touches k.

Theorem 5.3.1. *Let $(X, \mathcal{F})$ be a fuzzy c-space and f, g, and $k \in I^X$. If f touches g and $g \leq k$, then f touches k.*

Proof. Since f touches g, there exists a fuzzy point $p \leq f \vee g$ that touches both f and g. Now, p touches g implies that there exists some $g^{'} \leq g$ such that $g^{'} \vee p$ is fuzzy connected. Also, $g \leq k$ implies that $g^{'} \leq k$. It follows that p touches k, also $p \leq f \vee g \leq f \vee k$. Thus, f touches k.

□

Theorem 5.3.2. *Let $\mathcal{F}$ be a fuzzy c-structure on X and $f \in \mathcal{F}$. Then p touches f if and only if $f \vee p$ is fuzzy connected.*

Proof. Suppose that p touches f. Then there exists a $\underline{0} \neq f^{'} \leq f$ such that $f^{'} \vee p$ is fuzzy connected. Since $f^{'} \leq f$, $f \wedge (f^{'} \vee p) \neq \underline{0}$ which implies that $f \vee (f^{'} \vee p)$ is fuzzy connected. Consequently, $f \vee p$ is fuzzy connected.

Conversely, suppose that $f \vee p$ is fuzzy connected. Then it is clear that p touches f. This completes the proof. □

In the following theorems, we look at the characteristics of fuzzy touching points.

Theorem 5.3.3. *Let $(X, \mathcal{F})$ be a fuzzy c-space and $f,\ g \in I^X$. Then*

1. *If $f \leq g$, then $Ft(f) \subseteq Ft(g)$.*
2. *If f touches g, then g touches f.*
3. *Let $f,\ g \in \mathcal{F}$. If f and g touch, then $f \vee g \in \mathcal{F}$.*

Proof.

1. Suppose that p is a fuzzy touching point of f, then there exists some $\underline{0} \neq f' \leq f$ such that $f' \vee p \in \mathcal{F}$. Since $f' \leq f \leq g$, p is a fuzzy touching point of g also.
2. Trivial
3. Since f and g touches, there exists a fuzzy point $p \leq f \vee g$ which touches both f and g. Thus, $f \vee p$ and $g \vee p$ are fuzzy connected. Now, $(f \vee p) \wedge (g \vee p) \geq p \neq \underline{0}$ and so $(f \vee p) \vee (g \vee p) \in \mathcal{F}$. Hence $f \vee g \in \mathcal{F}$.

□

Proposition 5.3.1. *Let $(X, \mathcal{F})$ be a fuzzy c-space and $f,\ g \in I^X$ and $a \in I$. Then*

1. *If $f(x) = a \neq 0$ for some $x \in X$, then x_1 is a fuzzy touching point of f.*

2. *If* $f \leq g$, *then* $\bigvee Ft(f) \leq \bigvee Ft(g)$.

3. $f \leq \bigvee Ft(f)$.

4. *If* $f \in \mathcal{F}$, *then every fuzzy point in* f *is a fuzzy touching point of* f.

5. *If* $f(x) \neq 0$ *for all* $x \in X$, *then* $\bigvee Ft(f) = \underline{1}$.

Proof.

1. Since $f(x) = a \neq 0$, we have $x_a \leq f$. Let $p = x_1$. Then $x_1 \vee x_a = x_1$ is fuzzy connected. Thus, x_1 is a fuzzy touching point of f.

2. By Theorem 5.3.3 (1), every fuzzy touching point of f is a fuzzy touching point of g, the proof follows.

3. Let $x \in X$. If $f(x) = 0$, then $f(x) \leq \bigvee Ft(f)(x)$. Otherwise, x_1 is a fuzzy touching point of f. Thus, $f \leq \bigvee Ft(f)$.

4. Let $p = x_a \leq f$. Then $p \vee f = x_a \vee f = f \in \mathcal{F}$. Thus, every fuzzy point of f is a fuzzy touching point of f.

5. Since $f(x) \neq 0$ for all x, let $f(x) = a$ for some $x \in X$. Then $x_a \leq f$. Now, $x_1 \vee x_a = x_1 \in \mathcal{F}$. Thus, x_1 is a fuzzy touching point of f. Consequently $\bigvee Ft(f) = \underline{1}$. □

Theorem 5.3.4. *Let* $(X, \mathcal{F})$ *be a fuzzy c-space and* $f \in \mathcal{F}$ *where* $f(x) \neq 0$ *for all* $x \in X$. *Then* $\underline{1} \in \mathcal{F}$.

Proof. Since $f \in \mathcal{F}$ and $f(x) \neq 0$ for all $x \in X$, $x_1 \wedge f \neq \underline{0}$. Consequently, $x_1 \vee f \in \mathcal{F}$. Also, $f \leq x_1 \vee f$ implies $\bigwedge_{x \in X} (x_1 \vee f) \neq \underline{0}$. So $\bigvee_{x \in X} (x_1 \vee f) \in \mathcal{F}$. But $\bigvee_{x \in X} (x_1 \vee f) = \underline{1}$, it follows that $\underline{1} \in \mathcal{F}$. □

Definition 5.3.3. Let $(X, \mathcal{F})$ be a fuzzy c-space and $f \in I^X$. Then f is called fuzzy t-closed, if $\bigvee Ft(f) = f$. If f is fuzzy t-closed, then every fuzzy touching points f is contained in f.

We denote the set of all fuzzy points in f by $Fp(f)$.

Obviously, for any fuzzy c-space $(X, \mathcal{F})$, $\underline{1}$ and $\underline{0}$ are fuzzy t-closed.

Note that if $(X, \mathcal{F})$ is the trivial fuzzy c-space and $f \in I^X$, then f is a crisp subset of X if and only if f is fuzzy t-closed.

The following theorem asserts that all fuzzy t-closed sets in a fuzzy c-space are crisp.

Theorem 5.3.5. *Let $(X, \mathcal{F})$ be a fuzzy c-space. If $f \in I^X$ is fuzzy t-closed, then f is a crisp set.*

Proof. Since f is fuzzy t-closed, the only fuzzy touching points of f are the fuzzy points in f. For $a < 1$, suppose that $f(x) = a \neq 0$. Consider $x_a \leq f$. Then x_1 is a fuzzy touching point of f, since $x_1 \vee x_a = x_1 \in \mathcal{F}$. Hence f is not t-closed, a contradiction. □

But the converse is not true. For example, consider the fuzzy c-structure

$\mathcal{F} = D_X \cup \{f, g, k, l\}$ on X, where $X = \{a, b, c\}$ and

$$f(a) = 1, f(b) = 1, f(c) = 0$$

$$g(a) = 1, g(b) = 0, g(c) = 1$$

$$k(a) = 0, k(b) = 1, k(c) = 1$$

$$l(a) = 1, l(b) = 1, l(c) = 1$$

Let $p = c_1$. Then $f \vee p = \underline{1} \in \mathcal{F}$, which implies that p is a fuzzy touching point of f. But p is not a fuzzy point of f. So f is not fuzzy t-closed.

Remark 5.3.2. *If $\{f_i : i \in I\}$ is a collection of fuzzy t-closed sets in a fuzzy c-space $(X, \mathcal{F})$, then $\bigwedge_{i \in I} f_i$ is fuzzy t-closed.*

Proof. Let $f = \bigwedge_{i \in I} f_i$. By Proposition 5.3.1 (3) , $f \leq \bigvee Ft(f)$. Since $f \leq f_i$ for all i, $\bigvee Ft(f) \leq \bigvee Ft(f_i)$ which implies $\bigvee Ft(f) \leq f_i$ for all i, since f_i is fuzzy t-closed. It follows that $\bigvee Ft(f) \leq \bigwedge_{i \in I} f_i = f$. So $f = \bigvee Ft(f)$. Thus, f is fuzzy t-closed. □

Definition 5.3.4. The fuzzy connective closure of a fuzzy set f denoted by $\overline{f}$ is defined as the smallest fuzzy t-closed set containing f. That is, fuzzy connective closure of a fuzzy set f is the meet of all fuzzy t-closed sets that contain f.

Theorem 5.3.6. *Let $(X, \mathcal{F})$ be a fuzzy c-space and $f \in I^X$. Then*

1. *f is fuzzy t-closed if and only if $\overline{f} = f$.*

2. *$\bigvee Ft(f) \leq \overline{f}$.*

Proof. 1. Suppose that f is fuzzy t-closed. Then the smallest fuzzy t-closed set containing f is f itself, which implies $\overline{f} = f$.

Conversely, suppose that $\overline{f} = f$. That is, the meet of all fuzzy t-closed set containing f is f itself. So f is fuzzy t-closed.

2. Let $\mathbf{F}$ be a fuzzy t-closed set such that $f \leq \mathbf{F}$. Then $\bigvee Ft(\mathbf{F}) = \mathbf{F}$. Since $f \leq \mathbf{F}$, $\bigvee Ft(f) \leq \bigvee Ft(\mathbf{F})$ implies $\bigvee Ft(f) \leq \mathbf{F}$. This is true for all fuzzy t-closed set containing f, thus we can write $\bigvee Ft(f) \leq \bigwedge\{\mathbf{F} : f \leq \mathbf{F}, \bigvee Ft(\mathbf{F}) = \mathbf{F}\} = \overline{f}$. Hence $\bigvee Ft(f) \leq \overline{f}$.

□

Definition 5.3.5. A fuzzy set $f \in I^X$ is fuzzy c-dense if $\bigvee Ft(f) = \underline{1}$.

In an indiscrete fuzzy c-space, every fuzzy set is fuzzy c-dense. For any fuzzy c-space $(X, \mathcal{F})$, if $f(x) \neq 0$ for all $x \in X$, then f is fuzzy c-dense.

Next, we present the concept of fuzzy connective space.

Definition 5.3.6. Let X be a set. A fuzzy connective structure on X is a fuzzy c-structure $\mathcal{F}$ satisfying the following conditions

1. If $\underline{0} \neq f \in \mathcal{F}$ and $\underline{0} \neq g \in \mathcal{F}$ with $f \vee g \in \mathcal{F}$, then there exists a fuzzy point $p \leq f \vee g$ such that $p \vee f \in \mathcal{F}$ and $p \vee g \in \mathcal{F}$.

2. If f, g, $k_i \in \mathcal{F}$ with meet $\underline{0}$ and $f \vee g \vee (\bigvee_{i \in I} k_i) \in \mathcal{F}$, then there exists $J \subseteq I$, $f \vee (\bigvee_{j \in J} k_j) \in \mathcal{F}$ and $g \vee (\bigvee_{i \in I-J} k_i) \in \mathcal{F}$.

Then $(X, \mathcal{F})$ is called a fuzzy connective space.

Example 5.3.3. 1. The trivial fuzzy c-space (X, D_X) is a fuzzy connective space.

2. The indiscrete fuzzy c-space is a fuzzy connective space.

3. Let $X = \{a, b, c\}$. Then $\mathcal{F} = D_X \cup \{f, g, k\}$ is a fuzzy connective structure on X, where

$$f(a) = 1, f(b) = 0.4, f(c) = 0$$
$$g(a) = 0.4, g(b) = 1, g(c) = 0$$
$$k(a) = 1, k(b) = 1, k(c) = 0$$

The following is an example of a fuzzy c-structure that is not a fuzzy connective structure.

Example 5.3.4. Let $X = \{x, y, z, w\}$ and $\mathcal{F}$ be a fuzzy c-structure on X having base $\mathbf{B} = D_X \cup \{f, g, k\}$, where

$$f(x) = 0.5, f(y) = 0.8, f(z) = 0, f(w) = 0$$
$$g(x) = 0, g(y) = 0, g(z) = 0.4, g(w) = 0.7$$
$$k(x) = 0.5, k(y) = 0.8, k(z) = 0.4, k(w) = 0.7$$

But it is not a fuzzy connective structure. Here, $f \vee g = k \in \mathcal{F}$.

1. If $p = x_a$, $0 < a \leq 0.5$, then $p \vee f = f \in \mathcal{F}$ and $p \vee g \notin \mathcal{F}$.

2. If $p = y_a$, $0 < a \leq 0.8$, then $p \vee f = f \in \mathcal{F}$ and $p \vee g \notin \mathcal{F}$.

3. If $p = z_a$, $0 < a \leq 0.4$, then $p \vee f \notin \mathcal{F}$ and $p \vee g = g \in \mathcal{F}$.

3. If $p = w_a$, $0 < a \leq 0.7$, then $p \vee f \notin \mathcal{F}$ and $p \vee g = g \in \mathcal{F}$.

So there does not exist a fuzzy point $p \leq f \vee g$ with $p \vee f \in \mathcal{F}$ and $p \vee g \in \mathcal{F}$.

5.4 Fuzzy c-continuity

A function from a c-space X to another c-space Y is said to be c-continuous if it maps connected sets in X to connected sets in Y. In [23] J. Muscat and D. Buhagiar explore various aspects of c-continuous functions. In addition, they define c-automorphism. In an analogous way, we define a fuzzy c-continuous mapping and examine its properties.

Definition 5.4.1. Let $(X, \mathcal{F})$ and $(Y, \mathcal{F}')$ be two fuzzy c-spaces. Then a function $h : X \to Y$ is called fuzzy c-continuous if it maps fuzzy connected subsets of X into fuzzy connected subsets of Y.

Now, h is called a fuzzy c-isomorphism, if h is bijective and both h and h^{-1} are fuzzy c-continuous. A fuzzy c-isomorphism from X onto itself is called a fuzzy c-automorphism.

Example 5.4.1. Let $X = \{a, b, c\}$ and $\mathcal{F}$ is a fuzzy c-structure on X having base $\mathbf{B} = D_X \cup \{f, g, k\}$, where

$$f(a) = 1, f(b) = 0.5, f(c) = 0$$

$$g(a) = 0.5, g(b) = 0, g(c) = 1$$

$$k(a) = 0, k(b) = 1, k(c) = 0.5$$

If $h = I, (a, b, c)$ or (b, c, a), then h and h^{-1} maps fuzzy connected sets to fuzzy connected sets. So h is a fuzzy c-automorphism on X.

Theorem 5.4.1. *Let $(X, \mathcal{F})$ and $(Y, \mathcal{F}')$ be two fuzzy c-spaces and $h : X \to Y$ be fuzzy c-continuous and $g \in I^X$. If p touches g, then $h(p)$ touches $h(g)$.*

Proof. As p touches g, there exists a $\underline{0} \neq g' \leq g$ such that $g' \vee p \in \mathcal{F}$. Since h is fuzzy c-continuous, $h(g' \vee p) \in \mathcal{F}'$ it follows that $h(g') \vee h(p) \in \mathcal{F}'$. As $g' \leq g$, $g'(x) \leq g(x)$ for all $x \in X$. Now, for all $y \in Y$, $h(g')(y) = \bigvee\{g'(x) : h(x) = y\}$ $\leq \bigvee\{g(x) : h(x) = y\} = h(g)(y)$. Thus, $h(g') \leq h(g)$. So $h(p)$ is a fuzzy touching point of $h(g)$. □

Theorem 5.4.2. *Let $(X, \mathcal{F})$ and $(Y, \mathcal{F}')$ be two fuzzy c-spaces and $h : X \to Y$ be fuzzy c-continuous. If $\mathbf{F}$ is a fuzzy t-closed set in $\mathcal{F}'$, then $h^{-1}(\mathbf{F})$ is fuzzy t-closed set in $\mathcal{F}$.*

Proof. Since $\mathbf{F}$ is fuzzy t-closed, $\bigvee Ft(\mathbf{F}) = \mathbf{F}$. Then every fuzzy touching points of $\mathbf{F}$ is contained in $\mathbf{F}$. Now, let p be a fuzzy touching point of $h^{-1}(\mathbf{F})$ and $p \nleq h^{-1}(\mathbf{F})$ consequently, $h(p)$ is a fuzzy touching point of $\mathbf{F}$ and $h(p) \nleq \mathbf{F}$, which is a contradiction. As $\mathbf{F}$ is fuzzy t-closed by Theorem 5.3.5, $\mathbf{F}$ is a crisp set. So $h^{-1}(\mathbf{F})$ is also a crisp set. If $h^{-1}(\mathbf{F})(x) = 1$, then obviously x_1 is a fuzzy touching point of $h^{-1}(\mathbf{F})$. If $h^{-1}(\mathbf{F})(x) = 0$, then x_a cannot be a fuzzy touching point of $h^{-1}(\mathbf{F})$ for any $a \in (0,1]$. So $\bigvee Ft(h^{-1}(\mathbf{F})) = h^{-1}(\mathbf{F})$. Hence the theorem. □

Proposition 5.4.1. *(i) Any function from a trivial fuzzy c-space is fuzzy c-continuous.*

(ii) Any function into an indiscrete fuzzy c-space is fuzzy c-continuous.

(iii) Identity maps are fuzzy c-continuous.

Proof. (i) Let h be a function from a trivial fuzzy c-space $(X, \mathcal{F})$ to some fuzzy c-space to $(Y, \mathcal{G})$. Then h maps $\underline{0}$ to $\underline{0}$ and crisp points in X to crisp points in Y. So h maps every fuzzy connected sets of X in to fuzzy connected sets of Y. Thus, h is fuzzy c-continuous.

(ii) Let h be a function from an arbitrary fuzzy c-space $(X, \mathcal{F})$ to an indiscrete fuzzy c-space to $(Y, \mathcal{G})$. Thus, every $f \in I^Y$ is fuzzy connected. So every fuzzy connected set of X mapped onto fuzzy connected sets in Y. Thus, h is fuzzy c-continuous.

(iii) Let h be the identity function on $(X, \mathcal{F})$. Then h maps every fuzzy connected set f to f itself. Thus, h is fuzzy c-continuous.

□

Theorem 5.4.3. *The composition of fuzzy c-continuous function is fuzzy c-continuous.*

Proof. Let $h_1 : X \to Y$ and $h_2 : Y \to Z$ be fuzzy c-continuous and f be a fuzzy connected subset of X. Then $h_1(f)$ is fuzzy connected in Y. Now, consider $h_2 \circ h_1(f) = h_2(h_1(f))$, which is fuzzy connected in Z. Hence the proof. □

Remark 5.4.1. *The inverse of a fuzzy c-continuous function is not fuzzy c-continuous.*

This is demonstrated by the example below.

Example 5.4.2. Let $X = \{x, y, z\}$, $Y = \{a, b, c\}$. Let $h : X \to Y$ be defined by $h(x) = a$, $h(y) = b$, and $h(z) = c$.

Consider the fuzzy c-structure $\mathcal{F} = D_X \cup \{f, g, k, l\}$ on X, where

$$f(x) = 0.7, f(y) = 0.7, f(z) = 0$$

$$g(x) = 1, g(y) = 0.7, g(z) = 0$$

$$k(x) = 1, k(y) = 1, k(z) = 0$$

$$l(x) = 0.7, l(y) = 1, l(z) = 0$$

and $\mathcal{F}'$ is the fuzzy c-structure on Y generated by the base $\mathbf{B} = D_Y \cup \{f^{'}, g^{'}\}$, where

$$f^{'}(a) = 0.7, f^{'}(b) = 0.7, f^{'}(c) = 0$$

$$g^{'}(a) = 0, g^{'}(b) = 0.7, g^{'}(c) = 0.7$$

Since h maps fuzzy connected sets in X to fuzzy connected sets in Y, h is fuzzy c-continuous.

Now, let $k^{'}$ be a fuzzy subset of Y given by $k^{'}(a) = 0, k^{'}(b) = 0.7, k^{'}(c) = 1.$

Then $k^{'} \in \mathcal{F}'$, but $h^{-1}(k^{'}) = k^{'} \notin \mathcal{F}$. Therefore, h^{-1} is not fuzzy c-continuous.

5.5 Lattice of Fuzzy c-structures

Let X be any set. Then the collection of all fuzzy c-structures on X is denoted by $LFCS(X)$. In this section, we prove that $LFCS(X)$ forms a complete, bounded lattice. Also, explore the atomic and dually atomic properties of $LFCS(X)$.

The following theorem illustrates that the arbitrary intersection of fuzzy c-structures on a set results in another fuzzy c-structure.

Theorem 5.5.1. *Let X be any nonempty set and $\{\mathcal{F}_i : i \in I\}$ be a collection of fuzzy c-structures on X. Then $\bigcap\limits_{i \in I} \mathcal{F}_i$ is a fuzzy c-structure on X.*

Proof. Let $\mathcal{F} = \bigcap\limits_{i \in I} \mathcal{F}_i$. Since $\underline{0} \in \mathcal{F}_i$ for all i, $\underline{0} \in \bigcap\limits_{i \in I} \mathcal{F}_i = \mathcal{F}$. Also, $D_X \subseteq \mathcal{F}_i$ for all i implies $D_X \subseteq \mathcal{F}$. Let $\{f_k : k \in K\}$ be a sub collection of $\mathcal{F}$ such that $\bigwedge\limits_{k \in K} f_k \neq \underline{0}$. Then $\{f_k : k \in K\}$ is a sub collection of $\mathcal{F}_i$ for all i with $\bigwedge\limits_{k \in K} f_k \neq \underline{0}$. It follows that $\bigvee\limits_{k \in K} f_k \in \mathcal{F}_i$ for all i. Thus, $\bigvee\limits_{k \in K} f_k \in \mathcal{F}$. Hence $\mathcal{F}$ is a fuzzy c-structure. □

Remark 5.5.1. *The union of fuzzy c-structures on a set is not a fuzzy c-structure.*

Theorem 5.5.2. *For any set X, $LFCS(X)$ is a partially ordered set under the usual set inclusion.*

Proof. Consider $LFCS(X)$ with the relation $\mathcal{F}_1 \; R \; \mathcal{F}_2$ if and only if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ for any $\mathcal{F}_1$, $\mathcal{F}_2 \in LFCS(X)$. Then

1. R is reflexive, since for any $\mathcal{F} \in LFCS(X)$, $\mathcal{F} \subseteq \mathcal{F}$.

2. R is antisymmetric, since for any $\mathcal{F}_1$, $\mathcal{F}_2 \in LFCS(X)$, if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1$, then $\mathcal{F}_1 = \mathcal{F}_2$.

3. R is transitive, for any $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3 \in LFCS(X)$. If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{F}_2 \subseteq \mathcal{F}_3$, then $\mathcal{F}_1 \subseteq \mathcal{F}_3$.

Thus, R is a partial order on $LFCS(X)$. □

Remark 5.5.2. *For any $\mathcal{F}_1$, $\mathcal{F}_2 \in LFCS(X)$ if we define $\mathcal{F}_1 \wedge \mathcal{F}_2 = \mathcal{F}_1 \cap \mathcal{F}_2$ and $\mathcal{F}_1 \vee \mathcal{F}_2 = \langle \mathcal{F}_1 \cup \mathcal{F}_2 \rangle$, the fuzzy c-structure generated by $\mathcal{F}_1 \cup \mathcal{F}_2$. Then $\mathcal{F}_1 \wedge \mathcal{F}_2 \in LFCS(X)$ and $\mathcal{F}_1 \vee \mathcal{F}_2 \in LFCS(X)$. Thus, $LFCS(X)$ form a lattice. The trivial fuzzy c-structure is the least element and the indiscrete fuzzy c-structure is the greatest element in $LFCS(X)$.*

Thus, $LFCS(X)$ is a bounded lattice since it contains universal bounds.

Recall that a lattice $\mathcal{L}$ is said to be complete if every subset of $\mathcal{L}$ has a meet and join in $\mathcal{L}$. We prove that $LFCS(X)$ is a complete lattice using the following theorem.

Theorem 5.5.3. *[6] If $(X, \leq)$ is a partially ordered set containing the greatest element 1 and any subset of X has an infimum, then X is a complete lattice.*

Theorem 5.5.4. *$LFCS(X)$ is a complete lattice.*

Proof. Observe that $LFCS(X)$ contains 1. By Theorem 5.5.3, it is enough to prove that any subset of $LFCS(X)$ has an infimum. Let $\{\mathcal{F}_i : i \in I\}$ be a collection of fuzzy c-structures in $LFCS(X)$ and $\mathcal{F} = \bigcap\limits_{i\in I}\mathcal{F}_i$. Then $\mathcal{F}$ is a lower bound for each $\mathcal{F}_i$. Let $\mathcal{F}' \subseteq \mathcal{F}_i$ for all i. Then clearly $\mathcal{F}' \subseteq \mathcal{F}$. Thus, $\mathcal{F}$ is the greatest lower bound. Hence the proof. $\square$

Next, we determine the atoms and dual atoms of $LFCS(X)$. Consider the fuzzy c-structure, $\mathcal{F} = D_X \cup \{f\}$, where f is a crisp set or $f = x_a$. It is obvious that the only fuzzy c-structure that is strictly weaker than $\mathcal{F}$ is the trivial fuzzy c-structure. Hence the atoms in $LFCS(X)$ are of the form $D_X \cup \{f\}$, where f is a crisp set or $f = x_a$.

Now, consider the fuzzy c-structure $\mathcal{F} = I^X \backslash \{x_a\}$. Evidently, I^X covers $I^X \backslash \{x_a\}$. So the dual atoms in $LFCS(X)$ are of the form $I^X \setminus \{x_a\}$.

However, in general $LFCS(X)$ is neither atomic nor dually atomic.

Definition 5.5.1. Let X be any set and $\mathcal{F} \in LFCS(X)$. Then $\mathcal{F} \neq \underline{0}$ is said to be atomic if it can be written as the join of atoms in $LFCS(X)$ and $\mathcal{F} \neq I^X$ is said to be dually atomic if it can be written as the meet of dual atoms in $LFCS(X)$.

In the following theorem, we provide the necessary and sufficient conditions for a fuzzy c-structure to be atomic in $LFCS(X)$.

Theorem 5.5.5. *A fuzzy c-structure $\mathcal{F} \neq D_X$ is atomic if and only if every $f \in \mathcal{F} \setminus D_X$ is either a crisp set or a fuzzy point.*

Proof. Suppose that $\mathcal{F}$ is atomic. Let $\{D_X \cup \{f_i\} : i \in I\}$ be the collection of atoms in $LFCS(X)$, where f_i is either a crisp set or $f = x_a$. Since $\mathcal{F}$ is atomic, $\mathcal{F}$ can be written as $\mathcal{F} = \bigvee\limits_{j \in J} (D_X \cup \{f_j\})$ where $J \subseteq I$. Let $f \in \mathcal{F} \setminus D_X$. Then f can be written as $f = \bigvee f_k$, where $f_k \in D_X$ or $f_k = f_{j'}$ for some $j^{'} \in J$ and $\bigwedge f_k \neq \underline{0}$. Suppose that f is not a fuzzy point. Then each f_k is not a fuzzy point. Consequently, each f_k is a crisp set. So f is also a crisp set.

Conversely, let every $f \in \mathcal{F} \setminus D_X$ be either a crisp set or a fuzzy point. Then obviously $\mathcal{F}$ is atomic. □

The theorem that follows gives the equivalent condition for a fuzzy c-structure to be dually atomic in $LFCS(X)$.

Theorem 5.5.6. *A fuzzy c-structure $\mathcal{F} \neq I^X$ is dually atomic if and only if $\mathcal{F}$ contains all fuzzy sets other than the fuzzy points.*

Proof. Suppose that $\mathcal{F}$ is dually atomic. Then it can be written as the meet of dual atoms in $LFCS(X)$. The dual atoms in $LFCS(X)$ are of the form $I^X \setminus \{x_a\}$. So it contains all fuzzy sets other than the fuzzy points and so is their meet.

Conversely, suppose that $\mathcal{F}$ contains all fuzzy sets other than the fuzzy points. Let $\mathcal{P} = \{\{x_a\} : x_a \notin \mathcal{F}\}$. Then $\mathcal{F} = \bigwedge\limits_{x_a \in \mathcal{P}} I^X \setminus \{x_a\}$. Hence $\mathcal{F}$ is dually atomic. □

Chapter 6

Conclusion and Recommendations

This chapter summarizes the key findings of the thesis. It also briefly discusses the significant contributions made by the study. Additionally, it provides recommendations for further research within the context of c-spaces.

6.1 Conclusion

In this thesis, several characteristics of c-space were examined. The study primarily investigated the properties of the lattice of c-structures and homogeneous c-spaces. We defined the concept of simple expansion and c-spaces for which a complement exists were characterized. Additionally, the existence of an upper neighbor for any c-structure defined on a finite set was established. Moreover, the form of an upper neighbor of a c-space, whenever it exists, was determined. Furthermore, we identified the automorphism group of the lattice of c-structures.

A specific class of homogeneous c-structures on finite sets was then defined. We introduced and examined the characteristics of three different forms of homogeneity: n-homogeneity, strongly n-homogeneity, and local homogeneity in c-spaces. Next, various properties of hereditarily homogeneous c-spaces were studied and we provided a characterization for hereditarily homogeneous c-spaces. Moreover, the relationship between bihomogeneous c-spaces and hereditarily homogeneous c-spaces was investigated. Following this, hereditarily homogeneous connective spaces were examined.

Finally, we generalized the notion of c-spaces to fuzzy contexts. Fuzzy touching points and fuzzy c-continuous functions were defined, and their properties were studied. It was then proved that the collection of all fuzzy c-structures on a set forms a lattice. Additionally, the atomic and dually atomic properties of the lattice of fuzzy c-structures were discussed.

6.2 Recommendations

In this section, we address some unsolved problems in this area. We established the existence of an upper neighbor of a c-space when X is a finite set. However, in the infinite case, we provided an example of a c-space that does not possess an upper neighbor. The question of when an upper neighbor of a c-space exists in general remains unanswered. Although we have determined the automorphism group of the lattice of c-structures, the homomorphisms of the lattice of c-structures remain unknown.

We investigated several features of homogeneous c-spaces and provided a characterization of hereditarily homogeneous c-spaces. However, the characterization of homogeneous c-spaces remains an open problem. In particular, the characterization of homogeneous topological c-spaces and homogeneous graphical c-spaces is still unresolved. Additionally, we introduced the concept of fuzzy c-spaces, which opens many possibilities for further research in this area. The characterization of topological and graphical c-structures also remains an open question.

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Appendix

List of Publications

1. C. Darsana and P. Sini : *Hereditarily Homogeneous c-spaces*, Research in Mathematics, **10** (1), (2023) 1-6, https://doi.org/10.1080/27684830.2023.2231607

2. C. Darsana and P. Sini : *Lattice of c-structures*, Gulf Journal of Mathematics, **17** (1), (2024) 167-178, https://doi.org/10.56947/gjom.v17i1.2078

List of Presentations

1. C. Darsana: *On Simple Expansion of c-spaces.* International Conference on Recent Trends in Mathematics with AI, M.E.S College Mampad, Kerala, India.

2. C. Darsana: *A Study on Lattice of c-spaces.* International Conference on Algebra and Discrete Mathematics (ICADM-2024), Government College Kattappana, Kerala, India.

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