GOODNESS OF FIT TESTS FOR LIFETIME DISTRIBUTIONS USING STEIN'S TYPE CHARACTERIZATIONS

A thesis submitted during 2025 to the University of Calicut in partial fulfillment of the award of a Ph.D. degree in Statistics

by

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under the guidance of

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2026

DECLARATION

I hereby declare that the work presented in the thesis entitled 'Goodness of Fit Tests for Lifetime Distributions Using Stein's Type Characterization' is based on the original work done by me under the guidance of Dr.Sreedevi E P and has not been included in any other thesis submitted previously for the award of any degree. The contents of the thesis are undergone plagiarism check using iThenticate software at C.H.M.K. Library, University of Calicut, and the similarity index found within the permissible limit. I also declare that the thesis is free from AI generated contents.

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Place: Thrissur

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This is to certify that all the suggestions recommended by the adjudicators of the PhD thesis of Mr. Vaisakh K M have been incorporated and implemented in the thesis entitled, **"Goodness of fit tests for lifetime distributions using Stein's type characterizations"**. The content of the thesis in both hard copy and soft copy are one and same.

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Dr. Sreedevi E P

Supervising Teacher

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Contents

Abstract

This thesis focuses on the development of new goodness-of-fit tests for lifetime distributions, an area of importance in both reliability theory and survival analysis. Lifetime data often arise in engineering and biomedical studies, where assumptions regarding the underlying probability distribution form the basis for subsequent inference. Therefore, it is essential to validate these assumptions through appropriate goodness-of-fit procedures.

The central methodology of this work is based on Stein's type identity, which has proven to be a powerful tool in distributional characterizations. While Stein's identity is classical for the normal distribution, recent research has extended such characterizations to a broader class of probability distributions. Motivated by these developments, we construct goodness-of-fit tests for four important lifetime distributions: gamma, Rayleigh, inverse Gaussian, and Lindley. For each distribution, we employ characterization results based on Stein's type identity to define suitable departure measures and develop test statistics within the framework of U-statistic theory.

A key contribution of this study lies in addressing the problem of censoring, which is a common feature of lifetime data but not adequately handled in much of the existing literature. We extend the proposed test procedures to right-censored data, analyze their asymptotic properties using central limit theorems for U-statistics, and establish their validity under both censored and uncensored scenarios.

The performance of the proposed tests is investigated through extensive Monte Carlo simulation studies, where their power is compared with existing procedures. The results demonstrate that the developed tests provide reliable performance in finite samples. To illustrate practical utility, the methods are applied to real-life data sets from reliability and biomedical contexts.

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സംഗ്രഹം

ഈ പ്രബന്ധം പുതിയ ഗുഡ്നസ്-ഓഫ്-ഫിറ്റ് ടെസ്റ്റുകളുടെ വികസനത്തിലേക്ക് ശ്രദ്ധ കേന്ദ്രീകരിക്കുന്നു, ഇത് എഞ്ചിനിയറിംഗിലും ബയോമെഡിക്കൽ പഠനങ്ങളിലും ആയുസ് ഡാറ്റ സാധാരണയായി ഉണ്ടാകാറുണ്ട്, അപ്പോൾ അടിസ്ഥാന സാധ്യതാ വിതരണത്തെക്കുറിച്ചുള്ള ധാരണകൾ തുടർ അവലോകനങ്ങൾക്ക് ആധാരമാവുന്നു. അതിനാൽ, ഈ ധാരണകൾ ഉചിതമായ ഗുഡ്നസ്-ഓഫ്-ഫിറ്റ് നടപടികളിലൂടെ ശരിവെക്കുന്നത് നിർണായകമാണ്.

ഈ പഠനത്തിന്റെ മുഖ്യ രീതി സ്റ്റെയിൻ പോലെയുള്ള ഐഡന്റിറ്റിയിലാണ് ആധാരപ്പെടുത്തിയിരിക്കുന്നത്, ഇത് വിതരണം കഥാപാത്രീകരണത്തിൽ ശക്തമായ ഉപകരണം ആയി തെളിഞ്ഞിട്ടുണ്ട്. സ്റ്റെയിൻ ഐഡന്റിറ്റി സാധാരണ വിതരണത്തിന് പരമ്പരാഗതമായതിനാൽ, അടുത്തകാലത്തെ ഗവേഷണങ്ങൾ ഈ ഘടനയെ വ്യാപകമായ സാധ്യതാ വിതരണ ക്ലാസുകളിലേക്ക് വ്യാപിപ്പിച്ചിട്ടുണ്ട്. ഈ വികസനങ്ങളിൽ നിന്ന് പ്രചോദനം ഉൾക്കൊണ്ട്, ഞങ്ങൾ നാല് പ്രധാന ആയുസ് വിതരണങ്ങൾക്കായി ഗുഡ്നസ്-ഓഫ്-ഫിറ്റ് ടെസ്റ്റുകൾ നിർമ്മിച്ചിരിക്കുന്നു: ഗാമ, റെയ്‌ലി, ഇൻവേഴ്സ് ഗോസിയൻ, ലിൻഡ്ലി. ഓരോ വിതരണത്തിനും, സ്റ്റെയിൻ തരം ഐഡന്റിറ്റിയിൽ ആധാരിച്ചുള്ള കഥാപാത്രീകരണ ഫലങ്ങൾ ഉപയോഗിച്ച് പ്രസക്തമായ വ്യത്യസ്തതയുടെ അളവുകൾ നിർവചിക്കുകയും, U-സ്റ്റാറ്റിസ്റ്റിക് സിദ്ധാന്തത്തിന്റെ രൂപരേഖയിൽ ടെസ്റ്റ് സംഖ്യകൾ വികസിപ്പിക്കുകയും ചെയ്യുന്നു.

ഈ പഠനത്തിന്റെ ഒരു പ്രധാന സംഭാവനയാണ് "സെൻസറിംഗ്" എന്ന പ്രശ്നത്തെ അഭിമുഖീകരിക്കുന്നത് – ആയുസ് ഡാറ്റയിൽ സാധാരണമായ ഒരു ഘടകമായിട്ടുള്ളത്, എന്നാൽ നിലവിലുള്ള ലിറ്ററേച്ചറിൽ വേണ്ടത്ര പരിഗണന ലഭിച്ചിട്ടില്ല. ഞങ്ങൾ നിർദേശിച്ചിരിക്കുന്ന ടെസ്റ്റ് നടപടികളെ സെൻസർ ചെയ്ത ഡാറ്റയിലേക്ക് വികസിപ്പിക്കുകയും, U-സ്റ്റാറ്റിസ്റ്റിക്കിനായുള്ള സെൻട്രൽ ലിമിറ്റ് തിയറങ്ങളുപയോഗിച്ച് അവയുടെ ആസിംപ്ടോട്ടിക് ഗുണങ്ങൾ വിശകലനം ചെയ്യുകയും, സെൻസർ ചെയ്തതിന്റെയും ചെയ്യാത്തതിന്റെയും സാഹചര്യങ്ങളിൽ ഇവയുടെ ശുദ്ധി സ്ഥാപിക്കുകയും ചെയ്യുന്നു.

നാം നിർദ്ദേശിച്ച ടെസ്റ്റുകളുടെ പ്രകടനം വ്യാപകമായ മോണ്ടെ കാർലോ സിമുലേഷൻ പഠനങ്ങളിലൂടെ പരിശോധിച്ചിരിക്കുന്നു, അവയുടെ ശക്തി നിലവിലുള്ള നടപടികളുമായി താരതമ്യപ്പെടുത്തിയാണ് വിലയിരുത്തിയത്. ഫലങ്ങൾ സൂചിപ്പിക്കുന്നത്, വികസിപ്പിച്ചെടുത്ത ടെസ്റ്റുകൾ ചെറിയ സാമ്പിളുകളിൽ പോലും വിശ്വസനീയമായ പ്രകടനം കാഴ്ചവെക്കുന്നതാണ്. പ്രായോഗിക ഉപയോഗക്ഷമത തെളിയിക്കാൻ, വിശ്വാസ്യതയും ബയോമെഡിക്കൽ പശ്ചാത്തലങ്ങളുമുള്ള യഥാർത്ഥ ഡാറ്റ സെറ്റുകളിൽ ഈ രീതികൾ പ്രയോഗിച്ചിരിക്കുന്നു.

സെന്റ് തോമസ് കോളേജ് (ഓട്ടോണമസ്) ST THOMAS COLLEGE (AUTONOMOUS) ESTD 1889 THRISSUR തൃശ്ശൂർ

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Chapter 1

Introduction

1.1 Goodness of fit tests for lifetime distributions

Lifetime data analysis or survival analysis encompasses a collection of statistical methods for analyzing time-to-event data, where the attention is focused on the time to the occurrence of some event of interest. The modern era in lifetime data analysis started some decades back with engineering applications starting in 1970, the field expanded rapidly with respect to theory, methodology and fields of applications. The development proceeded to two intermingling streams, viz. reliability theory and survival analysis. The reliability theory deals with models and methods for the lifetime of components and systems in engineering and industrial fields whereas the survival analysis concerns the data arising from medical studies. In parametric lifetime data analysis, it is usually assumed that the data follows a specific probability distribution. When this assumption holds true, the subsequent analysis becomes more straightforward. Goodness of fit tests are employed to validate the assumption, whether the chosen distribution adequately represents the data. A detailed study of goodness of fit tests for lifetime data is given in Lawless (2011). In this thesis, we develop goodness of fit tests for different lifetime distributions using characterizations based on Stein's type identity.

Stein's identity for normal distribution and its applications has been well studied in statistical literature. Let X be a continuous random variable with finite mean μ and variance σ^2. Let $c(x)$ be a continuous function having a first derivative. Then X has a normal distribution with mean μ and variance σ^2 if and only if (Stein (1972))

$$E(c(X)(X-\mu)) = \sigma^2 E(c'(X)).$$

In literature, the above identity is known as Stein's identity or Stein's lemma. For a detailed discussion on Stein's type identity for the general class of probability distributions and related characterizations, one can refer to Kattumannil (2009), Kattumannil and Tibiletti (2012) and Kattumannil and Dewan (2016) and the references therein.

Using Stein's type identity Betsch and Ebner (2021) developed several characterizations of continuous probability distributions based on the distribution function of the random variable X. Betsch et al. (2022) gave similar results for discrete probability distributions also. These characterization results, pave the way to an area of research in the goodness of fit test problems as evident from the work of Betsch and Ebner (2019), Betsch et al. (2022), Anisha et al. (2023), Kumari et al. (2023), Sreedevi and Kattumannil (2023), Anastasiou et al. (2023) and Ebner et al. (2024). Motivated by the work of Betsch and Ebner (2021), we develop goodness of fit tests for four life distributions: gamma distribution, Rayleigh distribution, inverse Gaussian distribution and Lindley distribution. In all these cases, we developed tests for complete data as well as right censored data.

1.2 Nutshell of research work

In this work, we study the goodness of fit test for lifetime distributions. Next, we give the characterization results for non-negative continuous probability distributions; which is the main result that is used for our study.

Theorem 1.1. *Betsch and Ebner (2021) Assume that f is the probability density function of a random variable with $spt(f) = [0, \infty)$, that is positive and differentiable on $(0, \infty)$. Moreover, assume that $\int_0^\infty x \mid f'(x) \mid dx < \infty$. Let X be a positive random variable with*

$E|\frac{f'(X)}{f(X)}| < \infty$, then $X \sim F$ if and only if

$$F(t) = E\left[\frac{-f'(X)}{f(X)}\min(X,t)\right],$$

where $F(t)$ is the cumulative distribution function of X.

Next, we discuss the research problem in detail. Based on a random sample $X_1, ..., X_n$ from F, we are interested in testing the null hypothesis

$$H_0 : F \text{ follows a specific distribution}$$

against the alternative hypothesis

$$H_1 : F \text{ does not follow the specific distribution.}$$

For testing the above hypothesis, first we define a departure measure that discriminates between the null and alternative hypotheses. In view of Theorem 1.1, we consider the departure measure

$$\Delta(F) = \int_0^\infty \left(E\left[\frac{-f'(X)}{f(X)}\min\{X,t\}\right] - F(t)\right) \mathrm{d}F(t). \tag{1.1}$$

Then we simplify $\Delta(F)$, in terms of expectations of the function of random variables. This enables us to use the U-statistic theory to develop the test statistics. We study the asymptotic properties of the test statistics using the central limit theorem for U-statistics. We use the asymptotic distribution of the test statistics to obtain the normal-based critical region of the proposed tests. We then carry out a Monte Carlo simulation study to assess the performance of the proposed tests. We illustrate the test procedures using real data sets. We then discuss how to modify our test procedures to incorporate right-censored observations contained in the sample. This is very important as the occurrence of right-censored observations in lifetime data is common. We can also note that the goodness of fit test that accommodates right censored cases is not well-addressed in the literature. We aim to fill the gap in this study.

1.3 Organization of thesis

The thesis is organized into seven Chapters. In Chapter 2, we give a review of some important aspects that are used in the subsequent chapters. In Chapter 3, we develop a new goodness of fit test for the gamma distribution. We then discuss how the right censored observations are incorporated into the proposed methodology. The asymptotic properties of the test statistics in censored and uncensored cases are studied. We carry out a Monte Carlo simulation study to validate the finite sample performance of the proposed tests. We also illustrate the test procedures using real data sets. In Chapter 4, we develop new goodness fit tests for Rayleigh distribution for complete as well as right censored data. We use the U-statistics theory to derive the test statistic. First, we develop a test for complete data and then discuss, how the right censored observations can be incorporated in the testing procedure. The asymptotic properties of the test statistic in both uncensored and censored cases are studied in detail. Extensive Monte Carlo simulation studies are carried out to validate the performance of the proposed tests. We illustrate the procedures using real data sets. We also develop a jackknife empirical likelihood ratio test for the standard Rayleigh distribution.

In Chapter 5, making use of Stein's type identity for inverse Gaussian distribution, we develop a U-statistics-based goodness of fit test for inverse Gaussian distribution with shape parameter θ, when we have uncensored data. We also propose a jackknife empirical likelihood ratio test for the inverse Gaussian distribution when $\theta = 1$. We discuss how to modify the test statistic to accommodate right-censored observations. The asymptotic properties of the test statistics are studied in detail. Extensive Monte Carlo simulation studies are carried out to assess the finite sample performance of the proposed tests in the presence and absence of censoring. The practical applicability of the proposed tests is illustrated using real-life data sets. Finally, we conclude the study with a discussion on future work. Here as well, We develop a jackknife empirical likelihood ratio test for the standard inverse Gaussian distribution.

In Chapter 6, we develop a U-statistics-based goodness fit test for the Lindley distribution. Here as well, we discuss how to incorporate right-censored observations in the proposed testing methods. Extensive Monte Carlo Simulation studies are conducted to evaluate the performance of the test. The illustrations of the proposed methods using real data sets are reported. In Chapter 7, we highlight our major findings in this research work. We also give some open problems of research interest, identified during the present study, which opens up a broad area for future investigations.

Chapter 2

Basics concepts used in the proposed work

2.1 Introduction

Life time distributions typically refer to statistical models or descriptions that characterize the lifespan or survival times of individuals or entities in a population. These distributions are commonly used in fields such as reliability engineering, survival analysis, and actuarial science to model and analyze the longevity or failure times of components, organisms, systems, etc. Several parametric, nonparametric and semiparametric models are developed to study lifetime models. For using parametric models in lifetime studies, it is important to check the validity of an assumed parametric model. The goodness of fit test developed for specific distributions is used to validate the assumption that the given lifetime data follows a particular distribution. A detailed study of the goodness of fit tests for lifetime data is given by Lawless (2011). In this study, we develop new goodness of fit tests for commonly used lifetime models.

2.2 Basic concepts

Let X be a non-negative continuous random variable having probability density function $f(t)$ cumulative distribution function $F(t) = P(X \leq t)$. We assume that $\mu = E(X) < \infty$. Next, we define some basic qualities used in reliability analysis.

2.2.1 Reliability/Survival function

The probability that a given subject will survive beyond a specified time t is defined as the reliability/survival function of the lifetime variable T. Mathematically we define it as

$$R(t) = P(X > t) = \int_t^{\infty} f(y)dy.$$

The $R(t)$ also represents in alternate form as $S(t)$ or $\bar{F}(t)$. From the definition, it is clear that $R(t) = 1 - F(t)$. The survival function has the following properties:

1. $R(0) = 1$, indicating that the survival of a subject at time 0 is certain.
2. $\lim_{x\to\infty} R(t) = 0$, indicating that the survival probability approaches zero as the lifetime approaches infinity.
3. $R(t)$ is a non-increasing function of t.

The reliability/survival function can be estimated using parametric or nonparametric models.

2.2.2 Hazard rate function

The hazard rate quantifies the risk of failure for a specific item or system as it ages, useful for optimizing maintenance and reliability strategies in various fields. Mathematically, the hazard rate function denoted by $h(t)$ is defined as

$$h(t) = \lim_{\Delta_t \to 0} \frac{P(t \leq T < t + \Delta_t | T \geq t)}{\Delta_t}.$$

If the probability density function of T exist, we can write $h(t)$ as

$$h(t) = \frac{F(t)}{R(t)}.$$

The hazard rate function determines the distribution of T uniquely. That is, the survival function of T can be expressed in terms of the hazard function as follows:

$$R(t) = \exp\left(-\int_0^t h(u)du\right).$$

Classification of classes of lifetime distributions based on the notion of aging has an important role in reliability analysis as it helps in the identification of the underlying model. In this aspect, hazard function has a key role. Next, we give the definition of some aging notions which is used in classifying life distribution.

Constant hazard rate

A constant hazard rate describes a scenario where the likelihood of an event occurring at any given time is constant, regardless of how much time has passed. This idea is widely applied in survival analysis, reliability engineering, and actuarial science to model situations where the risk of failure or occurrence of an event remains unchanged over time.

Increasing hazard rate

An increasing hazard rate refers to a situation in which the probability of an event occurring becomes greater as time goes on. In this context, the chance of failure or occurrence escalates over time, typically due to aging, wear, or deterioration of a system, component, or individual. An increasing hazard rate refers to a situation in which the probability of an event occurring becomes greater as time goes on. In this context, the chance of failure or occurrence escalates over time, typically due to aging, wear, or deterioration of a system, component, or individual.

Decreasing hazard rate

A decreasing hazard rate describes a scenario where the probability of an event occurring decreases as time passes. In other words, the longer something continues, the less likely it is to fail soon. This pattern is often observed when initial failures are more frequent, but once a system or individual has moved beyond the early stages, its stability and reliability tend to increase.

Bathtub Shaped hazard rate

The bathtub-shaped hazard rate refers to a particular pattern of failure rate (or hazard rate) in reliability engineering and survival analysis, often used to describe the life cycle of mechanical systems, biological organisms, or products. The name comes from the shape of the hazard rate curve, which resembles a bathtub: high at the beginning, constant in the middle, and high again at the end of the lifetime.

2.3 Life distributions

Next we discuss some well-known parametric models that are used in reliability analysis.

2.3.1 Exponential distribution

This distribution describes the time between events in a Poisson process, where events occur continuously and independently at a constant average rate. It is often used to model the lifespan of components with a constant failure rate. The probability density function of exponential distribution is,

$$f(t;\lambda) = \lambda e^{-\lambda t}, \qquad t \geq 0.$$

The cumulative distribution function is,

$$F(t) = 1 - e^{-\lambda t}, \qquad t \geq 0.$$

The hazard function is given by,

$$h(t) = \lambda.$$

It means exponential distribution has a constant hazard rate.

2.3.2 Gamma distribution

The gamma distribution is a continuous probability distribution characterized by two parameters. It is widely used in probability theory, statistics, and numerous applied disciplines, including queuing theory, climatology, hydrology, and Bayesian analysis.
The PDF of a gamma-distributed random variable X with shape parameter $k > 0$ and scale parameter $\theta > 0$ is:

$$f(t; k, \theta) = \frac{1}{\Gamma(k)\theta} t^{\alpha-1} e^{(-t/\theta)^k}, t > 0.$$

The cumulative distribution function is

$$F(t) = \frac{1}{\Gamma(k)\theta} \gamma(\alpha, t/\theta).$$

The hazard function is given by

$$h(t) = \frac{\beta^\alpha t^{\alpha-1} e^{\beta t}}{\Gamma\alpha - \gamma(\alpha, \beta)}.$$

- If $\alpha < 1$: Decreasing hazard rate.
- If $\alpha = 1$: Constant hazard rate .
- If $\alpha > 1$: Increasing hazard rate.

2.3.3 Weibull distribution

The Weibull distribution is used extensively in reliability engineering, survival analysis. It can model various types of hazard functions so that suitable for modeling different types of failure mechanisms. The probability density function of the Weibull distribution is given by,

$$f(t;k,\lambda) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{(-t/\lambda)^k}, t > 0.$$

The cumulative distribution function is

$$F(t;k,\lambda) = 1 - e^{(-t/\lambda)^k}, t > 0$$

The hazard function is given by

$$h(x;k,\lambda) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1}.$$

The hazard function has different behavior based on the shape parameter.

- For $k = 1$, hazard rate is constant. Hence, it corresponds to exponential distributions.
- For $k > 1$ hazard rate increases over time.
- For $k < 1$ hazard rate decreases over time.

2.3.4 Rayleigh distribution

The Rayleigh distribution is a probability distribution that is commonly used to model the magnitude of vector quantities, such as the Euclidean norm of a two-dimensional vector or the magnitude of the amplitude of a wave after propagating through a random medium. The Rayleigh distribution is a special case of the Weibull distribution with shape parameter 2. The probability density function of the Rayleigh distribution is

$$f(t;\sigma) = \frac{t}{\sigma^2} e^{\frac{-t^2}{2\sigma^2}}, \qquad t \geq 0,$$

where σ is the scale parameter of the distribution. The cumulative distribution function is

$$F(t:\sigma) = 1 - e^{\frac{-t^2}{2\sigma^2}}, \qquad t > 0$$

The hazard function is given by

$$h(t;\sigma) = \frac{t}{\sigma^2}, \qquad t > 0.$$

The hazard rate increases linearly with time.

2.3.5 Inverse Gaussian distribution

The inverse Gaussian distribution is a probability distribution that generalizes the Gaussian (normal) distribution to accommodate positive and skewed data. Consider the inverse Gaussian distribution with parameters μ and λ, then the density function is given by

$$f(t;\mu,\mu^2) = \sqrt{\frac{\lambda}{2\pi t^3}}\exp\left(-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\right); \;\; t \geq 0, \mu, \lambda > 0 \tag{2.1}$$

and $g(t) = 0$ otherwise. If $X \sim \mathrm{IG}(\mu,\lambda)$, then $\frac{X}{\mu} \sim \mathrm{IG}\left(1, \frac{\lambda}{\mu}\right)$ and therefore the family of inverse Gaussian distributions is closed under scale transformations.

The cumulative distribution function is

$$F(t;\mu,\mu^2) = \Phi\left(\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}-1\right)\right) + e^{\frac{2\lambda}{\mu}}\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}+1\right)\right),$$

where Φ is the cumulative distribution function of standard normal distribution. The hazard function is given by

$$h(t;\mu,\mu^2) = \frac{\sqrt{\frac{\lambda}{2\pi t^3}}\exp\left(-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\right)}{1 + \left[\Phi\left(\sqrt{\frac{\lambda}{}}\left(\frac{t}{\mu}-1\right)\right) + e^{\frac{2\lambda}{\mu}}\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}+1\right)\right)\right]}.$$

- The hazard function for the inverse Gaussian distribution is non-monotonic.

- It typically starts low, increases to a peak, and then decreases, indicating that the probability of an event occurring increases, reaches a maximum, and then diminishes over time.

2.3.6 Lindley distribution

The Lindley distribution is a probability distribution that is used to model non-negative integer-valued data that are right-skewed. It has applications in various fields, including reliability analysis and survival analysis. Let X be a non-negative continuous random variable with distribution function F. The density and distribution function of Lindley distribution is given by

$$f(t, \theta) = \frac{\theta^2}{\theta + 1}(1 + t)e^{-\theta t},\ t > 0,\ \theta > 0$$

and the cumulative distribution function is

$$F(t, \theta) = 1 - \frac{\theta + 1 + \theta t}{\theta + 1}e^{-\theta t},\ t > 0,\ \theta > 0.$$

The hazard function is given by

$$h(t, \theta) = \frac{\theta^2(1 + t)}{1 + \theta + \theta t}.$$

Lindley distribution's hazard function models scenarios where the failure rate increases over time, which is typical in systems experiencing wear-out failures. The Lindley distribution has special attention in lifetime study as it is a mixture of exponential and gamma distributions.

$h(t)$ is an increasing function in t , $\theta$and $\theta^2/\theta + 1 < h(x) < \theta$

2.3.7 Pareto distribution

The Pareto distribution is used to model systems with high early failure rates. The Probability density function of the Pareto distribution is,

$$f(t, x, \alpha) = \frac{\alpha(x)^{\alpha}}{t^{\alpha+1}}, t \geq x.$$

The cumulative distribution function of the Pareto distribution is,

$$F(t, x, \alpha) = 1 - \left(\frac{x}{t}\right)^{\alpha}, t \geq x.$$

The hazard function is given by,

$$h(t, x, \alpha) = \frac{\alpha}{t}.$$

The hazard rate for the Pareto distribution decreases as t increases.

2.3.8 Log normal distribution

The probability density function of the log normal distribution is,

$$f(t; \mu, \sigma) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log t - \mu)^2}{2\sigma^2}\right), t > 0.$$

The cumulative distribution function is,

$$F(t; \mu, \sigma) = \phi\left(\frac{\log t - \mu}{\sigma}\right),\ t > 0.$$

The hazard function is given by,

$$h(t; \mu, \sigma) = \frac{\frac{1}{t\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log t - \mu)^2}{2\sigma^2}\right)}{1 - \phi\left(\frac{lnt - \mu}{\sigma}\right)}.$$

The hazard function for the log-normal distribution is not constant, and it can either increase or decrease depending on the values of μ and λ.

2.4 Different censoring schemes

Censoring is a common phenomenon in lifetime data analysis. The censoring occurs when the exact lifetime of the event of interest (eg. failure) is not known for some subjects under study. Different censoring schemes arise from various reasons leading to different censoring mechanisms. Different types of censoring that are common in lifetime studies are right censoring, left censoring, interval censoring, and double censoring.

2.4.1 Right censoring

Right censoring occurs when the exact value of an observation is unknown but is only known to exceed a certain value. It is common in survival analysis and reliability studies, especially when the event of interest (like death or failure) has not occurred by the end of the study or observation period.

2.4.2 Left censoring

Left censoring occurs in survival or time-to-event analysis when the event of interest takes place before a known observation time, but the exact timing of the event is unknown. In essence, we only know that the event occurred prior to a certain time point.

2.4.3 Interval censoring

Interval censoring refers to the type of censoring in statistical data where the exact time of an event is not known, but it is known to have occurred within a certain time interval. This means that the event occurred between two observations, but the exact time is unknown.

2.4.4 Double censoring

In many survival studies double censoring occurs when patients are only watched within a window of observational time. In such contexts the lifetimes of patients lie within the window of the observational time are only exactly known and remaining lifetimes will either be left censored or right censored.

2.5 U- statistics

Let $X_1, X_2, ..., X_n$ be a sample from $F \in \mathcal{F}$. We assume that the mean of X_i are finite. A parameter θ is said to be estimable of degree m for the family of distributions $\mathcal{F}$, if m is the smallest sample size for which there exists a function $h(x_1, x_2, ..., x_m)$ such that

$$E_F(h(X_1, X_2, ..., X_m)) = \theta, \, \forall F \in \mathcal{F}. \tag{2.2}$$

Definition 2.1. *Let $X_1, ..., X_n$ be a random sample of size n from a distribution F. Let $h(X_1, ..., X_m)$ be a symmetric kernel of degree m with $E\big(h(X_1, ..., X_m)\big) = \theta$, where $\theta(F)$ is a real valued parametric function. The U-statistics with symmetric kernel h is defined as*

$$U_n = \frac{1}{\binom{n}{m}} \sum_{1 \le i_1 < ... < i_m \le n} h(X_{i_1}, ..., X_{i_m}),$$

where the summation is taken over the set of all combinations of m integers, $(i_1, ..., i_m)$ chosen from $(1, \ldots, n)$.

By definition, U_n is an unbiased estimator of $\theta(F)$.

Example 2.2. *Let $\mathcal{F}$ denote the collection of all distributions with finite variance σ^2. Then*

$$E_F(X_1^2 - X_1X_2) = \sigma^2, \forall F \in \mathcal{F}.$$

Thus variance is estimable and of degree 2. The associated symmetric kernel is h(x_1, x_2) $=$ $\frac{1}{2}[(x_1^2 - x_1x_2) + (x_2^2 - x_1x_2)] = \frac{1}{2}(x_1 - x_2)^2$. Using the symmetric kernel, we find the

U-statistic estimator of σ^2 as

$$\begin{aligned} U(X_1, ..., X_n) &= \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \frac{(X_i - X_j)^2}{2} \\ &= \frac{1}{n-1} \left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2 \right] = S^2. \end{aligned} \tag{2.2}$$

Sen (1960) established some convergence properties of the U-statistics including the consistency. Moreover, the asymptotic distribution of U is that of a Gaussian random variable (see Theorem 1, Chapter 3 of Lee (2019)). To establish this result we need to find the variance of U-statistics.

2.5.1 Variance of U-statistics

Let $X_1, ..., X_n$ be a random sample of size n from a distribution F. Let $h(X_1, ..., X_m)$ be a symmetric kernel of degree m. For $c = 0, 1, ..., m$, let

$$h_c(x_1, ..., x_c) = E_F(h(x_1, ..., x_c, X_{c+1}, ..., X_m)),$$

where $X_{c+1}, ..., X_m$ are independent and identically distributed with distribution function F. Then $h_0 = \theta$ and $h_m(x_1, ..., x_m) = h(x_1, ..., x_m)$. Suppose $(\beta_1, \beta_2, ..., \beta_m)$ and $(\beta_1', \beta_2', ..., \beta_m')$ are subset of integers $\{1, 2, ..., n\}$ having exactly c integers in common. Then

$$\begin{aligned} \sigma_c^2 &= Cov(h(X_{\beta_1}, X_{\beta_2}, ..., X_{\beta_m}, h(X_{\beta_1'}, X_{\beta_2'}, ..., X_{\beta_m'})) \\ &= Var(h_c(X_1, X_2, \ldots, X_c)). \end{aligned}$$

If $c = 0$ then the kernels based on β and β' are independent and hence covariance is 0. The variance of $U-$statistic is given by

$$Var(U) = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \sigma_c^2.$$

Consider $\theta = \mu = E(X_1)$. Then $h_1(x_1) = E(X_1|X_1 = x_1) = x_1$. And

$$\sigma_1^2 = Var(X_1) = \sigma^2.$$

Then the variance is

$$Var(U) = \frac{1}{n}\binom{1}{1}\binom{n-m}{0}\sigma_1^2 = \frac{1}{n}VaR(t) = \frac{\sigma^2}{n}.$$

Example 2.3. *Consider the problem of estimating variance with the symmetric kernel $h(X_1, X_2) = \frac{1}{2}(X_1 - X_2)^2$. Then the corresponding U-statistic is (see Example 2.2)*

$$U = \frac{1}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(X_i - X_j)^2 = s^2.$$

Let

$$\begin{aligned} h_1(x_1) &= E(h(X_1, X_2)|X_1 = x_1) \\ &= \frac{1}{2}E(x_1 - X_2)^2 = \frac{\sigma^2}{2} + \frac{(x_1 - \mu)^2}{2}. \end{aligned}$$

Then

$$\begin{aligned} \sigma_1^2 &= V(h_1(X_1)) \\ &= V((X - \mu)^2/2) = \frac{1}{4}(\mu_4 - \sigma^4) \end{aligned}$$

and

$$\sigma_2^2 = V((X_1 - X_2)^2/4) = \frac{\mu_4 - \sigma^4}{2}.$$

The variance of U-statistic is then obtained as

$$\begin{aligned} Var(U) &= \frac{1}{\binom{n}{2}}\left[\binom{2}{1}\binom{n-2}{1}\sigma_1 + \binom{2}{2}\binom{n-2}{0}\sigma_2\right] \\ &= \frac{\mu_4 - \sigma^4}{n}. \end{aligned}$$

Next, theorem is very helpful in establishing the asymptotic distribution of U-statistics.

Theorem 2.4. *Let $U(X_1, ..., X_n)$ be the U-statistic for a symmetric kernel $h(X_1, ..., X_m)$. Assume $E(h^2(X_1, ..., X_m)) < \infty$, then*

$$\lim_{n\to\infty} nV(U(X_1, ..., X_n)) = m^2\sigma_1^2.$$

Given the above theorem, while finding the asymptotic variance of U-statistics, we need to find the first term in the variance of U-statistics given in the expression (2.3). Next we state the results which discuss the asymptotic distribution of U-statistics. In literature, we call this result a central limit theorem for U-statistics.

Theorem 2.5. *Suppose $E(h^2(X_1, ..., X_m)) < \infty$. Then as $n \to \infty$, $\sqrt{n}(U_n - \theta(F))$ converges in distribution to a Gaussian random variable with mean 0 and variance $m^2\sigma_1^2$ where $\sigma_1^2 = Var\big(h^{(1)}(X_1)\big)$ and $h^{(1)}(x_1) = E\big(h(X_1, ..., X_m)|X_1 = x_1\big)$.*

2.5.2 U- statistics for right censored data

Here, we discuss the results taken from Datta et al. (2010). Consider the right-censored data (Y, δ), with $Y = \min(X, C)$ and $\delta = I(X \leq C)$, where C is the censoring point. Here $\delta_i = 1$ means the ith item is not censored and $\delta_i = 0$ means the ith item is censored from the right. Also, let $K(x)$ be the survival function of C. The observed data consist of n independent and identical copies of (Y, δ). Under the above formulation, Datta et al. (2010) define a U-statistics for right censored data as

$$U_c = \frac{1}{\binom{n}{m}} \sum_{1\leq i_1<\ldots<i_m\leq n} \frac{h(Y_{i1}, ..., Y_{im}) \prod_{l\in \underline{i}} \delta_l}{\prod_{l\in \underline{i}} K(Y_l)} \tag{2.3}$$

provided $K(Y_i) > 0$, for each i, with probability one. Here, the notation $l \in \underline{i}$ is used to indicate that l is one of the integers $\{i_1, i_2 \ldots, i_m\}$ chosen from $(1, \ldots, n)$. Since $K(x)$ appeared in the equation (2.3) is not known, we need to estimate it. We estimate it by Kaplan-Meier estimator of the survival function. We denote it as $\widehat{K}_c(Y_i)$. Hence IPCW

U-statistics under right censoring is given by

$$U_c = \frac{1}{\binom{n}{m}} \sum_{1 \le i_1 < ... < i_m \le n} \frac{h(Y_{i1}, ..., Y_{im}) \prod_{l \in \underline{i}} \delta_l}{\prod_{l \in \underline{i}} \widehat{K}(Y_l)}. \tag{2.4}$$

For $m = 2$ we have

$$U_c = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{h(Y_i, Y_j)\delta_i \delta_j}{\widehat{K}(Y_i)\widehat{K}(Y_j)}.$$

When $m = 3$, we have

$$U_c = \frac{6}{n(n-1)(n-2)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \frac{h(Y_i, Y_j, Y_k)\delta_i \delta_j \delta_k}{\widehat{K}(Y_i)\widehat{K}(Y_j)\widehat{K}(Y_k)}.$$

The consistency of $\widehat{U}_c$ is established by Kattumannil et al. (2021). To obtain the asymptotic distribution of U_c, we define $N_i^c(t) = I(Y_i \le t, \delta_i = 0)$ as the counting process corresponding to censoring for the i-th individual and $R_i(u) = I(Y_i \ge u)$. Let $\lambda_c(t)$ be the hazard rate of the censoring variable C. By definition, the counting process $N_i^c(t)$ is a sub-martingale with appropriate filtration. The martingale associated with $N_i^c(t)$ is given by

$$M_i^c(t) = N_i^c(t) - \int_0^t R_i(u)\lambda_c(u)du. \tag{2.5}$$

Let $H_c(t) = P(Y_1 \le t, \delta = 1)$, $y(t) = P(Y_i > t)$ and

$$w(t) = \frac{1}{y(t)} \int \frac{h_1(x)}{\widehat{K}(x-)} I(x > t) dH_c(x), \tag{2.6}$$

where $h_1(y) = E(h(Y_1, Y_2)|Y_1 = y)$. Now, we are in a position to state the results for the asymptotic distribution of U_c.

Theorem 2.6. *Assume* $E(h^2(Y_1, Y_2, \ldots, Y_m)) < \infty$, $\int_0^\infty \frac{h_1(x)}{\widehat{K}^2(x-)} dH_c(x) < \infty$ *and* $\int_0^\infty w^2(t)\lambda_c(t)dt < \infty$.

As $n \to \infty$, *the distribution of* $\sqrt{n}(U_c - \theta)$ *is Gaussian with mean zero and variance* $4\sigma_{1c}^2$, *where σ_{1c}^2 is given by*

$$\sigma_{1c}^2 = Var\Big(\frac{h_1(X)\delta_1}{\widehat{K}(Y-)} + \int_0^\infty w(t)dM_1^c(t)\Big). \tag{2.7}$$

Theorem 2.6 can be used to construct normal-based tests when the data contain the right censored observation. For this purpose, we need to estimate the asymptotic variance given in the expression (2.7) and we estimate it using the reweighted technique. An estimator of σ_{c0}^2 is given by

$$\widehat{\sigma}_{c0}^2 = \frac{4}{(n-1)}\sum_{i=1}^{n}(V_i - \bar{V})^2,$$

where

$$V_i = \frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} + \widehat{w}(X_i)(1-\delta_i) - \sum_{j=1}^{n}\frac{\widehat{w}(X_i)I(X_i > X_j)(1-\delta_i)}{\sum_{i=1}^{n} I(X_i > X_j)},$$

$$\bar{V} = \frac{1}{n}\sum_{i=1}^{n} V_i, \quad \widehat{h}_1(X) = \frac{1}{n}\sum_{i=1}^{n}\frac{h(X, Y_i)\delta_i}{\widehat{K}_c(Y_i-)}, \quad R(t) = \sum_{i=1}^{n} I(Y_i > t)$$

and

$$\widehat{w}(t) = \frac{1}{R(t)}\sum_{i=1}^{n}\frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} I(X_i > t).$$

2.6 Goodness of fit test problem

Goodness of fit is a statistical concept that assesses how well an observed probability distribution fits a theoretical or expected distribution. It is essential in various fields, including statistics, engineering, and biology, where comparing observed data to theoretical models is crucial for making valid conclusions or predictions. That is we need to find the data-generating mechanism to make valid conclusions about the lifetime. The goodness of fit test for specific distributions is normally developed based on some characterization results on that distribution. In this thesis, we consider fixed point characterization based on Stein's type identity to develop the test. We have briefly mentioned this aspect in

Chapter 1. After developing the tests, we carry out a Monte Carlo simulation to assess the performance of the proposed tests. This is done by comparing the power of the tests with the other tests available in the literature. In all the problems considered here, apart from the specific tests developed for a particular distribution, we also considered well-known classical tests for power comparison. Next, we give a brief account of these tests.

2.6.1 Kolmogorov-Smirnov test (KS test)

The one-sample Kolmogorov-Smirnov (KS) test compares the empirical distribution function of a sample to a specified theoretical distribution. Hence, a one-sample KS test is used to determine if a sample comes from a specific distribution, such as normal, exponential, gamma and other distributions. The Kolmogorov-Smirnov test statistic is given by

$$KS = \max\{D^+, D^-\}$$

where

$$D^+ = \max_{i=1,2,\ldots,n}\left(\frac{i}{n} - \widehat{F}(X_{(i)})\right) \quad \text{and} \quad D^- = \max_{i=1,2,\ldots,n}\left(\widehat{F}(X_{(i)}) - \frac{i-1}{n}\right),$$

where $\widehat{F}(.)$ is the empirical distribution function given by $\widehat{F}(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq t)$.

2.6.2 Cramér-Von Mises test (CvM test)

The Cramér-Von Mises (CvM) test is another nonparametric test used to check whether a sample comes from a specified distribution. The test is based on the squared differences between the empirical distribution function and the cumulative distribution function of the specific distribution. The Cramer von Mises statistics is given by

$$CvM = \frac{1}{12n} + \sum_{i=1}^{n}\left(\widehat{F}(X_{(i)}) - \frac{2i-1}{2n}\right)^2.$$

2.6.3 Anderson Darling test (AD test)

The Anderson Darling test (AD test) is another nonparametric test used to see whether a sample comes from a specified distribution. The AD test is a modification of the Kolmogorov-Smirnov test and gives more weight to the tails of the distribution. The Anderson Darling test statistic is given by $AD = -n - S$, where

$$S = \sum_{i=1}^{n} \frac{2i-1}{n}[\ln \widehat{F}(X_{(i)}) + \ln(1 - \widehat{F}(X_{n+1-i}))].$$

2.6.4 Classical goodness of fit test for censored data

The goodness of fit tests discussed above consider the complete data only. Hence, to compare the efficiency of the proposed test in censored case, we generate a pseudo-complete random sample using the method proposed by Balakrishnan et al. (2015). Following Balakrishnan et al. (2015), to generate the pseudo-complete sample, for each Y_i with $\delta_i = 0$, a value $\hat{Y}_i$ is generated as $\hat{Y}_i = F_0^{-1}(\zeta_i)$, where $\zeta_i \sim U[F_0(C_i), 1)$, where $F_0(.)$ is the corresponding distribution function and C_i is the censoring time for $i = 1, 2, ..., n$. The corresponding values of censored observations in the original sample are then replaced by these generated values to obtain the pseudo-complete sample. The goodness of fit tests developed for the complete sample described above can be then applied to the pseudo-complete sample.

2.7 Steins identity and related characterization

Stein's identity for normal distribution and its applications has been well studied in statistical literature. Let X be a continuous random variable with finite mean μ and variance σ^2. Let $c(x)$ be a continuous function having first derivative. Then X has normal distribution with mean μ and variance σ^2 if and only if

$$E(c(X)(X - \mu)) = \sigma^2 E(c'(X)). \tag{2.8}$$

This has come to be known in literature as Stein's identity or Stein's lemma. Using Stein's type identity, Betsch and Ebner (2021) developed a fixed point characterization for lifetime distributions (See Theorem 1.2.1).

Using fixed point characterization, we developed goodness of fit tests for different life distributions. We discussed how the proposed tests can be modified to incorporate censored observations.

2.8 Jackknife empirical likelihood

Empirical likelihood is a non-parametric inference tool based on likelihood principle which was originally developed by Thomas and Grunkemeier (1975) for obtaining the confidence interval for survival probability in the presents of right censored observations. Pioneering papers by Owen (1988), Owen (1990) for finding the confidence interval of regression coefficients and several statistical functionals take the concept of empirical likelihood inference into a general methodology in statistical inference. The implementation of empirical likelihood inference is computationally challenging when the optimization of the non-parametric likelihood has nonlinear constraints such as U-statistics with higher degree (≥ 2) kernel. To overcome this computational difficulties, Jing et al. (2009) introduced the jackknife empirical likelihood (JEL) inference, which combines two popular non-parametric approaches, namely, the jackknife and the empirical likelihood method. Jing et al. (2009) illustrated the proposed methodology using one and two sample U-statistics. This approach has become popular due to its ability to effectively integrate the likelihood approach and the jackknife technique. We use JEL for developing goodness of fit test for standard distributions.

Chapter 3

Goodness of fit tests for the gamma distribution

3.1 Introduction

Lifetime data analysis involves the modeling of time-to-event data. Several parametric distributions are used to model lifetime data. Exponential, gamma, and Weibull distributions are some of the most commonly used models for analyzing lifetime data. In this context, it is important to check the validity of an assumed parametric model. The goodness of fit test is employed to validate the assumption that the lifetime data follows a particular distribution. A detailed study of goodness of fit tests for lifetime data is given in Lawless (2011) . The gamma distribution has great significance in lifetime distributions, due to its ability in modeling different aging patterns. It generalizes exponential, χ^2 and Erlang distributions. For applications of gamma distribution in lifetime data analysis one can refer to Barlow and Proschan (1996) and Deshpande and Purohit (2015) among many others. Some applications of gamma distribution include modeling rainfall data from Africa (Husak et al. (2007)) and analyzing vinyl chloride data from an environmental study Bhaumik et al. (2009) along with others. Several goodness of fit tests for the gamma distribution are available in the literature. Kallioras et al. (2006) proposed a method using the empirical moment-generating function to test the goodness of fit for the gamma distribution. A goodness of fit test based on the empirical Laplace

[1]The research work reported in this chapter is published in Vaisakh et al. (2025).

transform for the gamma distribution is developed by Henze et al. (2012). Villaseñor and González-Estrada (2015) suggested a variance ratio test for testing gamma distribution, while Baringhaus et al. (2017) proposed tests based on some independent properties of the gamma distribution. Recently, a new characterization for the gamma distribution and associated goodness of fit test has been developed by Betsch and Ebner (2019). Note that all these tests are developed for complete data. Anaya and O'Reilly (2001) developed goodness-of-fit tests for the inverse Gaussian and gamma distributions in the presence of censored data, extending classical empirical distribution function–based procedures to censored samples. As mentioned, censoring is a common phenomenon in lifetime analysis. Hence it is important to develop goodness-of-fit tests for the gamma distribution with the right-censored observations using characterization results. Using Stein's type identity, Betsch and Ebner (2019) developed a fixed point characterization for gamma distribution and then developed a goodness fit test for complete data. Using the same fixed point characterization, we develop U-statistic based goodness fit tests for gamma distribution for complete and right-censored data.

The rest of the chapter is organized as follows. In Section 3.2, we develop a new non-parametric test for the gamma distribution for complete data. In Section 3.3, we discuss how to incorporate right-censored observations in the proposed testing procedure. We obtain the asymptotic distribution of both test statistics. The results of the Monte Carlo simulation studies are presented in Section 3.4, which are carried to assess the finite sample performance of the proposed tests. In Section 3.5, the test procedures are illustrated using real data sets. Finally, Section 3.6 concludes the study.

3.2 Test statistic: complete data

In this section, we developed the test for complete data. Let X be a non-negative random variable having distribution function F. Then X has gamma distribution with parameter

k and λ (denoted as $\Gamma(k, \lambda)$), if its probability density function is given by

$$f(x) = \frac{\lambda^{-k}}{\Gamma(k)} x^{k-1} e^{-x/\lambda},\ k > 0,\ \theta > 0,\ x > 0.$$

We use the following fixed point characterization based on Steins's type identity for gamma distribution to develop the test.

Theorem 3.1. *The random variable X has a gamma distribution with parameter k and λ if and only if*

$$F(t) = E\left[\left(\frac{1-k}{X} + \frac{1}{\lambda}\right) \min(X, t)\right].$$

Next we discuss the test procedure. Consider a random sample $X_1, ..., X_n$ from F. Based on these sample we are interested in testing the null hypothesis

$$H_0 : F \in \Gamma(k, \lambda)$$

against the alternative hypothesis

$$H_1 : F \notin \Gamma(k, \lambda).$$

For testing the above hypothesis, using Theorem 3.2.1 we define a departure measure that discriminates between null and alternative hypotheses. Consider $\Delta_1(F)$ given by

$$\Delta_1(F) = \int_0^{\infty} \left(E\left[\left(\frac{1-k}{X} + \frac{1}{\lambda}\right) \min(X, t)\right] - F(t)\right) dF(t). \tag{3.1}$$

In view of Theorem 3.2.1, $\Delta_1(F)$ is zero under H_0 and non zero under H_1. Hence $\Delta_1(F)$ can be considered as a measure of departure from the null hypothesis H_0 towards the alternative hypothesis H_1. As we propose the test using the theory of U-statistics, first we simplify $\Delta_1(F)$ in terms of expectation of the function of random variables. For this purpose, consider

$$\begin{aligned}
\Delta_1(F) &= \int_0^\infty E\left[\left(\frac{1-k}{X}+\frac{1}{\lambda}\right)\min(X,t)\right]dF(t) - \int_0^\infty F(t)dF(t). \\
&= \int_0^\infty \int_0^\infty \left(\frac{1-k}{x}\min(x,t)+\frac{1}{\lambda}\min(x,t)\right)dF(x)dF(t) - 1/2 \\
&= \Delta_{11}(F) + \frac{1}{\lambda}E(\min(X_1,X_2)) - \frac{1}{2}, \qquad (3.2)
\end{aligned}$$

where $\Delta_{11}(F) = \int_0^\infty \int_0^\infty \frac{1-k}{x}\min(x,t)dF(x)dF(t)$.

Now

$$\begin{aligned}
\Delta_{11}(F) &= \int_0^\infty \int_0^\infty \frac{1-k}{x}\min(x,t)dF(x)dF(t). \\
&= (1-k)\int_0^\infty \int_0^\infty \frac{1}{x}\min(x,t)dF(x)dF(t). \\
&= (1-k)\int_0^\infty \left(\int_0^\infty I(x<t) + \frac{t}{x}I(t<x)\right)dF(x)dF(t). \\
&= (1-k)P(X_1<X_2) + (1-k)\int_0^\infty \int_t^\infty \frac{t}{x}dF(x)dF(t). \\
&= (1-k)\frac{1}{2} + (1-k)E\left[\frac{X_1}{X_2}I(X_1<X_2)\right], \qquad (3.3)
\end{aligned}$$

where $I(A)$ denote the indicator function of a set A. Substitute equation (5.5) in equation (6.2) we obtain

$$\begin{aligned}
\Delta_1(F) &= \frac{1-k}{2} - \frac{1}{2} + (1-k)E\left[\frac{X_1}{X_2}I(X_1<X_2)\right] + \frac{1}{\lambda}E(\min(X_1,X_2)) \\
&= \frac{1}{\lambda}E(\min(X_1,X_2)) + (1-k)E\left(\frac{X_1}{X_2}I(X_1<X_2)\right) - \frac{k}{2}. \qquad (3.4)
\end{aligned}$$

Next, we find the test statistic using U-statistics theory. Define $h_1(X_1,X_2) = \min(X_1,X_2)$. Then a U-statistic defined by

$$U_1 = \frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} h_1(X_i,X_j),$$

is an unbiased and consistent estimator of $E(\min(X_1, X_2))$. Define a symmetric kernel

$$h_2(X_1, X_2) = \frac{1}{2}\left(\frac{X_1}{X_2} I(X_1 < X_2) + \frac{X_2}{X_1} I(X_2 < X_1)\right).$$

Then a U-statistic defined by

$$U_2 = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} h_2(X_i, X_j),$$

is an unbiased and consistent estimator of $E\left(\frac{X_1}{X_2} I(X_1 < X_2)\right)$. Next we find moment estimators of k and λ. Denote $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{X}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Consistent estimators of λ and k are given by

$$\widehat{k} = \frac{(\bar{X})^2}{\bar{X}^2 - (\bar{X})^2}, \tag{3.5}$$

and

$$\widehat{\lambda} = \frac{\bar{X}^2 - (\bar{X})^2}{\bar{X}}. \tag{3.6}$$

Hence the test statistic is given by

$$\widehat{\Delta}_1 = \frac{U_1}{\widehat{\lambda}} + (1 - \widehat{k})U_2 - \frac{\widehat{k}}{2}.$$

We reject the null hypothesis H_0 against the alternative H_1 for large value of $\widehat{\Delta}_1$. Next, we obtain a normal-based critical region of the test.

Remark 3.2. *The test developed using the measure in (6.1) is not consistent. If the integrand in quantity (6.1) is squared, then the test would be consistent. However, proposing a test based on the U-statistic in that case is not easy.*

Next, we study the asymptotic properties of the test statistic. Since U_1 and U_2 are U-statistics they are consistent estimators of $E(\min(X_1, X_2))$ and $E\left(\frac{X_1}{X_2} I(X_1 < X_2)\right)$, respectively (Lehmann, 1951). Hence the following result is straightforward.

Theorem 3.3. *Let $\widehat{\lambda}$ and $\widehat{k}$ be the consistent estimators of λ and k, respectively. Under*

H_1, as $n \to \infty$, $\widehat{\Delta}_1$ converges in probability to $\Delta_1(F)$.

Next we obtain the asymptotic distribution of $\widehat{\Delta}_1$.

Theorem 3.4. *Let $\widehat{\lambda}$ and $\widehat{k}$ be the consistent estimators of λ and k, respectively. As $n \to \infty$, $\sqrt{n}(\widehat{\Delta}_1 - \Delta_1(F))$ converges in distribution to a normal random variable with mean zero and variance σ^2, where σ^2 is given by*

$$\sigma^2 = Var\Big(\frac{2X\bar{F}(X)}{\lambda}+\frac{2}{\lambda}\int_0^X ydF(y)+(1-k)X\int_X^\infty \frac{1}{y}dF(y)+\frac{(1-k)}{X}\int_0^X ydF(y)\Big). \quad (3.7)$$

Proof. Define

$$\tilde{\Delta}_1 = \frac{U_1}{\lambda} + (1-k)U_2.$$

Since $\widehat{\lambda}$ and $\widehat{k}$ are consistent estimators of λ and k, respectively, by Slutsky's theorem, the asymptotic distribution of $\sqrt{n}(\widehat{\Delta}_1 - \Delta_1(F))$ and $\sqrt{n}(\tilde{\Delta}_1 - E(\tilde{\Delta}_1))$ are same. Now we observe that $\tilde{\Delta}_1$ is a U-statistic with the symmetric kernel,

$$h(X_1, X_2) = \frac{1}{2}\left[\left(\frac{2\min(X_1, X_2)}{\lambda}\right) + (1-k)\left(\frac{X_1}{X_2}I(X_1 < X_2) + \frac{X_2}{X_1}I(X_2 < X_1)\right)\right].$$

Using the central limit theorem for U-statistics we have the asymptotic normality of $\tilde{\Delta}_1$. The asymptotic variance is $4\sigma_1^2$ where σ_1^2 is given by (Lee, 2019)

$$\sigma_1^2 = Var\left[E\left(h(X_1, X_2)|X_1\right)\right]. \quad (3.8)$$

Next we evaluate the variance expression in (6.9). Using indicator function we can express $\min(X_1, X_2)$ as $X_1 I(X_1 < X_2) + X_2 I(X_2 < X_1)$. Now, consider

$$\begin{aligned} E[2\min(x, X_2)] &= 2E[xI(x < X_2) + X_2 I(X_2 < x)] \\ &= 2xP(x < X_2) + 2\int_0^\infty yI(y < x)dF(y) \\ &= 2x\bar{F}(x) + 2\int_0^x ydF(y). \end{aligned} \quad (3.9)$$

Also

$$
\begin{aligned}
E[\frac{x}{X_2}I(x < X_2] + E[\frac{X_2}{x}I(X_2 < x)] &= xE[\frac{1}{X_2}I(x < X_2] + \frac{1}{x}E[X_2 I(X_2 < x)] \\
&= x\int_0^\infty \frac{1}{y}I(x < y)dF(y) + \frac{1}{x}\int_0^x ydF(y) \\
&= x\int_x^\infty \frac{1}{y}dF(y) + \frac{1}{x}\int_0^x ydF(y). \qquad (3.10)
\end{aligned}
$$

Substituting equations (6.10) and (6.12) in equation (6.9) we obtain the variance expression as specified in the theorem. □

Under the null hypothesis H_0, $\Delta_1(F) = 0$. Hence we have the following corollary.

Corollary 3.5. *Under H_0, as $n \to \infty$, $\sqrt{n}\widehat{\Delta}_1$ converges in distribution to a normal random variable with mean zero and variance σ_0^2, where σ_0^2 is obtained by evaluating (6.7) under H_0.*

An asymptotic critical region of the scale-invariant test can be obtained using Corollary 3.2.1. Let $\widehat{\sigma}_0^2$ be a consistent estimator of σ_0^2. We reject the null hypothesis H_0 against the alternative hypothesis H_1 at a significance level α, if

$$
\frac{\sqrt{n}|\widehat{\Delta}_1|}{\widehat{\sigma}_0} > Z_{\alpha/2},
$$

where Z_α is the upper α-percentile point of a standard normal distribution. Since it is difficult to find a consistent estimator of σ_0^2, we obtain the critical region of the test using the bootstrap procedure. We determine lower (c_1) and upper (c_2) quantiles in such a way that $P(\widehat{\Delta}_1 < c_1) = P(\widehat{\Delta}_1 > c_2) = \alpha/2$. The finite sample performance of the test is evaluated through a Monte Carlo simulation study and the results are reported in Section 4.

Remark 3.6. *A scale-invariant test can be developed using $X_i/\bar{X}$, $i = 1, 2, \ldots, n$. We can set $\lambda = 1$ while finding the asymptotic null variance in this case.*

3.3 Test statistic: censored data

Next, we discuss how the censored observations can be incorporated in the proposed testing procedure. Consider the right-censored data (Y, δ), with $Y = \min(X, C)$ and $\delta = I(X \le C)$, where C is the censoring time. We assume censoring times and lifetimes are independent. Now we develop test discussed in Section 3.1 based on n independent and identical observations on $\{(Y_i, \delta_i), 1 \le i \le n\}$. Since we consider U-statistics theory for right censored data, we use the same departure measure $\Delta_1(F)$ given in (6.1). Hence, we rewrite (3.4), a simplified form of (6.1) as

$$\Delta_1(F) = \frac{1}{\lambda}E(\min(X_1, X_2)) + (1-k)E\left(\frac{X_1}{X_2}I(X_1 < X_2)\right) - kP(X_1 < X_2). \quad (3.11)$$

To develop the test statistic for right censored case, we estimate each quantity in (5.16) using U-statistics for right censored data (Datta et al., 2010). An estimator of $E(\min(X_1, X_2))$ is given by

$$\widehat{\Delta}_{11c} = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\min(Y_i, Y_j)\delta_i\delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}, \quad (3.12)$$

provided $\widehat{K}_c(Y_i) > 0$ and $\widehat{K}_c(Y_j) > 0$, with probability 1 and $\widehat{K}_c$ is the Kaplan-Meier estimator of K_c, the survival function of C. Again, an estimator of $E\left(\frac{X_1}{X_2}I(X_1 < X_2)\right)$ is given by

$$\widehat{\Delta}_{12c} = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{(\frac{X_i}{X_j}I(X_i < X_j) + \frac{X_j}{X_i}I(X_j < X_i))\delta_i\delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}. \quad (3.13)$$

An estimator of $P(X_1 < X_2)$ is given by

$$\widehat{\Delta}_{13c} = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{(I(X_i < X_j) + I(X_j < X_i))\delta_i\delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}. \quad (3.14)$$

Similarly, the estimators of $E(X)$ and $E(X^2)$ are given by (Datta, 2005)

$$\bar{X}_c = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i\delta_i}{\widehat{K}_c(Y_i)}.$$

and

$$\bar{X}_c^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i^2\delta_i}{\widehat{K}_c(Y_i)}.$$

Using $\bar{X}_c$ and $\bar{X}_c^2$ we obtain the estimators of k and λ. Moment estimators of k and λ are given by

$$\widehat{k}_m = \frac{E^2(X)}{Var(X)},$$

and

$$\widehat{\lambda}_m = \frac{Var(X)}{E(X)}.$$

Hence under right censored case, we estimate these quantities by

$$\widehat{k}_c = \frac{(\bar{X}_c)^2}{\bar{X}_c^2 - (\bar{X}_c)^2}, \tag{3.15}$$

and

$$\widehat{\lambda}_c = \frac{\bar{X}_c^2 - (\bar{X}_c)^2}{\bar{X}_c}. \tag{3.16}$$

Since $\bar{X}_c$ and $\bar{X}_c^2$ are consistent estimators of $E(X)$ and $E(X^2)$, we can easily verify that $\widehat{k}_c$ and $\widehat{\lambda}_c$ are consistent estimators of k and λ, respectively. Using the estimators given in equations (5.16-3.16), we obtain the test statistic in right-censored case as

$$\widehat{\Delta}_{1c} = \frac{\widehat{\Delta}_{11c}}{\widehat{\lambda}_c} + (1-\widehat{k}_c)\widehat{\Delta}_{12c} - \widehat{k}_c\widehat{\Delta}_{13c}. \tag{3.17}$$

We reject H_0 in favour of H_1 for large values of $\widehat{\Delta}_{1c}$. We obtain an asymptotic critical region based on normal approximation. Next, we obtain the limiting distribution of $\widehat{\Delta}_{1c}$. Let $N_i^c(t) = I(Y_i \leq t, \delta_i = 0)$ be the counting process corresponds to the censoring variable C_i, $R_i(t) = I(Y_i \geq t)$. Also, let λ_c be the hazard rate of C. The martingale associated with this counting process $N_i^c(t)$ is given by

$$M_i^c(t) = N_i^c(t) - \int_0^t R_i(u)\lambda_c(u)du.$$

Let $G(x, y) = P(X_1 \le x, Y_1 \le y, \delta = 1), x \in \mathcal{X}$, $\bar{H}(t) = P(Y_1 > t)$ and

$$w(t) = \frac{1}{\bar{H}(t)} \int_{\mathcal{X}\times[0,\infty)} \frac{h_1(x)}{K_c(y-)} I(y > t)dG(x, y),$$

where $h_1(x) = E(h(X_1, X_2)|X_1 = x)$. The proof of next result follows from Theorem 1 of Datta et al. (2010) for a particular choice of the kernel.

Theorem 3.7. *Let*

$$h_1(x) = \frac{1}{2}E\left[\left(\frac{2\min(x, Y_2)}{\lambda}\right) + (1-k)\left(\frac{x}{Y_2}I(x < Y_2) + \frac{Y_2}{x}I(Y_2 < x)\right) - k\left(I(x < Y_2) + I(Y_2 < x)\right)\right].$$

Suppose the conditions

$$E\left[\frac{1}{2}\left(\left(\frac{2\min(Y_1, Y_2)}{\lambda}\right) + (1-k)E\left(\frac{Y_1}{Y_2}I(Y_1 < Y_2) + \frac{Y_2}{Y_1}I(Y_2 < Y_1)\right) - k\right)\right]^2 < \infty,$$

$\int_{\mathcal{X}\times[0,\infty)} \frac{h_1^2(x)}{K_c^2(y)} dG(x, y) < \infty$ *and* $\int_0^\infty w^2(t)\lambda_c(t)dt < \infty$ *holds. As* $n \to \infty$, $\sqrt{n}(\widehat{\Delta}_{1c} - \Delta_1(F))$ *converges in distribution to a Gaussian random variable with mean zero and variance* $4\sigma_c^2$, *where* σ_c^2 *is given by*

$$\sigma_c^2 = Var\Big(\frac{h_1(X)\delta_1}{K_c(Y_1-)} + \int w(t)dM_1^c(t)\Big).$$

Corollary 3.8. *Suppose the assumptions stated in Theorem 4 hold. Also, let* σ_{0c}^2 *be the value of* σ_c^2 *evaluated under* H_0. *Under* H_0, *as* $n \to \infty$, $\sqrt{n}\widehat{\Delta}_{1c}$ *converges in distribution to a Gaussian random variable with mean zero and variance* $4\sigma_{c0}^2$.

Next, we find an estimator of σ_{c0}^2 using the reweighed techniques. An estimator of σ_{c0}^2 is given by

$$\widehat{\sigma}_{c0}^2 = \frac{4}{(n-1)}\sum_{i=1}^{n}(V_i - \bar{V})^2,$$

where

$$V_i = \frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} + \widehat{w}(X_i)(1-\delta_i) - \sum_{j=1}^{n} \frac{\widehat{w}(X_i)I(X_i > X_j)(1-\delta_i)}{\sum_{i=1}^{n} I(X_i > X_j)},$$

$$\bar{V} = \frac{1}{n}\sum_{i=1}^{n} V_i, \quad \widehat{h}_1(X) = \frac{1}{n}\sum_{i=1}^{n} \frac{h(X,Y_i)\delta_i}{\widehat{K}_c(Y_i-)}, \quad R(t) = \sum_{i=1}^{n} I(Y_i > t)$$

and

$$\widehat{w}(t) = \frac{1}{R(t)}\sum_{i=1}^{n} \frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} I(X_i > t).$$

Under the right censored situation, we reject the null hypothesis H_0 against the alternative hypothesis H_1 at a significance level α, if

$$\frac{\sqrt{n}|\widehat{\Delta}_{1c}|}{\widehat{\sigma}_{c0}} > Z_{\alpha/2}.$$

The results of the Monte Carlo simulation, which assess the finite sample performance of the test are also reported in Section 3.4.

3.4 Empirical evidence

To evaluate the finite sample performance of the proposed test procedures, we conduct a Monte Carlo simulation study using R software. To show the competitiveness of our test with the existing test procedures for complete data, we compare the empirical powers of the same. In the censored case, we evaluate the power of our test against different alternatives.

3.4.1 Uncensored case

We find the empirical type I error and empirical power of the proposed test and other acknowledged tests. The algorithm used to find the empirical power of the proposed test can be summarized as follows.

1. Generate lifetime data from the desired alternative and calculate the value of $\widehat{\Delta}_1$.

2. Find the value of the estimators $\widehat{k}$ and $\widehat{\lambda}$ using the data obtained in Step 1.

3. Generate a sample of size n from $\Gamma(\widehat{k}, \widehat{\lambda})$.

4. Obtain the bootstrap distribution of the test statistic with 500 bootstrap samples obtained from the data generated in Step 3 and determine the critical point.

5. Repeat Steps 1-4 10000 times and calculate empirical power as the proportion of significant test statistics.

First we find the empirical type I error of the test. We generate lifetimes from a gamma distribution with different sample sizes $n = 25, 50, 75, 100, 200$ to calculate the empirical type I error. To find the empirical power, lifetime random variables are generated from different choices of alternatives including Weibull, lognormal and Pareto distributions where the distribution functions are;
Weibull distribution: $F(x) = 1 - e^{(-x/\alpha)^{\beta}}$, $x > 0, \alpha,\ \beta > 0,$
Pareto distributions: $F(x) = 1 - (\beta/x)^{\alpha},\ x \geq \beta, \alpha,\ \beta > 0,$
Lognormal distribution: $F(x) = \Phi(\frac{\ln x - \mu}{\sigma})$, $x > 0, -\infty < \mu < \infty, \sigma^2 > 0,$
where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable. We also find the empirical power for linear failure rate and Makeham distributions where the distribution functions are;
Linear failure rate distribution: $F(x) = 1 - e^{(-x - \frac{\alpha}{2}x^2)}$, $x > 0, \alpha > 0,$
Makeham distribution: $F(x) = 1 - e^{-\alpha x - \beta(e^{-\alpha x} + x - 1)}$, $x > 0, \alpha,\ \beta > 0.$

In our simulation, we consider the distributions having different types of hazard rates. Weibull distribution has increasing (decreasing) hazard rate for $\beta > 1 (< 1)$ and the hazard rate is constant for $\beta = 1$. We consider the Weibull distribution with $\alpha = 1$ and $\beta = 2$, which has an increasing hazard rate. Linear failure rate and Makeham distributions have increasing hazard rates. Pareto distribution always has a decreasing hazard rate. The hazard rate of log-normal distribution has a pattern: increasing, decreasing and upside-down bathtub. A detailed discussion of the hazard rate for lognormal distribution can be found in Kurniasari et al. (2019). We consider the lognormal distribution with parameters $m = 2$ and $\sigma = 1$, which corresponds to decreasing hazard rate.

We compare the performance of our test with the goodness of fit test for gamma distribution proposed by Henze et al. (2012) and Betsch and Ebner (2019) and also with the well-known Kolmogorov Smirnov (KS) test, Cramer von Mises (CvM) test and Anderson-Darling (AD) test. The test statistic by Henze et al. (2012) is given by

$$HME = \int_0^\infty Z_{1n}^2(t)w(t)dt,$$

where $Z_{1n}(t) = \sqrt{n}\left[(1+t)L_n^{'}(t) + \widehat{k}L_n(t)\right]$ with $L_n(t)$ as the empirical Laplace transform defined as $L_n(t) = \frac{1}{n}\sum_{i=1}^{n}\exp(-tY_j)$, $w(t)$ is a weight function satisfying some predefined conditions and $Y_i = \frac{X_i}{\widehat{\lambda}}, i = 1, 2, .., n.$ Test statistic proposed by Betsch and Ebner (2019) based on fixed point characterization is given by

$$BE = \int_0^\infty Z_{2n}^2(t)w(t)dt,$$

where $Z_{2n}(t) = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(\frac{1-\hat{k}}{Y_i}+1)\min(Y_i, t) - \frac{1}{n}\sum_{i=1}^{n} I(Y_i \leq t)\right]$. The value of the Henze tests and Betsch and Ebner tests are obtained using the R-Package ‘gofgamma’. Critical values of these tests are obtained using the parametric bootstrap procedure, as in the case of the proposed test. This guaranteed that the type I error of all the tests considered here attained a specified significance level.

We observe that the size can not be achieved, if we use the estimators of k and λ given in (3.5) and (3.6) to find $\widehat{\Delta}_1$. Hence as suggested by Henze and Klar (2002), we consider the approximate maximum likelihood estimators $\widehat{\lambda} = \bar{X}/\widehat{k}$ and

$$\widehat{k} = \begin{cases} \frac{0.5000876+0.1648852R_n-0.0544274R_n^2}{R_n}, & \text{if } 0 < R_n \leq 0.5772 \\ \frac{8.898919+9.059950R_n+09775373R_n^2}{R_n(17.79728+11.968477R_n+R_n^2)}, & \text{if } 0.5772 < R_n \leq 17 \\ \frac{1}{R_n}, & \text{if } 17 < R_n, \end{cases}$$

where $R_n = \log(\bar{X}) - \frac{1}{n}\sum_{i=1}^{n}\log(X_i)$.

The results of the simulation study are given in Tables 3.1-3.6. In Table 3.1, we

Table 3.1: Empirical power: Log normal distribution ($\mu = 2, \sigma = 1$)

	$\widehat{\Delta}_1$		HME		BE		WA		KS		CvM		AD	
n/α	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
25	1.0000	1.0000	0.1448	0.3342	0.1576	0.3329	0.9652	0.9987	0.8921	0.9978	0.9993	1.0000	0.9234	0.9989
50	1.0000	1.0000	0.4028	0.6456	0.4089	0.6226	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
75	1.0000	1.0000	0.5788	0.7923	0.5669	0.7621	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	0.9962	1.0000	0.9987	0.9991	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3.2: Empirical power: Pareto distribution ($\alpha = 2, \beta = 1$)

	$\widehat{\Delta}_1$		HME		BE		WA		KS		CvM		AD	
n/α	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
25	0.8128	0.8621	0.7089	0.8166	0.5828	0.8534	0.8109	0.9117	0.7986	0.8561	0.8732	0.9437	0.8871	0.9926
50	0.8621	0.9123	0.9824	0.9956	0.9717	0.9982	0.9498	0.9671	0.8901	0.9877	0.9658	0.9799	0.9677	0.9898
75	0.8911	0.9891	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	0.9822	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

report the empirical type I error, and in Tables 3.2-3.6, we present the empirical power against different alternatives. From Table 3.1, we observe that the size of the test attains the chosen level of significance. From Tables 3.2-3.6, we observe that the test has good power against all choices of alternatives. The empirical power increases as the sample size increases. We can see that the newly proposed test performs better than other tests in most of the cases we considered. When the alternative is a Pareto distribution, we note other tests perform better than our test for small sample sizes. The proposed test has high power even for small sample sizes, which affirms the efficiency of the test.

3.4.2 Censored case

We calculate empirical type I error and power of the test statistic proposed for right censored data using Monte Carlo simulation studies. To calculate the empirical type I error, lifetimes are generated from the gamma distribution. We considered the same alternatives as in the uncensored case for finding the empirical power. Here, the censoring percentages are chosen to be 20% and 40%. In all cases, the censoring random variable C is generated from an exponential distribution with parameter b, where b is chosen such that $P(T > C) = 0.2(0.4)$. The re-weighting technique explained in Section 3.2 is used to estimate the asymptotic null variance of $\widehat{\Delta}_{1c}$. The results of the simulation study are presented in Tables 3.7 and 3.8.

We can see that the empirical power of the test approaches the chosen level of signif-

Table 3.3: Empirical power: Weibull distribution ($\alpha = 1, \beta = 2$)

	$\widehat{\Delta}_1$		HME		BE		WA		KS		CvM		AD	
n/α	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
25	0.3542	0.6812	0.0761	0.1849	0.1289	0.1639	0.0712	0.4063	0.1271	0.3843	0.0785	0.4122	0.2569	0.5671
50	0.6791	0.8921	0.1821	0.2641	0.2312	0.2922	0.5720	0.8632	0.5133	0.8154	0.5821	0.8726	0.5809	0.8921
75	0.9409	9.9952	0.2302	0.3431	0.2654	0.3847	0.9002	0.9861	0.8289	0.9802	0.9143	0.9952	0.8901	0.9881
100	1.0000	1.0000	0.3236	0.4790	0.3852	0.4387	0.9763	0.9945	0.9442	0.9987	0.9845	1.0000	0.9872	1.0000
200	1.0000	1.0000	0.5305	0.6819	0.5121	0.6334	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3.4: Empirical power: Linear failure rate distribution ($\alpha = 3$)

	$\widehat{\Delta}_1$		HME		BE		WA		KS		CvM		AD	
n/α	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
25	0.3489	0.5076	0.0983	0.2176	0.2761	0.3560	0.0901	0.3185	0.1652	0.2508	0.0963	0.3216	0.2871	0.4522
50	0.4987	0.7184	0.1851	0.3672	0.3819	0.4802	0.3466	0.6151	0.2629	0.3876	0.3671	0.6231	0.5401	0.7833
75	0.6504	0.9080	0.3219	0.5802	0.5488	0.5757	0.4718	0.8301	0.7218	0.7987	0.4890	0.8752	0.8451	0.9101
100	0.9167	0.9781	0.4690	0.6600	0.6704	0.7621	0.8603	0.9811	0.9581	0.9987	0.8701	0.9832	0.9982	1.0000
200	1.0000	1.0000	0.5671	0.7297	0.7487	0.8732	0.9848	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

icance as n increases. The performance of the test is good in terms of empirical power. From Tables 3.7 and 3.8, we observe that the power of the test increases with sample size and decreases with the censoring percentage.

3.5 Data analysis

The proposed test procedures are illustrated using several real data sets.

3.5.1 Complete data

We consider two data sets for the analysis. To find the critical region, we use the following algorithm.

1. Estimate the parameters of gamma distribution k and λ from the observed data.

2. Generate a random sample from the gamma distribution using the estimated parameters in Step 1.

Table 3.5: Empirical power: Makeham distribution ($\alpha = 0.3, \beta = 0.1$)

	$\widehat{\Delta}_1$		HME		BE		WA		KS		CvM		AD	
n/α	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
25	0.9561	0.9989	0.7560	0.8561	0.6791	0.8828	0.8810	0.9022	0.7979	0.8890	0.8891	0.9128	0.8391	0.8905
50	0.9807	1.0000	0.9566	0.9898	0.8790	0.9732	0.9533	0.9750	0.9453	0.9871	0.9662	0.9985	0.9732	0.9899
75	1.0000	1.0000	0.9861	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3.6: Empirical type I error and power of the test when 20% of lifetimes are censored.

	Gamma (1,1)		Lognormal (2,1)		Weibull (2,1)		Pareto (2,1)		LFR(3)		Mak (0.3,0.1)	
n/α	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
50	0.0132	0.0461	0.9863	0.9916	0.9891	0.9948	0.8381	0.8892	0.6752	0.8901	0.9652	0.9871
75	0.0121	0.0489	0.9992	1.0000	1.0000	1.0000	0.8675	0.9543	0.9483	0.9888	0.9822	9.9983
100	0.0118	0.0508	1.0000	1.0000	1.0000	1.0000	0.9782	0.9999	0.9971	1.0000	1.0000	1.0000
200	0.0104	0.0497	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

3. Obtain the bootstrap distribution of the test statistic with 10000 bootstrap samples from the data generated in Step 2 and determine the critical points.

Table 3.7: Empirical type I error and power of the test when 40% of lifetimes are censored.

	Gamma(1,1)		Lognormal (2,1)		Weibull (2,1)		Pareto (2,1)		LFR(3)		Mak (0.3,0.1)	
n/α	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
50	0.0140	0.0514	0.5248	0.6069	0.9594	0.9847	0.7893	0.8456	0.5657	0.8219	0.6912	0.7341
75	0.0112	0.0489	0.6785	0.7494	0.9976	0.9995	0.8175	0.8893	0.9053	0.9511	0.8291	0.8908
100	0.0108	0.0507	0.8044	0.8798	1.0000	1.0000	0.9274	0.9632	0.9780	1.0000	0.8901	0.9755
200	0.0106	0.0505	0.9627	0.9778	1.0000	1.0000	0.9461	0.9874	1.0000	1.0000	0.9918	0.9997

Illustration 1: We consider the data on survival times in weeks for 20 male rats that were exposed to a high level of radiation. The data is discussed in Lawless (2011) in Example 4.2.1. We choose a data with small sample size to make sure that our test is suitable for any sample size. The test statistic is obtained as 0.0964 where the 5% level critical values are 0.0456 and 1.2664 and 1% level critical values are -0.0055 and 2.1042 respectively. Hence we accept H_0 that the data on survival times of male rats follow a gamma distribution.

Illustration 2 : We also examine the lifetime data on the number of millions of revolutions before failure for 23 ball bearings. The data is studied in Lawless (2011) in the context of the goodness of fit tests for parametric models and given in Example 3.3.1. We obtain the test statistic as -0.0379 where the 5% level critical values are 0.0956 and 1.2978 and 1% level critical values are 0.0385 and 2.0577. Hence we reject the null hypothesis that the data follows gamma distribution.

3.5.2 Censored data

Two real data sets are considered for illustrating the proposed test procedure. We use the normal-based critical region given in Section 3 to make a decision. The asymptotic null variance of $\widehat{\Delta}_{1c}$ is estimated using the re-weighting techniques explained in Section 3.3.

Illustration 1: We analyse stanford heart transplant data available in R software named 'stanford2' to test for gamma assumption. The data consist of 184 lifetimes where 72 of them are censored lifetimes. The censoring percentage of data is 38.5%. The test statistic is calculated as 0.3503. So we accept the null hypothesis that the data follow gamma distribution at both 1% and 5% levels of significance.

Illustration 2: We examine the data on the lifetimes of disk brake pads on 40 cars studied in Lawless (2011). The complete data set is given in Table 6.11, Page 337 of Lawless (2011). Out of the 40 observed lifetimes, 9 are censored lifetimes, hence data contains 22.5% of censored observations. The test statistic is obtained as 3.5142. Hence we reject the hypothesis of gamma distribution assumption for this data at both 1% and 5% levels of significance.

3.6 Concluding remarks

Based on fixed point characterization associated with Stein's type identity, we developed a new goodness of fit test for gamma distribution. We studied the asymptotic properties of the proposed test statistic. The proposed test has a well-controlled error rate. The power of the test is compared with the recently developed tests for gamma distribution. The applicability of the proposed test is illustrated using two real data sets.

Even though several tests are available for gamma distribution in literature, to our knowledge, all of these tests are developed for complete data. Motivated by this, we developed a new goodness of fit test for gamma distribution with the right censored data. We proved that the asymptotic distribution of the proposed test statistic for right censored data is normal. We also find a consistent estimator of the asymptotic variance. The finite sample performance of the test is evaluated through a Monte Carlo simulation study. Apart from right censoring, truncation and other types of censoring are common in lifetime data analysis. The proposed test can be modified to incorporate these situations.

Chapter 7

Conclusions and Future works

7.1 Conclusions

In this thesis, we addressed the problem of developing new goodness of fit tests for commonly employed lifetime distributions in two scenarios; when complete data is available as well when the observations are right censored. We consider gamma, Rayleigh, inverse Gaussian and Lindley distributions in our study. For each distribution, we investigated the Stein type identity and then employed the theory of U-statistics to construct the test statistics, investigated their asymptotic properties. We validated the performance of the proposed procedures through extensive Monte Carlo simulation studies, including comparison with the existing tests. The practical usefulness of the tests were further demonstrated with applications to real-life data sets. The main contributions of the thesis are summarized below.

We developed a new goodness-of-fit test for the Gamma distribution using the fixed-point characterization in the thirds chapter. The test statistic was derived based on the theory of U-statistics. We discussed how right-censored observations were incorporated into the proposed test and studied the asymptotic properties of the test statistic in both censored and uncensored cases. Extensive Monte Carlo simulations were carried out to assess the performance of the proposed test, and its application was illustrated using several real data sets.

We proposed a new goodness-of-fit test for the Rayleigh distribution for both complete and right-censored data in the fourth chapter. Using U-statistic theory, we first developed the test for complete data and then extended it to incorporate right-censored observations. The asymptotic properties of the test statistic were examined in detail for both censored and uncensored cases. Monte Carlo simulation studies were conducted to evaluate the performance of the test. Real data applications were presented, and in addition, we introduced a goodness of fit test based on the jackknife empirical likelihood for the standard Rayleigh distribution.

In the fifth chapter, we presented a novel goodness-of-fit test for the Inverse Gaussian distribution using the fixed-point characterization. The test statistic was derived using the theory of U-statistics. The performance of the proposed test was investigated through Monte Carlo simulation studies, and the procedure was illustrated with real-life datasets. In the sixth chapter, we developed a new goodness-of-fit test for the Lindley distribution, applicable to both complete and right-censored data. Using U-statistic theory, we first constructed the test for complete data and then extended the approach to handle right-censored observations. The asymptotic properties of the test statistic were established for both censored and uncensored cases. An extensive Monte Carlo simulation study was carried out to evaluate the performance of the proposed test, and its application was illustrated with real data sets.

7.2 Future works

In this thesis, we considered goodness of fit tests for lifetime data subject to right censoring. However, in lifetime data analysis, observations may also be subject to other forms of censoring, such as left censoring, double censoring, and interval censoring, as well as left and right truncation. Most of the existing goodness-of-fit tests available in the literature are designed for complete data only. Hence, it is essential to develop new goodness of fit tests that can incorporate these diverse censoring and truncation schemes. The proposed procedures in this thesis can be suitably modified to accommodate such cases, and further advancements in this direction provide a rich scope for future research.

In many situations, are observed in discrete units, where continuous lifetime models are inadequate. The commonly used discrete lifetime distributions include geometric. Discrete Weibull and Poisson distributions. Only sparse literature is available on goodness of fit tests for discrete lifetime distributions. U-statistics based goodness of fit tests, combined with bootstarp and other resampling techniques can be developed to fill this research gap. Theses tests can be extended to handle censoring and truncation, which points into a promising yet unexplored area of future research.

In lifetime studies it is common to observe a covariate vector corresponding to each individual, to represent the heterogeneity in the population. Covariate values often have a significant effect on lifetime and various regression models are used to study this relationship. In this thesis, we consider lifetimes without covariates only. Evaluating the goodness of fit of regression models is very much important to ensure the accuracy of the estimated regression parameters. In most of the available goodness of fit tests for regression models, it is assumes that the lifetime is completely observed, which highlights the importance of developing goodness of fit tests for different regression models under various censoring schemes and truncation. We can use martingale-type transformation of residual empirical process to develop the goodness of fit test under right censored situations. We can also proceed in similar ways to develop a goodness of fit test for general linear models under censoring and truncation.

The rapid advancement of computational techniques, opens up the opportunity to integrate modern machine learning and data driven approaches into goodness of fit test problems in lifetime data. Methods such as ensemble learning, neural networks, and Bayesian nonparametric models can be explored to develop adaptive and robust tests that perform well under complex data structures. Simulation based frameworks and scalable algorithms can be explored to enhance the applicability of such tests to large datasets encountered in biomedical studies, reliability engineering, and social sciences. The integration of classical statistical methodologies with modern computational tools offers promising areas for further advancement in the field.

Chapter 4

Goodness of fit tests for Rayleigh distribution

4.1 Introduction

Parametric approach for lifetime data often requires the assumption that the data follows a particular distribution. If this assumption about data can be proven to be right, it makes the further analysis easy. Goodness of fit tests are used to check whether the data follows a particular distribution. Some widely used parametric models in lifetime data analysis include Exponential, gamma and Weibull distributions. One other important lifetime distribution is Rayleigh distribution, which has its origin in an acoustics problem, where a study was conducted on the resultant of a large number of sound waves with differing phases (Rayleigh (1880)). We can note that Rayleigh distribution is derived in literature as a special case of Weibull distribution with shape parameter 2. The distribution has been implemented in the fields of astronomy, astrophysics and environmental sciences (see Fang and Margot (2012); Bovaird and Lineweaver (2017);Morgan et al. (2011); Celik (2004). The uses of Rayleigh distribution in survival analysis and reliability theory is studied inLawless (2011).

The goodness of fit tests for testing the hypothesis that the observed data follows Rayleigh distribution, have been developed and studied in literature. These include tests based on the empirical Laplace transform (Meintanis and Iliopoulos (2003)), entropy

[1]The research work reported in this chapter is published in Vaisakh et al. (2023).

(Baratpour and Khodadadi (2013);Alizadeh Noughabi et al. (2014)), the Hellinger distance (Jahanshahi et al. (2016)), a phi-divergence measure (Mahdizadeh and Zamanzade (2017)), moments (Best et al. (2010)) and the empirical likelihood ratio (Safavinejad et al. (2015)). The characterizations for Rayleigh distribution has also been studied by many authors which include, the conditional expectation characterization by Ahsanullah and Shakil (2013), the characterization based on record values by Nanda (2010) and the entropy based characterization by Baratpour and Khodadadi (2013). Recently, Ahrari et al. (2022) developed a quantile based test and and Liebenberg et al. (2022) proposed test based on conditional expectation characterization for Rayleigh distribution. However, all theses tests and studies were developed for complete data. Since, lifetime data involve censoring in most of the cases, it is important to develop a goodness of fit test for Rayleigh distribution, incorporating right censored observations. As mentioned, Betsch and Ebner (2021) developed a fixed point characterization for the continuous probability distribution. We use fixed point characterization for Rayleigh distribution to develop a goodness of fit test for same. Then we discuss, how the testing procedure can be extended to incorporate right-censored observations.

The rest of the article is organized as follows. In Section 2, making use of Stein's type identity for the Weibull distribution, we develop a U-statistic based goodness of fit test for Rayleigh distribution. In Section 3, we discuss how to incorporate the right censored observations in the proposed testing procedure. Extensive Monte Carlo Simulation studies are carried out in Section 4 to assess the performance of the tests in a finite sample. The illustrations of the proposed methods using real data sets are reported in Section 5. A jackknife empirical likelihood ratio test is developed for standard Rayleigh distribution in Section 6. Finally, we conclude the study with a discussion on future works in Section 7.

4.2 Test statistics: complete data

In this section, we develop a goodness of fit test for Rayleigh distribution. We use fixed point characterization based on Stein's type identity for Rayleigh distribution to develop the test.

Theorem 4.1. *The random variable X with distribution function F has Rayleigh distribution with parameter σ if and only if*

$$F(t) = E\left[\left(\frac{x^2 - \sigma^2}{x\sigma^2}\right)\min(X, t)\right]$$

4.2.1 U-statistics based test

Based on a random sample $X_1, ..., X_n$ from F, we are interested in testing the null hypothesis

$$H_0 : F \text{ has Rayleigh distribution.}$$

against

$$H_1 : F \text{ does not follow Rayleigh distribution.}$$

For testing the above hypothesis, first we define a departure measure that discriminates between the null and alternative hypotheses. Consider $\Delta_2(F)$ given by

$$\Delta_2(F) = \int_0^\infty \left(E\left[\left(\frac{x^2 - \sigma^2}{x\sigma^2}\right)\min(X, t)\right] - F(t)\right) dF(t). \tag{4.1}$$

In view of Theorem 4.2.1, $\Delta_2(F)$ is zero under H_0 and not zero under H_1. Hence $\Delta_2(F)$ can be considered as a measure of departure from the null hypothesis H_0 towards the alternative hypothesis H_1.

Now we simplify $\Delta_2(F)$ in terms of the expectation of the function of random variables to employ U-statistic theory. Consider

$$\begin{aligned}
\Delta_2(F) &= \int_0^\infty E\left[\left(\frac{x^2-\sigma^2}{x\sigma^2}\right)\min(X,t)\right]dF(t) - \int_0^\infty F(t)dF(t). \\
&= \int_0^\infty\int_0^\infty \left(\frac{x^2-\sigma^2}{x\sigma^2}\right)\min(x,t)dF(x)dF(t) - 1/2 \\
&= \frac{1}{\sigma^2}\int_0^\infty\int_0^t \left(x^2-\sigma^2\right)dF(x)dF(t) \\
&\qquad + \frac{1}{\sigma^2}\int_0^\infty\int_t^\infty \left(x^2-\sigma^2\right)\frac{t}{x}dF(x)dF(t) - 1/2 \\
&= \frac{1}{\sigma^2}\int_0^\infty\int_0^t x^2 dF(x)dF(t) \\
&\qquad + \frac{1}{\sigma^2}\int_0^\infty\int_t^\infty \left(x^2-\sigma^2\right)\frac{t}{x}dF(x)dF(t) - 1 \\
&= \frac{1}{\sigma^2}(\Delta_{21}(F)+\Delta_{22}(F)) - 1 \ (say). \qquad (4.2)
\end{aligned}$$

Now, changing the order of integration, we have

$$\begin{aligned}
\Delta_{21}(F) &= \int_0^\infty\int_0^t x^2 dF(x)dF(t) \\
&= \int_0^\infty x^2\int_x^\infty dF(t)dF(x) \\
&= \int_0^\infty x^2\bar{F}(x)dF(x) \\
&= \frac{1}{2}E\left(\min(X_1,X_2)^2\right). \qquad (4.3)
\end{aligned}$$

Again

$$\begin{aligned}
\Delta_{22}(F) &= \int_0^\infty\int_t^\infty \left(x^2-\sigma^2\right)\frac{t}{x}dF(x)dF(t) \\
&= \int_0^\infty\int_t^\infty tx dF(x)dF(t) - \sigma^2\int_0^\infty\int_t^\infty \frac{t}{x}dF(x)dF(t) \\
&= E\left((X_1X_2)I(X_2<X_1)\right) - \sigma^2 E\left(\frac{X_2}{X_1}I(X_2<X_1)\right) \qquad (4.4)
\end{aligned}$$

Substituting (5.5) and (6.3) in (6.2), we obtain

$$\Delta_2(F) = \frac{1}{\sigma^2}E\left(\min(X_1,X_2)^2 + X_1X_2I(X_2<X_1)\right) - E\left(\frac{X_2}{X_1}I(X_2<X_1)\right) - 1.(4.5)$$

We have for Rayleigh distribution with parameter σ

$$E(X^2) = 2\sigma^2.$$

Hence an unbiased and consistent estimator of σ^2 is given by

$$\widehat{\sigma}^2 = \frac{1}{2n}\sum_{i=1}^{n} X_i^2.$$

Consider a symmetric kernel

$$\begin{aligned} h_1(X_1, X_2) &= \frac{1}{2}\big(2\min(X_1, X_2)^2 + X_1X_2I(X_2 < X_1) + X_1X_2I(X_2 < X_1)\big) \\ &= \frac{1}{2}(2\min(X_1, X_2)^2 + X_1X_2). \end{aligned}$$

Then a U-statistic defined by

Next we study the asymptotic properties of the test statistic. Since $\widehat{\sigma}^2$, U_1 and U_2 are U-statistics they are consistent estimators of σ, $E(\min(X_1, X_2))$ and $E\left(\frac{X_1}{X_2}I(X_1 < X_2)\right)$, respectively (Lehmann, 1951). Hence the following result is straight forward.

Theorem 4.2. *Under H_1, as $n \to \infty$, $\widehat{\Delta}_2$ converges in probability to $\Delta_2(F)$.*

Theorem 4.3. *As $n \to \infty$, $\sqrt{n}(\widehat{\Delta}_2 - \Delta_2(F))$ converges in distribution to normal random variable with mean zero and variance σ^2, where σ^2 is given by*

$$\sigma^2 = Var\Big(\frac{2X^2\bar{F}(X)}{\sigma^2} + \frac{2}{\sigma^2}\int_0^X y^2 dF(y) + \frac{X\mu}{\sigma^2} - X\int_X^\infty \frac{1}{y}dF(y) - \frac{1}{X}\int_0^X ydF(y)\Big). \quad (4.6)$$

Proof: Define

$$\widehat{\Delta_2} = \frac{\widehat{\Delta_{21}}}{\widehat{\sigma}^2} - \widehat{\Delta_{22}} - 1. \quad (4.7)$$

Let $X_{(i)}$ be. Since $\widehat{\sigma}^2$ is a consistent estimator of σ^2, by Slutsky's theorem, the asymptotic distributions of $\sqrt{n}(\widehat{\Delta_2} - \Delta_2(F))$ and $\sqrt{n}(\widehat{\Delta_2}^* - E(\widehat{\Delta_2}^*))$ are the same. Now we observe that $\widehat{\Delta_2}^*$ is a U statistic with symmetric kernel,

$$h(X_1, X_2) = \frac{1}{2}\left(\frac{2\min(X_1, X_2^2)}{\sigma^2} + \frac{X_1X_2}{\sigma^2} - \frac{X_1}{X_2}I(X_1 < X_2) - \frac{X_2}{X_1}I(X_2 < X_1)\right).$$

Hence using the central limit theorem for U-statistics we have the asymptotic normality of $\widehat{\Delta_2}^*$. The asymptotic variance is $4\sigma_1^2$ where σ_1^2 is given by (Lee, 2019)

$$\sigma_1^2 = Var\left[E\left(h(X_1, X_2)|X_1\right)\right]. \tag{4.8}$$

Consider

$$\begin{aligned} E[2\min(x, X_2)^2 + xX_2] &= 2E[x^2 I(x < X_2) + X_2^2 I(X_2 < x) + xX_2] \\ &= 2x^2 P(x < X_2) + 2\int_0^\infty y^2 I(y < x)dF(y) + x\mu \\ &= 2x^2\bar{F}(x) + 2\int_0^x y^2 dF(y) + x\mu. \end{aligned} \tag{4.9}$$

Also

$$\begin{aligned} E[\frac{x}{X_2}I(x < X_2] + E[\frac{X_2}{x}I(X_2 < x)] &= xE[\frac{1}{X_2}I(x < X_2] + \frac{1}{x}E[X_2 I(X_2 < x)] \\ &= x\int_0^\infty \frac{1}{y}I(x < y)dF(y) + \frac{1}{x}\int_0^x ydF(y) \\ &= x\int_x^\infty \frac{1}{y}dF(y) + \frac{1}{x}\int_0^x ydF(y). \end{aligned} \tag{4.10}$$

Substituting equations (6.10) and (6.12) in equation (6.9) we obtain the variance expression as specified in the theorem.

Under the null hypothesis H_0, $\Delta_2(F) = 0$. Hence we have the following corollary. We obtain the value of σ_0^2 using the software Mathematica.

Corollary 4.4. *Under H_0, as $n \to \infty$, $\sqrt{n}\widehat{\Delta}_2$ converges in distribution to normal with mean zero and variance σ_0^2, where σ_0^2 is the value of σ^2 evaluated under H_0 and it is given by*

$$\sigma_0^2 = \frac{1}{4}E\left(8 - 6e^{\frac{-X^2}{2s^2}} - \frac{\sqrt{2\pi}(s^2 - X^2)erf\left[\frac{X}{\sqrt{2}s}\right]}{sX}\right)^2 - 9,$$

where $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

An asymptotic critical region of the test can be obtain using Corollary 1. Let $\widehat{\sigma}_0^2$ be a consistent estimator of σ_0^2. We reject the null hypothesis H_0 against the alternative hypothesis H_1 at a significance level α, if

$$\frac{\sqrt{n}|\widehat{\Delta}_2|}{\widehat{\sigma}_0} > Z_{\alpha/2},$$

where Z_α is the upper α-percentile point of the standard normal distribution.

Finding a consistent estimator of the null variance σ_0^2 is difficult. Hence we find the critical region of the proposed test using Monte Carlo simulation. We determine lower (c_1) and upper (c_2) quantiles in such a way that $P(\widehat{\Delta}_2 < c_1) = P(\widehat{\Delta}_2 > c_2) = \alpha/2$. Finite sample performance of the test is evaluated through Monte Carlo simulation study and the results are presented in Section 4.5.

4.2.2 Jackknife empirical likelihood (JEL) based test

Next we develop jackknife empirical likelihood ratio test for testing standard Rayleigh distribution. For standard Rayleigh distribution the departure given in (6.2) become

$$\Delta_{2s}(F) \;=\; \int_0^\infty \left(E\left[\left(\frac{x^2-1}{x} \right) \min(X,t) \right] - F(t) \right) dF(t). \tag{4.11}$$

Hence (4.11) become

$$\Delta_{2s}(F) = E\left(\min(X_1,X_2)^2 + X_1X_2 I(X_2 < X_1)\right) - E\left(\frac{X_2}{X_1} I(X_2 < X_1) \right) - 1. \tag{4.12}$$

Hence the U-statistics based test is given by

$$\widehat{\Delta}_{2s} = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(2\min(X_i,X_j)^2 + X_iX_j + \frac{X_i}{X_j} I(X_i < X_j) + \frac{X_j}{X_i} I(X_j < X_i) \right). \tag{4.13}$$

In term of order statistics we can write above test statistic as

$$\widehat{\Delta}_{2s}=\frac{2}{n(n-1)}\sum_{i=1}^{n}(n-i)X_{(i)}^{2}+\frac{1}{n(n-1)}\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}X_{(j)}\left(X_{(i)}-\frac{1}{X_{(i)}}\right)-1. \quad (4.14)$$

For developing JEL based goodness of fit test for Rayleigh distribution, first we construct jackknife pseudo values using the test statistic given (4.13). Suppose $\widehat{\Delta}_{ks}$, are the value of the test statistic given in (4.13) calculated using $(n-1)$ observations X_1, X_2,..., X_{k-1},X_{k+1},...,X_n; $k=1,2,...,n$. The jackknife pseudo values denoted by ν_k are defined as

$$\nu_k=n\widehat{\Delta}_2-(n-1)\widehat{\Delta}_{2k},\ k=1,\ldots,n.$$

Note that ν_k are asymptotically independent under some mild conditions (Shi, 1984) and this justify the use of ν_k in the empirical likelihood inference.

Let $p=(p_1,\ldots,p_n)$ be a probability vector. It is well-known that $\prod_{i=1}^{n}p_i$ subject to $\sum_{i=1}^{n}p_i=1$ attain its maximum value n^{-n} at $p_i=1/n$. Hence the jackknife empirical likelihood ratio for testing PED based on the departure measure $\Delta_{2s}(F)$ is defined as

$$R(\Delta)=\max\Big\{\prod_{i=1}^{n}np_i,\ \sum_{i=1}^{n}p_i=1,\ \sum_{i=1}^{n}p_i\nu_i=0\Big\},$$

where

$$p_i=\frac{1}{n}\frac{1}{1+\lambda\nu_i}$$

and λ satisfies

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\nu_i}{1+\lambda\nu_i}=0.$$

Hence the jackknife empirical log-likelihood ratio is given by

$$\log R(\Delta_2)=-\sum\log(1+\lambda\nu_i).$$

We find the limiting distribution of the jackknife empirical log-likelihood ratio to obtain the critical region of the JEL based test. Using Theorem 1 of Jing et al. (2009) we have the following result as an analogue of Wilk's theorem.

Theorem 4.5. *Let* $h(X_1, X_2) = \Big(\min(X_1, X_2)^2 + X_1X_2 + \frac{X_1}{X_2}I(X_1 < X_2) + \frac{X_2}{X_1}I(X_2 < X_1)\Big)$. *Suppose* $E(h^2(X_1, X_2)) < \infty$ *and* $\sigma^2 > 0$, *then as* $n \to \infty$, $-2\log R(\Delta_2)$ *converges in distribution to a* χ^2 *random variable with one degree of freedom.*

Using Theorem 4.6.1 we can obtain the critical region of the JEL based test. We reject the null hypothesis H_0 against the alternatives hypothesis H_1 at a significance level α, if

$$-2\log R(\Delta_2) > \chi^2_{1,\alpha}, \tag{4.15}$$

where $\chi^2_{1,\alpha}$ is the upper α-percentile point of the χ^2 distribution with one degree of freedom.

4.3 Test statistic: censored case

Next we discuss how the right censored observations can be incorporated in the proposed testing method. Consider the right-censored data (Y, δ), with $Y = \min(X, C)$ and $\delta = I(X \leq C)$, where C is the censoring time. We assume censoring times and lifetimes are independent. Now we are interested to test the hypothesis discussed in Section 4.2 based on n independent and identical observation $\{(Y_i, \delta_i), 1 \leq i \leq n\}$. As we developed the test based on U-statistics for right censored data, we use the same departure measure $\Delta_2(F)$ given in (6.1). For that purpose, we rewrite (6.5) as

$$\Delta_2(F) = E\left(\frac{\min(X_1, X_2)^2}{\sigma^2} + \frac{X_1X_2I(X_2 < X_1)}{\sigma^2} - \frac{X_2}{X_1}I(X_2 < X_1) - 2I(X_2 < X_1)\right).$$

To develop the test statistic for right censored case, we estimate each quantity in $\Delta_2(F)$ using U-statistics for right censored data (Datta et al. (2010)). An estimator of $E(\min(X_1, X_2)^2 + X_1X_2I(X_2 < X_1))$ is given by

$$\widehat{\Delta}_{21c} = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j<i;j=1}^{n}\frac{(2\min(Y_1, Y_2)^2 + X_1X_2)\delta_i\delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}, \tag{4.16}$$

provided $\widehat{K}_c(Y_i) > 0$ and $\widehat{K}_c(Y_j) > 0$, with probability 1 and $\widehat{K}_c$ is the Kaplan-Meier estimator of K_c, the survival function of C. Again, an estimator of $E\left(\frac{X_2}{X_1}I(X_2 < X_1) + 2I(X_2 < X_1)\right)$ is given by

$$\widehat{\Delta}_{22c} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j<i;j=1}^{n} \frac{(\frac{X_i}{X_j}I(X_i < X_j) + \frac{X_j}{X_i}I(X_j < X_i) + 2)\delta_i\delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}. \tag{4.17}$$

Similarly, an estimators of $\sigma^2 = E(X^2)/2$ is given by

$$\sigma_c^2 = \frac{1}{2n} \sum_{i=1}^{n} \frac{Y_i^2\delta_i}{\widehat{K}_c(Y_i)}. \tag{4.18}$$

Using the estimators given in equations (5.16-5.18), we obtain the test statistic as

$$\widehat{\Delta}_{2c} = \frac{\widehat{\Delta}_{21c}}{\widehat{\sigma}_c^2} - \widehat{\Delta}_{22c}. \tag{4.19}$$

Hence in the right censored case, we reject H_0 in favour of H_1 for large values of $\widehat{\Delta_{2c}}$.

To obtain the limiting distribution of $\widehat{\Delta_{2c}}$, let $N_i^c(t) = I(Y_i \leq t, \delta_i = 0)$ be the counting process corresponds to the censoring variable C_i. Denote $R_i(t) = I(Y_i \geq t)$.

Also let σ_c^2 be the hazard rate of C. The martingale associated with this counting process $N_i^c(t)$ is given by

$$M_i^c(t) = N_i^c(t) - \int_0^t R_i(u)\sigma_c^2(u)du.$$

Let $G(x,y) = P(X_1 \leq x, Y_1 \leq y,\, \delta = 1),\ x \in \mathcal{X}$, $\bar{H}(t) = P(Y_1 > t)$ and

$$w(t) = \frac{1}{\bar{H}(t)} \int_{\mathcal{X}\times[0,\infty)} \frac{h_1(x)}{K_c(y-)} I(y > t)dG(x,y),$$

where $h_1(x) = E(h(X_1, X_2)|X_1 = x)$. The proof of next result follows from Theorem 1 of Datta et al. (2010) for the choice of the kernel

$$h(X_1, X_2) = \frac{1}{2}\left(\frac{2\min(X_1,X_2)^2}{\sigma^2} + \frac{X_1X_2}{\sigma^2} - \frac{X_1}{X_2}I(X_1 < X_2) - \frac{X_2}{X_1}I(X_2 < X_1)\right).$$

Theorem 4.6. *Let*

$$h_1(x) = \frac{1}{2}E\left(\frac{2\min(x,Y_2)^2 + xX_2}{\sigma^2} - \frac{x}{Y_2}I(x<Y_2) - \frac{Y_2}{x}I(Y_2<x) - 2\right)$$

Suppose the conditions

$$E\left[\left(\frac{2\min(Y_1,Y_2)^2 + X_1X_2}{\sigma^2}\right) - E\left(\frac{Y_1}{Y_2}I(Y_1<Y_2) + \frac{Y_2}{Y_1}I(Y_2<Y_1) + 2\right)\right]^2 < \infty,$$

$\int_{\mathcal{X}\times[0,\infty)} \frac{h_1^2(x)}{K_c^2(y)} dG(x,y) < \infty$ *and* $\int_0^\infty w^2(t)\sigma_c^2(t)dt < \infty$ *holds.*
As $n \to \infty$, $\sqrt{n}(\widehat{\Delta_{2c}} - \Delta_2(F))$ *converges in distribution to a Gaussian random variable with mean zero and variance* $4\sigma_c^2$*, where* σ_c^2 *is given by*

$$\sigma_c^2 = Var\Big(\frac{h_1(X)\delta_1}{K_c(Y_1-)} + \int w(t)dM_1^c(t)\Big).$$

Next we find an estimator of σ_c^2 using the reweighed techniques. An estimator of σ_c^2 is given by

$$\widehat{\sigma}_c^2 = \frac{4}{(n-1)}\sum_{i=1}^{n}(V_i - \bar{V})^2,$$

where

$$V_i = \frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} + \widehat{w}(X_i)(1-\delta_i) - \sum_{j=1}^{n}\frac{\widehat{w}(X_i)I(X_i>X_j)(1-\delta_i)}{\sum_{i=1}^{n} I(X_i>X_j)},$$

$$\bar{V} = \frac{1}{n}\sum_{i=1}^{n} V_i, \quad \widehat{h}_1(X) = \frac{1}{n}\sum_{i=1}^{n}\frac{h(X,Y_i)\delta_i}{\widehat{K}_c(Y_i-)}, \quad R(t) = \sum_{i=1}^{n} I(Y_i>t)$$

and

$$\widehat{w}(t) = \frac{1}{R(t)}\sum_{i=1}^{n}\frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)}I(X_i>t).$$

Let $\widehat{\sigma}_{0c}^2$ be the value of $\widehat{\sigma}_c^2$ evaluated under H_0. Under right censored situation, we reject the null hypothesis H_0 against the alternative hypothesis H_1 at a significance level α, if

$$\frac{\sqrt{n}|\widehat{\Delta}_{2c}|}{\widehat{\sigma}_{0c}} > Z_{\alpha/2}.$$

The results of the Monte Carlo simulation to assess the finite sample performance of the test is also reported in Section 4.4.

4.4 Simulations

The finite sample performance of the proposed test procedure is evaluated through extensive Monte Carlo simulation studies using R software (version 4.1.3). We compare the empirical type I error and power of our test with the existing tests for complete data, to show the competitiveness. In censored case, we evaluate the empirical power of our test against different alternatives at various censoring percentages.

4.4.1 Uncensored case

We find the empirical type I error and empirical power of the proposed test and other tests used for comparison. To find the empirical type I error, we simulate random samples of different sizes ($n = 10, 20, 30, 40, 50, 100, 200, 400$) from Rayleigh distribution with parameter 1 and 2. Then, based on the calculated value of the test statistic, we accept or reject the null hypothesis. This procedure is repeated ten thousand times and the proportion of times the null hypothesis is rejected is observed. This values gives the empirical type I error. The empirical type I error is reported in Table 1. We can see that the empirical type I error is getting close to chosen level of significance as n increases. This observation is true for both choices of parameters 1 and 2. A similar procedure, by generating lifetimes from different alternative distributions is followed to calculate the empirical power of the proposed test. The algorithm used for finding the empirical power is summarized as follows;

1. Generate lifetime data from the desired alternative and calculate the test statistic.

2. Find the maximum likelihood estimator $\widehat{\sigma}^2$ of the parameter σ^2 using the data generated in Step 1.

3. Generate a sample of size n from Rayleigh distribution with parameter $\widehat{\sigma}^2$.

4. Repeat Step 3, 10000 times and determine the simulated critical points.

5. Based on the test statistic value and critical points obtained in Step 4, determine whether the null hypothesis is rejected or not.

6. Repeat Steps 1-5, 10000 times and calculate the empirical power as the proportion of rejections of the test.

To find the empirical power, lifetime random variables are generated from different choices of alternatives including Weibull, gamma, log-normal, Pareto and half-normal distributions, where the distribution functions are:

- Weibull distribution: $F_1(x) = 1 - e^{(-x/\lambda)^k}$, $x > 0$, k, $\lambda > 0$.

- Gamma distribution: $F_2(x) = \frac{1}{\Gamma(k)}\gamma(k, \frac{x}{\lambda})$, $k, \lambda > 0$ and $\gamma(k, \frac{x}{\lambda})$ is the lower incomplete gamma function.

- Pareto distributions: $F_3(x) = (\lambda/x)^\alpha$ $x > 0$, α, $\lambda > 0$.

- Lognormal distribution: $F_4(x) = \Phi(\frac{\ln x - \mu}{\sigma})$, $x > 0$, $-\infty < \mu < \infty, \sigma^2 > 0$ where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable.

- Half-normal distribution: $F_5(x) = erf(\frac{x}{\sigma\sqrt{2}})$, $\sigma > 0$, where *erf* is the error function.

We compare the performance of our test with goodness of fit test for Rayleigh distribution proposed by Ahrari et al. (2022), Meintanis and Iliopoulos (2003), Baratpour and Khodadadi (2013), Jahanshahi et al. (2016) as well as with the classical tests Kolmogorov Smirnov (KS), Cramer von Mises (CvM) and Anderson-Darling (AD) test.

We report the results of power comparison in Tables 4.2 and 4. 3. Table 2 gives the power of $\widehat{\Delta}_2$ for different alternatives at significance level $\alpha = 0.01$ and Table 3 represents the same when $\alpha = 0.05$. Tables 5.2 and 5.3 show that the developed test has good power against all choices of alternatives which increases with sample size. We can see that the newly proposed test performs better than other tests in almost all of the cases. The proposed test has high power even for small sample size, which affirms the efficiency of the test. When the alternative follows a Pareto distribution, then KS test and CvM test perform better than the developed test.

4.4.2 Censored case

We calculate the empirical type I error and the empirical power of the test statistic proposed for right censored data using Monte Carlo simulation studies. To calculate the empirical type I error, lifetimes are generated from Rayleigh distribution with parameter 1. The steps to calculate empirical type I error and power for the uncensored data is adapted to find the same in censored case also. We consider different choices of alternatives, as in uncensored case for finding the empirical power. Here, the censoring percentages are chosen to be 20% and 40%, to depict the effect of censoring on the proposed test statistic. In all cases, the censoring random variable C is generated from exponential distribution with parameter b, where b is chosen such that $P(T > C) = 0.2(0.4)$. The re-weighting technique explained in Section 4.3 is used to estimate the variance of $\widehat{\Delta}_{2c}$. The results of the simulation study are presented in Table 4. We can see that the empirical type I error of the test approaches the chosen level significance as n increases. Form Table 4.1, it is also clear that, the test has good empirical power for all choices of alternatives. Empirical power of the test increases with increase in n and decreases with increases in censoring percentage.

From Table 3, we also observe that the proportion of rejections out of 10000 increases with sample size and decreases with censoring percentage.

4.5 Data analysis

The proposed test procedures are illustrated using several real data sets.

Table 4.1: Comparison of empirical type I error at α=0.01 and α=0.05

	n	α	$\hat{\Delta}_2$	KS	CvM	AD	AH	MI	BK	JH
Rayleigh (1)	10	0.01	0.0116	0.0183	0.0186	0.0113	0.0095	0.0113	0.0111	0.0093
	20		0.0112	0.0136	0.0150	0.0099	0.0102	0.0086	0.0086	0.0106
	30		0.0089	0.0124	0.0137	0.0102	0.0104	0.0109	0.0093	0.0095
	40		0.0091	0.0117	0.0110	0.0094	0.0092	0.0093	0.0104	0.0096
	50		0.0095	0.0115	0.0112	0.0090	0.0095	0.0094	0.0107	0.0106
	100		0.0106	0.0110	0.0110	0.0087	0.0104	0.0110	0.0106	0.0105
	200		0.0104	0.0110	0.0108	0.0099	0.0094	0.0109	0.0104	0.0104
	400		0.0103	0.0095	0.0096	0.0084	0.0086	0.0093	0.0096	0.0105
Rayleigh (1)	10	0.05	0.0534	0.0582	0.0627	0.0492	0.0446	0.0495	0.0489	0.0542
	20		0.0487	0.0561	0.0597	0.0501	0.0478	0.0486	0.0516	0.0518
	30		0.0516	0.0566	0.0507	0.0512	0.0490	0.0536	0.0515	0.0514
	40		0.0513	0.0510	0.0545	0.0515	0.0505	0.0492	0.0476	0.0483
	50		0.0489	0.0487	0.0487	0.0483	0.0487	0.0510	0.0497	0.0492
	100		0.0492	0.0513	0.0531	0.0487	0.0583	0.0513	0.0518	0.0503
	200		0.0505	0.0490	0.0474	0.0493	0.0453	0.0496	0.0492	0.0494
	400		0.0497	0.0496	0.0460	0.0496	0.0522	0.0512	0.0512	0.0508
Rayleigh (2)	10	0.01	0.0111	0.0098	0.0113	0.0103	0.0104	0.0110	0.0109	0.0089
	20		0.0107	0.0120	0.0101	0.0099	0.0094	0.0093	0.0108	0.0116
	30		0.0092	0.0099	0.0097	0.0112	0.0098	0.0094	0.0088	0.0112
	40		0.0094	0.0094	0.0107	0.0098	0.0087	0.0109	0.0104	0.0096
	50		0.0104	0.0108	0.0102	0.0099	0.0094	0.0107	0.0137	0.0105
	100		0.0094	0.0105	0.0110	0.0098	0.0123	0.0092	0.0102	0.0094
	200		0.0097	0.0105	0.0115	0.0096	0.0099	0.0096	0.0093	0.0102
	400		0.0102	0.0096	0.0098	0.0098	0.0097	0.0098	0.0098	0.0097
Rayleigh (2)	10	0.05	0.0519	0.0464	0.0462	0.0518	0.0480	0.0456	0.0484	0.0519
	20		0.0492	0.0512	0.0524	0.0465	0.0461	0.0510	0.0532	0.0518
	30		0.0494	0.0481	0.0474	0.0468	0.0472	0.0506	0.0527	0.0510
	40		0.0515	0.0521	0.0479	0.0499	0.0520	0.0498	0.0515	0.0511
	50		0.0517	0.0514	0.0523	0.0494	0.0557	0.0491	0.0486	0.0495
	100		0.0513	0.0514	0.0504	0.0489	0.0465	0.0558	0.0513	0.0502
	200		0.0507	0.0498	0.0464	0.0529	0.0506	0.0497	0.0488	0.0487
	400		0.0505	0.0492	0.0502	0.0509	0.0486	0.0539	0.0490	0.0503

Table 4.2: Comparison of empirical power for different alternatives (α=0.01)

	n	$\hat{\Delta}_2$	KS	CvM	AD	AH	MI	BK	JH
Weibull (1, 1.5)	10	0.2267	0.0081	0.0017	0.0014	0.0509	0.0152	0.0482	0.0787
	20	0.6782	0.0411	0.0212	0.0144	0.3666	0.0535	0.0979	0.5235
	30	0.8932	0.1046	0.0885	0.1034	0.5684	0.1286	0.1456	0.7365
	40	0.9782	0.2219	0.2401	0.4532	0.8207	0.2741	0.2778	0.8850
	50	0.9954	0.3638	0.4628	0.6455	0.9398	0.9429	0.4228	0.9127
	100	1.0000	0.9481	0.9918	0.9999	0.9999	0.9977	0.9980	0.9998
	200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Gamma (1,1)	10	0.2894	0.0066	0.0022	0.0967	0.0649	0.0787	0.0357	0.1969
	20	0.6179	0.0412	0.0254	0.4758	0.2828	0.1018	0.2207	0.7463
	30	0.8893	0.1119	0.0946	0.8186	0.6613	0.2953	0.6872	0.9261
	40	0.9836	0.2225	0.2476	0.9572	0.8567	0.4564	0.9028	0.9758
	50	0.9949	0.3585	0.4613	0.9976	0.9462	0.7136	0.9818	0.9926
	100	1.0000	0.8525	0.5897	1.0000	1.0000	0.9935	0.9967	0.9990
	200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Log-normal (0,1)	10	0.1884	0.0129	0.0053	0.0013	0.0433	0.0026	0.0324	0.0847
	20	0.6478	0.0439	0.0276	0.0156	0.1878	0.0077	0.2372	0.5212
	30	0.9129	0.0917	0.0763	0.0685	0.4445	0.0311	0.6385	0.8843
	40	0.9918	0.1567	0.1548	0.1774	0.7057	0.2312	0.9599	0.9674
	50	0.9992	0.2353	0.2676	0.3378	0.8947	0.7072	0.9932	0.9888
	100	1.0000	0.7677	0.8629	0.9616	0.9999	1.0000	1.0000	0.9998
	200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Pareto (1,1)	10	0.0323	0.5922	0.7479	0.9945	0.0188	0.1942	0.0584	0.6869
	20	0.3733	0.9659	0.9925	1.0000	0.0906	0.3153	0.2791	0.9003
	30	0.9562	1.0000	1.0000	1.0000	0.2576	0.5060	0.9481	0.9825
	40	0.9998	1.0000	1.0000	1.0000	0.6516	0.5863	0.9975	0.9967
	50	1.0000	1.0000	1.0000	1.0000	0.8538	0.6507	0.9998	0.9994
	100	1.0000	1.0000	1.0000	1.0000	1.0000	0.8320	1.0000	1.0000
	200	1.0000	1.0000	1.0000	1.0000	1.0000	0.8887	1.0000	1.0000
	400	1.0000	1.0000	1.0000	1.0000	1.0000	0.9553	1.0000	1.0000
Half-normal (2)	10	0.0726	0.0126	0.0051	0.0007	0.0229	0.0107	0.0164	0.1198
	20	0.2334	0.0526	0.0267	0.0154	0.0744	0.0355	0.0354	0.4872
	30	0.3656	0.1450	0.0870	0.0819	0.1523	0.0682	0.0736	0.6372
	40	0.5155	0.2868	0.2122	0.2692	0.2793	0.1086	0.1441	0.7813
	50	0.6638	0.4551	0.3929	0.5076	0.3801	0.1612	0.2598	0.8316
	100	0.9716	0.9634	0.9709	0.9983	0.8466	0.8846	0.1229	0.9820
	200	0.9999	1.0000	1.0000	1.0000	0.9977	0.9454	0.7455	0.9999
	400	1.0000	1.0000	1.0000	1.0000	1.0000	0.9858	1.0000	1.0000

Table 4.3: Comparison of empirical power for different alternatives (α=0.05)

	n	$\hat{\Delta}_2$	KS	CvM	AD	AH	MI	BK	JH
Weibull (1, 1.5)	10	0.5189	0.0704	0.0471	0.0193	0.2365	0.0493	0.1465	0.3287
	20	0.8931	0.2084	0.2232	0.2277	0.6550	0.1294	0.2019	0.8849
	30	0.9783	0.4181	0.5205	0.6056	0.8812	0.3972	0.3282	0.9513
	40	0.9976	0.6145	0.7610	0.8718	0.9666	0.9595	0.5759	0.9754
	50	0.9992	0.8029	0.9093	0.9737	0.9929	0.9694	0.7971	0.9879
	100	1.0000	0.9996	1.0000	1.0000	1.0000	0.9909	1.0000	1.0000
	200	1.0000	1.0000	1.0000	1.0000	1.0000	0.9981	1.0000	1.0000
	400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Gamma (1, 1)	10	0.5191	0.4583	0.4721	0.3674	0.2381	0.3462	0.1835	0.3218
	20	0.8742	0.8160	0.8512	0.8101	0.6999	0.6432	0.6523	0.8749
	30	0.9787	0.9542	0.9721	0.9655	0.8882	0.8399	0.9317	0.9491
	40	0.9982	0.9911	0.9957	0.9938	0.9658	0.9461	0.9853	0.9776
	50	0.9994	0.9988	0.9997	0.9996	0.9949	0.9761	0.9958	0.9859
	100	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	1.0000
	200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Log-normal (0,1)	10	0.5236	0.0738	0.0560	0.0308	0.1613	0.0247	0.1434	0.1977
	20	0.9038	0.1852	0.1791	0.1589	0.5322	0.0869	0.6312	0.6867
	30	0.9934	0.2997	0.3369	0.3773	0.7819	0.5245	0.9696	0.8050
	40	0.9991	0.4396	0.5273	0.6166	0.9361	0.8887	0.9953	0.8377
	50	0.9999	0.5818	0.6795	0.8089	0.9852	0.9889	0.9990	0.9004
	100	1.0000	0.9871	0.9928	0.9999	1.0000	1.0000	1.0000	0.9941
	200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Pareto (1, 1)	10	0.2346	0.8488	0.9301	0.9954	0.0911	0.2529	0.1920	0.2517
	20	0.9345	0.9997	0.9996	1.0000	0.3454	0.4429	0.8611	0.4152
	30	0.9990	1.0000	1.0000	1.0000	0.7701	0.5525	0.9982	0.5478
	40	1.0000	1.0000	1.0000	1.0000	0.9682	0.6411	1.0000	0.6522
	50	1.0000	1.0000	1.0000	1.0000	0.9968	0.6962	1.0000	0.8092
	100	1.0000	1.0000	1.0000	1.0000	1.0000	0.8715	1.0000	0.9972
	200	1.0000	1.0000	1.0000	1.0000	1.0000	0.9173	1.0000	1.0000
	400	1.0000	1.0000	1.0000	1.0000	1.0000	0.9736	1.0000	1.0000
Half-normal (2)	10	0.4566	0.0804	0.0597	0.0490	0.0966	0.0485	0.0790	0.1015
	20	0.4698	0.2489	0.2118	0.1964	0.2618	0.1049	0.1532	0.3679
	30	0.6476	0.4642	0.4492	0.5074	0.4342	0.1789	0.2371	0.5122
	40	0.7875	0.6691	0.6873	0.7858	0.5776	0.2671	0.4078	0.5989
	50	0.8601	0.8154	0.8505	0.9279	0.7071	0.4160	0.5595	0.6772
	100	0.9915	0.9980	0.9996	1.0000	0.9760	0.9805	0.3634	0.9017
	200	1.0000	1.0000	1.0000	1.0000	0.9999	0.9650	0.9332	0.9988
	400	1.0000	1.0000	1.0000	1.0000	1.0000	0.9989	1.0000	1.0000

Table 4.4: Proportion of rejections out of 10000 using the test for censored data

	Rayleigh (1)		Exponential (1)		Lognormal (2,1)		Gamma (3,1)		Inv Gaussian (2,1)	
n	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$
					20% censoring					
50	0.0149	0.0474	0.9891	0.9967	0.8296	0.8791	0.4794	0.6735	0.8425	0.9037
75	0.0128	0.0488	0.9925	0.9974	0.6696	0.8098	0.6907	0.7738	0.9481	0.9745
100	0.0115	0.0491	1.0000	1.0000	0.9388	0.9635	0.7836	0.8432	0.9917	0.9986
200	0.0107	0.0496	1.0000	1.0000	0.9999	1.0000	0.8492	0.9071	1.0000	1.0000
400	0.0105	0.0499	1.0000	1.0000	1.0000	1.0000	0.9708	0.9944	1.0000	1.0000
500	0.0101	0.0501	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
					40% censoring					
50	0.0148	0.0524	0.7529	0.8403	0.4366	0.5358	0.1916	0.5607	0.4978	0.5812
75	0.0131	0.0517	0.9740	0.9849	0.5444	0.6476	0.3408	0.6628	0.5759	0.6654
100	0.0118	0.0512	0.9895	0.9963	0.6405	0.7346	0.7028	0.7929	0.7758	0.8326
200	0.0126	0.0507	0.9996	1.0000	0.6398	0.7618	0.8187	0.8790	0.9083	0.9326
400	0.0119	0.0506	1.0000	1.0000	0.9237	0.9726	0.9309	0.9745	0.9568	0.9789
500	0.0107	0.0503	1.0000	1.0000	0.9941	0.9987	0.9789	0.9983	0.9832	1.0000

4.5.1 Uncensored data

We consider two data sets for the analysis. To find the critical region, we use the following algorithm.

1. Obtain the maximum likelihood estimate the parameter of Rayleigh distribution from the observed data.
2. Generate a random sample from Rayleigh distribution using the estimated parameter in Step 1.
3. Repeat Step 2, 10000 times and determine the simulated critical points.
4. Based on the test statistic value and critical points obtained in Step 3, determine whether the null hypothesis is rejected or not.

Illustration 1: To investigate the performance of the proposed test, we analyze the data, given in Caroni (2002), represents the failure times of 23 ball bearings . These failure times are: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 67.80, 67.80, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40. This data set originally used by Lieblein and Zelen (1956), is also discussed in Caroni (2002). The observations are the number of million revolutions before failure for each 23 ball bearings ordered according to life endurance. Later, the data is analysed by many

Table 4.5: Critical values and test statistics for data on failure time of ball bearings

α	Test	Test statistic	LCV	UCV	Decision
0.01	$\hat{\Delta}_2$	0.4433	0.0944	0.8706	Accept H_0
	KS	0.1368	-	0.3297	Accept H_0
	CvM	0.0534	-	0.7320	Accept H_0
	AD	0.3272	-	3.8903	Accept H_0
	AH	0.0029	-	0.0160	Accept H_0
	MI	-3.2917	-3.3439	-3.2287	Accept H_0
	BK	0.5116	0.5085	0.5913	Accept H_0
	JH	0.0360	0.0330	0.0612	Accept H_0
0.05	$\hat{\Delta}_2$	0.4433	0.1927	0.7904	Accept H_0
	KS	0.1368	-	0.2750	Accept H_0
	CvM	0.0534	-	0.4586	Accept H_0
	AD	0.3272	-	2.5046	Accept H_0
	AH	0.0029	-	0.0115	Accept H_0
	MI	-3.2917	-3.3355	-3.2439	Accept H_0
	BK	0.5116	0.5097	0.5543	Accept H_0
	JH	0.0360	0.0346	0.0557	Accept H_0

authors including Kim and Han (2009) and Dey and Dey (2014) in studying Rayleigh distribution. Hence we fail to reject the null hypothesis H_0, that the data on failure times of ball bearings follow Rayleigh distribution. We also examine the performance of other tests used for comparison in simulation studies for the above data set. The results are summarized in Table 5.5. We can see that all the tests accept the assumption of Rayleigh distribution. Since KS, CvM and AH tests are one sided tests, lower critical values (LCV) for those tests are not given in Table 5.5 and Table 5.6, below.

Illustration 2: We also examine the time to breakdown of an electrical insulating fluid which was subject to a constant voltage stress of 32 kilo volts (kv). The data gives results for seven groups of specimens, tested at voltages ranging from 26 to 38 kv and we choose the readings from voltage level 32 kv in our analysis. The length of time until each specimen failed was observed for 15 specimens. This experiment was conducted to investigate the distribution of time to failure. The data is studied inLawless (2011) (page 3) and given in Example 1.1.5. We obtain the test statistic as -0.5643 where the 1% level critical values are -0.0119 and 0.9396 and 5% level critical value are 0.1109 and 0.8531. Hence we reject the null hypothesis that the data follows Rayleigh distribution. For this data set

Table 4.6: Critical values and test statistics for data on breakdown times for electrical insulating fluid

α	Test	Test statistic	LCV	UCV	Decision
0.01	$\hat{\Delta}_2$	-0.5644	-0.0156	0.9538	Reject H_0
	KS	0.5508	-	0.4045	Reject H_0
	CvM	1.2902	-	0.7249	Reject H_0
	AD	20.8404	-	3.8869	Reject H_0
	AH	0.7138	-	0.0240	Reject H_0
	MI	-2.5962	-3.0845	-2.7818	Reject H_0
	BK	1.1677	0.5129	0.6211	Reject H_0
	JH	0.1654	0.0368	0.0638	Reject H_0
0.05	$\hat{\Delta}_2$	-0.05644	0.1171	0.8571	Reject H_0
	KS	0.5508	-	0.3376	Reject H_0
	CvM	1.2902	-	0.4567	Reject H_0
	AD	20.8404	-	2.5068	Reject H_0
	AH	0.7138	-	0.0160	Reject H_0
	MI	-2.5962	-3.0674	-2.8190	Reject H_0
	BK	1.1677	0.5143	0.5737	Reject H_0
	JH	0.1654	0.0382	0.0638	Reject H_0

also, we compare our result with the other goodness of fit tests for Rayleigh distribution. Table 5.6 gives the results of comparison study. It can be noted that, in accordance with our test, all the other tests also reject the assumption of Rayleigh distribution.

4.5.2 Censored data

We consider two real data sets for illustration. We use the normal based critical region given in Section 3 to make a decision. The asymptotic null variance of $\widehat{\Delta}_{2c}$ is estimated using the re-weighting techniques explained in Section 3.

Illustration 1: We examine the data on lifetimes of disk break pads on 40 cars studied in Lawless (2011). The data gives the lifetimes (in km) of front disk brake pads on a randomly selected set of 40 cars of same model that were monitored by a dealer network. Some other factors like model year, driving conditions, geographic region are also given for each vehicle, which is not included in our analysis. Complete data set is given in Table 6.11, Page 337 of Lawless (2011). Out of the 40 observed lifetimes, 9 are censoring times, hence data contains 22.5% of censored observations. The test statistic is obtained as 0.7230. Hence we cannot reject the null hypothesis at both 1% and 5% level of significance. Accordingly, Rayleigh distribution assumption is valid for this data.

Illustration 2: We analyse Stanford heart transplant data available in R software named 'stanford2' to test for the assumption of Rayleigh distribution. The data consist of the survival time or censoring time of 184 patients waiting for the Stanford heart transplant program. There are 112 observed lifetimes and 72 censoring times. Covariate information on age of the patient, condition of surgery and transplant, patient id and year of acceptance is also available. The censoring percentage of data is 38.5%. The test statistic is calculated as 50.2974. So we reject the null hypothesis that the data follows Rayleigh distribution at both 1% and 5% level of significance.

4.6 Concluding remarks

Based on fixed point characterization for Rayleigh distribution, we developed new goodness of fit test for Rayleigh distribution. Even though many goodness of fit tests are available for Rayleigh distribution, as of our knowledge, all of these test are developed for complete data. Motivated by this we develop a new goodness of fit test for Rayleigh distribution for under right censored data. We showed that the asymptotic distribution of the proposed test statistic is normal. A consistent estimator of the asymptotic variance is also obtained. Monte Carlo simulation study is conducted to evaluate the finite sample performance of the test as well as to compare with other test available in literature. The proposed test is illustrated using two real data sets. We also developed JEL ratio test for testing standard Rayleigh distribution.

Our test deals with the right censoring scenario in lifetime data analysis. Apart from right censoring, left truncation, interval censoring and double censoring are common in such contexts. The proposed test can be modified to incorporates these situations. Goodness of fit tests for other important lifetime distributions can also be derived using similar fixed point characterizations. The goodness of fit tests for gamma distribution and inverse Gaussian distribution in presence of right censored observations will be reported in separate studies. Goodness of fit tests for discrete lifetime distributions can also be developed using Stein's type identity.

Chapter 5

Goodness of fit tests for inverse Gaussian distribution

Lifetime data can be analyzed using either parametric or non-parametric methods. When the data is known to follow a specific lifetime distribution, parametric methods typically provide more accurate and efficient results compared to non-parametric approaches. The goodness of fit tests are employed to test whether the data follows a specific lifetime model. The inverse Gaussian (IG) distribution is an important parametric model used for the analysis of lifetime data. A positive random variable X is said to follow an IG distribution, with parameters μ and θ, if the density function is given by

$$f(x) = \sqrt{\frac{\theta}{2\pi x^3}} \exp\left(-\frac{\theta(x-\mu)^2}{2\mu^2 x}\right); \quad x > 0, \mu, \theta > 0. \tag{5.1}$$

Here $\mu > 0$ represents the mean and $\theta > 0$ is the shape parameter of the distribution. Inverse Gaussian originates, as the first passage time of Brownian motion with positive drift, which later found applications in various fields. For example, IG distribution has been used in modeling data from stock market prices, submicroscopic particles, biology, hydrology, meteorology, labor relation, and reliability analysis among others (Folks and Chhikara, 1978; Seshadri, 2012). The lifetime data modeling with IG distribution is extensively studied by Chhikara and Folks (1977) and further explored by Bhattacharyya and Fries (1982) and Whitmore (1983). An important property of IG distribution is that it is closed under scale transformation as $X \sim \mathrm{IG}(\mu, \theta)$, implies $cX \sim \mathrm{IG}\left(c\mu, c\theta\right), c > 0$.

[1]The research work reported in this chapter is published in Xavier et al. (2025).

The extensive use of IG distribution in the modeling of lifetime data motivates researchers to develop goodness of fit tests for IG distribution. Some earlier tests for the same can be found in O'Reilly and Rueda (1992) and Pavur et al. (1992). Later Ducharme (2001) provided a smooth, consistent test statistic for testing the IG distribution. They also discussed the test statistic in right censored case without illustration. A test statistic based on empirical Laplace transform for testing IG distribution is developed by Henze and Klar (2002). Vexler et al. (2011) proposed an empirical likelihood ratio test for IG distribution. A comprehensive study of goodness of fit tests using R packages can be found in Rayner et al. (2009), which includes IG distribution as well. It is well known that lifetime data suffers from censoring, quite often. IG distribution is also used in literature, as a model to explore lifetime data subjected to right censoring (Folks and Chhikara, 1978; Whitmore, 1983; Ismail and Auda, 2006 and Jayalath and Chhikara, 2022). However, most of the tests, except Ducharme (2001) considered the complete data. Even though Ducharme (2001) addressed the problem of censoring, their test procedures is very complicated. The goodness of fit tests developed for complete data can not be directly applied to the right censored samples. One way to tackle this problem is to use the pseudo-complete data generation method proposed by Balakrishnan et al. (2015) or other similar techniques. Since these methods do not allow us to use the original data, information loss may occur. To fill this gap, we develop an easy-to-implement goodness of fit test for IG distribution, incorporating right-censored observations. Our test is based on a characterization that uses Stein's type identity. We use Stein's type identity for IG distribution, to develop a goodness of fit test for IG distribution. Then we discuss how the testing procedure can be extended to incorporate right-censored observations.

The rest of the article is organized as follows. In Section 5.1, Using Stein's type identity for IG distribution, we develop a U-statistic based goodness of fit test for IG distribution with shape parameter θ, when we have uncensored data. We also propose a jackknife empirical likelihood ratio test for the inverse Gaussian distribution when $\theta = 1$. In Section 5.2, we discuss how to modify the test statistic to accommodate right censored observations. The asymptotic properties of the test statistics are studied in detail. Extensive Monte Carlo simulation studies are carried out in Section 5.3 to assess the finite sample performance of the proposed tests in the presence and absence of censoring. The practical applicability of the proposed tests are illustrated using real life data sets in Section 5.4. Finally, we conclude the study with a discussion on future works in Section 6.

5.1 Test statistic: complete data

In this Section, we propose a new goodness of fit test to assess whether the data follows an IG distribution with shape parameter θ. We use the following characterization theorem stated in Betsch and Ebner (2021) for developing the test statistic. If $f(.)$ is a probability density function with the support $[0,\infty)$, and X is a real-valued random variable which is continuously differentiable on $[0,\infty)$ with $\int_0^\infty x|f'(x)|\mathrm{d}x < \infty$, then X is said to follow the distribution function $F(x)$ with density function $f(x)$ if and only if, $F(t) = E\left[-\frac{f'(X)}{f(X)}\min\{X,t\}\right]$.

In view of Theorem 1, a random variable $X \sim \mathrm{IG}(1,\theta)$ if and only if

$$F(t) = E\left[\frac{1}{2}\left(\theta + \frac{3}{X} - \frac{\theta}{X^2}\right)\min\{X,t\}\right], t > 0. \tag{5.2}$$

5.1.1 U-statistics based test

Based on a random sample $X_1, ..., X_n$ from F, we are interested in testing the null hypothesis

$$H_0 : F \text{ follows inverse Gaussian distribution}$$

against

$$H_1 : F \text{ does not follow inverse Gaussian distribution.}$$

For testing the above hypothesis first, we define a departure measure that discriminates between the null and the alternative hypothesis. In view of (5.2), we consider the departure measure

$$\Delta_3(F) = \int_0^\infty \left(E\left[\frac{1}{2}\left(\theta + \frac{3}{X} - \frac{\theta}{X^2}\right) \min\{X, t\} \right] - F(t) \right) \mathrm{d}F(t). \tag{5.3}$$

Our aim is to simplify $\Delta_3(F)$, in terms of expectations of the function of random variables. Consider

$$\begin{aligned}
\Delta_3(F) &= \int_0^\infty \left(E\left[\frac{1}{2}\left(\theta + \frac{3}{X} - \frac{\theta}{X^2}\right) \min\{X, t\} \right] - F(t) \right) \mathrm{d}F(t) \\
&= \frac{\theta}{2}\int_0^\infty \int_0^\infty \min\{x,t\}\mathrm{d}F(x)\mathrm{d}F(t) + \frac{3}{2}\int_0^\infty \int_0^\infty \frac{1}{x}\min\{x,t\}\mathrm{d}F(x)\mathrm{d}F(t) \\
&- \frac{\theta}{2}\int_0^\infty \int_0^\infty \frac{1}{x^2}\min\{x,t\}\mathrm{d}F(x)\mathrm{d}F(t) - \frac{1}{2} \\
&= \Delta_{31} + \Delta_{32} - \Delta_{33} - \frac{1}{2}, \text{(say)}.
\end{aligned} \tag{5.4}$$

Now,

$$c = \frac{\theta}{2}\int_0^\infty \int_0^\infty \min\{x,t\}\mathrm{d}F(x)\mathrm{d}F(t) = \frac{\theta}{2}E[\min\{X_1, X_2\}], \tag{5.5}$$

and

$$\begin{aligned}
\Delta_{32} &= \frac{3}{2}\int_0^\infty \int_0^\infty \frac{1}{x}\min\{x,t\}\mathrm{d}F(x)\mathrm{d}F(t) \\
&= \frac{3}{2}\left[\int_0^\infty \int_x^\infty \mathrm{d}F(t)\mathrm{d}F(x) + \int_0^\infty \int_0^\infty \frac{t}{x} I(t < x)\mathrm{d}F(x)\mathrm{d}F(t) \right] \\
&= \frac{3}{4} + \frac{3}{2}E\left[\frac{X_2}{X_1} I(X_2 < X_1) \right],
\end{aligned} \tag{5.6}$$

where $I(A)$ denotes the indicator function of a set A. Also

$$\begin{aligned}
\Delta_{33} &= \frac{\theta}{2}\int_0^\infty \int_0^\infty \frac{1}{x^2}\min\{x,t\}\mathrm{d}F(x)\mathrm{d}F(t) \\
&= \frac{\theta}{2}\left[\int_0^\infty \frac{1}{x}\int_x^\infty \mathrm{d}F(t)\mathrm{d}F(x) + \int_0^\infty \int_0^\infty \frac{t}{x^2} I(t < x)\mathrm{d}F(x)\mathrm{d}F(t) \right] \\
&= \frac{\theta}{4}E\left[\frac{1}{\min\{X_1, X_2\}} \right] + \frac{\theta}{2}E\left[\frac{X_2}{X_1^2} I(X_2 < X_1) \right].
\end{aligned} \tag{5.7}$$

Substituting (5.5)-(5.7) in (6.2), we obtain

$$\begin{aligned}\Delta_3(F) &= \frac{\theta}{2}E\left[\min\{X_1,X_2\}-\frac{1}{2\min\{X_1,X_2\}}\right]+\frac{3}{2}E\left[\frac{X_2}{X_1}I(X_2<X_1)\right]\\ &- \frac{\theta}{2}E\left[\frac{X_2}{X_1^2}I(X_2<X_1)\right]+\frac{1}{4}. \end{aligned} \tag{5.8}$$

Now we use U-statistics theory to develop the test statistic. Define a symmetric kernel $h_1(X_1, X_2) = \min\{X_1, X_2\} - \frac{1}{2\min\{X_1,X_2\}}$. Then a U-statistic defined as

$$\hat{\Delta}_{31} = \frac{1}{\binom{n}{2}}\sum_{i=1}^{n}\sum_{j=1,j<i}^{n} h_1(X_i, X_j)$$

is an unbiased estimate for $E\left[\min\{X_1, X_2\} - \frac{1}{2\min\{X_1,X_2\}}\right]$. Consider another symmetric kernel $h_2(X_1, X_2) = \frac{1}{2}\left[\frac{X_1}{X_2}I(X_1 < X_2) + \frac{X_2}{X_1}I(X_2 < X_1)\right]$, then a U-statistic defined by

$$\hat{\Delta}_{32} = \frac{1}{\binom{n}{2}}\sum_{i=1}^{n}\sum_{j=1,j<i}^{n} h_2(X_i, X_j) \tag{5.9}$$

is an unbiased estimate for $E\left[\frac{X_2}{X_1}I(X_2 < X_1)\right]$.

Let $h_3(X_1, X_2) = \frac{1}{2}\left[\frac{X_1}{X_2^2}I(X_1 < X_2) + \frac{X_2}{X_1^2}I(X_2 < X_1)\right]$, then a U-statistic defined by

$$\hat{\Delta}_{33} = \frac{1}{\binom{n}{2}}\sum_{i=1}^{n}\sum_{j=1,j<i}^{n} h_3(X_i, X_j) \tag{5.10}$$

is an unbiased estimate for $E\left[\frac{X_2}{X_1^2}I(X_2 < X_1)\right]$.

Let $\hat{\theta} = \left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{X_i} - 1\right)\right)^{-1}$ is an unbiased and consistent estimator of θ. Then, the test statistic $\hat{\Delta}$ is given by

$$\widehat{\Delta}_3 = \frac{\hat{\theta}}{2}\hat{\Delta}_{31} + \frac{3}{2}\hat{\Delta}_{32} - \frac{\hat{\theta}}{2}\hat{\Delta}_{33} + \frac{1}{4}. \tag{5.11}$$

We reject the null hypothesis H_0, against the alternative hypothesis H_1 for large values of $\widehat{\Delta}_3$. We now study the asymptotic properties of the statistic $\widehat{\Delta}_3$. Since U_1 and U_2 are consistent estimators, we have $\widehat{\Delta}_3$ converges in probability to Δ_3 as $n \to \infty$.

Next, we find the asymptotic distribution of the test statistics. Define

$$\widehat{\Delta}_3 = \frac{\hat{\theta}}{2}\hat{\Delta}_{31} + \frac{3}{2}\hat{\Delta}_{32} - \frac{\hat{\theta}}{2}\hat{\Delta}_{33} + \frac{1}{4}. \tag{5.12}$$

Since $\hat{\theta}$ is a consistent estimator of θ, then by Slutsky's theorem the asymptotic distributions of $\sqrt{n}(\widehat{\Delta}_3 - \Delta_3)$ and $\sqrt{n}(\hat{\Delta}'_3 - E(\Delta'_3))$ are the same. Also as $n \to \infty$, $\sqrt{n}(\hat{\Delta}_3 - \Delta_3)$ converges in distribution to normal random variable with mean 0 and variance σ^2, where $\sigma^2 = \mathrm{Var}(E[h(X_1, X_2)|X_1])$ and $h(X_1, X_2)$ is a symmetric kernel of degree 2 given by

$$\begin{aligned} h(X_1, X_2) &= \frac{1}{2}\left[2\min\{X_1, X_2\} - \frac{1}{\min\{X_1, X_2\}} + \frac{X_1}{X_2} I(X_1 < X_2) + \frac{X_2}{X_1} I(X_2 < X_1)\right. \\ &- \left.\frac{X_1}{X_2^2} I(X_1 < X_2) - \frac{X_2}{X_1^2} I(X_2 < X_1)\right]. \end{aligned} \tag{5.13}$$

To calculate σ^2, consider

$$\begin{aligned} E[\min\{X_1, X_2\}|X_1 = x] &= E[xI(x < X_2) + X_2 I(X_2 \le x)] \\ &= x\int_x^\infty \mathrm{d}F(y) + \int_0^x y\mathrm{d}F(y) \\ &= x\bar{F}(x) + \int_0^x y\mathrm{d}F(y). \end{aligned} \tag{5.14}$$

Also,

$$\begin{aligned} E\left[\frac{1}{\min\{X_1, X_2\}}|X_1 = x\right] &= \frac{1}{x}\bar{F}(x) + \int_0^x y\mathrm{d}F(y) \\ E\left[\frac{X_1}{X_2} I(X_1 < X_2)|X_1 = x\right] &= x\int_x^\infty \frac{1}{y}\mathrm{d}F(y) \\ E\left[\frac{X_2}{X_1} I(X_2 < X_1)|X_1 = x\right] &= \frac{1}{x}\int_0^x y\mathrm{d}F(y) \\ E\left[\frac{X_1}{X_2^2} I(X_1 < X_2)|X_1 = x\right] &= x\int_x^\infty \frac{1}{y^2}\mathrm{d}F(y) \\ E\left[\frac{X_2}{X_1^2} I(X_2 < X_1)|X_1 = x\right] &= \frac{1}{x^2}\int_0^x y\mathrm{d}F(y). \end{aligned}$$

Then, we have

$$\begin{aligned}
\sigma^2 &= Var\left[\frac{\theta}{2}\left(X\bar{F}(X)+\int_0^X y\mathrm{d}F(y)\right)-\frac{\theta}{4}\left(\frac{1}{X}\bar{F}(X)+\int_0^X y\mathrm{d}F(y)\right)\right.\\
&+ \frac{3}{4}\left(X\int_X^\infty \frac{1}{y}\mathrm{d}F(y)+\frac{1}{X}\int_0^X y\mathrm{d}F(y)\right)\\
&- \left.\frac{\theta}{4}\left(X\int_X^\infty \frac{1}{y^2}\mathrm{d}F(y)-\frac{1}{X^2}\int_0^X y\mathrm{d}F(y)\right)\right]. \qquad (5.15)
\end{aligned}$$

Note that $\Delta_3 = 0$ under null hypothesis H_0. Hence, under H_0, as $n \to \infty$, $\sqrt{n}\hat{\Delta}_3$ converges in distribution to a normal random variable with mean zero and variance σ_0^2, where σ_0^2 is the value of σ^2 evaluated under the null hypothesis.

Let $\hat{\sigma}_0^2$ be a consistent estimator of σ_0^2, then at a chosen significance level, α, we reject H_0 against H_1 if

$$\frac{\sqrt{n}|\widehat{\Delta}_3|}{\hat{\sigma}_0^2} > Z_{\frac{\alpha}{2}},$$

where Z_α is the upper α-percentile point for the standard normal distribution. Finding a consistent estimator of the null variance σ_0^2 is difficult. Hence we find the critical region of the proposed test using Monte Carlo simulation. We determine lower (c_1) and upper (c_2) quantiles in such a way that $P(\hat{\Delta}_3 < c_1) = P(\hat{\Delta}_3 > c_2) = \alpha/2$. The finite sample behavior of the test is evaluated through an extensive Monte Carlo simulation study. The results of the simulation study are reported in Section 4.

Remark 5.1. *We can note that, the assumption $\mu = 1$, does not make any impact on the testing procedure, since the family of inverse Gaussian distribution is closed under scale transformations as $X \sim \mathrm{IG}(\mu, \theta)$, implies $\frac{X}{\mu} \sim \mathrm{IG}\left(1, \frac{\theta}{\mu}\right)$.*

5.1.2 Jackknife empirical likelihood (JEL) based test

We also develop a JEL based goodness of fit test for IG distribution with unit parameters. For developing JEL based test, consider the jackknife pseudo values corresponding to the test statistic given in (5.12). The jackknife pseudo values, $\beta_i,\ i = 1, 2, ..., n$ are defined as

$$\beta_i = n\hat{\Delta}_3 - (n-1)\hat{\Delta}_{3i},\ i = 1, 2, ..., n$$

where $\hat{\Delta}_{3i}$ is the value of the test statistic by deleting the i^{th} observation from the sample $X_1, ..., X_n$. Let $q = (q_1, ..., q_n)$ be a probability vector, then $\prod_{i=1}^{n} q_i$ subject to $\sum_{i=1}^{n} q_i = 1$ attains its maximum value at n^{-n} at $q_i = \frac{1}{n}$. Then the JEL ratio for testing the IG distribution with parameters $(1, 1)$, based on the departure measure $\Delta(F)$ is defined as

$$l(\Delta_3) = \max\left\{\prod_{i=1}^{n} nq_i; \sum_{i=1}^{n} q_i = 1, \sum_{i=1}^{n} q_i\beta_i = 0\right\}.$$

Then using the Lagrange multipliers method, we obtain q_i as

$$q_i = \frac{1}{n(1+\nu\beta_i)},$$

and ν satisfies

$$\nu = \frac{1}{n}\sum_{i=1}^{n} \frac{\beta_i}{1+\nu\beta_i} = 0,$$

provided

$$\min_{1\leq k\leq n} \beta_k < \hat{\Delta}_3 < \min_{1\leq k\leq n} \beta_k.$$

For more details on this , see Jing et al. (2009). Hence the jackknife empirical log likelihood ratio is given by

$$\ln\ l(\Delta_3) = -\ln\sum_{i=1}^{n} \ln(1+\nu\beta_i).$$

Now we reject the null hypothesis against the alternative hypothesis for large values of $\ln\ l(\Delta)$. To construct the critical region of the JEL-based test, we find the asymptotic null distribution of the jackknife empirical log-likelihood ratio. Then using Wilk's theorem, we reject the null hypothesis against the alternative at a significance level, α, if

$$-2\ \ln\ l(\Delta_3) > \chi^2_{1,\alpha},$$

where $\chi^2_{1,\alpha}$ is the upper α-percentile point of the χ^2 distribution with one degree of freedom.

5.2 Test statistic: censored data

Next, we discuss how right-censored observations can be incorporated into the proposed testing method. Consider the right-censored data (Y, δ), with $Y = \min(X, C)$ and $\delta = I(X \leq C)$, where C is the censoring time. We assume censoring times and lifetimes are independent. Now we are interested to test the hypothesis discussed in Section 5.1 based on n independent and identical observations $\{(Y_i, \delta_i), 1 \leq i \leq n\}$. As we developed the test based on U-statistics for right censored data, we use the same departure measure $\Delta(F)$ given in equation (1). For that purpose, we consider equation (8) given in Section 2

$$\Delta_3(F) = \frac{\theta}{2} E\left[\min(Y_1, Y_2) - \frac{1}{2\min(Y_1, Y_2)}\right] + \frac{3}{2} E\left[\frac{Y_2}{Y_1} I(Y_2 < Y_1)\right] - \frac{\theta}{2} E\left[\frac{Y_2}{Y_1} I(Y_2 < Y_1)\right] + \frac{1}{4}.$$

To develop the test statistic for right censored case, we estimate each quantity in $\Delta_3(F)$ using U-statistics for right censored data (Datta et al., 2010).

An estimator of $E(\min(Y_1, Y_2) - \frac{1}{\min(Y_1, Y_2)}$ is given by

$$\widehat{U}_{1c} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j<i;j=1}^{n} \frac{(\min(Y_i, Y_j) - 1/\min(Y_i, Y_j))\, \delta_i \delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}, \tag{5.16}$$

provided $\widehat{K}_c(Y_i) > 0$ and $\widehat{K}_c(Y_j) > 0$, with probability 1 and $\widehat{K}_c$ is the Kaplan-Meier estimator of K_c, the survival function of C. Again, an estimator of $E\left[\frac{Y_2}{Y_1} I(Y_2 < Y_1)\right]$ is given by

$$\widehat{U}_{2c} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j<i;j=1}^{n} \frac{(\frac{Y_i}{Y_j} I(Y_i < Y_j) + \frac{Y_j}{Y_i} I(Y_j < Y_i))\delta_i \delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}. \tag{5.17}$$

Similarly, an estimators of θ in the right censored case is given by

$$\theta = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{\delta_i}{Y_i \widehat{K}_c(Y_i)} - 1 \right)^{-1}. \tag{5.18}$$

Using the estimators given in equations (5.16-5.18), we obtain the test statistic as

$$\widehat{\Delta}_c = \frac{\widehat{\theta}}{2}\widehat{U}_{1c} + \frac{2}{3}\widehat{U}_{2c} - \frac{\widehat{\theta}}{2}\widehat{U}_{1c} + \frac{1}{4}. \tag{5.19}$$

Hence in the right censored case, we reject H_0 in favor of H_1 for large values of $\widehat{\Delta}_c$.

To obtain the limiting distribution of $\widehat{\Delta}_c$, let $N_i^c(t) = I(Y_i \leq t, \delta_i = 0)$ be the counting process corresponds to the censoring variable C_i. Denote $R_i(t) = I(Y_i \geq t)$. Also let σ_c^2 be the hazard rate of C. The martingale associated with this counting process $N_i^c(t)$ is given by

$$M_i^c(t) = N_i^c(t) - \int_0^t R_i(u)\sigma_c^2(u)du.$$

Let $G(x,y) = P(X_1 \leq x, Y_1 \leq y,\, \delta = 1),\, x \in \mathcal{X},\, \bar{H}(t) = P(Y_1 > t)$ and

$$w(t) = \frac{1}{\bar{H}(t)} \int_{\mathcal{X}\times[0,\infty)} \frac{h_1(x)}{K_c(y-)} I(y > t) dG(x,y),$$

where $h_1(x) = E(h(X_1, X_2)|X_1 = x)$. The proof of next result follows from Theorem 1 of Datta et al. (2010) for a particular choice of the kernel

$$h(X_1, X_2) = \frac{\theta}{2}\left[\min(X_1, X_2) - \frac{1}{2\min(X_1, X_2)}\right] + \frac{3}{2}\left[\frac{X_2}{X_1} I(X_2 < X_1)\right] - \frac{\theta}{2}\left[\frac{X_2}{X_1} I(X_2 < X_1)\right] + \frac{1}{4}.$$

Theorem 5.2. *Let*

$$h_1(x) = \frac{\theta}{2}E\left[\min(x, Y_2) - \frac{1}{2\min(x, Y_2)}\right] + \frac{3}{2}E\left[\frac{Y_2}{x} I(Y_2 < x)\right] - \frac{\theta}{2}E\left[\frac{Y_2}{x} I(Y_2 < x)\right] + \frac{1}{4}.$$

Suppose the conditions

$$E\left[\frac{\theta}{2}E\left[\min(Y_1, Y_2) - \frac{1}{2\min(Y_1, Y_2)}\right] + \frac{3}{2}E\left[\frac{Y_2}{Y_1} I(Y_2 < Y_1)\right] - \frac{\theta}{2}E\left[\frac{Y_2}{Y_1} I(Y_2 < Y_1)\right] + \frac{1}{4}\right]^2 < \infty,$$

$\int_{\mathcal{X}\times[0,\infty)} \frac{h_1^2(x)}{K_c^2(y)} dG(x,y) < \infty$ and $\int_0^\infty w^2(t)\sigma_c^2(t)dt < \infty$ holds.

As $n \to \infty$, $\sqrt{n}(\widehat{\Delta_3}_c - \Delta_3(F))$ converges in distribution to Gaussian random variable with mean zero and variance $4\sigma_c^2$, where σ_c^2 is given by

$$\sigma_c^2 = Var\Big(\frac{h_1(X)\delta_1}{K_c(Y_1-)} + \int w(t)dM_1^c(t)\Big).$$

Next we find an estimator of σ_c^2 using the reweighed techniques. An estimator of σ_c^2 is given by

$$\widehat{\sigma}_c^2 = \frac{4}{(n-1)}\sum_{i=1}^{n}(V_i - \bar{V})^2,$$

where

$$V_i = \frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} + \widehat{w}(X_i)(1-\delta_i) - \sum_{j=1}^{n}\frac{\widehat{w}(X_i)I(X_i > X_j)(1-\delta_i)}{\sum_{i=1}^{n} I(X_i > X_j)},$$

$$\bar{V} = \frac{1}{n}\sum_{i=1}^{n} V_i, \quad \widehat{h}_1(X) = \frac{1}{n}\sum_{i=1}^{n}\frac{h(X,Y_i)\delta_i}{\widehat{K}_c(Y_i-)}, \quad R(t) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i > t)$$

and

$$\widehat{w}(t) = \frac{1}{R(t)}\sum_{i=1}^{n}\frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} I(X_i > t).$$

Let $\widehat{\sigma}_{0c}^2$ be the value of $\widehat{\sigma}_c^2$ evaluated under H_0. Under right censored situation, we reject the null hypothesis H_0 against the alternative H_1 at a significance level α, if

$$\frac{\sqrt{n}|\widehat{\Delta}_c|}{\widehat{\sigma}_{0c}} > Z_{\alpha/2}.$$

The results of the Monte Carlo simulation which assess the finite sample performance of the test is also reported in Section 5.3.

5.3 Simulation study

The finite sample performance of the proposed test procedures is evaluated through extensive Monte Carlo simulation studies using R software (version 4.1.3). We compare the empirical type I error and power of our test with the existing tests for complete data, to show the competitiveness of the proposed tests, in both uncensored and censored cases. The power of the tests is evaluated against various alternatives, which are the commonly employed parametric lifetime models. In the censored case, we evaluate the empirical power at various censoring percentages, to depict the effect of censoring on the proposed test.

5.3.1 Uncensored case

We find the empirical type I error and empirical power of the proposed test and other competitive tests to conduct a comprehensive study. We simulate random samples of different sizes ($n = 10, 20, 30, 40, 50$) from IG distribution with parameter value $\theta = 1$ or 2. We then calculate the test statistic value for the simulated random samples and based on the value of it, we accept or reject the null hypothesis. This procedure is repeated ten thousand times and the proportion of times, the null hypothesis rejected is observed. This value gives the empirical type I error of the proposed test.

We follow a similar procedure to calculate the empirical power of the proposed test. To find the empirical power, lifetime random variables are generated from different choices of alternatives including Weibull, gamma, log-normal, Pareto, and half-normal distributions. The choices of alternative distributions along with the corresponding cumulative distribution functions (CDF), employed in this study are listed in Table 6.2. To calculate the empirical power of the test, first, we generate lifetime data from the desired alternative and calculate the test statistic. We then estimate the maximum likelihood estimator $\hat{\theta}$ of the parameter θ, of the generated data. Next, to compute the simulated critical points, we generate a sample of size n from inverse Gaussian distribution with parameter $\hat{\theta}$, 10000 times. Based on the test statistic value and the simulated critical points, we determine whether the null hypothesis is to be rejected or not. The entire procedure is repeated 10000 times and the empirical power is computed as the proportion of rejections of the test.

Table 5.1: Choices of alternative distributions with the corresponding CDFs

Distribution	CDF
Weibull	$F_1(x) = 1 - e^{\left(-\frac{x}{\lambda}\right)^k}$, $x > 0, k, \lambda > 0$
Rayleigh	$F_2(x) = 1 - \exp\{\frac{-x^2}{2\lambda^2}\}$, $x > 0, \lambda > 0$
Chi-square	$F_3(x) = \frac{1}{\Gamma\left(\frac{k}{2}\right)}\gamma\left(\frac{k}{2}, \frac{x}{2}\right)$, $x > 0, k > 0$ and $\gamma(k, \frac{x}{\lambda})$ is the lower incomplete gamma function
Gamma	$F_4(x) = \frac{1}{\Gamma(k)}\gamma\left(k, \frac{x}{\lambda}\right)$, $x > 0, k, \lambda > 0$ and $\gamma\left(k, \frac{x}{\lambda}\right)$ is the lower incomplete gamma function
Log-normal	$F_5(x) = \Phi\left(\frac{\ln x - \mu}{\lambda}\right)$, $x > 0$, $-\infty < \mu < \infty$, $\lambda > 0$ where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable
Half-normal	$F_6(x) = erf\left(\frac{x}{\lambda\sqrt{2}}\right)$, $x > 0, \lambda > 0$, where *erf* is the error function
Pareto	$F_7(x) = (\lambda/x)^k$ $x > 0, k, \lambda > 0$

We test the competitiveness of our test with the classical tests for goodness of fit include Kolmogorov-Smirnov (KS), Cramer-von Mises (CvM) and Anderson-Darling (AD) tests. Moreover, we compare the competitiveness of our test with the another three tests proposed specifically for testing IG distribution, given by Henze and Klar (2002) and González-Estrada and Villaseñor (2018). In Henze and Klar (2002), they developed a test statistic for goodness of fit for IG distribution as the weighted integrals over the squared modulus of a measure of deviation of the empirical distribution of given data from the family of inverse Gaussian laws, expressed by means of the empirical Laplace transform. A variance ratio test for the goodness of fit test for IG distribution is proposed in Henze and Klar (2002) by exploiting the relation between gamma and IG distributions, while Villaseñor et al. (2019) developed by transforming IG variables to normal variables and then by employing Shapiro-Wilk test for normality. Illustration of both these tests are explained in González-Estrada and Villaseñor (2018). We use the above three tests to compare the performance of the proposed test statistic in simulation studies. We denote the test by Henze and Klar (2002) as 'HK' and the tests by Henze and Klar (2002) and Villaseñor et al. (2019) as 'EV1' and 'EV2' in the tables presenting simulation results. The expressions for the classical test statistics KS, CvM and AD are also summarized below.

The Kolmogorov-Smirnov test statistic is given by KS=max$\{D^+, D^-\}$ where

$$D^+ = \max_{i=1,2,\ldots,n}\left(\frac{i}{n} - F_0(X_{(i)})\right) \quad \text{and} \quad D^- = \max_{i=1,2,\ldots,n}\left(F_0(X_{(i)}) - \frac{i-1}{n}\right);$$

the Cramer-von Mises test statistic is given by

$$CvM = \frac{1}{12n} + \sum_{i=1}^{n}\left(F_0(X_{(i)}) - \frac{2i-1}{2n}\right)^2;$$

and the Anderson-Darling test statistic is given by $AD = -n - S$, where

$$S = \sum_{i=1}^{n}\frac{2i-1}{n}[\ln F_0(X_{(i)}) + \ln(1 - F_0(X_{n+1-i}))],$$

where $F_0(.)$ is the specified distribution function.

The calculated empirical type I error of the proposed test and other competing tests are reported in Table 5.2. The same is visually represented in Figure 5.1.

From Table 5.2 and Figure 5.1, we can observe that as n increases, the test $\widehat{\Delta}$ stabilizes near the chosen level of significance for both $\alpha = 0.05$ and $\alpha = 0.01$. This observation is true for both choices of parameters 1 and 2. The tests HK, EV1, and EV2 exhibit similar behavior. The classical tests KS, CvM, and AD provide type I error values slightly higher than the desired significance level for both choices of α.

The results of the simulation study for power comparison are tabulated in Tables 5.3 and 5.4. Table 5.3 gives the power of $\widehat{\Delta}$ for different alternatives at significance level $\alpha = 0.01$ and Table 5.4 represents the same when $\alpha = 0.05$. The results in Tables 5.3 and 5.4 are portrayed visually in Figures 5.2 and 5.3. Tables 5.3, 5.4 and Figures 5.2, 5.3 clearly show that, the newly developed test achieves a higher empirical power compared to all the other goodness of fit tests. Additionally, the power of all tests increases with sample size. Notably, the proposed test demonstrates high power even for small sample sizes, highlighting its efficiency. Furthermore the proposed test, attains the maximum power against the alternatives, Rayleigh, lognormal and Pareto distributions for the

Table 5.2: Comparison of empirical type I error at α=0.01 and α=0.05

	n	α	$\hat{\Delta}_3$	HK	EV1	EV2	KS	CvM	AD
InvG (1, 1)	10	0.01	0.0110	0.0094	0.0128	0.0077	0.0478	0.0323	0.0510
	20		0.0101	0.0093	0.0112	0.0110	0.0345	0.0214	0.0262
	30		0.0088	0.0096	0.0085	0.0100	0.0301	0.0178	0.0191
	40		0.0106	0.0109	0.0099	0.0104	0.0276	0.0153	0.0168
	50		0.0110	0.0094	0.0097	0.0112	0.0293	0.0146	0.0153
InvG (1, 1)	10	0.05	0.0519	0.0441	0.0559	0.0437	0.1067	0.1102	0.1259
	20		0.0495	0.0481	0.0549	0.0474	0.0945	0.0759	0.0909
	30		0.0582	0.0570	0.0502	0.0485	0.0945	0.0778	0.0775
	40		0.0455	0.0475	0.0492	0.0476	0.0857	0.0672	0.0747
	50		0.0486	0.0526	0.0474	0.0472	0.0810	0.0641	0.0666
InvG (1, 2)	10	0.01	0.0109	0.0108	0.0137	0.0093	0.0355	0.0265	0.0347
	20		0.0121	0.0093	0.0082	0.0095	0.0220	0.0176	0.0182
	30		0.0101	0.0088	0.0099	0.0110	0.0237	0.0164	0.0146
	40		0.0113	0.0113	0.0100	0.0093	0.0198	0.0122	0.0134
	50		0.0138	0.0102	0.0111	0.0109	0.0176	0.0139	0.0105
InvG (1, 2)	10	0.05	0.0526	0.0535	0.0476	0.0380	0.0903	0.0894	0.1017
	20		0.0522	0.0493	0.0532	0.0474	0.0745	0.0657	0.0681
	30		0.0531	0.0509	0.0513	0.0474	0.0716	0.0660	0.0631
	40		0.0538	0.0508	0.0460	0.0470	0.0635	0.0593	0.0586
	50		0.0499	0.0515	0.0543	0.0481	0.0668	0.0550	0.0591

chosen parameter settings. Among the tests, the HK test exhibits the lowest power. In contrast, EV1 and EV2 deliver comparable results. The classical goodness-of-fit tests generally have low power for small sample sizes ($n = 10$), though their performance improves significantly with larger sample sizes. This could be attributed to the higher rejection rates under the null hypothesis associated with these classical tests. The proposed test consistently demonstrates robust power, even with small sample sizes, confirming its effectiveness.

5.3.2 Censored case

We carry out Monte Carlo simulation studies to calculate the empirical type I error and the empirical power of the test statistic proposed for right censored data. We generate random samples of sizes $n = 50, 75, 100, 200$ and 400 to evaluate the empirical type I

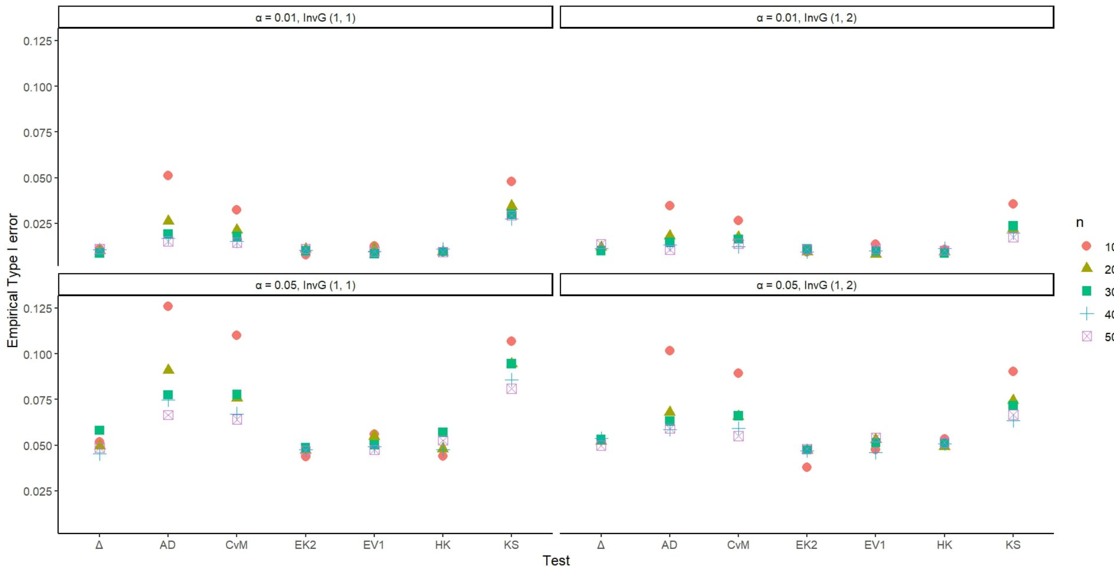

Figure 5.1: Plot of empirical type I error at α=0.01 and α=0.05

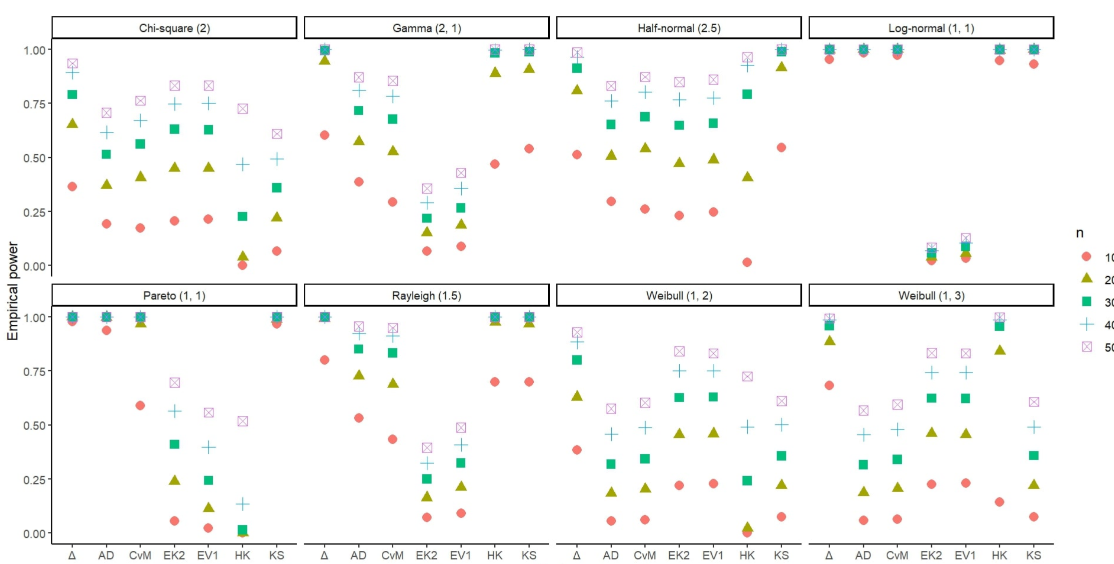

Figure 5.2: Plot of empirical power when α=0.01

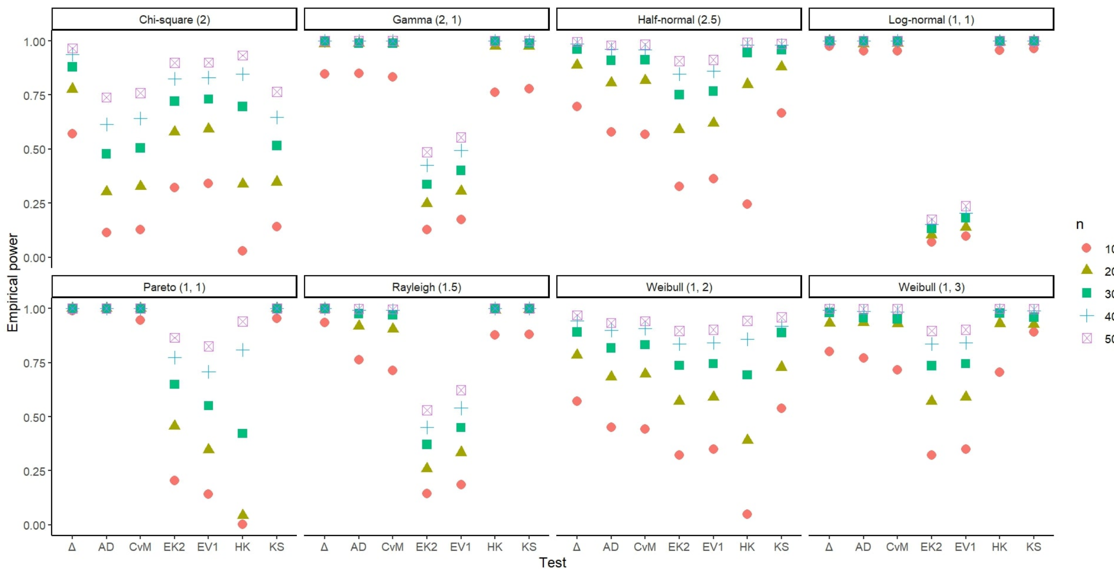

Figure 5.3: Plot of empirical power when α=0.05

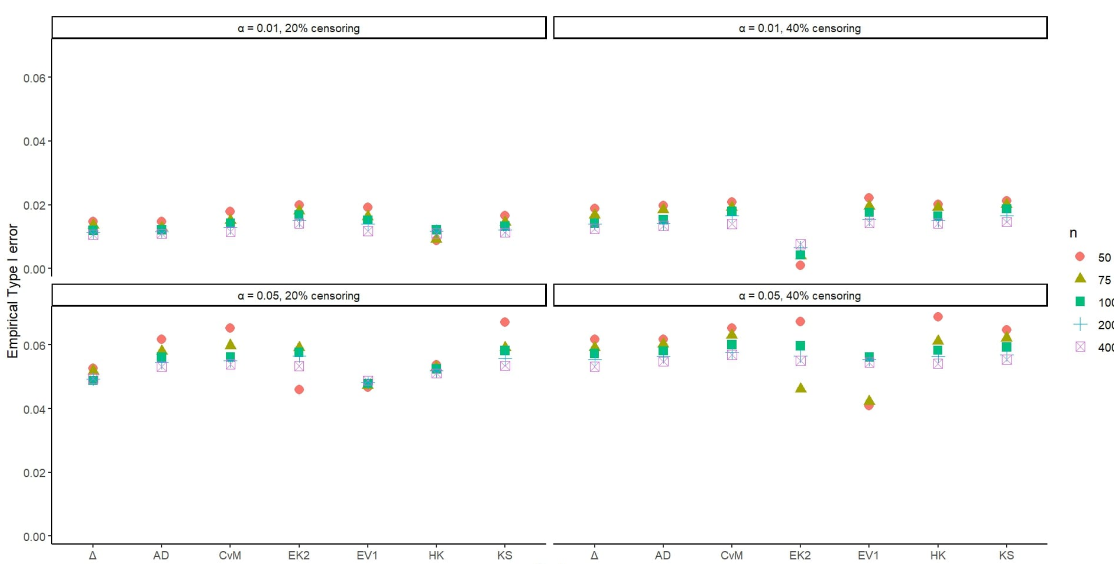

Figure 5.4: Plot of empirical type I error at α=0.01 and α=0.05 for IG (1, 1)

Table 5.3: Comparison of empirical power for different alternatives ($\alpha = 0.01$)

	n	$\hat{\Delta}_3$	HK	EV1	EVf2	KS	CvM	AD
Weibull (1,2)	10	0.3839	0.0008	0.2274	0.2208	0.0742	0.0621	0.0567
	20	0.6293	0.0235	0.4595	0.4557	0.2204	0.2045	0.1843
	30	0.7999	0.2424	0.6297	0.6269	0.3571	0.3438	0.3180
	40	0.8840	0.4924	0.7519	0.7507	0.5019	0.4876	0.4592
	50	0.9297	0.7253	0.8324	0.8409	0.6109	0.6032	0.5765
Weibull (1,3)	10	0.6825	0.1433	0.2305	0.2253	0.0747	0.0652	0.0579
	20	0.8860	0.8416	0.4546	0.4612	0.2202	0.2060	0.1879
	30	0.9584	0.9564	0.6230	0.6237	0.3586	0.3401	0.3158
	40	0.9878	0.9878	0.7440	0.7421	0.4914	0.4813	0.4560
	50	0.9939	0.9990	0.8314	0.8339	0.6078	0.5952	0.5684
Rayleigh (1.5)	10	0.8000	0.6986	0.0923	0.0736	0.6993	0.4328	0.5333
	20	0.9924	0.9749	0.2110	0.1639	0.9683	0.6887	0.7271
	30	0.9996	0.9972	0.3244	0.2504	0.9982	0.8341	0.8516
	40	1.0000	0.9998	0.4093	0.3248	0.9999	0.9133	0.9232
	50	1.0000	1.0000	0.4885	0.3958	1.0000	0.9493	0.9559
Chi-Square (2)	10	0.3654	0.0012	0.2141	0.2072	0.0665	0.1731	0.1915
	20	0.6525	0.0381	0.4507	0.4493	0.2203	0.4072	0.3699
	30	0.7911	0.2267	0.6271	0.6310	0.3594	0.5621	0.5137
	40	0.8928	0.4702	0.7516	0.7482	0.4946	0.6706	0.6156
	50	0.9353	0.7269	0.8323	0.8330	0.6099	0.7631	0.7071
Gamma (2, 1)	10	0.6020	0.4683	0.0895	0.0676	0.5404	0.2951	0.3868
	20	0.9444	0.8891	0.1869	0.1518	0.9076	0.5236	0.5732
	30	0.9942	0.9838	0.2672	0.2189	0.9883	0.6764	0.7168
	40	0.9996	0.9982	0.3564	0.2920	0.9990	0.7846	0.8103
	50	0.9998	0.9997	0.4291	0.3566	0.9998	0.8552	0.8706
Log-normal (1,1)	10	0.9531	0.9494	0.0344	0.0234	0.9303	0.9720	0.9823
	20	0.9979	0.9958	0.0571	0.0396	0.9995	0.9999	0.9999
	30	0.9999	0.9999	0.0881	0.0587	1.0000	1.0000	1.0000
	40	1.0000	1.0000	0.1051	0.0690	1.0000	1.0000	1.0000
	50	1.0000	1.0000	0.1286	0.0833	1.0000	1.0000	1.0000
Half-normal (2.5)	10	0.5140	0.0156	0.2468	0.2309	0.5447	0.2606	0.2957
	20	0.8094	0.4056	0.4894	0.4719	0.9152	0.5398	0.5058
	30	0.9122	0.7915	0.6580	0.6486	0.9898	0.6873	0.6525
	40	0.9653	0.9270	0.7764	0.7687	0.9989	0.8028	0.7615
	50	0.9857	0.9652	0.8599	0.8500	0.9999	0.8727	0.8321
Pareto (1, 1)	10	0.9791	0.0001	0.0240	0.0566	0.9675	0.5886	0.9372
	20	0.9994	0.0004	0.1123	0.2401	0.9892	0.9671	0.9988
	30	0.9999	0.0152	0.2431	0.4107	0.9999	0.9988	0.9999
	40	1.0000	0.1348	0.3972	0.5661	1.0000	1.0000	1.0000
	50	1.0000	0.5183	0.5582	0.6958	1.0000	1.0000	1.0000

Table 5.4: Comparison of empirical power for different alternatives ($\alpha = 0.05$)

	n	$\hat{\Delta}_3$	HK	EV1	EV2	KS	CvM	AD
Weibull (1,2)	10	0.5705	0.0501	0.3502	0.3234	0.5388	0.4438	0.4498
	20	0.7848	0.3908	0.5901	0.5709	0.7283	0.6976	0.6837
	30	0.8915	0.6932	0.7452	0.7360	0.8886	0.8320	0.8166
	40	0.9427	0.8578	0.8409	0.8375	0.9187	0.9080	0.8989
	50	0.9687	0.9426	0.9021	0.8969	0.9595	0.9420	0.9338
Weibull (1,3)	10	0.7995	0.7056	0.3496	0.3225	0.8905	0.7161	0.7696
	20	0.9321	0.9306	0.5900	0.5714	0.9262	0.9304	0.9357
	30	0.9816	0.9785	0.7450	0.7358	0.9598	0.9515	0.9568
	40	0.9919	0.9915	0.8407	0.8370	0.9881	0.9840	0.9873
	50	0.9988	0.9985	0.9020	0.8968	0.9971	0.9979	0.9980
Rayleigh (1.5)	10	0.9343	0.8786	0.1857	0.1435	0.8810	0.7122	0.7612
	20	0.9982	0.9935	0.3345	0.2599	0.9940	0.9049	0.9191
	30	1.0000	0.9995	0.4497	0.3713	0.9999	0.9703	0.9757
	40	1.0000	1.0000	0.5397	0.4517	1.0000	0.9909	0.9929
	50	1.0000	1.0000	0.6228	0.5300	1.0000	0.9965	0.9969
Chi-Square (2)	10	0.5721	0.0303	0.3422	0.3216	0.1431	0.1281	0.1140
	20	0.7773	0.3391	0.5935	0.5797	0.3482	0.3284	0.3034
	30	0.8804	0.6972	0.7304	0.7218	0.5167	0.5049	0.4781
	40	0.9385	0.8463	0.8313	0.8262	0.6473	0.6420	0.6152
	50	0.9656	0.9326	0.8999	0.8983	0.7643	0.7602	0.7388
Gamma (2, 1)	10	0.8481	0.7622	0.1732	0.1277	0.7783	0.8339	0.8493
	20	0.9859	0.9749	0.3063	0.2486	0.9766	0.9886	0.9886
	30	0.9985	0.9984	0.4016	0.3375	0.9889	0.9897	0.9896
	40	1.0000	0.9997	0.4937	0.4257	1.0000	1.0000	1.0000
	50	1.0000	1.0000	0.5545	0.4867	1.0000	1.0000	1.0000
Log-normal (1,1)	10	0.9749	0.9558	0.0976	0.0694	0.9638	0.9531	0.9549
	20	0.9987	0.9958	0.1384	0.1025	0.9986	0.9898	0.9857
	30	0.9999	1.0000	0.1812	0.1320	1.0000	1.0000	1.0000
	40	1.0000	1.0000	0.2055	0.1527	1.0000	1.0000	1.0000
	50	1.0000	1.0000	0.2366	0.1760	1.0000	1.0000	1.0000
Half-normal (2.5)	10	0.6978	0.2459	0.3618	0.3283	0.6665	0.5686	0.5780
	20	0.8884	0.7993	0.6206	0.5897	0.8799	0.8158	0.8066
	30	0.9606	0.9453	0.7670	0.7509	0.9579	0.9129	0.9100
	40	0.9864	0.9802	0.8612	0.8482	0.9798	0.9600	0.9606
	50	0.9960	0.9921	0.9127	0.9064	0.9872	0.9834	0.9797
Pareto (1, 1)	10	0.9886	0.0015	0.1406	0.2061	0.9528	0.9468	0.9980
	20	0.9998	0.0435	0.3462	0.4566	0.9999	0.9999	1.0000
	30	1.0000	0.4225	0.5504	0.6489	1.0000	1.0000	1.0000
	40	1.0000	0.8100	0.7077	0.7733	1.0000	1.0000	1.0000
	50	1.0000	0.9407	0.8247	0.8646	1.0000	1.0000	1.0000

error and power of the proposed test in the presence of censoring. Lifetimes are generated from inverse Gaussian distribution with parameters $(1, 1)$ to calculate the empirical type I error. The computational procedures given in the absence of censoring to calculate empirical type I error/ power is used to calculate the same in presence of censoring also. We consider different choices of alternatives, as in uncensored case for finding the empirical power. The percentage of censoring is chosen to be 20% or 40%, to examine the effect of censoring on the proposed test statistic. In all cases, the censoring random variable C is generated from an exponential distribution with parameter b, where b is chosen such that $P(T > C) = 0.2$ or 0.4. Re-weighting technique explained in Section 3 is used to estimate the variance of $\widehat{\Delta}_{2c}$.

As mentioned earlier, the existing goodness of fit tests in literature, consider complete data only. Hence, to compare the efficiency of the proposed test, we generate pseudo-complete random sample using the method proposed by Balakrishnan et al. (2015). Following Balakrishnan et al. (2015), to generate the pseudo-complete sample, for each Y_i with $\delta_i = 0$, a value $\hat{Y}_i$ is generated as $\hat{Y}_i = F_0^{-1}(\zeta_i)$, where $\zeta_i \sim U[F_0(C_i), 1)$, where $F_0(.)$ is the corresponding distribution function and C_i is the censoring time for $i = 1, 2, ..., n$. The corresponding values of censored observations in the original sample are then replaced by these generated values to obtain the pseudo-complete sample. Goodness of fit tests developed for complete sample can be then applied to the pseudo-complete sample.

To compare the proficiency of the proposed test for censored samples, we consider the same tests described in Section 4.2. We perform the simulation study for all the choices of alternatives, since the results are similar, we present the results for gamma, log-normal, Weibull and Rayleigh distributions only. First we generate a censored sample from the chosen alternative with 20% or 40% of censored observations, while the censored observations are generated from an exponential distribution as mentioned above. The censored sample is then converted to pseudo-complete sample using the method explained above. Results of the simulation study when the censoring percentage is chosen as 20% are presented in Table 5.5 and the same for 40% censoring is shown in Table 5.6. In Tables

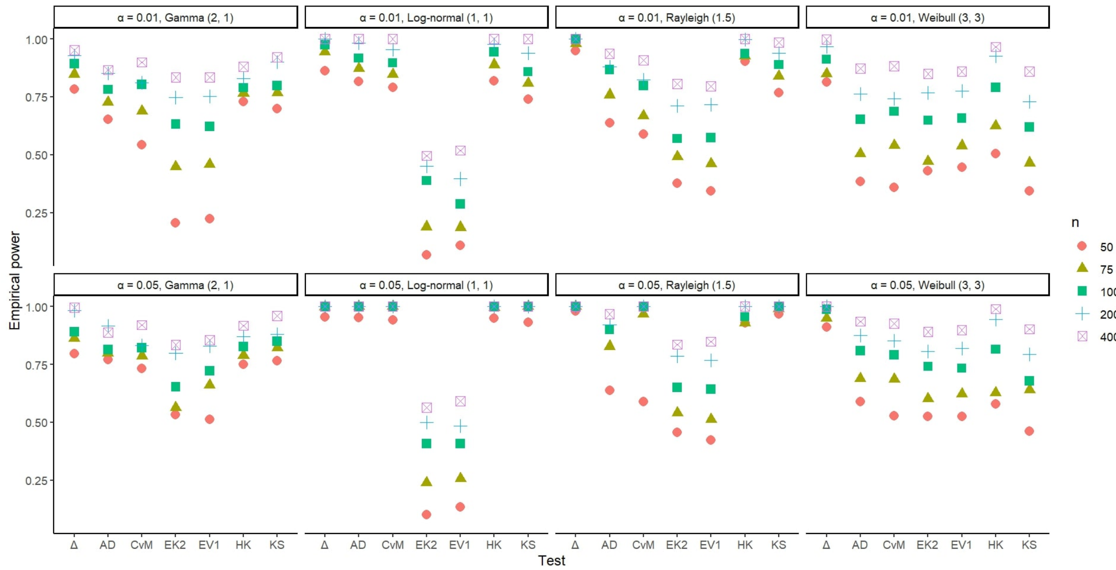

Figure 5.5: Plot of empirical power for 20% censoring

5.5 and 5.6, values corresponding to inverse Gaussian distribution give the empirical type I error and all the values corresponding to all other choices of alternatives give the empirical power of the test.

From Tables 5.5 and 5.6, it is evident that, as n increases, the empirical type I error of the test converges to the specified significance levels $\alpha = 0.01$ or $\alpha = 0.05$. Additionally, for all alternative distributions considered, the proposed test demonstrates superior power compared to tests based on the pseudo-complete sample. Specifically, the tests EV1 and EV2 exhibit lower power relative to others, while the HK test and the classical tests KS, CvM, and AD provide higher power than EV1 and EV2 but fall short of the performance of the newly proposed test. This pattern holds true across both significance levels and under various censoring scenarios. This can be explained by the fact that the proposed test incorporates the censored lifetimes as such in computing the test statistic, while the other tests based on pseudo-complete sample need to convert the censored lifetimes into observed lifetimes. The empirical power of the test increases with increase in n, while it shows a decreasing tendency with the increase in censoring percentage.

To depict the results of the simulation study, we plot the values of empirical type I error and power in Figures 5.4, 5.5 and 5.6. It is evident from Figure 5.4 that, the empirical type I error is approaching the chosen significance level as n increases. Figures 5.5 and

Table 5.5: Comparison of empirical power for different alternatives : 20% censoring

	n	α	$\hat{\Delta}_3$	HK	EV1	EV2	KS	CvM	AD
Inv Gauss (1,1)	50	0.01	0.0147	0.0088	0.0192	0.0199	0.0165	0.0178	0.0148
	75		0.0136	0.0092	0.0163	0.0181	0.0145	0.0151	0.0129
	100		0.0119	0.0121	0.0152	0.0168	0.0132	0.0142	0.0121
	200		0.0113	0.0118	0.0139	0.0150	0.0121	0.0129	0.0115
	400		0.0106	0.0110	0.0117	0.0141	0.0113	0.0115	0.0109
Inv Gauss (1,1)	50	0.05	0.0527	0.0537	0.0467	0.0460	0.0671	0.0652	0.0617
	75		0.0518	0.0530	0.0472	0.0592	0.0591	0.0598	0.0581
	100		0.0488	0.0525	0.0479	0.0577	0.0582	0.0562	0.0562
	200		0.0493	0.0519	0.0481	0.0565	0.0558	0.0551	0.0545
	400		0.0495	0.0511	0.0488	0.0534	0.0535	0.0540	0.0532
Gamma (2,1)	50	0.01	0.7823	0.7286	0.2248	0.2072	0.6993	0.5428	0.6533
	75		0.8482	0.7649	0.4586	0.4483	0.7683	0.6887	0.7271
	100		0.8921	0.7892	0.6221	0.6327	0.7982	0.8041	0.7816
	200		0.9282	0.8298	0.7518	0.7485	0.8999	0.8133	0.8532
	400		0.9521	0.8808	0.8345	0.8349	0.9213	0.8993	0.8656
Gamma (2,1)	50	0.05	0.7965	0.7512	0.5123	0.5336	0.7665	0.7311	0.7715
	75		0.8625	0.7891	0.6610	0.5639	0.8209	0.7872	0.7996
	100		0.8911	0.8267	0.7221	0.6527	0.8494	0.8221	0.8137
	200		0.9828	0.8701	0.8293	0.7985	0.8794	0.8307	0.9159
	400		0.9953	0.9169	0.8545	0.8349	0.9599	0.9201	0.8872
Log-normal (1,1)	50	0.01	0.8612	0.8183	0.1095	0.0676	0.7404	0.7915	0.8167
	75		0.9448	0.8891	0.1869	0.1896	0.8076	0.8461	0.8721
	100		0.9742	0.9439	0.2877	0.3889	0.8585	0.8962	0.9168
	200		0.9996	0.9772	0.3964	0.4520	0.9390	0.9546	0.9821
	400		1.0000	0.9998	0.5191	0.4961	0.9998	1.0000	1.0000
Log-normal (1,1)	50	0.05	0.9531	0.9494	0.1344	0.1024	0.9303	0.9422	0.9524
	75		0.9979	0.9958	0.2571	0.2398	0.9995	0.9999	0.9999
	100		0.9999	0.9999	0.4083	0.4087	1.0000	1.0000	1.0000
	200		1.0000	1.0000	0.4854	0.4990	1.0000	1.0000	1.0000
	400		1.0000	1.0000	0.5912	0.5632	1.0000	1.0000	1.0000
Weibull (3,3)	50	0.01	0.8145	0.5057	0.4465	0.4309	0.3447	0.3606	0.3857
	75		0.8498	0.6256	0.5394	0.4719	0.4652	0.5398	0.5058
	100		0.9125	0.7915	0.6580	0.6486	0.6198	0.6873	0.6525
	200		0.9659	0.9270	0.7764	0.7687	0.7289	0.7428	0.7615
	400		0.9971	0.9652	0.8599	0.8500	0.8599	0.8828	0.8722
Weibull (3,3)	50	0.05	0.9113	0.5801	0.5243	0.5266	0.4617	0.5286	0.5891
	75		0.9494	0.6281	0.6229	0.6012	0.6399	0.6872	0.6888
	100		0.9869	0.8152	0.7331	0.7410	0.6791	0.7911	0.8091
	200		0.9999	0.9448	0.8179	0.8068	0.7942	0.8522	0.8763
	400		1.0000	0.9883	0.8989	0.8901	0.9021	0.9265	0.9348
Rayleigh(1.5)	50	0.01	0.9492	0.9029	0.3440	0.3766	0.7672	0.5886	0.6384
	75		0.9795	0.9252	0.4608	0.4913	0.8382	0.6676	0.7590
	100		0.9997	0.9367	0.5731	0.5701	0.8890	0.7987	0.8677
	200		1.0000	0.9981	0.7172	0.7111	0.9387	0.8251	0.8801
	400		1.0000	1.0000	0.7952	0.8058	0.9843	0.9082	0.9356
Rayleigh(1.5)	50	0.05	0.9791	0.9283	0.4243	0.4566	0.9675	0.5886	0.6377
	75		0.9994	0.9288	0.5123	0.5405	0.9892	0.9671	0.8281
	100		0.9999	0.9557	0.6431	0.6507	0.9999	0.9988	0.8991
	200		1.0000	0.9992	0.7672	0.7861	1.0000	1.0000	0.9212
	400		1.0000	1.0000	0.8482	0.8358	1.0000	1.0000	0.9671

Table 5.6: Comparison of empirical power for different alternatives : 40% censoring

	n	α	$\hat{\Delta}_3$	HK	EV1	EV2	KS	CvM	AD
Inv Gauss (1,1)	50	0.01	0.0188	0.0201	0.0222	0.0009	0.0212	0.0209	0.0198
	75		0.0167	0.0191	0.0195	0.0039	0.0201	0.0192	0.0184
	100		0.0142	0.0165	0.0176	0.0042	0.0187	0.0179	0.0153
	200		0.0139	0.0150	0.0154	0.0065	0.0166	0.0166	0.0142
	400		0.0125	0.0142	0.0143	0.0076	0.0146	0.0140	0.0134
Inv Gauss (1,1)	50	0.05	0.0617	0.0688	0.0410	0.0674	0.0647	0.0652	0.0617
	75		0.0591	0.0612	0.0422	0.4612	0.0622	0.0631	0.0602
	100		0.0573	0.0583	0.0562	0.0597	0.0593	0.0601	0.0581
	200		0.0555	0.0564	0.0554	0.0565	0.0569	0.0577	0.0564
	400		0.0532	0.0541	0.0545	0.0551	0.0554	0.0569	0.0548
Gamma (2,1)	50	0.01	0.7203	0.6586	0.1928	0.1837	0.5826	0.5028	0.5228
	75		0.7965	0.7049	0.3512	0.3398	0.6683	0.5871	0.6841
	100		0.8341	0.7477	0.5644	0.4561	0.7188	0.6864	0.7054
	200		0.8867	0.7961	0.6987	0.6890	0.7699	0.7865	0.7751
	400		0.9054	0.8114	0.7854	0.7981	0.8298	0.8235	0.8310
Gamma (2,1)	50	0.05	0.6563	0.6989	0.4314	0.5089	0.6465	0.5721	0.6321
	75		0.7825	0.7437	0.4507	0.6193	0.7263	0.7174	0.7399
	100		0.8390	0.7983	0.6271	0.7110	0.7591	0.7622	0.7637
	200		0.8998	0.8341	0.7516	0.7182	0.8441	0.8206	0.8454
	400		0.9355	0.8862	0.8323	0.7901	0.8992	0.8733	0.8471
Log-normal (1,1)	50	0.01	0.6028	0.5672	0.0795	0.0467	0.6501	0.6551	0.6868
	75		0.7843	0.7802	0.1386	0.1518	0.7274	0.7235	0.7734
	100		0.8814	0.8486	0.2577	0.2881	0.7901	0.7847	0.8104
	200		0.9190	0.8910	0.3564	0.3922	0.8790	0.8671	0.9028
	400		0.9518	0.9522	0.4291	0.4264	0.9209	0.9152	0.9306
Log-normal (1,1)	50	0.05	0.7513	0.6954	0.1147	0.0935	0.7801	0.7620	0.7823
	75		0.8461	0.8231	0.1957	0.2096	0.8322	0.8451	0.8652
	100		0.9312	0.8992	0.2982	0.3105	0.9182	0.8851	0.9021
	200		0.9629	0.9337	0.3877	0.3869	0.9462	0.9342	0.9487
	400		0.9987	0.9765	0.5183	0.5333	0.9712	0.9763	0.9877
Weibull (3,3)	50	0.01	0.7245	0.4391	0.3826	0.3708	0.3289	0.3060	0.3289
	75		0.7991	0.5473	0.4994	0.4114	0.4015	0.4622	0.4461
	100		0.8622	0.6311	0.6078	0.5889	0.4980	0.5672	0.5921
	200		0.9320	0.8192	0.7216	0.6467	0.6500	0.6892	0.6981
	400		0.9561	0.8892	0.8014	0.7901	0.8109	0.8321	0.7905
Weibull (3,3)	50	0.05	0.8213	0.4981	0.4788	0.4562	0.3981	0.4590	0.4781
	75		0.8618	0.5790	0.5671	0.5712	0.5899	0.6172	0.6380
	100		0.9347	0.7234	0.6891	0.6902	0.6399	0.7241	0.7691
	200		0.9785	0.8214	0.7966	0.7709	0.7645	0.8129	0.8435
	400		0.9941	0.9023	0.8691	0.8501	0.8792	0.8763	0.9021
Rayleigh(1.5)	50	0.01	0.8346	0.7928	0.2549	0.2671	0.6355	0.4238	0.5567
	75		0.9049	0.8562	0.3890	0.3875	0.8382	0.6092	0.6792
	100		0.9562	0.9021	0.4904	0.4860	0.8692	0.7333	0.7826
	200		0.9766	0.9673	0.5891	0.6233	0.9077	0.7905	0.8409
	400		0.9912	0.9873	0.7561	0.7348	0.9561	0.8762	0.9045
Rayleigh(1.5)	50	0.05	0.8993	0.8567	0.3879	0.3907	0.7874	0.7423	0.7278
	75		0.9456	0.8934	0.4677	0.4992	0.9092	0.9230	0.7972
	100		0.9879	0.9310	0.5875	0.5881	0.9578	0.9651	0.8588
	200		0.9993	0.9762	0.6981	0.6861	0.9802	0.9908	0.8921
	400		1.0000	0.9995	0.7652	0.7881	0.9982	1.0000	0.9499

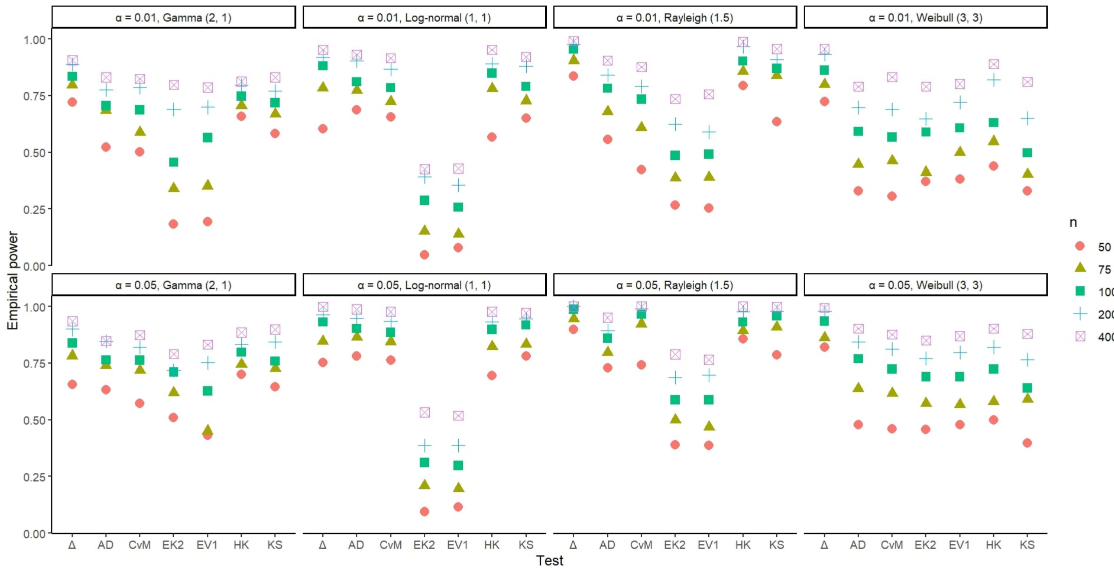

Figure 5.6: Plot of empirical power for 40% censoring

5.6 clearly depicts the supremacy of the proposed in test in terms of power, compared to the competitors.

5.4 Data analysis

We illustrate the applicability of the proposed test procedures using several real data sets in this Section.

5.4.1 Complete data

Illustration 1

We consider the data on active repair times in hours for an airborne communication transceiver reported in Chhikara and Folks (1977). The data consists of 46 observations. Jayalath and Chhikara (2022) used this data set to illustrate the Gibbs sampling approach for IG distribution in the presence of right censoring. We now standardize the data to apply the testing procedures discussed in Section 3, to test whether this data set follows an IG distribution. We obtain the test statistic as $\hat{\Delta}_3 = 0.1870$. Accordingly, we accept the null hypothesis that the data follows an inverse Gaussian distribution at both 1% (critical values are -0.0.5583 and 0.5583) and 5% (critical values are -0.4248 and 0.4248) level of significance.

Illustration 2:

Here the data of age at death of 141 Roman era Egyptian mummies is considered. The data can be found in *egypt* data in library *univariateML* of statistical package R. After standardizing the data, we obtain the test statistic as $\hat{\Delta}_3 = 0.3245$. Accordingly, we reject the null hypothesis that the data follows an inverse Gaussian distribution at both 1% (critical values are -0.2869 and 0.2869) and 5% (critical values are -0.2183 and 0.2183) level of significance.

5.4.2 Censored case

Illustration 1:

We consider the same data used for the uncensored case and following Jayalath and Chhikara (2022), we randomly select 6 observations out 46, to be randomly right censored and the right censoring times are generated using a random number generator. This generates, randomly right censored data where 13% of observations are censored. We now standardize the data to apply the testing procedures discussed in Section 3, to test whether this data set follows an IG distribution. When we use the same censoring points as in Jayalath and Chhikara (2022) We obtain the test statistic as $\hat{\Delta}_{3c} = -1.7341$. Accordingly, we accept the null hypothesis that the data follows an inverse Gaussian distribution at both 1% and 5% level of significance. We can also note that, when we use random censoring points also, we accept H_0.

Illustration 2:

We examine the data on survival times for patients with bile duct cancer who took part in a study to determine whether a combination of radiation treatment (RoRx) and the drug 5-fluorouracil (5-FU) prolonged survival (Fleming et al. (1980)). Survival times in days for 47 patients are given, which include 3 censoring times. The complete data is given Lawless (2011) (Page 372, Example 7.4). We analyse the data using the proposed procedures and the test statistic is obtained as -3.6553. The obtained test statistic value suggests that, we can reject the null hypothesis that the data follows inverse Gaussian distribution at both 5% and 1% level of significance.

5.5 Concluding remarks

In this article, we developed new goodness of fit tests for the inverse Gaussian distribution in the presence and absence of censoring. We used the fixed point characterization for inverse Gaussian distribution by Betsch and Ebner (2021) to develop the tests. First, we proposed a test static based on U-Statistic theory for IG distribution, when the data is complete, and studied the asymptotic properties of the test statistic in detail. Several tests are available for IG distribution when the data is complete. For right censored data, Ducharme (2001) proposed a goodness of fit test for IG distribution, for which the computational procedures are complex. Inspired by this, we developed a new goodness of fit test for IG distribution for right censored data which is easy to implement. We showed that the asymptotic distribution of the proposed test statistic is normal. A consistent estimator of the asymptotic variance is also obtained. Further, we also developed a JEL ratio test to test IG distribution with unit parameters.

An extensive Monte Carlo simulation study is conducted to evaluate the finite sample performance of the proposed tests. In the complete data scenario, we compared the performance of our test with the other tests available in the literature for IG distribution. For right censored data, we generated pseudo-complete sample from the original right censored data, using the method proposed by Balakrishnan et al. (2015) and then applied the classical goodness of fit tests and other tests proposed for inverse Gaussian distribution. The proposed test procedures have well-controlled type I error rate and good power against different alternatives. The practical applicability of the proposed procedures is well exemplified by applying it to four real data sets.

The test considered in this article deals with the right censoring scenario. In lifetime data analysis, the data may also suffer from several other forms of censoring and truncation. The proposed procedures can be modified to accommodate left truncated data. Further, we can develop goodness of fit tests using similar fixed point characterization for other important lifetime distributions, in both continuous and discrete cases.

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Chapter 6

Goodness of fit tests for Lindley distribution

6.1 Introduction

Modeling and analyzing lifetime data is of paramount importance across a wide range of applied sciences, including medicine, engineering, insurance, and finance. Among the various statistical models used for such analyses, the Lindley distribution has emerged as a valuable and flexible alternative, particularly in the context of reliability modeling. The Lindley distribution, originally introduced by D.V. Lindley in 1958, has gained attention as an alternative to exponential and gamma distributions for modeling positively skewed lifetime data. It is particularly suitable in scenarios where the failure rate initially decreases and then increases, making it relevant for reliability and survival analysis. Given its specialized characteristics, it is important to assess whether a given dataset follows a Lindley distribution, which motivates the development of goodness-of-fit tests tailored to this family.

The Lindley has been effectively applied in numerous practical scenarios, such as load-sharing systems Singh and Gupta (2012), and stress-strength reliability models Al-Mutairi et al. (2013) .One of its significant advantages lies in its flexible shape, which allows for better fitting of real-life lifetime data compared to more traditional models. **?**

[1]The research work reported in this chapter is communicated in the research article entitled ‘Goodness of fit tests for Lindley distribution with complete and right censored data’ by Vaisakh K.M. and Sreedevi E. P.

presented an extensive study on the Lindley distribution, exploring its mathematical and statistical properties, parameter estimation techniques, and applications.

Despite its potential, the Lindley distribution remained relatively underutilized for several years, partly due to the dominance of the exponential distribution in reliability analysis. However, in recent years, there has been a renewed interest in the Lindley distribution for survival analysis, competing risks models, and stress-strength reliability studies. Key contributions in this regard include works by **?**, Mazucheli and Achcar (2011). Recently Alizadeh Noughabi and Shafaei Noughabi (2024) developed Gini index based goodness-of-fit test for the Lindley distribution. Their test overestimates the type I error for small and moderate sample sizes. Also, they don't consider censored cases in their work. Motivated by this, we develop goodness-of-fit test for the Lindley distribution for censored and complete data.

The rest of the auricle is organized as follows. In Section 2, making use of Stein's type identity for the Lindley distribution, we develop a U-statistics-based goodness-of-fit test for the Rayleigh distribution. In Section 3, we discuss how to incorporate right-censored observations in our test procedure. We carry out Monte Carlo simulation studies in Section 4 to assess the performance of the proposed procedures. The illustrations of the proposed methods using real data sets are reported in Section 5. Finally, we conclude the study in Section 6.

6.2 Test statistic: complete case

Let X be a non-negative continuous random variable with distribution function F. The density and distribution function of the Lindley distribution are given by

$$f(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x},\ x>0,\, \theta>0$$

and

$$F(x,\theta) = 1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x},\ x>0,\, \theta>0.$$

In this section, we develop a goodness of fit test for the Lindley distribution. We use fixed point characterization based on Steins's type identity for the Lindley distribution to develop the test.

Theorem 6.1. *The random variable X has a Lindley distribution with parameter θ if and only if*

$$F(t) = E\left[\left(\theta - \frac{1}{1+X}\right)\min(X,t)\right].$$

Based on a random sample $X_1, ..., X_n$ from F, we are interested in testing the null hypothesis

$$H_0 : F \text{ has Lindley distribution.}$$

against

$$H_1 : F \text{ does not follow Lindley distribution.}$$

For testing the above hypothesis, we define a departure measure that discriminates between null and alternative hypotheses. Consider $\Delta_4(F)$ given by

$$\Delta_4(F) \;=\; \int_0^\infty \left(E\left[\left(\theta - \frac{1}{1+X}\right)\min(X,t)\right] - F(t)\right) dF(t). \tag{6.1}$$

In view of Theorem 1, $\Delta_4(F)$ is zero under H_0 and not zero under H_1. Hence $\Delta_4(F)$ can be considered as a measure of departure from the null hypothesis H_0 towards the alternative hypothesis H_1. As we find the test statistics using the theory of U-statistics, we simplify $\Delta_4(F)$ in terms of the expectation of the function of random variables. Consider

$$\begin{aligned}
\Delta_4(F) &= \int_0^\infty E\left[\left(\theta - \frac{1}{1+X}\right)\min(X,t)\right]dF(t) - \int_0^\infty F(t)dF(t). \\
&= \int_0^\infty \int_0^\infty \left(\theta - \frac{1}{1+X}\right)\min(x,t)dF(x)dF(t) - 1/2 \\
&= \int_0^\infty \int_0^\infty \theta \min(x,t)dF(x)dF(t) - \int_0^\infty \int_0^t \left(\frac{1}{1+X}\right) xdF(x)dF(t) \\
&\quad - \int_0^\infty \int_t^\infty \left(\frac{1}{1+X}\right) tdF(x)dF(t) - 1/2 \\
&= \theta E\left(\min(X_1,X_2)\right) - \int_0^\infty \int_0^t \frac{x}{1+x}dF(x)dF(t) \\
&\quad - \int_0^\infty \int_t^\infty \frac{t}{1+x}dF(x)dF(t) - 1/2 \\
&= \theta E\left(\min(X_1,X_2)\right) - \Delta_{42}(F) - \Delta_{43}(F) - 0.5.\ (say) \qquad (6.2)
\end{aligned}$$

Now, changing the order of integration, we have

$$\begin{aligned}
\Delta_{42}(F) &= \int_0^\infty \int_0^t \frac{x}{1+x}dF(x)dF(t) \\
&= \int_0^\infty \frac{x}{1+x}\int_x^\infty dF(t)dF(x) \\
&= \int_0^\infty \frac{x}{1+x}\bar{F}(x)dF(x) \\
&= \frac{1}{2}\int_0^\infty \frac{x}{1+x}2\bar{F}(x)dF(x) \\
&= \frac{1}{2}E\left(\frac{\min(X_1,X_2)}{1+\min(X_1,X_2)}\right). \qquad (6.3)
\end{aligned}$$

Again

$$\begin{aligned}
\Delta_{43}(F) &= \int_0^\infty \int_t^\infty \frac{t}{1+x}dF(x)dF(t) \\
&= E\left(\frac{X_2}{1+X_1}I(X_2 < X_1)\right). \qquad (6.4)
\end{aligned}$$

Substituting (6.3) and (6.4) in (6.2), we obtain

$$\Delta_4(F) = \theta E\left(\min(X_1,X_2)\right) - \frac{1}{2}E\left(\frac{\min(X_1,X_2)}{1+\min(X_1,X_2)}\right) - E\left(\frac{X_2}{1+X_1}I(X_2 < X_1)\right) - \frac{1}{2} \quad (6.5)$$

The maximum likelihood estimator of θ is given by

$$\widehat{\theta} = \frac{(1-\bar{X}) + \sqrt{(1-\bar{X})^2 + 8\bar{X}}}{2\bar{X}}.$$

We find the test statistics using the U-statistics theory. An unbiased estimator of $E\left(\min(X_1, X_2)\right)$ based on U statistics is given by

$$\widehat{\Delta}_{41} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j<i}^{n} \min(X_i, X_j).$$

Again an unbiased estimator of $E\left(\frac{\min(X_1,X_2)}{1+\min(X_1,X_2)}\right)$ based on U statistics is given by

$$\widehat{\Delta}_{42} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j<i}^{n} \frac{\min(X_i, X_j)}{1+\min(X_I, X_J)}.$$

An unbiased estimator of $E\left(\frac{X_2}{1+X_1} I(X_2 < X_1)\right)$ based on U statistics is given by

$$\widehat{\Delta}_{43} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j<i}^{n} h(X_i, X_j),$$

where $h(X_1, X_2) = \frac{1}{2}\left(\frac{X_2}{1+X_1} I(X_2 < X_1) + \frac{X_1}{1+X_2} I(X_1 < X_2)\right)$. Hence the test statistic is given by

$$\widehat{\Delta}_4 = \widehat{\theta}\widehat{\Delta}_{41} - \frac{1}{2}\widehat{\Delta}_{42} - \widehat{\Delta}_{43} - \frac{1}{2}. \tag{6.6}$$

We reject H_0 against H_1 for large value of $\widehat{\Delta}_4$.

Next we study the asymptotic properties of the test statistic. Being a maximum likelihood estimator, $\widehat{\theta}$ is a consistent estimator of θ. Since $\widehat{\Delta}_{41}$, $\widehat{\Delta}_{42}$ and $\widehat{\Delta}_{43}$ are U-statistics they are consistent estimators of $E(\min(X_1, X_2))$, $E\left(\frac{\min(X_1,X_2)}{1+\min(X_1,X_2)}\right)$ and $E\left(\frac{X_2}{1+X_1} I(X_2 < X_1)\right)$, respectively (Lehmann, 1951). Hence the following result is straightforward.

Theorem 6.2. *Under H_1, as $n \to \infty$, $\widehat{\Delta}_4$ converges in probability to $\Delta_4(F)$.*

Next, we find the asymptotic distribution of $\widehat{\Delta}_4$.

Theorem 6.3. *As $n \to \infty$, $\sqrt{n}(\widehat{\Delta_4} - \Delta_4(F))$ converges in distribution to a normal* *random variable with mean zero and variance σ^2, where σ^2 is given by*

$$\sigma^2 = Var\Big(\frac{2\theta X^2\bar{F}(X)}{1+X} + \int_0^X \frac{y^2}{1+y}dF(y) - \frac{1}{1+X}\int_0^X ydF(y) - X\int_X^\infty \frac{1}{1+y}dF(y)\Big). \tag{6.7}$$

Proof:

Define

$$\widehat{\Delta}_4 = \widehat{\theta}\widehat{\Delta}_{41} - \frac{1}{2}\widehat{\Delta}_{42} - \widehat{\Delta}_{43} - \frac{1}{2}. \tag{6.8}$$

Since $\widehat{\theta}$ is a consistent estimators of θ, by Slutsky's theorem, the asymptotic distribution of $n \to \infty$, $\sqrt{n}(\widehat{\Delta_4} - \Delta_4(F))$ and $\sqrt{n}(\widehat{\Delta}_4^* - E(\widehat{\Delta}_4^*))$ are same. We observe that $\widehat{\Delta}^*$ is a U statistic with the symmetric kernel,

$$h(X_1,X_2) = \frac{1}{2}\left(2\theta\min(X_1,X_2) - \frac{\min(X_1X_2)}{1+\min(X_1X_2)} - \frac{X_2}{1+X_1}I(X_2<X_1) - \frac{X_1}{1+X_2}I(X_1<X_2)\right).$$

Hence using the central limit theorem for U-statistics we have the asymptotic normality of $\widehat{\Delta_4}^*$. The asymptotic variance is $4\sigma_1^2$ where σ_1^2 is given by (Lee, 2019)

$$\sigma_1^2 = Var\left[E\left(h(X_1,X_2)|X_1\right)\right]. \tag{6.9}$$

Consider

$$\begin{aligned} E[\min(x,X_2)] &= E[xI(x<X_2) + X_2I(X_2<x)] \\ &= xP(x<X_2) + E[X_2I(X_2<x)] \\ &= x\bar{F}(x) + \int_0^x ydF(y). \end{aligned} \tag{6.10}$$

$$\begin{aligned} E\left[\frac{\min(x,X_2)}{1+\min(x,X_2)}\right] &= \int_0^x \frac{y}{1+y}dF(y) + \int_x^\infty \frac{x}{1+x}dF(y) \\ &= \int_0^x \frac{y}{1+y}dF(y) + \frac{x}{1+x}\bar{F}(x). \end{aligned} \tag{6.11}$$

Also

$$E[\frac{X_2}{1+x}I(X_2 < x] + E[\frac{x}{1+X_2}I(x < X_2)] = \int_0^x \frac{y}{1+x}dF(y) + \int_x^\infty \frac{x}{1+y}dF(y). \tag{6.12}$$

Substituting equations (6.10), (6.11) and (6.12) in (6.9) we obtain the variance expression as specified in equation (6.7).

Under the null hypothesis H_0, $\Delta_4(F) = 0$. Hence we have the following corollary.

Corollary 6.4. *Under H_0, as $n \to \infty$, $\sqrt{n}\widehat{\Delta_4}$ converges in distribution to a normal randome varaible with mean zero and variance σ_0^2, where σ_0^2 is the value of σ^2 evaluated under H_0 and it is given by*

Proof. Under H_0, we have

$$\int_X^\infty \frac{1}{1+y}dF(y) = \frac{\theta e^{-\theta X}}{1+\theta},$$

$$\int_0^X \frac{y^2}{1+y}dF(y) = \frac{e^{-\theta X}[-\theta X(\theta X+2)-2]+2}{\theta(\theta+1)}$$

and

$$\int_0^X ydF(y) = \frac{\theta - e^{-\theta X}[\theta + \theta X(\theta X + \theta + 2) + 2] + 2}{\theta(\theta+1)}.$$

Hence the variance expression in (6.7) reduces to

$$\begin{aligned}
\sigma^2 &= Var\Bigg(\frac{2\theta X^2(1+\theta+\theta X)e^{-\theta X}}{(1+\theta)(1+X)} + \frac{(-\theta X(\theta X+2)-2)e^{-\theta X}}{\theta(\theta+1)} \\
&\quad - \frac{\theta - e^{-\theta X}[\theta + \theta X(\theta X+\theta+2)+2)]+2}{\theta(\theta+1)(1+X)} - \frac{\theta X e^{-\theta X}}{1+\theta}\Bigg) \\
&= Var\Bigg(\frac{(2\theta^3X^2 + 2\theta^3X^3 + \theta^2X^2 - \theta^2X^3 + 2\theta X^2 + \theta + 2)e^{-\theta X} - (\theta+2)}{\theta(1+\theta)(1+X)}\Bigg)
\end{aligned} \tag{6.13}$$

After simplifying we obtain

$$\begin{aligned}\sigma^2 &= 1/8\theta(1+\theta)^2 - (1/(1+\theta)^6)\Big(-91/486 + 16/243\theta^2 - 67/243\theta + 32\theta/81 \\ &\quad -14\theta^2/9 - 4\theta^3/3 + e^{\theta}(2+\theta)^2\Gamma[0,\theta] - 2e^{2\theta}(4+8\theta+7\theta^2+2\theta^3)\Gamma[0,2\theta] \\ &\quad +4e^{3\theta}\Gamma[0,3\theta] + 12e^{3\theta}\theta\Gamma[0,3\theta] + 17e^{3\theta}\theta^2\Gamma[0,3\theta] + 12e^{3\theta}\theta^3\Gamma[0,3\theta] \\ &\quad +4e^{3\theta}\theta^4\Gamma[0,3\theta]\Big)^2. \end{aligned} \tag{6.14}$$

□

An asymptotic critical region of the test can be obtained using Corollary 1. Let $\widehat{\sigma}_0^2$ be a consistent estimator of σ_0^2. We reject the null hypothesis H_0 against the alternative hypothesis H_1 at a significance level α, if

$$\frac{\sqrt{n}|\widehat{\Delta_4}|}{\widehat{\sigma}_0} > Z_{\alpha/2},$$

where Z_α is the upper α-percentile point of the standard normal distribution.

Finding a consistent estimator of the null variance σ_0^2 is difficult. Therefore, we find the critical region of the proposed test using Monte Carlo simulation. We determine lower (c_1) and upper (c_2) quantiles in such a way that $P(\widehat{\Delta}_4 < c_1) = P(\widehat{\Delta}_4 > c_2) = \alpha/2$.

6.3 Test statistic: censored case

Next we discuss how the proposed testing method can be modified to incorporate the right-censored observations. Let X and C be the lifetime and the censoring time, respectively. Then the right-censored data can be represented by (Y, δ), with $Y = \min(X, C)$ and $\delta = I(X \leq C)$. We assume that the censoring times and the lifetimes are independent. Now, based on n independent and identical observations $\{(Y_i, \delta_i), 1 \leq i \leq n\}$ we are interested in developing a goodness of fit test for the Lindley distribution. As our test is based on U-statistics, we use the same departure measure $\Delta_4(F)$ given in (6.1). For that purpose, we rewrite (6.5) as

$$\begin{aligned}\Delta_4(F) &= \theta E\left(\min(X_1, X_2)\right) - \frac{1}{2}E\left(\frac{\min(X_1, X_2)}{1+\min(X_1, X_2)}\right) \\ &\quad -E\left(\frac{X_2}{1+X_1}I(X_2 < X_1)\right) - E(I(X_2 < X_1). \qquad (6.15)\end{aligned}$$

Next, we estimate each quantity in $\Delta_4(F)$ using the U-statistics for right censored data (Datta et al., 2010). An estimator of $E(\min(X_1, X_2))$ is given by

$$\widehat{\Delta}_{41c} = \frac{2}{n(n-1)}\sum_{i=1}^{n}\sum_{j<i;j=1}^{n}\frac{\min(Y_1, Y_2)\delta_i\delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}, \qquad (6.16)$$

provided $\widehat{K}_c(Y_i) > 0$ and $\widehat{K}_c(Y_j) > 0$, with probability 1 and $\widehat{K}_c$ is the Kaplan-Meier estimator of K_c, the survival function of C. Also, an estimator of $E(\min(X_1, X_2)/(1 + \min(X_1, X_2)))$ is given by

$$\widehat{\Delta}_{42c} = \frac{2}{n(n-1)}\sum_{i=1}^{n}\sum_{j<i;j=1}^{n}\frac{\min(Y_1, Y_2)\delta_i\delta_j}{(1+\min(X_1, X_2))\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}. \qquad (6.17)$$

Again, an estimator of $E\left(\frac{X_2}{X_1}I(X_2 < X_1) + \frac{X_1}{X_2}I(X_2 < X_1)\right)$ is given by

$$\widehat{\Delta}_{43c} = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j<i;j=1}^{n}\frac{(\frac{X_i}{X_j}I(X_i < X_j) + \frac{X_j}{X_i}I(X_j < X_i) + 1)\delta_i\delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}. \qquad (6.18)$$

An estimator of $E(I(X_2 < X_1)$ is given

$$\widehat{\Delta}_{44c} = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j<i;j=1}^{n}\frac{I(X_j < X_i)\delta_i\delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}. \qquad (6.19)$$

Similarly, an estimator of $\bar{X}$ is given by

$$\bar{X}_c = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i\delta_i}{\widehat{K}_c(Y_i)}.$$

Hence, an estimators of θ under right censored case is given by

$$\widehat{\theta}_c = (1-\bar{X}_c) + \frac{\sqrt{(1-\bar{X}_c)^2 + 8\bar{X}_c}}{2\bar{X}_c}. \tag{6.20}$$

In the right censored case, using the estimators given in equations (6.16-6.20), from (6.15) we obtain the test statistic as

$$\widehat{\Delta}_{4c} = \widehat{\theta}_c \widehat{\Delta}_{41c} - \frac{1}{2}\widehat{\Delta}_{42c} - \widehat{\Delta}_{43c} - \widehat{\Delta}_{44c}. \tag{6.21}$$

In the right censored case, the test procedure is to reject null hypothesis H_0 in favour of alternative hypothesis H_1 for large values of $\widehat{\Delta}_{4c}$. Next, we provided

Theorem 6.5. *Let*

$$h_1(x) = \frac{1}{2}E\left(2\theta\min(x,Y_2) - \frac{\min(x,Y_2)}{(1+\min(x,Y_2))} - \left(\frac{x}{Y_2}I(x<Y_2) + \frac{Y_2}{x}I(Y_2<x) + 1\right)\right)$$

Suppose the conditions

$$E\left[2\theta\min(Y_1,Y_2) - \frac{\min(Y_1,Y_2)}{(1+\min(Y_1,Y_2))} - E\left(\frac{Y_1}{Y_2}I(Y_1<Y_2) + \frac{Y_2}{Y_1}I(Y_2<Y_1) + 1\right)\right]^2 < \infty,$$

$\int_{\mathcal{X}\times[0,\infty)} \frac{h_1^2(x)}{K_c^2(y)} dG(x,y) < \infty$ *and* $\int_0^\infty w^2(t)dt < \infty$ *holds. As* $n \to \infty$, $\sqrt{n}(\widehat{\Delta}_{4c} - \Delta_4(F))$ *converges in distribution to Gaussian random variable with mean zero and variance* $4\sigma_c^2$, *where* σ_c^2 *is given by*

$$\sigma_c^2 = Var\Big(\frac{h_1(X)\delta_1}{K_c(Y_1-)} + \int w(t)dM_1^c(t)\Big).$$

Next we find an estimator of σ_c^2 using the reweighed techniques. An estimator of σ_c^2 is given by

$$\widehat{\sigma}_c^2 = \frac{4}{(n-1)}\sum_{i=1}^{n}(V_i - \bar{V})^2,$$

where

$$V_i = \frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} + \widehat{w}(X_i)(1-\delta_i) - \sum_{j=1}^{n} \frac{\widehat{w}(X_i)I(X_i > X_j)(1-\delta_i)}{\sum_{i=1}^{n} I(X_i > X_j)},$$

$$\bar{V} = \frac{1}{n}\sum_{i=1}^{n} V_i, \quad \widehat{h}_1(X) = \frac{1}{n}\sum_{i=1}^{n} \frac{h(X, Y_i)\delta_i}{\widehat{K}_c(Y_i-)}, \quad R(t) = \sum_{i=1}^{n} I(Y_i > t)$$

and

$$\widehat{w}(t) = \frac{1}{R(t)} \sum_{i=1}^{n} \frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} I(X_i > t).$$

Let $\widehat{\sigma}^2_{0c}$ be the value of $\widehat{\sigma}^2_c$ evaluated under H_0. Under right censored situation, we reject the null hypothesis H_0 against the alternative hypothesis H_1 at a significance level α, if

$$\frac{\sqrt{n}|\widehat{\Delta_{4c}}|}{\widehat{\sigma}_{0c}} > Z_{\alpha/2}.$$

6.4 Empirical evidence

We conduct a Monte Carlo simulation study using R software for evaluating the finite sample performance of the proposed test procedure. In uncensored cases, to demonstrate the competitiveness of our test with existing test procedures, we compare the empirical powers of these tests. In the censored case, we evaluate the empirical power of our test against different alternatives at different censoring percentages.

6.4.1 Simulation: uncensored case

First, we find the empirical type I error of the proposed test and other tests used for comparison. Then we compare the performance of the proposed test with other goodness of fit tests available for Lindly distribution. We used the following algorithm for finding the empirical:

1. Generate a sample of size n from the desired alternative and calculate the test statistic.

2. Find an estimator $\widehat{\theta}$ of the parameter θ using the sample obtained in Step 1.

3. Generate a sample of size n from Lindley distribution with parameter $\widehat{\theta}$.

4. Generate 500 bootstrap samples using the data obtained in Step 3. Then obtain the bootstrap distribution of the test statistic and determine the critical point.

5. Repeat Steps 1-4 ten thousand times and calculates the empirical power as the proportion of significant test statistics.

First we find the empirical type I error of the test. We generate observation from Lindley distribution with different choices of the parameters with different sample sizes $n = 25, 50, 75, 100, 200$ to calculate the empirical type I error. To find the empirical power, lifetime random variables are generated from different choices of alternatives including Weibull, lognormal and Pareto distributions where the distribution functions are;
Weibull distribution: $F(x) = 1 - e^{(-x/\lambda)^k}$, $x > 0, k,\ \lambda > 0$,
Pareto distributions: $F(x) = (\lambda/x)^\alpha\ x > 0, \alpha,\ \lambda > 0$,
Lognormal distribution: $F(x) = \Phi(\frac{\ln x - \mu}{\sigma})$, $x > 0, -\infty < \mu < \infty, \sigma^2 > 0$,
where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable. We compare the performance of our test with some other goodness of fit tests for Lindley distribution as well as with the well-known Anderson Darling test (AD), Kolmogorov Smirnov (KS) test, and Cramer von Mises (CvM) test. We consider the Gini index based goodness of fit tests for the Lindley distribution, proposed by Hadi Alizadeh Noughabis.

The Gini index based test statistic is given by

$$F_0(x, \theta) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x}$$

and $\hat{\theta}$ is the maximum likelihood estimate (MLE) of the parameter θ, i.e.

$$\widehat{\theta} = (1 - \bar{X}) + \frac{\sqrt{(1 - \bar{X})^2 + 8\bar{X}}}{2\bar{X}}.$$

Table 6.1: Empirical type I error of different tests at α=0.01 and α=0.05

n	$\hat{\Delta}_4$		Gini		KS		AD		CvM	
α	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
10	0.006	0.047	0.070	0.162	0.009	0.055	0.013	0.066	0.013	0.062
20	0.012	0.073	0.034	0.104	0.007	0.038	0.009	0.034	0.006	0.037
30	0.009	0.049	0.028	0.086	0.016	0.053	0.013	0.067	0.016	0.066
40	0.013	0.076	0.019	0.069	0.010	0.044	0.011	0.049	0.010	0.050
50	0.012	0.059	0.017	0.061	0.009	0.044	0.006	0.042	0.007	0.042
100	0.005	0.049	0.015	0.053	0.010	0.045	0.011	0.050	0.012	0.052

Table 6.2: Comparison of empirical power at α=0.01 and α=0.05

	n	$\hat{\Delta}_4$		Gini		KS		AD		CvM	
	α	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
Pareto(2,1)	10	0.839	0.975	0.000	0.000	0.210	0.308	0.219	0.288	0.199	0.276
	20	1.000	1.000	0.000	0.000	0.473	0.624	0.449	0.598	0.445	0.589
	30	1.000	1.000	0.000	0.000	0.678	0.810	0.674	0.806	0.662	0.796
	40	1.000	1.000	0.000	0.000	0.816	0.913	0.813	0.912	0.805	0.896
	50	1.000	1.000	0.000	0.000	0.879	0.968	0.882	0.970	0.868	0.959
Weibull (2,0.5)	10	1.000	1.000	0.000	0.019	0.003	0.075	0.000	0.012	0.000	0.034
	20	1.000	1.000	0.000	0.000	0.082	0.362	0.014	0.287	0.027	0.327
	30	1.000	1.000	0.000	0.001	0.233	0.622	0.160	0.664	0.168	0.663
	40	1.000	1.000	0.000	0.000	0.482	0.854	0.444	0.921	0.424	0.896
	50	1.000	1.000	0.000	0.000	0.695	0.949	0.766	0.985	0.723	0.973
lognormal(1,2)	10	0.992	1.000	0.047	0.113	0.400	0.610	0.595	0.750	0.382	0.589
	20	1.000	1.000	0.015	0.041	0.830	0.930	0.931	0.968	0.832	0.937
	30	1.000	1.000	0.009	0.025	0.964	0.989	0.988	0.997	0.974	0.990
	40	1.000	1.000	0.002	0.012	0.998	1.000	1.000	1.000	0.999	1.000
	50	1.000	1.000	0.001	0.012	0.999	1.000	1.000	1.000	0.999	1.000

6.4.2 Censored case

We use the general wrap-bootstrap procedure for right-censored data to the find the power of the proposed test statistic. The following algorithm to evaluate the power.

Algorithm 1 General Wrap-Bootstrap Procedure for Right-Censored Data

1. Consider the observed data $\{(Y_i, \delta_i) : i = 1, \ldots, n\}$, where $Y_i = \min(X_i, C_i)$ and $\delta_i = I(X_i \le C_i)$.
2. Choose a test statistic $T_n = T_n(Y, \delta)$ appropriate for the null model H_0.
3. Estimate the model parameter(s) under H_0, say $\hat{\theta}$, using censored MLE or estimating equations.
4. Compute the observed statistic $T_{\text{obs}} = T_n(Y, \delta)$.
5. Bootstrap resampling: $b = 1$ to B
6. Generate bootstrap survival times $X_1^{*(b)}, \ldots, X_n^{*(b)}$ from the fitted null model $F_{\hat{\theta}}$.
7. Generate censoring times $C_1^{*(b)}, \ldots, C_n^{*(b)}$ from the estimated censoring distribution $\hat{G}$.
8. Construct $Y_i^{*(b)} = \min(X_i^{*(b)}, C_i^{*(b)})$, $\delta_i^{*(b)} = I(X_i^{*(b)} \le C_i^{*(b)})$.
9. Compute bootstrap statistic $T_n^{*(b)} = T_n(Y^{*(b)}, \delta^{*(b)})$.
10. Use the empirical distribution of $\{T_n^{*(b)} : b = 1, \ldots, B\}$ to obtain bootstrap based critical region.
11. Find the power of the test as the proportion of significant test statistics.

Table 6.3: Power Comparison: Censoring rate =0.2

n	Lindley	weibull	pareto	lognormal
25	0.056	0.604	0.123	0.889
50	0.059	0.771	0.170	0.925
75	0.048	0.811	0.164	0.946
100	0.054	0.851	0.184	0.948
200	0.052	0.900	0.254	0.957

6.5 Data analysis

The proposed test procedures are illustrated using various real data sets.

Table 6.4: Power Comparison: Censoring rate =0.4

n	Lindley	weibull	pareto	lognormal
25	0.053	0.052	0.188	0.185
50	0.046	0.191	0.240	0.437
75	0.039	0.279	0.266	0.591
100	0.050	0.367	0.260	0.618
200	0.075	0.579	0.265	0.738

6.5.1 Complete data

To find the critical region, we use a parametric bootstrap and we use the following algorithm.

1. Estimate the parameter of Lindley distribution from the observed data.
2. Generate a random sample from Lindley distribution using the estimated parameter in Step 1.
3. Obtain the bootstrap distribution of the test statistic with ten thousand bootstrap samples using the data generated in Step 2 and find the corresponding bootstrap based p-value.
4. Accept or reject the null hypothesis according to the test static value and the obtained p-value.

Illustration 1: We consider the real-life data from Lawless (2011). The data set consists of $n = 15$ electronic components in an accelerated life test. The data are as follows.

1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23.0, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2.

The proposed procedures are applied to investigate whether the data come from a Lindley distribution. The ML estimate of θ is $\widehat{\theta} = 0.0702$. The value of the proposed test statistic is obtained as 0.04, where the bootstrap two sided p-value is 0.586. This also aligns with the inference using the proposed test statistic and critical values. Therefore, there is evidence to suggest that the data follow a Lindley distribution. We can note that Alizadeh Noughabi and Shafaei Noughabi (2024) also arrived at the same conclusion for the discussed data set in their study on the Lindley distribution.

Illustration 2: We also consider the data on survival times in weeks for 20 male rats that were exposed to a high level of radiation. The data is discussed in Lawless (2011) in Example 4.2.1. After applying the proposed testing procedures, we obtain a test statistic value of 1.204 with a bootstrap p-value of 0.02. Hence, we reject the null hypothesis that the data follow the Lindley distribution. We can note that the above data is tested to follow a Rayleigh distribution in Vaisakh et al. (2023). This ensured the factuality of the proposed procedure.

6.5.2 Censored data

Here, to find the critical region, we use the parametric wrap-bootstrap method for right censored data and the algorithm is given below.

1. We have the observed data

$$\{(T_i, \delta_i) : i = 1, \ldots, n\},$$

   where $T_i = \min(X_i, C_i)$, $\delta_i = \mathbf{1}(X_i \leq C_i)$, with X_i survival time and C_i censoring time.

2. Fit the model under H_0 and compute the statistic T_n using the observed sample.

3. Draw

$$(W_1, \ldots, W_n) \sim \text{Multinomial}\big(n; 1/n, \ldots, 1/n\big).$$

4. Form the bootstrap estimating equations. For this purpose, replace sample averages

$$\frac{1}{n} \sum_{i=1}^{n} \psi(T_i, \delta_i)$$

   with the weighted version

$$\frac{1}{n} \sum_{i=1}^{n} W_i \, \psi(T_i, \delta_i).$$

5. Solve the weighted estimating equations to obtain the bootstrap parameter estimate $\hat{\theta}^*$ and compute the bootstrap statistic T_n^*.

6. Repeat. Repeat Steps 3–5 for $b = 1, \ldots, B$ to generate

$$\{T_n^{*(1)}, T_n^{*(2)}, \ldots, T_n^{*(B)}\}.$$

7. Compute the wrap-bootstrap p-value as

$$p = \frac{1}{B} \sum_{b=1}^{B} \mathbf{1}\big(T_n^{*(b)} \geq T_n\big)\,.$$

   Alternatively, use the empirical quantiles of $\{T_n^{*(b)}\}$ as critical values.

Illustration 1: To illustrate the procedures, we use the data set of the lifetimes of 15 electronic components in an accelerated life test discussed in illustration 1 for complete data. As the data is complete , we modify the dat by introducing 20% of censoring, randomly. Thus 3 lifetimes are randomly converted to censoring times. After applying the proposed procedures, the test statistic value is obtained as 0.0286, while the critical region is estimated to be 0.0357, corresponding to $\alpha = 0.05$. The p-value corresponding to $\alpha = 0.05$ is also obtained as 0.138, grater than 0.05. Based on theses observations, we accept the null hypothesis that the data comes from a Lindley distribution.

Illustration 2 : Consider a study on remission times (in months) for 137 bladder cancer patients, as reported in Lee and Wang (2003), which includes right-censored data. Out of 137 observations, 9 are right-censored (patients lost to follow-up or no recurrence at study end), and the rest show actual remission times. The observed test statistic (0.0427) exceeds the critical value (0.0357) from the bootstrap null distribution, and the p-value of 0.0215 is less than the conventional 0.05 threshold for statistical significance. The data provides strong statistical evidence that the remission times of bladder cancer patients do not follow a Lindley distribution. This suggests that alternative distributions should be considered for modeling this survival data.

6.6 Concluding remarks

Based on fixed point characterization for Lindley distribution, we developed new goodness of fit test for Lindley distribution. We studied the asymptotic properties of the proposed test statistic. Monte Carlo simulation study is conducted to evaluate the finite sample performance of the test as well as to compare with other test available in literature. The proposed procedures are illustrated using two real data sets.

Even though some goodness of fit tests are available for the Lindley distribution in literature, as our knowledge, all of these test are developed for complete data. Motivated by this we developed a new goodness of fit test for the Lindley distribution under right censored data. We proved that the asymptotic distribution of the proposed test statistic is normal. We also proposed a consistent estimator of the asymptotic variance. The finite sample performance of the test is evaluated through Monte Carlo simulation study. Apart from right censoring, truncation and other types of censoring are common in lifetime data analysis. The proposed test can be modified to incorporates these situations.

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