D 11216 (Pages : 2) Name..............................

Reg. No..........................

THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2021

(CCSS)

Mathematics

MAT 3C 11—COMPLEX ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Define Complex exponential function and where it is defined.
2. Define a path in a region G subset of C and give an example of a path joining 1 + i1 and 0 + i0.
3. What are the zeros of $\cos\left(\frac{1+z}{1-z}\right), |z|<1$?
4. Evaluate $\int_{|z|=1} \frac{\sin(z)}{z}\, dz$.
5. When we say that two closed rectifiable curves are homotopic ?
6. Evaluate $\int_{\gamma} \frac{2z+1}{z^2+z+1}\, dz$, where γ is the curve $|z|=2$.
7. State Schwarz's lemma.
8. State and prove Maximum modulus principle (First version)

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let S be the unit sphere in R^3 centred at origin. Then define a bijection between the extended complex plane C_∞ and the sphere S and verify.
10. Define branch of logarithm and give a branch of log z. Also show that it is analytic.
11. Show that if γ is a piecewise smooth and $f : [a\ b] \to C$ continuous, then $\int_a^b f d\gamma = \int_a^b f(t)\gamma'(t)\, dt$.

Turn over

12. If G is simply connected and $f : G \to C$ is analytic in G, $\to$ then show that f has a primitive in G.
13. State and prove open mapping theorem of analytic functions.
14. State and prove Argument principle.

(4 × 4 = 16 marks)

Part C

Answer either A **or** *B of each of the four questions.*
Each question carries 12 marks.

Unit I

15. A. (i) Prove : If G is open and connected and $f : G \to C$ is differentiable with $f'(z) = 0$ for all z in G, then f is constant.

(ii) Prove : Let G be either the whole plane C or some open disk. If $u : G \to R$ is a harmonic function, then u has a harmonic conjugate.

B. (i) Show that Mobius transformation is a bijection and it is the composition of translation, dilation and modulation.

(ii) Define cross ratio and show that cross ratio of four points is real number if and only if all four points lies on a circle.

Unit II

16. A. Let $f : G \to C$ be analytic and suppose $\overline{B(a,r)}$ subset of G. If $\gamma(t) = a + re^{it}, 0 \le t \le 2\pi$.

Then show that $f(z) = \frac{1}{2\pi} \int_\gamma \frac{f(w)}{w-z} dw$ for $|z-a| < r$.

B. Let γ be a closed rectifiable curve in C. Then show that $n(\gamma, a)$ is constant for a belong to the component of $G = C - \{\gamma\}$ and $n(\gamma, a) = 0$ for a belong to the unbounded component of G.

Unit III

17. A. State and prove Morera's theorem.

B. Let G be an open set and $f : G \to C$ be a differentiable function ; then f is analytic on G.

Unit IV

18. A. If f has an isolated singularity at a then the point $z = a$ is a removable singularity if and only if $\lim_{z \to a} (z-a) f(z) = 0$.

B. Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

(4 × 12 = 48 marks)

D 11214 (Pages : 3) Name..............................

Reg. No...........................

THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2021

(CCSS)

Mathematics

MAT 3C 12—FUNCTIONAL ANALYSIS

(2017 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Define a Cauchy sequence in a metric space and show that a bounded sequence in a metric space need not be a Cauchy sequence.
2. Let X be a normed space. Show that if E_1 is open in X and $E_1 \subset X$, then $E_1 + E_2$ is open in X.
3. Let F be a linear map from a normed space X to a normed space Y and $\beta\|x\| \leq \|F(x)\| \leq \alpha\|x\|$ for some $\alpha, \beta > 0$ and all $x \in X$. Show that F is a homeomorphism.
4. Let X be the normed linear space K^n with norm $\| \|_\infty$ and let an $n \times n$ diagonal matrix diag $(k_1, \ldots, k_n)$ define a linear map $M : K^n \to K^n$. Determine $\|M\|$.
5. Let X be a normed space and let $\{a_1, \ldots, a_m\}$ be a linearly independent set in X. Show that there are $f_1, f_2, \ldots, f_m$ in X′ such that $f_j(a_i) = \delta_{ij}, i, j = 1, 2, \ldots, m.$
6. Show that the linear space c_{00} cannot be a Banach space in any norm.
7. Let X, Y and Z be metric spaces. Show that if $F : X \to Y$ is continuous and $G : Y \to Z$ is closed then $GoF : X \to Z$ is closed.
8. Let X and Y be Banach spaces and let F ε BL (X, Y) be bijective. Show that F^{-1} ε BL (Y, X).

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Show that l^∞ is not separable.
10. Let $\| \|$ and $\| \|'$ be norms on a linear space X. Show that the two norms are equivalent iff there are $\alpha > 0$ and $\beta > 0$ such that $\beta\|x\| \leq \|x\|' \leq \alpha\|x\|$ for all x ε X.

Turn over

11. Let $X = c_{00}$ with the norm $\| \ \|_1$ and let $f : X \to K$ be defined by $f(x) = \sum_{j=1}^{\infty} \frac{x(j)}{j}$ for $x \varepsilon X$. Show that f in a continuous linear functional and find $\|f\|$.

12. Let Y be a subspace of a normed space X and $a \varepsilon X$ but $a \notin \overline{Y}$. Show that there is some $f \varepsilon X'$ such that $f / Y = 0$, $f(a) = dist\left(a, \overline{Y}\right)$ and $\|f\| = 1$.

13. Let X and Y be normed spaces and let $F : X \to Y$ be a linear map such that the subspace Z(F) is closed in X. Define $\tilde{F} : \frac{X}{Z(F)} \to Y$ by $\tilde{F}\left(x + Z(F)\right) = F(x)$ for all $x \varepsilon X$. Show that F is an open map iff $\tilde{F}$ is an open map.

14. Let X denote the sequence space l^2. Let $\| \ \|'$ be a complete norm on X such that if $\|x_n - x\|' \to 0$ then $x_n^{(j)} \to x(j)$ for every $j = 1, 2, \ldots$. Show that $\| \ \|'$ is equivalent to the norm $\| \ \|_2$ on X.

(4 × 4 = 16 marks)

Part C

Answer either A **or** *B part of the following questions.*
Each question carries 12 marks.

15. A. (a) Show that the intersection of a finite number of dense open subsets of a metric space X is dense in X.

(b) Let E be a measurable subset of R and $m(E) < \infty$. Show that if $1 \le p < \infty$, then the set of all bounded continuous functions on E is dense in L^p (E).

B. Let Y be a closed subspace of a normed space X for $x + Y$ in X/Y, let $\|x + Y\| = \inf\{\|x + y\| : y \varepsilon Y\}$.

(i) Show that $\| \ \|$ is a norm on X/Y.

(ii) Show that a sequence $(x_n + Y)$ converges to x + Y in X/Y iff there is a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X.

16. A. (a) Let X and Y be normed spaces and $F : X \to Y$ be a linear map. Show that F is continuous on X iff $\|F(x)\| \le \alpha \|x\|$ for all $x \varepsilon X$ and some $\alpha > 0$.

(b) Show that a linear functional f on a normed linear space X is continuous iff the zero space Z (f) is closed in X.

B. (a) State and prove Hahn-Banach separation theorem.

(b) Let E be a non-empty convex subset of a normed space X over K. Show that if $E^\circ \neq \phi$ and b belongs to the boundary of E in X, then there is a nonzero $f \varepsilon X'$ such that $\text{Re} f(x) \le \text{Re} f(b)$ for all $x \varepsilon \overline{E}$.

17. A. Let X be a normed space. Show that for every sub-space Y of X and every $g \varepsilon Y'$, there is a unique Hahn-Banach extension of g to X iff X′ is strictly convex.

B. (a) Show that a normed space X is a Banach space iff every absolutely summable series of elements in X is summable in X.

(b) Let X be a normed space and Y be a Banach space. Let X_0 be a dense subspace of X and $F_0 \varepsilon BL(X_0, Y)$. Show that there is a unique $F \varepsilon BL(X, Y)$ such that $F/X_0 = F_0$ and $\|F\| = \|F_0\|$.

18. A. (a) Let E be a subset of a normed space X. Show that E is bounded in X iff f (E) is bounded in k for every $f \varepsilon X'$.

(b) Let X and Y be normed spaces and $F : X \to Y$ be linear. Show that F is an open map iff there exists some $r > 0$ such that for every $y \varepsilon Y$, there is some $x \varepsilon X$ with $F(x) = y$ and $\|x\| \leq \gamma \|y\|$.

B. Show that there are scalars $k_n \varepsilon K, n = 0, \pm 1, \pm 2, \ldots$ such that $k_n \to 0$ as $n \to \pm\infty$, but there is no $x \varepsilon L^1([-\pi, \pi)$ such that $\hat{x}(n) = k_n$ for all $n = 0, \pm 1, \pm 2, \ldots$, where $\hat{x}(n)$ is the nth Fourier coefficient of x.

(4 × 12 = 48 marks)

D 11217 **(Pages : 3)** **Name..............................**

Reg. No...........................

THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2021

(CCSS)

Mathematics

MAT 3C 12—FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Define stronger metric and equivalent metric on a metric space X. Give examples to each in $\mathbb{K}^n$.
2. Show that every closed and bounded subset of a finite dimensional normed space X is compact.
3. If X is a finite dimensional normed space, prove that any linear map from X to a normed space Y is continuous.
4. Let X be a normed space over $\mathbb{K}$, and f be a non-zero linear functional on X. If E is an open subset of X, then prove that f(E) is an open subset of $\mathbb{K}$.
5. Let X be a normed space over $\mathbb{K}$. If $0 \neq a \in X$, show that there is some $f \in X'$ such that $f(a) = \|a\|$ and $\|f\| = 1$.
6. Let $X_1, X_2, \ldots, X_m$ are Banach spaces and $X = X_1 \times X_2 \times \ldots \times X_m$. Is X is complete ? If yes, prove it.
7. If P is a projection map on a normed space X, show that R(P) = Z(1 – P) and Z(P) = R(1 – P), where I is the identity map on X.
8. State Bounded Inverse Theorem. Show by an example that completeness of the spaces cannot be omitted.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let Y be a closed subspace of a normed space X. For x + Y in the quotient space X/Y, define quotient norm and prove that it is a norm on X/Y.
10. Prove that every finite dimensional normed linear space X is complete.

Turn over

11. Show by an example that, a linear map on a linear space X may be continuous with respect to some norm on X but discontinuous with respect to another norm on X.

12. Consider $Z = \{(x(1), x(2)) \in X : x(1) = x(2)\}$, and define $g \in Z'$ by $g(x(1), x(2)) = x(1)$. Find any two Hahn Banach extensions of g.

13. Let X be a Banach space. Y be a normed space and (F_n) be a sequence in BL (X, Y) such that the sequence $(F_n(x))$ converges in Y for every $x \in X$. For $x \in X$, define $F(x) = \lim_{n\to\infty} F_n(x)$. Prove that $(F_n(x))$ converges to $F(x)$ uniformly for all x in a totally bounded subset E of X.

14. X be a normed space and $P : X \to X$ be a projection. Show that P is a closed map if and only if the subspaces R(P) and Z(P) are closed in X.

(4 × 4 = 16 marks)

Part C

Answer either A **or** *B of each question.*
Each question carries 12 marks.

15. A. (a) State and prove Baire's theorem for metric spaces.

(b) Show that the metric space l^∞ is not separable.

B. (a) Consider a measurable subset $E = [a, b]$ of $\mathbb{R}$ and $1 \le p \le \infty$. Show that the set of all step functions on E is dense in $L^p([a,b])$.

(b) Let $\| \ \|_j$ be a norm on a linear space X_j, $j = 1, 2, \ldots, m$. Fix p such that $1 \le p < \infty$. For $x = (x(1), \ldots, x(m))$ in the product space $X = X_1 \times X_2 \times \ldots \times X_m$, let $\|x\|_p = \left(\|x(1)\|_1^p + \ldots + \|x(m)\|_m^p\right)^{\frac{1}{p}}$. Prove that $\| \ \|_p$ is a norm on X.

16. A. (a) Show that a linear map F from a normed space X to a normed space Y is a homeomorphism if and only if there are $\alpha, \beta > 0$ such that $\beta\|x\| \le \|F(x)\| \le \alpha\|x\|$ for all $x \in X$.

(b) There exists a discontinuous linear map on an infinite dimensional normed linear space X. Prove or disprove.

B. (a) Let $M = (K_{ij})$ be an infinite matrix with scalar entries such that $\sup\left\{\sum_{i=1}^{\infty}|K_{ij}| : j = 1, 2, 3 \ldots\right\} < \infty$. For $x \in l^1$, let $M(x) \in l^1$ be defined by $(Mx)(i) = \sum_{j=1}^{\infty} K_{ij}x(j)$. Show that M is a continuous linear map and find its norm.

(b) State and prove Hahn Banach separation theorem.

17. A. (a) Let X and Y be normed space and $X \neq \{0\}$. Prove that BL (X, Y) is a Banach space in the operator norm if and only if Y is a Banach space.

    (b) State and prove Hahn Banach extension theorem.

    B. (a) Show that a normed space X is a Banach space if and only if every absolutely summable series of elements in X is summable in X.

    (b) If Y is a proper dense subspace of a Banach space X, then prove that Y is not a Banach space in the induced norm.

18. A. State and prove closed graph theorem.

    B. (a) Let X and Y be normed spaces and $F : X \to Y$ is a linear map. Then prove that F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma \|y\|$.

    (b) Give an example to show that open mapping theorem may not hold if the normed spaces X and/or Y are not complete.

(4 × 12 = 48 marks)

D 11218 **(Pages : 4)** Name.............................

Reg. No...........................

THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2021

(CCSS)

Mathematics

MAT 3C 13—PDE AND INTEGRAL EQUATIONS

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. What are the disadvantages of Lagrange method to solve Partial differential equations.
2. State the existence and uniqueness theorem for Partial Differential Equations.
3. Consider the equation $xu_{xx} - yu_{yy} + \frac{1}{2}(u_x - u_y) = 0$. Find the domain where the equation is elliptic, and the domain where it is hyperbolic.
4. State the Cauchy problem for the nonhomogeneous wave equation.
5. State Neumann problem for the vibrtating string.
6. State Dirichlet problem.
7. Define Fredholm equation and Volterra equation.
8. Define Separable kernels. Give an example.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let p be a real number. Consider the PDEs $xu_x + yu_y = pu, -\infty < x < \infty, -\infty < y < \infty$.
   (a) Find the characteristic curves for the equations.
   (b) Let $p = 4$. Find an explicit solution that satisfies $u = 1$ on the circle $x^2 + y^2 = 1$.
10. Consider the Tricomi equation : $u_{xx} + xu_{yy} = 0, x > 0$. Find a canonical transformation $q = q(x, y), r = r(x, y)$ and the corresponding canonical form.
11. Let $u(x,t)$ be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$, which is defined in the whole plane. Assume that u is constant on the line $x = 2 + ct$. Prove that $u_t + cu_x = 0$.

Turn over

12. Find a formal solution of the following problem and show that the solution is classical :

$$\begin{aligned} u_{tt} &= u_{xx},\ 0 < x < \pi,\ t > 0 \\ u(0, t) = u(\pi, t) &= 0,\ t \geq 0 \\ u(x, 0) &= \sin^3 x,\ 0 \leq x \leq \pi, \\ u_t(x, 0) &= \sin 2x,\ 0 \leq x \leq \pi. \end{aligned}$$

13. Let u be a function in $C^2(D)$ satisfying the mean value property at every point in D. Then prove that u is harmonic in D.

14. Find the resolvent kernel associated with the kernel $K(x,\xi) = 1 - 3x\xi$, in the interval (0, 1).

(4 × 4 = 16 marks)

Part C

Answer either A **or** *B of each question.*
Each question carries 12 marks.

Unit I

15. A. (a) Consider the Cauchy problem $u_x + u_y = 1,\ u(x,x) = x$. Show that it has infinitely many solutions.

(5 marks)

(b) (1) Find a function $u(x, y)$ that solves the Cauchy problem :

$$\left(x + y^2\right)u_x + yu_y + \left(\frac{x}{y} - y\right)u = 1,\ u(x,1) = 0,\ x \in \mathbb{R}.$$

(2) Check whether the transversality condition holds.

(3) Is the solution you obtained is defined at the origin $(x, y) = (0, 0)$? Explain your answer in light of the existence-uniqueness theorem.

(7 marks)

B. (a) Solve the equation $u_x + u_y + u = 1$, subject to the initial condition $u = \sin x$, on $y = x + x^2, x > 0$.

(6 marks)

(b) Solve the equation $\left(y^2 + u\right)u_x + yu_y = 0$ in the domain $y > 0$ under the initial condition $u = 0$ on the planar curve $x = \frac{y^2}{2}$.

(6 marks)

Unit II

16. A. (a) If $L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$, where $a, b, \ldots, f, g$ are given functions of x, y and u (x, y) is the unknown function, is elliptic in a planar domain D and if the coefficients a, b, c are real analytic functions in D, then prove that there exists a co-ordinate system (ξ, η) in which the equation has the canonical form $w_{\xi\xi} + w_{\eta\eta} + l_1[w] = G(\xi, \eta)$, where l_1 is a first-order linear differential operator, and G is a function which depends on the second-order linear equation $L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$.

(6 marks)

(b) Prove that the equation $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$ is parabolic and find its canonical form, find the general solution on the half-plane $x > 0$.

(6 marks)

B. (a) Derive the d'Alembert's formula for the Cauchy problem for the one-dimensional homogeneous wave equation.

(8 marks)

(b) Solve the Cauchy problem :

$$\begin{aligned} u_{tt} - 9u_{xx} &= e^x - e^{-x}, -\infty < x < \infty, t > 0 \\ u(x, 0) &= x, -\infty < x < \infty, \\ u_t(x, 0) &= \sin x, -\infty < x < \infty. \end{aligned}$$

using the d'Alembert formula.

(4 marks)

Unit III

17. A. Using method of separation of variables solve the heat equation :

$$\begin{aligned} u_t - ku_{xx} &= 0, 0 < x < L, t > 0, \\ u(0, t) = u(L, t) &= x, t \geq 0, \\ u(x, 0) &= f(x), 0 \leq x \leq L, \end{aligned}$$

where f is a given initial condition, and k is a positive constant. The compatibility conditions are given by $f(0) = f(L) = 0$.

(12 marks)

B. (a) Find a harmonic function u (x, y) in the square $0 < x, y < \pi$ satisfying the Neumann boundary conditions, $u_y(x, \pi) = x - \frac{\pi}{2}, u_x(0, y) - u_x(\pi, y) = u_y(x, 0) = 0$.

(6 marks)

(b) Solve the Poisson equation $\Delta w = 8r\cos\theta, 0 \leq r < 1, 0 \leq \theta \leq 2\pi$, subject to the boundary conditions $w(1, \theta) = \cos 2\theta$.

(6 marks)

Turn over

Unit IV

18. A. Solve the integral equation $y(x) = \lambda \int_0^1 (1 - 3x\xi)\, y(\xi)\, d\xi + F(x)$. (12 marks)

B. (a) Solve the integral equation $y(x) = 1 + \lambda \int_0^1 (1 - 3x\xi)\, y(\xi)\, d\xi$ by a method of successive approximations.

(6 marks)

(b) Let $y_m(x)$ and $y_n(x)$ be characteristic functions corresponding respectively to two different characteristic numbers λ_m and λ_n on the homogeneous Fredholm equation $y(x) = \lambda \int_a^b K(x, \xi)\, y(\xi)\, d\xi$, and suppose that the kernal $K(x, \xi)$ is symmetric, then prove that then $y_m(x)$ and $y_n(x)$ are orthogonal over the interval (a, b).

(6 marks)

[4 × 12 = 48 marks]

D 11219 **(Pages : 3)** **Name..............................**

Reg. No............................

THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2021

(CCSS)

Mathematics

MAT 3E 01—ADVANCED TOPOLOGY

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Prove that a compact subset in a Hausdorff space is closed.
2. Let A, B be subsets of a space X and suppose there exists a continuous function $f : X \to [0,1]$, such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Prove that there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$.
3. Prove that every path connected space is connected.
4. Show that T_2 property is productive.
5. Prove that the intersection of any family of filters on a set is again a filter on that set.
6. Let X and Y be topological spaces, $x \in X$ and $f : X \to Y$ a function. If f is continuous at x, then show that a net S converges to x in X implies the net $f \circ S$ converges to $f(x)$ in Y.
7. Prove or disprove. Every complete metric space is compact.
8. Give an example to show that weak contraction need not imply contraction.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Prove that every regular, Lindeloff space is normal.
10. Show that the product topological space is completely regular if each co-ordinate space is completely regular.

Turn over

11. Prove that a topological space is Hausdorff if the limits of all nets in it are unique.

12. Let $\mathcal{S}$ be a family of subsets of a set X. Prove that there exists a filter on X having $\mathcal{S}$ as a sub-base if and only if $\mathcal{S}$ has the finite intersection property.

13. Let A be a subset of a metric space (X, d) such that A is complete w.r.t. the metric induced on it. Prove that A is closed in X.

14. Prove that in a locally compact Hausdorff space, a subset of a first category can have no interior points.

(4 × 4 = 16 marks)

Part C

Answer **either** *A* **or** *B of each question.*
Each question carries 12 *marks.*

Unit I

15. A. Let A be a closed subset of a normal space X and suppose $f : \to [-1, 1]$ is a continuous function. Then show that there exists a continuous extension of f on X.

B. (a) Let A and B be compact subsets of topological spaces X and Y respectively. Let W be an open subset of X × Y containing the rectangle A × B. Show that there exist open sets U, V in X, Y respectively such that $A \subset U, B \subset V$ and $U \times V \subset W$.

(b) Suppose $\mathcal{D}$ is a decomposition of a space X each of whose members is compact and suppose the projection $p : X \to \mathcal{D}$ is closed. Prove that the quotient space $\mathcal{D}$ is Hausdorff of regular according as X is Hausdorff or regular.

Unit II

16. A. (a) Prove that the following statements are equivalent in a topological space X :

(i) X is locally connected.

(ii) Components of open subsets of X are open.

(iii) X has a base consisting of connected subsets.

(iv) For every $x \in X$ and every neighbourhood N of x there exists a connected open neighbourhood M of x such that $M \subset N$.

(b) Give an example of a space which is connected but not locally connected.

B. Prove that metrisability is a countably productive property.

Unit III

17. A. (a) Define cluster point of a net and subnet of a net.

(b) Let $S : D \to X$ be a net in a topological space and let $x \in X$. Then prove that x is a cluster point of S if and only if there exists a subnet of S which converges to x in X.

B. (a) Let $\mathcal{F}$ be a filter in a space X and $\mathcal{S}$ be the associated net in X. Prove that $\mathcal{F}$ converges to a point $x \in X$ as a filter if and only if $\mathcal{S}$ converges to x as a net.

(b) Let X be the topological product of an indexed family of spaces $\{X_i : i \in I\}$. Let $\mathcal{F}$ be a filter on X and let $x \in X$. Then show that $\mathcal{F}$ converges to x in X if the filter $\pi_{i\#}(\mathcal{F})$ converges to $\pi_i(x)$ in X_i.

Unit IV

18. A. Define totally bounded set. Prove that a metric space is compact if and only if it is complete and totally bounded.

B. When will you say that a topological space is an absolute G_δ. Prove that a topological space is metrically topologically complete if and only if it is an absolute G_δ set.

(4 × 12 = 48 marks)

D 11215 **(Pages : 2)** **Name..............................**

Reg. No...........................

THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2021

(CCSS)

Mathematics

MAT 3E 01—ADVANCED TOPOLOGY

(2017 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 2 marks.

1. Prove that a compact subset of a Hausdorff space is closed.
2. Define Urysohn function. State the result that guarantees the existence of such a function.
3. Define path-connected topological space. Verify whether the real line (in the usual topology) is path-connected or not.
4. When do we say that a topological property is productive ? Is normality is productive property ? Justify your claim.
5. Prove that if a subset A of a topological space is closed then limits of nets in A are in A.
6. Define filter and ultrafilter. Prove that every filter is contained in an ultrafilter.
7. Prove that compact metric space is totally bounded.
8. Is it true that, in general a complete metric space is compact ? Justify your claim.

$(8 \times 2 = 16$ marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Prove that a continuous bijection from a compact space onto a Hausdorff space is a homeomorphism.
10. Prove that every compact Hausdorff space is T_4.
11. Prove that a product a topological spaces is path-connected if and only if each co-ordinate space is path-connected.
12. Prove that second countability is preserved under continuous open functions.

Turn over

13. Prove that if limits of all nets in a topological space are unique, then the space is Hausdorff.

14. Define confinal subset of a directed set. Let F be a confinal subset of a directed set (D, $\geq$) in a topological space X. Then prove that if a net $S : D \rightarrow X$ converges to a point $x \in X$, then so does its restriction $S/F : F \rightarrow X$.

(4 × 4 = 16 marks)

Part C

Answer either A **or** *B part of the following questions.*
Each question carries 12 marks.

15. A. (a) Let X be a Hausdorff space, $x \in X$ and F a compact subset of X not containing x. Then prove that there exists open sets U, V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$.

(b) Prove that regular, second countable space is normal.

B. (a) Let A, B be subsets of a space X and suppose there exists a continuous function $f : X \rightarrow [0, 1]$, such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Then prove that there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$.

(b) Suppose a topological space X has the property that for every closed subset A of X, every continuous real valued functin on A has a continuous extension to X. Prove that X is normal.

16. A. (a) Prove that a space X is locally connected at a point $x \in X$ if and only if for every neighbourhood N of x, the component of N containing x is a neighbourhood of x.

(b) Prove that every open subset of the real line (in the usual topology) can be expressed as the union of mutually disjoint open intervals.

B. (a) Prove that a product of space is connected if and only if each space is connected.

(b) Prove that every subspace of the Hilbert cube is a second countable metric space.

17. A. (a) Prove that in an indiscrete space, every net converges to every point of the space.

(b) Prove that a subset B of a space X is open if and only if no net in the complement X – B can converges to a point in B.

B. (a) Prove that the intersection of any family of filters on a set is again a filter on that set.

(b) Prove that any family which does not contain the empty set and which is closed under finite intersections is a base for a unique filter.

18. A. (a) Prove that every compact metric space is complete.

(b) Prove that an open subspace of a metrically topologically complete space is metrically topologically complete.

B. Prove that a topological space is metrically topologically complete if and only if it is an absolute G_δ.

(4 × 12 = 48 marks)