STUDY ON AUTOREGRESSIVE MODELS

Thesis submitted

to the University of Calicut

for the degree of

DOCTOR OF PHILOSOPHY

Under the Faculty of Science

By

THOMAS MATHEW

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

KERALA, INDIA

Pin. 673 635

MAY, 2004

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DEDICATED TO MY BELOVED PARENTS

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Dr. K. Jayakumar
Lecturer(Senior Scale)
Department of Statistics

University of Calicut
Calicut University P.O.
Kerala- 673 635.

May 27, 2004

CERTIFICATE

This is to certify that the work reported in this Thesis entitled **STUDY ON AUTOREGRESSIVE MODELS** submitted to the University of Calicut for the award of degree of Doctor of Philosophy in Statistics is a bona fide research work carried out by Sri. Thomas Mathew under my supervision and guidance in the Department of Statistics, University of Calicut. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

Dr. K. JAYAKUMAR
(Supervising Teacher)

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DECLARATION

I here by declare that the matter embodied in this thesis is the result of investigations carried out by me in the Department of Statistics, University of Calicut, under the supervision and guidance of Dr. K. Jayakumar, Lecturer (Senior Scale), Department of Statistics, University of Calicut. This thesis contains no material which has been accepted for award of any other degree or diploma in any University or Institute and to the best of my knowledge and belief, it contains no material previously published by any other person, except where the due references are made in the text of the thesis.

Calicut University Campus
May 27, 2004.

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I am grateful to Dr. M. Manoharan, Head of the Department of Statistics, University of Calicut for his extensive help in my research work during the past three years. I express my sincere gratitude to all the faculty members and staff of the Department of Statistics, University of Calicut for their wholehearted cooperation and consideration extended to me during the period of study.

My personal thanks are due to my parents, wife and kids for remaining before and behind me as a lodestar throughout.

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CHAPTER I

INTRODUCTION

1.1. Introduction

A time series is a set of observations generated sequentially in time. The primary objective of time series modeling is to develop a sample model capable of forecasting. Examples of time series are annual yield of crop for a particular period, population of a country during a specified time, total industrial production per month of a country, monthly rainfall at a place, etc. Time series has an important place in the field of economic and business statistics. The most important use of time series analysis is to provide an aid to forecasting. For this the methodology developed is to decompose a time series in to trend, seasonal, cyclical and irregular components. An important feature of time series is that successive observations are usually dependent. When successive observations are dependent, future values may be predicted from the past observations.

A statistical phenomenon that evolves in time according to probability laws is called stochastic process. The time series to be analyzed may be thought of as one particular realization of the stochastic process. A stochastic model, which can be extremely useful in the representation of certain practically occurring series, is the socalled autoregressive model. In this model, the current value of the process is expressed as a finite linear aggregate of previous values of the process and a shock

ε_n. Let us denote the values of the process at equally spaced times n, n-1, n-2,... by $X_n, X_{n-1}, X_{n-2}, \ldots$. Then

$$X_n = a_1 X_{n-1} + a_2 X_{n-2} + \ldots + a_p X_{n-p} + \varepsilon_n \tag{1.1.1}$$

is called a p^{th} order autoregressive process. In particular, autoregressive processes of first order (p = 1) and second order (p = 2) are

$$X_n = a_1 X_{n-1} + \varepsilon_n$$

and

$$X_n = a_1 X_{n-1} + a_2 X_{n-2} + \varepsilon_n$$

are of considerable practical importance (see Box and Jenkins (1970)). Another kind of model of great practical importance in the representation of observed time series is the socalled finite moving average process. Here we make X_n linearly dependent on a finite number q of previous ε's. Thus

$$X_n = \varepsilon_n - b_1 \varepsilon_{n-1} - b_2 \varepsilon_{n-2} - \ldots - b_q \varepsilon_{n-q} \tag{1.1.2}$$

is called a moving average process of order q.

To achieve grater flexibility in fitting of actual time series, it is sometimes advantageous to include both the autoregressive and moving average terms in the model. This leads to the mixed autoregressive-moving average model

$$X_n = a_1 X_{n-1} + a_2 X_{n-2} + \ldots + a_p X_{n-p} + \varepsilon_n - b_1 \varepsilon_{n-1} - \ldots - b_q \varepsilon_{n-q} \tag{1.1.3}$$

In practice, it is frequently true that adequate representation of actually occurring stationary time series can be obtained with autoregressive, moving average or mixed models, in which p and q are not grater than two and often less than two.

The models used in classical analysis of time series are linear in nature. Moreover the time series $\{X_n\}$ is assumed to be Gaussian sequence (see Box and Jenkins (1970)). One of the linear stochastic models used in time series analysis is the p^{th} order autoregressive model defined in (1.1.1) where $\{\varepsilon_n\}$ is an innovation sequence of independent and identically distributed random variables following normal distribution. But there are situations where the naturally occurring series do not fit to Gaussian models. In modeling of such non-Gaussian models, the standard technique is to make a suitable transformation to remove the skewness of the data and then fit a Gaussian model. But the stringent condition that the transformed sequence must be Gaussian is very unlikely to be true in practice (see Sim (1990)). As a result, the time series that does not fit to the Gaussian setup needs a separate treatment. Recently a number of different models have been introduced for generating non-Gaussian time series.

In the case of Gaussian first order autoregressive model both $\{X_n\}$ and $\{\varepsilon_n\}$ have normal distribution, which is not the case in non-Gaussian models. Identifying the stationary distribution of X_n and innovation sequence $\{\varepsilon_n\}$ is the problem in non-Gaussian time series model building. Some of the examples of stationary solutions having continuous distributions are given in Dewald and Lewis (1985), Jayakumar et al. (1995), Jayakumar (1997), Jayakumar and Pillai (1993), Lawrance and Lewis (1980, 1985) and Sim (1986,1987, 1990).

1.2. Minification processes

The study on minification processes began with the work of Tavares (1980). He developed a first order autoregressive exponential minification process. In his work, the observations are generated by the equation

$$X_n = k\min(X_{n-1}, \varepsilon_n), \qquad n \geq 1 \tag{1.2.1}$$

where $k > 1$ is a constant and $\{\varepsilon_n\}$ is an innovation process of independent and identically distributed random variables chosen to ensure that $\{X_n\}$ is a stationary Markov process with a given marginal distribution. Because of the structure of (1.2.1) the process $\{X_n\}$ is called minification process. Sim (1986) developed a first order autoregressive Weibull process and studied its properties. Arnold (1993) developed a logistic process involving Markovian minimization.

Giving slight modifications to (1.2.1), several other minification models have been constructed so far. Yeh et al. (1988) considered a first order autoregressive minification process having Pareto marginal distribution. Pillai (1991) extended this to obtain a first order autoregressive semi-Pareto process. Arnold and Robertson (1989) considered a minification process having logistic marginal distribution. Such minification processes in general have the structure given by

$$X_n = \begin{cases} kX_{n-1} & \text{w.p. } p \\ k\min(X_{n-1}, \varepsilon_n) & \text{w.p. } 1-p \end{cases}, \quad 0 < p < 1,$$

where ‘w.p.’ stands for ‘with probability’. Pillai, Jose and Jayakumar (1995) introduced another minification process having the form

$$X_n = \begin{cases} \varepsilon_n & \text{w.p. } p \\ k\min(X_{n-1}, \varepsilon_n) & \text{w.p. } 1-p \end{cases}, \quad 0 < p < 1.$$

Jayakumar (1995c) obtained the marginal distribution of a p^{th} order integer valued autoregressive process having minification structure and characterized discrete Pareto type III distribution. Lewis and McKenzie (1991) obtained necessary and sufficient conditions on the hazard rate of the marginal distributions for a minification process to exist.

1.3. Integer valued processes

Statistical data which are expressed in terms of counts taken sequentially in time and which are correlated arise in many settings. Examples of these processes are the number of patients in a hospital at specific point of time or the number of persons in a queue waiting for service at a certain moment. In each of these examples, an element of the process at time t can be either the survival of an element of the process at previous times or an arrival or innovation sequence, which has certain discrete distribution.

Al-Osh and Alzaid (1987) introduced integer valued autoregressive time series model analogous to the standard time series model in which they used the definition of binomial thinning operator introduced by Steutel and van Harn (1979). Let X be a non-negative integer valued random variable. Then for any $\alpha \in (0, 1)$, the binomial thinning operator '$\oplus$' is defined by

$$\alpha \oplus X = \sum_{i=1}^{X} Y_i \tag{1.3.1}$$

where $\{Y_i\}$ is a sequence of independent and identically distributed random variables independent of X such that

$$P(Y_i = 1) = 1 - P(Y_i = 0) = \alpha .$$

The first order integer valued autoregressive process is defined as

$$X_n = \alpha \oplus X_{n-1} + \varepsilon_n \quad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{1.3.2}$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables and $\alpha \oplus X_{n-1}$ is as in (1.3.1).

The first order integer valued autoregressive process defined above simply states that the component of the process at time n, X_n are (i) the survival of the element of the process at time n-1, X_{n-1} each with probability of survival α and (ii) elements which entered the system in the interval $(n-1, n]$ as innovation term ε_n.

Alzaid and Al-Osh (1988) developed first order integer valued autoregressive process having geometric marginal distributions and studied its properties. Alzaid and Al-Osh (1990) introduced a p^{th} order integer valued autoregressive model and studied its structural properties. Jin Guan and Yuan (1991) developed another p^{th} order integer valued autoregressive time series model. Al-Osh and Alzaid (1988) developed integer valued moving average process and discussed its applications. They have developed first order integer valued moving average Poisson process and studied its properties such as regression behavior, time reversibility, conditional and partial correlations. Also they have extended the integer valued moving average Poisson process to the q^{th} order case. Pillai and Jayakumar (1995) introduced first order autoregressive discrete Mittag-Leffler process and studied its properties. Jayakumar (1995a) developed a p^{th} order integer valued autoregressive time series model having discrete Mittag-Leffler marginal distribution. Jayakumar (1995b) obtained the stationary solution of a first order integer valued autoregressive process. Jayakumar (1995c) introduced a p^{th} order integer valued autoregressive process having minification structure and characterized discrete Pareto type III distribution. For a detailed account of the works on integer valued models, see McKenzie (2003).

1.4. Mittag-Leffler distribution

Pillai (1990) introduced the Mittag-Leffler distribution and studied its properties including geometric infinite divisibility and attraction to stable laws. We say that a random variable X on (0, ∞) has Mittag-Leffler distribution and write X $\underline{d}$ ML (σ, α) if its distribution function is

$$F_{\sigma,\alpha}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\sigma^{k\alpha}x^{k\alpha}}{\Gamma(1+k\alpha)}, \quad \sigma > 0,\ 0 < \alpha \leq 1. \tag{1.4.1}$$

For α = 1, (1.4.1) reduces to exponential distribution. The Laplace transform of the ML (σ, α) random variable X is

$$\phi_X(\lambda) = \int_0^\infty e^{-\lambda X} dF_{\sigma,\alpha}(x) = \frac{1}{1+\left(\frac{\lambda}{\sigma}\right)^\alpha}, \quad \lambda, \sigma > 0, 0 < \alpha \le 1. \quad (1.4.2)$$

For the cumulative probability plots of the Mittag-Leffler distribution, see Jayakumar and Pillai (1993). Lin (1998) proved that the Mittag-Leffler distribution belongs to the class of distributions with complete monotone derivative. Kozubowski (1998) showed that the Mittag-Leffler random variable X having distribution function $F_{1,\alpha}(x)$ admits the representation $X = ZW_\alpha^{1/\alpha}$ where Z is standard exponential and W_α is the positive valued random variable with density $f_\alpha(x) = \frac{\text{Sin}(\alpha\pi)}{\alpha\pi(x^\alpha + 2x\text{Cos}\alpha\pi + 1)}$ (see also Pillai (1988)). The Mittag-Leffler distribution has been found to be useful in a variety of situations. For example, Weron and Kotulski (1996) used the Mittag-Leffler distribution to describe the Cole-Cole relaxation phenomena in Physics. Jayakumar (2003) used the Mittag-Leffler distribution to model the rate of flow of water in Kallada River, Kerala, India. For the applications of Mittag-Leffler distribution in random summation, see Gnedenko and Korolev (1996). For various properties of Mittag-Leffler distribution, see Jayakumar and Suresh (2003).

The cumulative probability plot of the Mittag-Leffler distribution ML (σ, α) in (1.4.1) for $\alpha = 0.3, 0.5, 0.7, 0.9$ and 1 (with $\sigma = 2$) is presented in Figure 1.4.1 below.

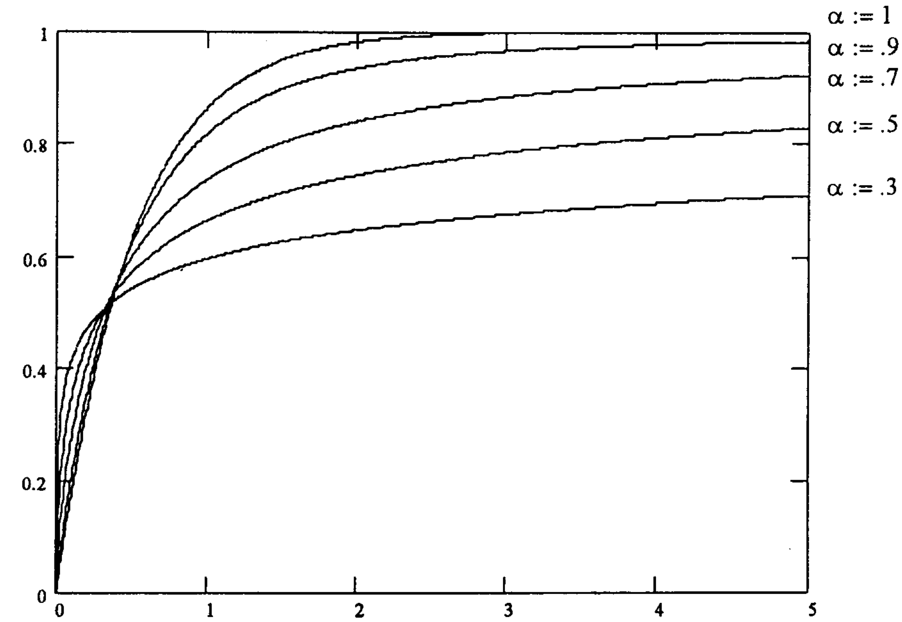

Figure 1.4.1

Cumulative distribution function of ML (σ, α) for σ = 2

The probability density function of ML (σ, α) for α = 0.3, 0.5, 0.7, 0.9 and 1 (with σ = 2) is presented in Figure 1.4.2 below.

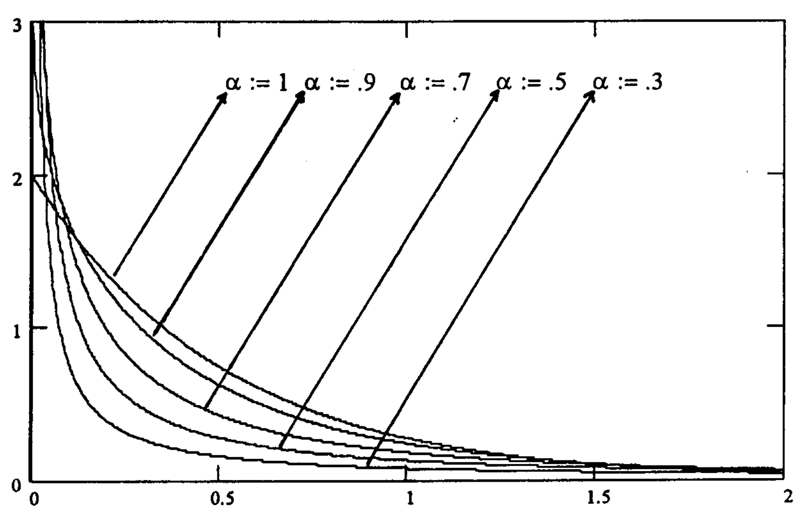

Figure 1.4.2

Probability density function of ML (σ, α) for σ = 2

1.5. Discrete Mittag-Leffler distribution

Pillai and Jayakumar (1995) introduced discrete Mittag-Leffler distribution and studied its properties including geometric infinite divisibility, discrete self-decomposability and attraction to stable laws. The mathematical origin of discrete Mittag-Leffler distribution can be formulated as follows:

Consider a sequence of independent Bernoulli trials in which the k^{th} trial has probability of success $\frac{\alpha}{k}, 0 < \alpha < 1$, $k = 1,2,3,\ldots$. Let N be the trial number in which the first success occurs. Then the probability that $N = r$ is given by

$$p_r = (1-\alpha)\left(1-\frac{\alpha}{2}\right)\ldots\left(1-\frac{\alpha}{r-1}\right)\frac{\alpha}{r}$$

$$= \frac{(-1)^{r-1}\alpha(\alpha-1)\ldots(\alpha-r+1)}{r!}.$$

The probability generating function of N is given by $G(z) = 1-(1-z)^{\alpha}$.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed as N. Let M be geometric with parameter p, that is, $P(M = k) = q^k p$, $k = 0,1,2,\ldots$, $0 < p < 1$, $q = 1$-p. Then $X_1 + X_2 + \ldots + X_M$ has generating function

$$P(z) = \frac{1}{1+\sigma(1-z)^{\alpha}}, 0 < \alpha \leq 1, \sigma = \frac{q}{p}. \tag{1.5.1}$$

The distribution with probability generating function (1.5.1) is called the discrete Mittag-Leffler distribution and is denoted as DML (σ, α). Note that for α =1, DML (σ, α) reduces to geometric distribution. Pillai and Jayakumar (1995) showed that the discrete Mittag-Leffler distributions are discrete self decomposable and are normally attracted to stable law. For other properties of discrete Mittag-Leffler distribution, see Christoph and Schreiber (2000a, 2000b).

Pillai and Jayakumar (1995) derived an expression for calculating the probabilities of the discrete Mittag-Leffler distribution. If $p_k = P(X = k)$, where X is the discrete Mittag-Leffler random variable, then

$$p_0 = \frac{1}{1+\sigma}, \quad p_1 = \frac{\sigma\alpha}{(1+\sigma)^2} \text{ and}$$

$$p_n = \frac{\sigma}{1+\sigma}\left[\binom{\alpha}{1}p_{n-1} - \binom{\alpha}{2}p_{n-2} + \binom{\alpha}{3}p_{n-3} + \ldots + (-1)^{n-1}\binom{\alpha}{n}p_0\right].$$

1.6. Symmetric Linnik distribution

For each $\alpha \in (0,2]$, the function

$$\phi(t) = \frac{1}{1+\sigma\,|\,t\,|^{\alpha}}, \quad t \in R, \quad \sigma > 0 \tag{1.6.1}$$

is the characteristic function of a symmetric probability distribution called the symmetric Linnik distribution ($SL(\sigma,\alpha)$) (see Linnik (1963)). A Linnik random variable Y defined on R = (-∞, ∞) with characteristic function (1.6.1) admits the representation $Y \stackrel{d}{=} W^{1/\alpha}X$, where X is symmetric stable with characteristic function

$\phi(t) = \exp\left\{-\sigma|t|^{\alpha}\right\}$ and W is a standard exponential random variable, independent of X. Kozubowski (2000) suggested a method of generating Linnik random variables. Kotz and Ostrovskii (1996) proved the following mixture representation for one Linnik law in terms of other. If $0 < \beta < \alpha \leq 2$, there is a positive valued random variable Z whose density is

$$g(x,\alpha,\beta) = \left(\frac{\alpha}{\pi}\right)\sin\left(\frac{\pi\beta}{\alpha}\right)\frac{x^{\beta-1}}{1+x^{2\beta}+2x^{\beta}\cos\left(\frac{\pi\beta}{\alpha}\right)},$$

such that $Y = XZ$, where $X \stackrel{d}{=} SL(1, \alpha)$ and $Y \stackrel{d}{=} SL(1, \beta)$. Lin (1994) proved that $SL(\sigma,\alpha)$ distributions are self-decomposable and are geometrically infinitely divisible. He obtained a characterization of the same using closure under geometric summation and obtained an expression for the density of $SL(\sigma,\alpha)$ random variables in terms of the Meijer's G-function (see also, Sabu George and Pillai (1987)). Kozubowski (2001) discussed the fractional moment estimation of Linnik parameters. Kotz et al. (2001) discussed the characterizations, stability properties, representations and methods of generation of Linnik random variables. For the applications of Linnik laws in modeling financial data, quality control, astronomy, biological and environmental sciences, see Kotz et al. (2001).

The probability density function of the symmetric Linnik distribution for α = 0.3, 1.2 and 1.9 (with σ = 1.5) is presented in Figure 1.6.1. In Figure 1.6.2, the probability density function of the symmetric Linnik distribution for $\sigma = 0.5$, 1 and 6 (with $\alpha = 1.4$) is presented.

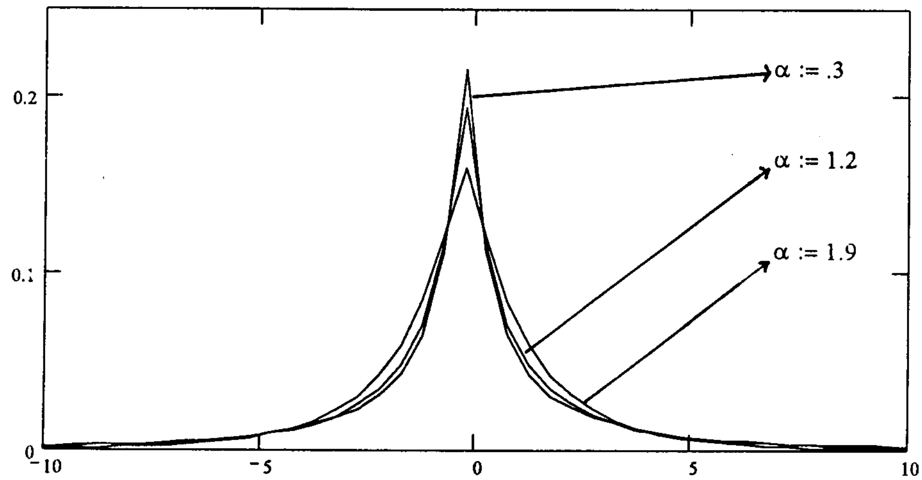

Figure 1.6.1.

Probability density function of symmetric Linnik distribution for $\sigma = 1.5$.

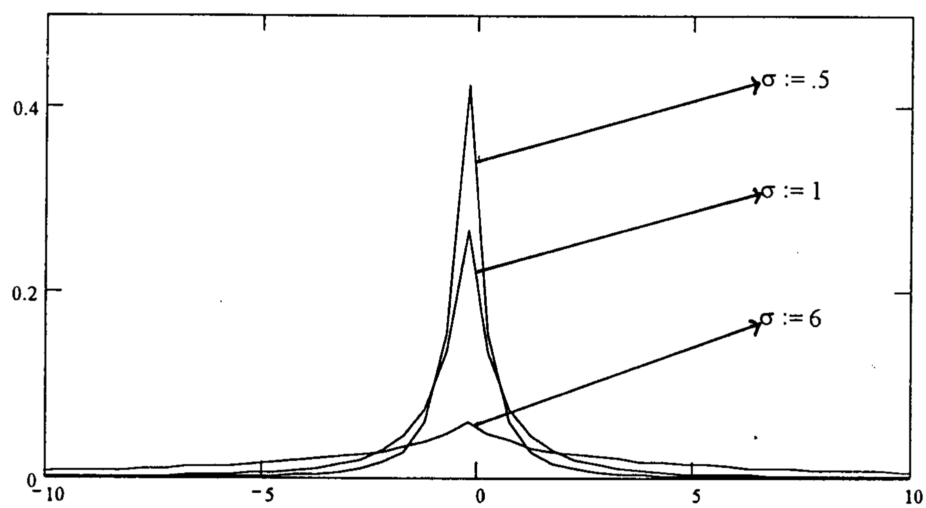

Figure 1.6.2.

Probability density function of symmetric Linnik distribution for $\alpha = 1.4$.

1.7. Symmetric generalized Linnik distribution

Pakes (1998) obtained mixture representations for symmetric generalized Linnik distribution defined by the characteristic function

$$\phi(t) = \left[\frac{1}{1+\sigma|t|^{\alpha}}\right]^{\beta+1}, \quad 0<\alpha\leq 2, \quad \beta\geq 0, \sigma>0. \tag{1.7.1}$$

We denote the random variable X defined on R = (-∞, ∞) having characteristic function (1.7.1) as $SGL(\sigma,\alpha,\beta)$. If $X_{\alpha,\beta}$ denote the random variable with characteristic function $\phi(t)$ and $\gamma(\beta)$ the random variable having gamma distribution with Laplace transform $\frac{1}{(1+\lambda)^{\beta+1}}$, then $X_{\alpha,\beta} \underline{\underline{d}} (\gamma(\beta))^{1/\alpha} S_{\alpha}$ where S_{α} is symmetric stable with characteristic function $e^{-|t|^{\alpha}}$. Note that $(\gamma(\beta))^{1/\alpha} = U^{1/(\beta+1)\alpha}$ where U has the density $\frac{e^{-x^{1/(\beta+1)}}}{\Gamma(2+\beta)}$ (see also, Devroye (1990)). Jacques et al. (1999) proved that the generalized Linnik laws belong to Paretian family. For estimation of parameters of symmetric generalized Linnik laws, see Jacques et al. (1999).

1.8. Discrete Linnik distribution

Devroye (1993) studied the properties of the class of distributions with support on Z = {0, 1, ...} having probability generating function

$$P(z) = \left[\frac{1}{1+\sigma(1-z)^{\alpha}}\right]^{\beta+1}, \quad \sigma>0, \ \beta\geq 0, \ 0<\alpha\leq 1, |z|\leq 1. \tag{1.8.1}$$

He named the distribution with probability generating function (1.8.1) as discrete Linnik distribution. Devroye (1993) showed that the distribution with probability generating function $\left[\dfrac{1}{1+(1-z)^{\alpha}}\right]^{\beta+1}$ is distributed as Poisson with parameter $(\gamma(\beta))^{1/\alpha} S_\alpha$ where $\gamma(\beta)$ is a gamma (β+1) random variable independent of S_α and S_α is a positive stable random variable having characteristic function $e^{-t^{\alpha}}$ with parameter α. Note that when $\beta = 0$, (1.8.1) becomes the discrete Mittag-Leffler distribution DML (σ, α) in (1.5.1). Christoph and Schreiber (2000b) studied the properties of discrete Linnik laws in (1.8.1). They have obtained explicit formula for probabilities of discrete Linnik distribution. Bouzar (2002) obtained certain mixture representations for discrete Linnik laws. He showed that the discrete Linnik distribution is a mixture of negative binomial distribution.

1.9. Burr distribution

Burr (1942) has suggested a number of forms of cumulative distribution functions, which are useful for fitting data (see also Burr (1968, 1973), Burr and Cislak (1968), Rodriguez (1977) devoted special attention to one of the forms denoted by type XII whose distribution function is given by

$$F_X(x) \;=\; 1-\left[\frac{1}{1+\sigma x^{\alpha}}\right]^{\gamma+1}, x>0, \quad \sigma,\alpha,\gamma>0 \tag{1.9.1}$$

where both α and γ are shape parameters. The probability density function corresponding to (1.9.1) is

$$f_X(x) = \sigma(\gamma+1)\alpha x^{\alpha-1}\frac{1}{\left(1+\sigma x^{\alpha}\right)^{\gamma+2}}.$$

We denote the random variable X on $(0, \infty)$ having distribution function (1.9.1) as Burr (σ, α, γ).

The Burr type XII distribution, which gives a wide range of values of skewness and kurtosis, can be used to fit almost any given set of unimodal data (see Tadikamalla (1980)). For the relationship between Burr type XII distribution and various other distributions, namely the Lomax, the compound Weibull, the Weibull-exponential, the logistic and the Kappa family of distributions, see Tadikamalla (1980). The distribution function and the inverse distribution function for the Burr type XII distribution exist in simple closed form. This fact plays an important role in the selection of a particular family of distributions as a stochastic model in simulation studies. Since the inverse of the Burr type XII distribution exist in simple closed form, random samples from this distribution can be obtained by the so-called 'direct method' or the 'inverse transformation method'. For the applications of the Burr type XII distribution, see Tadikamalla (1980). Takashi (1965) introduced a multivariate Burr type XII distribution and studied its properties.

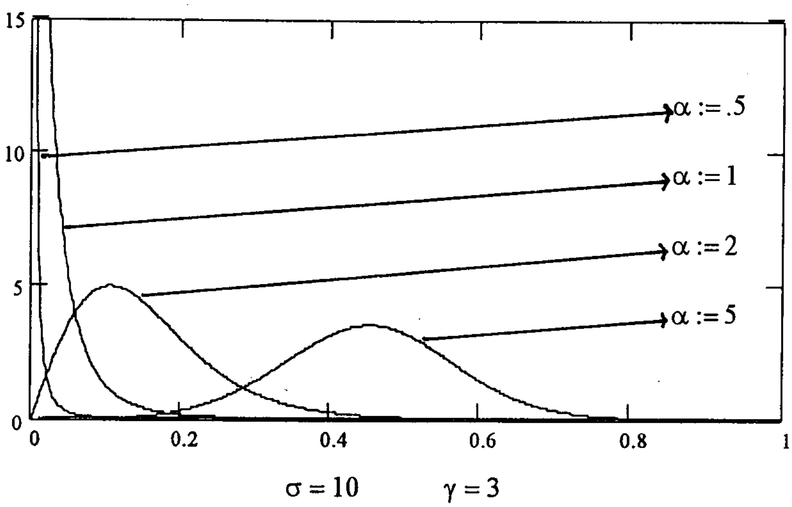

Figure 1.9.1.

Density function of Burr (σ, α, γ) distribution for σ = 10 and γ = 3

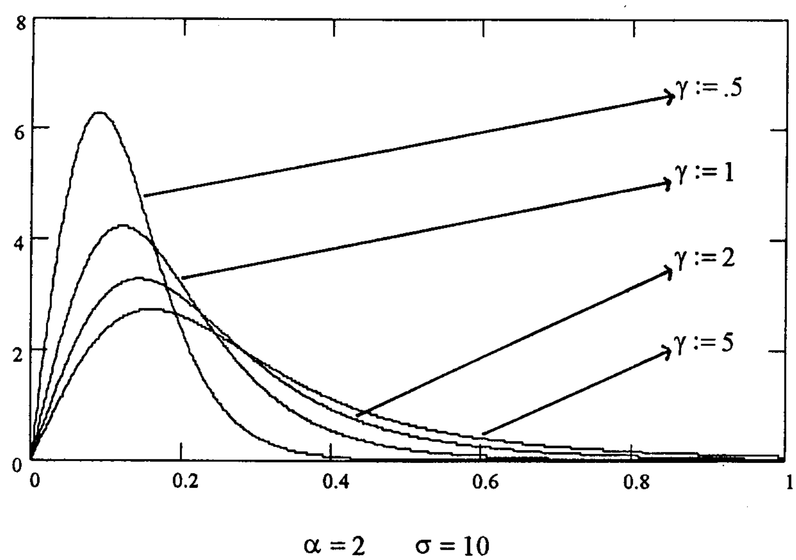

Figure 1.9.2

Density function of Burr (σ, α, γ) distribution for α = 2 and σ = 10

1.10. Random coefficient autoregressive model

Nicholls and Quinn (1982) generalized the model (1.1.1) by allowing a_i's to be random variables to define a random coefficient autoregressive model. The sequence $\{X_n\}$ is said to follow the p^{th} order random coefficient autoregressive model if

$$X_n = \sum_{i=1}^{p}(b_i + V_{i,n})X_{n-i} + \varepsilon_n, \quad n = 1,2,... \tag{1.10.1}$$

The following assumptions are made on this model.

- $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables with mean 0 and variance σ^2.
- $\{Z_n = (V_{1,n}, V_{2,n}, ...V_{p,n})\}$ is sequence of independent and identically distributed random vectors with mean zero and dispersion matrix Γ.
- $\{\varepsilon_n\}$ and $\{Z_n\}$ are statistically independent.
- $b = (b_1, b_2, ...b_p)$ is a vector of real constants.

Lawrance and Lewis (1985) used the random coefficient autoregressive model to generate exponential random variables. Liu (1990) noted that any time series model exhibiting sharp spikes or occasional bursts of outlying observations suggest the use of an infinite variance model. In such situations if the state of nature frequently changes owing to some other influential factors, then it would be more plausible to use random coefficient autoregressive models with infinite variance. Hutton (1990), Liu (1990) and Basu and Das (1993) considered some situations where random coefficient autoregressive models are useful.

Use of infinite variance model can be found in the stock market prices. The factors influencing the stock price like the turn over of a company, its working environment, its position relative to other competing companies etc. do not always have the same relative impact on the stock prices. Hence the use of fixed coefficient model would be untenable and recourse must be taken for random coefficient autoregressive models.

Basu and Das (1993) discussed different aspects of random coefficient autoregressive models with infinite variance. Given n data points, the least square estimator $\hat{b}_i$, i = 1, 2 ,..., p of the autoregressive constants for a stationary random coefficient autoregressive model was obtained by them, when the error ε_n has a distribution attracted to a symmetric stable law with a finite first absolute moment.

The first order random coefficient autoregressive model is given by

$$X_n = (b_1 + V_{1,n})X_{n-1} + \varepsilon_n, \quad n = 1,2,\ldots \tag{1.10.2}$$

We say that model (1.10.2) has a stationary solution if there exists a proper probability distribution for $\{\varepsilon_n\}$. A standard technique developed is, obtaining solution of (1.10.2) using characteristic function or Laplace transform. A general discussion on conditions for existence of the model can be found in Dewald and Lewis (1985), Gaver and Lewis (1980) and Sim (1986, 1990).

If $b_1 = 0$, by taking $V_{1,n}$ as V_n, (1.10.2) becomes

$$X_n = V_n X_{n-1} + \varepsilon_n, \quad n = 1, 2, \ldots \tag{1.10.3}$$

1.11. Summary of the Thesis

In the present work, we develop and study nine classes of random coefficient autoregressive models having symmetric generalized Linnik, symmetric Linnik, Mittag-Leffler, Burr, Pareto, generalized logistic, logistic, discrete Linnik and discrete Mittag-Leffler marginal distributions. Applications are also discussed.

The model (1.10.3) is used to construct random coefficient first order autoregressive symmetric generalized Linnik process. Random coefficient symmetric Linnik and Mittag-Leffler processes are constructed using the model

$$X_n = V_n(X_{n-1} + \varepsilon_n). \quad n = 1, 2, \ldots \tag{1.11.1}$$

Random coefficient first order autoregressive Pareto process is developed using the equation

$$X_n = V_n^{-1} \min(X_{n-1}, \varepsilon_n). \tag{1.11.2}$$

The model

$$X_n = \min(V_n^{-1} X_{n-1}, \varepsilon_n) \tag{1.11.3}$$

is used to develop random coefficient first order autoregressive Burr process.

Logistic and generalized logistic processes are constructed using the random coefficient first order autoregressive models

$$X_n = -\ln V_n + \min(X_{n-1}, \varepsilon_n) \tag{1.11.4}$$

and

$$X_n = \min(-\ln V_n + X_{n-1}, \varepsilon_n) \tag{1.11.5}$$

respectively.

Random coefficient integer valued first order autoregressive processes with discrete Linnik and discrete Mittag-Leffler marginal distributions are developed using the equations

$$X_n = V_n \oplus X_{n-1} + \varepsilon_n \tag{1.11.6}$$

and

$$X_n = V_n \oplus (X_{n-1} + \varepsilon_n). \tag{1.11.7}$$

In Chapter II, random coefficient first order autoregressive symmetric generalized Linnik, symmetric Linnik and Mittag-Leffler models are constructed and their properties are studied. Random coefficient first order autoregressive moving average process with mixed symmetric generalized Linnik distribution as marginal is introduced. The symmetric Linnik and Mittag-Leffler processes are extended to define random coefficient first order autoregressive moving average processes. Generalizations to multivariate distributions are also done.

Random coefficient first order autoregressive minification processes with Burr and Pareto distributions as marginals are introduced in Chapter III and their properties are studied. Extensions to multivariate distributions are also done. First order autoregressive moving average processes with mixed Burr distribution and Pareto distribution as marginals are also developed.

In Chapter IV, random coefficient first order autoregressive logistic and generalized logistic processes are developed and their properties are studied. Random coefficient first order autoregressive moving average processes with logistic and mixed generalized logistic distributions as marginals are also developed.

Two random coefficient first order integer valued autoregressive models, one with discrete Linnik marginal distribution and other with discrete Mittag-Leffler marginal distribution are introduced and studied in Chapter V. Extensions to multivariate distributions are also done. Random coefficient first order integer valued autoregressive processes with mixed discrete Linnik and discrete Mittag-Leffler distribution as marginals are also developed.

In Chapter VI, the random coefficient first order autoregressive symmetric generalized Linnik process is fitted to the total industrial production index per month in USA and 10-month ahead forecasts are obtained.

Algorithms for generating Mittag-Leffler, symmetric Linnik, symmetric generalized Linnik, discrete Mittag-Leffler and discrete Linnik random variables are given in the Appendix. Also algorithms for generation of first order autoregressive symmetric generalized Linnik, symmetric Linnik, Mittag-Leffler, Burr, Pareto, generalized logistic, logistic, discrete Mittag-Leffler and discrete Linnik processes are presented in the Appendix. Simulated numerical cumulative distribution function table of the symmetric generalized Linnik distribution and the density plot of the symmetric generalized Linnik distribution in comparison with standard normal distribution are also given in the Appendix.

STUDY ON AUTOREGRESSIVE MODELS

Thesis submitted

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By

THOMAS MATHEW

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

KERALA, INDIA

Pin. 673 635

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CHAPTER II

SYMMETRIC GENERALIZED LINNIK, SYMMETRIC LINNIK AND MITTAG-LEFFLER PROCESSES

2.1. Introduction

Many non-Gaussian autoregressive processes were introduced and studied during the past two decades. Lawrance (1978) developed an autoregressive process with Laplace marginals. Dewald and Lewis (1985) developed a second order autoregressive Laplace process. Damsleth and El-Shaarawi (1989) used autoregressive moving average model with Laplace noise for modeling environmental time series. Anderson and Arnold (1993) developed an autoregressive process with Linnik marginals. They have discussed the application of Linnik distribution in the study of stock price changes. Jayakumar et al. (1995) developed a first order autoregressive process with Linnik marginals. Pillai (1985) introduced the semi-Linnik distribution having the characteristic function $\phi(t) = \frac{1}{1+\psi(t)}$ where $\psi(t)$ satisfies the functional equation $\psi(t) = a\psi(bt)$, $0 < b < 1$, where a is the unique solution of $a b^{\alpha} = 1, 0 < \alpha \leq 2$. It can be seen that the solution of the functional equation $\psi(t) = \frac{1}{p}\psi(p^{1/\alpha}t)$ is $\psi(t) = |t|^{\alpha} h(t)$ where h(t) is periodic in ln |t| with period $\frac{-2\pi\alpha}{\ln p}$ (see Pillai (1985)).

This Chapter is based on Jayakumar and Thomas Mathew (2002a), Thomas Mathew and Jayakumar (2003a).

When h(t) is a constant, semi–Linnik becomes Linnik. Jayakumar (1997) developed first order autoregressive semi-Linnik process and extended it to higher orders. Jayakumar and Pillai (1993) introduced and studied first order autoregressive Mittag-Leffler process. Pillai and Jayakumar (1994) characterized a p^{th} order autoregressive Mittag-Leffler process using specialized class L property. Jayakumar (2003) studied a first order autoregressive Mittag-Leffler process and it is fitted to weakly stream flows of Kallada River, Kerala, India. Even though various authors constructed a number of autoregressive processes using heavy tailed distributions, not much attention has been devoted for the study of random coefficient autoregressive models using heavy tailed distributions as marginals. Thomas Mathew and Jayakumar (2003a) introduced a random coefficient autoregressive symmetric generalized Linnik model and studied its properties. They have also studied some properties of symmetric generalized Linnik distribution. Jayakumar and Thomas Mathew (2002a) introduced a random coefficient autoregressive model with Mittag-Leffler marginals and studied its properties.

In this Chapter we consider the random coefficient autoregressive model defined in (1.10.3), that is,

$$X_n = V_n X_{n-1} + \varepsilon_n \quad n = 1, 2, \dots \tag{2.1.1}$$

A general discussion on the condition for existence of solution of the model (2.1.1) can be found in Paulson and Uppuluri (1972). They claimed that the model (2.1.1) considered above arises in the study of retention of a substance in a system when the substance is periodically introduced in random quantities and the system periodically eliminates a random proportion of this substance. Then one may be interested in the behavior of the quantity of substance present in the system at the end of epoch n-1, n = 1,2,.... Suppose a quantity ε_n of this substance is introduced during the time interval (n-1, n] and during the same interval a modification of the quantity X_{n-1} to V_nX_{n-1} takes place. Hence the total quantity of the substance present at epoch n is $V_nX_{n-1}+\varepsilon_n$. More specific examples are described in Vervaat (1979). He considered the following examples: Let X_n denote the balance of a saving account, ε_n denote the deposit made just before time n and V_n the interest factor which may fluctuate stochastically with time. In another example, ε_n is the quantity of radioactive material added or taken away just before time n and V_n the natural decay of radioactivity. Sim (1986) has derived some of the properties of the model (2.1.1) and used the model in modeling hydrological data. Now we look in to some properties of the model (2.1.1).

Consider the model (2.1.1),

$$
\begin{aligned}
X_n &= V_nX_{n-1}+\varepsilon_n \qquad n = 1, 2, \ldots. \\
E(X_n) &= E(V_nX_{n-1}+\varepsilon_n) \\
&= E(V_n)E(X_{n-1})+E(\varepsilon_n) \\
Cov(X_n, X_{n-j}) &= E(X_nX_{n-j})-E(X_n)E(X_{n-j}) \\
E(X_nX_{n-j}) &= E((V_nX_{n-1}+\varepsilon_n)X_{n-j})
\end{aligned}
$$

$$
\begin{aligned}
&= E(V_n)E(X_{n-1}X_{n-j}) + E(\varepsilon_n)E(X_{n-j}) \\
&= E(V_n)\big(E(X_{n-1}X_{n-j}) - E(X_{n-1})E(X_{n-j})\big) + \\
&\quad E(V_n)E(X_{n-1})E(X_{n-j}) + E(\varepsilon_n)E(X_{n-j}).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}(X_n, X_{n-j}) &= E(V_n)\mathrm{Cov}(X_{n-1}, X_{n-j}) + E(V_n)E(X_{n-1})E(X_{n-j}) \\
&\quad + E(\varepsilon_n)E(X_{n-j}) - E(X_n)E(X_{n-j}) \\
&= E(V_n)\mathrm{Cov}(X_{n-1}, X_{n-j}) + E(V_n)E(X_{n-1})E(X_{n-j}) \\
&\quad + E(X_{n-j})[E(\varepsilon_n) - E(X_n)] \\
&= E(V_n)\mathrm{Cov}(X_{n-1}, X_{n-j}) + E(V_n)E(X_{n-1})E(X_{n-j}) \\
&\quad + E(X_{n-j})[E(\varepsilon_n) - (E(V_n)E(X_{n-1}) + E(\varepsilon_n))].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{Cov}(X_n, X_{n-j}) &= E(V_n)\mathrm{Cov}(X_{n-1}, X_{n-j}) \\
&= (E(V_n))^j \mathrm{Cov}(X_{n-j}, X_{n-j})
\end{aligned}
$$

The correlation coefficient,

$$
\mathrm{Corr}(X_n, X_{n-j}) = \rho_j = (E(V_n))^j, \quad j > 0. \tag{2.1.2}
$$

The joint characteristic function of (X_n, X_{n+1}) is

$$
\begin{aligned}
\phi_{X_n X_{n+1}}(t_1, t_2) &= E\left(e^{it_1 X_n + it_2 X_{n+1}}\right) \\
&= E\left(e^{(V_{n+1}X_n + \varepsilon_{n+1})it_2 + it_1 X_n}\right)
\end{aligned}
$$

$$= \phi_{\varepsilon_{n+1}}(t_2)\int_v \phi_{X_n}(t_1 + vt_2)dG(v). \tag{2.1.3}$$

The random coefficient autoregressive model defined in (1.11.1) can be considered as a modification to the model (2.1.1) and may be more suitable for the data considered in Paulson and Uppuluri (1972) because the quantity of substance ε_n introduced during the time interval (n-1, n] is also viable to the same modification as that is taking place for the existing quantity of substance X_{n-1}. That is, the total quantity of substance X_n at the epoch n will be $V_n X_{n-1} + V_n \varepsilon_n$. The same argument may also applicable to the examples considered in Vervaat (1979). The quantity of radioactive material added during the time interval (n-1, n] will decay with the same half life period as that of the existing quantity of radioactive material present there. In this situation, the new model suggested may be more suitable than the existing one because the half life period of a radioactive material is the time elapsed for the material to become the half of the existing total of the radioactive material present in the system.

Now we consider the model (1.11.1),

$$X_n = V_n(X_{n-1} + \varepsilon_n). \tag{2.1.4}$$

$$E(X_n) = E(V_n)(E(X_{n-1}) + E(\varepsilon_n))$$

$$Cov(X_n, X_{n-j}) = E(X_n X_{n-j}) - E(X_n)E(X_{n-j})$$

$$E(X_n X_{n-j}) = E(V_n(X_{n-1} + \varepsilon_n)X_{n-j})$$

$$= E(V_n)\big(E(X_{n-1}X_{n-j}) + E(\varepsilon_n X_{n-j})\big)$$

$$= E(V_n)\big(E(X_{n-1}X_{n-j}) - E(X_{n-1})E(X_{n-j})\big) +$$

$$E(V_n)\big(E(X_{n-1})E(X_{n-j}) + E(\varepsilon_n X_{n-j})\big).$$

$$= E(V_n)Cov(X_{n-1}X_{n-j}) +$$

$$E(V_n)\big(E(X_{n-1})E(X_{n-j}) + E(\varepsilon_n X_{n-j})\big).$$

$$Cov(X_n, X_{n-j}) = E(V_n)Cov(X_{n-1}, X_{n-j}) + E(V_n)E(X_{n-1})E(X_{n-j})$$

$$+ E(V_n)E(\varepsilon_n X_{n-j}) - E(X_n)E(X_{n-j})$$

$$= E(V_n)Cov(X_{n-1}, X_{n-j}) + E(V_n)E(X_{n-1})E(X_{n-j})$$

$$+ E(V_n)E(\varepsilon_n X_{n-j}) - \big(E(V_n)\big(E(X_{n-1}) + E(\varepsilon_n)\big)\big)E(X_{n-j}).$$

$$Cov(X_n, X_{n-j}) = E(V_n)Cov(X_{n-1}, X_{n-j}).$$

$$Cov(X_n, X_{n-j}) = (E(V_n))^j V(X_{n-j}).$$

The correlation coefficient

$$\rho_j = (E(V_n))^j, \quad j > 0. \tag{2.1.5}$$

The joint characteristic function of (X_n, X_{n+1})

$$\phi_{X_n, X_{n+1}}(t_1, t_2) = E\left(e^{it_1 X_n + it_2 X_{n+1}}\right)$$

$$= E\left(e^{V_{n+1}(X_n + \varepsilon_{n+1})it_2 + it_1 X_n}\right)$$

$$= E\left(e^{(it_1 + V_{n+1}it_2)X_n}\right)E\left(e^{V_{n+1}\varepsilon_{n+1}it_2}\right)$$

$$= \int_{V} \phi_{X_n}(t_1 + vt_2)\phi_{\varepsilon_{n+1}}(vt_2)dG(v). \tag{2.1.6}$$

In this work, we consider V_n as power function random variable.

In Section 2, we introduce a first order autoregressive symmetric generalized Linnik model and study some of its properties. A multivariate generalization is done. A first order autoregressive moving average mixed symmetric generalized Linnik model is introduced and a multivariate generalization is done. In Section 3, first order autoregressive symmetric Linnik process is introduced and studied. A first order autoregressive multivariate symmetric Linnik process is developed. First order autoregressive moving average symmetric Linnik model is introduced. A first order autoregressive Mittag-Leffler process is introduced in Section 4. A multivariate generalization is considered. First order autoregressive moving average process with Mittag-Leffler marginals is introduced.

2.2. First order autoregressive symmetric generalized Linnik process

Consider the $SGL(\sigma, \alpha, \beta)$ random variable having characteristic function,

$\phi(t) = \left[\dfrac{1}{1+\sigma|t|^{\alpha}}\right]^{\beta+1}$, $0 < \alpha \leq 2$, $\sigma, \beta > 0$ in (1.7.1) and $SL(\sigma, \alpha)$ random variable having characteristic function

$$\phi(t) = \frac{1}{1+\sigma|t|^{\alpha}}, \quad t \in R, \quad 0 < \alpha \leq 2, \sigma > 0 \text{ in } (1.6.1).$$

Theorem 2.2.1.

Let $$X_n = V_n X_{n-1} + \varepsilon_n \tag{2.2.1}$$

where $\{V_n\}$ is a sequence of independent and identically distributed random variables with probability density function

$$f_{V_n}(v) = \alpha\beta\, v^{\alpha\beta-1}, 0 < v < 1,\ 0 < \alpha \le 2,\ \beta > 0 \tag{2.2.2}$$

and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables independent of $\{V_n\}$. Suppose $\{X_n\}$ is stationary. Then $X_n \stackrel{d}{=} SGL(\sigma, \alpha, \beta)$ if and only if $\varepsilon_n \stackrel{d}{=} SL(\sigma, \alpha)$.

Proof:

Consider $X_n = V_n X_{n-1} + \varepsilon_n$.

In terms of characteristic functions, we have

$$\begin{aligned}
\phi_{X_n}(t) &= \phi_{V_n X_{n-1}}(t)\phi_{\varepsilon_n}(t) \\
&= \phi_{\varepsilon_n}(t)\int_0^1 \phi_{X_{n-1}}(vt)\alpha\beta\, v^{\alpha\beta-1}dv.
\end{aligned}$$

Since $\{X_n\}$ is stationary, we get

$$\phi_X(t) = \phi_{\varepsilon_n}(t)\int_0^1 \phi_X(vt)\alpha\beta\, v^{\alpha\beta-1}dv. \tag{2.2.3}$$

That is, $$t^{\alpha\beta}\phi_X(t) = \alpha\beta\phi_{\varepsilon_n}(t)\int_0^t \phi_X(z)z^{\alpha\beta-1}dz.$$

$$\frac{\phi'_X(t)}{\phi_X(t)} = \frac{\phi'_{\varepsilon_n}(t)}{\phi_{\varepsilon_n}(t)} - \frac{\alpha\beta}{t}\left(1-\phi_{\varepsilon_n}(t)\right).$$

Hence

$$\phi_X(t) = \phi_{\varepsilon_n}(t)\exp\left\{-\alpha\beta\int_0^t \frac{1-\phi_{\varepsilon_n}(u)}{u}du\right\}.$$

If $\{\varepsilon_n\}$is a sequence of independent and identically distributed $SL(\sigma,\alpha)$ random variables, we get

$$\phi_X(t) = \left(\frac{1}{1+\sigma|t|^{\alpha}}\right)^{\beta+1}.$$

Conversely, assume that $\{X_n\}$ is stationary $SGL(\sigma,\alpha,\beta)$ as marginal. Then from (2.2.3), we have

$$\frac{1}{\phi_{\varepsilon_n}(t)} = \int_0^1 \frac{\phi_X(vt)\alpha\beta\, v^{\alpha\beta-1}}{\phi_X(t)}dv.$$

$$= \int_0^1 \left(\frac{1+\sigma|t|^{\alpha}}{1+\sigma\omega|t|^{\alpha}}\right)^{\beta+1} \beta\omega^{\beta-1}d\omega$$

$$= \frac{\left(1+\sigma|t|^{\alpha}\right)^{\beta+1}}{|t|^{\alpha\beta}} \int_0^{|t|^{\alpha}} \frac{\beta z^{\beta-1}}{(1+\sigma z)^{\beta+1}}dz.$$

Taking $u=\dfrac{z}{1+\sigma z}$, we get

$$\frac{1}{\phi_{\varepsilon_n}(t)} = 1+\sigma|t|^{\alpha}.$$

This completes the proof. □

Theorem 2.2.2.

Let $X_0 = SGL(\sigma, \alpha, \beta)$ and for n= 1,2,3,...

$$X_n = V_n X_{n-1} + \varepsilon_n$$

where $\{V_n\}$ is a sequence of independent and identically distributed random variables with probability density function

$$f_{V_n}(v) = \alpha\beta v^{\alpha\beta-1}, 0 < v < 1, 0 < \alpha \leq 2, \beta > 0$$

and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables having $SL(\sigma, \alpha)$ distribution, independent of $\{V_n\}$. Then, the process $\{X_n\}$ is stationary with $SGL(\sigma, \alpha, \beta)$ marginals.

Proof:

Consider the process

$$X_n = V_n X_{n-1} + \varepsilon_n.$$

In terms of characteristic functions, we have

$$\begin{aligned} \phi_{X_n}(t) &= \phi_{V_n X_{n-1}}(t)\phi_{\varepsilon_n}(t) \\ &= \phi_{\varepsilon_n}(t)\int_0^1 \phi_{X_{n-1}}(vt)\alpha\beta v^{\alpha\beta-1}dv. \end{aligned}$$

For n = 1, we get

$$\phi_{X_1}(t) = \phi_{\varepsilon_1}(t)\int_0^1 \phi_{X_0}(vt)\alpha\beta v^{\alpha\beta-1}dv.$$

Since $X_0 \stackrel{d}{=} SGL(\sigma,\alpha,\beta)$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables having $SL(\sigma,\alpha)$ we get

$$\phi_{X_1}(t) = \frac{1}{1+\sigma|t|^\alpha}\int_0^1\left(\frac{1}{1+v^\alpha\sigma|t|^\alpha}\right)^{\beta+1} \alpha\beta v^{\alpha\beta-1}dv.$$

Put $v^\alpha = u$. Then

$$\phi_{X_1}(t) = \frac{\beta}{1+\sigma|t|^\alpha}\int_0^1\left(\frac{1}{1+u\sigma|t|^\alpha}\right)^{\beta+1} u^{\beta-1}du.$$

Take $u\sigma|t|^\alpha = w$. Then,

$$\phi_{X_1}(t) = \frac{\beta}{1+\sigma|t|^\alpha}\int_0^{\sigma|t|^\alpha}\left(\frac{w}{1+w}\right)^{\beta-1}\left(\frac{1}{1+w}\right)^2\frac{dw}{|t|^{\alpha\beta}\sigma^\beta}.$$

$$\phi_{X_1}(t) = \frac{\beta}{1+\sigma|t|^\alpha}\frac{1}{|t|^{\alpha\beta}\sigma^\beta}\int_0^{\frac{|t|^\alpha\sigma}{1+\sigma|t|^\alpha}} u^{\beta-1}du$$

$$= \frac{\beta}{1+\sigma|t|^\alpha}\frac{1}{|t|^{\alpha\beta}\sigma^\beta\beta}\left(\frac{|t|^\alpha\sigma}{1+|t|^\alpha}\right)^\beta$$

$$= \left(\frac{1}{1+\sigma|t|^\alpha}\right)^{\beta+1}.$$

That is, $X_1 \stackrel{d}{=} SGL(\sigma,\alpha,\beta)$.

If $X_{n-1} \stackrel{d}{=} SGL(\sigma, \alpha, \beta)$, then we get $X_n \stackrel{d}{=} SGL(\sigma, \alpha, \beta)$. Thus the process $\{X_n\}$ is stationary with symmetric generalized Linnik marginals. □

Based on the above Theorem, we define first autoregressive symmetric generalized Linnik process as follows:

Let $\quad X_0 = SGL(\sigma, \alpha, \beta)$ and for $\quad n = 1,2,3,\ldots$

$$X_n \quad = \quad V_n X_{n-1} + \varepsilon_n$$

where $\{V_n\}$ is a sequence of independent and identically distributed random variables with probability density function

$$f_{V_n}(v) \quad = \quad \alpha\beta\, v^{\alpha\beta-1}, 0 < v < 1,\ 0 < \alpha \leq 2,\ \beta > 0$$

and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables having $SL(\sigma, \alpha)$ distribution, independent of $\{V_n\}$.

Now we study some properties of the stationary first order autoregressive symmetric generalized Linnik process.

2.2.1. Properties of the first order autoregressive symmetric generalized Linnik process

The simulated sample path of the first order autoregressive symmetric generalized Linnik process is presented in Figure 2.2.1.

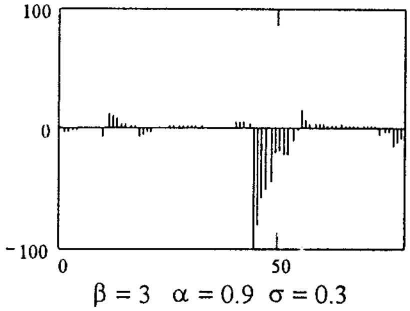


Figure 2.2.1a.

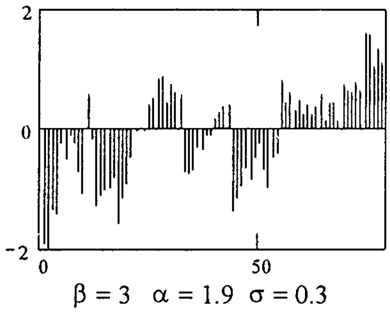


Figure 2.2.1b.

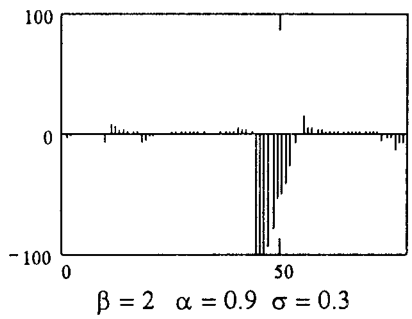


Figure 2.2.1c.

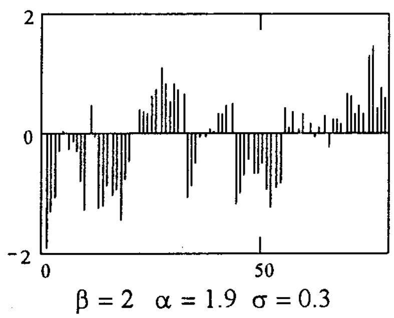


Figure 2.2.1d.

Figure 2.2.1

Sample path behavior of first order autoregressive symmetric generalized Linnik process

The joint characteristic function of (X_n, X_{n+1}) is

$$\phi_{X_n X_{n+1}}(t_1, t_2) = E\left(e^{it_1 X_n + it_2 X_{n+1}}\right)$$

$$= \frac{1}{1+|t_2|^\alpha}\int_0^1 \left[\frac{1}{1+|t_1+vt_2|^\alpha}\right]^{\beta+1} \alpha\beta v^{\alpha\beta-1} dv.$$

This expression is not symmetric in t_1, t_2 and hence the process is not time reversible.

The empirical joint distribution of (X_n, X_{n+1}) of the first order autoregressive symmetric generalized Linnik process for $\alpha = 1.8$, $\sigma = 3$ and $\beta = 4$ is presented in Figure 2.2.2.

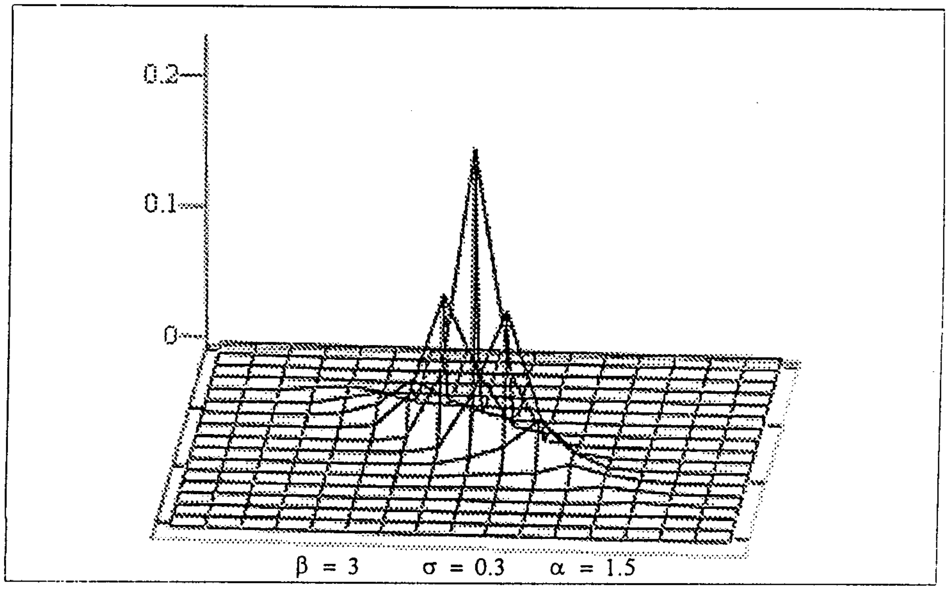


Figure 2.2.2.

Empirical joint probability density function of (X_n, X_{n+1}) of first order autoregressive symmetric generalized Linnik process.

2.2.2. First order autoregressive multivariate symmetric generalized Linnik process

Consider the multivariate symmetric generalized Linnik ($SGL(\underline{\sigma}, \alpha, \beta)$) distribution with characteristic function

$$\phi_{\underline{X}}(\underline{t}) = \left[\frac{1}{1+\sigma_1|t_1|^{\alpha}+\sigma_2|t_2|^{\alpha}+\ldots+\sigma_k|t_k|^{\alpha}}\right]^{\beta+1}, \tag{2.2.4}$$

where $\underline{t} = (t_1, t_2, \ldots, t_k)' \in R_k$, $0 < \alpha \leq 2$, $\beta > 0$, $\sigma_j > 0$, $j = 1, 2, \ldots, k$, and multivariate symmetric Linnik (SL($\underline{\sigma}$, α)) distribution with characteristic function

$$\phi_{\underline{\varepsilon}_n}(\underline{t}) = \frac{1}{1+\sigma_1|t_1|^\alpha+\sigma_2|t_2|^\alpha+\ldots+\sigma_k|t_k|^\alpha}. \qquad (2.2.5)$$

Theorem 2.2.3.

Consider the process $\{X_n\}$ be defined by

$$\underline{X}_n = V_n\underline{X}_{n-1}+\underline{\varepsilon}_n, \qquad n = 1, 2, \ldots \qquad (2.2.6)$$

where $\{V_n\}$ is sequence of independent and identically distributed random variables having distribution function

$$F_{V_n}(v) = v^{\alpha\beta}, \alpha\beta > 0, \quad 0 < v < 1, \quad 0 < \alpha \le 2, \quad \beta > 0$$

and $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors on R_k. Suppose the process $\{\underline{X}_n\}$ is stationary. Then $\underline{X}_n \stackrel{d}{=}$ multivariate SGL($\underline{\sigma}$, α, β) if and only if $\underline{\varepsilon}_n \stackrel{d}{=}$ multivariate SL($\underline{\sigma}$, α).

Proof:

Equation (2.2.6) in terms of characteristic function is

$$\phi_{\underline{X}_n}(\underline{t}) = \phi_{\underline{\varepsilon}_n}(\underline{t})\int_0^1 \phi_{\underline{X}_{n-1}}(v\underline{t})\, f_{V_n}(v)dv.$$

That is,

$$\phi_{\underline{X}_n}(\underline{t}) = \phi_{\underline{\varepsilon}_n}(\underline{t})\int_0^1 \phi_{\underline{X}_{n-1}}(v\underline{t})\alpha\beta v^{\alpha\beta-1}dv.$$

If the process is stationary, then

$$\phi_{\underline{X}}(\underline{t}) = \phi_{\underline{\varepsilon}_n}(\underline{t})\int_0^1 \phi_{\underline{X}}(v\underline{t})\alpha\beta\, v^{\alpha\beta-1}dv. \tag{2.2.7}$$

That is,

$$\phi_{\underline{X}}(t_1, t_2, \ldots, t_k) = \phi_{\underline{\varepsilon}_n}(t_1, t_2, \ldots, t_k)\int_0^1 \phi_{\underline{X}}(t_1 v, t_2 v, \ldots, t_k v)\alpha\beta v^{\alpha\beta-1}dv.$$

Taking $z_j = \delta_j^{1/\alpha} z$ for j = 1, 2,.., k, we get

$$\phi_{\underline{X}}(\underline{\delta}\, t) = \phi_{\underline{\varepsilon}_n}(\underline{\delta}\, t)\int_0^1 \phi_{\underline{X}}(\underline{\delta}\, t\, v)\alpha\beta v^{\alpha\beta-1}dv$$

$$\text{where } \underline{\delta} = (\sigma_1\delta_1, \sigma_2\delta_2, \ldots, \sigma_k\delta_k)'.$$

If tv = u, then

$$t^{\alpha\beta}\phi_{\underline{X}}(\underline{\delta}\, t) = \alpha\beta\phi_{\underline{\varepsilon}_n}(\underline{\delta}\, t)\int_0^t \phi_{\underline{X}}(\underline{\delta}\, u)u^{\alpha\beta-1}du.$$

Differentiating with respect to t, gives

$$\frac{\underline{\delta}\,\phi'_{\underline{X}}(\underline{\delta}\, t)}{\phi_{\underline{X}}(\underline{\delta}\, t)} = \frac{\underline{\delta}\,\phi'_{\underline{\varepsilon}_n}(\underline{\delta}\, t)}{\phi_{\underline{\varepsilon}_n}(\underline{\delta}\, t)} - \frac{\alpha\beta}{t}\left(1 - \phi_{\underline{\varepsilon}_n}(\underline{\delta}\, t)\right).$$

That is,

$$\phi_{\underline{X}}(\underline{\delta}\, t) = \phi_{\underline{\varepsilon}_n}(\underline{\delta}\, t)\exp\left(-\alpha\beta\int_0^t \frac{1-\phi_{\underline{\varepsilon}_n}(\underline{\delta} u)}{u}du\right).$$

Since ε_n has the multivariate $SL(\underline{\sigma}, \alpha)$ distribution with characteristic function (2.2.5), we get

$$\phi_{\underline{X}}(\underline{\delta}\, t) = \left[\frac{1}{1+\underline{\delta}|t|^{\alpha}}\right]^{\beta+1}.$$

That is,

$$\phi_{\underline{X}}(\underline{t}) = \left[\frac{1}{1+\sigma_1|t_1|^{\alpha}+\sigma_2|t_2|^{\alpha}+\ldots+\sigma_k|t_k|^{\alpha}}\right]^{\beta+1}.$$

Conversely, suppose $\{X_n\}$ has stationary multivariate $\mathrm{SGL}(\underline{\sigma},\alpha,\beta)$. Then from (2.2.7)

$$\frac{1}{\phi_{\underline{\varepsilon}_n}(\underline{t})} = \int_0^1\left(\frac{\phi_{\underline{X}}(v\underline{t})\alpha\beta\, v^{\alpha\beta-1}}{\phi_{\underline{X}}(\underline{t})}\right)dv$$

$$= \int_0^1\left(\frac{1+\underline{\delta}|t|^{\alpha}}{1+\underline{\delta}|vt|^{\alpha}}\right)^{\beta+1}\alpha\beta v^{\alpha\beta-1}dv$$

$$= 1+\underline{\delta}|t|^{\alpha}.$$

That is, $\quad \phi_{\varepsilon_n}(\underline{t}) = \dfrac{1}{1+\underline{\delta}\,|t|^{\alpha}}.$

Therefore,

$$\phi_{\underline{\varepsilon}_n}(\underline{t}) = \frac{1}{1+\sigma_1|t_1|^{\alpha}+\sigma_2|t_2|^{\alpha}+\ldots+\sigma_k|t_k|^{\alpha}}. \qquad \square$$

2.2.3. First order autoregressive moving average mixed symmetric generalized Linnik model

Consider the process $\{X_n\}$ defined by

$$\begin{aligned} X_n &= K_n\varepsilon_n + V_n Y_{n-1} \\ Y_n &= U_n Y_{n-1} + \varepsilon_n \end{aligned} \tag{2.2.8}$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed SL(σ, α) random variables. $\{K_n\}$'s are i.i.d Bernoulli random variables such that $P(K_n = 0) = \theta = 1\text{-}P(K_n = 1)$, $\{V_n\}$ and $\{U_n\}$ are two independent sequences of independent and identically distributed random variables defined on the interval (0,1) with distribution functions $F_{V_n}(v) = v^{\alpha\beta}$, $F_{W_n}(w) = w^{\alpha\beta}$, $0 < v, w < 1$, $0 < \alpha < 2, \beta > 0$ and $Y_n \underset{=}{d} SGL(\sigma, \alpha, \beta)$ then $\{X_n\}$ is distributed as mixed symmetric generalized Linnik.

We have from (2.2.8),

$$\begin{aligned} \phi_{X_n}(t) &= E\left(e^{(K_n\varepsilon_n + V_n Y_{n-1})}\right) \\ &= \theta E\left(e^{V_n Y_{n-1}}\right) + (1-\theta)E\left(e^{\varepsilon_n + V_n Y_{n-1}}\right). \end{aligned}$$

That is, $$\phi_{X_n}(t) = \left[\theta + (1-\theta)\phi_{\varepsilon_n}(t)\right]\int_0^1 \phi_{Y_{n-1}}(vt)\alpha\beta v^{\alpha\beta-1}dv.$$

$$\phi_{X_n}(t) = \left[\theta + (1-\theta)\phi_{\varepsilon_n}(t)\right]\int_0^1 \phi_{Y_{n-1}}(vt)\alpha\beta v^{\alpha\beta-1}dv.$$

Put vt = u. Then

$$\phi_{X_n}(t) = \left[\theta + (1-\theta)\phi_{\varepsilon_n}(t)\right]\int_0^t \frac{\phi_{Y_{n-1}}(u)\alpha\beta u^{\alpha\beta-1}}{u^{\alpha\beta}}du.$$

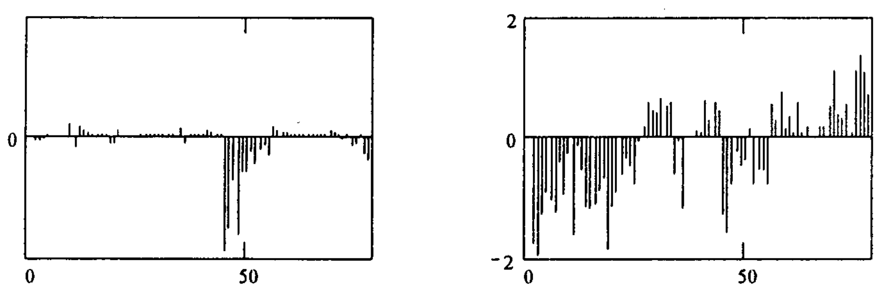
$$= \frac{1}{t^{\alpha\beta}}\left[\theta+(1-\theta)\frac{1}{1+\sigma|t|^{\alpha}}\right]\int_0^t \phi_{Y_{n-1}}(u)\alpha\beta\, u^{\alpha\beta-1}du.$$

$$= \theta\frac{1}{(1+\sigma|t|^{\alpha})^{\beta}}+(1-\theta)\frac{1}{(1+\sigma|t|^{\alpha})^{\beta+1}}.$$

$$\phi_{X_n}(t) = \theta\left[\frac{1}{1+\sigma|t|^{\alpha}}\right]^{\beta}+(1-\theta)\left[\frac{1}{1+\sigma|t|^{\alpha}}\right]^{\beta+1}.$$

Thus $\{X_n\}$ has mixed $SGL(\sigma,\alpha,\beta)$ distribution. □

Now we shall study some properties of the first order autoregressive moving average mixed symmetric generalized Linnik model.



$\beta = 3$ $\alpha = 0.9$ $\sigma = 0.3$ $\theta = 0.5$

$\beta = 3$ $\alpha = 1.9$ $\sigma = 0.3$ $\theta = 0.5$

Figure 2.2.3.

Sample path behavior of first order autoregressive moving average mixed symmetric generalized Linnik process

2.2.4. First order autoregressive moving average multivariate mixed symmetric generalized Linnik model

Consider the process $\{\underline{X}_n\}$ defined by

$$\begin{aligned} \underline{X}_n &= K_n\underline{\varepsilon}_n + V_n\underline{Y}_{n-1} \\ \underline{Y}_n &= U_n\underline{Y}_{n-1} + \underline{\varepsilon}_n \end{aligned} \qquad (2.2.9)$$

where $\{\underline{\varepsilon}_n\}$ is sequence of independent and identically distributed multivariate SL($\underline{\sigma}$, α) random vectors, $\{K_n\}$'s are independent and identically distributed Bernoulli random variables such that $P(K_n = 0) = \theta = 1-P(K_n = 1)$, $\{V_n\}$ and $\{U_n\}$ are two independent sequences of independent and identically distributed random variables defined on the interval $(0,1)$ with distribution functions $F_{V_n}(v) = v^{\alpha\beta}$, $F_{W_n}(w) = w^{\alpha\beta}$, $0 < v, w < 1$, $0 < \alpha \leq 2$, $\beta > 0$ then $\underline{X}_n$ $\underline{d}$ Multivariate mixed symmetric generalized Linnik.

We have from (2.2.9),

$$\begin{aligned}\phi_{\underline{X}_n}(\underline{t}) &= E\left(e^{(K_n\underline{\varepsilon}_n + V_n\underline{Y}_{n-1})}\right)\\ &= \theta E\left(e^{V_n\underline{Y}_{n-1}}\right) + (1-\theta)E\left(e^{\underline{\varepsilon}_n + V_n\underline{Y}_{n-1}}\right).\end{aligned}$$

That is,

$$\phi_{\underline{X}_n}(\underline{t}) = \left[\theta + (1-\theta)\phi_{\underline{\varepsilon}_n}(\underline{t})\right]\int_0^1 \phi_{\underline{Y}_{n-1}}(v\underline{t})\alpha\beta v^{\alpha\beta-1}dv.$$

$$\phi_{\underline{X}_n}(\underline{t}) = \left[\theta + (1-\theta)\phi_{\underline{\varepsilon}_n}(\underline{t})\right]\int_0^1 \phi_{\underline{Y}_{n-1}}(v\underline{t})\alpha\beta v^{\alpha\beta-1}dv.$$

It can be easily shown that

$$\phi_{\underline{X}_n}(\underline{t}) = \theta\frac{1}{(1+\underline{\delta}|t|^{\alpha})^{\beta}} + (1-\theta)\frac{1}{(1+\underline{\delta}|t|^{\alpha})^{\beta+1}}.$$

That is,

$$\phi_{\underline{X}_n}(\underline{t}) = \theta\left[\frac{1}{1+\sigma_1|t_1|^\alpha+\sigma_2|t_2|^\alpha+\ldots+\sigma_k|t_k|^\alpha}\right]^\beta +$$

$$(1-\theta)\left[\frac{1}{1+\sigma_1|t_1|^\alpha+\sigma_2|t_2|^\alpha+\ldots+\sigma_k|t_k|^\alpha}\right]^{\beta+1}.$$

Thus $\{\underline{X}_n\}$ has multivariate mixed symmetric generalized Linnik distribution. □

2.3. First order autoregressive symmetric Linnik process

Now we consider the model (2.1.4).

Theorem 2.3.1.

Let the process $\{X_n\}$ be defined as

$$\text{for } n = 1,2,\ldots \qquad \begin{aligned} X_0 &= \varepsilon_1 \quad \text{and} \\ X_n &= V_n(X_{n-1}+\varepsilon_n) \end{aligned} \qquad (2.3.1)$$

where $\{V_n\}$ and $\{\varepsilon_n\}$are two independent sequences of independent and identically distributed random variables such that V_n has the distribution function $F_{V_n}(v) = v^\alpha, \; 0<\alpha\le 2, \; 0<v<1$. Then the process $\{X_n\}$ is stationary if and only if $\varepsilon_1 \underline{\underline{d}}\ SL(\sigma,\alpha)$.

Proof:

Denoting the characteristic function of X_n and ε_n by $\phi_{X_n}(t)$ and $\phi_{\varepsilon_n}(t)$ respectively, (2.3.1) in terms of characteristic functions is

$$\phi_{X_n}(t) = \int_0^1 \phi_{X_{n-1}}(tv)\phi_{\varepsilon_n}(tv) f_{V_n}(v)dv, \tag{2.3.2}$$

where $f_{V_n}(v)$ is the probability density function of V_n.

Since $F_{V_n}(v) = v^{\alpha}, 0 < \alpha \le 2,\ 0 < v < 1$, (2.3.2) becomes

$$\phi_{X_n}(t) = \int_0^1 \phi_{X_{n-1}}(tv)\phi_{\varepsilon_n}(tv)\alpha v^{\alpha-1}dv.$$

For n = 1,

$$\phi_{X_1}(t) = \int_0^1 \phi_{X_0}(tv)\phi_{\varepsilon_1}(tv)\alpha v^{\alpha-1}dv.$$

Since $X_0 \stackrel{d}{=} \varepsilon_1$, we have

$$\phi_{X_1}(t) = \int_0^1 \phi^2_{X_0}(tv)\alpha v^{\alpha-1}dv.$$

Assume that the process $\{X_n\}$ is stationary. Then

$$\phi_X(t) = \int_0^1 \phi^2_X(tv)\alpha v^{\alpha-1}dv,$$

Taking tv = z and on simplification, we get

$$\phi_X(t) = \frac{1}{1+\sigma|t|^{\alpha}}.$$

Hence $\varepsilon_1 \stackrel{d}{=} SL(\sigma,\alpha)$.

Conversely, assume that $\varepsilon_1 \stackrel{d}{=} SL(\sigma,\alpha)$.

From (2.3.2), we have

$$\phi_{X_n}(t) = \int_0^1 \phi_{X_{n-1}}(tv)\phi_{\varepsilon_n}(tv)f_{V_n}(v)dv.$$

Since $X_0 \stackrel{d}{=} \varepsilon_1$ and $\{V_n\}$ has the power function distribution with probability density function $f_{V_n}(v) = \alpha v^{\alpha-1}$, $0 < \alpha \le 2$, $0 < v < 1$, for n = 1, we get

$$\begin{aligned}\phi_{X_1}(t) &= \int_0^1 \phi_{X_0}(tv)\phi_{\varepsilon_1}(tv)\alpha v^{\alpha-1}dv \\ &= \frac{1}{1+\sigma|t|^\alpha}.\end{aligned}$$

Assuming $X_{n-1} \stackrel{d}{=} SL(\sigma,\alpha)$, it can be shown that $X_n \stackrel{d}{=} SL(\sigma,\alpha)$. Hence the process $\{X_n\}$ is stationary. This completes the proof. □

Based on Theorem 2.3.1, we define first order autoregressive symmetric Linnik process as follows:

$$\begin{aligned}&\text{Let} & X_0 &= \varepsilon_1 \quad \text{and} \\ &\text{for } n = 1,2,\ldots & X_n &= V_n(X_{n-1}+\varepsilon_n),\end{aligned}$$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables with $F_{V_n}(v) = v^\alpha$, $0 < \alpha \le 2$, $0 < v < 1$ and ε_1 is distributed as symmetric Linnik $SL(\sigma,\alpha)$.

2.3.1. Properties of the first order autoregressive symmetric Linnik process

Simulated sample path using 80 observations generated from first order autoregressive symmetric Linnik process with $\sigma = .2$ and $\alpha = 1.9$ and 0.9 are presented in Figure 2.3.1.

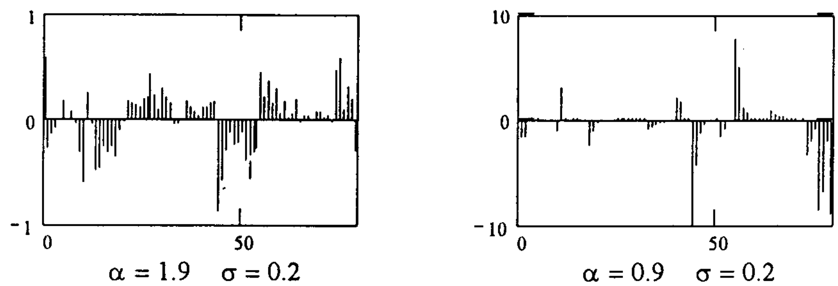


Figure 2.3.1.

Sample path behavior of first order autoregressive symmetric Linnik process

The joint distribution of (X_n, X_{n+1}) for first order autoregressive symmetric Linnik model is obtained with the use of characteristic function as

$$
\begin{aligned}
\phi_{X_n,X_{n+1}}(t_1,t_2) &= E\left(e^{it_1X_n+it_2X_{n+1}}\right) \\
&= E\left(e^{it_1X_n+it_2(V_nX_n+\varepsilon_{n+1})}\right) \\
&= \int_0^1 \phi_{X_n}(t_1+vt_2)\phi_{\varepsilon_{n+1}}(vt_2)\alpha v^{\alpha-1}dv \\
&= \int_0^1 \frac{1}{1+\sigma|t_1+vt_2|^{\alpha}}\frac{1}{1+\sigma|vt_2|^{\alpha}}\alpha v^{\alpha-1}dv.
\end{aligned}
$$

Thus like the first order autoregressive α-Laplace process of Jayakumar et al.(1995), the first order autoregressive symmetric Linnik process is not time reversible.

The graph of the empirical joint distribution of (X_n, X_{n+1}) is presented in Figure 2.3.2 below using simulated series of 10000 observations from the first order autoregressive symmetric Linnik process.

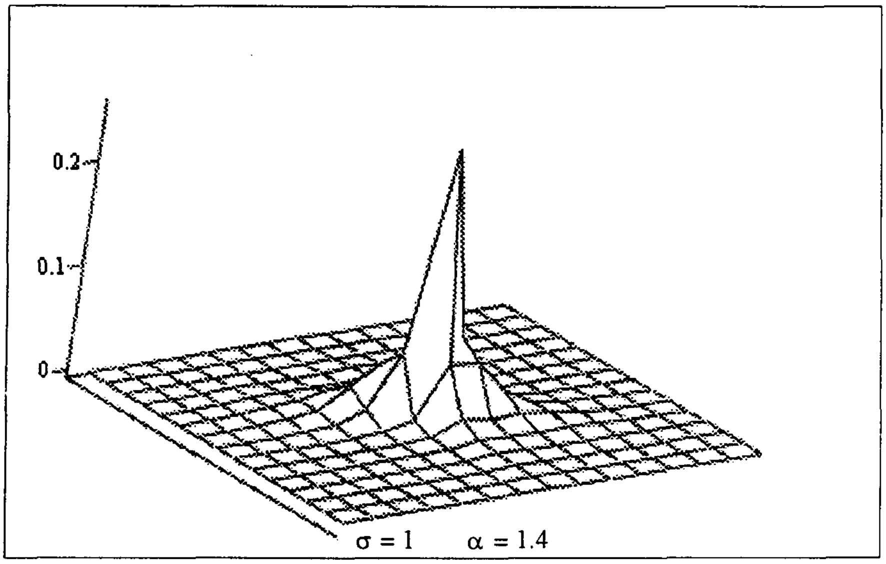


Figure 2.3.2.

Empirical joint distribution of (X_n, X_{n+1}) of first order autoregressive symmetric Linnik process

2.3.2. First order autoregressive moving average symmetric Linnik process

The first order moving average symmetric Linnik process is built using a sequence $\{\varepsilon_n\}$ of independent and identically distributed symmetric Linnik random variables in the following manner:

$$X_n \quad = \quad W_n(\varepsilon_n + \varepsilon_{n-1}) \tag{2.3.3}$$

where $\{W_n\}$ is a sequence of independent and identically distributed random variables independent of $\{\varepsilon_n\}$ with distribution function $F_{W_n}(w) = w^{\alpha}, \; 0 < w \leq 1, 0 < \alpha \leq 2$.

Combining (2.3.1) and (2.3.3) and using Theorem 2.3.1 we get.

Theorem 2.3.2.

Let the process $\{X_n\}$ be defined by

$$X_n = W_n(\varepsilon_n + Y_{n-1}),$$
$$Y_n = V_n(Y_{n-1} + Z_n)$$

where $\{Z_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables and $\{W_n\}$ and $\{V_n\}$ are also two independent sequences of independent and identically distributed random variables with distribution function $F_{W_n}(w) = w^{\alpha}, 0 < w \le 1, \quad 0 < \alpha \le 2, \quad F_{V_n}(v) = v^{\alpha}, 0 < v \le 1, 0 < \alpha \le 2$ with $\varepsilon_1 \stackrel{d}{=} Y_0 \stackrel{d}{=} Z_1$. Then the process $\{X_n\}$ is stationary if and only if $\varepsilon_1 \stackrel{d}{=} SL(\sigma, \alpha)$.

Proof:

Proof follows easily using the arguments similar to those in Theorem 2.3.1. □

2.3.3. Properties of the first order autoregressive moving average symmetric Linnik process

The simulated sample path using 80 observations generated from first order autoregressive moving average symmetric Linnik process is presented in Figure 2.3.3.

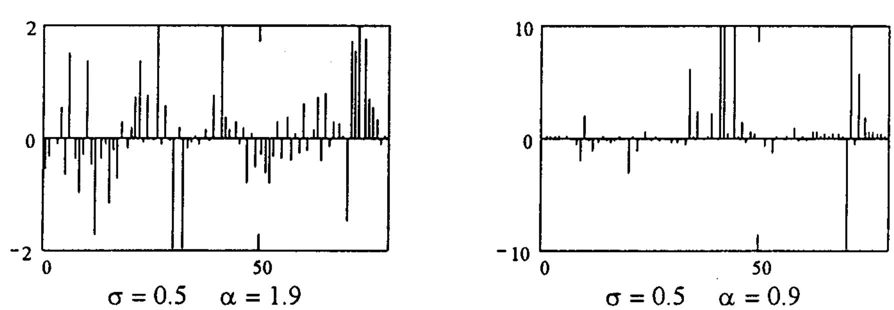


Figure 2.3.3.

Sample path behavior of first order autoregressive moving average symmetric Linnik process

The empirical joint distribution of (X_n, X_{n+1}) of the first order autoregressive moving average symmetric Linnik process is presented in Figure 2.3.4.

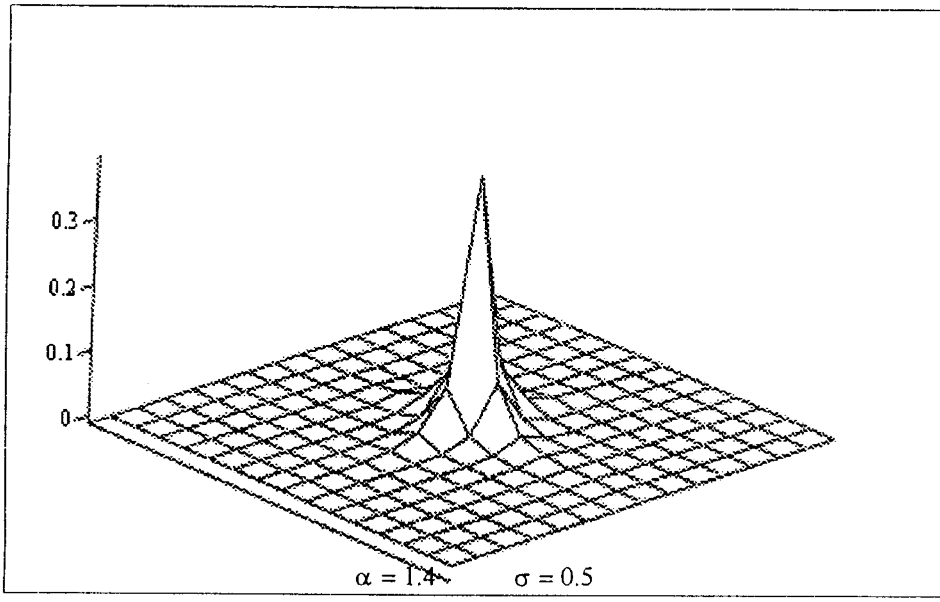


Figure 2.3.4.

Empirical joint distribution of (X_n, X_{n+1}) of first order autoregressive moving average symmetric Linnik process

2.3.4. First order autoregressive multivariate symmetric Linnik process

We consider the multivariate symmetric Linnik distribution with characteristic function

$$\phi(t_1, t_2, \dots t_k) = \frac{1}{1+\sigma_1|t_1|^{\alpha}+\sigma_2|t_2|^{\alpha}+\dots+\sigma_k|t_k|^{\alpha}}, \qquad (2.3.4)$$

$$(t_1, t_2, \dots t_k) \in R_k,\ 0<\alpha\le 2,\ \sigma_i>0,\ i=1,2,3,\dots,k.$$

Consider the process $\{\underline{X}_n\}$ defined by

$$\underline{X}_0 = \underline{\varepsilon}_1 \quad \text{and}$$

$$\underline{X}_n = V_n(\underline{X}_{n-1} + \underline{\varepsilon}_n), \quad n \geq 1 \tag{2.3.5}$$

where $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors defined on R_k, $\{V_n\}$ is a sequence of independent and identically distributed random variables defined on $(0,1)$ such that V_n has distribution function, $F_{V_n}(v) = v^{\alpha}, 0 < v \leq 1, 0 < \alpha \leq 2$.

Equation (2.3.5) in terms of characteristic functions is

$$\phi_{\underline{X}_n}(\underline{t}) = E\left(e^{i\underline{t}V_n(\underline{X}_{n-1} + \underline{\varepsilon}_n)}\right)$$

$$= \int_0^1 \phi_{\underline{X}_{n-1}}(\underline{t}v)\phi_{\underline{\varepsilon}_n}(\underline{t}v)f(v)dv. \tag{2.3.6}$$

If the process is stationary with $X_0 \overset{d}{=} \varepsilon_1$, we have

$$\phi_{\underline{X}}(t) = \int_0^1 \phi^2_{\underline{X}}(\underline{t}v)f(v)dv$$

That is,

$$\phi(t_1, t_2, \ldots, t_k) = \int_0^1 \phi^2(t_1 v, t_2 v, \ldots, t_k v)\alpha v^{\alpha-1} dv.$$

Take $t_j = \delta_j t$, we get

$$\phi(\delta_1 t, \delta_2 t, \ldots, \delta_k t) = \int_0^1 \phi^2(\delta_1 tv, \delta_2 tv, \ldots, \delta_k tv)\alpha v^{\alpha-1} dv.$$

Substituting tv = s, we get

$$\phi_{\underline{\delta}}(t) = \int_0^1 \phi^2_{\underline{\delta}}(s)\alpha s^{\alpha-1} ds,$$

$$\phi_{\underline{\delta}}(t) = \frac{1}{1+|t|^{\alpha} c(\underline{\delta})},$$

where $c(\underline{\delta}) = c(\delta_1, \delta_2, \ldots \delta_k)$.

But $$|t|^{\alpha} c(\underline{\delta}) = |t|^{\alpha} c(\delta_1, \delta_2, \ldots, \delta_k) = c(\delta_1 t, \delta_2 t, \ldots, \delta_k t).$$

Hence there exist $\delta_1, \delta_2, \ldots, \delta_k$ such that

$$\phi_{\underline{\delta}}(t) = \frac{1}{1+|t|^{\alpha} c(\underline{\delta})}.$$

Thus, there exist $a_1, a_2, \ldots, a_k$ such that

$$\phi_{\underline{\delta}}(t) = \frac{1}{1+a_1|t_1|^{\alpha}+a_2|t_2|^{\alpha}+\ldots+a_k|t_k|^{\alpha}}, \quad a_i > 0, \quad i = 1,2,\ldots k.$$

If $\underline{X}_0 \stackrel{d}{=} \varepsilon_1$ and ε_1 is multivariate symmetric Linnik, it can be seen that the process is stationary.

From (2.3.5),

$$\begin{aligned}
\phi_{\underline{X}_1}(t_1, t_2, \ldots, t_k) &= \int_0^1 \phi^2_{\underline{X}_0}(t_1 v, t_2 v, \ldots t_k v)\alpha v^{\alpha-1} dv. \\
&= \int_0^1 \frac{1}{\left[1+\sigma_1|vt_1|^{\alpha}+\sigma_1|vt_2|^{\alpha}+\ldots+\sigma_k|vt_k|^{\alpha}\right]^2} \alpha v^{\alpha-1} dv \\
&= \frac{1}{1+\sigma_1|t_1|^{\alpha}+\sigma_2|t_2|^{\alpha}+\ldots+\sigma_k|t_k|^{\alpha}}.
\end{aligned}$$

Therefore $\underline{X}_1$ is multivariate symmetric Linnik. By induction, we get $\{\underline{X}_n\}$ is multivariate symmetric Linnik.

Thus we have proved the following Theorem.

Theorem 2.3.3.

Let the process $\{\underline{X}_n\}$ be defined by

$$\text{for } n \geq 1 \qquad \begin{aligned} \underline{X}_0 &= \underline{\varepsilon}_0 \qquad \text{and} \\ \underline{X}_n &= V_n(\underline{X}_{n-1} + \underline{\varepsilon}_n) \end{aligned}$$

where $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors defined on R_k, $\{V_n\}$ is a sequence of independent and identically distributed random variables defined on $(0,1)$ such that V_n has distribution function $F_{V_n}(v) = v^{\alpha}, 0 < v \leq 1, 0 < \alpha \leq 2$. Then the process $\{\underline{X}_n\}$ is stationary if and only if $\underline{\varepsilon}_1$ is multivariate symmetric Linnik.

2.4. First order autoregressive Mittag-Leffler process

In Section 1.4, we have discussed the applications of the Mittag-Leffler as an alternative to exponential distribution. It can be used in a variety of situations where the exponential distribution is unrealistic. Also, for various values of α in (1.4.1), we get a number of distributions that are heavy tailed as compared to exponential. These facts are utilized by various authors for describing some real life situations with Mittag-Leffler distribution (see Weron and Kotulski (1996), Jayakumar (2003), Kotz et al (2001)). Here using the model (2.1.4), we introduce a new first order autoregressive process having Mittag-Leffler distribution as marginal and study its properties.

We have from (1.4.2), the Laplace transform of Mittag-Leffler distribution $ML(\sigma, \alpha)$ in (1.4.1) is

$$\phi_X(\lambda) = \frac{1}{1+\left(\frac{\lambda}{\sigma}\right)^{\alpha}}, \quad \lambda, \sigma > 0, 0 < \alpha \leq 1.$$

As in Theorem 2.3.1 we get the following result, in the case of first order autoregressive Mittag-Leffler process.

Theorem 2.4.1.

Let the process $\{X_n\}$ be defined as

$$X_0 = \varepsilon_1 \quad \text{and for } n = 1,2,\ldots$$

$$X_n = V_n(X_{n-1} + \varepsilon_n) \tag{2.4.1}$$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables such that V_n has the distribution function $F_{V_n}(v) = v^{\alpha}, \quad 0 < \alpha \leq 1, \quad 0 < v < 1$. Then the process $\{X_n\}$ is stationary if and only if $\varepsilon_1 \stackrel{d}{=} ML(\sigma, \alpha)$.

Based on Theorem 2.4.1, we define the first order autoregressive Mittag-Leffler process as follows:

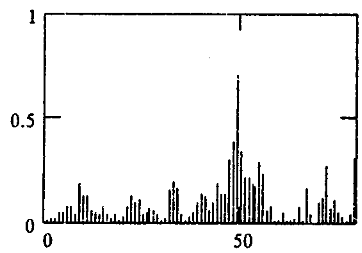
Let $\quad X_0 = \varepsilon_1$ and

for n= 1,2, ... $\quad X_n = V_n(X_{n-1} + \varepsilon_n)$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables with $f_{V_n}(v) = \alpha v^{\alpha-1}, \quad 0 < \alpha \leq 1, \quad 0 < v < 1,$ and $\varepsilon_n \stackrel{d}{=} ML(\sigma, \alpha)$.

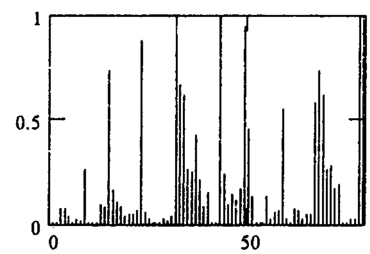
2.4.1. Properties of the first order autoregressive Mittag-Leffler process

The simulated sample paths of the first order autoregressive Mittag-Leffler model (2.4.1) are given in Figure 2.4.1.



$\alpha = 0.9 \qquad \sigma = 0.1$

Figure 2.4.1a



$\alpha = 0.7 \qquad \sigma = 0.1$

Figure 2.4.1b

Figure 2.4.1.

Sample path behavior of first order autoregressive Mittag-Leffler process

The joint distribution of (X_n, X_{n+1}) for first order autoregressive Mittag-Leffler process is obtained with the use of Laplace transform as

$$\phi_{X_n, X_{n+1}}(\lambda_1, \lambda_2) \quad = \quad E\left(e^{-\lambda_1 X_n - \lambda_2 X_{n+1}}\right).$$

That is,

$$\phi_{X_n, X_{n+1}}(\lambda_1, \lambda_2) \quad = \quad \int_0^1 \phi_{X_n}(\lambda_1 + v\lambda_2)\phi_{\varepsilon_{n+1}}(v\lambda_2)\alpha\, v^{\alpha-1}dv$$

$$= \quad \int_0^1 \frac{1}{1+\sigma(\lambda_1 + v\lambda_2)^\alpha} \frac{1}{1+\sigma(v\lambda_2)^\alpha} \alpha v^{\alpha-1} dv.$$

Thus like the first order autoregressive α-Laplace process of Jayakumar et al.(1995), the first order autoregressive Mittag-Leffler process is not time reversible.

The graph of the empirical joint density of (X_n, X_{n+1}) is given in Figure 2.4.2. below using simulated series of 10000 observations from the first order autoregressive Mittag-Leffler process.

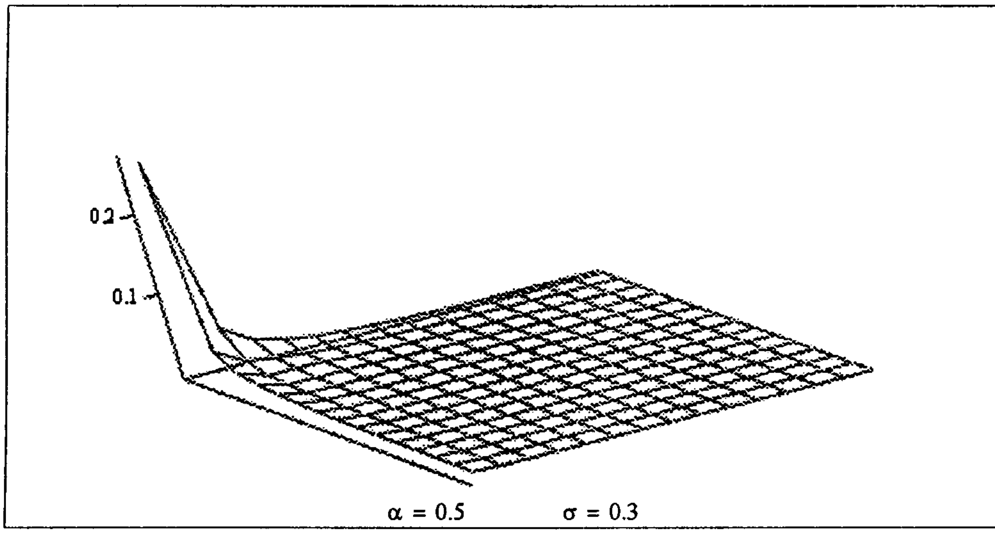


Figure 2.4.2.

Empirical joint density of (X_n, X_{n+1}) of first order autoregressive Mittag-Leffler process

2.4.2. First order autoregressive moving average Mittag-Leffler process

The first order autoregressive moving average Mittag-Leffler process is built using a sequence $\{\varepsilon_n\}$ of independent and identically distributed Mittag-Leffler random variables in the following manner.

$$X_n = W_n(\varepsilon_n + \varepsilon_{n-1}) \qquad (2.4.2)$$

where $\{W_n\}$ is a sequence of independent and identically distributed random variables independent of $\{\varepsilon_n\}$ with distribution function $F_{W_n}(w) = w^{\alpha}$ $0 < \alpha \leq 1$, $0 < w \leq 1$.

Combining (2.4.1) and (2.2.2) we define the first order autoregressive moving average process $\{X_n\}$ as follows.

$$\begin{aligned} X_n &= W_n(\varepsilon_n + Y_{n-1}) \\ Y_n &= V_n(Y_{n-1} + Z_n) \end{aligned}$$

where $\{Z_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables and $\{W_n\}, \{V_n\}$ are two independent sequences of independent and identically distributed random variables such that both have the power function distribution with distribution functions $F_{W_n}(w) = w^{\alpha}$, $F_{V_n}(v) = v^{\alpha}$, $0 < w, v < 1$, $0 < \alpha \leq 1$.

Theorem 2.4.2.

Let the process $\{X_n\}$ be defined by

$$\begin{aligned} X_n &= W_n(\varepsilon_n + Y_{n-1}) \\ Y_n &= V_n(Y_{n-1} + Z_n) \end{aligned}$$

where $\{Z_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables and, $\{W_n\}$ and $\{V_n\}$ are also two independent sequences of independent and identically distributed random variables with distribution function $F_{W_n}(w) = w^{\alpha}$, $F_{V_n}(v) = v^{\alpha}$ $0 < v, w < 1, 0 < \alpha \leq 1$ with $\varepsilon_1 \overset{d}{=} Y_0 \overset{d}{=} Z_1$. Then the process $\{X_n\}$ is stationary if and only if $\varepsilon_1 \overset{d}{=} ML(\sigma, \alpha)$.

Proof:

Proof follows easily using the arguments similar to those in Theorem 2.3.1. □

2.4.3. First order autoregressive multivariate Mittag-Leffler process

We consider the multivariate Mittag-Leffler distribution with Laplace transform

$$\phi(\lambda_1,\lambda_2,\ldots\lambda_k) = \frac{1}{1+\left(\frac{\lambda_1}{\sigma_1}\right)^{\alpha}+\left(\frac{\lambda_2}{\sigma_2}\right)^{\alpha}+\ldots+\left(\frac{\lambda_k}{\sigma_k}\right)^{\alpha}},$$

$$(\lambda_1,\lambda_2,\ldots\lambda_k)\in R_k^+,\ 0<\alpha\le 1, \sigma_i>0, i=1,2,3,\ldots,k.$$

Theorem 2.4.3.

Let the process $\{\underline{X}_n\}$ be defined by

$$\underline{X}_n = V_n(\underline{X}_{n-1}+\underline{\varepsilon}_n),\ n\ge 1$$

where $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors defined on R_k^+, $\{V_n\}$ is a sequence of independent and identically distributed random variables defined on (0, 1) such that V_n has power function distribution, $F_{V_n}(v)=v^{\alpha}, 0<v<1, 0<\alpha\le 1$, with $\underline{X}_0 \stackrel{d}{=} \underline{\varepsilon}_1$. Then the process $\{\underline{X}_n\}$ is stationary if and only if $\underline{\varepsilon}_1$ is multivariate Mittag-Leffler.

Proof:

Proof follows using the arguments similar to those in Theorem 2.3.3. □

Remark 2.4.1.

The processes defined in this Chapter can be easily extended to higher order cases. □

STUDY ON AUTOREGRESSIVE MODELS

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By

THOMAS MATHEW

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

KERALA, INDIA

Pin. 673 635

MAY, 2004

CHAPTER III

BURR AND PARETO PROCESSES

3.1. Introduction

The methods used for finding solution to autoregressive models are by using the generating functions such as Laplace transform and characteristic functions. If these generating functions do not have a closed form expression then this method of finding solutions fails. But if the survival function of the distribution has a closed form expression such as Pareto, logistic, Weibull etc. then the autoregressive nature of the sequence $\{X_n\}$ can be studied by using equation (1.2.1), that is,

$$X_n = k \min(X_{n-1}, \varepsilon_n), \qquad n \geq 1.$$

Properties of this model are discussed in Alpium (1989), Arnold and Hallett (1989) and Lewis and McKenzie (1991).

Yeh, et al. (1988) introduced first order autoregressive Pareto model $\{X_n, n = 0,1,2,\ldots\}$ defined by

$$X_0 = \varepsilon_1 \qquad \text{and} \quad \text{for } n = 1,2,\ldots$$

$$X_n = \begin{cases} \beta^{-\gamma} X_{n-1} & \text{w.p. } \beta \\ \min(\beta^{-\gamma} X_{n-1}, \varepsilon_n) & \text{w.p. } 1-\beta \end{cases} \tag{3.1.1}$$

where 'w.p.' stands for 'with probability', $0 < \beta < 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed Pareto random variables having survival

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This Chapter is based on Jayakumar and Thomas Mathew (2002b) function

$$\bar{F}_{\varepsilon_n}(x) = \frac{1}{1+\left(\frac{x}{\sigma}\right)^{1/\gamma}}, \quad \sigma, \gamma > 0, x > 0 . \tag{3.1.2}$$

The model (3.1.1) can be written as

$$X_n = \min(\beta^{-\gamma} X_{n-1}, U_n \varepsilon_n)$$

where $\{U_n\}$, is a sequence of independent and identically distributed Bernoulli random variables with $P(U_n = 0) = \beta = 1 - P(U_n = 1)$. Pillai (1991) extended the Pareto process of Yeh et al.(1988) and studied first order autoregressive semi Pareto process. Balakrishna (1998) discussed the estimation of the first order autoregressive semi Pareto and Pareto processes. Balakrishna and Jayakumar (1997) introduced bivariate semi-Pareto distribution and studied the properties of autoregressive minification models for bivariate random vectors with bivariate semi Pareto and bivariate Pareto distributions. Alpium and Athayde (1990) characterized the class of stationary distributions arising out of the model

$$X_n = Z_n \max(X_{n-1}, Y_n), \quad n \geq 1, \tag{3.1.3}$$

X_0 being any random variable and $\{Y_n\}$, $\{Z_n\}$ being any sequences of independent random variables also independent of each other and of X_0 and identically distributed with Y and Z, respectively. This sequence is a generalized version of

$$X_n = k \max(X_{n-1}, Y_n), n \geq 1,$$

where k is a real constant, $0 < k < 1$, and $\{Y_n\}$ any independent and identically distributed random variables studied in Alpium (1989).

In this Chapter, we introduce and study two random coefficient autoregressive minification processes, one is with Burr type XII marginal distribution and the other with Pareto type III marginal distribution.

We say that a random variable X on $(0, \infty)$ has Pareto distribution and write $X \underset{=}{d} P(\sigma, \alpha)$ if it has the survival function

$$P(X > x) = \bar{F}_X(x) = \frac{1}{1+\left(\frac{x}{\sigma}\right)^{\alpha}}, \quad \alpha > 0, \sigma > 0. \tag{3.1.4}$$

Even though the Burr type XII distribution (1.9.1) can be used to fit almost any given set of unimodal data, not much work has been done on time series models with Burr type XII marginal distributions. Jayakumar and Thomas Mathew (2002b) introduced a random coefficient autoregressive process with Burr type XII marginal distribution and studied its properties. In Section 2, we introduce and study first order autoregressive Burr process. A multivariate generalization is also considered. First order autoregressive moving average mixed Burr process is developed. In Section 3, first order autoregressive Pareto process is introduced and studied. A multivariate generalization is done. First order autoregressive moving average Pareto process is also developed.

3.2. First order autoregressive Burr process

Theorem 3.2.1.

Let the process $\{X_n\}$ be defined as

$$X_n = \min(V_n^{-1} X_{n-1}, \varepsilon_n), \quad n = 1,2, \ldots \tag{3.2.1}$$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables such that $\{V_n\}$ has distribution function $F_{V_n}(v) = v^{\alpha\gamma}, 0 < v < 1, \ \alpha, \gamma > 0$. Suppose the process $\{X_n\}$ is stationary. Then $X_n \underset{=}{d} \text{Burr}(\sigma, \alpha, \gamma)$ if and only if $\varepsilon_n \underset{=}{d} P(\sigma, \alpha)$.

Proof :

Denoting the survival function of X_n and ε_n by $\bar{F}_{X_n}(x)$ and $\bar{F}_{\varepsilon_n}(x)$ respectively, (3.2.1) in terms of survival functions is

$$\bar{F}_{X_n}(x) = \bar{F}_{\varepsilon_n}(x)\int_0^1 \bar{F}_{X_{n-1}}(xv) f_{V_n}(v)dv,$$

where $f_{V_n}(v) = \alpha\gamma\, v^{\alpha\gamma-1}, \quad 0 < v < 1, \quad \alpha, \gamma > 0$ is the probability density function of V_n.

That is ,

$$\bar{F}_{X_n}(x) = \bar{F}_{\varepsilon_n}(x)\int_0^1 \bar{F}_{X_{n-1}}(xv)\alpha\gamma\, v^{\alpha\gamma-1}dv.$$

Assume that the process $\{X_n\}$ is stationary. Then

$$\bar{F}_X(x) = \bar{F}_{\varepsilon_n}(x)\int_0^1 \bar{F}_X(xv)\alpha\gamma\, v^{\alpha\gamma-1}dv.$$

By taking xv = t, we get

$$x^{\alpha\gamma}\bar{F}_X(x) = \alpha\gamma\bar{F}_{\varepsilon_n}(x)\int_0^x \bar{F}_X(t)\,t^{\alpha\gamma-1}dt.$$

Differentiating with respect to x on both sides and simplifying, we get

$$\frac{\bar{F}'_X(x)}{\bar{F}_X(x)} = \frac{\bar{F}'_{\varepsilon_n}(x)}{\bar{F}_{\varepsilon_n}(x)} - \frac{\alpha\gamma}{x}\left(1-\bar{F}_{\varepsilon_n}(x)\right).$$

Thus,

$$\bar{F}_X(x) = \bar{F}_{\varepsilon_n}(x)\exp\left(-\alpha\gamma\int_0^x \frac{1-\bar{F}_{\varepsilon_n}(u)}{u}du\right).$$

Since ε_n is Pareto,

$$\bar{F}_X(x) = \frac{1}{\left(1+\left(\frac{x}{\sigma}\right)^{\alpha}\right)^{\gamma+1}}.$$

Conversely, if $\{X_n\}$ is stationary with Burr (σ,α,γ) distribution as the marginal, then $\{\varepsilon_n\}$ is P (σ, α).

$$\bar{F}_{X_n}(x) = \bar{F}_{\varepsilon_n}(x)\int_0^1 \bar{F}_{X_{n-1}}(xv)f_{V_n}(v)dv.$$

$$\frac{1}{\bar{F}_{\varepsilon_n}(x)} = \int_0^1 \frac{\bar{F}_X(xv)\alpha\gamma\, v^{\alpha\gamma-1}}{\bar{F}_X(x)}dv.$$

$$= \int_0^1 \left(\frac{1+\left(\frac{x}{\sigma}\right)^{\alpha}}{1+v^{\alpha}\left(\frac{x}{\sigma}\right)^{\alpha}} \right)^{\gamma+1} \alpha\,\gamma\, v^{\alpha\gamma-1} dv.$$

If $v^{\alpha} = w$, then

$$\frac{1}{\overline{\overline{F}}_{\varepsilon_n}(x)} = \int_0^1 \left(\frac{1+\left(\frac{x}{\sigma}\right)^{\alpha}}{1+w\left(\frac{x}{\sigma}\right)^{\alpha}} \right)^{\gamma+1} \gamma\, w^{\gamma-1} dw.$$

Taking $w\dfrac{x^{\alpha}}{\sigma^{\alpha}} = z$, we get

$$\frac{1}{\overline{\overline{F}}_{\varepsilon_n}(x)} = \frac{(1+\left(\frac{x}{\sigma}\right)^{\alpha})^{\gamma+1}}{\left(\frac{x}{\sigma}\right)^{\alpha\gamma}} \int_0^{\left(\frac{x}{\sigma}\right)^{\alpha}} \frac{\gamma z^{\gamma-1}}{(1+z)^{\gamma+1}} dz.$$

Put $t = \dfrac{z}{1+z}$. Thus,

$$\frac{1}{\overline{\overline{F}}_{\varepsilon_n}(x)} = \frac{(1+\left(\frac{x}{\sigma}\right)^{\alpha})^{\gamma+1}}{\left(\frac{x}{\sigma}\right)^{\alpha\gamma}} \int_0^{\frac{\left(\frac{x}{\sigma}\right)^{\alpha}}{1+\left(\frac{x}{\sigma}\right)^{\alpha}}} \gamma\, t^{\gamma-1} dt.$$

$$= 1+\left(\frac{x}{\sigma}\right)^{\alpha}.$$

Therefore,

$$\overline{F}_{\varepsilon_n}(x) = \frac{1}{1+\left(\frac{x}{\sigma}\right)^{\alpha}}.$$

Hence $\varepsilon_n \underset{=}{d} P(\sigma, \alpha)$. This completes the proof. $\square$

Theorem 3.2.2.

Let the process $\{X_n\}$ be defined as

$$X_n = \min(V_n^{-1} X_{n-1}, \varepsilon_n), \quad n = 1, 2, \ldots$$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables such that $\{V_n\}$ has distribution function $F_{V_n}(v) = v^{\alpha\gamma}, 0 < v < 1, \ \alpha, \gamma > 0$. If $X_0 \underset{=}{d} \text{Burr}(\sigma, \alpha, \gamma)$ and $\varepsilon_n \underset{=}{d} P(\sigma, \alpha)$, then the process $\{X_n\}$ is stationary with $\text{Burr}(\sigma, \alpha, \gamma)$ marginals.

Proof:

We have

$$\bar{F}_{X_n}(x) = \bar{F}_{\varepsilon_n}(x) \int_0^1 \bar{F}_{X_{n-1}}(xv) \alpha\gamma \, v^{\alpha\gamma - 1} dv$$

Putting n = 1, we get

$$\bar{F}_{X_1}(x) = \bar{F}_{\varepsilon_1}(x) \int_0^1 \bar{F}_{X_0}(xv) \alpha\gamma \, v^{\alpha\gamma - 1} dv.$$

$$\bar{F}_{X_1}(x) = \frac{1}{1 + \left(\frac{x}{\sigma}\right)^{\alpha}} \int_0^1 \left(\frac{1}{1 + \left(\frac{xv}{\sigma}\right)^{\alpha}} \right)^{\gamma + 1} \alpha\gamma \, v^{\alpha\gamma - 1} dv$$

$$\bar{F}_{X_1}(x) = \frac{1}{1+\left(\frac{x}{\sigma}\right)^{\alpha}} \int_0^1 \left(\frac{1}{1+v^{\alpha}\left(\frac{x}{\sigma}\right)^{\alpha}} \right)^{\gamma+1} \alpha\gamma\, v^{\alpha\gamma-1} dv.$$

If $v^{\alpha} = w$, then

$$\bar{F}_{X_1}(x) = \frac{1}{1+\left(\frac{x}{\sigma}\right)^{\alpha}} \int_0^1 \left(\frac{1}{1+w\left(\frac{x}{\sigma}\right)^{\alpha}} \right)^{\gamma+1} \gamma\, w^{\gamma-1} dw.$$

Taking $w\dfrac{x^{\alpha}}{\sigma^{\alpha}} = z$, we get

$$\bar{F}_{X_1}(x) = \frac{\gamma}{1+\left(\frac{x}{\sigma}\right)^{\alpha}} \int_0^{\left(\frac{x}{\sigma}\right)^{\alpha}} \left(\frac{z}{1+z}\right)^{\gamma-1} \left(\frac{1}{1+z}\right)^{2} \frac{dz}{\left(\frac{x}{\sigma}\right)^{\alpha\gamma}}.$$

Put $t = \dfrac{z}{1+z}$. Thus,

$$\bar{F}_{X_1}(x) = \frac{1}{\left(1+\left(\frac{x}{\sigma}\right)^{\alpha}\right)\left(\frac{x}{\sigma}\right)^{\alpha\gamma}} \int_0^{\frac{\left(\frac{x}{\sigma}\right)^{\alpha}}{1+\left(\frac{x}{\sigma}\right)^{\alpha}}} \gamma\, t^{\gamma-1}\, dt$$

$$= \frac{1}{\left(1+\left(\frac{x}{\sigma}\right)^{\alpha}\right)^{\gamma+1}}.$$

That is, $X_1 \underset{=}{d} \mathrm{Burr}(\sigma, \alpha, \gamma)$.

If $X_{n-1} \stackrel{d}{=} \text{Burr}(\sigma,\alpha,\gamma)$, then we get $X_n = \text{Burr}(\sigma,\alpha,\gamma)$. Thus the process $\{X_n\}$ is stationary with Burr marginals. □

Based on Theorem 3.2.2, we define the first order autoregressive Burr process $\{X_n\}$ as follows:

$$\begin{aligned} X_0 &= \text{Burr}(\sigma,\alpha,\gamma) \quad \text{and} \\ \text{for } n = 1,2,\ldots \quad X_n &= \min(V_n^{-1}X_{n-1}, \varepsilon_n) \end{aligned} \tag{3.2.2}$$

where $\{V_n\}$ is a sequence of independent and identically distributed power function random variables with distribution function $F_{V_n}(v) = v^{\alpha\gamma}$, $0 < v < 1$, $\alpha, \gamma > 0$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $P(\sigma,\alpha)$ random variables independent of $\{V_n\}$.

Now we study some properties of the stationary Burr process 3.2.2.

3.2.1. Properties of the first order autoregressive Burr process

The joint survival function of (X_n, X_{n+1}) is of the form

$$\begin{aligned} \bar{F}_{X_n,X_{n+1}}(x,y) &= P(X_n > x, X_{n+1} > y). \\ &= P\left(X_n > x, \min(V_{n+1}^{-1}X_n, \varepsilon_{n+1}) > y\right) \\ &= P\left(X_n > x, V_{n+1}^{-1}X_n > y, \varepsilon_{n+1} > y\right) \\ &= P(\varepsilon_{n+1} > y)\int_0^1 P(X_n > x, X_n > vy)\alpha\gamma v^{\alpha\gamma-1}dv \\ &= P(\varepsilon_{n+1} > y)\int_0^1 P(X_n > \max(x, vy))\alpha\gamma v^{\alpha\gamma-1}dv. \end{aligned}$$

$$\bar{F}_{X_n,X_{n+1}}(x,y) = \frac{1}{1+\left(\frac{y}{\sigma}\right)^{\alpha}} \int_0^1 \frac{1}{\left(1+\max\left(\left(\frac{x}{\sigma}\right)^{\alpha}, v^{\alpha}\left(\frac{y}{\sigma}\right)^{\alpha}\right)\right)^{\gamma+1}} \alpha\gamma v^{\alpha\gamma-1} dv.$$

The joint survival function of (X_n, X_{n+1}) of the first order autoregressive Burr process for $\alpha = 5$, $\sigma = 1$ and $\gamma = .5$ is given in Figure 3.2.1 below.

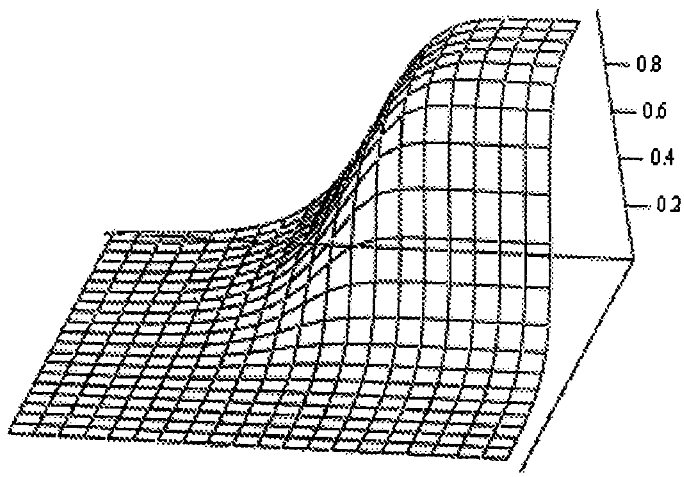


Figure 3.2.1.

Joint survival function of (X_n, X_{n+1}) of the first order autoregressive Burr process.

Now consider

$$P(X_{n+1} > X_n) = P\left(\min(V_{n+1}^{-1}X_n, \varepsilon_{n+1}) > X_n\right)$$

$$= \int_0^1 P(X_n > vX_n)\alpha\gamma v^{\alpha\gamma-1} P(\varepsilon_{n+1} > X_n)dv$$

$$= \int_x P(\varepsilon_{n+1} > x / X_n = x)f(x)dx$$

$$= \int_0^\infty P(\varepsilon_{n+1} > x) f(x) dx$$

$$= \int_0^\infty \frac{1}{1+\left(\frac{x}{\sigma}\right)^\alpha} \frac{(\gamma+1)\alpha\left(\frac{x}{\sigma}\right)^{\alpha-1}\frac{1}{\sigma}}{\left(1+\left(\frac{x}{\sigma}\right)^\alpha\right)^{\gamma+2}} dx$$

$$= \int_0^\infty \frac{(\gamma+1)\alpha\left(\frac{x}{\sigma}\right)^{\alpha-1}\frac{1}{\sigma}}{\left(1+\left(\frac{x}{\sigma}\right)^\alpha\right)^{\gamma+3}} \frac{(\gamma+2)}{(\gamma+2)} dx.$$

$$= \frac{\gamma+1}{\gamma+2}.$$

Now,

$$E(X_{n+1}X_n) = E\left(\min(V_{n+1}^{-1}X_n, \varepsilon_{n+1})X_n\right)$$

$$= \int_v E\left(\min(v^{-1}X_n, \varepsilon_{n+1})X_n\right) dG(v)$$

where G(.) is the distribution function of V_n.

That is,

$$E(X_{n+1}X_n) = \int_v E\left(\min\left(v^{-1}X_n, \varepsilon_{n+1}\right)X_n\right) dG(v)$$

$$= \int_v \int_x \left(\min\left(v^{-1}x, \varepsilon_{n+1}\right) x\right) dF(x) dG(v)$$

where F(x) is the distribution function of $\text{Burr}(\sigma, \alpha, \gamma)$.

$$= \int_v \int_x \int_y \left(\min\left(v^{-1}x, y\right) x\right) dH(y) dF(x) dG(v)$$

where H(y) is the distribution function of $P(\sigma, \alpha)$.

Autocorrelation coefficients of various orders for various values of γ (for $\alpha = 2.5$ and $\sigma = 1$) are computed using simulation of equation (3.2.2) (100 runs of length 5000) and are presented in Table 3.2.1. From the table it can be observed that as the value of γ increases autocorrelation increases.

j\γ	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	.44	.67	.76	.82	.85	.88	.89	.91	.92	.92	.92	.93	.94	.94
2	.25	.46	.58	.66	.72	.77	.80	.82	.84	.85	.86	.87	.88	.89
3	.11	.29	.44	.54	.61	.67	.71	.75	.77	.79	.80	.82	.83	.84
4	.04	.18	.32	.43	.52	.59	.64	.67	.70	.73	.75	.76	.78	.79
5	.02	.12	.25	.36	.45	.52	.57	.61	.65	.67	.70	.72	.78	.75
6	.01	.08	.19	.29	.38	.45	.51	.55	.59	.62	.65	.67	.69	.71
7	.01	.05	.15	.24	.33	.40	.45	.50	.54	.57	.60	.62	.65	.67
8	.01	.04	.11	.20	.28	.35	.40	.45	.49	.52	.55	.58	.60	.63
9	0	.03	.08	.16	.23	.30	.36	.41	.45	.48	.51	.54	.56	.59
10	0	.02	.06	.13	.20	.26	.32	.36	.41	.44	.47	.5	.53	.55
11	0	.01	.04	.10	.16	.22	.28	.32	.36	.40	.43	.46	.49	.52
12	0	0	.03	.07	.13	.19	.24	.29	.33	.36	.40	.43	.46	.48
13	0	0	.02	.05	.10	.16	.21	.25	.29	.33	.36	.39	.42	.45
14	0	0	0	.03	.08	.13	.17	.21	.26	.29	.33	.36	.39	.42
15	0	0	0	.02	.06	.10	.14	.18	.22	.26	.3	.33	.36	.39

Table 3.2.1.

Autocorrelation of various orders 'j' for various values of γ ($\alpha = 2.5$ and $\sigma = 1$).

The sample path behavior of the first order autoregressive Burr process is presented in Figures 3.2.1.

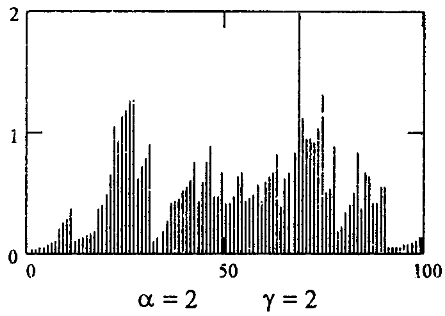


Figure 3.2.1a.

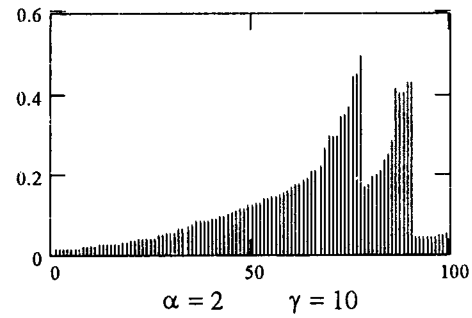


Figure 3.2.1b.

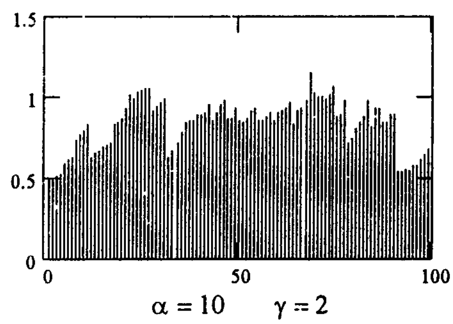


Figure 3.2.1c.

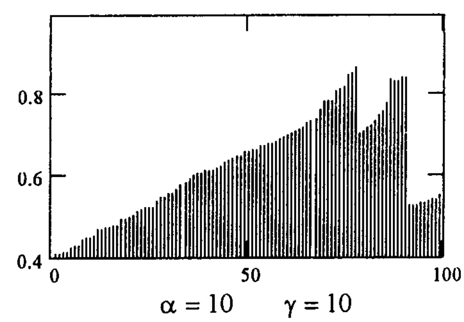


Figure 3.2.1d.

Figure 3.2.1.

Sample path of the Burr process

3.2.2. First order autoregressive multivariate Burr process

Consider the multivariate Burr distribution with survival function

$$P(X_1 > x_1, X_2 > x_2, \ldots X_k > x_k) = \frac{1}{\left(1+\left(\frac{x_1}{\sigma_1}\right)^{\alpha}+\left(\frac{x_2}{\sigma_2}\right)^{\alpha}+\ldots+\left(\frac{x_k}{\sigma_k}\right)^{\alpha}\right)^{\gamma+1}}, \tag{3.2.3}$$

$$(x_1, x_2, \ldots x_k) \in R_k^+, \quad \sigma_i, \alpha, \gamma > 0, i = 1,2,\ldots k.$$

Let the process $\{\underline{X}_n\}$ defined by

$$\underline{X}_n = \min(V_n^{-1}\underline{X}_{n-1}, \underline{\varepsilon}_n), n > 1, \tag{3.2.4}$$

where the minimum is component wise, $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors defined on R_k^+, $\{V_n\}$ is a sequence of independent and identically distributed random variables defined on (0, 1) such that $\{V_n\}$ has power function distribution $F_{V_n}(v) \; = \; v^{\alpha\gamma}, 0 < v < 1, \quad \alpha, \gamma > 0$. Suppose $\{\underline{\varepsilon}_n\}$ has multivariate Pareto distribution with survival function

$$\bar{F}_{\underline{\varepsilon}_n}(\underline{x}) \;=\; \frac{1}{1+\left(\frac{x_1}{\sigma_1}\right)^{\alpha}+\left(\frac{x_2}{\sigma_2}\right)^{\alpha}+\ldots+\left(\frac{x_k}{\sigma_k}\right)^{\alpha}}.$$

Then, equation (3.2.4) in terms of survival functions is

$$\begin{aligned}\bar{F}_{\underline{X}_n}(\underline{x}) &= P\left(\min(V_n^{-1}\underline{X}_{n-1}, \underline{\varepsilon}_n) > \underline{x}\right),\ \underline{x} \in (x_1, x_2, \ldots x_k)' \in R_k^+. \\ &= \bar{F}_{\underline{\varepsilon}_n}(\underline{x}) \int_0^1 \bar{F}_{\underline{X}_{n-1}}(\underline{x}v) f(v) dv.\end{aligned}$$

If the process is stationary, then

$$\bar{F}_{\underline{X}}(\underline{x}) \;=\; \bar{F}_{\underline{\varepsilon}_n}(\underline{x}) \int_0^1 \bar{F}_{\underline{X}}(\underline{x}v) f(v) dv.$$

That is,

$$\bar{F}_{\underline{X}}(x_1, x_2, \ldots, x_k) \;=\; \bar{F}_{\underline{\varepsilon}_n}(x_1, x_2, \ldots, x_k) \int_0^1 \bar{F}_{\underline{X}}(x_1 v, x_2 v, \ldots, x_k v) \alpha\gamma\, v^{\alpha\gamma-1} dv.$$

Taking $\frac{x_j}{\sigma_j} = \delta_j^{1/\alpha}\, x$, for j = 1, 2, ...k, we get

$$\bar{F}_{\underline{X}}(\underline{\delta}x) = \bar{F}_{\underline{\varepsilon}_n}(\underline{\delta}x)\int_0^1 \bar{F}_{\underline{X}}(\underline{\delta}xv)\alpha\gamma\, v^{\alpha\gamma-1}dv,$$

$$\text{where} \qquad \underline{\delta} = (\delta_1^{1/\alpha}, \delta_2^{1/\alpha}, \ldots, \delta_k^{1/\alpha})'.$$

If xv = t, then

$$x^{\alpha\gamma}\, \bar{F}_{\underline{X}}(\underline{\delta}x) = \alpha\gamma\, \bar{F}_{\underline{\varepsilon}_n}(\underline{\delta}x)\int_0^x \bar{F}_{\underline{X}}(\underline{\delta}\, t)\, t^{\alpha\gamma-1}dt.$$

Differentiating with respect to x gives

$$\frac{\underline{\delta}\,\bar{F}'_{\underline{X}}(\underline{\delta}x)}{\bar{F}_{\underline{X}}(\underline{\delta}x)} = \frac{\underline{\delta}\,\bar{F}'_{\underline{\varepsilon}_n}(\underline{\delta}x)}{\bar{F}_{\underline{\varepsilon}_n}(\underline{\delta}x)} - \alpha\gamma\left(\frac{1-\bar{F}_{\underline{\varepsilon}_n}(\underline{\delta}x)}{x}\right).$$

That is,

$$\bar{F}_{\underline{X}}(\underline{\delta}x) = \bar{F}_{\underline{\varepsilon}_n}(\underline{\delta}x)\exp\left(-\alpha\gamma\int_0^x\left(\frac{1-F_{\underline{\varepsilon}_n}(\underline{\delta}u)}{u}\right)du\right).$$

Since $\underline{\varepsilon}_n \overset{d}{=}$ multivariate P($\underline{\sigma}$, α), we get

$$\bar{F}_{\underline{X}}(\underline{\delta}\,x) = \frac{1}{\left(1+\underline{\delta}x^{\alpha}\right)^{\gamma+1}}.$$

$$= \frac{1}{\left(1+\left(\frac{x_1}{\sigma_1}\right)^{\alpha}+\left(\frac{x_2}{\sigma_2}\right)^{\alpha}+\ldots+\left(\frac{x_k}{\sigma_k}\right)^{\alpha}\right)^{\gamma+1}}.$$

Suppose $\{\underline{X}_n\}$ in (3.2.4) is stationary multivariate Burr.

Then,

$$\bar{F}_{\underline{X}}(\underline{x}) = \bar{F}_{\underline{\varepsilon}_n}(\underline{x}) \int_0^1 \bar{F}_{\underline{X}}(\underline{x}v) f(v) dv .$$

$$\frac{1}{\bar{F}_{\underline{\varepsilon}_n}(\underline{x})} = \int_0^1 \frac{\bar{F}_{\underline{X}}(\underline{x}v) \alpha \gamma v^{\alpha\gamma-1}}{\bar{F}_{\underline{X}}(\underline{x})} dv .$$

$$= \int_0^1 \left(\frac{1+\underline{\delta x}^{\alpha}}{1+v^{\alpha}\underline{\delta x}^{\alpha}} \right)^{\gamma+1} \alpha \gamma v^{\alpha\gamma-1} dv .$$

$$= 1+\underline{\delta} x^{\alpha} .$$

That is,

$$\bar{F}_{\underline{\varepsilon}_n}(\underline{\delta x}) = \frac{1}{1+\underline{\delta} x^{\alpha}} .$$

Therefore

$$\bar{F}_{\underline{\varepsilon}_n}(\underline{x}) = \frac{1}{1+\left(\frac{x_1}{\sigma_1}\right)^{\alpha}+\left(\frac{x_2}{\sigma_2}\right)^{\alpha}+\ldots+\left(\frac{x_k}{\sigma_k}\right)^{\alpha}} .$$

Thus, we have proved the following theorem

Theorem 3.2.3.

Let the process $\{\underline{X}_n\}$ be defined as

$$\underline{X}_n = \min(V_n^{-1} \underline{X}_{n-1}, \underline{\varepsilon}_n), \quad n = 1,2, \ldots$$

where $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors defined on R_k^+ and $\{V_n\}$ is a sequence of independent and identically distributed univariate random variables having distribution function $F_{V_n}(v) = v^{\alpha\gamma}$, $0 < v < 1,\ \alpha, \gamma > 0$. Suppose the process $\{\underline{X}_n\}$ is stationary. Then $\underline{X}_n \stackrel{d}{=} \text{Burr}(\underline{\sigma}, \alpha, \gamma)$ if and only if $\underline{\varepsilon}_n \stackrel{d}{=} P(\underline{\sigma}, \alpha)$. □

Based on this result, we define the first order autoregressive multivariate Burr process as follows:

$$\begin{aligned} \underline{X}_0 &= \text{Burr}(\underline{\sigma}, \alpha, \gamma) \qquad \text{and} \\ \text{for } n = 1,2,\ldots\ \underline{X}_n &= \min(V_n^{-1}\underline{X}_{n-1}, \underline{\varepsilon}_n) \end{aligned}$$

where $\{V_n\}$ is a sequence of independent and identically distributed power function random variables with distribution function $F_{V_n}(v) = v^{\alpha\gamma}$, $0 < v < 1,\ \alpha, \gamma > 0$ and $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed $P(\underline{\sigma}, \alpha)$ random variables.

3.2.3. First order autoregressive moving average mixed Burr model

Consider the process $\{X_n\}$, n=1,2,3, ... defined by

$$\begin{aligned} X_n &= \min(K_n\varepsilon_n, V_n^{-1}Y_{n-1}) \\ Y_n &= \min(U_n^{-1}Y_{n-1}, \varepsilon_n) \end{aligned} \qquad (3.2.5)$$

where $\{\varepsilon_n\}$ is a sequences of independent and identically distributed $P(\sigma, \alpha)$ random variables, $\{K_n\}$ is a sequence of independent and identically distributed Bernoulli random variables such that $P(K_n = 0) = \theta = 1 - P(K_n = 1)$, $\{V_n\}$ and $\{U_n\}$ are two independent sequences of independent and identically distributed random variables defined on the interval (0, 1) with distribution function $F_{V_n}(v) = v^{\alpha\gamma}, 0 < v < 1$, $F_{U_n}(u) = u^{\alpha\gamma}, 0 < u < 1$, $\alpha, \gamma > 0$ and $Y_n \overset{d}{=} \text{Burr}(\sigma, \alpha, \gamma)$.

We have from (3.2.5),

$$\bar{F}_{X_n}(x) = P(\min(K_n Z_n, V_n^{-1} Y_{n-1}) > x).$$

$$= [\theta + (1-\theta)P(Z_n > x)] \int_0^1 P(Y_{n-1} > xv)\alpha\gamma v^{\alpha\gamma-1} dv.$$

If $xv = t$, then

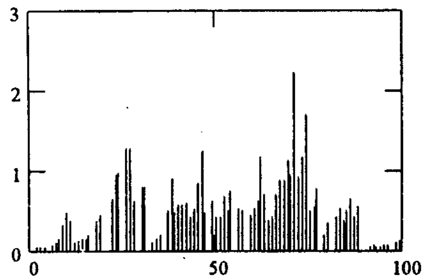
$$\bar{F}_{X_n}(x) = [\theta + (1-\theta)P(Z_n > x)] \int_0^x \frac{P(Y_{n-1} > t)\alpha\gamma t^{\alpha\gamma-1}}{x^{\alpha\gamma}} dt.$$

$$= \frac{1}{x^{\alpha\gamma}} \left[\theta + (1-\theta)\frac{1}{(1+\left(\frac{x}{\sigma}\right)^{\alpha})}\right] \int_0^x P(Y_{n-1} > t)\alpha\gamma t^{\alpha\gamma-1} dt.$$

$$= \theta \frac{1}{\left(1+\left(\frac{x}{\sigma}\right)^{\alpha}\right)^{\gamma}} + (1-\theta)\frac{1}{\left(1+\left(\frac{x}{\sigma}\right)^{\alpha}\right)^{\gamma+1}}.$$

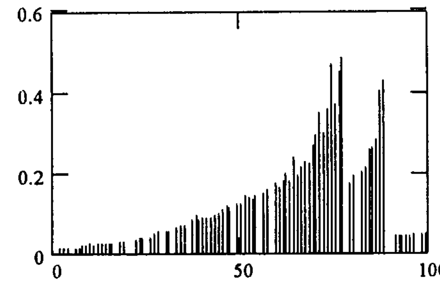
Thus $\{X_n\}$ has mixed Burr distribution.

The sample path behavior of mixed $\text{Burr}(\sigma, \alpha, \gamma)$ process is given in Figure 3.2.2.



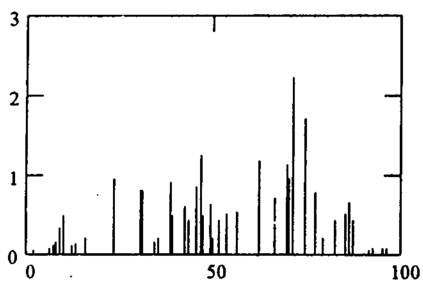
$\alpha = 2 \quad \gamma = 2 \quad \theta = 0.1$

Figure 3.2.2a.



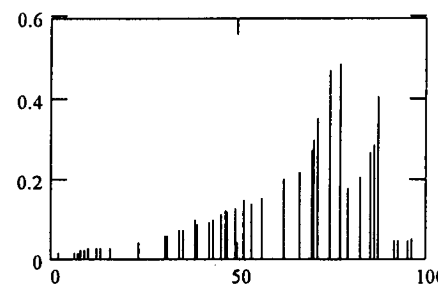
$\alpha = 2 \quad \gamma = 10 \quad \theta = 0.1$

Figure 3.2.2b.



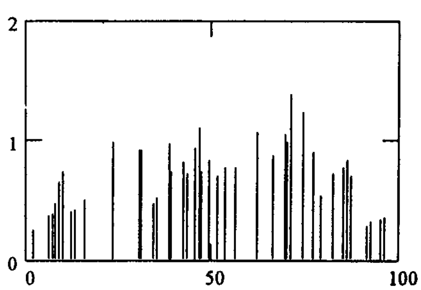
$\alpha = 2 \quad \gamma = 2 \quad \theta = 0.5$

Figure 3.2.2c.



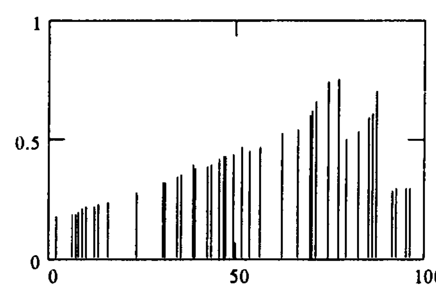
$\alpha = 2 \quad \gamma = 10 \quad \theta = 0.5$

Figure 3.2.2d.



$\alpha = 5 \quad \gamma = 2 \quad \theta = 0.5$

Figure 3.2.2e.



$\alpha = 5 \quad \gamma = 10 \quad \theta = 0.5$

Figure 3.2.2f.

Figure 3.2.2.

Sample path behavior of the first order autoregressive moving average mixed Burr process

3.3. First order autoregressive Pareto process

Here we develop a first order autoregressive Pareto process and study its properties.

Theorem 3.3.1.

Let the process $\{X_n\}$ be defined as

$$X_0 = \varepsilon_1 \qquad \text{and}$$

for n= 1,2, ... $X_n = V_n^{-1} \min(X_{n-1}, \varepsilon_n)$ (3.3.1)

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables such that V_n has the distribution function $F_{V_n}(v) = v^{\alpha}, \ \alpha > 0, \ 0 < v < 1$. Then the process $\{X_n\}$ is stationary if and only if $\varepsilon_1 \stackrel{d}{=} P(\sigma, \alpha)$.

Proof:

Denoting the survival function of X_n and ε_n by $\bar{F}_{X_n}(x)$ and $\bar{F}_{\varepsilon_n}(x)$ respectively, (3.3.1) in terms of survival function is

$$\bar{F}_{X_n}(x) = \int_0^1 \bar{F}_{X_{n-1}}(xv)\, \bar{F}_{\varepsilon_n}(xv)\, f_{V_n}(v)dv \tag{3.3.2}$$

where $f_{V_n}(v)$ is the probability density function of V_n.

Since $F_{V_n}(v) = v^{\alpha}, \alpha > 0, 0 < v < 1$, (3.3.2) becomes

$$\bar{F}_{X_n}(x) = \int_0^1 \bar{F}_{X_{n-1}}(xv)\,\bar{F}_{\varepsilon_n}(xv)\,\alpha\, v^{\alpha-1}dv$$

Assume that the process $\{X_n\}$ is stationary. Then

$$\bar{F}_X(x) = \int_0^1 \bar{F}_X(xv)\,\bar{F}_{\varepsilon_n}(xv)\,\alpha\, v^{\alpha-1}dv .$$

Since $X_0 \overset{d}{=} \varepsilon_1$, we have

$$\bar{F}_X(x) = \int_0^1 \bar{F}_X^2(xv)\,\alpha\, v^{\alpha-1}dv .$$

That is,

$$x^{\alpha}\bar{F}_X(x) = \int_0^x \bar{F}_X^2(t)\,\alpha\, t^{\alpha-1}dt .$$

Differentiating both sides with respect to x and rearranging the terms, we get

$$\frac{x}{\alpha}\frac{\bar{F}_X'(x)}{\bar{F}_X^2(x)} + \frac{1}{\bar{F}_X(x)} = 1 . \tag{3.3.3}$$

Writing $\bar{F}_X(x) = \dfrac{1}{1+h(x)}$, $h(x)$ is monotone increasing with $h(0) = 0$ and $h(\infty) = \infty$, we get

$$\frac{h'(x)}{h(x)} = \frac{\alpha}{x}$$

and hence $\quad h(x) = c\,x^{\alpha},\ c > 0.$

Therefore $\quad \bar{F}_X(x) = \dfrac{1}{1+c\,x^{\alpha}}$. Take $c = \dfrac{1}{\sigma^{\alpha}}$.

Hence $\varepsilon_1 \stackrel{d}{=} P(\sigma, \alpha)$.

Conversely, assume that $\varepsilon_1 \stackrel{d}{=} P(\sigma, \alpha)$. Using the method of induction, we can see that the process $\{X_n\}$ is stationary.

From (3.3.2), we have

$$\bar{F}_{X_n}(x) = \int_0^1 \bar{F}_{X_{n-1}}(xv)\,\bar{F}_{\varepsilon_n}(xv)\,f_{V_n}(v)dv \; .$$

Since $X_0 \stackrel{d}{=} \varepsilon_1$ and $\{V_n\}$ has the Power function distribution with probability density function $f_{V_n}(v) = \alpha\, v^{\alpha-1}, \alpha > 0, 0 < v < 1,$ for n = 1 we get

$$\bar{F}_{X_1}(x) = \int_0^1 \bar{F}_{X_0}(xv)\,\bar{F}_{\varepsilon_1}(xv)\,\alpha\, v^{\alpha-1}dv \, .$$

If $X_0 \stackrel{d}{=} \varepsilon_1$, then

$$\bar{F}X_1(x) = \int_0^1 \frac{1}{\left(1+v^{\alpha}\left(\frac{x}{\sigma}\right)^{\alpha}\right)^2}\alpha v^{\alpha-1}dv$$

$$= \frac{1}{1+\left(\frac{x}{\sigma}\right)^{\alpha}} \, .$$

Assuming $X_{n-1} \stackrel{d}{=} P(\sigma, \alpha)$, it can be shown that $X_n \stackrel{d}{=} P(\sigma, \alpha)$. Hence the process $\{X_n\}$ is stationary. This completes the proof. □

Based on Theorem 3.3.1, we define the first order autoregressive Pareto process as follows:

$$\text{for } n = 1,2,\ldots \quad \begin{aligned} X_0 &= \varepsilon_1 \quad \text{and} \\ X_n &= V_n^{-1}\min(X_{n-1}, \varepsilon_n) \end{aligned} \tag{3.3.4}$$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables with $F_{V_n}(v) = v^{\alpha}$, $0 < v < 1$, $\alpha > 0$ and $\varepsilon_n \underset{=}{d} P(\sigma, \alpha)$.

3.3.1. Properties of the first order autoregressive Pareto process

The sample path behavior of the first order autoregressive Pareto process with and $\alpha = 10$, $\sigma = 1$ presented in Figure 3.3.1.

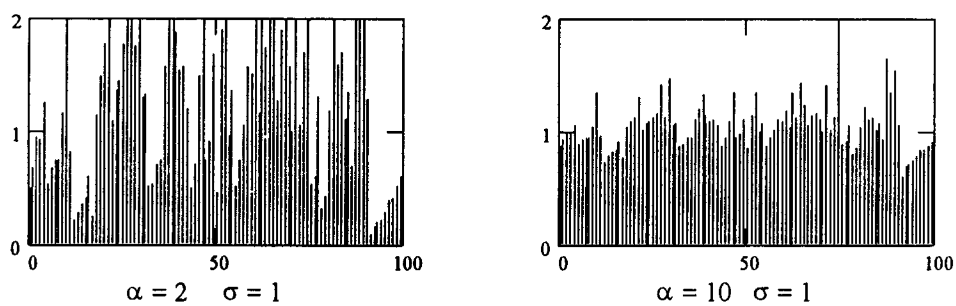


Figure 3.3.1.

Sample path behavior of first order autoregressive Pareto process

From (3.3.4), The joint survival function of (X_n, X_{n+1}) of the first order autoregressive Pareto process is

$$\begin{aligned} \bar{F}_{X_n X_{n+1}}(x,y) &= P(X_n > x, X_{n+1} > y). \\ &= P\left(X_n > x, \min V_{n+1}^{-1}(X_n, \varepsilon_{n+1}) > y\right) \\ &= P\left(X_n > x, V_{n+1}^{-1}X_n > y, V_n^{-1}\varepsilon_{n+1} > y\right) \end{aligned}$$

$$= \int_0^1 P(\varepsilon_{n+1} > vy)P(X_n > x, X_n > vy)\alpha v^{\alpha-1} dv$$

$$= \int_0^1 P(\varepsilon_{n+1} > vy)P(X_n > \max(x, vy)\alpha v^{\alpha-1} dv.$$

$$\bar{F}_{X_n X_{n+1}}(x, y) = \int_0^1 \frac{\alpha\, v^{\alpha-1} dv}{\left[1 + \max\left(\left(\frac{x}{\sigma}\right)^{\alpha}, \left(\frac{yv}{\sigma}\right)^{\alpha}\right)\right]\left[1 + \left(\frac{yv}{\sigma}\right)^{\alpha}\right]}.$$

The joint survival function of (X_n, X_{n+1}) of the first order autoregressive Pareto process for $\alpha = 8$, $\sigma = 1$ is presented in Figure 3.3.2. below.

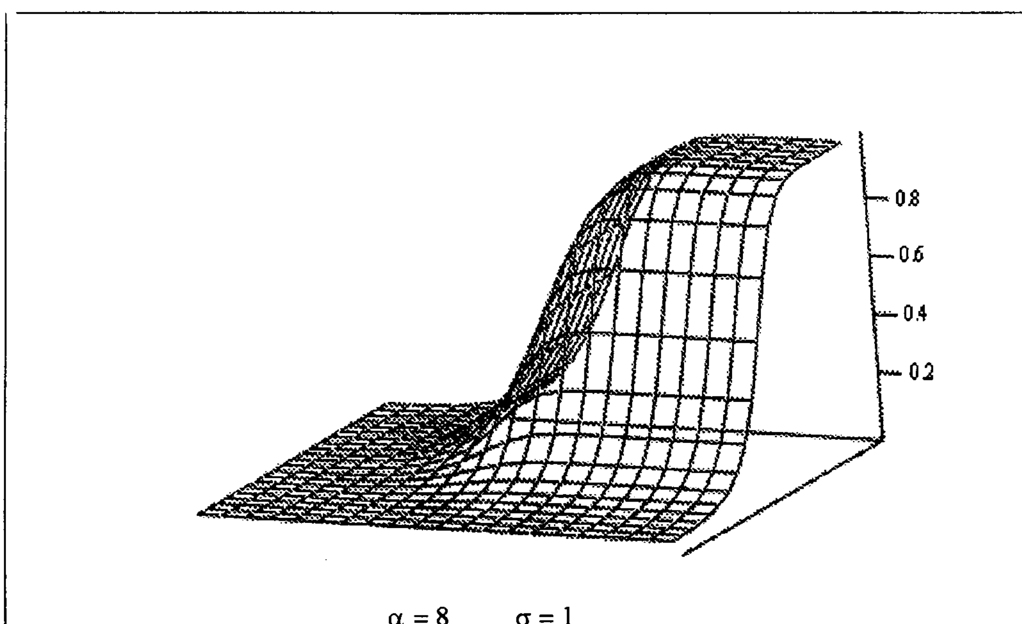


Figure 3.3.2.

Joint survival function of (X_n,X_{n+1}) of the first order autoregressive Pareto process.

Consider

$$
\begin{aligned}
P(X_{n+1} > X_n) &= P\left(V_{n+1}^{-1}\min(X_n, \varepsilon_{n+1}) > X_n\right) \\
&= \int_v P\left(v^{-1}\min(X_n, \varepsilon_{n+1}) > X_n\right) dG(v) \\
&= \int_v P\left(v^{-1}X_n > X_n, v^{-1}\varepsilon_{n+1} > X_n\right) dG(v) \\
&= \int_v P\left(v^{-1}\varepsilon_{n+1} > X_n\right) dG(v) \\
&= \int_x \int_v P(\varepsilon_{n+1} > xv)\, dG(v)\, dF(x)
\end{aligned}
$$

where G(v) and F(x) are the distribution functions of V_{n+1} and ε_{n+1} respectively.

Thus,

$$
\begin{aligned}
P(X_{n+1} > X_n) &= \int_0^\infty \int_0^1 \frac{1}{1+v^\alpha\left(\frac{x}{\sigma}\right)^\alpha} \frac{\alpha\left(\frac{x}{\sigma}\right)^{\alpha-1}\frac{1}{\sigma}}{\left(1+\left(\frac{x}{\sigma}\right)^\alpha\right)^2} \alpha v^{\alpha-1}\, dv\, dx \\
&= \int_0^\infty \left(\frac{\alpha \ln\left(1+\left(\frac{x}{\sigma}\right)^\alpha\right)}{x\left(1+\left(\frac{x}{\sigma}\right)^\alpha\right)^2} \right) dx.
\end{aligned}
$$

Numerical evaluation of the integral gives the result that $P(X_{n+1} > X_n)$ converges to 0.645 for all values of σ and α.

$$E(X_{n+1}X_n) = E\left(V_{n+1}^{-1}\min(X_n,\varepsilon_{n+1})X_n\right)$$

That is,

$$= E\left(V_{n+1}^{-1}\right)E\left(\min(X_n,\varepsilon_{n+1})X_n\right)$$

$$= E\left(V_{n+1}^{-1}\right)\int_x E\left(\min(x,\varepsilon_{n+1})x\right)dF(x)$$

$$= E\left(V_{n+1}^{-1}\right)\int_x\int_y \left(\min(x,y)x\right)dF(x)dH(y)$$

$$= E\left(V_{n+1}^{-1}\right)\int_x\left(\int_{x<y} x^2 dF(x)dH(y) + \int_{x>y} xy\,dF(x)dH(y)\right).$$

where F(x) and H(y) are the distribution functions of X_n and ε_{n+1} respectively.

That is,

$$E(X_{n+1}X_n) = E\left(V_{n+1}^{-1}\right)\int_x\int_y \left(\min(x,y)x\right)\frac{\alpha\left(\frac{x}{\sigma}\right)^{\alpha-1}\frac{1}{\sigma}}{\left(1+\left(\frac{x}{\sigma}\right)^{\alpha}\right)^2}\frac{\alpha\left(\frac{y}{\sigma}\right)^{\alpha-1}\frac{1}{\sigma}}{\left(1+\left(\frac{y}{\sigma}\right)^{\alpha}\right)^2}dxdy$$

Autocorrelation coefficient ρ of various orders for different values of α, are computed using 10000 simulated observations from first order autoregressive Pareto process and is presented in Table 3.3.1.

α \ ρ_j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	0.26	0.13	0.05	0.03	0.02	0.01	-0.02	-0.01	0.00	-0.01	0.00	-0.01	0.00	-0.01	-0.01
5	0.38	0.20	0.09	0.05	0.02	0.01	-0.01	-0.01	-0.01	-0.01	0.00	-0.01	-0.01	-0.01	-0.01
7	0.43	0.22	0.11	0.06	0.03	0.01	-0.01	-0.01	-0.01	-0.01	0.00	-0.01	-0.01	-0.02	-0.01
10	0.46	0.24	0.13	0.07	0.03	0.01	-0.01	-0.01	-0.01	0.00	0.00	-0.01	-0.01	-0.02	-0.01
15	0.48	0.26	0.14	0.08	0.03	0.02	0.00	-0.01	-0.01	0.00	0.00	-0.01	-0.01	-0.02	-0.02
20	0.49	0.27	0.14	0.08	0.03	0.02	0.00	-0.01	-0.01	0.00	0.00	-0.01	-0.01	-0.02	-0.02
30	0.50	0.27	0.15	0.09	0.03	0.02	0.00	-0.01	-0.01	0.00	0.00	-0.01	-0.01	-0.02	-0.02
50	0.50	0.28	0.15	0.09	0.04	0.02	0.00	-0.01	-0.01	0.00	0.00	-0.01	-0.01	-0.02	-0.02
70	0.51	0.28	0.16	0.09	0.04	0.02	0.00	-0.01	-0.01	0.00	0.00	-0.01	-0.01	-0.02	-0.02

Table 3.3.1.

Autocorrelations $\rho_j = Corr(X_n, X_{n+j})$ for j = 1 to 15 for different values of α and for $\sigma = 1$

3.3.2. First order autoregressive moving average Pareto process

The first order moving average Pareto process is built using a sequence $\{\varepsilon_n\}$ of independent and identically distributed Pareto random variables in the following manner.

$$X_n = W_n^{-1} \min(\varepsilon_n, \varepsilon_{n-1}) \tag{3.3.5}$$

where $\{W_n\}$ is a sequence of independent and identically distributed random variables independent of $\{\varepsilon_n\}$ with distribution function $F_{W_n}(v) = w^{\alpha},\ 0 < v < 1,\ \alpha > 0$.

Combining (3.3.1) and (3.3.5) we define the first order autoregressive moving average process $\{X_n\}$ as follows:

$$\begin{aligned} X_n &= W_n^{-1}\min(\varepsilon_n, Y_{n-1}) \\ Y_n &= V_n^{-1}\min(Y_{n-1}, Z_n) \end{aligned} \tag{3.3.6}$$

where $\{Z_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables. $\{W_n\}$ and $\{V_n\}$ are two independent sequences of independent and identically distributed random variables distribution function $F_{W_n}(w) = w^{\alpha}$, $F_{V_n}(v) = v^{\alpha}$, $0 < w, v < 1$, $\alpha > 0$.

Theorem 3.3.2.

Let the process $\{X_n\}$ be defined as

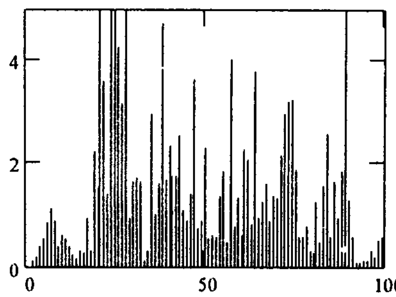
$$\begin{aligned} X_n &= W_n^{-1}\min(\varepsilon_n, Y_{n-1}) \\ Y_n &= V_n^{-1}\min(Y_{n-1}, Z_n) \end{aligned}$$

where $\{Z_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables and, $\{W_n\}$ and $\{V_n\}$ are two independent sequences of independent and identically distributed random variables with distribution function $F_{W_n}(w) = w^{\alpha}$, $F_{V_n}(v) = v^{\alpha}$, $0 < w, v < 1, \alpha > 0$ with $\varepsilon_1 \stackrel{d}{=} Y_0 \stackrel{d}{=} Z_1$. Then the process $\{X_n\}$ is stationary if and only if $\varepsilon_1 \stackrel{d}{=} P(\sigma, \alpha)$.

Proof:

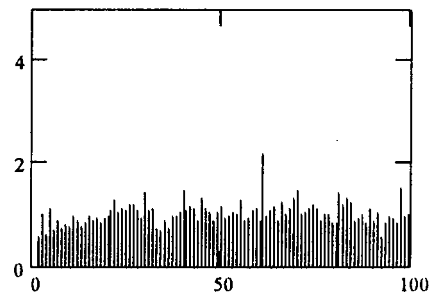
Follows easily using the arguments similar to those in Theorem 3.3.1. □

The sample path behavior of (3.3.6) is presented in Figure 3.3.3 for α =2, α =10 (with $\sigma = 1$).



$\alpha = 2$

Figure 3.3.3a.



$\alpha = 10$

Figure 3.3.3b.

Figure 3.3.3.

Simulated sample path of the first order autoregressive moving average Pareto process for $\alpha = 2$ and 10 with $\sigma = 1$

3.3.4. The first order autoregressive multivariate Pareto process

We consider the multivariate Pareto distribution with survival function

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_k > x_k) \quad = \quad \frac{1}{1+\left(\frac{x_1}{\sigma_1}\right)^{\alpha}+\left(\frac{x_2}{\sigma_2}\right)^{\alpha}+\ldots\ldots\left(\frac{x_k}{\sigma_k}\right)^{\alpha}},$$

$$(x_1, x_2, \ldots x_k) \in R_k^+, \alpha > 0, \sigma_i > 0, i = 1,2,3,..k.$$

Consider the process $\{\underline{X}_n\}$ defined by

$$\underline{X}_n \quad = \quad V_n^{-1} \min(\underline{X}_{n-1}, \underline{\varepsilon}_n), \quad n > 0 \tag{3.3.7}$$

where the minimum is component wise, $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors defined on R_k^+, $\{V_n\}$ is a sequence of independent and identically distributed random variables defined on (0, 1) such that V_n has Power function distribution, $F_{V_n}(v) = v^{\alpha},\ 0 < v < 1, \alpha > 0$.

Equation (3.3.7) in terms of survival function is

$$\bar{F}_{\underline{X}_n}(\underline{x}) = P\left(V_n^{-1}\min(\underline{X}_{n-1}, \underline{\varepsilon}_n) > \underline{x}\right), \quad \underline{x} = \left(x_1, x_2, \ldots x_k\right) \in R_n^{+}.$$

$$= \int_0^1 P\left(\underline{X}_{n-1} > \underline{x}v\right)P\left(\underline{\varepsilon}_n > \underline{x}v\right)f(v)\,dv\,.$$

If the process is stationary with $\underline{X}_0 \stackrel{d}{=} \underline{\varepsilon}_1$, we have

$$\bar{F}_{\underline{X}}(\underline{x}) = \int_0^1 \bar{F}_{\underline{X}}^2(\underline{x}v)f(v)\,dv\,.$$

That is,

$$\bar{F}(x_1, x_2, \ldots x_k) = \int \bar{F}^2(x_1v, x_2v, \ldots x_kv)\alpha\, v^{\alpha-1}dv\,.$$

Take $x_j = \delta_j x$, we get

$$\bar{F}(\delta_1x, \delta_2x, \ldots \delta_kx) = \int_0^1 \bar{F}^2(\delta_1xv, \delta_2xv, \ldots \delta_kxv)\,\alpha\, v^{\alpha-1}dv\,.$$

Putting $xv = t$,

$$\bar{F}(\delta_1x, \delta_2x, \ldots \delta_kx) = \frac{1}{x^{\alpha}}\int_0^x \bar{F}(\delta_1t, \delta_2t, \ldots \delta_kt)\,\alpha\, t^{\alpha-1}dt\,.$$

Take $\bar{F}(\delta_1x, \delta_2x, \ldots \delta_kx) = \bar{F}_{\underline{\delta}}(x)$ for all $\underline{\delta} \in R_k^{+}$.

Hence

$$x^{\alpha} F_{\underline{\delta}}(x) \;=\; \int_0^x F_{\underline{\delta}}(t)\,\alpha\, t^{\alpha-1} dt \,. \tag{3.3.8}$$

Differentiating with respect to x, (3.3.8) becomes

$$\frac{x}{\alpha}\frac{\bar{F}'_{\underline{\delta}}(x)}{\bar{F}^2_{\underline{\delta}}(x)} + \frac{1}{\bar{F}_{\underline{\delta}}(x)} \;=\; 1\,.$$

Writing $\bar{F}_{\underline{\delta}}(x) \;=\; \dfrac{1}{1+\eta_{\underline{\delta}}(x)}$, we get

$$\eta_{\underline{\delta}}(x) \;=\; x^{\alpha} c(\underline{\delta}) \quad \text{where} \quad c(\underline{\delta}) = c(\delta_1, \delta_2, \ldots, \delta_k)\,.$$

That is,

$$\bar{F}(\delta_1 x, \delta_2 x, \ldots \delta_k x) \;=\; \bar{F}_{\underline{\delta}}(x) \;=\; \frac{1}{1+x^{\alpha} c(\underline{\delta})}\,.$$

But $\quad x^{\alpha} c(\underline{\delta}) = x^{\alpha} c(\delta_1, \delta_2, \ldots, \delta_k) = c(\delta_1 x, \delta_2 x, \ldots, \delta_k x)$

Hence there exist $\delta_1, \delta_2, \ldots, \delta_k$ such that

$$\bar{F}_{\underline{\delta}}(x) \;=\; \frac{1}{1+x^{\alpha} c(\underline{\delta})}$$

Thus there exist $a_1, a_2, \ldots, a_k$ such that

$$\bar{F}_{\underline{\delta}}(x) \;=\; \frac{1}{1+a_1 x_1^{\alpha} + a_2 x_2^{\alpha} + \ldots + a_k x_k^{\alpha}}, \qquad a_i > 0, \quad i = 1,2, \ldots k.$$

If $\underline{X}_0 \stackrel{d}{=} \underline{\varepsilon}_1$ and $\underline{\varepsilon}_1$ is multivariate Pareto, it can be seen that the process is stationary.

From (3.3.7),

$$\bar{F}_{\underline{X}_1}(x_1, x_2, \dots x_k) = \int_0^1 \bar{F}^2(x_1 v, x_2 v, \dots x_k v)\,\alpha\, v^{\alpha-1} dv\ .$$

$$= \int_0^1 \frac{1}{\left[1+\left(\frac{x_1 v}{\sigma_1}\right)^{\alpha}+\left(\frac{x_2 v}{\sigma_2}\right)^{\alpha}+\dots+\left(\frac{x_n v}{\sigma_k}\right)^{\alpha}\right]^2}\alpha\, v^{\alpha-1} d\, v$$

$$= \frac{1}{1+\left(\frac{x_1}{\sigma_1}\right)^{\alpha}+\left(\frac{x_2}{\sigma_2}\right)^{\alpha}+\dots+\left(\frac{x_k}{\sigma_k}\right)^{\alpha}}.$$

Therefore $\underline{X}_1$ is Pareto. By induction we get $\{\underline{X}_n\}$ is Pareto.

Thus we have proved the following Theorem.

Theorem 3.3.3.

Let the process $\{\underline{X}_n\}$ be defined by

$$\underline{X}_n = V_n^{-1} \min(\underline{X}_{n-1}, \underline{\varepsilon}_n\ ), \quad n > 1$$

where $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors defined on R_k^+, $\{V_n\}$ is a sequence of independent and identically distributed random variables defined on (0, 1) such that V_n has distribution function, $F_{V_n}(v) = v^{\alpha}$, $0 < v < 1, \alpha > 0$, with $\underline{X}_0 \overset{d}{=} \underline{\varepsilon}_1$. Then the process $\{\underline{X}_n\}$ is stationary if and only if $\underline{\varepsilon}_1$ is multivariate Pareto.

Based on Theorem 3.3.3, we define the first order autoregressive multivariate Pareto process as follows.

$$\begin{aligned} & \underline{X}_0 = \underline{\varepsilon}_1 \quad \text{and} \\ \text{for } n = 1,2, \ldots \quad & \underline{X}_n = V_n^{-1} \min(\underline{X}_{n-1}, \underline{\varepsilon}_n) \end{aligned} \tag{3.3.4}$$

where $\{V_n\}$ is a sequences of independent and identically distributed random variables with distribution function $F_{V_n}(v) = v^{\alpha}$, $0 < v < 1$, $\alpha > 0$ and $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors defined on R_k^+ and $\underline{\varepsilon}_n \overset{d}{=} P(\underline{\sigma}, \alpha)$.

STUDY ON AUTOREGRESSIVE MODELS

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By

THOMAS MATHEW

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

KERALA, INDIA

Pin. 673 635

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CHAPTER IV

GENERALIZED LOGISTIC AND LOGISTIC PROCESSES

4.1. Introduction

Logistic distribution has attracted the attention of many researchers due to the application of this distribution in various fields. Balakrishnan (1992) discussed the application of logistic distribution in population growth, medical diagnosis and public health. He also discussed the analysis of bio availability data when successive samples are from logistic distribution. Arnold and Robertson (1989) studied first order autoregressive logistic process. Arnold (1993) constructed a logistic process using Markovian minimization. Sim (1993) have developed additive autoregressive models with logistic marginals. Voorn (1987) obtained characterizations of logistic distribution using maximum stability. For the properties of logistic distribution, see Balakrishnan (1992) and Johnson et al. (1995). Even though logistic distribution has many applications in real life situations not much work has been done on autoregressive processes with logistic marginals.

Similar to the models studied in Chapter III, we introduce and study the following two models

$$X_n = \min(X_{n-1} - \ln V_n, \varepsilon_n) \text{ and}$$

$$X_n = \min(X_{n-1}, \varepsilon_n) - \ln V_n .$$

This is actually a minification version of the models proposed by Helland and Nilsen (1976) to describe the water density in a sill fjord as well as other phenomenon such as, for instance, the utility of certain industrial equipments.

Definition 4.1.1.

We say that a random variable X on $(-\infty, \infty)$ has logistic distribution and write $X \underset{=}{d} L(\sigma, \alpha)$ if it has the survival function

$$\bar{F}_X(x) = P(X > x) = \frac{1}{1+\sigma e^{\alpha x}}, \sigma > 0, \alpha > 0. \tag{4.1.1}$$

□

Definition 4.1.2.

We say that a random variable X on $(-\infty, \infty)$ has generalised logistic distribution and write $X \underset{=}{d} GL(\sigma, \alpha, \gamma)$ if it has the survival function

$$\bar{F}_X(x) = \frac{1}{\left[1+\sigma e^{\alpha x}\right]^{\gamma+1}}, \quad \alpha, \sigma, \gamma > 0. \tag{4.1.2}$$

□

In Section 2, a random coefficient first order autoregressive process with generalised logistic marginal distribution is introduced and its properties are studied. A first order autoregressive moving average process with generalised logistic marginals is developed. A random coefficient first order autoregressive process with logistic marginals is introduced and studied in Section 3 and it is extended to define first order autoregressive moving average logistic process.

4.2. First order autoregressive generalized logistic process

Theorem 4.2.1.

Let the process $\{X_n\}$ be defined as

$$X_n = \min(X_{n-1} - \ln V_n, \varepsilon_n), \quad n = 1,2, \ldots \tag{4.2.1}$$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables such that V_n has distribution function $F_{V_n}(v) = v^{\alpha\gamma}, \; \alpha,\gamma > 0, \; 0 < v < 1$. Suppose the process $\{X_n\}$ is stationary. Then $X_n \overset{d}{=} GL(\sigma,\alpha,\gamma)$ if and only if $\varepsilon_n \overset{d}{=} L(\sigma,\alpha)$.

Proof:

Denoting the survival function of X_n and ε_n by $\bar{F}_{X_n}(x)$ and $\bar{F}_{\varepsilon_n}(x)$ respectively, (4.2.1) in terms of survival functions is

$$\bar{F}_{X_n}(x) = \bar{F}_{\varepsilon_n}(x)\int_0^1 \bar{F}_{X_{n-1}}(x + \ln v) f_{V_n}(v)dv,$$

where $f_{V_n}(v) = \alpha\gamma v^{\alpha\gamma-1}, \; \alpha,\gamma > 0, \; 0 < v < 1$ is the probability density function of V_n.

That is,

$$\bar{F}_{X_n}(x) = \bar{F}_{\varepsilon_n}(x)\int_0^1 \bar{F}_{X_{n-1}}(x+\ln v)\alpha\gamma\, v^{\alpha\gamma-1}dv. \tag{4.2.2}$$

Assume that the process $\{X_n\}$ is stationary. Then

$$\bar{F}_X(x) = \bar{F}_{\varepsilon_n}(x)\int_0^1 \bar{F}_X(x+\ln v)\alpha\gamma\, v^{\alpha\gamma-1}dv. \tag{4.2.3}$$

By taking $x + \ln v = t$, we get

$$e^{\alpha\gamma x}\bar{F}_X(x) = \alpha\,\gamma\,\bar{F}_{\varepsilon_n}(x)\int_{-\infty}^{x} \bar{F}_X(t)\,e^{\alpha\gamma t}dt.$$

Differentiating with respect to x on both sides and simplifying, we get

$$\frac{\bar{F}'_X(x)}{\bar{F}_X(x)} = \frac{\bar{F}'_{\varepsilon_n}(x)}{\bar{F}_{\varepsilon_n}(x)} - \alpha\,\gamma\left(1-\bar{F}_{\varepsilon_n}(x)\right).$$

Thus,

$$\bar{F}_X(x) = \bar{F}_{\varepsilon_n}(x)\exp\left(-\alpha\gamma\int_{-\infty}^{x}\left(1-\bar{F}_{\varepsilon_n}(u)\right)du\right).$$

Since $\varepsilon_n \stackrel{d}{=} L(\sigma,\alpha)$, we get

$$\bar{F}_X(x) = \frac{1}{\left(1+\sigma e^{\alpha x}\right)^{\gamma+1}}.$$

Conversely, let $\{X_n\}$ is stationary and $X_n \stackrel{d}{=} GL(\sigma,\alpha,\gamma)$.

Then from (4.2.3), we have

$$\frac{1}{\bar{F}_{\varepsilon_n}(x)} = \int_0^1 \frac{\bar{F}_X(x+\ln v)\alpha\,\gamma\, v^{\alpha\gamma-1}}{\bar{F}_X(x)}dv$$

$$= \int_0^1 \left(\frac{1+\sigma e^{\alpha x}}{1+\sigma e^{\alpha(x+\ln v)}} \right)^{\gamma+1} \alpha \gamma v^{\alpha\gamma-1} dv$$

$$= \int_0^1 \left(\frac{1+\sigma e^{\alpha x}}{1+\sigma v^{\alpha} e^{\alpha x}} \right)^{\gamma+1} \alpha \gamma v^{\alpha\gamma-1} dv.$$

$$= 1+\sigma e^{\alpha x}.$$

Hence $\varepsilon_n \underset{=}{d} L(\sigma, \alpha)$. This completes the proof. □

Theorem 4.2.2.

Let the process $\{X_n\}$ be defined as

$$X_n = \min(X_{n-1} - \ln V_n, \varepsilon_n), \quad n = 1,2, \ldots$$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables such that $\{V_n\}$ has distribution function $F_{V_n}(v) = v^{\alpha\beta}, 0 < v < 1, \ \alpha, \beta > 0$. If $X_0 \underset{=}{d} GL(\sigma, \alpha, \gamma)$ and $\varepsilon_n \underset{=}{d} L(\sigma, \alpha)$, then the process $\{X_n\}$ is stationary with $GL(\sigma, \alpha, \gamma)$ marginals.

Proof:

From (4.2.2), we have

$$\bar{F}_{X_1}(x) = \bar{F}_{\varepsilon_1}(x) \int_0^1 \bar{F}_{X_0}(x + \ln v) \alpha\gamma v^{\alpha\gamma-1} dv$$

$$= \frac{1}{1+\sigma e^{\alpha x}} \int_0^1 \left(\frac{1}{1+\sigma e^{\alpha(x+\ln v)}} \right)^{\gamma+1} \alpha\gamma v^{\alpha\gamma-1} dv$$

$$= \frac{1}{1+\sigma e^{\alpha x}} \int_0^1 \left(\frac{1}{1+\sigma v^{\alpha} e^{\alpha x}} \right)^{\gamma+1} \alpha\gamma v^{\alpha\gamma-1} dv.$$

$$= \frac{1}{\left(1+\sigma e^{\alpha x}\right)^{\gamma+1}}.$$

That is, $X_1 \stackrel{d}{=} GL(\sigma, \alpha, \gamma)$.

If $X_{n-1} \stackrel{d}{=} GL(\sigma, \alpha, \gamma)$, then we get $X_n = GL(\sigma, \alpha, \gamma)$. Thus the process $\{X_n\}$ is stationary with generalized logistic marginals. □

Based on Theorem 4.2.2, we define the first order autoregressive generalized logistic process $\{X_n\}$, $n = 0,1,2,\ldots$ as follows:

$$\begin{aligned} X_0 &= GL(\sigma, \alpha, \gamma) \quad \text{and} \\ \text{for } n = 1,2,\ldots \quad X_n &= \min(X_{n-1} - \ln V_n, \varepsilon_n) \end{aligned} \tag{4.2.4}$$

where $\{V_n\}$ is a sequence of independent and identically distributed power function random variables with distribution function $F_{V_n}(v) = v^{\alpha\gamma}, 0 < v < 1, \ \alpha, \gamma > 0$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $L(\sigma, \alpha)$ random variables independent of $\{V_n\}$.

4.2.1. Properties of first order autoregressive generalized logistic process

Now we study some properties of the first order autoregressive generalized logistic process.

$$\bar{F}_{X_n, X_{n+1}}(x, y) = P(X_n > x, X_{n+1} > y).$$

$$= P(X_n > x, \min(X_n - \ln V_{n+1}, \varepsilon_{n+1}) > y)$$

$$= P(X_n > x, X_n - \ln V_{n+1} > y, \varepsilon_{n+1} > y)$$

$$= P(\varepsilon_{n+1} > y)\int_0^1 P(X_n > x, X_n > y + \ln v)\alpha\gamma v^{\alpha\gamma-1} dv$$

$$= P(\varepsilon_{n+1} > y)\int_0^1 P(X_n > \max(x, y + \ln v))\alpha\gamma v^{\alpha\gamma-1} dv .$$

$$\bar{F}_{X_n X_{n+1}}(x, y) = \frac{1}{1+\sigma e^{\alpha y}} \int_0^1 \frac{1}{\left(1+\max\left(\sigma e^{\alpha x}, v^{\alpha}\sigma e^{\alpha y}\right)\right)^{\gamma+1}} \alpha\gamma v^{\alpha\gamma-1} dv.$$

The joint survival function of (X_n, X_{n+1}) of the first order autoregressive generalized logistic process is presented in Figure 4.2.1 below.

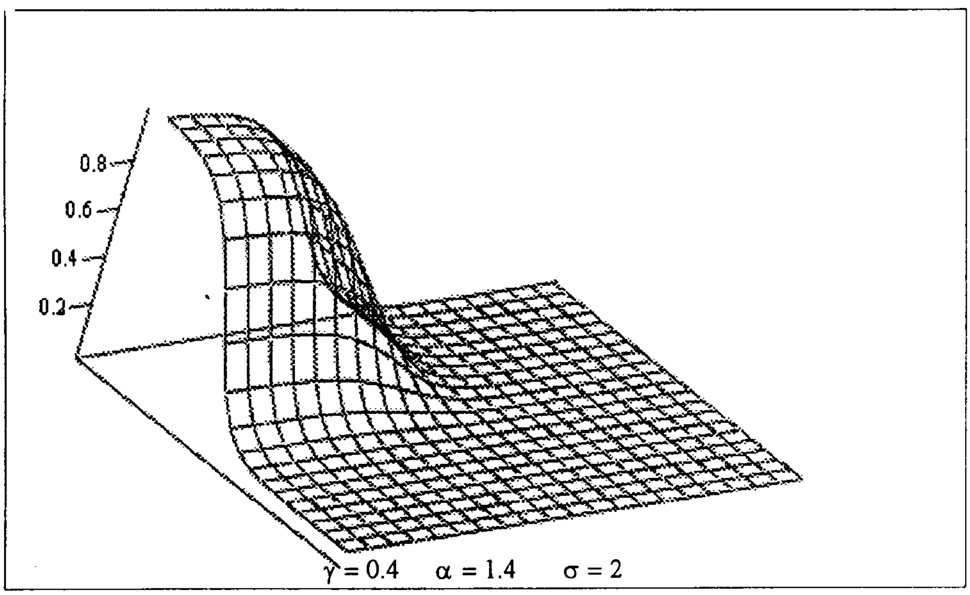


Figure 4.2.1

Joint survival function of (X_n, X_{n+1}) of the first order autoregressive generalised logistic process

Consider

$$P(X_{n+1} > X_n) = P(\min(X_n - \ln V_{n+1}, \varepsilon_{n+1}) > X_n)$$

$$= \int_0^1 P(X_n > X_n + \ln v)\alpha\gamma v^{\alpha\gamma - 1} P(\varepsilon_{n+1} > X_n)dv$$

$$= \int_x P(\varepsilon_{n+1} > x / X_n = x)f(x)dx$$

$$= \frac{\gamma + 1}{\gamma + 2}.$$

Now

$$E(X_{n+1}X_n) = E(\min(X_n - \ln V_{n+1}, \varepsilon_{n+1})X_n)$$

$$= \int_v E(\min(X_n - \ln v, \varepsilon_{n+1})X_n)\, dG(v)$$

where G(.) is the distribution function of V_{n+1}.

We have,

$$E(X_{n+1}X_n) = \int_v E(\min(X_n - \ln v, \varepsilon_{n+1})X_n)dG(v)$$

$$= \int_v\int_x \min(x - \ln v, \varepsilon_{n+1})\, xdF(x)dG(v)$$

where F(x) is the distribution function of $GL(\sigma, \alpha, \gamma)$.

$$= \int_v\int_x\int_y \min(x - \ln v, y)\, xdH(y)dF(x)dG(v)$$

where H(y) is the distribution function of. $L(\sigma, \alpha)$.

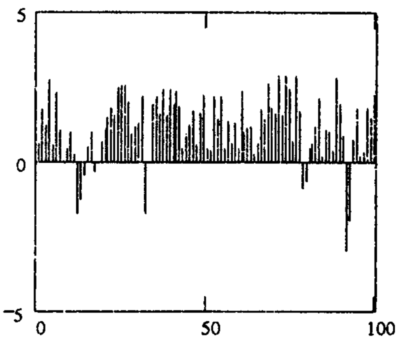
10000 observations are simulated from the first order autoregressive generalized logistic process for $\alpha = 1.4$ and $\sigma = 2$, and autocorrelation of lag up to 20 for various values of γ is calculated. This is presented in Table 4.2.1 given below.

ρ_j \ γ	1	2	3	4	5	6	8	10	14	20	24	30	50	80
1	0.566	0.729	0.817	0.855	0.882	0.902	0.925	0.943	0.959	0.969	0.975	0.980	0.989	0.993
2	0.318	0.532	0.668	0.728	0.778	0.812	0.855	0.888	0.920	0.940	0.950	0.961	0.977	0.985
3	0.188	0.389	0.548	0.619	0.686	0.732	0.791	0.836	0.882	0.911	0.925	0.942	0.966	0.978
4	0.105	0.280	0.447	0.523	0.603	0.659	0.730	0.787	0.845	0.883	0.902	0.924	0.955	0.971
5	0.045	0.197	0.359	0.439	0.529	0.592	0.673	0.742	0.809	0.856	0.879	0.906	0.944	0.963
6	0.016	0.140	0.286	0.364	0.462	0.531	0.620	0.697	0.774	0.829	0.856	0.888	0.934	0.956
7	0.000	0.098	0.231	0.299	0.405	0.477	0.571	0.655	0.741	0.803	0.833	0.870	0.923	0.949
8	0.014	0.070	0.185	0.240	0.355	0.428	0.527	0.616	0.709	0.778	0.811	0.853	0.913	0.942
9	0.011	0.051	0.150	0.194	0.313	0.385	0.487	0.580	0.679	0.755	0.790	0.837	0.902	0.935
10	0.003	0.044	0.124	0.159	0.278	0.346	0.451	0.546	0.651	0.732	0.769	0.820	0.892	0.928
11	0.003	0.037	0.103	0.130	0.247	0.312	0.418	0.514	0.623	0.709	0.748	0.804	0.882	0.921
12	0.001	0.028	0.084	0.107	0.218	0.281	0.387	0.484	0.596	0.688	0.729	0.789	0.872	0.914
13	0.003	0.026	0.073	0.091	0.197	0.255	0.360	0.457	0.571	0.667	0.710	0.774	0.862	0.907
14	0.011	0.017	0.058	0.076	0.176	0.231	0.335	0.431	0.547	0.648	0.692	0.759	0.853	0.900
15	0.023	0.012	0.049	0.068	0.162	0.211	0.313	0.408	0.524	0.629	0.675	0.745	0.843	0.894
16	0.017	0.018	0.045	0.064	0.148	0.196	0.293	0.387	0.503	0.611	0.658	0.731	0.834	0.887
17	0.007	0.028	0.048	0.064	0.139	0.184	0.275	0.368	0.483	0.593	0.642	0.718	0.825	0.880
18	0.004	0.035	0.045	0.064	0.130	0.172	0.258	0.350	0.463	0.577	0.626	0.704	0.816	0.873
19	0.007	0.033	0.040	0.061	0.120	0.160	0.241	0.332	0.445	0.561	0.611	0.691	0.806	0.867
20	0.011	0.029	0.033	0.059	0.110	0.146	0.225	0.314	0.426	0.545	0.596	0.678	0.797	0.860

Table 4.2.1

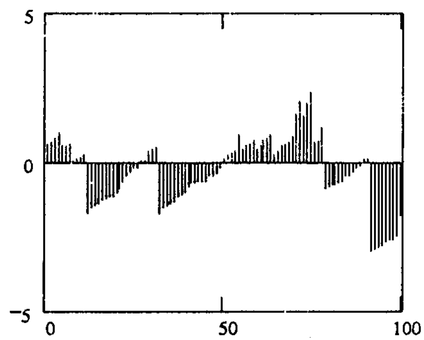
Autocorrelation (ρ_j) of lag up to 20 for various values of γ for $\alpha = 1.4$ and $\sigma = 2$

The simulated sample path behaviour of the first order autoregressive generalized logistic process is presented in Figure 4.2.2.



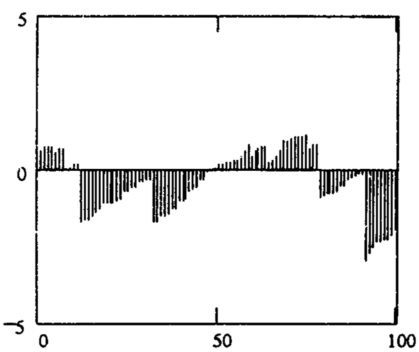
$\alpha = 1.4 \quad \gamma = 0.5 \quad \sigma = 0.1$

Figure 4.2.2a



$\alpha = 1.4 \quad \gamma = 5 \quad \sigma = 0.1$

Figure 4.2.2b



$\alpha = 1.4 \quad \gamma = 9 \quad \sigma = 0.1$

Figure 4.2.2c

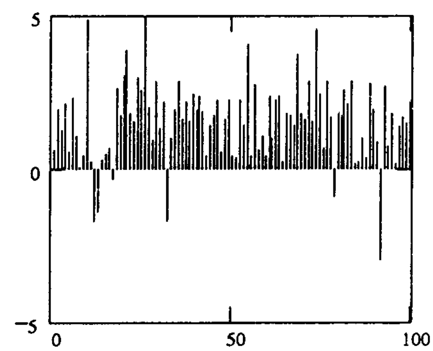


Figure 4.2.2d

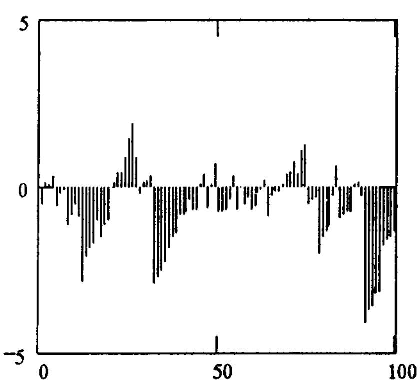
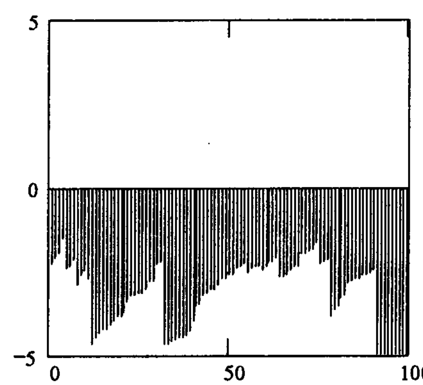


Figure 4.2.2e



$\alpha = 1.4 \quad \gamma = 5 \quad \sigma = 6$

Figure 4.2.2f

Figure 4.2.2.

Sample path behavior of the generalized logistic process

4.2.2. First order autoregressive moving average mixed generalized logistic process

Theorem 4.2.3.

Let

$$\begin{aligned} X_n &= \min(K_n Z_n, Y_{n-1} - \ln V_n) \\ Y_n &= \min(Y_{n-1} - \ln U_n, \ Z_n) \end{aligned} \qquad (4.2.5)$$

where $\{Z_n\}$ are independent and identically distributed $L(\sigma, \alpha)$ random variables, K_n 's are independent and identically distributed Bernoulli random variables such that $P(K_n = 0) = \theta = 1 - P(K_n = 1)$, $\{V_n\}$ and $\{U_n\}$ are two independent sequences of independent and identically distributed random variables with distribution function $F(v) = v^{\alpha\gamma}$, $F(u) = u^{\alpha\gamma}$, $\alpha, \gamma > 0$, $0 < v, u < 1$ and $Y_n \overset{d}{=} GL(\sigma, \alpha, \gamma)$. Then the process $\{X_n\}$ is stationary with mixed generalized logistic distribution.

Proof:

The equation $X_n = \min(Y_{n-1} - \ln V_n, K_n Z_n)$ in terms of survival functions is

$$\bar{F}_X(x) = [\theta + (1-\theta)P(Z_n > x)] \int_0^1 P(Y_{n-1} > x + \ln v)\alpha\gamma v^{\alpha\gamma - 1} dv.$$

If $x + \ln v = t$, then

$$\bar{F}_X(x) = [\theta + (1-\theta)P(Z_n > x)] \int_{-\infty}^{x} \frac{P(Y_{n-1} > t)\alpha\gamma e^{t\alpha\gamma}}{e^{\alpha\gamma x}} dt$$

$$= \frac{1}{e^{\alpha\gamma x}} \left[\theta + (1-\theta)\frac{1}{(1+\sigma e^{\alpha x})}\right] \int_{-\infty}^{x} P(Y_{n-1} > t)\alpha\gamma e^{\alpha\gamma t} dt$$

$$= \theta \frac{1}{\left(1+\sigma e^{\alpha x}\right)^{\gamma}} + (1-\theta)\frac{1}{\left(1+\sigma e^{\alpha x}\right)^{\gamma+1}}. \qquad \square$$

4.3. First order autoregressive logistic process

Theorem 4.3.1.

Let $X_0 = \varepsilon_1$ and

and for n = 1,2,... $X_n = \min\left(X_{n-1} - \ln(V_n), \varepsilon_n - \ln(V_n)\right)$,

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables such that V_n has the distribution function $F_{V_n}(v) = v^{\alpha}$, $\alpha > 0$, $0 < v < 1$. Then the process $\{X_n\}$ is stationary if and only if $\varepsilon_1 \underline{\underline{d}}\, L(\sigma, \alpha)$.

Proof:

Consider

$$X_n = \min\left(X_{n-1} - \ln(V_n), \varepsilon_n - \ln(V_n)\right) \tag{4.3.1}$$

Denoting the survival function of X_n and ε_n by $\bar{F}_{X_n}(x)$ and $\bar{F}_{\varepsilon_n}(x)$ respectively, (4.3.1) in terms of survival function is

$$\bar{F}_{X_n}(x) = \int_0^1 \bar{F}_{X_{n-1}}(x + \ln v)\, \bar{F}_{\varepsilon_n}(x + \ln v)\, f_{V_n}(v) dv$$

where $f_{V_n}(v)$ is the probability density function of V_n.

Since $F_{V_n}(v) = v^{\alpha}, \alpha > 0, 0 < v < 1$, we get

$$\bar{F}_{X_n}(x) = \int_0^1 \bar{F}_{X_{n-1}}(x+\ln v)\,\bar{F}_{\varepsilon_n}(x+\ln v)\,\alpha\, v^{\alpha-1} dv \tag{4.3.2}$$

Assume that the process $\{X_n\}$ is stationary. Then

$$\bar{F}_X(x) = \int_0^1 \bar{F}_X(x+\ln v)\,\bar{F}_{\varepsilon_n}(x+\ln v)\,\alpha\, v^{\alpha-1} dv.$$

Since $X_0 \overset{d}{=} \varepsilon_1$, we have

$$\bar{F}_X(x) = \int_0^1 \bar{F}_X^2(x+\ln v)\,\alpha\, v^{\alpha-1} dv.$$

That is, $$e^{\alpha x}\bar{F}_X(x) = \int_{-\infty}^{x} \bar{F}_X^2(t)\,\alpha\, e^{\alpha t} dt.$$

Differentiating both sides with respect to x and rearranging the terms, we get

$$\frac{1}{\alpha}\frac{\bar{F}_X'(x)}{\bar{F}_X^2(x)} + \frac{1}{\bar{F}_X(x)} = 1. \tag{4.3.3}$$

Writing $\bar{F}_X(x) = \dfrac{1}{1+h(x)}$, $h(x)$ is monotone increasing with $h(-\infty) = -\infty$ and $h(\infty) = \infty$, we get

$$\frac{h'(x)}{h(x)} = \alpha$$

and hence $h(x) = ce^{\alpha x}$, $c > 0$ is a constant.

Therefore. $$\bar{F}_X(x) = \frac{1}{1+c\,e^{\alpha x}}.$$

Taking $c = \sigma_{\cdot\cdot}$, we get $X \overset{d}{=} L(\sigma, \alpha)$.

Conversely, assume that $\varepsilon_1 \underline{\underline{d}} L(\sigma,\alpha)$. Using the method of induction, we show that the process $\{X_n\}$ is stationary.

Since $X_0 \underline{\underline{d}} \varepsilon_1$, from (4.3.2) we get

$$\bar{F}_{X_1}(x) = \int_0^1 \bar{F}_{X_0}(x+\ln v)\, \bar{F}_{\varepsilon_1}(x+\ln v)\, \alpha\, v^{\alpha-1} dv.$$

If $X_0 \underline{\underline{d}} \varepsilon_1$, then

$$\bar{F}_{X_1}(x) = \int_0^1 \frac{1}{\left(1+\sigma e^{\alpha(x+\ln v)}\right)^2}\, \alpha\, v^{\alpha-1} dv$$

$$= \int_0^1 \frac{1}{\left(1+v^{\alpha}\sigma e^{\alpha x}\right)^2}\, \alpha\, v^{\alpha-1} dv$$

$$= \frac{1}{1+\sigma e^{\alpha x}}.$$

Assuming $X_{n-1} \underline{\underline{d}} L(\sigma,\alpha)$, it can be shown that $X_n \underline{\underline{d}} L(\sigma,\alpha)$. Hence the process $\{X_n\}$ is stationary. This completes the proof. □

Based on theorem 4.3.1, the first order autoregressive logistic process is defined as

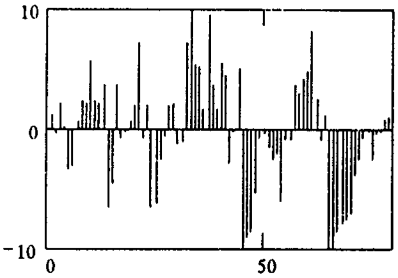
$$\begin{aligned} X_0 &= \varepsilon_1 \quad \text{and} \\ \text{for } n = 1,2,\ldots \quad X_n &= \min\left(X_{n-1} - \ln(V_n), \varepsilon_n - \ln(V_n)\right), \end{aligned}$$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables such that V_n has the distribution function

$F_{V_n}(v) = v^{\alpha}, \quad \alpha > 0, \quad 0 < v < 1$ and $\varepsilon_1 \underline{\underline{d}} L(\sigma,\alpha)$.

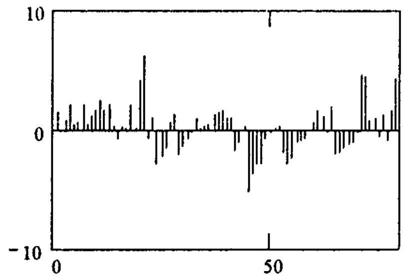
4.3.1. Properties of first order autoregressive logistic process

Simulated sample path behavior of 80 observations from the first order autoregressive logistic process is presented in Figures 4.3.1.



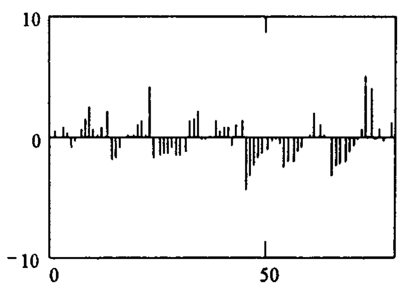
$\alpha = 0.4 \quad \sigma = 1$

Figure 4.3.1a.



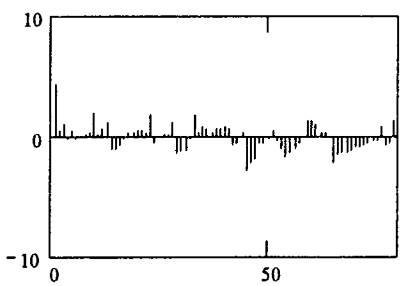
$\alpha = 0.9 \quad \sigma = 1$

Figure 4.3.1b.



$\alpha = 1.4 \quad \sigma = 1$

Figure 4.3.1c.



$\alpha = 1.9 \quad \sigma = 1$

Figure 4.3.1d.

Figure 4.3.1.

The simulated sample path behavior of the first order autoregressive Logistic process

The joint survival function of (X_n, X_{n+1}) is

$$\begin{aligned}\bar{F}_{X_n, X_{n+1}}(x, y) &= P(X_n > x, X_{n+1} > y) \\ &= P(X_n > x, \min(X_n, \varepsilon_{n+1}) - \ln V_{n+1} > y)\end{aligned}$$

$$= P\left(X_n > x, X_n > y + \ln V_{n+1}, \varepsilon_{n+1} > y + \ln V_{n+1}\right)$$

$$= \int_0^1 P(\varepsilon_{n+1} > y + \ln v) P(X_n > x, X_n > y + \ln v) \alpha v^{\alpha-1} dv$$

$$= \int_0^1 P(\varepsilon_{n+1} > y + \ln v) P(X_n > \max(x, y + \ln v)) \alpha v^{\alpha-1} dv .$$

$$\bar{F}_{X_n X_{n+1}}(x, y) = \int_0^1 \frac{\alpha\, v^{\alpha-1} dv}{\left[1 + \max\left(\sigma e^{\alpha x}, v^{\alpha} \sigma e^{\alpha x}\right)\right]\left[1 + \sigma v^{\alpha} e^{\alpha x}\right]} .$$

The joint survival function of (X_n, X_{n+1}) of the first order autoregressive logistic process for $\alpha = 8$, $\sigma = 1$ is given in Figure 4.3.2. below.

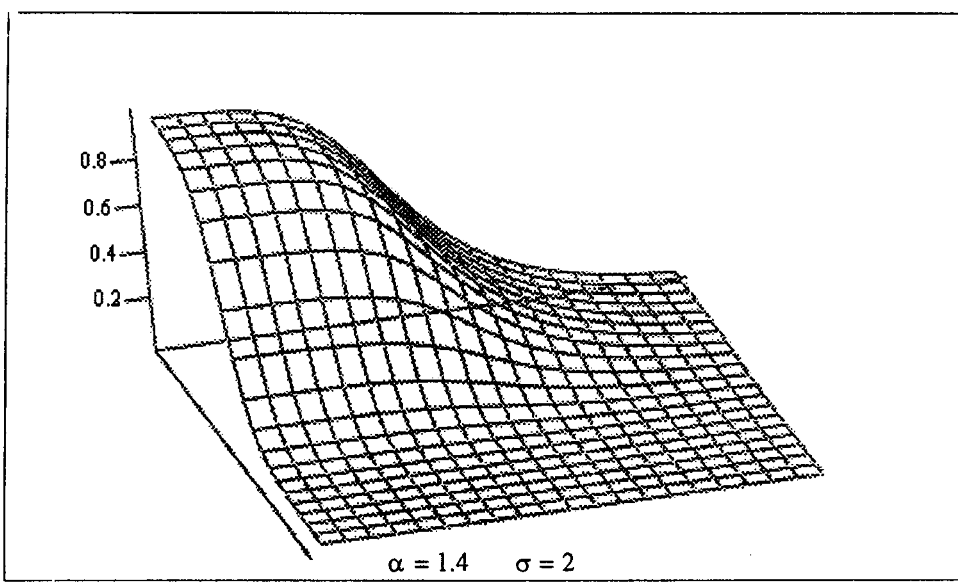


Figure 4.3.2.

Joint survival function of (X_n,X_{n+1}) of first order autoregressive logistic process

Consider,

$$P(X_{n+1} > X_n) = P\left(\min(X_n, \varepsilon_{n+1}) - \ln V_{n+1} > X_n\right)$$

$$= \int_v P\left(\min(X_n, \varepsilon_{n+1}) - \ln v > X_n\right) dG(v)$$

$$= \int_v P\left(X_n - \ln v > X_n, \varepsilon_{n+1} - \ln v > X_n\right) dG(v)$$

$$= \int_v P\left(\varepsilon_{n+1} - \ln v > X_n\right) dG(v)$$

$$= \int_x \int_v P\left(\varepsilon_{n+1} > x + \ln v\right) dG(v)\, dF(x).$$

where G(v) and F(x) are the distribution functions of V_{n+1} and ε_{n+1} respectively.

$$P(X_{n+1} > X_n) = \int_{-\infty}^{\infty}\int_0^1 \frac{1}{1+v^{\alpha}\sigma e^{\alpha x}} \frac{\alpha\sigma e^{\alpha x}}{\left(1+\sigma e^{\alpha x}\right)^2} \alpha v^{\alpha-1} dv dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{\alpha \ln\left(1+\sigma e^{\alpha x}\right)}{\left(1+\sigma e^{\alpha x}\right)^2} \right) dx.$$

4.3.2. First order autoregressive moving average logistic process

The first order autoregressive moving average logistic process is built using a sequence $\{\varepsilon_n\}$ of independent and identically distributed logistic random variables in the following manner:

$$X_n = \min(\varepsilon_n - \ln W_n, \varepsilon_{n-1} - \ln W_n) \tag{4.3.4}$$

where $\{W_n\}$ is a sequence of independent and identically distributed random variables independent of $\{\varepsilon_n\}$ with distribution function $F_{W_n}(w) = w^{\alpha}, \alpha > 0, 0 < w < 1$.

Combining (4.3.1) and (4.3.4) we define the first order autoregressive moving average logistic process $\{X_n\}$ as follows:

$$\begin{aligned} X_n &= \min(\varepsilon_n - \ln W_n, Y_{n-1} - \ln W_n) \\ Y_n &= \min(Y_{n-1} - \ln V_n, Z_n - \ln V_n) \end{aligned}$$

where $\{Z_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables and $\{V_n\}, \{W_n\}$ are two independent sequences of independent and identically distributed random variables such that both have the power function distribution $F_{W_n}(w) = w^{\alpha}$, $F_{V_n}(v) = v^{\alpha}$, $0 < v, w < 1$, $\alpha > 0$.

Theorem 4.3.2.

Let the process $\{X_n\}$ be defined by

$$\begin{aligned} X_n &= \min(\varepsilon_n - \ln W_n, Y_{n-1} - \ln W_n) \\ Y_n &= \min(Y_{n-1} - \ln V_n, Z_n - \ln V_n) \end{aligned} \tag{4.3.5}$$

where $\{Z_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables and, $\{W_n\}$ and $\{V_n\}$ are also two independent sequences of independent and identically distributed random variables with distribution function $F_{W_n}(w) = w^{\alpha}$, $F_{V_n}(v) = v^{\alpha}$, $0 < v, w < 1$, $\alpha > 0$ with $\varepsilon_1 \underline{\underline{d}}\ Y_0 \underline{\underline{d}}\ Z_1$. Then the process $\{X_n\}$ is stationary if and only if $\varepsilon_1 \underline{\underline{d}}\ L(\sigma, \alpha)$.

Proof:

Proof follows using the arguments similar to those in Theorem 4. 3.1. □

The sample path behavior of the first order autoregressive moving average logistic process is given in Figures 4.3.3.

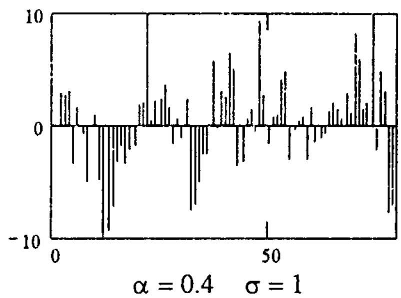


Figure 4.3.3a.

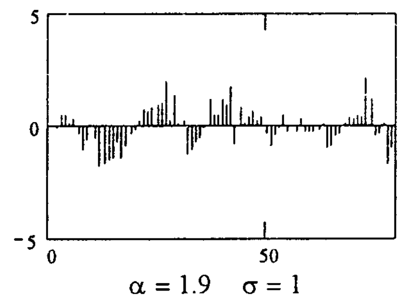


Figure 4.3.3b.

Figure 4.3.3.

Sample path behavior of the first order autoregressive moving average logistic process

STUDY ON AUTOREGRESSIVE MODELS

Thesis submitted

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By

THOMAS MATHEW

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

KERALA, INDIA

Pin. 673 635

MAY, 2004

CHAPTER V

DISCRETE LINNIK AND DISCRETE MITTAG-LEFFLER PROCESSES

5.1. Introduction

One of the assumptions underlying linear stochastic difference equation models of time series is that the noise process generating the observed series is composed of independent and identically distributed random shocks which are usually, for statistical convenience, taken to be normally distributed (see Al-Osh and Alzaid (1988)). It is well known that a linear combination of normal random variables is also normal. However, observed time series often differ considerably from the familiar Gaussian realizations, especially when the observations are discrete. In recent years there has been growing interest in non-Gaussian processes having discrete marginal distributions.

Discrete variate time series occur in many contexts, often as counts of events, objects or individuals in consecutive intervals or at consecutive points of time. Some simple examples are the numbers of accidents in a manufacturing plant each month, the numbers of patients treated in a hospitals emergency unit each hour, the number of fish caught in a particular area of sea each week, the number of busy lines in a telephone network noted every thirty minutes and the number of lifts in a tall office building which are fully operational at the start of business each day. Such data may

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This Chapter is based on Thomas Mathew and Jayakumar (2003b), Jayakumar and Thomas Mathew (2004).

also be arise from the discretization of continuous variate time series. An example of this is the reduction of daily rainfall volumes to a binary series of ones and zeros, that is, wet and dry days. See for example, Phatarfod and Srikanthan (1981).

Construction of discrete time series models with Markovian dependence structure gained importance with the work of Al-Osh and Alzaid (1987). Based on the concept of discrete self-decomposability, they introduced the first order integer valued autoregressive model defined in (1.3.2). They studied the distributional properties of the model and discussed methods of estimation of the parameters of the model. Alzaid and Al-Osh (1990) and Jin Guan and Yuan (1991) extended the first order integer valued autoregressive model to p^{th} order case and studied the properties of the model.

Pillai and Jayakumar (1995) introduced the first order autoregressive discrete Mittag-Leffler distribution. They defined the first order autoregressive discrete Mittag-Leffler process as follows:

Let $\quad X_0 \stackrel{d}{=} \varepsilon_1$

and for n = 1,2,3, ...

$$X_n = \begin{cases} \rho \oplus X_{n-1} & \text{w.p.} \quad \rho^{\alpha} \\ \rho \oplus X_{n-1} + \varepsilon_n & \text{w.p.} \quad 1-\rho^{\alpha} \end{cases}$$

where $\rho \oplus X$ is defined (in distribution) by its probability generating function $P(1-\rho+\rho z)$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed discrete Mittag-Leffler random variables. Thomas Mathew and Jayakumar (2003b) introduced the concept of random coefficient integer valued autoregressive processes and developed a random coefficient first order autoregressive discrete Mittag-Leffler process. Jayakumar and Thomas Mathew (2004) constructed a random coefficient first order autoregressive discrete Linnik model and studied its properties.

In the present study, we consider a generalization of the first order integer valued autoregressive model (1.3.2). Here we replace α in (1.3.2) by a random variable. In practical situations, we can see that it is reasonable to replace α in (1.3.2) by a non-degenerate random variable.

For a non-degenerate random variable V defined on (0,1), the operator $\oplus$ defined in (1.3.1) is as follows: Let X be a non-negative integer valued random variable. Then, for any given V, the binomial thinning operator '$\oplus$' is defined by

$$V \oplus X = \sum_{i=1}^{X} Y_i$$

where $\{Y_i\}$ is a sequence of independent and identically distributed random variables independent of X such that

$$P(Y_i = 1) = 1 - P(Y_i = 0) = V.$$

Let $P_X(z) = E\left(z^X\right)$ is the probability generating function of the non-negative integer valued random variable X. Then the probability generating function of $V \oplus X = \sum_{i=1}^{X} Y_i$ is

$$P_{V \oplus X}(z) = \int_v E\left(z^{v \oplus X}\right) dG(v)$$

$$= \int_v E\left(z^{\sum_{i=1}^{X} Y_i} \right) dG(v).$$

But $V \oplus X \;=\; \sum_{i=1}^{X} Y_i$ is binomial random variable with $n = X$ and $p = V$.

$$P_{V\oplus X}(z) \;=\; \int_v \left(\sum_{y=0}^{X} E\left(z^{\sum_{i=1}^{X} Y_i} \Big/ \sum_{i=1}^{X} Y_i = y \right) f(y) \right) dG(v)$$

where $\qquad f(y) = \binom{X}{y} V^y (1-V)^{X-y},\ y = 0, 1, 2, \ldots., X.$

Therefore,

$$P_{V\oplus X}(z) \;=\; \int_v \left(\sum_{y=0}^{X} E\left(z^y\right) f(y) \right) dG(v)$$

$$= \int_v \left(E\left(\sum_{y=0}^{X} z^y f(y) \right) \right) dG(v)$$

$$= \int_v E\left(E\left(z^Y\right)\right) dG(v)$$

$$= \int_v E\left((1 - v + vz)^X\right) dG(v)$$

$$= \int_v P_X(1 - v + vz)\, dG(v).$$

Definition 5.1.1.

The generalized integer valued first order autoregressive model $\{X_n, n = 0, \pm 1, \pm 2, \ldots\}$ is defined as

$$X_n \;=\; V_n \oplus X_{n-1} + \varepsilon_n, \;\; n = 1, 2, \ldots \tag{5.1.1}$$

where $\{V_n\}$ is a sequence of independent and identically distributed random variables defined on (0, 1) and $\{\varepsilon_n\}$is another sequence of independent and identically distributed non negative integer valued random variables. □

Similar to Al-Osh and Alzaid (1987), the properties of this model can be studied.

$$E(V \oplus X) \;=\; \int_0^1 E\left(\sum_{z=0}^{X} (v \oplus X / v \oplus X = z) f(z) \right) dG(v),$$

where Z is binomial random variable with parameters X and V.

$$\begin{aligned} E(V \oplus X) &= \int_0^1 E\left(\sum_{z=0}^{X} z f(z) \right) dG(v) \\ &= \int_0^1 E(E(Z))dG(v) \\ &= \int_0^1 E(XV)dG(v) \\ &= \int_0^1 E(X)v dG(v) \end{aligned}$$

$$
\begin{aligned}
&= E(V)E(X). \\
\mathrm{Var}(V \oplus X) &= \int_0^1 \left(E(v \oplus X)^2 - (E(v \oplus X))^2\right) dG(v) \\
&= \int_0^1 \left(\sum_{z=0}^{X} E\left((v \oplus X)^2 / v \oplus X = z\right) f(z) - (vE(X))^2 \right) dG(v) \\
&= \int_0^1 \left(E\left(\sum_{z=0}^{X} (z^2) f(z) \right) - (vE(X))^2 \right) dG(v) \\
&= \int_0^1 \left(E\left(E(Z^2)\right) - (vE(X))^2\right) dG(v) \\
&= \int_0^1 \left(E\left(X(X-1)v^2 + Xv\right) - (vE(X))^2\right) dG(v) \\
&= \int_0^1 \left(v^2 \mathrm{Var}(X) + v(1-v)E(X)\right) dG(v) \\
&= E\left(V^2\right)\mathrm{Var}(X) + E\left(V(1-V)\right)E(X). \\
E(X_n) &= E\left(V_n \oplus X_{n-1} + \varepsilon_n\right) \\
&= E(V)E(X_{n-1}) + E(\varepsilon_n). \\
\mathrm{Var}(X_n) &= E\left(V^2\right)\mathrm{Var}(X_{n-1}) + E\left(V(1-V)\right)E(X_{n-1}) + \mathrm{Var}(\varepsilon_n). \\
E\left(X_{n-1}X_n\right) &= E\left(X_{n-1}(V_n \oplus X_{n-1} + \varepsilon_n)\right)
\end{aligned}
$$

$$= \int_0^1 E\left(X_{n-1}\left(v \oplus X_{n-1}\right) + X_{n-1}\varepsilon_n\right)dG(v)$$

$$= \int_0^1 \left(E\left(X_{n-1}\left(v \oplus X_{n-1}\right)\right) + E\left(X_{n-1}\varepsilon_n\right)\right)dG(v).$$

$$\begin{aligned}
\mathrm{Cov}\left(X_{n-1}, X_n\right) &= \int_0^1 (E\left(X_{n-1}\left(v \oplus X_{n-1}\right)\right) - E(X_{n-1})E\left(v \oplus X_{n-1}\right) + \\
&\quad E\left(X_{n-1}\varepsilon_n\right) - E\left(X_{n-1}\right)E\left(\varepsilon_n\right) + \\
&\quad E(X_{n-1})E\left(v \oplus X_{n-1}\right) + E\left(X_{n-1}\right)E\left(\varepsilon_n\right) - \\
&\quad E(X_{n-1})E(X_n))dG(v) \\
&= \int_0^1 \left(\mathrm{Cov}\left(X_{n-1}, v \oplus X_{n-1}\right) + \mathrm{Cov}\left(X_{n-1}, \varepsilon_n\right)\right)dG(v) \\
&= \int_0^1 v\mathrm{Var}\left(X_{n-1}\right)dG(v).
\end{aligned}$$

That is,

$$\mathrm{Cov}\left(X_{n-1}, X_n\right) = E(V)\mathrm{Var}\left(X_{n-1}\right).$$

Hence,

$$\mathrm{Corr}(X_{n-1}, X_n) = \rho_1 = E(V).$$

Proceeding like this, we get

$$\mathrm{Corr}(X_{n-1}, X_n) = \rho_j = \left(E(V)\right)^j.$$

The joint probability generating function of $\left(X_{n-1}, X_n\right)$ is

$$P_{X_{n-1},X_n}(z_1,z_2) = \int_v E\left(\left(z_1^{X_{n-1}}\right)\left(z_2^{v\oplus X_{n-1}+\varepsilon_n}\right)\right)dG(v)$$

$$= \int_v E\left(\left(z_2^{\varepsilon_n}\right)\left(z_1^{X_{n-1}} z_2^{v\oplus X_{n-1}}\right)\right)dG(v)$$

$$= \int_v E\left(\left(z_2^{\varepsilon_n}\right)\left(z_1^{X_{n-1}}(1-v+vz)^{X_{n-1}}\right)\right)dG(v)$$

$$= P_{\varepsilon_n}(z_2)\int_v P_{X_{n-1}}(z_1(1-v+vz_2))\,dG(v).$$

Next we consider the generalized integer-valued first order autoregressive model $\{X_n, n=0,\pm1,\pm2,\ldots\}$ defined by

$$X_n = V_n \oplus (X_{n-1}+\varepsilon_n),\ n=1,2,\ldots \tag{5.1.2}$$

where $\{V_n\}$ is a sequence of independent and identically distributed random variables defined on (0, 1) and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed non-negative integer valued random variables.

$$E(X_n) = E(V_n \oplus (X_{n-1}+\varepsilon_n))$$

$$= E(V_n)(E(X_{n-1})+E(\varepsilon_n)).$$

$$\mathrm{Var}(X_n) = E\left(V_n^2\right)\mathrm{Var}(X_{n-1}) + E\left(V_n(1-V_n)\right)E(X_{n-1})$$

$$+E\left(V_n^2\right)\mathrm{Var}(\varepsilon_n) + E\left(V_n(1-V_n)\right)E(\varepsilon_n).$$

$$E(X_{n-1}X_n) = E(X_{n-1}(V_n \oplus X_{n-1} + V_n \oplus \varepsilon_n))$$

$$= \int_0^1 E\left(X_{n-1}\left(v \oplus X_{n-1}\right) + X_{n-1}\left(v \oplus \varepsilon_n\right)\right) dG(v)$$

$$= \int_0^1 \left(E\left(X_{n-1}\left(v \oplus X_{n-1}\right)\right) + E\left(X_{n-1}\left(v \oplus \varepsilon_n\right)\right)\right) dG(v).$$

$$\begin{aligned}
\operatorname{Cov}\left(X_{n-1}, X_n\right) = & \int_0^1 (E\left(X_{n-1}\left(v \oplus X_{n-1}\right)\right) - E(X_{n-1})E\left(v \oplus X_{n-1}\right) + \\
& E\left(X_{n-1}\left(v \oplus \varepsilon_n\right)\right) - E\left(X_{n-1}\right)E\left(v \oplus \varepsilon_n\right) \\
& E(X_{n-1})E\left(v \oplus X_{n-1}\right) + E\left(X_{n-1}\right)E\left(v \oplus \varepsilon_n\right) - \\
& E(X_{n-1})E(X_n))dG(v) \\
= & \int_0^1 \left(\operatorname{Cov}\left(X_{n-1}, v \oplus X_{n-1}\right) + \operatorname{Cov}\left(X_{n-1}, v \oplus \varepsilon_n\right)\right) dG(v). \\
= & E(V_n)\operatorname{Var}\left(X_{n-1}\right).
\end{aligned}$$

The joint probability generating function of $\left(X_{n-1}, X_n\right)$ is

$$\begin{aligned}
P_{X_{n-1}, X_n}(z_1, z_2) = & \int_v E\left(\left(z_1^{X_{n-1}}\right)\left(z_2^{v \oplus X_{n-1} + v \oplus \varepsilon_n}\right)\right) dG(v) \\
= & \int_v E\left(\left(z_2^{v \oplus \varepsilon_n}\right)\left(z_1^{X_{n-1}} z_2^{v \oplus X_{n-1}}\right)\right) dG(v) \\
= & \int_v E\left(\left((1 - v + vz_2)^{\varepsilon_n}\right)\left(z_1^{X_{n-1}} (1 - v + vz_2)^{X_{n-1}}\right)\right) dG(v)
\end{aligned}$$

$$= \int_v P_{\varepsilon_n}(1-v+vz_2)P_{X_{n-1}}(z_1(1-v+vz_2))\,dG(v).$$

In Section 2, we introduce and study a first order integer valued autoregressive model with discrete Linnik marginals. A first order integer valued autoregressive multivariate discrete Linnik model is developed. First order integer valued autoregressive moving average mixed discrete Linnik process is introduced. In Section 3, a first order integer valued autoregressive process with discrete Mittag-Leffler marginals is developed and extended it to define a first order integer valued autoregressive moving average discrete Mittag-Leffler process. A first order integer valued autoregressive multivariate discrete Mittag-Leffler process is introduced.

5.2. First order autoregressive discrete Linnik process

Consider the $DL(\sigma,\alpha,\beta)$ random variable having probability generating function

$P(z) = \left[\dfrac{1}{1+\sigma(1-z)^{\alpha}}\right]^{\beta+1}, \quad |z|\le 1, 0<\alpha\le 1, \quad \sigma>0, \beta\ge 0$ in (1.8.1) and

$DML(\sigma,\alpha)$ random variable having probability generating function

$P(z) = \dfrac{1}{1+\sigma(1-z)^{\alpha}}, \quad |z|\le 1, \; 0<\alpha\le 1, \; \sigma>0$ in (1.5.1).

Theorem 5.2.1.

Let $\quad X_n = V_n \oplus X_{n-1} + \varepsilon_n \qquad (5.2.1)$

where $\{V_n\}$ is a sequence of independent and identically distributed random variables with probability density function

$$f_{V_n}(v) = \alpha\beta\, v^{\alpha\beta-1}, 0 < v < 1, 0 < \alpha \le 1, \beta > 0$$

and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed non negative integer valued random variables independent of $\{V_n\}$. Suppose $\{X_n\}$ is stationary. Then $X_n \underline{\underline{d}}\ DL(\sigma,\alpha,\beta)$ if and only if $\varepsilon_n \underline{\underline{d}}\ DML(\sigma,\alpha)$.

Proof:

Denoting the probability generating function of X_n and ε_n by $P_{X_n}(z)$ and $P_{\varepsilon_n}(z)$ respectively, (5.2.1) in terms of probability generating function is

$$P_{X_n}(z) = P_{\varepsilon_n}(z)\int_0^1 P_{X_{n-1}}(1-v+vz) f_{V_n}(v)dv.$$

Writing $h(z) = P(1-z)$, we get

$$h_{X_n}(z) = h_{\varepsilon_n}(z)\int_0^1 h_{X_{n-1}}(vz)\alpha\beta v^{\alpha\beta-1}dv. \qquad (5.2.2)$$

The result now follows using the arguments similar to those in Theorem 2.2.1. □

Theorem 5.2.2.

Let $X_0 = DL(\sigma,\alpha,\beta)$ and

for n= 1,2,3,... $X_n = V_n \oplus X_{n-1} + \varepsilon_n$

where $\{V_n\}$ is a sequence of independent and identically distributed random variables with probability density function.

$$f_{V_n}(v) = \alpha\beta\, v^{\alpha\beta-1}, 0 < v < 1, 0 < \alpha \le 1, \beta > 0$$

and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables having $DML(\sigma, \alpha)$ distribution, independent of $\{V_n\}$. Then, the process $\{X_n\}$ is stationary with $DL(\sigma, \alpha, \beta)$ marginals.

Proof:

Proof follows using the arguments similar to those in Theorem 2.2.2. □

Based on the above Theorem, we define the first autoregressive discrete Linnik process $\{X_n\}$, n = 0, 1, 2, .. as follows:

$$X_0 = DL(\sigma, \alpha, \beta) \quad \text{and for} \quad n = 1, 2, 3, \ldots$$

$$X_n = V_n \oplus X_{n-1} + \varepsilon_n$$

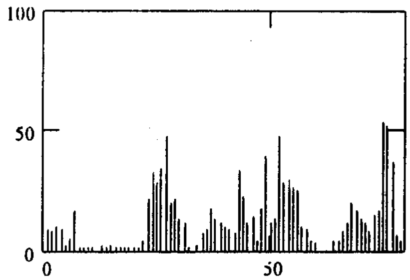
where $\{V_n\}$ is a sequence of independent and identically distributed random variables with probability density function

$$f_{V_n}(v) = \alpha\beta\, v^{\alpha\beta-1}, 0 < v < 1, 0 < \alpha \le 1, \beta > 0$$

and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables having $DML(\sigma, \alpha)$ distribution.

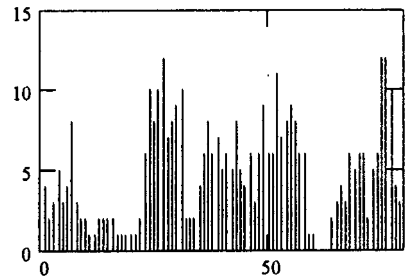
5.2.1. Properties of the first order autoregressive discrete Linnik process

The simulated sample path behavior of the first order autoregressive discrete Linnik process is presented in Figure 5.2.1.



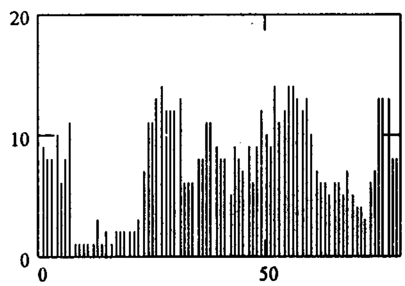
$\alpha = 0.6 \;\; \beta = 3 \;\; \sigma = 1.2$

Figure 5.2.1a.



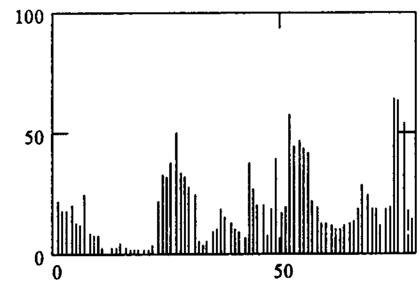
$\alpha = 0.9 \;\; \beta = 3 \;\; \sigma = 1.2$

Figure 5.2.1b.



$\alpha = 0.9 \;\; \beta = 5 \;\; \sigma = 1.2$

Figure 5.2.1c.



$\alpha = 0.6 \;\; \beta = 5 \;\; \sigma = 1.2$

Figure 5.2.1d.

Figure 5.2.1.

Sample path behavior of the first order autoregressive discrete Linnik process

The joint probability generating function of (X_{n-1}, X_n) is

$$P_{X_{n-1},X_n}(z_1, z_2) = \int_V E\left(\left(z_1^{X_{n-1}}\right)\left(z_2^{v \oplus X_{n-1} + \varepsilon_n}\right)\right) dG(v)$$

$$= P_{\varepsilon_n}(z_2) \int_V P_{X_{n-1}}(z_1(1 - v + vz_2))\, dG(v)$$

$$= \frac{1}{1 + \sigma(1 - z_2)^{\alpha}} \int_0^1 \left(\frac{1}{1 + \sigma\left((1 - z_1(1 - v + vz_2))^{\alpha}\right)}\right)^{\beta+1} \alpha\beta v^{\alpha\beta - 1} dv.$$

5.2.2. First order autoregressive multivariate discrete Linnik process

Consider the multivariate $DL(\underline{\sigma}, \alpha, \beta)$ distribution with probability generating function

$$P_{\underline{X}}(\underline{z}) = \left[\frac{1}{1+\sigma_1(1-z_1)^{\alpha}+\sigma_2(1-z_2)^{\alpha}+\ldots+\sigma_k(1-z_k)^{\alpha}}\right]^{\beta+1}, \tag{5.2.3}$$

where $\underline{z} = (z_1, z_2, \ldots, z_k)'$, $0 < \alpha \leq 1$, $\beta \geq 0$, $\sigma_j > 0$, $|z_j| \leq 1$, $j = 1, 2, \ldots, k$.

Theorem 5.2.3.

Let the process $\{\underline{X}_n\}$ be defined as

$$\underline{X}_n = V_n \oplus \underline{X}_{n-1} + \underline{\varepsilon}_n, \qquad n = 1, 2, \ldots$$

where $\{V_n\}$ is a sequence of independent and identically distributed random variables having distribution function

$$F_{V_n}(v) = v^{\alpha\beta}, \quad 0 < v < 1, \quad 0 < \alpha \leq 1, \quad \beta > 0,$$

$\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors. Suppose the process $\{\underline{X}_n\}$ is stationary. Then $\underline{X}_n \stackrel{d}{=}$ multivariate $DL(\underline{\sigma}, \alpha, \beta)$ if and only if $\underline{\varepsilon}_n \stackrel{d}{=}$ multivariate $DML(\underline{\sigma}, \alpha)$.

Proof follows from the arguments used in Theorem 2.2.3. □

Based on the above Theorem, we define first autoregressive multivariate discrete Linnik process $\{\underline{X}_n, n=0,1,2,\ldots\}$ as follows:

$$\underline{X}_0 = DL(\underline{\sigma}, \alpha, \beta) \quad \text{and for} \quad n = 1,2,3,\ldots$$

$$\underline{X}_n = V_n \oplus \underline{X}_{n-1} + \underline{\varepsilon}_n$$

where $\{V_n\}$ is a sequence of independent and identically distributed random variables with probability density function

$$f_{V_n}(v) = \alpha\beta\, v^{\alpha\beta-1}, 0 < v < 1, 0 < \alpha \le 1, \beta > 0$$

and $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors having multivariate DML($\underline{\sigma}, \alpha$) distribution, independent of $\{V_n\}$.

5.2.3. First order autoregressive moving average mixed discrete Linnik model

Consider the process $\{X_n\}$ defined by

$$\begin{aligned} X_n &= K_n\varepsilon_n + V_n \oplus Y_{n-1} \\ Y_n &= U_n \oplus Y_{n-1} + Z_n \end{aligned} \tag{5.2.4}$$

where $\{\varepsilon_n\}$ and $\{Z_n\}$ are two independent sequences of independent and identically distributed DML(σ, α) random variables, $\{K_n\}$ is sequence of independent and identically distributed Bernoulli random variables such that $P(K_n = 0) = \theta = 1 - P(K_n = 1)$, $\{V_n\}$ and $\{U_n\}$ are two independent sequences of independent and identically distributed random variables defined on the interval $(0,1)$ with distribution function

$F_{V_n}(v) = v^{\alpha\beta}$, $F_{U_n}(u) = u^{\alpha\beta}$, $0 < v, u < 1$, $0 < \alpha \le 1$, $\beta > 0$ and $Y_n \stackrel{d}{=} DL(\sigma, \alpha, \beta)$.

We have from (5.2.4),

$$\begin{aligned} P_{X_n}(z) &= E\left(z^{(K_n\varepsilon_n + V_n \oplus Y_{n-1})}\right) \\ &= \theta E\left(z^{V_n \oplus Y_{n-1}}\right) + (1-\theta) E\left(z^{\varepsilon_n + V_n \oplus Y_{n-1}}\right). \end{aligned}$$

That is,

$$P_{X_n}(z) = \left[\theta + (1-\theta)P_{\varepsilon_n}(z)\right]\int_0^1 P_{Y_{n-1}}(1-v+vz)\alpha\beta v^{\alpha\beta-1}dv.$$

Taking $P(1-z) = h(z)$,

$$h_{X_n}(z) = \left[\theta + (1-\theta)h_{\varepsilon_n}(z)\right]\int_0^1 h_{Y_{n-1}}(vz)\alpha\beta v^{\alpha\beta-1}dv.$$

Put zv = t. Then

$$h_{X_n}(z) = \left[\theta + (1-\theta)h_{\varepsilon_n}(z)\right]\int_0^1 \frac{h_{Y_{n-1}}(t)\alpha\beta\, t^{\alpha\beta-1}}{z^{\alpha\beta}}dt.$$

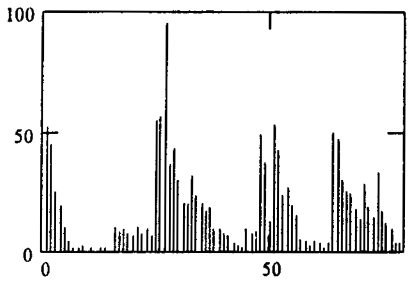
$$= \frac{1}{z^{\alpha\beta}}\left[\theta + (1-\theta)\frac{1}{1+\sigma z^{\alpha}}\right]\int_0^1 h_{Y_{n-1}}(t)\alpha\beta\, t^{\alpha\beta-1}dt.$$

$$= \theta\frac{1}{(1+\sigma z^{\alpha})^{\beta}} + (1-\theta)\frac{1}{(1+\sigma z^{\alpha})^{\beta+1}}.$$

$$P_{X_n}(z) = \theta\left[\frac{1}{1+\sigma(1-z)^{\alpha}}\right]^{\beta} + (1-\theta)\left[\frac{1}{1+\sigma(1-z)^{\alpha}}\right]^{\beta+1}.$$

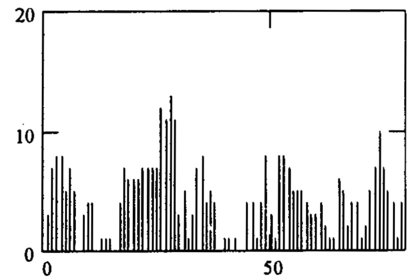
Thus $\{X_n\}$ has mixed $DL(\sigma,\alpha,\beta)$ distribution.

Simulated sample path behavior of the first order autoregressive moving average mixed discrete Linnik process is presented in Figure 5.2.2.



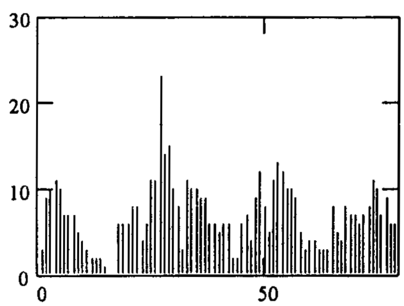
$\alpha = 0.6 \quad \beta = 3 \quad \sigma = 1.2 \quad \theta = 0.5$

Figure 5.2.2a.



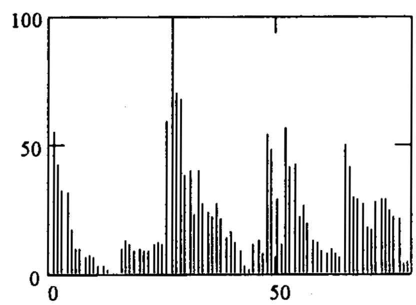
$\alpha = 0.9 \quad \beta = 3 \quad \sigma = 1.2 \quad \theta = 0.5$

Figure 5.2.2b.



$\alpha = 0.9 \quad \beta = 5 \quad \sigma = 1.2 \quad \theta = 0.5$

Figure 5.2.2c.



$\alpha = 0.6 \quad \beta = 5 \quad \sigma = 1.2 \quad \theta = 0.5$

Figure 5.2.2d.

Figure 5.2.2.

Sample path behavior of the first order autoregressive moving average mixed discrete Linnik Process

5.3. First order autoregressive discrete Mittag-Leffler process

Consider the generalized integer valued first order autoregressive process defined in (5.1.2).

Let $\{X_n, n = 0, \pm1, \pm2, \ldots\}$ is defined as

$$X_n \;=\; V_n \oplus (X_{n-1} + \varepsilon_n) \tag{5.3.1}$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed non negative integer valued random variables and $\{V_n\}$ is a sequence of independent and identically distributed random variable defined in (0, 1). Note that when V_n is a degenerate random variable, which degenerate at α, the generalized integer valued first order autoregressive model reduces to first order integer valued autoregressive model. We can see that it is more appropriate to take V_n as a non-degenerate random variable.

Theorem 5.3.1.

Let the process $\{X_n\}$ be defined as

$$X_0 = \varepsilon_1 \quad \text{and}$$

$$\text{for } n = 1, 2, \ldots \qquad X_n = V_n \oplus (X_{n-1} + \varepsilon_n) \tag{5.3.2}$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed non negative integer valued random variables and $\{V_n\}$ is a sequence of independent and identically distributed random variable defined in (0, 1) such that $\{V_n\}$ has power function distribution $F_{V_n}(v) = v^{\alpha}$, $0 < \alpha \le 1$, $0 < v < 1$. Then the process $\{X_n\}$ is stationary if and only if $\varepsilon_n \stackrel{d}{=} DML(\sigma, \alpha)$.

Proof:

Let $P_{X_n}(z)$ and $P_{\varepsilon_n}(z)$ be the probability generating function of X_n and ε_n respectively. We have,

$$E\left(z^{X_n}\right) = E\left(z^{(V_n \oplus X_{n-1} + V_n \oplus \varepsilon_n)}\right)$$

$$= \int_0^1 P_{X_{n-1}}(1-v+vz)\, P_{\varepsilon_n}(1-v+vz)\, f(v)dv.$$

That is,

$$P_{X_n}(z) = \int_0^1 P_{X_{n-1}}(1-v+vz)\, P_{\varepsilon_n}(1-v+vz)\alpha v^{\alpha-1}dv. \qquad (5.3.3)$$

For n = 1, we get

$$P_{X_1}(z) = \int_0^1 P_{X_0}(1-v+vz)\, P_{\varepsilon_1}(1-v+vz)\alpha v^{\alpha-1}dv$$

If $\{ X_n \}$ is stationary, then

$$P_X(z) = \int_0^1 P_X^2(1-v+vz)\,\alpha v^{\alpha-1}dv.$$

Let $\quad P_X(1-z) = h_X(z).$

Then $\quad h_X(z) = \int_0^1 h_X^2(vz)\alpha v^{\alpha-1}dv.$

Proof now follows using the arguments similar to those in Theorem 2.3.1. □

Based on Theorem 5.3.1, we define the first order autoregressive discrete Mittag-Leffler process as follows

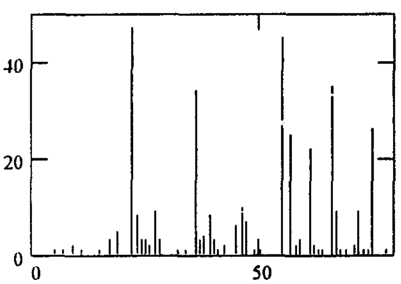
$$X_0 = \varepsilon_1 \quad \text{and}$$

for n = 1, 2, ... $\quad X_n = V_n \oplus (X_{n-1} + \varepsilon_n)$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed random variables with V_n having distribution function $F_{V_n}(v) = v^{\alpha}$, $0 < v < 1, 0 < \alpha \leq 1$ and ε_1 distributed as DML(σ, α).

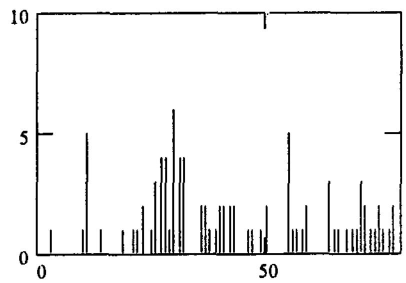
5.3.1. Properties of the first order autoregressive discrete Mittag-Leffler process

The simulated sample path behavior of the first order autoregressive discrete Mittag-Leffler process is given in Figure 5.3.1.



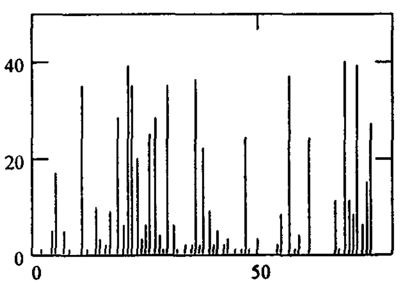
$\alpha = 0.4 \quad \sigma = 1.2$

Figure 5.3.1a.



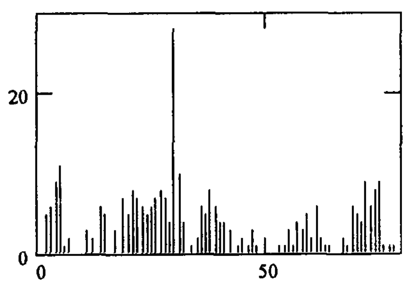
$\alpha = 0.9 \quad \sigma = 1.2$

Figure 5.3.1b.



$\alpha = 0.4 \quad \sigma = 2.2$

Figure 5.3.1c.



$\alpha = 0.9 \quad \sigma = 2.2$

Figure 5.3.1d.

Figure 5.3.1.

Sample path behavior of the first order autoregressive discrete Mittag-Leffler process

The joint probability generating function of (X_{n-1}, X_n) is

$$P_{X_{n-1},X_n}(z_1, z_2) = \int_V E\left(\left(z_1^{X_{n-1}}\right)\left(z_2^{v \oplus X_{n-1} + v \oplus \varepsilon_n}\right)\right) dG(v)$$

$$= \int_V P_{\varepsilon_n}(1 - v + vz_2) P_{X_{n-1}}(z_1(1 - v + vz_2))\, dG(v).$$

$$= \int_0^1 \frac{1}{1 + \sigma v^\alpha (1 - z_2)^\alpha} \left(\frac{1}{1 + \sigma(1 - z_1(1 - v + vz_2))^\alpha}\right)^{\beta+1} \alpha v^{\alpha-1} dv.$$

5.3.2. First order autoregressive moving average discrete Mittag-Leffler process

The first order autoregressive moving average discrete Mittag-Leffler process is built using a sequence $\{\varepsilon_n\}$ of independent and identically distributed discrete Mittag-Leffler random variables in the following manner:

$$X_n = W_n \oplus (\varepsilon_n + \varepsilon_{n-1}) \tag{5.3.4}$$

where $\{W_n\}$ is a sequence of independent and identically distributed random variables with distribution functions $F_{W_n}(w) = w^\alpha, 0 < \alpha \le 1,\ 0 < w < 1$.

Combining (5.3.2) and (5.3.4) we define first order autoregressive moving average discrete Mittag-Leffler process $\{X_n\}$ as follows:

$$\begin{aligned} X_n &= W_n \oplus (\varepsilon_n + Y_{n-1}) \\ Y_n &= V_n \oplus (Y_{n-1} + Z_n) \end{aligned} \tag{5.3.5}$$

where $\{Z_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed non negative integer valued random variables and $\{W_n\}$, $\{V_n\}$ are also two independent sequences of independent and identically distributed random variables with distribution function $F_{W_n}(w) = w^{\alpha}, F_{V_n}(v) = v^{\alpha}$, $0 < w, v < 1, 0 < \alpha \leq 1$.

Theorem 5.3.2.

Let the process $\{X_n\}$ be defined by

$$\begin{aligned} X_n &= W_n \oplus (\varepsilon_n + Y_{n-1}) \\ Y_n &= V_n \oplus (Y_{n-1} + Z_n) \end{aligned}$$

where $\{Z_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of independent and identically distributed non negative integer valued random variables and $\{W_n\}$, $\{V_n\}$ are also two independent sequences of independent and identically distributed random variables with distribution function $F_{W_n}(w) = w^{\alpha}, F_{V_n}(v) = v^{\alpha}$, $0 < w, v < 1$, $0 < \alpha \leq 1$ with $\varepsilon_1 \overset{d}{=} Y_0 \overset{d}{=} Z_1$. Then the process $\{X_n\}$ is stationary if and only if $\varepsilon_1 \overset{d}{=} DML(\sigma, \alpha)$.

Proof:

Proof follows easily using arguments similar those in Theorem 5.3.1. □

5.3.3. First order autoregressive multivariate discrete Mittag-Leffler process

Consider the multivariate discrete Mittag-Leffler distribution with probability generating function

$$P_{\underline{X}}(z_1, z_2, \ldots, z_n) = \frac{1}{1 + \sigma_1(1-z_1)^{\alpha} + \sigma_2(1-z_2)^{\alpha} + \ldots + \sigma_k(1-z_k)^{\alpha}}, \tag{5.3.6}$$

$$|z| \le 1,\ \sigma_j > 0, j = 1,2, \quad ..., k\ ; \quad 0 < \alpha \le 1.$$

Define $\quad h_{\underline{X}}(z_1, z_2, ..., z_k) \;=\; \dfrac{1}{1+\sigma_1 z_1^{\alpha} + \sigma_2 z_2^{\alpha} + ... + \sigma_k z_k^{\alpha}}$, where

$h_{\underline{X}}(\underline{z}) \;=\; P_{\underline{X}}(\underline{1}-\underline{z}),\ \underline{z} = (z_1, z_2, ... z_k), |z_j| \le 1, j = 1,2,..k$ and $\underline{1} = (1,1,...,1)'$.

Consider the process $\{\underline{X}_n\}$ defined by

$$\begin{aligned} \underline{X}_0 &= \underline{\varepsilon}_1 \quad \text{and} \\ \underline{X}_n &= V_n \oplus (\underline{X}_{n-1} + \underline{\varepsilon}_n), \quad n \ge 1 \end{aligned} \tag{5.3.7}$$

where $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors defined on Z_k^+, $\{V_n\}$ is a sequence of independent and identically distributed random variables defined on (0, 1) such that $\{V_n\}$ has power function distribution

$F_{V_n}(v) \;=\; v^{\alpha}, 0 < v < 1, 0 < \alpha \le 1.$

Representing the probability generating function of $\{\underline{X}_n\}$ and $\{\underline{\varepsilon}_n\}$ by $P_{\underline{X}_n}(\underline{z})$ and $P_{\underline{\varepsilon}_n}(\underline{z})$ respectively, equation (5.3.7) in terms of probability generating function is

$$P_{\underline{X}_n}(\underline{z}) \;=\; \int_0^1 P_{\underline{X}_{n-1}}((1-v)\underline{1} + v\underline{z}) P_{\underline{\varepsilon}_n}((1-v)\underline{1} + v\underline{z}) \alpha v^{\alpha-1} dv. \tag{5.3.8}$$

As in Theorem 5.3.1, we get the following result.

Theorem 5.3.3.

Let the process $\{\underline{X}_n\}$ be defined by

$$\left.\begin{aligned} \underline{X}_0 &= \underline{\varepsilon}_1 \quad \text{and} \\ \underline{X}_n &= V_n \oplus (\underline{X}_{n-1} + \underline{\varepsilon}_n) \end{aligned}\right., \quad n \ge 1$$

where $\{\underline{\varepsilon}_n\}$ is a sequence of independent and identically distributed random vectors defined on Z_k^+, $\{V_n\}$ is a sequence of independent and identically distributed random variables defined on (0, 1) such that V_n has power function distribution $F_{V_n}(v) = v^{\alpha}, \quad 0 < v < 1, 0 < \alpha \leq 1$. Then the process $\{\underline{X}_n\}$ is stationary if and only if $\underline{\varepsilon}_1$ is multivariate discrete Mittag-Leffler. $\square$

STUDY ON AUTOREGRESSIVE MODELS

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By

THOMAS MATHEW

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

KERALA, INDIA

Pin. 673 635

MAY, 2004

CHAPTER VI
APPLICATIONS

6.1. Introduction

Forecasting is very important in many type of organizations since prediction of future events must be incorporated in to the decision making process. The government of a country must be able to forecast such things as air quality, water quality, unemployment rate, inflation rate and welfare payments in order to formulate its policy. University must be able to forecast students enrolment in order to make decisions concerning the faculty resources and housing availability. Business firms in particular require forecast of many events and conditions in the phase of their operation. In marketing departments, reliable forecast of demand must be available so that sales strategies can be planned. In finance, interest rate must be predicted so that new capital acquisitions must be planned and financed. In personal management, forecasting of number of workers needed in different job categories are required in order to plan job recruiting and training programmes. In production scheduling prediction of demand for each product line are needed. Production process control requires forecast of future behavior of the process. Strategic management requires forecast of general economic conditions such as price and cost changes, technological change, market growth, etc.

In forecasting events that will occur in future, a forecaster must relay on information concerning events that have occurred in the past. That is, in order to prepare a forecast, the forecaster must analyze past data and must base the forecast on the result of this analysis. Forecasters use past data in the following way: First the forecaster analyses the data in order to identify a pattern that can be used to describe it. Then this pattern is extrapolated or extended in to future in order to prepare a forecast. A forecasting technique cannot be expected to give good predictions unless this assumption is valid. If the data pattern that has been identified does not persist in future, the forecasting technique being used is likely to produce inaccurate predictions. A forecaster should not be surprised by such situations, but must try to anticipate when such a change in pattern will take place so that appropriate changes in the forecasting system can be made before the prediction become too inaccurate. Lawrance and Lewis (1985) used nonlinear time series models in exponential variables for modeling wind velocity data. Sim (1987) fitted a first order autoregressive moving average mixed gamma process to monthly stream flows of the Perak River in Malaysia. Sim (1994) developed a model building approach that consists of model identification, estimation, diagnostic checking and forecasting and used the same for modeling some non-normal time series.

The time series data are often examined in hopes of discovering a historical pattern that can be exploited in the preparation of a forecast. It is often convenient to think of a time series consisting of several components, trend, cyclical variation, seasonal variation and irregular fluctuations. Trend refers to upward or down ward movements that characterize a time series. Thus trend reflects the long run growth or decline in time series. Cycle refers to recurring up and down movements around trend levels. These fluctuations can have a duration of anywhere from two to ten years or even longer measured from peak to peak or trough-to-trough. Cyclical fluctuations especially in agricultural field might reflect changes in weather cycle, which repeat yearly. Irregular fluctuations are erotic movements in a time series that follows no recognizable regular pattern. Such movements represent what is left over in the time sires after eliminating trend, cyclical and seasonal variations. It may also be caused by the errors on the part of time series analyst. If the data considered is a time series data the time ordered fluctuations might be autocorrelated. In such cases we shall remedy the problem by modeling the autocorrelated error terms using autoregressive processes.

In Section 2, the total industrial production index in USA is considered and it is fitted to random coefficient first order autoregressive symmetric generalized Linnik process developed in Chapter II and forecasting is done for the next 10 months.

6.2. Industrial production index in USA

We consider total industrial production index per month, in USA (Data is taken from the web sight www.economagic.com). The data consist of 1022 observations from January 1919 to March 2004. Time series plot of the data ($\{A_n\}$) is provided in Figure 6.2.1 below. From the graph it can be seen that the production per month increases very slowly then a little fast and then faster. Thus we can notice an upward trend in the time series data.

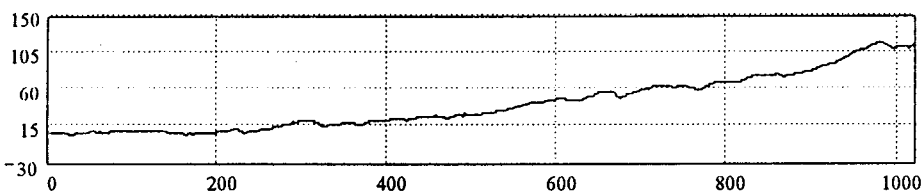

Figure 6.2.1.
Time series of the total industrial production per month in USA

The non stationary historical series is made stationary by taking the first order difference of the historical data $\{A_n\}$ under consideration and denote it by $\{B_n\}$. A time series plot of the first order difference of the historical data is given in Figure 6.2.2.

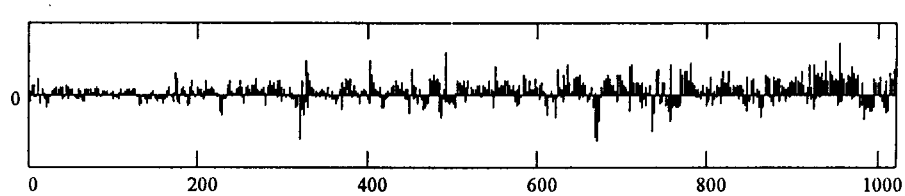

Figure 6.2.2.
Time series of the first order difference of the total industrial production per month in USA

The average of the data is found to be 0.106 and the standard deviation 0.402. By subtracting the mean and dividing by standard deviation, we standardize the data. The plot of the standardized data ($\{X_n\}$) is given in Figure 6.2.3.

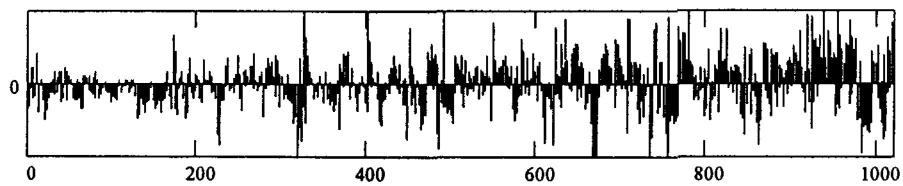

Figure 6.2.3.

Time series plot of the standardized data

We can see that at the initial stage the fluctuations are small but as time increases larger fluctuations are observed. It indicates the growth of the series and consecutively the growth in fluctuations. One more thing that can be observed is the occasional large fluctuations and often very small fluctuations, which indicate heavy tail characteristic. Therefore we can conclude that the data can be modeled by a heavy tailed distribution. The autocorrelation structure of the sequence $\{X_n\}$ indicates that an autoregressive process can model the data. Next we shall examine whether the data can be modeled by a heavy tailed distribution. For that we construct the histogram of the data and is given in Figure 6.2.4a and the cumulative frequency curve in Figure 6.2.4b. Now we consider the random coefficient first order autoregressive generalized Linnik process in Chapter II.

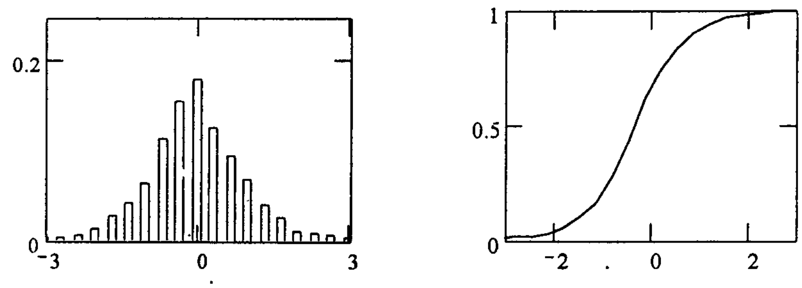

Figure 6.2.4a. **Figure 6.2.4b.**

Figure 6.2.4.

Histogram and empirical cumulative distribution of the standardized historical data $\{X_n\}$

We propose a method for fitting the process and it is as follows: The value of α, σ and β are estimated from the data using the procedure suggested in Jacques et al. (1999) and is obtained as α = 1.99, σ = 0.65 and β = 0.34. The correctness of the procedure can be examined as follows:

Our symmetric generalized Linnik model is

$$X_0 = SGL(\sigma, \alpha, \beta) \text{ and}$$

$$X_n = V_n X_{n-1} + \varepsilon_n, \; n = 1, 2, 3, \ldots \qquad (6.2.1)$$

where $\{V_n\}$ is a sequence of independent and identically distributed power function random variables with probability density function $f_{V_n}(v) = \alpha\beta v^{\alpha\beta-1}$, $0 < v < 1$, $0 < \alpha \leq 2$ and $\beta > 0$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed symmetric Linnik $SL(\sigma, \alpha)$ random variables independent of V_n.

If we consider the standardized sequence as $\{X_n\}$, then its distribution should be symmetric generalized Linnik. If the distribution of X_n is symmetric generalized Linnik then by Theorem 2.2.1 the distribution of ε_n should be symmetric Linnik. From $\{X_n\}$, $\{\varepsilon_n\}$ can be obtained as $\varepsilon_n = X_n - V_n X_{n-1}$. If we prove that the distribution of ε_n is symmetric Linnik then by Theorem 2.2.1 the distribution of X_n is symmetric generalized Linnik. Therefore for fitting the first order autoregressive symmetric generalized Linnik process it is enough to prove that the distribution of ε_n is symmetric Linnik.

1020 power function random variables with parameter $\alpha\beta$ are generated. Sequences $\{\varepsilon_n\}$ of 1020 random variables are generated using the equation $\varepsilon_n = X_n - V_n X_{n-1}$. A time series plot of the $\{\varepsilon_n\}$ is given in Figure 6.2.5. Histogram of ε_n is constructed in Figure 6.2.6a.

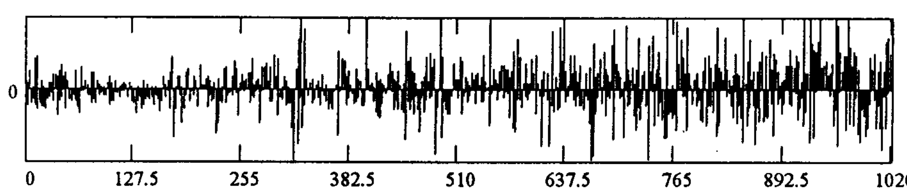

Figure 6.2.5.

Time series plot of the error sequence $\{\varepsilon_n\}$

The parameters of the proposed symmetric Linnik distribution are estimated by the methods suggested in Jacques et al.(2000) and are obtained as $\alpha = 1.99$, $\sigma = 0.65$. The histogram of the $\{\varepsilon_n\}$ is constructed in Figure 6.2.6a with the symmetric Linnik distribution having the estimated parameters is embedded on it. The Q Q Plot, P P Plot and distribution function plots are also presented in Figures 6.2.6b, 6.2.6c and 6.2.6d respectively.

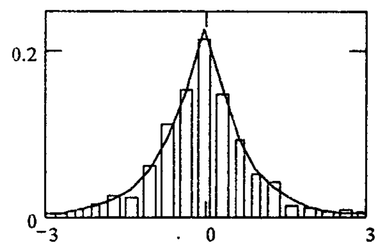

Figure 6.2.6a.

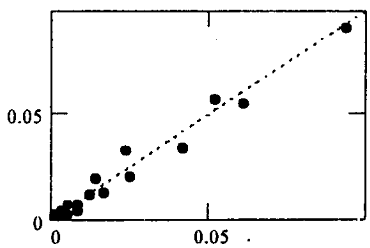

Figure 6.2.6b.

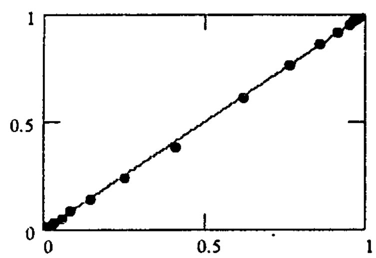

Figure 6.2.6c.

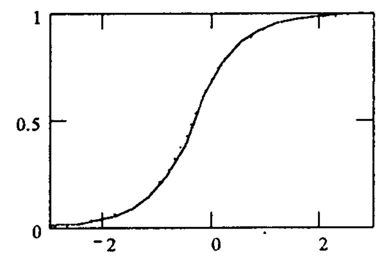

Figure 6.2.6d.

Figure 6.2.6.

Histogram, QQ plot, PP plot and distribution function plot of the error sequence $\{\varepsilon_n\}$

From the figures it is very clear that the error sequence ε_n is distributed as symmetric Linnik. Hence by Theorem 2.2.1, the sequence $\{X_n\}$ is distributed as symmetric generalized Linnik with parameters $\alpha = 1.99$, $\sigma = 0.65$ and $\beta = 0.34$.

By Theorem 2.2.2 if $X_0 = SGL(\sigma, \alpha, \beta)$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $SL(\sigma, \alpha)$ then $X_n = V_n X_{n-1} + \varepsilon_n$ is $SGL(\sigma, \alpha, \beta)$. Using this result we can generate dependent random variables from

SGL(σ, α, β) and verify that the sequence $\{X_n\}$ is SGL(σ, α, β). Histogram of the $\{X_n\}$ is drawn in Figure 6.2.7a with the generated SGL(σ, α, β) distribution embedded on it. The Q Q Plot, P P Plot and distribution function plots are also presented in Figures 6.2.7b, 6.2.7c and 6.2.7d.

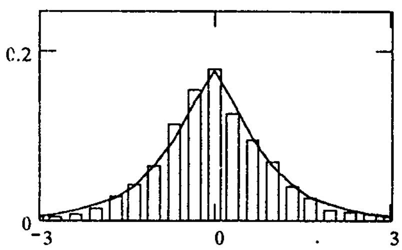

Figure 6.2.7a.

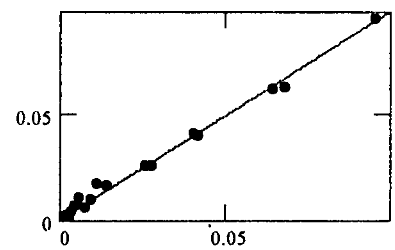

Figure 6.2.7b.

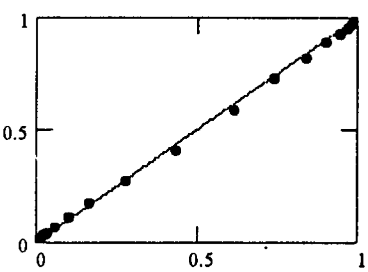

Figure 6.2.7c.

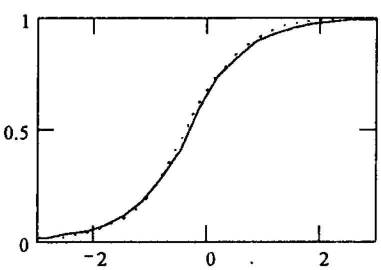

Figure 6.2.7d.

Figure 6.2.7.

Histogram, QQ plot, PP plot and distribution function plot of the sequence $\{X_n\}$

Next we shall verify whether the fitted model generate the historical sequence. For that generate 1022 independent and identically distributed power function random variables $(V_1, V_2, \ldots, V_{1022})$ using the parameter $\alpha\beta = 0.677$. Generate 1022 independent and identically distributed symmetric Linnik random variables $(\varepsilon_1, \varepsilon_2, \ldots,$

ε_{1022}) with parameters $\alpha = 1.99$, $\sigma = 0.65$. Generate a single symmetric generalized Linnik random variable X_0 with parameters $\alpha = 1.99$, $\sigma = 0.65$ and $\beta = 0.34$. Generate the remaining 1021 generalized symmetric Linnik random variables by the equation $X_n = V_n X_{n-1} + \varepsilon_n$. A time series plot of the simulated sequence $\{\varepsilon_n\}$ and $\{X_n\}$ is given in Figure 6.2.8 and Figure 6.2. 9. Remove the standardization of the sequence $\{X_n\}$ by using the formula $Y_n = 0.106 + 0.402 X_n$. The series $\{Y_n\}$ is given in Figure 6.2.10. Generate the trend included sequence $\{Z_n\}$ by the formula, $Z_{i+1} = Z_i + Y_i$, $i = 0, 1, 2, \ldots, 1021$ where Z_0 = the first data of the historical sequence. The generated sequence is plotted in Figure 6.2.11.

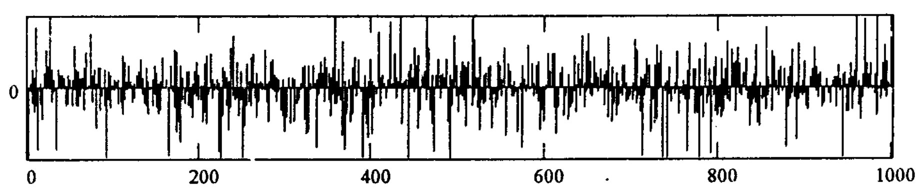

Figure 6.2.8.

Time series plot of the simulated error sequence $\{\varepsilon_n\}$

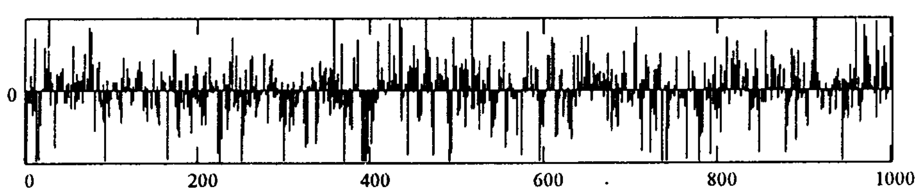

Figure 6.2.9.

Time series plot of the simulated standardized data

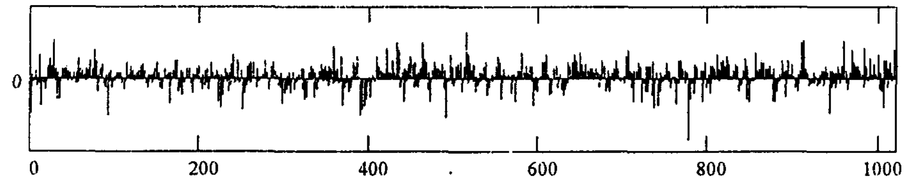

Figure 6.2.10.

Time series of the simulated first order difference of the total industrial production per month $\{Y_n\}$ in USA

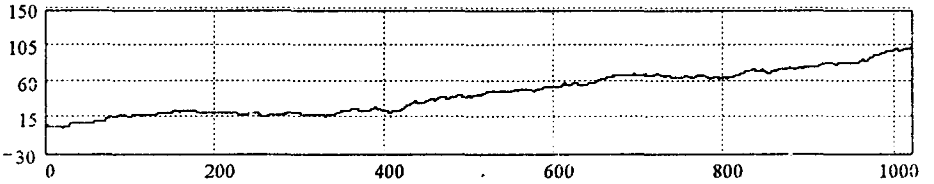

Figure 6.2.11.

Time series of the simulated total industrial production per month $\{Z_n\}$ in USA

Here the forecast can be done very easily. Taking X_0 as the last observation in the historical data, we have generated the next 10 observations using the same technique adopted in simulation and the point forecast is given in Table 6.2.1.

Mar 04	Apr 04	May 04	Jun 04	Jul 04	Aug 04	Sep 04	Oct 04	Nov 04	Dec 04	Jan 05
114.52	114.6	114.48	115.04	115.1	115.54	115.43	115.6	115.64	115.81	116.07

Table 6.2.1.

Point forecast of the next 10 observations.

Interval estimation of the next 10 forecast is done at 5% level of significance (using symmetric generalized Linnik distribution) and is given in Table 6.2.2. A plot of the

5% level of confidence interval is given in Figure 6.2.12 where U represent upper limit, L represent lower limit and A represent point forecast.

Month	Apr 04	May 04	Jun 04	Jul 04	Aug 04	Sep 04	Oct 04	Nov 04	Dec 04	Jan 05
U	115.67	115.55	116.11	116.17	116.61	116.5	116.67	116.71	116.88	117.14
A	114.6	114.48	115.04	115.1	115.54	115.43	115.6	115.64	115.81	116.07
L	113.74	113.62	114.18	114.24	114.68	114.57	114.74	114.78	114.95	115.22

Table 6.2.2.

Interval forecast of the next 10 observations.

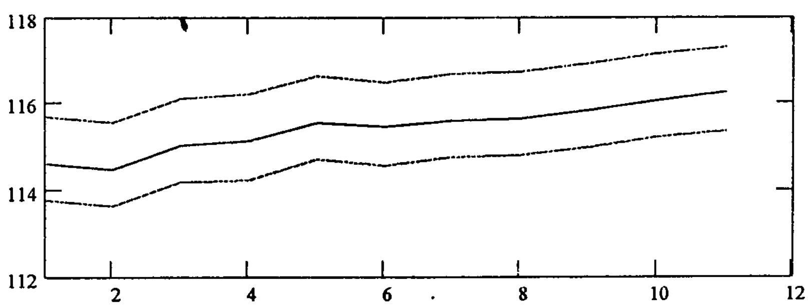

Figure 6.2.12.

5% level of confidence interval for the forecast

STUDY ON AUTOREGRESSIVE MODELS

Thesis submitted

to the University of Calicut

for the degree of

DOCTOR OF PHILOSOPHY

Under the Faculty of Science

By

THOMAS MATHEW

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

KERALA, INDIA

Pin. 673 635

MAY, 2004

APPENDIX

1. Generation of Mittag-Leffler random variable

- Generate a uniform [0, 1] random variable U.
- Generate a standard exponential random variable Z, independent of U.
- Set $\rho \leftarrow \alpha$.
- Set $W \leftarrow \mathrm{Sin}(\pi\rho)\mathrm{Cot}(\pi\rho U) - \mathrm{Cos}(\pi\rho)$.
- Set $Y \leftarrow Z(\sigma W)^{\frac{1}{\alpha}}$.
- RETURN Y.

2. Generation of symmetric Linnik random variable

- Generate a random variable Z from standard Laplace distribution with location 0 and scale 1.
- Generate a uniform [0, 1] random variable U independent of Z.
- Set $\rho \leftarrow \frac{\alpha}{2}$.
- Set $W \leftarrow \mathrm{Sin}(\pi\rho)\mathrm{Cot}(\pi\rho U) - \mathrm{Cos}(\pi\rho)$.
- Set $Y \leftarrow Z(\sigma W)^{\frac{1}{\alpha}}$.
- RETURN Y.

3. Generation of symmetric generalized Linnik random variable

- Set β as integers from 1…n.
- Generate β number of symmetric Linnik random variables, $X_1, X_2, \ldots, X_\beta$.

- Set $Y \leftarrow X_1 + X_2 + \ldots + X_\beta$.
- Return Y.

4. **Generation of discrete Mittag-Leffler random variable**

- Let σ be the scale parameter and α be the index parameter and $p_i = P(X = i)$, where X is the discrete Mittag-Leffler random variable.
- Set $p_0 = \frac{1}{1+\sigma}$.
- Set $p_1 = \frac{\sigma\alpha}{(1+\sigma)^2}$.
- Set $p_n = \frac{\sigma}{1+\sigma}\left[\binom{\alpha}{1}p_{n-1} + \binom{\alpha}{2}p_{n-2} + \binom{\alpha}{3}p_{n-3} + \ldots + (-1)^{n-1}\binom{\alpha}{n}p_0\right]$. (For n = 2, 3, ...)
- Construct the distribution function $F(x)$, $x = 0, 1, 2, \ldots$
- Generate a uniform random variable U.
- The random variable Y is such that $F(Y) < U \le F(Y+1)$.
- RETURN Y.

5. **Generation of discrete Linnik random variable**

- Set β as integers from 1...n.
- Generate β number of discrete Mittag-Leffler random variables, $X_1, X_2, \ldots, X_\beta$.
- Set $Y \leftarrow X_1 + X_2 + \ldots + X_\beta$.
- RETURN Y.

6. **Generation of first order autoregressive symmetric generalized Linnik process**

- Generate a random variable X_0 from symmetric generalized Linnik distribution.
- Generate k independent power function random variables, $V_1, V_2, \ldots, V_k$.
- Generate k independent symmetric Linnik random variables, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$.
- Set $X_n = V_n X_{n-1} + \varepsilon_n$ for n = 1, 2, …,k.
- RETURN X_n.

7. **Generation of first order autoregressive symmetric Linnik process**

- Generate k independent symmetric Linnik random variables, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$.
- Set $X_0 \leftarrow \varepsilon_1$.
- Generate k independent power function random variables, $V_1, V_2, \ldots, V_k$.
- Set $X_n = V_n(X_{n-1} + \varepsilon_n)$, for n = 1, 2, …,k.
- RETURN X_n.

8. **Generation of first order autoregressive Mittag-Leffler process**

- Generate k independent Mittag-Leffler random variables, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$.
- Set $X_0 \leftarrow \varepsilon_1$.
- Generate k independent power function random variables, $V_1, V_2, \ldots, V_k$.
- Set $X_n = V_n(X_{n-1} + \varepsilon_n)$, for n = 1, 2, …,k.

- RETURN X_n.

9. **Generation of first order autoregressive Burr process**

- Generate a random variable X_0 from Burr distribution.
- Generate k independent power function random variables, $V_1, V_2, \ldots, V_k$.
- Generate k independent Pareto random variables, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$.
- Set $X_n = \min(V_n^{-1} X_{n-1}, \varepsilon_n)$ for n = 1, 2, …,k.
- RETURN X_n.

10. **Generation of first order autoregressive Pareto process**

- Generate k independent Pareto random variables, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$.
- Set $X_0 \leftarrow \varepsilon_1$.
- Generate k independent power function random variables, $V_1, V_2, \ldots, V_k$.
- Set $X_n = V_n^{-1} \min(X_{n-1}, \varepsilon_n)$, for n = 1, 2, …,k.
- RETURN X_n.

11. **Generation of first order autoregressive generalized logistic process**

- Generate a random variable X_0 from generalized logistic distribution.
- Generate k independent power function random variables, $V_1, V_2, \ldots, V_k$.
- Generate k independent logistic random variables, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$.
- Set $X_n = \min(X_{n-1} - \ln V_n, \varepsilon_n)$, for n = 1, 2, …,k.

- RETURN X_n.

12. Generation of first order autoregressive logistic process

- Generate k independent logistic random variables, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$.
- Set $X_0 \leftarrow \varepsilon_1$.
- Generate k independent power function random variables, $V_1, V_2, \ldots, V_k$.
- Set $X_n = \min(X_{n-1}, \varepsilon_n) - \ln V_n$, for n = 1, 2, …,k.
- RETURN X_n.

13. Generation of a discrete random variable using binomial thinning operator $\oplus$, that is, generation the random variable $Y = V \oplus X$ where 0 < V < 1

- Generate a random variable V in the interval [0, 1].
- Generate a binomial random variable Y with n = X and p = V.
- RETURN Y.

14. Generation of first order autoregressive discrete Linnik process

- Generate a random variable X_0 from discrete Linnik distribution.
- Generate k power function random variables, $V_1, V_2, \ldots, V_k$.
- Generate k discrete Mittag-Leffler random variables, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$.
- Set $X_n = V_n \oplus X_{n-1} + \varepsilon_n$ for n = 1, 2, …,k.
- RETURN X_n.

15. Generation of first order autoregressive discrete Mittag-Leffler process

- Generate k discrete Mittag-Leffler random variables, $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$.
- Set $X_0 \leftarrow \varepsilon_1$.
- Generate k power function random variables, $V_1, V_2, \dots, V_k$.
- Set $X_n = V_n \oplus X_{n-1} + V_n \oplus \varepsilon_n$, for n = 1, 2, ...,k.
- RETURN X_n.

In Table A.1a, the simulated numerical cumulative distribution function of SGL (σ, α, β) distribution is given for various values of α with β = 2 and σ = 2. In Table A.1b, the simulated numerical cumulative distribution function of SGL (σ, α, β) for various values of α, with β = 4 and σ = 2 is given.

X/α	0.200	0.400	0.600	0.800	1.000	1.200	1.400	1.600	1.800	2.000
0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
1	0.607	0.585	0.569	0.559	0.554	0.551	0.549	0.547	0.546	0.545
2	0.649	0.651	0.648	0.643	0.640	0.638	0.636	0.634	0.633	0.633
3	0.675	0.699	0.711	0.717	0.719	0.720	0.721	0.721	0.721	0.721
4	0.695	0.736	0.760	0.773	0.783	0.789	0.792	0.794	0.796	0.798
5	0.710	0.765	0.798	0.819	0.832	0.841	0.847	0.851	0.854	0.856
6	0.723	0.788	0.828	0.853	0.869	0.880	0.887	0.893	0.898	0.901
7	0.734	0.807	0.852	0.880	0.897	0.909	0.918	0.924	0.929	0.932
8	0.743	0.824	0.872	0.901	0.919	0.931	0.940	0.946	0.951	0.954
9	0.752	0.837	0.888	0.917	0.936	0.948	0.956	0.962	0.966	0.970
10	0.759	0.849	0.901	0.930	0.948	0.960	0.968	0.973	0.977	0.980
22	0.813	0.920	0.965	0.984	0.993	0.997	0.998	0.999	1.000	1.000
24	0.818	0.926	0.969	0.987	0.994	0.997	0.999	0.999	1.000	1.000
26	0.823	0.931	0.973	0.989	0.995	0.998	0.999	1.000	1.000	1.000
28	0.828	0.936	0.976	0.991	0.996	0.998	0.999	1.000	1.000	1.000
30	0.832	0.940	0.978	0.992	0.997	0.999	0.999	1.000	1.000	1.000
32	0.836	0.943	0.980	0.993	0.997	0.999	1.000	1.000	1.000	1.000
34	0.839	0.946	0.982	0.994	0.998	0.999	1.000	1.000	1.000	1.000
36	0.842	0.949	0.983	0.995	0.998	0.999	1.000	1.000	1.000	1.000
38	0.846	0.951	0.984	0.995	0.998	0.999	1.000	1.000	1.000	1.000
40	0.849	0.954	0.986	0.996	0.999	0.999	1.000	1.000	1.000	1.000
42	0.852	0.956	0.987	0.996	0.999	1.000	1.000	1.000	1.000	1.000
44	0.854	0.958	0.988	0.996	0.999	1.000	1.000	1.000	1.000	1.000
46	0.857	0.959	0.988	0.997	0.999	1.000	1.000	1.000	1.000	1.000
48	0.859	0.961	0.989	0.997	0.999	1.000	1.000	1.000	1.000	1.000
50	0.861	0.962	0.990	0.997	0.999	1.000	1.000	1.000	1.000	1.000
52	0.864	0.964	0.991	0.997	0.999	1.000	1.000	1.000	1.000	1.000
54	0.866	0.965	0.991	0.998	0.999	1.000	1.000	1.000	1.000	1.000
56	0.868	0.966	0.992	0.998	0.999	1.000	1.000	1.000	1.000	1.000
58	0.869	0.967	0.992	0.998	0.999	1.000	1.000	1.000	1.000	1.000
60	0.871	0.968	0.992	0.998	0.999	1.000	1.000	1.000	1.000	1.000
62	0.873	0.969	0.993	0.998	0.999	1.000	1.000	1.000	1.000	1.000
64	0.874	0.970	0.993	0.998	1.000	1.000	1.000	1.000	1.000	1.000
66	0.876	0.971	0.993	0.999	1.000	1.000	1.000	1.000	1.000	1.000
68	0.877	0.972	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000
70	0.879	0.973	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000
72	0.880	0.973	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000
74	0.881	0.974	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000
76	0.883	0.975	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000
78	0.884	0.975	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000
80	0.885	0.976	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000
82	0.886	0.976	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000
84	0.887	0.977	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000
86	0.888	0.977	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000
88	0.889	0.978	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000
500	0.948	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1000	0.962	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
5000	0.983	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
10000	0.988	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
900000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
90000000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table A.1a

Simulated numerical c.d.f. Table of SGL Distribution for various values of α with β = 2, σ = 2

X/α	0.200	0.400	0.600	0.800	1.000	1.200	1.400	1.600	1.800	2.000
0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
1	0.517	0.507	0.503	0.502	0.502	0.501	0.501	0.501	0.501	0.501
2	0.535	0.525	0.519	0.515	0.513	0.512	0.511	0.510	0.510	0.510
3	0.549	0.549	0.545	0.541	0.538	0.536	0.535	0.534	0.533	0.533
4	0.562	0.575	0.577	0.576	0.575	0.574	0.573	0.573	0.572	0.571
5	0.574	0.599	0.611	0.617	0.619	0.621	0.621	0.621	0.622	0.622
6	0.585	0.623	0.645	0.658	0.665	0.670	0.672	0.674	0.676	0.677
7	0.595	0.645	0.677	0.697	0.709	0.718	0.723	0.727	0.731	0.733
8	0.604	0.665	0.706	0.733	0.750	0.762	0.770	0.776	0.781	0.784
9	0.612	0.683	0.733	0.765	0.787	0.801	0.812	0.819	0.825	0.829
10	0.619	0.700	0.757	0.794	0.819	0.835	0.847	0.856	0.863	0.868
22	0.681	0.825	0.907	0.950	0.972	0.984	0.990	0.994	0.996	0.998
24	0.688	0.837	0.919	0.959	0.978	0.988	0.993	0.996	0.998	0.999
26	0.694	0.847	0.928	0.966	0.983	0.991	0.996	0.998	0.999	0.999
28	0.700	0.857	0.936	0.971	0.987	0.994	0.997	0.998	0.999	1.000
30	0.706	0.866	0.942	0.975	0.989	0.995	0.998	0.999	1.000	1.000
32	0.711	0.873	0.948	0.979	0.991	0.996	0.998	0.999	1.000	1.000
34	0.717	0.880	0.953	0.981	0.993	0.997	0.999	0.999	1.000	1.000
36	0.721	0.886	0.957	0.984	0.994	0.998	0.999	1.000	1.000	1.000
38	0.726	0.892	0.961	0.986	0.995	0.998	0.999	1.000	1.000	1.000
40	0.730	0.897	0.964	0.988	0.996	0.998	0.999	1.000	1.000	1.000
42	0.734	0.901	0.967	0.989	0.996	0.999	0.999	1.000	1.000	1.000
44	0.738	0.905	0.969	0.990	0.997	0.999	1.000	1.000	1.000	1.000
46	0.742	0.909	0.971	0.991	0.997	0.999	1.000	1.000	1.000	1.000
48	0.745	0.913	0.973	0.992	0.997	0.999	1.000	1.000	1.000	1.000
50	0.749	0.917	0.975	0.993	0.998	0.999	1.000	1.000	1.000	1.000
52	0.752	0.920	0.977	0.993	0.998	0.999	1.000	1.000	1.000	1.000
54	0.755	0.923	0.978	0.994	0.998	0.999	1.000	1.000	1.000	1.000
56	0.758	0.926	0.980	0.994	0.998	0.999	1.000	1.000	1.000	1.000
58	0.761	0.928	0.981	0.995	0.998	1.000	1.000	1.000	1.000	1.000
60	0.764	0.930	0.982	0.995	0.999	1.000	1.000	1.000	1.000	1.000
62	0.766	0.932	0.983	0.996	0.999	1.000	1.000	1.000	1.000	1.000
64	0.769	0.935	0.984	0.996	0.999	1.000	1.000	1.000	1.000	1.000
66	0.771	0.937	0.985	0.996	0.999	1.000	1.000	1.000	1.000	1.000
68	0.774	0.939	0.985	0.996	0.999	1.000	1.000	1.000	1.000	1.000
70	0.776	0.940	0.986	0.997	0.999	1.000	1.000	1.000	1.000	1.000
72	0.778	0.942	0.986	0.997	0.999	1.000	1.000	1.000	1.000	1.000
74	0.780	0.943	0.987	0.997	0.999	1.000	1.000	1.000	1.000	1.000
76	0.782	0.945	0.988	0.997	0.999	1.000	1.000	1.000	1.000	1.000
78	0.784	0.946	0.988	0.997	0.999	1.000	1.000	1.000	1.000	1.000
80	0.786	0.947	0.989	0.998	0.999	1.000	1.000	1.000	1.000	1.000
82	0.788	0.949	0.989	0.998	0.999	1.000	1.000	1.000	1.000	1.000
84	0.790	0.950	0.990	0.998	0.999	1.000	1.000	1.000	1.000	1.000
86	0.792	0.951	0.990	0.998	1.000	1.000	1.000	1.000	1.000	1.000
88	0.793	0.952	0.990	0.998	1.000	1.000	1.000	1.000	1.000	1.000
500	0.899	0.992	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1000	0.926	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
5000	0.965	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
10000	0.975	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400000	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
900000	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
90000000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table A.1b

Simulated numerical c.d.f. Table of SGL Distribution for various values of α with β = 4, $\sigma = 2$

Density plot of the symmetric generalized Linnik distribution for $\beta = 0,1,2$ with $\alpha = 1.5$ and $\sigma = 3$ is presented in Figure A.1. Comparison of symmetric generalized Linnik distribution for $\beta = 0,1,2$ (with $\alpha = 1.5$ and $\sigma = .8$) and standard normal distribution is presented in Figure A.2.

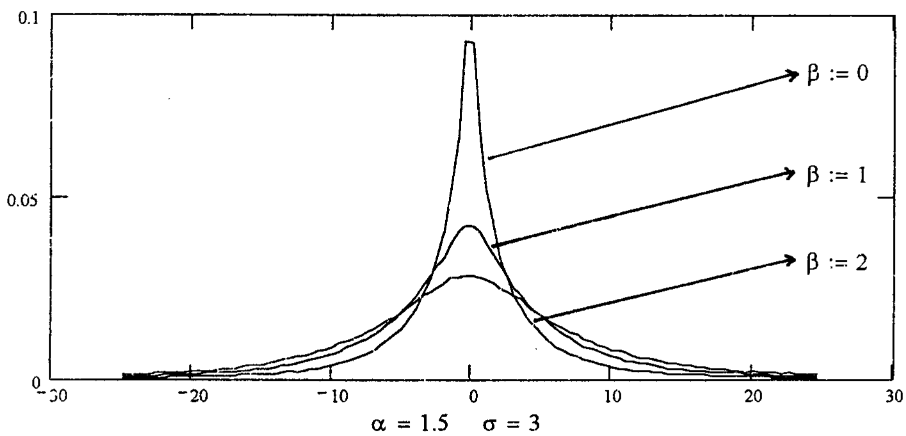

Figure A.1.

Density function of the symmetric generalized Linnik distribution for $\beta = 0$, 1 and 2 with $\alpha = 1.5$ and $\sigma = 3$

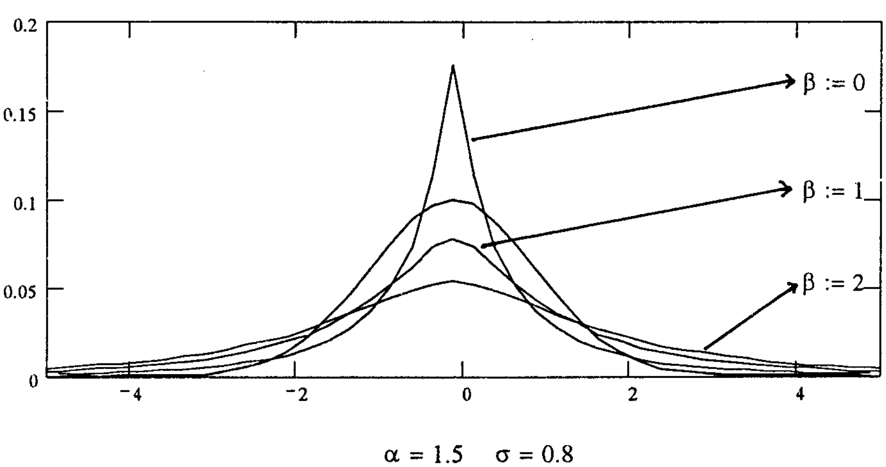

Figure A.2

Comparison of SGL(σ, α, β) distribution for $\beta = 0$, 1 and 2 ($\alpha = 1.5$ and $\sigma = 0.8$) and standard normal

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By

THOMAS MATHEW

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

KERALA, INDIA

Pin. 673 635

MAY, 2004

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